FOUR YEAR DEGREE COURSE

(CBCS in Accordance with NEP 2020)

SEMESTER I

CLASSICAL ALGEBRA AND TRIGONOMETRY

MATHEMATICS

Hiranya Chandra Bhuyan School of Science and Technology

KRISHNA KANTA HANDIQUI STATE OPEN UNIVERSITY

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June, 2026 ISBN: .....

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Printed and published by Registrar on behalf of the Krishna Kanta Handiqui State Open University.

The University acknowledges with thanks the financial support provided by the **Distance Education Bureau, UGC** for the preparation of this study material.

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COURSE INTRODUCTION

Classical Algebra and Trigonometry is the first Discipline-Specific Course (DSC) in the B.Sc. Mathematics (Honours) programme. The course consists of 14 units.

The course Classical Algebra and Trigonometry is designed to provide students with a strong foundation in fundamental algebraic concepts and advanced trigonometric techniques that are essential for higher studies in mathematics and its applications. This course systematically develops the theoretical and computational skills required to understand equations, matrices, determinants, and complex numbers, along with their wide-ranging applications in science, engineering, and mathematical analysis.

The course Classical Algebra and Trigonometry is structured to provide a systematic and comprehensive understanding of algebraic methods, matrix theory, and trigonometric concepts. Each unit is carefully designed to build conceptual clarity and computational skills, progressing from fundamental algebraic ideas to advanced applications in complex analysis and trigonometric series.

UNIT 1: Theory of Equations – I : This unit introduces the basic concepts of polynomial equations and their roots. Students learn the Factor Theorem, relationships between roots and coefficients, and symmetric functions of roots, which are essential tools in algebra. The method of synthetic division is also discussed to simplify polynomial computations efficiently.

UNIT 2: Theory of Equations – II : This unit extends the theory of equations by introducing advanced techniques such as Cardan's method for solving cubic equations. It also covers Descartes' Rule of Signs, which helps in determining the nature and number of roots of polynomial equations.

UNIT 3: Inequalities : In this unit, students study the concept of inequalities and the rules governing them. Important means such as Arithmetic Mean (AM), Geometric Mean (GM), and Harmonic Mean (HM) are introduced, along with their relationships, which are widely used in problem-solving and optimization.

UNIT 4: Determinants : This unit focuses on determinants of order 2 and 3. Students learn various properties of determinants and techniques for their evaluation, forming the foundation for solving systems of equations and understanding matrix theory.

UNIT 5: Matrices : This unit introduces matrices, their types, and fundamental operations such as addition, subtraction, and multiplication. Concepts like transpose, symmetric and skew-symmetric matrices, and adjoint and inverse of a matrix are discussed in detail.

UNIT 6: Adjoint and Inverse of a Matrix : This unit provides a deeper understanding of the adjoint and inverse of matrices, along with their computation. It also revisits transpose and special types of matrices, strengthening the foundation for solving matrix equations.

UNIT 7: Rank and Elementary Operations : Students learn about the rank of a matrix and how it is determined using elementary row operations. The unit also explains how to find the inverse of a matrix using these operations.

UNIT 8: System of Linear Equations : This unit deals with solving systems of linear equations using matrix methods. Techniques such as the matrix method and Cramer's Rule are introduced for efficient problem-solving.

UNIT 9: Eigenvalues and Eigenvectors : This unit introduces the concept of eigenvalues and eigenvectors through the characteristic equation of a matrix. It also covers their properties and the Cayley-Hamilton Theorem, which has important theoretical and practical applications.

UNIT 10: De Moivre's Theorem and Its Applications : Students are introduced to the complex number plane and De Moivre's Theorem, which is used to find powers and roots of complex numbers. This unit builds a bridge between algebra and trigonometry.

UNIT 11: Expansion of Trigonometric Functions : This unit focuses on the expansion of trigonometric functions in different forms. Students learn how to express trigonometric expressions in terms of series and other functions, which is useful in higher mathematics.

UNIT 12: Hyperbolic Functions : This unit introduces hyperbolic functions and their relationship with trigonometric functions. Topics include Euler's formula, expansions of hyperbolic functions, and properties such as periodicity and inverse functions.

UNIT 13: Logarithm of Complex Numbers : Students learn about the logarithm of complex quantities, including principal and general values. The unit also explains how to compute logarithms of complex numbers and their properties.

UNIT 14: Expansion and Summation of Trigonometric Series : The final unit deals with advanced topics such as Gregory's series and various methods for summing trigonometric series. Techniques involving binomial, exponential, and logarithmic series are discussed, enhancing analytical skills.

While going through a unit, you will notice some along-side boxes, which have been included to help you know some of the difficult, unseen terms. Again, we have included some relevant concepts in ''LET US KNOW'' along with the text. And, at the end of each section, you will get "CHECK YOUR PROGRESS" questions. These have been designed to self-check your progress of study. It will be better if you solve the problems put in these boxes immediately after you go through the sections of the units and then match your answers with "ANSWERS TO CHECK YOUR PPROGRESS" given at the end of each unit.

UNIT 1: THEORY OF EQUATIONS-I

UNIT STRUCTURE

1.1 Learning Objectives
1.2 Introduction
1.3 Polynomial Equation and its Roots
1.4 Factor Theorem
1.5 Relation between Roots and Coefficients
1.6 Symmetric Functions of Roots
1.7 Synthetic Division
1.8 Let Us Sum Up
1.9 Answers to Check Your Progress
1.10 Further Reading
1.11 Model Questions

1.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- know about factor theorem of a polynomial equation
- relate between roots and co-efficients of a polynomial equation
- define synthetic division for polynomial equation.

1.2 INTRODUCTION

In this unit, we will first introduce polynomial equation and factor theorem of a polynomial equation. We will also discuss relation between roots and coefficients of polynomial equation.

1.3 POLYNOMIAL EQUATION AND ITS ROOTS

Let f(x) be a polynominal of degree ≥ 1 whose co-efficients are real or complex numbers. Then f(x) = 0 is said to be a polynomial equation.

polynomial equation of degree 1 is $a_0x + a_1 = 0$, $a_0 \neq 0$

polynomial equation of degree 2 is $a_0x^2 + a_1x + a_2 = 0$, $a_0 \neq 0$

polynomial equation of degree 4 is $a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0$, $a_0 \neq 0$

polynomial equation of degree n is $a_0x^n + a_1x^{n-1} + \ldots\ldots + a_{n-1}x + a_n = 0$, $a_0 \neq 0$

Root: If α is a value of x such that $f(\alpha) = 0$, then α is said to be a root of the equation f(x) = 0.

1.4 FACTOR THEOREM

Theorem 1: If α is a root of the polynomial equation f(x) = 0 then the polynomial f(x) is divisible by $(x - \alpha)$ and conversely.

Proof: Let Q(x) be the quotient and R be the remainder when f(x) is divided by $(x-\alpha)$ Then,

$f(x) = (x-\alpha)\,Q(x) + R$ (1.1)

and putting $x = \alpha$ in (1.1),

one can have $f(\alpha) = R$

Since α is a root of $f(x) = 0$

$\Rightarrow f(\alpha) = 0$

$\Rightarrow R = 0$

$f(x)$ is divisible by $x - \alpha$

Conversely, if $f(x)$ is divisible by $(x - \alpha)$,

then $f(x) = (x - \alpha)\,Q(x)$.

So, putting $x = \alpha$,

we have $f(\alpha) = 0$ implies

$\Rightarrow \alpha$ is a root of the equation $f(x) = 0$

Theorem 2: Every equation of n^{th} degree has n roots and no more.

Proof: Let the polynomial equation of n^{th} degree be

$f(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n = 0$ (1)

Let α_1 be a root of (1).

So, $f(x) = (x - \alpha_1)\,Q_1(x)$ (2)

where $Q_1(x)$ is of degree $(n - 1)$.

Again, let α_2 be a root of $Q_1(x) = 0$

then, $Q_1(x) = (x - \alpha_2)\,Q_2(x)$ (3)

where $Q_2(x)$ is of degree (n-2).

By (2) and (3)

$f(x) = (x - \alpha_1)(x - \alpha_2)Q_2(x)$

Proceeding as above, we get

$f(x) = (x-\alpha_1)(x-\alpha_2) \ldots (x - \alpha_n)Q_n(x)$ (4)

where $Q_n(x)$ is of degree $(n - n) = 0$ i.e. $Q_n(x)$ does not contain x.

Comparing the coefficients of x^n in (1) and (4)

$Q_n(x) = 1$ (5)

$\therefore$ By (4) and (5)

$f(x) = (x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n)$.

Hence $f(x) = 0$ has n roots.

If possible, let β be another root.

Then $f(\beta) = (\beta-\alpha_1)(\beta-\alpha_2) \ldots (\beta - \alpha_n)$

Since β is assumed to be different from $\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n$, $f(\beta) \neq 0$

$\therefore \beta$ cannot be a root.

Hence, the equation can have only n roots.

1.5 RELATION BETWEEN ROOTS AND COEFFICIENTS

Let $f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n$ be a polynomial of degree n

where $a_0 \neq 0$

Let $\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n$ be the roots of the equation $f(x) = 0$.

Then, $f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n$

$= a_0(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n)$

$= a_0[x^n - \sum \alpha_1 x^{n-1} + \sum \alpha_1 \alpha_2 x^{n-2} \ldots\ldots + (-1)^n(\alpha_1 \alpha_2 \ldots \alpha_n)]$

where $\sum \alpha_1$ = sum of the roots and

$\sum \alpha_1 \alpha_2$ = sum of the products of the roots taken two at a time.

Equating the co-efficients of the various powers of x on both side–

$a_1 = a_0(-\sum \alpha_1)$ i.e., $\sum \alpha_1 = -\frac{a_1}{a_0}$

$a_2 = a_0 \sum \alpha_1 \alpha_2$ i.e., $\sum \alpha_1 \alpha_2 = \frac{a_2}{a_0}$

$a_3 = a_0 \sum \alpha_1 \alpha_2 \alpha_3$ i.e., $\sum \alpha_1 \alpha_2 \alpha_3 = -\frac{a_3}{a_0}$

In this way

$a_n = a_0(-1)^n \alpha_1 \alpha_2 \alpha_3 \cdots \alpha_n$ i.e., $\alpha_1 \alpha_2 \alpha_3 \cdots \alpha_n = (-1)^n \frac{a_n}{a_0}$

Above relations represent the relation between roots and coefficients of a polynomial equation.

Note: 1) Polynomial equation of degree $n \geq 1$ is

$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots + a_{n-1}x + a_n = 0$, $a_0 \neq 0$

2) If α be a root of f(x) =0; then

i) $f(\alpha) = 0$

ii) f(x) is exactly divisible by $(x-\alpha)$

iii) If $\alpha_1, \alpha_2, \alpha_3, \cdots \alpha_n$ be the roots of $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots + a_{n-1}x + a_n = 0$ then

$$\sum \alpha_1 = -\frac{a_1}{a_0}$$

$$\sum \alpha_1 \alpha_2 = (-1)^2 \frac{a_2}{a_0}$$

$$\sum \alpha_1 \alpha_2 \alpha_3 = (-1)^3 \frac{a_3}{a_0}$$

.....

$$\alpha_1 \alpha_2 \alpha_3, \cdots \alpha_n = (-1)^n \frac{a_n}{a_0}$$

Particular Case: Relation between the roots and co efficients of quadratic, cubic and biquadratic equations are deduced as follows:

Quadratic: For quadratic equation,

$a_0x^2 + a_1x + a_2 = 0$, $a_0 \neq 0$

$\sum \alpha_1 = \alpha + \beta = -\frac{a_1}{a_0}$

$\alpha_1 \alpha_2 = \alpha\beta = \frac{a_2}{a_0}$

Cubic: $a_0x^3 + a_1x^2 + a_2x + a_3 = 0, a_0 \neq 0$

$$\sum \alpha_1 = \alpha + \beta + \gamma = -\frac{a_1}{a_0}$$

$$\sum \alpha_1\alpha_2 = \alpha\beta + \beta\gamma + \gamma\alpha = \frac{a_2}{a_0}$$

$$\alpha_1\alpha_2\alpha_3 = \alpha\beta\gamma = -\frac{a_3}{a_0}$$

Biquadratic:

$a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0, a_0 \neq 0$

$$\sum \alpha_1 = \alpha + \beta + \gamma + \delta = -\frac{a_1}{a_0}$$

$$\sum \alpha_1\alpha_2 = \alpha\beta + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{a_2}{a_0}$$

$$\sum \alpha_1\alpha_2\alpha_3 = \alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \gamma\beta\delta = -\frac{a_3}{a_0}$$

$$\alpha_1\alpha_2\alpha_3\alpha_4 = \alpha\beta\gamma\delta = \frac{a_4}{a_0}$$

IIUSTRATIVE EXAMPLES:

Example 1: Solve $x^3 - 13x^2 + 15x + 189 = 0$ given that one root exceeds the other by 2.

Solution: Let the roots of the equation be $\alpha, \alpha + 2, \beta$.

$\therefore \sum \alpha = \alpha + \alpha + 2 + \beta = 13$

i.e., $2\alpha + 2 + \beta = 13$

i.e., $2\alpha + \beta = 11$

i.e., $\beta = 11 - 2\alpha$

and $\sum \alpha\beta = \alpha(\alpha + 2) + (\alpha + 2)\beta + \alpha\beta = 15$

i.e., $\alpha(\alpha + 2) + (\alpha + 2)(11 - 2\alpha) + \alpha(11 - 2\alpha) = 15$

i.e., $\alpha^2 + 2\alpha + 11\alpha - 2\alpha^2 + 22 - 4\alpha + 11\alpha - 2\alpha^2 = 15$

i.e., $-3\alpha^2 + 20\alpha + 7 = 0$

i.e., $3\alpha^2 - 20\alpha - 7 = 0$

i.e., $(3\alpha + 1)(\alpha - 7) = 0$

$$\therefore \alpha = -\frac{1}{3}, 7$$

When $\alpha = -\frac{1}{3}$, $\beta = 11 - 2 \times (-\frac{1}{3}) = 11 + \frac{2}{3} = 11\frac{2}{3}$

$\alpha = 7$, $\beta = 11 - 2 \times 7 = -3$

Now, the third relation is

$\alpha\beta\gamma = \alpha(\alpha + 2)\beta = -189$

i.e., $\alpha(\alpha + 2)(11 - 2\alpha) = -189$

This relation is satisfied only by the set of values

7, 7 + 2 = 9, 11 – 2x7= –3 i.e., 7, 9, –3.

Hence the roots of the equation are 7, 9, –3

Example 2: Solve the equation $16x^4 – 64x^3 + 56x^2 + 16x – 15 = 0$ whose roots are in arithmetic progression.

Solve: Let the roots of the equation be $\alpha – 3d, \alpha – d, \alpha + d, \alpha + 3d$, where 2d is the common difference.

$\sum\alpha = \alpha – 3d + \alpha – d + \alpha + d + \alpha + 3d = 4\alpha$

$i.e., 4\alpha = \frac{64}{16} = 4$

i.e., $\alpha = 1$ (1)

$\sum\alpha\beta = (\alpha – 3d)(\alpha – d) + (\alpha – d)(\alpha + d) + (\alpha + d)(\alpha + 3d) +$

$(\alpha – 3d)(\alpha + d) + (\alpha – 3d)(\alpha + 3d) + (\alpha – d)(\alpha + 3d) = \frac{56}{16}$

i.e., $\alpha^2 – 4\alpha d + 3d^2 + \alpha^2 – d^2 + \alpha^2 + 4\alpha d + 3d^2 + \alpha^2 – 2\alpha d – 3d^2$

$+ \alpha^2 – 9d^2 + \alpha^2 + 2\alpha d – 3d^2 = \frac{7}{2}$

i.e., $6\alpha^2 – 10d^2 = \frac{7}{2}$ (2)

By (1) and (2)

$6 \times 1 – 10d^2 = \frac{7}{2}$

i.e., $– 10d^2 = \frac{7}{2} – 6 = \frac{-5}{2}$

i.e., $d^2 = \frac{5}{20} = \frac{1}{4}$

Hence, $d = \pm\frac{1}{2}$

So, roots are $1–\frac{3}{2}, 1–\frac{1}{2}, 1+\frac{1}{2}, 1+\frac{3}{2}$, i.e, $–\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$

and $1+\frac{3}{2}, 1+\frac{1}{2}, 1–\frac{1}{2}, 1–\frac{3}{2}$, i.e, $\frac{5}{2}, \frac{3}{2}, \frac{1}{2}, –\frac{1}{2}$

Since both the sets are same, the required roots are– $\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$

CHECK YOUR PROGRESS

Q.1: Solve the equation $3x^3 –26x^2 + 52x – 24 = 0$, whose roots are in geometrical progression.

Q.2: Solve the equation $x^3 – 7x^2 – 5x + 75 =0$, two of its roots being equal.

Q.3: Solve the equation $x^3 – 7x^2 – 36 = 0$, if one root is double of another.

1.6 SYMMETRIC FUNCTION OF ROOTS

A function of the roots of an equation is said to be symmetric if it remains unaltered by an interchange of any two of its roots.

For example, if α, β, γ be the roots of a cubic equation then

$\sum\alpha = \alpha + \beta + \gamma$, $\sum\alpha^2 = \alpha^2 + \beta^2 + \gamma^2$, $\sum\alpha\beta = \alpha\beta + \beta\gamma + \gamma\alpha$ etc.

are symmetric function.

IIUSTRATIVE EXAMPLES:

Example 3: If α, β, γ be the roots of $x^3 - px^2 + qx - r = 0$ find the value of

i) $\sum\alpha^2$ ii) $\sum\frac{1}{\alpha}$ iii) $\sum\frac{1}{\alpha^2\beta^2}$

iv) $\sum\alpha^2\beta$ v) $\sum\frac{1}{\alpha^2}$ vi) $\sum\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right)$

Solution: Since α, β, γ are the roots of $x^3 - px^2 + qx - r = 0$

$\sum\alpha = \alpha + \beta + \gamma = -(-p) = p$

$\sum\alpha\beta = \alpha\beta + \beta\gamma + \gamma\alpha = q$

$\alpha\beta\gamma = -(-r) = r$

i) $\sum\alpha^2 = \alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) = p^2 - 2q$

ii) $\sum\frac{1}{\alpha} = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \gamma\alpha + \alpha\beta}{\alpha\beta\gamma} = \frac{q}{r}$

iii) $\sum\frac{1}{\alpha^2\beta^2} = \frac{1}{\alpha^2\beta^2} + \frac{1}{\beta^2\gamma^2} + \frac{1}{\gamma^2\alpha^2} = \frac{\gamma^2 + \alpha^2 + \beta^2}{\alpha^2\beta^2\gamma^2} = \frac{p^2 - 2q}{r^2}$ (By (i))

iv) $\sum\alpha\sum\alpha\beta$

$= (\alpha + \beta + \gamma)(\alpha\beta + \beta\gamma + \gamma\alpha)$

$= \alpha^2\beta + \alpha\beta\gamma + \alpha^2\gamma + \alpha\beta^2 + \beta^2\gamma + \alpha\beta\gamma + \alpha\beta\gamma + \beta\gamma^2 + \gamma^2\alpha$

$= \sum\alpha^2\beta + 3\alpha\beta\gamma$

$\therefore \sum\alpha^2\beta = \sum\alpha\sum\alpha\beta - 3\alpha\beta\gamma = pq - 3r$

v) $\sum\frac{1}{\alpha^2} = \left(\sum\frac{1}{\alpha}\right)^2 - 2\sum\frac{1}{\alpha\beta}$

$= \left(\frac{q}{r}\right)^2 - 2\frac{p}{r}$ [By (ii)]

$= \frac{q^2}{r^2} - 2\frac{p}{r} = \frac{q^2 - 2pr}{r^2}$

vi) $\sum\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right) = \sum\frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{\alpha^2 + \beta^2}{\alpha\beta} + \frac{\beta^2 + \gamma^2}{\beta\gamma} + \frac{\gamma^2 + \alpha^2}{\gamma\alpha}$

$= \frac{1}{\alpha\beta\gamma}\left(\alpha^2\gamma + \beta^2\gamma + \beta^2\alpha + \gamma^2\alpha + \gamma^2\beta + \alpha^2\beta\right)$

$$= \frac{1}{\alpha\beta\gamma}\{(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha)-3\alpha\beta\gamma\} = \frac{1}{r}(pq-3r)$$

Example 4: If α,β,γ are the roots of $x^3 + px^2 + qx + r = 0$, find the value of $(\beta+\gamma-\alpha)^2 + (\gamma+\alpha-\beta)^2 + (\alpha+\beta-\gamma)^2$

Solution: Since α, β, γ are the roots of the equation $x^3 + px^2 + qx + r = 0$

We have $\sum\alpha = \alpha + \beta + \gamma = -p$

$\sum\alpha\beta = \alpha\beta + \beta\gamma + \gamma\alpha = q$

$\alpha\beta\gamma = -r$

Now $\beta + \gamma - \alpha = \alpha + \beta + \gamma - 2\alpha = -p - 2\alpha = -(p + 2\alpha)$.

Similarly, $\gamma + \alpha - \beta = -(p + 2\beta)$

and $\alpha + \beta - \gamma = -(p + 2\gamma)$

$\therefore (\beta+\gamma-\alpha)^2 + (\gamma+\alpha-\beta)^2 + (\alpha+\beta-\gamma)^2$

$= (p + 2\alpha)^2 + (p + 2\beta)^2 + (p + 2\gamma)^2$

$= 3p^2 + 4p(\alpha + \beta + \gamma) + 4(\alpha^2 + \beta^2 + \gamma^2)$

$= 3p^2 + 4p(-p) + 4[(\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)]$

$= 3p^2 - 4p^2 + 4\,[(-p)^2 - 2q]$

$= 3p^2 - 4p^2 + 4p^2 - 8q$

$= 3p^2 - 8q.$

CHECK YOUR PROGRESS

Q.4: If α , β,γ are the roots of $x^3 - px^2 + qx - r = 0$ find the value of

i) $\sum\alpha^2\beta$ ii) $\sum\alpha^2\beta^2$ iii) $\sum\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right)$

Q.5: If α,β,γ are the roots of $x^3+qx+r=0$ find

i) $\sum\frac{2\beta\gamma-\alpha^2}{\beta+\gamma-\alpha}$ $ii)\sum\frac{1}{\alpha^2-\beta\gamma}$

Q.6: If α , β, γ, δ are the roots of $x^4 + px^3 + qx^2 + rx + s = 0$, find the values of

i) $\sum\alpha^2$ ii) $\sum\alpha^2\beta$ iii) $\sum\alpha^2\beta\gamma$

Q.7: If $\alpha_1,\alpha_2,\alpha_3,\cdots,\alpha_n$ are the roots of the equation

$x^n + p_1x^{n-1} + p_2x^{n-2} + ... + p_n = 0$ $(p_n \neq 0)$ find the value of

i) $\sum\frac{\alpha_1}{\alpha_2}$ ii) $\sum\frac{\alpha_1^2+\alpha_2^2}{\alpha_1\alpha_2}$

1.7 SYNTHETIC DIVISION

To find the quotient and remainder when a given polynomial is divided by a binomial easily, we may use synthetic division.

The method of synthetic division is discussed below.

Let f(x) be a polynomial of n th degree given by–

$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ...+ a_{n-1}x + a_n$ and the binomial is $x - h$.

When f(x) is divided by x–h, let the quotient be

$Q(x) = b_0 x^{n-1} + b_1x^{n-2} + ... + b_{n-1}$ and the remainder be R.

Then, $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n$

$= (x - h)(b_0x^{n-1} + b_1x^{n-2} + ...+ b_{n-1}) + R$

Comparing the coefficients of same powers of x in both sides

$a_0 = b_0$	$\Rightarrow b_0 = a_0$
$a_1 = b_1 - b_0.h$	$\Rightarrow b_1 = a_1 + b_0.h$
$a_2 = b_2 - b_1.h$	$\Rightarrow b_2 = a_2 + b_1.h$
.....	.....
$a_{n-1} = b_{n-1} - b_{n-2}.h$	$\Rightarrow b_{n-1} = a_{n-1} + b_{n-2}.h$
$a_n = R - b_{n-1}.h$	$\Rightarrow R = a_n + b_{n-1}.h$

The calculation of $b_0, b_1, b_2, ..., b_{n-1}$ and R are shown below in a very simple way.

h	a_0	a_1	a_2	... ...	a_{n-1}	a_n
		$(+)b_0h$	$(+)b_1h$	... ...	$(+)b_{n-2}.h$	$(+)b_{n-1}.h$
	$a_0 = b_0$	b_1	b_2	... ...	b_{n-1}	R

IIUSTRATIVE EXAMPLES:

Example 5: Find the value of $x^4 - 8x^2 + 8x + 6$ when x = 2.

Solution: The problem is equivalent to find the remainder when the polynomial $x^4 + 0x^3 - 8x^2 + 8x + 6$ is divided by x – 2

By synthetic division–

2	1	0	–8	8	6
		(+)2	(+)4	(–)8	(+)0
	1	2	–4	0	6 = R

Hence quotient is $Q(x) = x^3 + 2x^2 - 4x$

and the remainder is R = 6.

Note: When $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n$ is divided by (x–h), the coefficient of different terms of the quotient and the remainder R are–

$b_0 = a_0$

$b_1 = a_1 + b_0.h$

$b_2 = a_2 + b_1.h$

.....

$b_{n-1} = a_{n-1} + b_{n-2}.h$

and $R = a_n + b_{n-1}.h$

Example 6: Solve $x^3 - 5x^2 - 4x + 20 = 0$ given that the product of two roots is 10.

Solution: Let α, β, γ be the roots of the equation

$\alpha\beta\gamma = -20$ and $\alpha\beta = 10$ (given)

$\therefore \gamma = -2$

Hence x + 2 is a factor of $x^3 - 5x^2 - 4x + 20$.

By synthetic division,

–2	1	–5	–4	20
		–2	14	–20
	1	–7	10	0 = R

$\therefore Q(x) = x^2 - 7x + 10$

$= x^2 - 5x - 2x + 10$

= (x – 5)(x–2)

Now $x^2 - 7x + 10 = 0$

$\Rightarrow (x-5)(x-2) = 0$

$\therefore$ other roots are 2, 5.

1.8 LET US SUM UP

- Every equation of the nth degree has n roots and no more.
- If α is a root of the polynomial equation f(x) = 0, then by Factor theorem f(x) is exactly divisible by $(x - \alpha)$ and conversly.
- If $\alpha_1, \alpha_2, \cdots, \alpha_n$ be the roots of $a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n = 0$, then relation between roots and coefficients are given by

$$\sum \alpha_1 = -\frac{a_1}{a_0}, \sum \alpha_1 \alpha_2 = (-1)^2 \frac{a_2}{a_{01}}, \sum \alpha_1 \alpha_2 \alpha_3 = (-1)^3 \frac{a_3}{a_0}, , \cdots\cdots,$$

$$\alpha_1 \alpha_2 \cdots \alpha_n = (-1)^n \frac{a_n}{a_0}.$$

- A symmetric function of the roots of an equation is a function in which all the roots are involved alike,so that the expression remains unaltered when two of the roots are interchanged.
- We may use synthetic division to find the quotient and remainder when a given polynomial is divided by a binomial easily.

1.9 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1: Given equation is–

$f(x) = 3x^3 - 26x^2 + 52x - 24 = 0$

Since the roots of the equation are in G.P, we take the roots as $ar, a, \frac{a}{r}$.

$\therefore$ Product of the roots $= \alpha\beta\gamma = ar.a.\frac{a}{r} = \frac{24}{3}$

$\Rightarrow a^3 = 8 \quad \therefore a = 2$

Since a = 2 is a root of the equation, hence f(x) is divisible by x – 2.

By synthetic division–

2	3	–26	52	–24
		6	–40	–24
	3	–20	12	0 = R

$\therefore$ We get the quadratic equation–

$3x^2 - 20x + 12 = 0$

$\Rightarrow 3x^2 - 18x - 2x + 12 = 0$

$\Rightarrow (x - 6)(3x - 2) = 0$

$\therefore$ x = 6, $\frac{2}{3}$

Hence, roots of the equation are–

$\frac{2}{3}$, 2, 6 which are in G.P.

Ans. to Q. No. 2: Given equation is–

$x^3 - 7x^2 - 5x + 75 = 0$

Let the roots of the equation be α, α, β.

$\sum \alpha = \alpha + \alpha + \beta = 2\alpha + \beta = 7 \Rightarrow \beta = 7 - 2\alpha$

$\sum \alpha\beta = \alpha^2 + \alpha\beta + \beta\alpha = -5$

$\Rightarrow \alpha^2 + 2\alpha\beta = -5$

$\Rightarrow \alpha^2 + 2\alpha(7 - 2\alpha) + 5 = 0$

$\Rightarrow -3\alpha^2 + 14\alpha + 5 = 0$

$\Rightarrow 3\alpha^2 - 14\alpha - 5 = 0$

$\Rightarrow 3\alpha^2 - 15\alpha + \alpha - 5 = 0$

$\Rightarrow (\alpha - 5)(3\alpha + 1) = 0$

$\therefore \alpha = 5, \frac{1}{3}$

when $\alpha = 5, \beta = 7 - 2 \times 5 = -3$

when $\alpha = \frac{1}{3}, \beta = 7 - 2 \times \frac{1}{3} = \frac{19}{3}$

$\therefore$ roots are 5, 5, –3 and $\frac{1}{3}, \frac{1}{3}, \frac{19}{3}$.

Product of the roots = $\alpha\beta\gamma = \alpha.\alpha.\beta = \alpha^2\beta = -75$

Since $5 \times 5 \times (-3) = -75$ and $\frac{1}{3} \times \frac{1}{3} \times \frac{19}{3} \neq -75$

$\therefore$ Required roots of the equations are 5, 5, –3.

Ans. to Q. No. 3: Given equation is–

$x^3 - 7x^2 + 36 = 00$

Let the roots of the given equation be–

α, 2α, β.

$\therefore \sum \alpha\beta = \alpha.2\alpha + 2\alpha.\beta + \beta.\alpha = 0$

$\Rightarrow 2\alpha^2 + 3\alpha\beta = 0$

$\Rightarrow 2\alpha^2 + 3\alpha(7 - 3\alpha) = 0$

$\Rightarrow 2\alpha^2 + 21\alpha - 9\alpha^2 = 0$

$\Rightarrow -7\alpha^2 + 21\alpha = 0$

$\Rightarrow \alpha(21 - 7\alpha) = 0$

$\therefore \alpha = 0$ and 3.

when $\alpha = 0$, $\beta = 7 - 3 \times 0 = 7$

when $\alpha = 3$, $\beta = 7 - 3 \times 3 = -2$

$\therefore$ roots are 0, 0, 7 and 3, 6, –2.

Product of the roots = $\alpha.2\alpha.\beta = -36$

Since, $0 \times 0 \times 7 \neq -36$ and $3 \times 6 \times (-2) = -36$

$\therefore$ The required roots are 3, 6, –2

Ans. to Q. No. 4: Given equation is–

$$x^3 - px^2 + qx - r = 0$$

Let α, β, γ be its roots.

$\sum\alpha = p, \sum\alpha\beta = q, \alpha\beta\gamma = r$

i) $\sum\alpha \sum\alpha\beta = pq$

$\Rightarrow (\alpha + \beta + \gamma)(\alpha\beta + \beta\gamma + \gamma\alpha) = pq$

$\Rightarrow \sum\alpha^2\beta + 3\alpha\beta\gamma = pq$

$\Rightarrow \sum\alpha^2\beta + 3r = pq$

$\Rightarrow \sum\alpha^2\beta = pq - 3r.$

ii) $\sum\alpha^2\beta^2 = \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2$

$= (\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2(\alpha\beta^2\gamma + \beta\gamma^2\alpha + \gamma\beta\alpha^2)$

$= q^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma)$

$= q^2 - 2pr$

iii) $\sum\left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right)$

$$= \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{\beta}{\gamma} + \frac{\gamma}{\alpha} + \frac{\gamma}{\alpha} + \frac{\alpha}{\gamma}$$

$$= \frac{\alpha^2\gamma + \beta^2\gamma + \alpha\beta^2 + \alpha\gamma^2 + \beta\gamma^2 + \alpha^2\beta}{\alpha\beta\gamma}$$

$$= \frac{\alpha\beta(\alpha + \beta) + \beta\gamma(\beta + \gamma) + \gamma\alpha(\gamma + \alpha)}{r}$$

$$= \frac{\alpha\beta(p - \gamma) + \beta\gamma(p - \alpha) + \gamma\alpha(p - \beta)}{r}$$

$$= \frac{p(\alpha\beta + \beta\gamma + \gamma\alpha) - 3\alpha\beta\gamma}{r}$$

$$= \frac{pq - 3r}{r}$$

Ans. to Q. No. 5: Let α, β, γ be its roots of the equation $x^3 + qx + r = 0$

$\therefore \sum\alpha = 0, \sum\alpha\beta = q, \alpha\beta\gamma = -r$

i) $\frac{2\beta\gamma - \alpha^2}{\beta + \gamma - \alpha} = \frac{2\beta\gamma - (\beta + \gamma)^2}{\alpha + \beta + \gamma - 2\alpha}$

$$= \frac{2\beta\gamma - (\beta + \gamma)^2}{-2\alpha}$$

$$= \frac{\beta^2 + \gamma^2}{2\alpha}$$

$$= \frac{\alpha^2 + \beta^2 + \gamma^2 - \alpha^2}{2\alpha}$$

$$= \frac{(\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha) - \alpha^2}{2\alpha}$$

$$= \frac{-2q-\alpha^2}{2\alpha}$$

$$= -\frac{q}{\alpha} - \frac{1}{2}\alpha$$

$$\therefore \sum \frac{2\beta\gamma - \alpha^2}{\beta+\gamma-\alpha}$$

$$= \sum \frac{q}{\alpha} - \sum \frac{1}{2}\alpha$$

$$= q\left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right) - \frac{1}{2}(\alpha+\beta+\gamma)$$

$$= \frac{q(\beta\gamma+\gamma\alpha+\alpha\beta)}{\alpha\beta\gamma}$$

$$= \frac{q^2}{r}$$

ii) $\alpha \cdot \alpha - \beta\gamma = \alpha \cdot \alpha - \beta\gamma$
$= -\alpha(\beta+\gamma) - \beta\gamma \qquad \because \sum\alpha = \alpha+\beta+\gamma = 0$
$= -(\alpha\beta+\beta\gamma+\gamma\alpha) \qquad \therefore \alpha = -(\beta+\gamma)$
$= -\sum\alpha\beta$
$= -q$

$$\therefore \sum \frac{1}{\alpha^2 - \beta\gamma} = -\frac{1}{q}$$

Ans. to Q. No. 6: Given equation is–

$x^4 + px^3 + qx^2 + rx + s = 0$

$\alpha, \beta, \gamma, \delta$ be its roots

$\therefore \sum\alpha = -p, \sum\alpha\beta = q, \sum\alpha\beta\gamma = -r, \alpha\beta\gamma\delta = s$

i) $\sum\alpha^2 = \alpha^2 + \beta^2 + \gamma^2 + \delta^2$
$= (\alpha+\beta+\gamma+\delta)^2 - 2\sum\alpha\beta$
$= (\sum\alpha)^2 - 2\sum\alpha\beta$
$= (-p)^2 - 2q$
$= p^2 - 2q$

ii) $\sum\alpha\sum\alpha\beta = \sum\alpha^2\beta + 3\sum\alpha\beta\gamma$
$\Rightarrow \sum\alpha^2\beta = \sum\alpha\sum\alpha\beta - 3\sum\alpha\beta\gamma$
$= (-p)q - 3(-r)$
$= -pq + 3r$

iii) $\sum\alpha\sum\alpha\beta\gamma = \sum\alpha^2\beta\gamma + 4\alpha\beta\gamma\delta$
$\Rightarrow \sum\alpha^2\beta\delta = \sum\alpha\sum\alpha\beta\delta - 4\alpha\beta\gamma\delta$
$= (-p)(-r) - 4s$
$= pr - 4s$.

Ans. to Q. No. 7: Given equation is–

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots\ \ldots\ \ldots + p_n = 0.$$

$\alpha_1, \alpha_2, \ldots\ \ldots\ \ldots\ \alpha_n$ be its roots.

$$\Sigma\alpha_1 = -p_1,\ \Sigma\alpha_1\alpha_2$$
$$= p_2,\ \Sigma\alpha_1\alpha_2\alpha_3$$
$$= -p_3,\ \ldots\ \ldots,\ \alpha_1, \alpha_2 \ldots\ \ldots \alpha_n$$
$$= (-1)^n p_n.$$

i) $$\Sigma\frac{\alpha_1}{\alpha_2} = \Sigma\alpha_1\Sigma\frac{1}{\alpha_1} - n$$

$$= -p_1\frac{\Sigma\alpha_1\alpha_2\ldots\alpha_{n-1}}{\alpha_1\alpha_2\ldots\alpha_n} - n$$

$$= -p_1\frac{(-1)^{n-1}p_{n-1}}{(-1)^n p_n} - n$$

ii) $$\Sigma\frac{\alpha_1^2 + \alpha_2^2}{\alpha_1\alpha_2}$$

$$= \Sigma\left(\frac{\alpha_1}{\alpha_2} + \frac{\alpha_2}{\alpha_1}\right)$$

$$= \alpha_1\left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \ldots\ \ldots\ \ldots + \frac{1}{\alpha_n}\right) - 1 + \alpha_2\left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \ldots\ \ldots\ \ldots + \frac{1}{\alpha_n}\right) - 1$$

$$+ \ldots\ \ldots\ \ldots + \alpha_n\left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \ldots\ldots\ldots + \frac{1}{\alpha_n}\right) - 1$$

$$= \Sigma\alpha_1\Sigma\frac{1}{\alpha_1} - n$$

$$= \frac{p_1p_{n-1}}{p_n} - n$$

1.10 FURTHER READING

- University Algebra, N S Gopala Krishnan, New Age International (P) Limited.
- Linear Algebra: Schum Solved Problems Series; Seymour Lipschitz: Tata McGraw Hill.
- Text Book of Algebra, A.K Sharma, DPH, Mathematics Series.

1.11 MODEL QUESTIONS

Q.1: Solve the equation $x^3 - 3x^2 - 6x + 8 = 0$ such that the roots of the equation are in arithmetic progression.

Q.2: Solve the equation $3x^3 + 14x^2 + 17x + 6 = 0$ given that two roots α and β are

connected by the relation $2\alpha + \beta = 1$

Q.3: Find the condition that the cubic $x^3 - px^2 + qx - r = 0$ should have roots in G.P.

Q.4: Find the equation whose roots are the squares of the roots of $x^3 - x^2 + 8x - 6 = 0$

Q.5: Calculate the values of the following symmetric function for the cubic $x^3 + px^2 + qx + r = 0$ whose roots are α, β, γ.

i) $\sum \alpha^2$ ii) $\sum \alpha^3$ iii) $\sum \alpha^2\beta$

Q.6: Express $\sum \alpha\beta$ and $\alpha\beta\gamma$ in terms of p, q, and r where $\sum \alpha = p$, $\sum \alpha^2 = q$ and $\sum \alpha^3 = r$.

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UNIT 2: THEORY OF EQUATIONS-II

UNIT STRUCTURE

2.1 Learning Objectives
2.2 Introduction
2.3 Cardan's Method of Solution of Cubic Equation
2.4 Descartes' Rule of Sign
2.5 Let Us Sum Up
2.6 Answers to Check Your Progress
2.7 Further Reading
2.8 Model Questions

2.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- learn about Cardan's method of solution of cubic equation
- Descartes' rule of sign.

2.2 INTRODUCTION

In this unit, we will discuss Cardan's method for solution of a cubic equation and Descartes' rule of sign.

2.3 CARDAN'S METHOD OF SOLUTION OF CUBIC EQUATION

Let the general cubic equation be

$a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0, a_0 \neq 0$ (1)

and let the roots of this equation be α, β and γ.

To diminish the roots of (i) by h, let

$y = x - h$, i.e. $x = y + h$

Putting the value of x in (i), we get

$a_0(y+h)^3 + 3a_1(y+h)^2 + 3a_2(y+h) + a_3 = 0$

$\Rightarrow a_0y^3 + 3(a_0h + a_1)y^2 + 3(a_0h^2 + 2a_1h + a_2)y$

$+ (a_0h^3 + 3a_1h^2 + 3a_2h + a_3) = 0$

Which is in the form $A_0y^3 + 3A_1y^2 + 3A_2y + A_3 = 0$

Choosing h so that the 2nd term be removed, we must have

$A_1 = 0$, i.e. $a_0h + a_1 = 0$

$\Rightarrow h = \frac{-a_1}{a_0}$

Then, $A_2 = a_0h^2 + 2a_1h + a_2 = a_0(\frac{-a_1}{a_0})^2 + 2a_1(\frac{-a_1}{a_0}) + a_2$

$$= \frac{a_0a_2 - a_1^2}{a_0} = \frac{H}{a_0}$$

$A_3 = a_0h^3 + 3a_1h^2 + 3a_2h + a_3$

$$= a_0(\frac{-a_1}{a_0})^3 + 3a_1(\frac{-a_1}{a_0})^2 + 3a_2(\frac{-a_1}{a_0}) + a_3$$

$$= \frac{a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3}{a_0^2} = \frac{G}{a_0^2}$$

Hence the transformed equation is–

$$a_0y^3 + 3\frac{H}{a_0}y + \frac{G}{a_0^2} = 0$$

$$\Rightarrow y^3 + \frac{3H}{a_0^2}y + \frac{G}{a_0^3} = 0 \qquad (2)$$

The roots of the equation (2) are–

$\alpha - h, \beta - h, \gamma - h$

i.e. $\alpha + \frac{a_1}{a_0}, \beta + \frac{a_1}{a_0}, \gamma + \frac{a_1}{a_0}$

i.e. $\frac{a_0\alpha + a_1}{a_0}, \frac{a_0\beta + a_1}{a_0}, \frac{a_0\gamma + a_1}{a_0}$

If the roots of this equation (2) be multiplied by a_0, the transformed equation will be–

$$z^3 + 0.a_0z^2 + a_0^2\frac{3H}{a_0^2}z + a_0^3\frac{G}{a_0^3} = 0$$

$$\Rightarrow z^3 + 3Hz + G = 0 \qquad (3)$$

whose roots are–

$$a_0(\frac{a_0\alpha + a_1}{a_0}), a_0(\frac{a_0\beta + a_1}{a_0}), a_0(\frac{a_0\gamma + a_1}{a_0})$$

i.e. $a_0\alpha + a_1,\ a_0\beta + a_1,\ a_0\gamma, + a_1$

Hence, $z = a_0y = a_0x + a_1$

Let us assume that $z = u + v$

Cubing both sides,

$$z^3 = (u+v)^3 = u^3 + v^3 + 3uv(u+v) = u^3 + v^3 + 3uvz$$

$$\Rightarrow z^3 - 3uvz - (u+v)^3 = 0 \qquad (4)$$

Comparing (3) and (4)–

$$uv = -H \Rightarrow u^3v^3 = -H^3 \qquad (5)$$

and $u^3 + v^3 = -G \qquad (6)$

Form (5) and (6), it is seen that u^3 and v^3 are the roots of the quadratic equation–

$t^2 + Gt - H^3 = 0$ (7)

$$\therefore t_1 = u^3 = \frac{-G + \sqrt{G^2 + 4H^3}}{2} \quad (8)$$

and $$t_2 = v^3 = \frac{-G - \sqrt{G^2 + 4H^3}}{2} \quad (9)$$

From (8) and (9), we will get three values of u and three values of v. Since the root of the equation in $u + v$ will be obtained from three values of u i.e,. u, $u\omega$, $u\omega^2$ and three values of v. i.e, v, $v\omega$, $v\omega^2$.

Only three values of $u + v$ are the roots of the equation. Six values will be eliminated by applying the relation $uv = -H$.

Hence, the required roots of (3) are $u + v, u\omega + v\omega^2, u\omega^2 + v\omega$ where ω is the cube root of unity.

Having found z = $u + v$, we can easily find x by the relation $z=a_0x+a_1$.

Note: To find the roots of the cubic– $a_0x^3 + 3a_1x^2 + 3a_2x + a^3 = 0$, removal of 2nd term is necessary. Removing the 2nd term, put z = $u + v$ out of which only three values are the actual roots. Other six values of $u + v$ will be eliminated if $uv = -H$.
Finding z = $u + v$ find the values of x by the relation $z = a_0x + a_1$.

Example 1: Solve: $x^3 - 9x - 28 = 0$

Solution: The given equation is a cubic equation whose 2nd term is eliminated.

Putting $x = u + v$

$$x^3 = (u+v)^3 = u^3 + v^3 + 3uv(u+v) = u^3 + v^3 + 3uvx$$

$$\Rightarrow x^3 - 3uvx - (u+v)^3 = 0 \quad (1)$$

Comparing the co efficients of same powers of x for the given equation and (1)

$-3uv = -9$ and

$\Rightarrow uv = 3$ (2) $\quad -(u^3 + v^3) = -28$

$\Rightarrow u^3v^3 = 27$ $\quad \Rightarrow (u^3 + v^3) = 28$ (3)

From (2) and (3), i.e. it is obvious that u^3 and v^3 are the roots of the quadratic equation–

$t^2 - 28t + 27 = 0$

$\Rightarrow (t - 1)(t - 27) = 0$

$\therefore u^3 = 1$ and $v^3 = 27$

$\Rightarrow u = 1, \omega, \omega^2$ and $v = 3, 3\omega, 3\omega^2$

$\therefore x = 1+3, \omega+3\omega^2, \omega^2+3\omega$

i.e, $4, \omega+3\omega^2, \omega^2+3\omega$

Example 2: Solve: $x^3 - 15x^2 - 33x + 847 = 0$

Solution: The given equation is $x^3 - 3 \times 5x^2 - 3 \times 11x + 847 = 0$

Here $a_0 = 1$, $a_1 = -5$, $a_2 = -11$, $a_3 = 847$.

$$\therefore h = \frac{-a_1}{a_0} = \frac{-(-5)}{1} = 5$$

Diminishing the roots of the equation by 5 by synthetic division, we get the

transformed

5	1	–15	–33	847
		5	–50	–415
	1	–10	–83	–432
		5	–25	
	1	–5	–108	
		5		
	1	0		

Equation becomes:

$$z^3 - 108z + 432 = 0 \qquad (1)$$

Let $z = u + v$

$$\therefore z^3 = (u+v)^3 = u^3 + v^3 + 3uv(u+v)$$

$$= u^3 + v^3 + 3uvz$$

$$\Rightarrow z^3 - 3uvz - (u^3 + v^3) = 0 \qquad (2)$$

Comparing (1) and (2)–

$-3uv = -108$ and $-(u^3 + v^3) = 432$

i.e, $uv = 36$ and $u^3 + v^3 = -432$

i.e, $u^3 v^3 = (36)^3$

$\therefore u^3$ and v^3 are the roots of

$$t^2 + 432t + (36)^3 = 0$$

$$\Rightarrow t^2 + 2.t.6^3 + (6^3)^2 = 0$$

$$\Rightarrow (t + 6^3)^2 = 0$$

Giving $u^3 = -6^3$, $v^3 = -6^3$

$\therefore u = -6$ $\therefore v = -6$

and hence $x = u + v = -6 - 6 = -12$ is as root of the cubic equation.

By synthetic division–

–12	1	0	–108	432
		–12	144	–432
	1	–12	36	0

$\therefore z^3 - 108z + 432 = 0$

$\Rightarrow (z + 12)(z^2 - 12z + 36) = 0$

$\Rightarrow (z + 12)(z-6)^2 = 0$

$\Rightarrow z = -12, 6, 6$

$\therefore$ Roots of z, are –12, 6, 6 and hence roots of x are –12 + 5, 6 + 5, 6 + 5 i.e. –7, 11, 11.

CHECK YOUR PROGRESS

Q.1: Solve:

a) $x^3 + 6x^2 + 9x + 4 = 0$

b) $x^3 - 12x - 344 = 0$

c) $x^3 - 27x - 54 = 0$

d) $x^3 + x^2 - 16x + 20 = 0$

2.4 DESCARTES RULE OF SIGN

Descartes' rule of sign gives the information regarding the character and position of the roots of an equation

Let f(x) = 0 be an equation with real coefficients.

Positive Roots: An equation f(x) = 0 with real co efficients cannot have more positive roots than the number of changes of signs from +ve to –ve or from –ve to +ve.

Negative Roots: An equation f(x)=0 cannot have more negative roots than the number of changes of sign in f(–x) = 0.

Example 9: Apply Descartes' rule of sign to examine the nature of roots of the equation–

$x^6 + x^4 + x^2 + x + 3 = 0$

Solve: Let $f(x) = x^6 + x^4 + x^2 + x + 3$

$\therefore f(-x) = x^6 + x^4 + x^2 - x + 3$

Since f(x) has no change of sign, it has no positive root.

f(-x) has two changes of sign.So,the number of negative roots of f(x) = 0 is either 2 or 0.

The equation being of sixth degree, it has 6 roots.

$\therefore$ The number of complex root is either 4 or 6.

Example 10: Find the minimum number of complex roots of the equation–

$2x^7 - x^4 + 4x^3 - 5 = 0$.

Solve: Let $f(x) = 2x^7 - x^4 + 4x^3 - 5$

f(x) has three changes of signs.

$\therefore f(x) = 0$ cannot have more than three possitive roots.

Again $f(-x) = -2x^7 - x^4 - 4x^3 - 5$. f(–x) has no change of sign.

$\therefore f(x) = 0$ will have no –ve root.

Hence, f(x) = 0 will have at most three positive real roots and the minimum number complex roots is 7 – 3 = 4

Note: 1) If the signs of the terms of an equation are all +ve, then the equation cannot have a +ve root.

2) If the equation is of even powers of x and if all the co efficients have +ve signs, it cannot have any real root.

3) If the equation is of odd powers of x and if all the terms are +ve, it has the root zero and no other real root.

4) Descartes rule of sign gives the maximum number of +ve or –ve roots of an equation and not the actual number.

CHECK YOUR PROGRESS

Q.2: Apply Descartes' rule of signs to examine the nature of roots of–

i) $x^4 + 2x^2 + 3x - 1 = 0$

ii) $x^5 + x^4 - 6x^3 - 8x^2 - 15x - 10 = 0$

iii) $x^{10} - 1 = 0$

Q.3: Find an upper limit of the real roots of the equation–

i) $x^4-5x^3+40x^2-8x+24=0$

ii) $x^4+4x^3-11x^2-9x-50=0$

Q.4: Apply Descartes' rule of signs to ascertain the minimum number of complex roots of–

i) $x^6 - 3x^2 - 2x - 3 = 0$

ii) $x^7 - 3x^3 + x^2 = 0$

where a, b, c are positive integer.

2.5 LET US SUM UP

- Cardan's method is used to solve a cubic equation.
- Descartes' rule of sign gives the information regarding the character and position of the roots of an equation.

2.6 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1: a) Given equation is–

$$x^3 + 6x^2 + 9x + 4 = 0 \qquad (1)$$

$\Rightarrow x^3 + 3.2x^2 + 3.3x + 4 = 0$

$a_0 = 1, a_1 = 2, a_2 = 3, a_3 = 4$

$$\therefore h = -\frac{a_1}{a_0} = -\frac{2}{1} = -2$$

Diminishing the roots of (1) by –2, we get by synthetic division–

–2	1	6	9	4
		–2	–8	–2
	1	4	1	$2 = A_4$
		–2	–4	
	1	2	$-3 = A_3$	
		–2		
	1	$0 = A_2$		
	$1 = A_1$			

$\therefore$ the transformed equation is–

$$z^3 - 3z + 2 = 0 \qquad (2)$$

Whose roots are z = x – h

Putting $z = u + v$

$$z^3 = (u+v)^3 = u^3 + v^3 + 3uv(u+v)$$
$$= u^3 + v^3 + 3uvz$$
$$\Rightarrow z^3 - 3uvz - (u^3 + v^3) = 0 \qquad (3)$$

Comparing (2) and (3)

$-3uv = -3 \quad \Rightarrow uv = 1$

$\therefore u^3v^3 = 1 \qquad (4)$

$-(u^3 + v^3) = 2 \quad \Rightarrow u^3 + v^3 = -2 \qquad (5)$

By (4) and (5), u^3 and v^3 are the roots of the quadratic equation.

$t^2 + 2t + 1 = 0$

$\Rightarrow (t+1)^2 = 0$

$\therefore t = -1, -1$

i.e. $u^3 = -1$ and $v^3 = -1$

$\therefore u = -1 \qquad \therefore v = -1$

$\therefore z = u + v = -1 - 1 = -2$ is a root of (2)

$\therefore z + 2$ is a factor of L.H.S. of (2)

$(z+2)(z^2 - 2z + 1) = 0$

$z = -2, 1, 1$

$\therefore x = z + h = z - 2 = -4, -2, -1$

b) Given equation is–

$x^3 - 21x - 344 = 0 \qquad (1)$

Let $x = u + v$

$\therefore x^3 = (u+v)^3 = u^3 + v^3 + 3uv(u+v)$
$= u^3 + v^3 + 3uvx$

$\Rightarrow x^3 - 3uvx - (u^3 + v^3) = 0 \qquad (2)$

Comparing (1) and (2)

$-3uv = -21 \Rightarrow uv = 7$

$\therefore u^3v^3 = 7^3 \qquad (3)$

and $-(u^3 + v^3) = -344$

$\Rightarrow u^3 + v^3 = 344 \quad (4)$

By (3) and (4), u^3 and v^3 are the roots of the quadratic equation.

$t^2 - 344t + 7^3 = 0$

$\Rightarrow (t - 343)(t - 1) = 0$

$\therefore t = 1, 343$

i.e. $u^3 = 1$ and $v^3 = 343 = 7^3$

$\Rightarrow u = 1 \qquad \Rightarrow v = 7$

$\therefore x = u + v = 1 + 7 = 8$ is a root of (1)

$\therefore x - 8$ is a factor of L.H.S. of (1)

$\therefore (x - 8)(x^2 + 8x + 43) = 0$

$\Rightarrow x^2 + 8x + 43 = 0$

$$\therefore x = \frac{-8 \pm \sqrt{64 - 172}}{2}$$

$$= \frac{-8 \pm \sqrt{-108}}{2}$$

$$= -8 \pm 6\sqrt{3}\,i$$

$\therefore$ roots of (1) are 8, –8 $\pm 6\sqrt{3}$ i

c) Given equation is–

$x^3 - 27x - 54 = 0$ (1)

Putting $x = u + v$

$\therefore x^3 = (u+v)^3 = u^3 + v^3 + 3uv(u+v) = u^3 + v^3 + 3uvx$

$\Rightarrow x^3 - 3uvx - (u^3 + v^3) = 0$ (2)

Comparing (1) and (2)

$-3uv = -27 \Rightarrow uv = 9 \quad \therefore u^3v^3 = 9^3$ (3)

$-(u^3 + v^3) = -54 \quad \Rightarrow u^3 + v^3 = 54$ (4)

By (3) and (4) u^3 and v^3 are the roots of the quadratic equation.

$t^2 - 54t + 9^3 = 0$

$\Rightarrow t^2 - 2 \times 27t + (27)^2 = 0$

$\Rightarrow (t-27)^2 = 0$

$\therefore$ t = 27, 27

$\therefore u^3 = 27 = 3^3$ and $v^3 = 27 = 3^3$

$\therefore u = 3 \quad \therefore v = 3$

$\therefore$ x = $u + v$ = 3 + 3 = 6 is a root of (1) i.e. x – 6 is a factor of the L.H.S. of (1)

$\therefore (x-6)(x^2 + 6x + 9) = 0$

$\Rightarrow x^2 + 6x + 9 = 0$

$\Rightarrow (x+3)^2 = 0$

$\therefore$ x= –3, –3

$\therefore$ Root of (1) are 6, –3, –3.

d) Given equation is–

$x^3 + x^2 - 16x + 20 = 0$ (1)

Here, $a_0 = 1$, $a_1 = 1$, $a_2 = -16$, $a_3 = 20$

To remove the 2nd term, we put

$na_0h + a_1 = 0$

$\Rightarrow 3 \times 1 \times h + 1 = 0$

$\Rightarrow h = -\frac{1}{3}$

By synthetic division–

$-\frac{1}{3}$	1	1	–16	20
		$-\frac{1}{3}$	$-\frac{2}{9}$	$\frac{146}{27}$
	1	$\frac{2}{3}$	$-\frac{146}{9}$	$\frac{686}{27}$
		$\frac{1}{3}$	$\frac{1}{9}$	
	1	$\frac{1}{3}$	$\frac{147}{9}$	
		$-\frac{1}{3}$		
	1	0		

$\therefore$ The transformed equation is–

$$z^3 - \frac{147}{9}z + \frac{686}{27} = 0 \qquad (2)$$

$$\Rightarrow z^3 - \frac{49}{3}z + \frac{686}{27} = 0$$

Let $z^3 = (u+v)^3 = u^3 + v^3 + 3uv(u+v) = u^3 + v^3 + 3uvz$

$$\Rightarrow z^3 - 3uvz - (u^3 + v^3) = 0 \qquad (3)$$

Comparing (2) and (3)

$$-3uv = -\frac{49}{3} \Rightarrow uv = \frac{49}{9}$$

$$\therefore u^3v^3 = \left(\frac{49}{9}\right)^3 \qquad (4)$$

$$-(u^3 + v^3) = \frac{686}{27}$$

$$\therefore u^3 + v^3 = -\frac{686}{27} \qquad (5)$$

By (4) and (5) u^3 and v^3 are the roots of

$$t^2 + \frac{686}{27}t + \left(\frac{49}{9}\right)^3 = 0$$

$$\Rightarrow t^2 - 2.\frac{343}{27}t + \left(\frac{73}{33}\right)^2 = 0 \Rightarrow \left(t + \frac{7^3}{3^3}\right)^2 = 0$$

$$\therefore t = -\left(\frac{7}{3}\right)^3, -\left(\frac{7}{3}\right)^3$$

i.e. $u^3 = -\left(\frac{7}{3}\right)^3$ and $v^3 = -\left(\frac{7}{3}\right)^3$

$$\therefore u = -\frac{7}{3} \qquad \therefore v = -\frac{7}{3}$$

$$\therefore z = u + v = -\frac{7}{3} - \frac{7}{3} = -\frac{14}{3}$$

$\therefore z + \frac{14}{3}$ is a factor of L.H.S. of (2)

$$\left(z + \frac{14}{3}\right)\left(z^2 - \frac{14}{3}z + \frac{49}{9}\right) = 0$$

$$\Rightarrow z^2 - \frac{14}{3}z + \frac{49}{9} = 0$$

$$\Rightarrow \left(z - \frac{7}{3}\right)^3 = 0$$

$\therefore z = \frac{7}{3}, \frac{7}{3}$

$\therefore$ roots of (2) are

$z = -\frac{14}{3}, \frac{7}{3}, \frac{7}{3}$

$\therefore x = z + h = z - \frac{1}{3} = \frac{-15}{3}, \frac{6}{3}, \frac{6}{3}$ i.e. - 5, 2, 2.

i.e. roots of (1) are – 5, 2, 2.

Ans. to Q. No. 2: i) Given equation is–

$x^4 + 2x^2 + 3x - 1 = 0$

Let $f(x) = x^4 + 2x^2 + 3x - 1 = 0$

It has only one change of sign and hence the equation has at most one +ve root.

$f(-x) = x^4 + 2x^2 - 3x - 1 = 0$

has only one change of sign and hence f(x) has at most one –ve root.

Again $f(0) \neq 0$ i.e. the equation has no zero root.

$\therefore$ the maximum number of real roots is 2.

Since the equation is of degree 4, it has 4 roots. Hence the minimum number of complex root ,of the equation is 4 – 2 = 2

ii) The given equation is –

$x^5 + x^4 - 6x^3 - 8x^2 - 15x - 10 = 0$

Let $f(x) = x^5 + x^4 - 6x^3 - 8x^2 - 15x - 10 = 0$.

It has only one change of sign and hence the equation has one +ve root. $f(-x) = -x^5 + x^4 + 6x^3 - 8x^2 + 15x - 10 = 0$. It has 4 changes of sign and hence the given equation has at most four –ve roots.

$\because$ $f(0) \neq 0$ i.e. the equation has no zero root.

$\therefore$ the maximum number of real roots is 1+4=5.

iii) The given equation is–

$x^{10} - 1 = 0$

Let $f(x) = x^{10} - 1 = 0$

The equation has only one change of sign.

$\therefore$ It has at most one +ve root.

$f(-x) = x^{10} - 1 = 0$

It has only one change of sign.

$\therefore$ the given equation has atmost one –ve root.

$\because$ $f(0) \neq 0$, The equation has no zero root and hence, the equation has 1+1=2 real roots.

$\therefore$ Other 8 roots are complex roots.

Ans. to Q. No. 3: i) The given equation is–

$x^4 - 5x^3 + 40x^2 - 8x + 24 = 0$

Let $f(x) = x^4 - 5x^3 + 40x^2 - 8x + 24 = 0$

It has four changes of sign.

$\therefore$ The equation has maximum number +ve roots is 4.

$\because$ The equation is of degree 4, the upper limit of real roots is 4.

ii) The given equation is –

$x^4 + 4x^3 – 11x^2 – 9x – 50 = 0$

Let $f(x) = x^4 + 4x^3 – 11x^2 – 9x – 50 = 0$

The equation has only one change of sign.

$\therefore$ It has only one +ve root.

Again, $f(–x) = x^4 + 4x^3 – 11x^2 + 9x – 50 = 0$

It has one change of sign.

$\therefore$ The given equation has only one –ve roots.

$\because$ The upper limit of real roots is 1+1=2

Ans. to Q. No. 4: i) The given equation is–

$x^6 – 3x^2 – 2x – 3 = 0$

Let $f(x) = x^6 – 3x^2 – 2x – 3 = 0$

It has only one change of sign.

$\therefore$ The equation has maximum one +ve root.

$f(–x) = x^6 – 3x^2 + 2x – 3 = 0$

It has three changes of sign

$\therefore$ the given equation has at most three –ve roots.

Hence, the given equation has at most 1 + 3 = 4 real roots.

$\therefore$ Minimum number of complex root is 6 – 4 = 2

ii) The given equation is

$x^7 – 3x^3 + x^2 = 0$

Let $f(x) = x^7 – 3x^3 + x^2 = 0$

It has two changes of sign.

$\therefore$ the equation has at most two +ve roots.

$f(–x) = –x^7 + 3x^3 + x^2 = 0$

has only one change of sign.

$\therefore$ the given equation has only one –ve root.

Hence the equation has at most 2 + 1 = 3 real roots.

Since the equation is of degree 7, it has at least 4 complex roots.

2.7 FURTHER READING

- University Algebra, N S Gopala Krishnan, New Age International (P) Limited.
- Linear Algebra: Schum Solved Problems Series; Seymour Lipschitz: Tata McGraw Hill.
- Text Book of Algebra, A.K Sharma, DPH, Mathematics Series.

2.8 MODEL QUESTIONS

Q.1: Solve: i) $27x^3 + 54x^2 + 198x – 73 = 0$

ii) $28x^3 – 9x^2 + 1 = 0$

ii) $x^3 – 18x – 35 = 0$

Q.2: Applying Descartes' rule of sign to find the minimum number of complex roots of $2x^7 - x^4 + 4x^2 - 5 = 0$

Q.3: Show that $x^3 + 2x^2 - 2x - 1 = 0$ has one +ve and two –ve roots.

Q.4: Show that $x^5 + x^3 - 8x - 5 = 0$ cannot have more than three real roots.

Q.5: If p, q, r be +ve, then show that $x^4 + px^3 + qx - r = 0$ has one +ve, one –ve and two imaginary roots.

Q.6: Show that the equation $x^6 - 3x^2 - x + 1 = 0$ has at least two imaginary roots.

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UNIT 3: INEQUALITIES

UNIT STRUCTURE

3.1 Learning Objectives
3.2 Introduction
3.3 Inequalities and Rules of Inequalities
3.4 Arithmetic Mean, Geometric Mean and Harmonic Mean
3.5 Let Us Sum Up
3.6 Answers to Check Your Progress
3.7 Further Reading
3.8 Model Questions

3.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- define Inequalities and rules of inequalities
- Inequalities involving arithmetic,geometric and harmonic means.

3.2 INTRODUCTION

In this unit, we will discuss inequalities, rules of inequalities and inequalities involving arithmetic arithmetic,geometric and harmonic means.

3.3 INEQUALITIES AND RULES OF INEQUALITIES

Any quantity a is said to be greater than another quantity b when $a - b > 0$ and smaller (less) when $a - b < 0$.

The statements $x > y$ and $x < y$ are called inequalities.

Rules of Inequalities:

1) If each side of an inequality is increased or decreased by the same positive quantity, inequalities remain unaltered.
   i.e, if a>b, to prove that
   i) $a + c > b + c$
   ii) $a - c > b - c$, c being positive quantity.

Proof: i) $a + c) - (b + c) = a - b > 0 \Rightarrow a+c > b+c$
ii) $(a-c) - (b-c) = a-b > 0 \Rightarrow a - c > b - c$

2) If each side of an inequality is multiplied or divided by the same positive quantity, inequality remains unaltered.
   i.e., If a > b, then i) $am > bm$

   ii) $\frac{a}{m} > \frac{b}{m}$, m being positve quantity.

Proof: $a > b \Rightarrow a - b > 0$ and $m > 0$
i) $am - bm = (a - b)m > 0$
$\Rightarrow am > bm$

ii) $\frac{a}{m} - \frac{b}{m} = \frac{a-b}{m} > 0$

$\Rightarrow \frac{a}{m} > \frac{b}{m}$ Proved.

3) In an inequality any term may be transposed from one side to the other side by changing its sign.

Proof: Let a – c > b

Adding c to each side,

a – c + c > b + c

$\Rightarrow$ a > b + c Proved

4) If the side of an inequality be transposed the symbol of the inequality is reversed.

Proof: If a>b, then it is evident that b<a.

5) If the signs of all the terms on both sides of an inequality are changed, the symbol of inequality is changed.

Proof : Let a + b > c + d

$\Rightarrow$ a + b – c > d [By (3)]

$\Rightarrow$ a + b – c – d > 0 [By (3)]

$\Rightarrow$ b – c – d > –a [By (3)]

$\Rightarrow$ –c–d > –a–b

i.e, – a – b < –c – d Proved

6) If the sides of an inequality are multiplied by the same negative quantity, the sign of the inequality is reversed.

Proof: Let a > b

$\Rightarrow$ ma > mb [By (2)]

Where m > 0

$\Rightarrow$ –ma < –mb [By (5)]

i.e, (–m)a < (–m)b Proved

7) If $a_1 > b_1$, $a_2 > b_2$, $a_3 > b_3$, ... $a_n > b_n$, then

i) $a_1 + a_2 + a_3 + \ldots \ldots + a_n > b_1 + b_2 + b_3 + \ldots \ldots + b_n$

ii) $a_1 a_2 a_3 \cdots a_n > b_1 b_2 b_3 \cdots b_n, a_i > 0, bi > 0$

Proof: $a_1 > b_1$, $a_2 > b_2$

$\therefore a_1 + a_2 > b_1 + a_2 > b_1 + b_2$ [By (1)]

Similarly

$a_1 + a_2 + a_3 > b_1 + b_2 + a_3 > b_1 + b_2 + b_3$

Proceeding as above, we can prove that

i) $a_1 + a_2 + a_3 + \ldots \ldots + a_n > b_1 + b_2 + b_3 + \ldots \ldots + b_n$

Again, $a_1 > b_1$, $a_2 > b_2$

$\therefore a_1 a_2 > b_1 . a_2 > b_1 . b_2$ [By (2)]

Similarly

Proceeding as above

ii) $a_1 a_2 a_3 \ldots \ldots a_n > b_1 b_2 b_3 \ldots \ldots b_n$ Proved

8) If a > b and n > 0, then (a > 0, b > 0) $a^n > b^n$

Proof: In 7(ii) Putting $a_1 = a_2 = a_3 = \ldots \ldots = a_n = a$

and $b_1 = b_2 = b_3 = \ldots \ldots = b_n = b$

then a.a.a. ... ... a > b.b.b. ... ... b

$\therefore a^n > b^n$

Example 1: Prove that: $a^2 + b^2 > 2ab$

Proof: $a^2 + b^2 - 2ab = (a - b)^2 > 0$

$\therefore a^2 + b^2 > 2ab$

Example 2: If the sum of two positive quantities is given, their product is greater when they are equal.

Proof: Let a and b be two positive quantities, S be the sum and P be their product.

Now, $(a + b)^2 - (a - b)^2 = 4ab$

$\Rightarrow S^2 - (a - b)^2 = 4P$

If S is fixed then P will be maximum (greatest) when $(a - b)^2$ will be minimum.

i.e, $(a - b)^2 = 0$

$\Rightarrow a - b = 0$

$\Rightarrow a = b.$

3.4 ARITHMETIC MEAN, GEOMETRIC MEAN AND HARMONIC MEAN

If $a_1, a_2, ..., a_n$ be n positive numbers, then

Arithmetic Mean (A.M) = $A = \frac{a_1 + a_2 + ... + a_n}{n}$

Geometric Mean (G.M) = $G = (a_1 a_2 \cdots a_n)^{\frac{1}{n}}$

Harmonic Mean (H.M)= $H = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}$

Theorem 1: The A.M of any number of unequal positive quantities is greater than their G.M.

Proof: $\left(\frac{a_1 + a_2}{2}\right)^2 - \left(\frac{a_1 - a_2}{2}\right)^2 = a_1 a_2$

$$\therefore \left(\frac{a_1 + a_2}{2}\right)^2 \geq a_1 a_2 \text{ since } \left(\frac{a_1 - a_2}{2}\right)^2 > 0 \qquad (1)$$

The equality occurs when $a_1 = a_2$

Similarly

$$\left(\frac{a_3 + a_4}{2}\right)^2 \geq a_3 a_4 \qquad (2)$$

The equality occurs when $a_3 = a_4$

By (1) and (2)

$$\left(\frac{a_1 + a_2}{2}\right)^2 \left(\frac{a_3 + a_4}{2}\right)^2 \geq a_1 a_2 a_3 a_4 \qquad (3)$$

But $\left(\frac{a_1+a_2+a_3+a_4}{4}\right)^2$

$$= \left(\frac{\frac{a_1+a_2}{2}+\frac{a_3+a_4}{2}}{2}\right)^2 \geq \frac{a_1+a_2}{2}\cdot\frac{a_3+a_4}{2} \qquad (4)$$

The equality occurs when $\frac{a_1+a_2}{2} = \frac{a_3+a_4}{2}$

By (4), we get

$$\left(\frac{a_1+a_2+a_3+a_4}{4}\right)^4 \geq \left(\frac{a_1+a_2}{2}\right)^2 \cdot \left(\frac{a_3+a_4}{2}\right)^2 \geq a_1 a_2 a_3 a_4$$

[By (1) and (2)]
The equality occurs when $a_1 = a_2 = a_3 = a_4$

Similarly, $\left(\frac{a_5+a_6+a_7+a_8}{4}\right)^4 \geq a_5 a_6 a_7 a_8$

Proceeding as above

$$\left(\frac{a_1+a_2+.....+a_8}{8}\right)^8 \geq a_1 a_2 .....a_8$$

Repeating this process when $n=2^m$ where m is a positive integer.

$$\left(\frac{a_1+a_2+.....+a_n}{n}\right)^n \geq a_1 a_2 .....a_n$$

The equality occurs when $a_1 = a_2 = a_3 = \cdots\cdots = a_n$.

when $2^{m-1} < n < 2^m$ for $m > 1$ i.e, when n is not a power of 2, then we consider a positive integer p such that $n + p = 2^m$

Let us consider n + p positive integers $a_1, a_2, ... a_n, a, a, ...a$.

Here a is repeated p-times and

$$a = \frac{a_1+a_2+.....+a_n}{n}$$

$\therefore n + p = 2^m$,

By above procedure, we can prove

$$\left(\frac{a_1+a_2+.....+a_n+pa}{n+p}\right)^{n+p} \geq a_1 a_2 .....a_n a^p$$

and the equality occurs when $a_1 = a_2 = ... = a_n = a$

$$\Rightarrow \left(\frac{na+pa}{n+p}\right)^{n+p} \geq a_1 a_2 \ldots\ldots a_n . a^p$$

$$\Rightarrow a^n \geq a_1 a_2 \ldots\ldots a_n .$$

$$\Rightarrow \left(\frac{a_1 + a_2 + \ldots\ldots\ldots + a_n}{n}\right)^n \geq a_1 a_2 \ldots\ldots a_n$$

$$\Rightarrow \frac{a_1+a_2+\ldots\ldots\ldots+a_n}{n} \geq \sqrt[n]{a_1 . a_2 \ldots . a_n}$$

i.e $A \geq G$

Theorem 2: If $a_1, a_2, \ldots \ldots a_n$ be n positive numbers and G and H be their geometric mean and harmonic mean respectively, then $G \geq H$.

Proof: Since $a_1, a_2, \ldots \ldots \ldots a_n$ are all positive numbers, $\frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_n}$ are also positive numbers.

Now, arithmetic mean of $\frac{1}{a_1}, \frac{1}{a_2} \ldots \ldots \frac{1}{a_n}$

$$= \frac{\frac{1}{a_1} + \frac{1}{a_2} + \ldots\ldots\ldots + \frac{1}{a_n}}{n}$$

and their geometric mean $= \sqrt[n]{\frac{1}{a_1} . \frac{1}{a_2} \ldots\ldots \frac{1}{a_n}}$

$$\therefore \frac{\frac{1}{a_1} + \frac{1}{a_2} + \ldots\ldots + \frac{1}{a_n}}{n} \geq \sqrt[n]{\frac{1}{a_1}, \frac{1}{a_2}, \ldots\ldots \frac{1}{a_n}}$$

The equality occurs when

$\frac{1}{a_1} = \frac{1}{a_2} = \ldots = \frac{1}{a_n} \Rightarrow a_1 = a_2 = \ldots \ldots \ldots \ldots \ldots \ldots = a_n$

or $(a_1 a_2 \ldots\ldots a_n)^{\frac{1}{n}} \geq \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \ldots\ldots\ldots + \frac{1}{a_n}}$

i.e $G \geq H$.

Note: For n positive numbers $A \geq G \geq H$.

Example 3: If a, b, c are positive real numbers.

Prove that $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3$

Solutive: Since a, b, c are all positive real numbers, $\frac{a}{b}$, $\frac{b}{c}$, $\frac{c}{a}$ are also real positive numbers.

$\because A \geq G$

$$\therefore \frac{\frac{a}{b}+\frac{b}{c}+\frac{c}{a}}{3} \geq \sqrt[3]{\frac{a}{b}.\frac{b}{c}.\frac{c}{a}}$$

$$\Rightarrow \frac{\frac{a}{b}+\frac{b}{c}+\frac{c}{a}}{3} \geq 1$$

$$\Rightarrow \frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq 3$$

Example 4: Prove that $a^2 + b^2 + c^2 > ab + bc + ca$

Hence deduce that $a^3 + b^3 + c^3 > 3abc$, a, b, c are positive and unequal.

Solution: We have

$$\frac{a^2+b^2}{2} > \sqrt{a^2b^2} = ab$$

$\Rightarrow a^2 + b^2 > 2ab$ (1)

Similarly $b^2 + c^2 > 2bc$ (2)

$c^2 + a^2 > 2ca$ (3)

(1) + (2) + (3)

$\Rightarrow (a^2 + b^2 + c^2) > 2(ab + bc + ca)$

$\Rightarrow a^2 + b^2 + c^2 > ab + bc + ca.$

Again, $a^3 + b^3 + c^3 – 3abc = (a + b + c)(a^2 + b^2 + c^2 – ab – bc – ca)$

> 0 $(\because a^2+b^2+c^2>ab+bc+ca)$

$\Rightarrow a^3+b^3+c^3 > 3abc.$

Example 5: Prove that $n^n > 1. 3. 5. ..... (2n–1)$

Solution: Considering n positive numbers

1, 3, 5, ... (2n–1), and since A > G

$$\therefore \frac{1+3+5+....+(2n-1)}{n} > \{1.3.5.....(2n-1)\}^{\frac{1}{n}}$$

$$\Rightarrow \frac{n}{2}.\frac{[2\times 1+(n-1).2]}{n} > \{1.3.5.....(2n-1)\}^{\frac{1}{n}}$$

$$\Rightarrow n > \{1.3.5.....(2n-1)\}^{\frac{1}{n}}$$

$$\Rightarrow n^n > 1.3.5.....(2n-1)$$

CHECK YOUR PROGRESS

Q.1: Prove that: (a + b + c)(bc + ca + ab) > 9abc,
a > 0, b > 0, c > 0, $a \neq b \neq c$

Q.2: Prove that $\left\{\frac{1}{2}(n+1)\right\}^n > n!$

Q 3: If x, y, z are positive and x + y + z = 1, Prove that:

$8xyz \leq (1-x)(1-y)(1-z) \leq \frac{8}{27}$

Q.4: Prove that: $(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) > 4$, a > 0, b > 0, $a \neq b$

Q.5: Prove that: $\left(\frac{a+b+c}{3}\right)^{a+b+c} < a^a b^b c^c < \left(\frac{a^2+b^2+c^2}{a+b+c}\right)^{a+b+c}$

where a, b, c are unequal positive integers.

3.5 LET US SUM UP

- Any quantity a is said to be greater than another quantity b when a–b > 0 and smaller (less) when a–b < 0.The statements x>y and x<y are called inequalities.
- **Rules of Inequalities:**
  a) If each side of an inequality is increased or decreased by the same positive quantity, inequality remains unaltered.
  b) If each side of an inequality is multiplied or divided by the same positive quantity, inequality remains unaltered.
  c) In an inequality any term may be transposed from one side to the other side by changing its sign.
  d) If the side of an inequality be transposed the symbol of the inequality is reversed.
  e) If the signs of all the terms on both sides of an inequality are changed. the symbol of inequality is changed.
  8) If $a_1, a_2, ..., a_n$ be n positive numbers then Their

  Arithmetic Mean (A.M) = A = $\frac{a_1+a_2+.....+a_n}{n}$

  Geometric Mean (G.M) = G = $(a_1 a_2 \cdots a_n)^{\frac{1}{n}}$

  Harmonic Mean (H.M) = $H = \frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}}$

  and $A \geq G \geq H$.

3.6 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1: Since A > G

$\therefore \frac{a+b+c}{3} > (abc)^{\frac{1}{3}}$ (1)

$\frac{bc+ca+ab}{3} > (bc.ca.ab)^{\frac{1}{3}}$ (2)

From (1) and (2),

$$\frac{a+b+c}{3}.\frac{bc+ca+ab}{3} > (abc)^{\frac{1}{3}}.(bc.ca.ab)^{\frac{1}{3}}$$

$$\Rightarrow \frac{(a+b+c)(bc+ca+ab)}{9} > (abc)^{\frac{1}{3}}.(a^2b^2c^2)^{\frac{1}{3}}$$

$$\Rightarrow (a+b+c)(bc+ca+ab) > 9abc \text{ Proved.}$$

Ans. to Q. No. 2: Since $A \geq G$

$$\frac{1+2+3+.....+n}{n} > (1.2.3.....n)^{\frac{1}{n}}$$

$$\Rightarrow \frac{n(n+1)}{2n} > (n!)^{\frac{1}{n}}$$

$$\Rightarrow \frac{1}{2}(n+1) > (n!)^{\frac{1}{n}}$$

$$\Rightarrow \left\{\frac{1}{2}(n+1)\right\}^n > n!$$

Ans. to Q. No. 3: x + y + z = 1

$\therefore\ x+y=1-z,\ y+z=1-x, z+x=1-y$

Since $A \geq G$

$$\therefore \frac{x+y}{2} \geq \sqrt{xy}, \frac{y+z}{2} \geq \sqrt{yz}, \frac{z+x}{2} \geq \sqrt{zx}$$

$$\therefore \frac{(x+y)(y+z)(z+x)}{8} \geq \sqrt{xy}.\sqrt{yz}.\sqrt{zx} = xyz$$

$$\Rightarrow \frac{(1-z)(1-x)(1-y)}{8} \geq xyz$$

$\Rightarrow 8xyz \leq (1–x)(1–y)(1–z)$

Ans. to Q. No. 4: Since A > G

$$\therefore \frac{a+b}{2} > \sqrt{ab} \qquad \text{and} \qquad \frac{\frac{1}{a}+\frac{1}{b}}{2} > \sqrt{\frac{1}{a}.\frac{1}{b}} = \sqrt{\frac{1}{ab}}$$

$$\Rightarrow a+b > 2\sqrt{ab} \quad (1) \qquad \Rightarrow \frac{1}{a}+\frac{1}{b} > 2\sqrt{\frac{1}{ab}} \quad (2)$$

$$\therefore (1) \times (2) \Rightarrow (a+b)\left(\frac{1}{a}+\frac{1}{b}\right) > 2\sqrt{ab}.2\sqrt{\frac{1}{ab}} = 4$$

$$\Rightarrow (a+b)\left(\frac{1}{a}+\frac{1}{b}\right) > 4$$

Ans. to Q. No. 5: Taking a times a, b time b and c times c and applying A > G

$\frac{a.a+b.b+c.c}{a+b+c}$ > {(a. a. ... ... a factors)(b. b. ... ... b factors)

(c.c. ... ... c factors)}$^{\frac{1}{a+b+c}}$

$$\Rightarrow \frac{a^2+b^2+c^2}{a+b+c} > \left(a^a\, b^b\, c^c\right)^{\frac{1}{a+b+c}}$$

$$\Rightarrow a^a\, b^b\, c^c < \left(\frac{a^2+b^2+c^2}{a+b+c}\right)^{a+b+c} \qquad (1)$$

Taking a times $\frac{a+b+c}{3a}$, b times $\frac{a+b+c}{3b}$ and c times $\frac{a+b+c}{3c}$

and applying A > G

$$\frac{a.\frac{a+b+c}{3a}+b.\frac{a+b+c}{3b}+c.\frac{a+b+c}{3c}}{a+b+c} > \left\{\left(\frac{a+b+c}{3a}\right)^a.\left(\frac{a+b+c}{3b}\right)^b.\left(\frac{a+b+c}{3c}\right)^c\right\}^{\frac{1}{a+b+c}}$$

$$\Rightarrow \frac{3.\frac{a+b+c}{3}}{a+b+c} > \left\{\left(\frac{a+b+c}{3}\right)^{a+b+c}.\frac{1}{a^a\, b^b\, c^c}\right\}^{\frac{1}{a+b+c}}$$

$$\Rightarrow 1 > \left\{\left(\frac{a+b+c}{3}\right)^{a+b+c}.\frac{1}{a^a\, b^b\, c^c}\right\}^{\frac{1}{a+b+c}}$$

$$\Rightarrow 1 > \left(\frac{a+b+c}{3}\right)^{a+b+c}.\frac{1}{a^a\, b^b\, c^c}$$

$$\Rightarrow \left(a^a\, b^b\, c^c\right) > \left(\frac{a+b+c}{3}\right)^{a+b+c} \qquad (2)$$

From (1) and (2)

$$\left(\frac{a+b+c}{3}\right)^{a+b+c} < a^a\, b^b\, c^c < \left(\frac{a^2+b^2+c^2}{a+b+c}\right)^{a+b+c}$$

3.7 FURTHER READING

- University Algebra, N S Gopala Krishnan, New Age International (P) Limited.
- Linear Algebra: Schum Solved Problems Series; Seymour Lipschitz: Tata McGraw Hill.
- Text Book of Algebra, A.K Sharma, DPH, Mathematics Series.

? 3.8 MODEL QUESTIONS

Q.1: If a,b,c are all real, prove that:

$$(a+b-c)^2+(b+c-a)^2+(c+a-b)^2 \geq ab+bc+ca$$

Q.2: If a,b,c are positive numbers,prove that

i) $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq 3$

ii) $a^4+b^4+c^4 \geq abc(a+b+c)$

iii) $a^5+b^5+c^5 \geq abc(a^2+b^2+c^2)$

iv) $\left(\frac{a+b+c}{3}\right)^3 \geq a\left(\frac{b+c}{2}\right)^2$

Q.3: If n be a positive integer,greater than 1,prove that

i) $n^n > 1.3.5\cdots(2n-1)$

ii) $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} > n(n+1)^{\frac{1}{n}}$

Q.4: If a,b,c be all postive and a + b + c = 1, prove that:

$$\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}\geq\frac{9}{2}$$

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UNIT 4: DETERMINANT

UNIT STRUCTURE

4.1 Learning Objectives
4.2 Introduction
4.3 Determinant of Order 2
4.4 Determinant of Order 3
4.5 Properties of Determinants
4.6 Let Us Sum Up
4.7 Answers to Check Your Progress
4.8 Further Reading
4.9 Model Questions

4.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- define determinant
- evaluate determinants of order 2 and 3
- use the properties of determinants for evaluation of determinants.

4.2 INTRODUCTION

The concept of determinants is a useful tool in solving system of linear equations in two or three variables.In this unit,we shall discusss the concept of determinants.We shall also study many properties of determinants which help in evaluation of determinants. We may use determinants to solve a system of linear equations by a method known as Cramer's rule.

4.3 DETERMINANT OF ORDER 2

Let us consider the equations $a_1x+b_1y=0$ and $a_2x+b_2y=0$.

Eliminating x and y from the equations, we have $-\frac{y}{x}=\frac{a_1}{b_1}=\frac{a_2}{b_2}$

$$\Rightarrow a_1b_2-a_2b_1=0$$

which is written in compact form as $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}=0$

In other words, we have $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}=a_1b_2-a_2b_1=0$.

The expression $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is called a **determinant of the second order.** It has two

horizotal lines $a_1\ b_1$ and $a_2\ b_2$ and two vertical lines $\begin{matrix} a_1 & b_1 \\ a_2 & b_2 \end{matrix}$. The horizontal lines are called rows and the vertical lines are called columns.A determinant of second order has two rows and two columns.The numbers a_1, b_1, a_2, b_2 are called the elements of the determinant. $a_1b_2 - a_2b_1$ is called the expansion or the value of the determinant.

Example 1: Evaluate: i) $\begin{vmatrix} 1 & 3 \\ -2 & -4 \end{vmatrix}$ ii) $\begin{vmatrix} x+1 & x \\ x & x-1 \end{vmatrix}$

iii) $\begin{vmatrix} x-1 & 1 \\ x^3 & x^2+x+1 \end{vmatrix}$ iv) $\begin{vmatrix} a^2 & ab \\ ab & b^2 \end{vmatrix}$

Solution: i) We have $\begin{vmatrix} 1 & 3 \\ -2 & -4 \end{vmatrix} = 1.(-4) - (-2).3 = -4 + 6 = 2$.

ii) $\begin{vmatrix} x+1 & x \\ x & x-1 \end{vmatrix} = (x+1).(x-1) - x.x = x^2 - 1 - x^2 = -1$.

iii) $\begin{vmatrix} x-1 & 1 \\ x^3 & x^2+x+1 \end{vmatrix} = (x-1)(x^2+x+1) - x^3$

$= x^3 - 1 - x^3$

$= -1$.

iv) $\begin{vmatrix} a^2 & ab \\ ab & b^2 \end{vmatrix} = a^2b^2 - (ab)^2 = 0$

Example 2: Find the value of x if

i) $\begin{vmatrix} x-3 & x \\ x+1 & x+3 \end{vmatrix} = 6$ ii) $\begin{vmatrix} 2x-1 & 2x+1 \\ x+1 & 4x+2 \end{vmatrix} = 0$

Solution: i) We have $\begin{vmatrix} x-3 & x \\ x+1 & x+3 \end{vmatrix} = (x-3)(x+3) - x(x+1)$

$= (x^2 - 9) - x^2 - x$

$= -x - 9$

$\therefore -x - 9 = 6$

$\Rightarrow -x = 15$

$\Rightarrow x = -15$

ii) We have $\begin{vmatrix} 2x-1 & 2x+1 \\ x+1 & 4x+2 \end{vmatrix} = (2x-1)(4x+2) - (x+1)(2x+1)$

$= 8x^2 + 4x - 4x - 2 - 2x^2 - x - 2x - 1$

$= 6x^2 - 3x - 3$

$$= 3\left(2x^2 - x - 1\right)$$

$$\therefore 3\left(2x^2 - x - 1\right) = 0$$

$$\Rightarrow 2x^2 - x - 1 = 0$$

$$\Rightarrow x = \frac{1 \pm \sqrt{1+8}}{4}$$

$$= \frac{1 \pm 3}{4}$$

$$= 1, -\frac{1}{2}$$

Example 3: Solve for x if $\begin{vmatrix} x & 5 \\ 7 & x \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ -1 & 1 \end{vmatrix} = 0$.

Solution: Given, $\begin{vmatrix} x & 5 \\ 7 & x \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ -1 & 1 \end{vmatrix} = 0$

$$\Rightarrow \left(x^2 - 35\right) + \left(1 - 2\right) = 0$$

$$\Rightarrow x^2 - 35 - 1 = 0$$

$$\Rightarrow x^2 - 36 = 0$$

$$\Rightarrow x^2 = 36$$

$$\Rightarrow x = \pm 6$$

CHECK YOUR PROGRESS

Q.1: Evaluate: i) $\begin{vmatrix} 2 & -3 \\ 7 & 11 \end{vmatrix}$

ii) $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$ iii) $\begin{vmatrix} 2+\sqrt{3} & 3+\sqrt{11} \\ 3-\sqrt{11} & 2-\sqrt{3} \end{vmatrix}$

Q.2: Solve for x:

i) $\begin{vmatrix} 2 & 3 \\ 1 & 4x \end{vmatrix} = \begin{vmatrix} 2x & -1 \\ 5 & x \end{vmatrix}$ ii) $\begin{vmatrix} x & 3 \\ 4 & x \end{vmatrix} = \begin{vmatrix} 0 & -2 \\ 2x & 5 \end{vmatrix}$

Q.3: If $\begin{vmatrix} 3 & p \\ 4 & 6 \end{vmatrix} = 6$, find the value of p.

Q.4: Prove that: $\begin{vmatrix} a+ib & c+id \\ c-id & a-ib \end{vmatrix} = a^2 + b^2 - c^2 - d^2$

4.4 DETERMINANT OF ORDER 3

In 4.3, We have already discussed determinant of order 2. Now, we think of a determinant which has 3 rows and 3 colums.

Let us consider the equations

$$a_1x + b_1y + c_1z = 0 \quad (1)$$

$$a_2x + b_2y + c_2z = 0 \quad (2)$$

and $a_3x + b_3y + c_3z = 0 \quad (3)$

Soving (2) and (3), we get $\frac{x}{b_2c_3 - b_3c_2} = \frac{y}{c_2a_3 - c_3a_2} = \frac{z}{a_2b_3 - a_3b_2}$

Substituting these proportional values of x, y and z in (1), we get

$$a_1(b_2c_3 - b_3c_2) + b_1(c_2a_3 - c_3a_2) + c_1(a_2b_3 - a_3b_2) = 0$$

Which can be written in compact form as:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

In other words, we have

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) + b_1(a_3c_2 - a_2c_3) + c_1(a_2b_3 - a_3b_2)$$

The expression $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is called a determinant of the third order. Since it has 3 rows and 3 columns, it is called a determinant of order 3.

$a_1(b_2c_3 - b_3c_2) + b_1(a_3c_2 - a_2c_3) + c_1(a_2b_3 - a_3b_2)$
is called the expansion or the value of the determinant.

Note: A determinant of order 3 has 9 elements.

Value of a Determinant: Determinant of order three can be determined by expressing it in terms of second order determinants. This is known as expansion of a determinant along a row (or a column). There are six ways of expanding a determinant of order 3 corresponding to each of three rows (R_1, R_2 and R_3) and three columns (C_1, C_2 and C_3) giving the same value.

Consider the determinant of order 3

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

First we expand the given determinant along first row (R_1):

Step 1: We multiply element a_1 of R_1 by $(-1)^{1+1}$ [$(-1)^{\text{position of the element } a_1}$] and with the second order determinant obtained by deleting the elements of first row (R_1) and first column (C_1) as a_1 lies in R_1 and C_1.

i.e., $(-1)^{1+1} a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$

Step 2: We multiply second element b_1 of R_1 by $(-1)^{1+2}$ [$(-1)^{\text{position of the element } b_1}$] and with the second order determinant obtained by deleting the elements of first row (R_1) and second column (C_2) as b_1 lies in R_1 and C_2.

i.e., $(-1)^{1+2} b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$

Step 3: We multiply third element c_1 of R_1 by $(-1)^{1+3}$ [$(-1)^{\text{position of the element } c_1}$] and with the second order determinant obtained by deleting the elements of first row(R_1) and third column (C_3) as c_1 lies in R_1 and C_3.

i.e., $(-1)^{1+3} c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$

Step 4: The expansion of determinant is written as the sum of all the three terms obtained in step 1, 2 and 3 above and is given by

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = (-1)^{1+1} a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + (-1)^{1+2} b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + (-1)^{1+3} c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

$$= a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 .$$

Expansion along second row (R_2):

Consider the determinant $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Expanding along R_2, we get

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = (-1)^{2+1} a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + (-1)^{2+2} b_2 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} + (-1)^{2+3} c_2 \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$$

$$= -a_2(b_1c_3 - b_3c_1) + b_2(a_2c_3 - a_3c_1) - c_2(a_1b_3 - a_3b_1)$$

$$= -a_2b_1c_3 + a_2b_3c_1 + a_1b_2c_3 - a_3b_2c_1 - a_1b_3c_2 + a_3b_1c_2 .$$

Expanding along C_1, we get

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = (-1)^{1+1} a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + (-1)^{2+1} a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + (-1)^{3+1} a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

$$= a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1$$

Similarly, we can expand the determinant along R_3, C_2 and C_3.

Note: Expanding a determinant along any row or column gives same value.

Example 1: Evaluate the determinant $\begin{vmatrix} 1 & 2 & 4 \\ -1 & 3 & 0 \\ 4 & 1 & 0 \end{vmatrix}$

Solution: Expanding along R_1 we get

$$\begin{vmatrix} 1 & 2 & 4 \\ -1 & 3 & 0 \\ 4 & 1 & 0 \end{vmatrix} = 1\begin{vmatrix} 3 & 0 \\ 1 & 0 \end{vmatrix} - 2\begin{vmatrix} -1 & 0 \\ 4 & 0 \end{vmatrix} + 4\begin{vmatrix} -1 & 3 \\ 4 & 1 \end{vmatrix}$$

$$= 1(0-0) - 2(0-0) + 4(-1-12)$$

$$= -52$$

Example 2: Evaluate the determinant $\Delta = \begin{vmatrix} 1 & -2 & 3 \\ -2 & 3 & 4 \\ 1 & -1 & -2 \end{vmatrix}$

i) by expanding about any row

ii) by expanding about any column.

Solution : i) a) Expanding about the first row

$$\Delta = 1\begin{vmatrix} 3 & 4 \\ -1 & -2 \end{vmatrix} - (-2)\begin{vmatrix} -2 & 4 \\ 1 & -2 \end{vmatrix} + 3\begin{vmatrix} -2 & 3 \\ 1 & -1 \end{vmatrix}$$

$$= 1[3(-2) - (-1)4] + 2[(-2)(-2) - (1)(4)] + 3[(-2)(-1) - (3)(1)]$$

$$= -6 + 4 + 2(4-4) + 3(2-3)$$

$$= -2 - 3$$

$$= -5$$

b) Expanding about the second row

$$\Delta = (-1)(-2)\begin{vmatrix} -2 & 3 \\ -1 & -2 \end{vmatrix} + 3\begin{vmatrix} 1 & 3 \\ 1 & -2 \end{vmatrix} - 4\begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix}$$

$$= 2(4+3) + 3(-2-3) - 4(-1+2)$$

$$= 14 - 15 - 4$$

$$= -5$$

c) Expanding about the third row

$$\Delta = 1\begin{vmatrix} -2 & 3 \\ 3 & 4 \end{vmatrix} - (-1)\begin{vmatrix} 1 & 3 \\ -2 & 4 \end{vmatrix} + 2\begin{vmatrix} 1 & -2 \\ -2 & 3 \end{vmatrix}$$

$$= -8 - 9 + 4 + 6 - 2(3 - 4)$$

$$= -17 + 10 + 2$$

$$= -5$$

ii) a) Expanding about first column

$$\Delta = 1\begin{vmatrix} 3 & 4 \\ -1 & -2 \end{vmatrix} + 2\begin{vmatrix} -2 & 3 \\ -1 & -2 \end{vmatrix} + 1\begin{vmatrix} -2 & 3 \\ 3 & 4 \end{vmatrix}$$

$$= -6 + 4 + 2(4 + 3) + 1(-8 - 9)$$

$$= -2 + 14 - 17$$

$$= -5$$

b) Expanding about the second column

$$\Delta = (-1)(-2)\begin{vmatrix} -2 & 4 \\ 1 & -2 \end{vmatrix} + 3\begin{vmatrix} 1 & 3 \\ 1 & -2 \end{vmatrix} + (-1)(-1)\begin{vmatrix} 1 & 3 \\ -2 & 4 \end{vmatrix}$$

$$= 2(4 - 4) + 3(-2 - 3) + (4 + 6)$$

$$= -15 + 10$$

$$= -5$$

c) Expanding about the third column

$$\Delta = 3\begin{vmatrix} -2 & 3 \\ 1 & -1 \end{vmatrix} - 4\begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix} + (-2)\begin{vmatrix} 1 & -2 \\ -2 & 3 \end{vmatrix}$$

$$= 3(2 - 3) - 4(-1 + 2) - 2(3 - 4)$$

$$= -3 - 4 + 2$$

$$= -5$$

4.5 PROPERTIES OF DETERMINANTS

There are some properties of determinants, which are very much useful in solving problems. Here we are going to discuss the properties only for the determinant of order 3.

Property 1: The value of determinant remains unchanged if a rows and columns are interchanged.

i.e., $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Note: If R_i = *i*th row and C_i = *i*th column, then for interchange of row and columns, we will symbolically write $C_i \leftrightarrow R_i$.

Property 2: If any two adjacent rows (or columns) of a determinant are interchanged, then sign of determinant changes.

i.e., $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Note: We can denote the interchange of rows by $R_i \leftrightarrow R_j$ and interchange of columns by $C_i \leftrightarrow C_j$.

Property 3: If any two rows (or columns) of a determinant are identical (all corresponding elements are same), then value of determinant is zero.

i.e., $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

Property 4: If each element of a row (or a column) of a determinant is multiplied by a constant *k*, then its value gets multiplied by *k*.

i.e., $\begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = k\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Note: The value of the determinant remains unchanged.by applying $R_i \leftrightarrow kR_i$ or $C_i \leftrightarrow kC_i$ to the determinant .

Property 5: If to any row(or column) is added k times the corresponding elements of another row (or column), the value of the determinant remains unchanged.

i.e., $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + ka_2 & b_1 + kb_2 & c_1 + kc_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Note: The value of determinant remain same if we apply the operation $Ri \leftrightarrow R_i + kR_j$ or $C_i \leftrightarrow C_i + k\,C_j$.

Property 6: If any row (or column) is the sum of two or more elements, then the determinant can be expressed as sum of two or more determinants.

i.e., $\begin{vmatrix} a_1 + d_1 & b_1 + d_2 & c_1 + d_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} d_1 & d_2 & d_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Illustrative Examples:

Example 1: Prove that: $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

Solution : $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix}$

(Apply $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$)

$$= (b-a)(c-a)\begin{vmatrix} 1 & 0 & 0 \\ a & 1 & 1 \\ a^2 & b+a & c+a \end{vmatrix}$$

$$= (b-a)(c-a)(c+a-b-a)$$
$$= (b-a)(c-a)(c-b)$$
$$= (a-b)(b-c)(c-a).$$

Example 2: Show that: $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = 0$

Solution: $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$

Apply $C_3 \to C_3 + C_2$

$$= \begin{vmatrix} 1 & a & a+b+c \\ 1 & b & b+c+a \\ 1 & c & c+a+b \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & a & 1 \\ 1 & b & 1 \\ 1 & c & 1 \end{vmatrix}$$

$= (a+b+c).0$ (C_1 and C_3 are identical)

$= 0$.

Example 3: Find x if $\begin{vmatrix} 1 & x & -4 \\ 5 & 3 & 0 \\ -2 & -4 & 8 \end{vmatrix} = 0$.

Solution: Expanding by 1st row

$$\begin{vmatrix} 1 & x & -4 \\ 5 & 3 & 0 \\ -2 & -4 & 8 \end{vmatrix} = 1\begin{vmatrix} 3 & 0 \\ -4 & 8 \end{vmatrix} - x\begin{vmatrix} 5 & 0 \\ -2 & 8 \end{vmatrix} + (-4)\begin{vmatrix} 5 & 3 \\ -2 & -4 \end{vmatrix}$$

$$= 1(24) - x(40) - 4(-20+6)$$
$$= 24 - 40x + 56$$
$$= -40x + 80$$
$$\Rightarrow -40x + 80 = 0$$
$$\Rightarrow x = 2 .$$

Example 4: Solve for x if $\begin{vmatrix} 0 & 1 & 0 \\ x & 2 & x \\ 1 & 3 & x \end{vmatrix} = 0$.

Solution: Given, $\begin{vmatrix} 0 & 1 & 0 \\ x & 2 & x \\ 1 & 3 & x \end{vmatrix} = 0$

$$\Rightarrow 0\begin{vmatrix} 2 & x \\ 3 & x \end{vmatrix} - 1\begin{vmatrix} x & x \\ 1 & x \end{vmatrix} + 0\begin{vmatrix} x & 2 \\ 1 & 3 \end{vmatrix} = 0$$
$$\Rightarrow 0 - 1(x^2 - x) + 0 = 0$$
$$\Rightarrow -x^2 + x = 0$$
$$\Rightarrow x(1-x) = 0$$
$$\Rightarrow x = 0,1 .$$

CHECK YOUR PROGRESS

Q.5: Find the value of the following determinants:

a) $\begin{vmatrix} 2 & 1 & -3 \\ 1 & -2 & 1 \\ 2 & 1 & -1 \end{vmatrix}$ b) $\begin{vmatrix} 4 & -1 & 5 \\ 5 & 7 & -3 \\ 3 & 4 & 1 \end{vmatrix}$

c) $\begin{vmatrix} 1 & -1 & 2 \\ 4 & 1 & 1 \\ 5 & -1 & 8 \end{vmatrix}$

i) by expanding about any one row

ii) by expanding about any one column.

Q.6: Factorise the determinants and solve the equations :

a) $\begin{vmatrix} x & x^2 & x^2 \\ 3 & 9 & 27 \\ -1 & -2 & -3 \end{vmatrix} = 0$ b) $\begin{vmatrix} 1 & x^3 & 2 \\ 1 & x^2 & 3 \\ 1 & x & 4 \end{vmatrix} = 0$

4.6 LET US SUM UP

- The value of determinant is unaltered, when its rows and columns are interchanged.
- If any two adjacent rows (columns) of a determinant are interchanged, then the value of the determinant changes only in sign.
- If the determinant has two identical rows (columns), then the value of the determinant is zero.
- If all the elements in a row or in a (column) of a determinant are multiplied by a constant k(k > 0) then the value of the determinant is multiplied by k.
- The value of the determinant is unaltered when a constant multiple of the elements of any row (column), is added to the corresponding elements of a different row (column) in a determinant.
- If each element of a row (column) of a determinant is expressed as the sum of two or more terms, then the determinant is expressed as the sum of two or more determinants of the same order.

4.7 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1: i) $\begin{vmatrix} 2 & -3 \\ 7 & 11 \end{vmatrix} = 22 + 21 = 43$

ii) $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1$

iii) $\begin{vmatrix} 2+\sqrt{3} & 3+\sqrt{11} \\ 3-\sqrt{11} & 2-\sqrt{3} \end{vmatrix} = \left(2+\sqrt{3}\right)\left(2-\sqrt{3}\right) - \left(3+\sqrt{11}\right)\left(3-\sqrt{11}\right)$

$= (4-3) - (9-11) = 3$

Ans. to Q. No. 2: i) Given, $\begin{vmatrix} 2 & 3 \\ 1 & 4x \end{vmatrix} = \begin{vmatrix} 2x & -1 \\ 5 & x \end{vmatrix}$

$\Rightarrow 8x - 3 = 2x^2 + 5$

$\Rightarrow 2x^2 - 8x + 8 = 0$

$\Rightarrow x^2 - 4x + 4 = 0$

$\Rightarrow (x-2)^2 = 0$

$\Rightarrow x = 2,2$.

ii) Given, $\begin{vmatrix} x & 3 \\ 4 & x \end{vmatrix} = \begin{vmatrix} 0 & -2 \\ 2x & 5 \end{vmatrix}$

$\Rightarrow x^2 - 12 = 0 + 4x$

$\Rightarrow x^2 - 4x - 12 = 0$

$$\Rightarrow x^2 - 6x + 2x - 12 = 0$$
$$\Rightarrow x(x-6) + 2(x-6) = 0$$
$$\Rightarrow (x-6)(x+2) = 0$$
$$\Rightarrow x = -2, 6$$

Ans. to Q. No. 3: Given, $\begin{vmatrix} 3 & p \\ 4 & 6 \end{vmatrix} = 6$

$$\Rightarrow 18 - 4p = 6$$
$$\Rightarrow -4p = -12$$
$$\Rightarrow p = 3$$

Ans. to Q. No. 4: L.H.S $= \begin{vmatrix} a+ib & c+id \\ c-id & a-ib \end{vmatrix}$

$$= (a+ib)(a-ib) - (c+id)(c-id)$$
$$= \{a^2 - (ib)^2\} - \{c^2 - (id)^2\}$$
$$= a^2 + b^2 - c^2 - d^2$$

Ans. to Q. No. 5: a) i) Expanding about the first row

$$\begin{vmatrix} 2 & 1 & -3 \\ 1 & -2 & 1 \\ 2 & 1 & -1 \end{vmatrix} = 2\begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} - 3\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix}$$

$$= 2(2-1) - 1(-1-2) - 3(1+4)$$
$$= 2 + 3 - 15$$
$$= -10$$

ii) Expanding about the first column

$$\begin{vmatrix} 2 & 1 & -3 \\ 1 & -2 & 1 \\ 2 & 1 & -1 \end{vmatrix} = 2\begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} - 1\begin{vmatrix} 1 & -3 \\ 1 & -1 \end{vmatrix} + 2\begin{vmatrix} 1 & -3 \\ -2 & 1 \end{vmatrix}$$

$$= 2(2-1) - 1(-1+3) + 2(1-6)$$
$$= 2 - 2 - 10$$
$$= -10$$

b) Do yourself

c) Do yourself.

Ans. to Q. No. 6: a) Given, $\begin{vmatrix} x & x^2 & x^2 \\ 3 & 9 & 27 \\ -1 & -2 & -3 \end{vmatrix} = 0$

$$\Rightarrow 3x\begin{vmatrix}1 & x & x^2\\ 1 & 3 & 9\\ -1 & -2 & -3\end{vmatrix}=0$$

$$\Rightarrow 3x\begin{vmatrix}3 & 9\\ -2 & -3\end{vmatrix}-3x\begin{vmatrix}x & x^2\\ -2 & -3\end{vmatrix}-3x\begin{vmatrix}x & x^2\\ 3 & 9\end{vmatrix}=0$$

$$\Rightarrow 3x(-9+18)-3x(-3x+2x^2)-3x(9x-3x^2)=0$$

$$\Rightarrow 3x(9+3x-2x^2-9x+3x^2)=0$$

$$\Rightarrow 3x(x^2-6x+9)=0$$

$$\Rightarrow 3x(x-3)^2=0$$

$$\Rightarrow x=0,3$$

b) Do yourself.

4.8 FURTHER READING

- University Algebra, N S Gopala Krishnan, New Age International (P) Limited.
- Theory of Matrices, B S Vatsa, New Age International Publishers..
- Linear Algebra: Schum Solved Problems Series; Seymour Lipschitz: Tata McGraw Hill.
- Text Book of Algebra, A.K Sharma, DPH, Mathematics Series.
- A Text Book of Matrices, Shanti Narayan and P K Mittal.

4.9 MODEL QUESTIONS

Q.1: Evaluate: i) $\begin{vmatrix}4 & 6\\ -2 & 3\end{vmatrix}$ ii) $\begin{vmatrix}3 & 2\\ 4 & 5\end{vmatrix}$ iii) $\begin{vmatrix}-2 & -4\\ -1 & -6\end{vmatrix}$

Q.2: Evaluate: i) $\begin{vmatrix}1 & 2 & 0\\ 3 & -1 & 4\\ 1 & 2 & 4\end{vmatrix}$ ii) $\begin{vmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{vmatrix}$

Q 3: Show that: $\begin{vmatrix}a+b & b+c & c+a\\ b+c & c+a & a+b\\ c+a & a+b & b+c\end{vmatrix}=2\begin{vmatrix}a & b & c\\ b & c & a\\ c & a & b\end{vmatrix}$.

Q 4: Show that: $\begin{vmatrix}1 & 1 & 1\\ a & b & c\\ a^3 & b^3 & c^3\end{vmatrix}=(b-a)(c-a)(a-b)(a+b+c)$

Q.5: Show that: $\begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix} = 0$

Q.6: Show that $\begin{vmatrix} a & b & c \\ a-c & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix} = a^3+b^3+c^3-3abc$

Q.7: Prove that $\begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix} = x^2(x+a+b+c)$

Q.8: Show that $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix} = xy$

Q.9: Without expanding the determinant, prove that:

$$\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$$

*** ***** ***