UNIT 5: MATRICES

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5.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- understand the definition of a matrix
- describe different types of matrix
- learn matrix operations e.g. addition, subtraction and multiplication.

5.2 INTRODUCTION

Matrix is one of the most powerful tools of modern mathematics. Matrix notations and operations are used in Computer graphics, programming and implementation of electronic spreadsheet programmes.

Matrix was introduced by the English mathematician Arthur Cayley in 1858. Matrices are initially connected with linear transformation. Matrix theory now occupies important position in Physics, Economics, Statistics, Engineering etc.

5.3 DEFINITION OF MATRIX

A set of mn numbers (real or complex) arranged in the form of a rectangular array having m rows and n columns is called an mxn matrix (to be read as 'm' by 'n' matrix).

An m x n matrix is usually written as–

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

The compact representation of the above matrix is–

$A = [a_{ij}]$, $i = 1, 2, ..., ..., m$, $j = 1, 2, ..., ..., n$.

or, simply, $A = [a_{ij}]_{mxn}$.

Sometimes, the bracket [ ] can be replaced by ( ).

The general element of the matrix A is a_{ij} which belongs to the i-th row and j-th column. a_{ij} is sometimes denoted by (i, j)th element of the matrix. Here i, the first suffix denotes the number of row and j, the second suffix denotes the number of column in which the element a_{ij} occurs.

Note:

1) A matrix having m-rows and n-columns is called a matrix of order m x n (read as 'm' by 'n') or simply m x n matrix.

2) In a matrix, the number of rows need not be equal to the number of columns.

3) A matrix is denoted by the capital letters A, B, C, … etc. whereas any elements of a matrix is denoted by small letters such as– $a_{ij}, b_{ij}, c_{ij}, \cdots\cdots$ etc.

Illustrative example:

$A = \begin{bmatrix} 1 & 2 & 0 \\ 4 & 3 & 5 \end{bmatrix}_{2\times3}$ or $\begin{pmatrix} 1 & 2 & 0 \\ 4 & 3 & 5 \end{pmatrix}_{2\times3}$

is a 2 x3 matrix i.e. it has 2 rows and 3 columns. For A,

a_{12} = (1, 2)th element = 2

a_{23} = (2,3)th element = 5

a_{13} = (1, 3)th element = 0

is a 3 x 3 matrix. For B,

$b_{11} = 1 + i$, $b_{31} = \sqrt{3}$, $b_{22} = 0$ etc.

5.4 TYPES OF MATRIX

There are different types of matrices. We will discuss one by one.

i) Row Matrix: Any 1x n matrix which has only one row and n column is called a row matrix.

For example, $A = [2 \;\; 5 \;\; -3 \;\; 0 \;\; \sqrt{3}]$ is a row matrix order 1×5.

In general, $B = [a_{ij}]_{1\times n}$ is a row matrix of order 1 x n

ii) Column Matrix: Any m x 1 matrix which has m rows and only one column is called a column matrix.

For example, $B = \begin{bmatrix} 6 \\ 0 \\ -3 \\ \sqrt{2} \end{bmatrix}$ is a column matrix of order 4 × 1

In general, $B = [b_{ij}]_{m\times1}$ is a column matrix of order $m \times 1$.

iii) Square Matrix: A matrix whose numbers of rows is equal to the number of columns is called a square matrix.

For the matrix $A = [a_{ij}]_{m\times n}$, if m = n, then the matrix A is said to be square matrix of order m.

Example: a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ is a square matrix of order 2.

b) $B = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 3 & 1 \\ 5 & 0 & 1 \end{bmatrix}$ is a square matrix of order 3

In a square matrix $A = [a_{ij}]_{m\times m}$, the elements a_{11}, a_{22}, a_{33} ..... are called the **Diagonal Elements** and the line along which they lie is called the **Principal Diagonal** of the matrix.

iv) Diagonal Matrix: A square matrix A is said to be a diagonal matrix if all its non-diagonal elements are zero.

Examples: $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

are diagonal matrices of order 2 and 3 respectively.

v) Scalar Matrix: A diagonal matrix (i.e. all non-diagonal elements being zero) where all the diagonal elements are equal is called a Scalar Matrix.

Examples : $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, B = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$

are two scalar matrices of order 2 and 3 respectively.

Thus, the square matrix $A = [a_{ij}]_{m\times m}$ is a scalar matrix if

a_{ij} = 0, when $i \neq j$

a_{ij} = k(say) when i = j

vi) Unit (or Identity) Matrix : A square matrix in which the diagonal elements are unity and non-diagonal elements are all zero is called a unit (or Identity)matrix.

Examples: $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Unit matrices are denoted by I.

I_2 and I_3 are Unit matrices of order 2 and 3 respectively.

Thus, the square matrix $A = [a_{ij}]_{m\times m}$ is a unit matrix if

a_{ij} = 0 when $i \neq j$

a_{ij} = 1 when i = j

vii) Null Matrix or Zero Matrix: A matrix of order m x n in which all the elements are

zero is called a null matrix (a zero matrix). It is denoted by O.

Thus, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

are all zero matrices of order 2 x 2 and 4 x 3 respectively.

viii) Upper Triangular Matrix and Lower Triangular Matrix: Let A be a square matrix of order n.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \ldots a_{1n} \\ a_{21} & a_{22} & a_{23} \ldots a_{2n} \\ a_{31} & a_{32} & a_{33} \ldots a_{3n} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1} & a_{n2} & a_{n3} \ldots a_{nn} \end{bmatrix}$$

The diagonal elements of A are $a_{11}, a_{12}, \ldots\ldots, a_{nn}$ i.e. a_{ij} for i = j.
All those elements of A above the principal diagonal are $a_{12}, a_{13} \ldots\ldots a_{1n}, a_{23} \ldots\ldots a_{2n} \ldots\ldots$ etc. i.e. a_{ij} for i < j
All those elements of A below the principal diagonal are $a_{21}, a_{31}, a_{32}, \ldots\ldots a_{n1}, a_{n2}, \ldots\ldots$ i.e. a_{ij} for i > j.
If all a_{ij} for i > j be zeros in a square matrix A, then A is called **Upper Triangular matrix.**

Example : $A = \begin{bmatrix} 3 & 1 & 2 & 6 \\ 0 & 2 & -4 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 5 \end{bmatrix}$ is Upper Triangular Matrix

If all a_{ij} for i < j be zeros is a square matrix A, the A is called **Lower Triangular Matrix.**

Example: $A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 2 & -1 & 4 & 0 \\ -2 & 1 & 3 & 5 \end{bmatrix}$ is Lower Triangular Matrix

Remarks: A diagonal matrix is both upper and lower triangular.

5.5 EQUALITY OF MATRICES

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal if

i) they are of the same size or order

ii) each element of one matrix is equal to the corresponding element of the other matrix i.e. $a_{ij} = b_{ij}$ for all i and j.

If A and B are equal, then we write A = B.If A and B are not equal, then we write $A \neq B$.

Example: 1) If $A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}_{2\times2}, B = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}_{2\times2}$ then A = B .

2) If $A = \begin{pmatrix} 1^2 & 3^2 \\ 2^2 & 4^2 \end{pmatrix}_{2\times2}, B = \begin{pmatrix} 1 & 9 \\ 4 & 16 \end{pmatrix}_{2\times2}$ then A = B.

3) If $A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}_{2\times2}, B = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}_{2\times3}$ then $A \neq B$.

4) If $\begin{pmatrix} x & a \\ y & b \\ z & c \end{pmatrix}_{3\times2} = \begin{pmatrix} -3 & 0 \\ 1 & 2 \\ 4 & \sqrt{3} \end{pmatrix}_{3\times2}$

then, x = - 3, y = 1, z = 4, a = 0, b = 2, c = $\sqrt{3}$

Remarks: The inequality of two matrices arise due to the following reasons:

i) This orders may not be equal.

ii) Elements in the corresponding places may not be equal.

CHECK YOUR PROGRESS

Q.1: If a matrix has 10 elements what are the possible orders it can have?

Q.2: Find the values of a, b, c, d when

$$\begin{pmatrix} 2a+b & 5c-d \\ a-2b & 4c+d \end{pmatrix} = \begin{pmatrix} 4 & 11 \\ -3 & 25 \end{pmatrix}$$

Q.3: Are the following matrices equal where

$$A = \begin{pmatrix} 2 & 4 & 3 \\ 1 & 0 & 5 \\ 0 & 0 & 6 \end{pmatrix}, B = \begin{pmatrix} 2 & 4 & 3 \\ 1 & 0 & 5 \\ 0 & 0 & 0 \end{pmatrix} ?$$

5.6 ADDITION AND SUBTRACTION OF MATRICES

Let $A = [a_{ij}]_{mxn}$ and $B = [b_{ij}]_{mxn}$ be two matrices of the same order m x n. Then their sum(or difference) denoted by A+B (or A – B) is defined as another matrix $C = [c_{ij}]$ of the order m xn such that any element of C is the sum (or difference) of the corresponding element of A and B.

i.e. $c_{ij} = a_{ij} + b_{ij}$ for all i, j

Thus, $C = [c_{ij}] = [a_{ij}+b_{ij}]_{mxn}$

Example: If $A=\begin{pmatrix}5 & 1\\ 3 & 0\end{pmatrix}, B=\begin{pmatrix}2 & -5\\ -2 & 1\end{pmatrix}$

Then, $C = A + B = \begin{pmatrix}5+2 & 1-5\\ 3+(-2) & 0+1\end{pmatrix} = \begin{pmatrix}7 & -4\\ 1 & 1\end{pmatrix}$

Similarly, $D = A - B = \begin{pmatrix}5-2 & 1-(-5)\\ 3-(-2) & 0-1\end{pmatrix} = \begin{pmatrix}3 & 6\\ 5 & -1\end{pmatrix}$

Remark: Two matrices A and B are said to be conformable for addition or subtraction if they are of the same order.

Negative of a Matrix: If A be a given matrix, then the negative of A denoted by – A is the matrix whose elements are the negative of the corresponding elements of A.

Example: If $A=\begin{pmatrix}2 & -1\\ 0 & 4\\ -5 & 6\end{pmatrix}$ then $-A=\begin{pmatrix}-2 & 1\\ 0 & -4\\ 5 & -6\end{pmatrix}$

Properties of Matrix Addition: The addition of matrices satisfies the following properties–

i) Commutative Law: If $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices of the same order, say m x n, then, A + B = B + A

Proof: $A+B = [a_{ij}] + [b_{ij}]$

$= [a_{ij} + b_{ij}]$

$= [b_{ij} + a_{ij}]$ ($\because$ addition of numbers is commutative)

$= [b_{ij}] + [a_{ij}]$

$= B + A$

ii) Associative Law: If $A = [a_{ij}]$, $B = [b_{ij}]$ and $C = [c_{ij}]$ are matrices of the same order mxn, then (A + B) + C = A + (B + C)

Proof: $(A + B) + C = ([a_{ij}] + [b_{ij}]) + [c_{ij}]$

$= [a_{ij} + b_{ij}] + [c_{ij}]$

$= [(a_{ij} + b_{ij}) + c_{ij}]$

$= [a_{ij}] + [(b_{ij} + c_{ij})]$

$= [a_{ij}] + ([b_{ij}] + [c_{ij}])$

$= A + (B+C)$

i.e. $(i,j)^{th}$ element of (A+B)+C = $(i,j)^{th}$ element of A+(B+C)

iii) Existence of Additive Identity: If O be the zero matrix of order $m\times n$ and $A = [a_{ij}]$ be an $m\times n$ matrix, then A + O = O + A = A

Proof: $A + O = [a_{ij}] + [0]$

$= [a_{ij} + 0] = [a_{ij}] = A$

and $O + A = [0] + [a_{ij}]$

$= [0+a_{ij}]$

$= [a_{ij}]$

$= A$

$\therefore A + O = O + A = A$

Hence, the zero matrix of the type $m \times n$ plays the role of additive identity.

iv) Existence of Additive Inverse: If $A = [a_{ij}]$ be a matrix of order $m \times n$, then we have another matrix $-A = [-a_{ij}]$ such that–

$$A + (-A) = (-A) + A = 0$$

Proof: $A + (-A) = [a_{ij}] + [-a_{ij}]$

$= [a_{ij} - a_{ij}]$

$= [0] = 0$

$(-A) + A = [-a_{ij}] + [a_{ij}]$

$= [-a_{ij} + a_{ij}] = [0] = 0$

Hence, $A + (-A) = (-A) + A = 0$

Here, $-A$ acts as the additive inverse of A.

v) Subtraction of two Matrices: If A and B be two matrices of order $m \times n$, then we define $A - B = A + (-B)$.

In other words, the difference $A - B$ is obtained by subtracting from each element of A the corresponding element of B.

vi) Cancellation Laws for addition of Matrices: If A, B, C are three matrices of the same order say $m \times n$, then–

$A + B = A + C \Rightarrow B = C$ [Left Cancellation law]

$B + A = C + A \Rightarrow B = C$ (Right Cancellation Law)

Proof: We have $A + B = A + C$

$\Rightarrow -A + (A + B) = -A + (A + C)$ [Adding $-A$ to both sides]

$\Rightarrow (-A + A) + B = (-A+A) + C$ [$\because$ matrix addition is associative]

$\Rightarrow O + B = O + C$

$\Rightarrow B = C$

Similarly: $B + A = C + A$

$\Rightarrow (B + A) + (-A) = (C + A) + (-A)$

$\Rightarrow B + (A - A) = C + (A - A)$

$\Rightarrow B + O = C + O$

$\Rightarrow B = C$

Example:

1) If $A = \begin{pmatrix} 5 & 0 \\ 3 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}$, find A + B and A – B

Solution: $A + B = \begin{pmatrix} 5 & 0 \\ 3 & 1 \end{pmatrix} + \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix} = \begin{pmatrix} 8 & 1 \\ 9 & 3 \end{pmatrix}$

$$A - B = \begin{pmatrix} 5 & 0 \\ 3 & 1 \end{pmatrix} - \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 5-3 & 0-1 \\ 3-6 & 1-2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -1 \\ -3 & -1 \end{pmatrix}$$

2) If $A = \begin{pmatrix} 1 & 0 \\ 4 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 3 \\ 5 & -2 \end{pmatrix}, C = \begin{pmatrix} 3 & 5 \\ 2 & 0 \end{pmatrix}$

Show that (A + B) + C = A + (B + C)

Solution: $A + B = \begin{pmatrix} 1 & 0 \\ 4 & 3 \end{pmatrix} + \begin{pmatrix} 0 & 3 \\ 5 & -2 \end{pmatrix} = \begin{pmatrix} 1+0 & 0+3 \\ 4+5 & 3-2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 9 & 1 \end{pmatrix}$

$$(A+B)+C = \begin{pmatrix} 1 & 3 \\ 9 & 1 \end{pmatrix} + \begin{pmatrix} 3 & 5 \\ 2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1+3 & 3+5 \\ 9+2 & 1+0 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 8 \\ 11 & 1 \end{pmatrix}$$

Again, $B + C = \begin{pmatrix} 0 & 3 \\ 5 & -2 \end{pmatrix} + \begin{pmatrix} 3 & 5 \\ 2 & 0 \end{pmatrix}$

$$= \begin{pmatrix} 0+3 & 3+5 \\ 5+2 & -2+0 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 8 \\ 7 & -2 \end{pmatrix}$$

$$\therefore A + (B+C) = \begin{pmatrix} 1 & 0 \\ 4 & 3 \end{pmatrix} + \begin{pmatrix} 3 & 8 \\ 7 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 1+3 & 0+8 \\ 4+7 & 3-2 \end{pmatrix} = \begin{pmatrix} 4 & 8 \\ 11 & 1 \end{pmatrix}$$

$$\therefore (A+B)+C = A+(B+C)$$

3) Find the additive inverse of $A = \begin{pmatrix} 2 & 3 & 1 \\ 0 & -1 & 5 \end{pmatrix}_{2\times 3}$

Solution: A is a matrix of order 2×3.

–A will also be a matrix of order 2×3.

$$-A = \begin{pmatrix} 2 & 3 & 1 \\ 0 & -1 & 5 \end{pmatrix} = \begin{pmatrix} -2 & -3 & -1 \\ 0 & 1 & -5 \end{pmatrix}_{2\times 3}$$

CHECK YOUR PROGRESS

Q.4: Is it possible to define A + B when–

i) A has 3 rows, 2 columns and B has 2 rows, 2 columns.

ii) A has 3 rows ,3 columns and B has 3 rows, 3 columns.

iii) A and B are square matrices of same order?

Q.5: What is the additive identity of 2 x 3 matrices?

5.7 MULTIPLICATION OF A MATRIX BY A SCALAR

Multiplication of a matrix by a scalar is defined as follows:

If $A = [a_{ij}]_{m\times n}$ is a matrix of order m×n and k is a scalar,

then $kA = k[a_{ij}]_{m\times n} = [ka_{ij}]_{m\times n}$

i.e., (i, j)th element of kA is k times the (i, j) th element of A .

Example: If k = 3 and $A = \begin{pmatrix} 2 & -1 \\ -2 & 3 \\ 0 & 1 \end{pmatrix}_{3\times 2}$, then

$$3A = 3\begin{pmatrix} 2 & -1 \\ -2 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3\times 2 & 3\times(-1) \\ 3\times(-2) & 3\times 3 \\ 3\times 0 & 3\times 1 \end{pmatrix} = \begin{pmatrix} 6 & -3 \\ -6 & 9 \\ 0 & 3 \end{pmatrix}_{3\times 2}$$

and, $$\frac{1}{2}A = \frac{1}{2}\begin{pmatrix} 2 & -1 \\ -2 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\times 2 & \frac{1}{2}\times(-1) \\ \frac{1}{2}\times(-2) & \frac{1}{2}\times 3 \\ \frac{1}{2}\times 0 & \frac{1}{2}\times 1 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{2} \\ -1 & \frac{3}{2} \\ 0 & \frac{1}{2} \end{pmatrix}_{3\times 2}$$

Properties of Multiplication of a Matrix by a Scalar: If $A = [a_{ij}]$ and $B = [b_{ij}]$ are two matrices of the same order m x n; If k and l are two scalars, then–

i) k(A + B) = kA + kB

ii) (k + l)A = kA + lA

iii) k(lA) = (kl)A

iv) (–k)A = – (kA) = k(–A)

Proof of (i): Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$

Then $k(A + B) = k([a_{ij}] + [b_{ij}])$

$= k([a_{ij} + b_{ij}])$

$= ([ka_{ij} + kb_{ij}])$

$= [ka_{ij} + kb_{ij}]$

$= [ka_{ij}] + [kb_{ij}]$

$= k[a_{ij}] + k[b_{ij}]$

$= kA + kB.$

Example: If $A = \begin{pmatrix} 3 & 9 \\ 1 & 8 \end{pmatrix}_{2\times2}$, $B = \begin{pmatrix} 4 & 0 \\ 7 & 2 \end{pmatrix}_{2\times2}$

Then verify that $2(A + B) = 2A + 2B$

Solution: We have $A + B = \begin{bmatrix} 3 & 9 \\ 1 & 8 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 7 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 3+4 & 9+0 \\ 1+7 & 8+2 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 9 \\ 8 & 10 \end{bmatrix}$$

$$\therefore 2\,(A+B) = 2\begin{bmatrix} 7 & 9 \\ 8 & 10 \end{bmatrix} = \begin{bmatrix} 2\times7 & 2\times9 \\ 2\times8 & 2\times10 \end{bmatrix} = \begin{bmatrix} 14 & 18 \\ 16 & 20 \end{bmatrix}$$

Again, $2A + 2B = 2\begin{bmatrix} 3 & 9 \\ 1 & 8 \end{bmatrix} + 2\begin{bmatrix} 4 & 0 \\ 7 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 2\times3 & 2\times9 \\ 2\times1 & 2\times8 \end{bmatrix} + \begin{bmatrix} 2\times4 & 2\times0 \\ 2\times7 & 2\times2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 18 \\ 2 & 16 \end{bmatrix} + \begin{bmatrix} 8 & 0 \\ 14 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 6+8 & 18+0 \\ 2+14 & 16+4 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 18 \\ 16 & 20 \end{bmatrix}$$

$\therefore 2(A + B) = 2A + 2B$ verified.

Proof of (ii): Let $A = [a_{ij}]$, then

$(k + l)A = (k+l)[a_{ij}]$

$$= \left[ka_{ij} + la_{ij}\right]$$

$$= \left[ka_{ij}\right] + \left[la_{ij}\right]$$

$$= k\left[a_{ij}\right] + l\left[a_{ij}\right]$$

$= kA + lA$

Example: If k = 3, l = –2 and A = $\begin{bmatrix} 3 & 2 \\ 1 & 0 \\ 2 & 1 \end{bmatrix}$

Then verify that (k+l) A = kA + lA

$(k + l)A = (3–2)\begin{bmatrix} 3 & 2 \\ 1 & 0 \\ 2 & 1 \end{bmatrix} = (1)\begin{bmatrix} 3 & 2 \\ 1 & 0 \\ 2 & 1 \end{bmatrix}$

Again, $kA + lA = 3A – 2A = 3\begin{bmatrix} 3 & 2 \\ 1 & 0 \\ 2 & 1 \end{bmatrix} - 2\begin{bmatrix} 3 & 2 \\ 1 & 0 \\ 2 & 1 \end{bmatrix}$

$= \begin{bmatrix} 9 & 6 \\ 3 & 0 \\ 6 & 3 \end{bmatrix} - \begin{bmatrix} 6 & 4 \\ 2 & 0 \\ 4 & 2 \end{bmatrix}$

$= \begin{bmatrix} 9 - 6 & 6 - 4 \\ 3 - 2 & 0 - 0 \\ 6 - 4 & 3 - 2 \end{bmatrix}$

$= \begin{bmatrix} 3 & 2 \\ 1 & 0 \\ 2 & 1 \end{bmatrix}$

$\therefore (3 + (-2))A = 3A + (-2)A$. Verified

Proof of (iii): Let A = $[a_{ij}]$, then

$k(lA) = k(l[a_{ij}])$

$= k[l\,a_{ij}]$

$= [k(l\,a_{ij})]$

$= [(kl)\,a_{ij}]$

$= (kl)\,[a_{ij}] = (kl)\,A.$

Example: If k = 4, l = 2 and A = $\begin{bmatrix} 3 & 2 \\ 1 & 0 \\ 2 & 1 \end{bmatrix}$, then Verify that k(lA) = (kl)A.

Solution: $k(lA) = 4(2A) = 4\begin{bmatrix} 2\times 3 & 2\times 2 \\ 2\times 1 & 2\times 0 \\ 2\times 2 & 2\times 1 \end{bmatrix}$

$$= \begin{bmatrix} 4\times2\times3 & 4\times2\times2 \\ 4\times2\times1 & 4\times2\times0 \\ 4\times2\times2 & 4\times2\times1 \end{bmatrix}$$

$$= \begin{bmatrix} 8\times3 & 8\times2 \\ 8\times1 & 8\times0 \\ 8\times2 & 8\times1 \end{bmatrix}$$

$$= 8\begin{bmatrix} 3 & 2 \\ 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$= 8A$

$= (4 \times 2)A$

$= (kl)A.$ verified

Proof of (iv): If $A = [a_{ij}]$ then

$(-k)\,A = [(-k)\,a_{ij}]$

$= [-(k\ a_{ij})]$

$= -[k\ a_{ij}] = -(kA)$

Also $k(-A) = k[-a_{ij}]$

$= [k.-(a_{ij})]$

$= -[k\ a_{ij}]$

$= -(kA)$

$\therefore (-k)\,A = -(kA) = k(-A)$

Example: If $k = 5$ and $A = \begin{bmatrix} 3 & 2 \\ 1 & 0 \\ 2 & 1 \end{bmatrix}$ Verify that $(-k)A = -(kA) = k\,(-A)$

Solution: $(-k)\,A = (-5)\begin{bmatrix} 3 & 2 \\ 1 & 0 \\ 2 & 1 \end{bmatrix}$

$$= \begin{bmatrix} (-5)\times3 & (-5)\times2 \\ (-5)\times1 & (-5)\times0 \\ (-5)\times2 & (-5)\times1 \end{bmatrix}$$

$$= -\begin{bmatrix} (5)\times3 & (5)\times2 \\ (5)\times1 & (5)\times0 \\ (5)\times2 & (5)\times1 \end{bmatrix} = -(kA)$$

Also, $k(-A) = 5\begin{bmatrix} -3 & -2 \\ -1 & 0 \\ -2 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 5\times(-3) & 5\times(-2) \\ 5\times(-1) & 5\times 0 \\ 5\times(-2) & 5\times(-1) \end{bmatrix}$$

$$= -\begin{bmatrix} 5\times 3 & 5\times 2 \\ 5\times 1 & 5\times 0 \\ 5\times 2 & 5\times 1 \end{bmatrix}$$

$$= -5\begin{bmatrix} 3 & 2 \\ 1 & 0 \\ 2 & 1 \end{bmatrix} = -(kA) \quad \text{Verified.}$$

Example: If $A = \begin{bmatrix} 0 & 3 & 2 \\ 1 & 4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 0 & -3 \\ 0 & 1 & 3 \end{bmatrix}$, find 3A – 2B.

Solution: $3A - 2B = 3\begin{bmatrix} 0 & 3 & 2 \\ 1 & 4 & 2 \end{bmatrix} - 2\begin{bmatrix} 4 & 0 & -3 \\ 0 & 1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 9 & 6 \\ 3 & 12 & 6 \end{bmatrix} - \begin{bmatrix} 8 & 0 & -6 \\ 0 & 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0-8 & 9-0 & 6-(-6) \\ 3-0 & 12-2 & 6-4 \end{bmatrix} = \begin{bmatrix} -8 & 9 & 12 \\ 3 & 10 & 2 \end{bmatrix}$$

5.8 MULTIPLICATION OF MATRICES

Two matrices A and B are conformable for the product AB if and only if the number of columns of A is equal to the number of rows of B.

Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{jk}]_{n\times p}$

Then the product AB is the matrix of order m × p

i.e. $AB = C = [c_{jk}]_{m\times p}$ where

$$C_{jk} = \sum_{j=1}^{n} a_{ij}\, b_{jk} = a_{i1}\, b_{1k} + a_{i2}\, b_{2k} + \ldots\ldots\ldots\ldots + a_{in}\, b_{nk}$$

i.e. the (j, k) th element of the matrix C = AB is found by multiplying the corresponding element of the j-th row of A and the k-th column of B and then, adding the product.

The rule of multiplication is *row by column multiplication*.

Example: If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}_{3\times 2}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}_{2\times 2}$

Then $AB = \begin{bmatrix} a_{11}\, b_{11} + a_{12}\, b_{21} & a_{11}\, b_{12} + a_{12}\, b_{22} \\ a_{21}\, b_{11} + a_{22}\, b_{21} & a_{21}\, b_{12} + a_{22}\, b_{22} \\ a_{31}\, b_{11} + a_{32}\, b_{21} & a_{31}\, b_{12} + a_{32}\, b_{22} \end{bmatrix}_{3\times 2}$

Note: 1) If the product AB exists, then it is not necessary that the product BA will also exist.

Example: $A = [2\ \ 3]_{1\times 2}$ and $B = \begin{bmatrix} 4 & 0 & 5 \\ 1 & 2 & 3 \end{bmatrix}_{2\times 3}$

Here, $AB = \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} 4 & 0 & 5 \\ 1 & 2 & 3 \end{bmatrix}$

$= \begin{bmatrix} 8+3 & 0+6 & 10+9 \end{bmatrix} = \begin{bmatrix} 11 & 6 & 19 \end{bmatrix}$

But $BA = \begin{bmatrix} 4 & 0 & 5 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 3 \end{bmatrix}$ which does not exist

$\therefore AB \neq BA$

Matrix multiplication is not commutative.

2) In the case when both A and B are square matrices of the same order then also both AB and BA are defined but still AB may not be equal to BA.

Example: If $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 \\ 4 & 1 \end{bmatrix}$

Then $AB = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 4 & 1 \end{bmatrix}$

$= \begin{bmatrix} 2x3+1x4 & 2x5+1x1 \\ 0x3+2x4 & 0x5+2x1 \end{bmatrix}$

$= \begin{bmatrix} 10 & 11 \\ 8 & 2 \end{bmatrix}$

$BA = \begin{bmatrix} 3 & 5 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

$= \begin{bmatrix} 3\times 2+5\times 0 & 3\times 1+5\times 2 \\ 4\times 2+1\times 0 & 4\times 1+1\times 2 \end{bmatrix}$

$= \begin{bmatrix} 6 & 133 \\ 8 & 6 \end{bmatrix}$

It is seen that $AB \neq BA$

Example: If $A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 & 0 \\ -2 & 1 & 3 \end{bmatrix}$

Can you find AB and BA?

Solution: Here, A is of order 2 × 2

B is of order 2 × 3

$\therefore$ AB will be order 2 × 3

Again, B is of order 2 × 3

A is of order 2 × 2

$\therefore$BA does not exist because the number of columns of B is 3 and the number of rows of A is 2.

So, they are not conformable for the product BA.

Properties of Matrix Multiplication: The multiplication of matrices possesses the following properties:

1) The multiplication of matrices is not always commutative.
   a) Whenever AB exists, it is not necessary that BA should exist.
   b) Whenever AB and BA both exist, it is not always necessary that they should be matrices of same order.
   c) Whenever AB and BA exists and are of the same order, it is not necessary that AB = BA.
   d) In some cases, AB = BA.
   e) Matrix multiplication is associative i.e. A(BC) = (AB)C
      where A, B, C are matrices of order m×n, n×p, p×q respectively.
   f) Matrix multiplication is distributive w. r. t. addition of matrix. i.e, A(B + C) = AB + AC
      where, A, B, C are matrices of order m×n, n×p and n × p respectively.
   g) For every square matrix A, there exists the identity matrix I of same order of A such that AI = IA = A.

Matrix Multiplication is Associative: If A, B, C are matrices of order m × n, n × p, p × q, then A(BC) = (AB)C

Proof: Let $A= [a_{ij}]_{m\times n}$, $B = [b_{jk}]_{n\times p}$, $C = [c_{kl}]_{p\times q}$

Then $AB = U = [u_{ik}]_{m\times p}$ where $u_{ik} = \sum_{j=1}^{n} a_{ij}\, b_{jk}$

Also, $BC = V = \left[v_{jl}\right]_{n\times q}$ where $v_{jl} = \sum_{k=1}^{n} b_{jk} c_{kl}$

$\therefore A(BC) = W = [w_{il}]_{m\times q}$ when (i, j)th element of A(BC)

$$= W_{il} = \sum_{j=1}^{n} a_{ij}\, v_{jl}$$

$$= \sum_{j=1}^{n}\left[a_{ij}\left\{\sum_{K=1}^{n} b_{jk} c_{kl}\right\}\right]$$

$$= \sum_{K=1}^{n}\left[\left\{\sum_{j=1}^{n} a_{ij}\, b_{kl}\right\} C_{kl}\right]$$

$$= \sum_{k=1}^{n} u_{ik} c_{kl}$$

= $(i, j)^{th}$ element of (AB) C

$\therefore$ A(BC) = (AB) \C

Multiplication of Matrices is Distributive with Respect to Addition of Matrices: If A, B and C are matrices of order m×n, n×p, n×p respectively, then A(B + C) = AB + AC

Proof : Let $A = [a_{ij}]_{m\times n}$, $B = [b_{jk}]_{n\times p}$, $C = [c_{jk}]_{nxp}$

We have $B + C = [b_{jk} + c_{jk}]_{n\times p}$

$\therefore$ (i, k)th element of $A(B + C) = \sum_{j=1}^{n} a_{ij}(b_{jk} + c_{jk})$

$= \sum_{j=1}^{n} a_{ij}\, b_{jk} + \sum_{j=1}^{n} a_{ij}\, c_{jk}$

= (i, k)th element of AB + (i, k)th element of AC

= (i, k)th element of AB + AC

Here, A(B + C) = AB + AC

Similarly, we can show that, (B + C) D = BD + CD

where B, C and D are matrices of suitable types.

Example: If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2}$, $B = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}_{2\times 3}$

$C = \begin{bmatrix} 5 & 0 & 1 \\ 4 & 6 & 0 \end{bmatrix}_{2\times 3}$ then show that A(B + C) = AB + AC

Solution: $B+C = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 1 \\ 4 & 6 & 0 \end{bmatrix} = \begin{bmatrix} 8 & 1 & 3 \\ 4 & 8 & 1 \end{bmatrix}_{2\times 3}$

A(B + C) =

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2} \begin{bmatrix} 8 & 1 & 3 \\ 4 & 8 & 1 \end{bmatrix}_{2\times 3} = \begin{bmatrix} 1\times 8 + 2\times 4 & 1\times 1 + 2\times 8 & 1\times 3 + 2\times 1 \\ 3\times 8 + 4\times 4 & 3\times 1 + 4\times 8 & 3\times 3 + 4\times 1 \end{bmatrix}_{2\times 3}$$

$$= \begin{bmatrix} 16 & 17 & 5 \\ 40 & 35 & 13 \end{bmatrix}$$

Again, $AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 3 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1\times 3 + 2\times 0 & 1\times 1 + 2\times 2 & 1\times 2 + 2\times 1 \\ 3\times 3 + 4\times 0 & 3\times 1 + 4\times 2 & 3\times 2 + 4\times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 5 & 4 \\ 9 & 11 & 10 \end{bmatrix}$$

$$AC = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 0 & 1 \\ 4 & 6 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times 5 + 2\times 4 & 1\times 0 + 2\times 6 & 1\times 1 + 2\times 0 \\ 3\times 5 + 4\times 4 & 3\times 0 + 4\times 6 & 3\times 1 + 4\times 0 \end{bmatrix}$$

$$= \begin{bmatrix} 13 & 12 & 1 \\ 31 & 24 & 3 \end{bmatrix}$$

$$AB + AC = \begin{bmatrix} 3 & 5 & 4 \\ 9 & 11 & 10 \end{bmatrix} + \begin{bmatrix} 13 & 12 & 1 \\ 31 & 24 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 17 & 5 \\ 40 & 35 & 13 \end{bmatrix}$$

$\therefore A(B + C) = AB + AC$

Example: If $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{bmatrix}_{2\times2}$, $B = \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ -1 & 4 \end{bmatrix}_{3\times2}$, $C = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}_{2\times2}$

Find A(BC) and (AB)C and show that A(BC) $\neq$ (AB) C.

Solution: $BC = \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ -1 & 4 \end{bmatrix}_{3\times2} \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}_{2\times2}$

$$= \begin{bmatrix} 1\times1+3\times2 & 1\times2+3\times0 \\ 0\times1+2\times2 & 0\times2+2\times0 \\ -1\times1+4\times2 & -1\times2+4\times0 \end{bmatrix} = \begin{bmatrix} 7 & 2 \\ 4 & 0 \\ 7 & 2 \end{bmatrix}$$

$$A(BC) = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} 7 & 2 \\ 4 & 0 \\ 7 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times7+1\times4+(-1)\times7 & 1\times2+1\times0+(-1)\times(-2) \\ 2\times7+0\times4+3\times7 & 2\times2+0\times0+3\times(-2) \\ 3\times7+(-1)\times4+2\times7 & 3\times2+(-1)\times0+2\times(-2) \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 4 \\ 35 & -2 \\ 31 & 2 \end{bmatrix}$$

$$\text{and } AB = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ -1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times1+1\times0+(-1)(-1) & 1\times3+1\times2+(-1)4 \\ 2\times1+0\times0+3(-1) & 2\times3+0\times2+3\times4 \\ 3\times1+(-1)\times0+2\times(-1) & 3\times3+(-1)\times2+2\times4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ -1 & 18 \\ 1 & 15 \end{bmatrix}$$

$$(AB)C = \begin{bmatrix} 2 & 1 \\ -1 & 18 \\ 1 & 15 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2\times1+1\times2 & 2\times2+1\times0 \\ (-1)\times1+18\times2 & (-1)\times2+18\times0 \\ 1\times1+15\times2 & 1\times2+15\times0 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 35 & -2 \\ 31 & 2 \end{bmatrix}$$

CHECK YOUR PROGRESS

Q.6: Given: $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$, $D = \begin{bmatrix} 5 & 1 \\ 7 & 4 \end{bmatrix}$

Verify that $B(C + D) = BC + BD$

$(B + C)D = BD + CD$

Q.7: For the matrices A,B,C put the numerical values of their orders, so that (A + B)C = AC + BC

Q.8: Find A and B if $A + B = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}$, $A - B = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

Q.9: Given $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$,

Show that $A^2 = B^2 = I$ (unit matrix)

Q.10: If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = 0$

Q.11: If $A_\alpha = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, then prove that

$$(A_\alpha)^3 = \begin{bmatrix} \cos3\alpha & \sin3\alpha \\ -\sin3\alpha & \cos3\alpha \end{bmatrix}$$

5.9 LET US SUM UP

- A matrix with exactly one row is called a row matrix. The order of a row matrix is of the type 1 × n.
- A matrix with exactly one column is called a column matrix. The order of a column matrix is of the type m × 1.
- A matrix $A = [a_{ij}]_{m\times n}$ is a square matrix if m = n.
- A square matrix $a = [a_{ij}]$ is a scalar matrix if $a_{ij} = k$ where i = j = 0 where $i \neq j$.
- A square matrix $A = [a_{ij}]$ is a diagonal matrix if $a_{ij} = 0$ where $i \neq j$.
- A square matrix $A = [a_{ij}]$ is a unit matrix if $a_{ij} = 1$ where i = j = 0 where $i \neq j$; A unit

matrix of order n is denoted by I or I_n.

- A matrix in which all elements are zero is called a null matrix or zero matrix. A null matrix is denoted by O.
- A matrix is said to be upper triangular if $a_{ij} = 0$ when $i > j$.
- A matrix is said to be lower triangular if $a_{ij} = 0$ when $i < j$.
- Two matrices are said to be equal if and only if they are of same order and the corresponding elements are equal.
- If A and B are two matrices of same order then their sum or difference denoted by A + B, A – B or B – A can be obtained.
- Matrix addition is commutative i.e. A + B = B + A.
- Matrix addition is associative i.e. (A + B) + C = A + (B + C)
- Existence of additive identity i.e. A + O = A = O + A
- Existence of additive inverse i.e. A + (–A) = (–A) + A
- Scalar multiplication of matrix i.e

  if a) $A = [a_{ij}]$ then $kA = [k a_{ij}]$
  b) k(A + B) = kA + kB
  c) (k + l) A = kA + lA
  d) k(lA) = (kl)A
  e) (–k)A= k(–A)
- Matrix multiplication is not commutative i.e. AB ≠ BA (in general) Matrix multiplication is associative i.e. (AB)C = A(BC)
- Matrix multiplication is distributive over addition i.e.

  i) A(B + C) = AB + AC

  ii) (A + B) C = AC + BC

✓ 5.10 ANSWER TO CHECK YOUR PROGRESS

Ans. to Q. No. 1: Find all possible orders of a matrix with 10 elements.

$10 = 1 \times 10$ $\quad 10 = 5 \times 2$

$10 = 2 \times 5$ $\quad 10 = 10 \times 1$

Possible orders of matrices are 1 × 10, 2 × 5, 5 × 2 and 10 × 1.

Ans. to Q. No. 2: By equality of two matrices $\begin{pmatrix} 2a+b & 5c+d \\ a-2b & 4c+d \end{pmatrix} = \begin{pmatrix} 4 & 11 \\ 3 & 25 \end{pmatrix}$

$2a + b = 4$ $\quad 5c – d = 11$

$a – 2b = –3$ $\quad 4c + d = 25$

Solving these we get a = 1, b = 2, c = 4, d = 9

Ans. to Q. No. 3: For equality of two matrices, their orders and the co-rresponding elements must be same. Here, the order of the matrices are same. But (3,3)th element of $A \neq (3,3)^{th}$ element of B $\therefore A \neq B$

Ans. to Q. No. 4: i) It is not possible to define A + B, since A is of order 3 × 2 and B is of order 2 × 2.

ii) It is possible to define A + B, since A is of order 3 × 3 and B is of order 3 × 3.

iii) It is possible to define A + B, since A and B are square matrices of the same order.

Ans. to Q. No. 5: $0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is the additive identity of 2 x 3 matrices.

Ans. to Q. No. 6: $B(C+D) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\left(\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 5 & 1 \\ 7 & 4 \end{bmatrix}\right)$

$$= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 8 & 2 \\ 11 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1x8+2x11 & 1x2+2x6 \\ 3x8+4x11 & 3x2+4x6 \end{bmatrix}$$

$$= \begin{bmatrix} 30 & 14 \\ 68 & 30 \end{bmatrix}$$

$$BC + BD = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ 7 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1x3+2x4 & 1x1+2x2 \\ 3x3+4x4 & 3x1+4x2 \end{bmatrix} + \begin{bmatrix} 1x5+2x7 & 1x1+2x4 \\ 3x5+4x7 & 3x1+4x4 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 5 \\ 25 & 11 \end{bmatrix} + \begin{bmatrix} 19 & 9 \\ 43 & 19 \end{bmatrix}$$

$$= \begin{bmatrix} 11+19 & 5+9 \\ 25+43 & 11+19 \end{bmatrix} = \begin{bmatrix} 30 & 14 \\ 68 & 30 \end{bmatrix}$$

$\therefore B(C+D) = BC + CD$

$$\text{Again } (B+C)D = \left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}\right)\begin{bmatrix} 5 & 1 \\ 7 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1+3 & 2+1 \\ 3+4 & 4+2 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ 7 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 3 \\ 7 & 6 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ 7 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4x5+3x7 & 4x1+3x4 \\ 7x5+6x7 & 7x1+6x4 \end{bmatrix}$$

$$= \begin{bmatrix} 41 & 16 \\ 77 & 31 \end{bmatrix}$$

$$BD + CD = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ 7 & 4 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ 7 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1x5+2x7 & 1x1+2x4 \\ 3x5+4x7 & 3x1+4x4 \end{bmatrix} + \begin{bmatrix} 3x5+1x7 & 3x1+1x4 \\ 4x5+2x7 & 4x1+2x4 \end{bmatrix}$$

$$= \begin{bmatrix} 19 & 9 \\ 43 & 19 \end{bmatrix} + \begin{bmatrix} 22 & 7 \\ 34 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 19+22 & 9+7 \\ 43+34 & 19+12 \end{bmatrix} = \begin{bmatrix} 41 & 16 \\ 77 & 31 \end{bmatrix}$$

$\therefore (B+C)D = BD + CD$

Ans. to Q. No. 7: If, A is of order 4×3

Then, B is of order 4×3

and C is of order 3×2

Ans. to Q. No. 8: Given, $A + B = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}$, $A - B = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

$$\therefore A + B + A - B = 2A = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 7+3 & 0+0 \\ 2+0 & 5+3 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 2 & 8 \end{bmatrix}$$

$$A = \frac{1}{2} \begin{bmatrix} 10 & 0 \\ 2 & 8 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix}$$

Again, $A + B - (A - B) = A + B - A + B = 2B$

$$= \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 7-3 & 0-0 \\ 2-0 & 5-3 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 2 & 2 \end{bmatrix}$$

$$\therefore B = \frac{1}{2} \begin{bmatrix} 4 & 0 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

Ans. to Q. No. 9: $A^2 = AA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 0x0+1x1 & 0x1+1x0 \\ 1x0+0x1 & 1x1+0x0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$B^2 = BB = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0x0 + (-i)i & 0x(-i) + (-i)x0 \\ ix0 + 0xi & ix(-i) + 0x0 \end{bmatrix}$$

$$= \begin{bmatrix} -i^2 & 0 \\ 0 & -i^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Ans. to Q. No. 10: $A^2 - 5A + 7I = A.A - 5A + 7I$

$$= \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} - 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3x3 + 1(-1) & 3x1 + 1X2 \\ (-1)x3 + 2(-1) & (-1)x1 + 2x2 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 8 - 15 + 7 & 5 - 5 + 0 \\ -5 + 5 + 0 & 3 - 10 + 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

Ans. to Q. No. 11: $A_\alpha = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

$$A_\alpha^{\,2} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\alpha - \sin^2\alpha & \cos\alpha\sin\alpha + \sin\alpha\cos\alpha \\ -\sin\alpha\cos\alpha - \cos\alpha\sin\alpha & -\sin^2\alpha + \cos^2\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos2\alpha & \sin2\alpha \\ -\sin2\alpha & \cos2\alpha \end{bmatrix}$$

$$A_\alpha^{\,3} = A_\alpha^{\,2}A_\alpha = \begin{bmatrix} \cos2\alpha & \sin2\alpha \\ -\sin2\alpha & \cos2\alpha \end{bmatrix}\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos2\alpha.\cos\alpha - \sin2\alpha.\sin\alpha & \cos2\alpha.\sin\alpha + \sin2\alpha.\cos\alpha \\ -\sin2\alpha.\cos\alpha - \cos2\alpha\sin\alpha & -\sin2\alpha.\sin\alpha + \cos2\alpha\cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos(2\alpha + \alpha) & \sin(\alpha + 2\alpha) \\ -\sin(2\alpha + \alpha) & \cos(2\alpha + \alpha) \end{bmatrix}$$

$$= \begin{bmatrix} \cos 3\alpha & \sin 3\alpha \\ -\sin 3\alpha & \cos 3\alpha \end{bmatrix}$$

5.11 FURTHER READING

- University Algebra, N S Gopala Krishnan, New Age International (P) Limited.
- Theory of Matrices, B S Vatsa, New Age International Publishers..
- Linear Algebra: Schum Solved Problems Series; Seymour Lipschitz: Tata McGraw Hill.

- Text Book of Algebra, A.K Sharma, DPH, Mathematics Series.
- A Text Book of Matrices, Shanti Narayan and P K Mittal.

5.12 MODEL QUESTIONS

Q.1: In a Matrix $\begin{bmatrix} 4 & -3 & 19 & \sqrt{2} \\ 11 & 24 & -7 & -5 \\ 1 & \sqrt{3} & 0 & 1 \end{bmatrix}$

Find i) Order of the matrix

ii) The number of elements

iii) Write the elements a_{23}, a_{12}, a_{32}, $a_{34,}$,$a_{44.}$

Q.2: If A = $\begin{bmatrix} 4 & 5 \\ 6 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 3 & -4 \end{bmatrix}$, find $A + B$ and $A - B$

Q.3: If $A = \begin{bmatrix} 3 & 2 \\ 0 & -2 \\ 1 & 0 \end{bmatrix}$, find $3A$ and $-5A$

Q.4: Find AB and BA if $A = \begin{bmatrix} 2 & 3 \\ 0 & -1 \\ 5 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 1 & 2 \\ -1 & 5 & 3 \end{bmatrix}$

Q.5: What is the additive identity of $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$?

Q.6: If A = $\begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 3 \\ -1 & 3 & 5 \end{pmatrix}$,$B = \begin{pmatrix} -1 & 2 \\ 1 & 3 \\ 0 & 2 \end{pmatrix}$,

C = $\begin{pmatrix} 3 & 1 & 2 \\ -2 & 2 & 0 \end{pmatrix}$,find A(BC)and(AB)C and show that A(BC) = (AB)C

Q.7: Show that A(B + C) = AB + AC

where $A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 4 & 5 \\ -1 & 0 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 1 & -3 \\ 6 & 0 & 4 \end{bmatrix}$

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UNIT 6: ADJOINT AND INVERSE OF A MATRIX

UNIT STRUCTURE

6.1 Learning Objectives
6.2 Transpose of a Matrix
6.3 Symmetric and Skew-Symmetric Matrices
6.4 Adjoint of a Square Matrix
6.5 Inverse of a Matrix
6.6 Let us Sum up
6.7 Answers to Check Your Progress
6.8 Further Reading
6.9 Model Questions

6.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- learn transpose of a matrix
- learn adjoint of a matrix
- learn inverse of a matrix.

6.2 INTRODUCTION

In unit 5, we have discussed the basic concept of matices.In this unit we will discuss the transpose of a matrix , symmetric matrix and skew-symmetric matrix.In this unit, we will also discuss the adjoint and inverse of a matrix.

6.3 TRANSPOSE OF A MATRIX

Let $A = [a_{ij}]_{m\times n}$. Then the n×m matrix obtained from A by changing the rows of A into columns and columns into rows is called the Transpose of matrix A and is denoted by $A^{/}$ or A^{T}.

Therefore, $A^{/}$ or A^{T} will have n-rows and m-columns.

Example: If $A = \begin{bmatrix} 2 & 0 & 5 \\ 1 & 2 & 4 \end{bmatrix}_{2\times 3}$ Then, $A^{/} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \\ 5 & 4 \end{bmatrix}_{3\times 2}$

Some Properties of Transpose of Matrices:

i) The transpose of transpose of a matrix is the matrix itself i.e. if A is a matrix, then $(A^{/})^{/} = A$

ii) If A is a matrix f order mxn and k is a scalar, then $(kA)^{/} = kA^{/}$

iii) If A and B are matrices of same order, then $(A+B)^{/} = A^{/} + B^{/}$

iv) If A and B are two matrices conformable for multiplication, the $(AB)^{/} = B^{/}A^{/}$.

This law is known as "reversal Law for transpose".

Example: If $A = \begin{bmatrix} 3 & 4 \\ 4 & 2 \\ 5 & 3 \end{bmatrix}$ then show that $\left(A^{/}\right)^{/} = A$

Solution: $A = \begin{bmatrix} 3 & 1 \\ 4 & 2 \\ 5 & 3 \end{bmatrix}$, then $A^{/} = \begin{bmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \end{bmatrix}$ $(A^{/})^{/} = \begin{bmatrix} 3 & 1 \\ 4 & 2 \\ 5 & 3 \end{bmatrix}$

Hence, $(A^{/})^{/} = A$

Note: The reversal law for transpose can be extended for more than two matrices.

$(ABC)^{/} = C^{/}B^{/}A^{/}$

$(ABCD)^{/} = D^{/}C^{/}B^{/}A^{/}$ etc.

Example: If $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Show that i) $(2A)^{/} = 2A^{/}$ ii) $(A+B)^{/} = A^{/} + B^{/}$ iii) $(AB)^{/} = B^{/}A^{/}$

Solution: i) $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$

$$2A = \begin{bmatrix} 4 & 6 \\ 8 & 2 \end{bmatrix}, A^{/} = \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}$$

$$\therefore (2A)^{/} = \begin{bmatrix} 4 & 8 \\ 6 & 2 \end{bmatrix}, 2A^{/} = 2\begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 6 & 2 \end{bmatrix}$$

$$\therefore \left(2A\right)^{/} = 2A^{/}$$

Note :

i) If A is a symmetric matrix, then $A^{/} = A$

ii) A is a skew-symmetric matrix then $A^{/} = -A$

ii) $A + B = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$= \begin{bmatrix} 2+1 & 3+2 \\ 4+3 & 1+4 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 7 & 5 \end{bmatrix}$$

$$(A+B)^{/} = \begin{bmatrix} 3 & 7 \\ 5 & 5 \end{bmatrix}$$

Again, $A^{/} + B^{/} = \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

$$= \begin{bmatrix} 2+1 & 4+3 \\ 3+2 & 1+4 \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ 5 & 5 \end{bmatrix}$$

Hence, $(A+B)^{/} = A^{/} + B^{/}$

ii) $AB = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$= \begin{bmatrix} 2\times1+3\times3 & 2\times2+3\times4 \\ 4\times1+1\times3 & 4\times2+1\times4 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 16 \\ 7 & 12 \end{bmatrix} \text{ and } (AB)^{/} = \begin{bmatrix} 11 & 7 \\ 16 & 12 \end{bmatrix}$$

Again, $B^{/}A^{/} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1\times2+3\times3 & 1\times4+3\times1 \\ 2\times2+4\times3 & 2\times4+4\times1 \end{bmatrix} = \begin{bmatrix} 11 & 7 \\ 16 & 12 \end{bmatrix}$$

Hence, $(AB)^{/} = B^{/}A^{/}$

CHECK YOUR PROGRESS

Q.1: If A= $\begin{bmatrix} -4 & 3 \\ 1 & 1 \\ 0 & 2 \end{bmatrix}_{3\times2}$ and $B = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 2 \end{bmatrix}_{2\times3}$

Show that $(AB)^{/} = B^{/}A^{/}$

Q.2: With suitable example show that $(A^{/})^{/} = A$

Q.3: With suitable example, show that $A^{/} = -A$

6.4 SYMMETRIC AND SKEW-SYMMETRIC MATRICES

Symmetric Matrix: A square matrix A = $\lfloor a_{ij} \rfloor$ is said to be symmetric if for all values of i and j, $a_{ij} = a_{ji}$

i.e. $(i, j)^{th}$ element is same as the $(j, i)^{th}$ element of A

For example: $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ are symmetric matrices.

Theorem: The necessary and sufficient condition for a matrix A to be symmetric is that $A = A^{/}$

Proof: Let A=$[a_{ij}]_{nxn}$ is a n-rowed square matrix.

Necessary: $a_{ij} = a_{ji}$

Now, A/ which is the transpose of A is also a square matrix.

$\therefore$ $(i, j)^{th}$ element of **A**/ = $(j, i)^{th}$ element of A

$= a_{ji}$ = $(j\ i)^{th}$ element of **A**

Hence A is symmetric.

Sufficient: Let $A^{/} = A$

$\therefore$ $(i, j)^{th}$ elements of **A** = (j, i) th element of A/

= $(j, i)^{th}$ element of A

Hence A is symmetric matrices.

Skew Symmetric Metrix: A square matrix $A = [a_{ij}]$ is said to be skew-synmetric if for all value of i and j, $a_{ij} = -a_{ji}$

i.e. $(i, j)^{th}$ element is the negative of $(j, i)^{th}$ element of A

For example: $\begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix}, \begin{bmatrix} 0 & -2 & -3 \\ 2 & 0 & -4 \\ 3 & 4 & 0 \end{bmatrix}$

are skew symmetric matrices.

If A is skew-symmetric then–

$a_{ij} = -a_{ji} \quad \forall$ i, j

$\therefore \ a_{ii} = -a_{ii} \quad \forall$ i

$\Rightarrow 2a_{ii} = 0$

$\Rightarrow a_{ii} = 0$

Thus the diagonal elements a_{11}, a_{22}, a_{33} ..., ... of a skew-symmetric matrix are all zero.

Theorem: Necessary and sufficient condition for a matrix A to be skew-symmetric is that $A^{/} = -A$.

Proof: Necessary: Let $A = [a_{ij}]_{nxn}$ be an n rowed skew symmetric matrix. Then $a_{ij} = -a_{ji}$

$\therefore$ (i, j) th element of $A^{/}$ = (j, i) th element of A

$= a_{ji}$

$= -a_{ij}$

= –$(i, j)^{th}$ element of A

$\therefore \ A^{/} = -A$

Sufficiency Part: Let $A^{/} = -A$

$\therefore (i, j)^{th}$ element of $A^{/}$ = –$(i, j)^{th}$ element of **A**

$\Rightarrow$ (j, i)th element of A = –(i, j)th element of **A**

$\Rightarrow a_{ji} = -a_{ij} \Rightarrow a_{ij} = -a_{ji}$

Hence A is skew-symmetric.

Example 1: If A is symmetric (skew symmetric) matrix then kA is also symmetric (skew symmetric).

Solution: i) Let A be symmetric. Then $A^{/} = A$

Now $(kA)^{/} = kA^{/} = kA$

$\Rightarrow$ kA is also symmetric.

ii) Let A be skew symmetric. Then $A^{/} = -A$

Now, $(kA)^{/} = kA^{/} = k(-A) = -(kA)$

$\Rightarrow$ kA is also skew symmetric.

Example 2: Show the matrix A + B is symmetric or skew-symmetric according as A and B are symmetric or skew-symmetric matrices.

Solution: i) Let A and B be two symmetric matrices

$\therefore A^{/} = A, B^{/} = B$

Now, $(A + B)^{/} = A^{/} + B^{/} = A + B$

$\therefore$ A + B is symmetric

ii) Let A and B be to skew-symmetric matrices

$\therefore A' = -A,\ B' = -B$

Now, $A' + B' = A' + B' = -A - B = -(A + B)$

$\therefore$ A+B is skew-symmetric.

Example 3: If A and B are symmetric matices, then AB is symmetic if and only if A and B commute i.e. AB = BA

Solution: A and B are symmetric. To prove that $(AB)' = AB$

$\therefore A' = A,\ B' = B$

Also, AB = BA

Now, $(AB)' = B'A'$

$= BA$

$= AB$

$\therefore$ AB is symmetric.

Conversely, Let AB be symmetric. To prove that AB = BA

$\therefore (AB)' = AB$ (1)

Also $(AB)' = B'A'$

$= BA$ (2)

From (1) and (2), AB = BA Proved

Example 4: If A is an n-square matrix, then AA' and $A'A$ are both symmetric matrices.

Solution: We have $(AA')' = (A')'A'$

$= AA' \quad \because (A')' = A$

Also $(A'A)' = A'(A')' = A'A$

$\therefore A'A$ is symmetric.

$A'A$ is symmetric.

Example 5: Show that every square matrix is uniquely expressible as the sum of a symmetric matrix and a skew-symmetric metrix.

Solution: Let A be any square matrix.

We can write $A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$

$= P + Q$

Where $P = \frac{1}{2}(A + A')$ and $Q = \frac{1}{2}(A - A')$

Now, $P' = \left\{\frac{1}{2}(A + A')\right\}' = \frac{1}{2}(A + A')' \quad \because (kA)' = kA'$

$= \frac{1}{2}\{A' + (A')'\}$

$= \frac{1}{2}(A' + A)$

$= \frac{1}{2}(A + A') = P$

$\therefore$ P is a symmetric matrix.

$$Q' = \left\{\frac{1}{2}(A - A')\right\}'$$

and $$= \frac{1}{2}(A - A')'$$

$$= \frac{1}{2}\{A' - (A')'\}$$

$$= \frac{1}{2}(A' - A)$$

$$= -\frac{1}{2}(A - A')$$

$$= -Q$$

$\therefore$ Q is a skew-symmetric matrix.

Uniqueness: To prove that the representation is unique,let us take another representation, A = R + S, where R is symmetric and S is skew-symmetric.

To prove that R = P, S = Q.

$$A' = (R+S)' = R' + S'$$
$$= R - S$$

$$\therefore A + A' = R + S + R - S = 2R$$

$$\Rightarrow R = \frac{1}{2}(A + A')$$

and $A - A' = R + S - R + S = 2S$

$$\Rightarrow S = \frac{1}{2}(A - A')$$

Thus R = P and S = Q.

$\therefore$ The representation is unique.

CHECK YOUR PROGRESS

Q.4: $A = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$, find AA' and $A'A$

Q.5: Let $A = \begin{bmatrix} 1 & 6 & 3 \\ 2 & 8 & 5 \\ 4 & 1 & 7 \end{bmatrix}$, then find $\frac{1}{2}(A + A')$ and

$\frac{1}{2}(A - A')$. show that $\frac{1}{2}(A + A')$ is symmetric

$\frac{1}{2}(A - A')$ is skew $-$ symmetric.

Q.6: Show that $\begin{bmatrix} 0 & -3i & -4 \\ 3i & 0 & 8 \\ 4 & -8 & 0 \end{bmatrix}$ is skew-symmetric matrix.

6.5 ADJOINT OF A SQUARE MATRIX

Let $A = \begin{bmatrix} a_{11} & a_{12.} \ldots & a_{1m} \\ a_{21} & a_{22.} \ldots & a_{2m} \\ \ldots\ldots\ldots\ldots\ldots\ldots & & \\ a_{m1} & a_{m2.} \ldots & a_{mm} \end{bmatrix} = [a_{ij}]_{mxm}$

The determinant of the matrix A,

$$|A| = \begin{bmatrix} a_{11} & a_{12.} \ldots & a_{1m} \\ a_{21} & a_{22.} \ldots & a_{2m} \\ \ldots\ldots\ldots\ldots\ldots\ldots & & \\ a_{m1} & a_{m2.} \ldots & a_{mm} \end{bmatrix}$$

Let the co-factors of the element a_{ij} be A_{ij}. Considering the matrix whose elements are the co-factors of the corresponding elements of A, we get

$$\Delta = \begin{bmatrix} A_{11} & A_{12.} & \ldots & A_{1m} \\ A_{21} & A_{22.} & \ldots & A_{2m} \\ . & \ldots\ldots\ldots\ldots\ldots\ldots & & \\ A_{m1} & A_{m2.} \ldots & & A_{mm} \end{bmatrix}$$

$$\Delta' = \begin{bmatrix} A_{11} & A_{21.} & \ldots & A_{m1} \\ A_{12} & A_{22.} & \ldots & A_{m2} \\ . & \ldots\ldots\ldots\ldots\ldots\ldots & & \\ A_{1m} & A_{2m.} . & .. & A_{mm} \end{bmatrix} = B(\text{say})$$

$B = \Delta'$ is called the adjoint of the matrix A and is denoted by adj A.

i.e. B = adj. A

Theorem: If A be any n-rowed square matrix, then (adj A) A= A(adj A) = |A| I_n where I_n is the n-rowed unit matrix.

Proof: Let A= $[a_{ij}]_{n\times n}$ be any n-rowed square matrix and let adj A = B = $[b_{ij}]_{n\times n}$

Then b_{ij} = co-factor of a_{ij} in |A|

= A_{ji}

Since A and adj A are both n-rowed square matrices, therefore they are conformable for multiplication

i.e. Both A (adj A) and (adj A) A exist and are of the type n × n.

Now, (i, j)th element of A (adj A)

$$= \sum_{k=1}^{n} a_{ik} b_{kj}$$

$$= \sum_{K=1}^{n} a_{ik} \; A_{jk}$$

$$= \begin{cases} 0 & \text{if } i \neq j \\ |A| & \text{if } i = j \end{cases}$$

In other words, all the elements of A(adj A) along the principal diagonal are equal to |A| and the non-diagonal elements are all zero.

$$\therefore A(\text{adj } A) = \begin{bmatrix} |A| & 0 & 0 \ldots\ldots\ldots\ldots 0 \\ 0 & |A| & 0 \qquad\quad 0 \\ 0 & 0 & |A| \ldots\ldots\ldots 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & & \\ 0 & 0 & 0 \ldots\ldots\ldots |A| \end{bmatrix}$$

$$= |A| \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ . & . & . & \cdots & . \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} A^{I}$$

$$= |A| \; I_n$$

Similarly, $(i, j)^{th}$ element of (adj A)A

$$= \sum_{K=1}^{n} b_{ik} a_{kj}$$

$$= \sum_{k=1}^{n} A_{ki} a_{kj}$$

$$= \begin{cases} 0 & \text{if } i \neq j; \\ |A| & \text{if } i = j \end{cases}$$

Therefore, $(\text{adj } A)\, A = |A| \, I_n$

Hence, $A(\text{adj } A) = (\text{adj } A)\, A = |A| I_n$

Example: Find the adjoint of the matrix A

where $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \\ 2 & 0 & 1 \end{bmatrix}$

Solution: We have $|A| = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \\ 2 & 0 & 1 \end{bmatrix}$

Now, A_{11} = cofactor of $(1,1)^{th}$ element of |A|

$$= \begin{vmatrix} -1 & 4 \\ 0 & 1 \end{vmatrix} = -1 - 0 = -1$$

A_{12} = cofactor of (1,2)th element of |A|

$$= \begin{vmatrix} 3 & 4 \\ 2 & 1 \end{vmatrix} = -(3-8) = 5$$

Similarly, $A_{13} = \begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix} = -0-(-1)\,2 = 2$

$$A_{21} = -\begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} = -2$$

$$A_{22} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1, A_{23} = -\begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} = 4$$

$$A_{31} = \begin{vmatrix} 2 & 0 \\ -1 & 4 \end{vmatrix} = 8, A_{32} = -\begin{vmatrix} 1 & 0 \\ 3 & 4 \end{vmatrix} = -4,\ A_{33} = \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} = -1-6 = -7$$

$$\therefore \text{adj } A = \begin{bmatrix} -1 & -2 & 8 \\ 5 & 1 & -4 \\ 2 & 4 & -7 \end{bmatrix}$$

Note: 1) To find the Adj A, write down the determinant |A| and find the co-factor of various element of |A|. Replace each element in A by its co-factor and then take the transpose to get adj A.

2) If A is an n-rowed square matrix, then A is said to be singular matrix if its determinant is zero i.e |A| = 0.

$\therefore$ A(adj A) = (Adj A)A = |A| I_n = 0 (null matrix)

Hence, in this case all the elements of A(adj A) and (adj A)A will be zero's.

3) If $|A| \neq 0$, then $|A\,\text{Adj}\,A| = |A|\,I_n| = \begin{vmatrix} |A| & 0 & 0 \ldots & 0 \\ 0 & |A| & 0 \ldots & 0 \\ 0 & 0 & |A| \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 \ldots & |A| \end{vmatrix}$

$$= |A|^n$$

$$\Rightarrow |\text{Adj } A| = |A|^{n-1}$$

6.6 INVERSE OF A MATRIX

Let A and B any n-squared matrices such that AB = BA = I_n

Then B is called inverse of A and is denoted by A^{-1} i.e. B = A^{-1}

Note: It is to be noted that every square matrix does not possess its inverse.

Properties of the Inverse:

a) The inverse of a matrix is unique.

Let A be n-squared matrix whose inverse exists.

If possible,let B and C be two inverses of A.

Then $AB = BA = I_n$ (1)

$AC = CA = I_n$ (2)

For (1) $C(AB) = C\,I_n = C$ (3)

For (2) $(CA)B = I_nB = B$ (4)

$\because$ $C(AB) = (CA)B$

$\because$ $C = B$

which shows that an invertible matrix possesses a unique inverse.

b) The necessary and sufficient condition for a square matrix A to possess inverse is that A is non-singular i.e. $|A| \neq 0$

Proof: Necessary: Let A be a square matrix of order n and B be its inverse.

Then $AB = I_n$

$\therefore$ $|AB| = |I_n|$

$\Rightarrow |A||B| = 1$

$\Rightarrow |A| \neq 0$

Sufficient: Let $|A| \neq 0$, we define a matrix B so that $B = \frac{1}{|A|}(\text{adj.A.})$

Then $AB = A\,\frac{1}{|A|}\text{adj A}$

$= \frac{1}{|A|}A(\text{adj A})$

$= \frac{1}{|A|}|A|\,I_n = I_n$

Similarly, $BA = \left(\frac{1}{|A|}\text{adj A}\right)A$

$= \frac{1}{|A|}(\text{adj A})\,A = \frac{1}{|A|}.|A|\,I_n = I_n$

Thus, $AB = BA = I_n$

Hence, A is invertible and B, is the inverse of A.

$\therefore B = A^{-1} = \frac{1}{|A|}\text{adj A}$

c) **Reversal Law for the Inverse of a Product:** If A, B be two n-rowed non-singular matrices,

then AB is also non-singular and $(AB)^{-1} = B^{-1}A^{-1}$

i.e. the inverse of the product is the product of the inverses in the reverse order.

Proof: Let A and B be two non-singular matrices i.e. $|A| \neq 0$, $|B| \neq 0$,

So their inverses exist.

We have $|AB| = |A||B| \neq 0$

Hence, the matrix AB is also invertible .

Let A^{-1} and B^{-1} be the inverses of A and B respectively and

$AA^{-1} = A^{-1}A = I_n$ and $B\ B^{-1} = B^{-1}B = I_n$

Choosing $C = B^{-1} A^{-1}$ we get

$C(AB) = (B^{-1} A^{-1})(AB)$

$= B^{-1} (A^{-1}A)B$

$= B^{-1} I_n B$

$= B^{-1} B$

$= I_n$ (1)

Again, $(AB) C = (AB) (B^{-1} A^{-1})$

$= A(BB^{-1}) A^{-1}$

$= AI_nA^{-1}$

$= AA^{-1}$

$= I_n$ (2)

By (1) and (2) $C(AB) = (AB)C = I_n$

Hence, C is the inverse of AB

or $B^{-1} A^{-1}$ is the inverse of AB

Thus, $(AB)^{-1} = B^{-1} A^{-1}$

d) The operation of transposing and inverting are commutative

i.e. $(A')^{-1} = (A^{-1})'$

Proof: Let A be a non-singular square matrix i.e. $|A| \neq 0$

$\therefore$ A^{-1} exists

We get $AA^{-1} = A^{-1}A = I_n$

$\therefore (AA^{-1})' = (A^{-1}A)' = I'_n$

$\Rightarrow (A^{-1})' A' = A' (A^{-1})' = I_n$

which shows that $(A^{-1})'$ is the inverse of A'.

Hence, $(A')^{-1} = (A^{-1})'$

Note: If A, B are two n-rowed non-singular matrices.

Then $(AB)^{-1} = B^{-1}A^{-1}$.

Example: Find the adjoint and inverse of the matrix $\begin{bmatrix} 0 & 1 & 3 \\ 3 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$

Solution: Let $A = \begin{bmatrix} 0 & 1 & 3 \\ 3 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$

$$\therefore |A| = \begin{vmatrix} 0 & 1 & 3 \\ 3 & 1 & 1 \\ 1 & 2 & 0 \end{vmatrix}$$

$$= 0 + (-1)\begin{vmatrix} 3 & 1 \\ 1 & 0 \end{vmatrix} + 3\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix}$$

$$= (-1)(-1) + 3(6-1)$$

$= 1 + 15 = 16 \neq 0$

$\therefore A^{-1}$ exists

Now, Co-factors of the elements of 1st row are–

$$\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix}, -\begin{vmatrix} 3 & 1 \\ 1 & 0 \end{vmatrix}, \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} \text{ i.e. } -2,\ 1, 5$$

Co-factors of the elements of 2nd row are–

$$-\begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix}, \begin{vmatrix} 0 & 3 \\ 1 & 0 \end{vmatrix}, -\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} \text{ i.e. } 6, -3, 1$$

Co-factors of the elements of 3rd row are–

$$\begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix}, -\begin{vmatrix} 0 & 3 \\ 3 & 1 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix} \text{ i.e. } -2, 9, -3$$

Matrix formed by the corresponding co-factors of the elements of |A| is–

$$C = \begin{bmatrix} -2 & 1 & 5 \\ 6 & -3 & 1 \\ -2 & 9 & -3 \end{bmatrix}$$

$\therefore$ adj A = transpose of C = $\begin{bmatrix} -2 & 6 & -2 \\ 1 & -3 & 9 \\ 5 & 1 & -3 \end{bmatrix}$

$$\therefore A^{-1} = \frac{1}{|A|}(\text{adj } A)$$

$$= \frac{1}{16}\begin{bmatrix} -2 & 6 & -2 \\ 1 & -3 & 9 \\ 5 & 1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{8} & \frac{3}{8} & \frac{-1}{8} \\ \frac{1}{16} & \frac{-3}{16} & \frac{9}{16} \\ \frac{5}{16} & \frac{1}{16} & \frac{-3}{16} \end{bmatrix}$$

Example: Find the inverse of the matrix $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

Solution: $|A| = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

$$= \begin{vmatrix} 4 & 3 \\ 3 & 4 \end{vmatrix} - 3\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} + 3\begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix}$$

$$= (16-9) - 3(4-3) + 3(3-4)$$
$$= 7 - 3 - 3 = 7 - 6 = 1 \neq 0$$

Now, Co-factors of the elements of the 1st row are–

$$\begin{vmatrix} 4 & 3 \\ 3 & 4 \end{vmatrix}, -\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix}, \begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} \text{ i.e. } 7, -1, -1$$

Co-factors of the elements of the 2nd row are–

$$-\begin{vmatrix} 3 & 3 \\ 3 & 4 \end{vmatrix}, \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix}, -\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} \text{ i.e. } 3, 1, 0$$

Co-factors of the elements of the 3rd row are–

$$\begin{vmatrix} 3 & 3 \\ 4 & 3 \end{vmatrix}, -\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix}, \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} \text{ i.e. } -3, 0, 1$$

$$\therefore \text{adj } A = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{adj } A$$
$$= \frac{1}{1} \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 7 & -3 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

CHECK YOUR PROGRESS

Q.7: Find adj A if $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 10 & 1 \end{bmatrix}$

Q.8: Find A^{-1} if $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & -\cos\alpha \end{bmatrix}$, α is a real number.

Q.9: Show that if $|A| \neq 0$, then $|A^{-1}| = |A|^{-1}$

Q.10: If AB = BA, then prove that $A^{-1} B^{-1} = B^{-1}A^{-1}$

Q.11: If $|A| \neq 0$ and A is symmetric, show that A^{-1} is also symmetric.

Q.12: If A be a square matrix, then prove that $|\text{adj } A| = |A|^{n-1}$.

Q.13: If $|A| \neq 0$, then show that adj adj $A = |A|^{n-2}A$.

6.7 LET US SUM UP

- For matrices A and B of the same order $(A + B)^{/} = A^{/} + B^{/}$
- For matrices A and B conformable to multiplication $(AB)^{/} = B^{/}A^{/}$
- The square matrix A is called symmetric if $A^{/} = A$
- The square matrix A is called skew symmetric if $A^{/} = -A$
- Every square matrix A can uniquely be expressed as A = P + Q where P is a symmetric matrix and Q is a skew-symmetric matrix.
- For a square matrix A (adj A) A = A (adj A) = |A|I
- For a square non-singular matrix A, $A^{-1} = \frac{\text{adj } A}{|A|}$
- For two non-singular matrices A and B of the same order, $(AB)^{-1} = B^{-1}A^{-1}$.

6.8 ANSWER TO CHECK YOUR PROGRESS

Ans. to Q. No. 1: $A^{/} = \begin{bmatrix} -4 & 1 & 0 \\ 3 & 1 & 2 \end{bmatrix}_{2x3}$, $B^{/} = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 4 & 2 \end{bmatrix}_{3x2}$

$$AB = \begin{bmatrix} -4 & 3 \\ 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -4+6 & -12+3 & -16+6 \\ 1+2 & 3+1 & 4+2 \\ 0+4 & 0+2 & 0+4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -9 & -10 \\ 3 & 4 & 6 \\ 4 & 2 & 4 \end{bmatrix}$$

$$(AB)^{/} = \begin{bmatrix} 2 & 3 & 4 \\ -9 & 4 & 2 \\ -10 & 6 & 4 \end{bmatrix}$$

$$B^{/}A^{/} = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} -4 & 1 & 0 \\ 3 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -4+6 & 1+2 & 0+4 \\ -12+3 & 3+1 & 0+2 \\ -16+6 & 4+2 & 0+4 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 \\ -9 & 4 & 2 \\ -10 & 6 & 4 \end{bmatrix}$$

$\therefore (AB)^{/} = B^{/}A^{/}$.

Ans. to Q. No. 2: Let $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ $A^{/} = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$

$$(A^{/})^{/} = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} = A$$

Ans. to Q. No. 3: $A = \begin{bmatrix} o & h & g \\ -h & o & f \\ -g & -f & o \end{bmatrix}$ is a skew symmetric matrix

$$A^{/} = \begin{bmatrix} o & -h & -g \\ h & o & -f \\ g & f & o \end{bmatrix} = -\begin{bmatrix} o & h & g \\ -h & o & f \\ -g & -f & o \end{bmatrix} = -A$$

Ans. to Q. No. 4: $A = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$ $\therefore$ $A^{/} = \begin{bmatrix} 1 & 3 \\ 3 & 2 \\ 2 & 1 \end{bmatrix}$

$$AA^{/} = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 3 & 2 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1x1+3x3+2x2 & 1x3+3x2+2x1 \\ 3x1+2x3+1x2 & 3x3+2x2+1x1 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 11 \\ 11 & 14 \end{bmatrix}$$

$$A^{/}A = \begin{bmatrix} 1 & 3 \\ 3 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1x1+3x3 & 1x3+3x2 & 1x2+3x1 \\ 3x1+2x3 & 3x3+2x2 & 3x2+2x1 \\ 2x1+1x3 & 2x3+1x2 & 2x2+1x1 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 9 & 5 \\ 9 & 13 & 8 \\ 5 & 8 & 5 \end{bmatrix} \quad \therefore AA^{/} \neq A^{/}A$$

Ans. to Q. No. 5: $A=\begin{bmatrix}1 & 6 & 3\\ 2 & 8 & 5\\ 4 & 1 & 7\end{bmatrix} \quad \therefore A'=\begin{bmatrix}1 & 2 & 4\\ 6 & 8 & 1\\ 3 & 5 & 7\end{bmatrix}$

$$\frac{1}{2}(A+A')=\frac{1}{2}\left(\begin{bmatrix}1 & 6 & 3\\ 2 & 8 & 5\\ 4 & 1 & 7\end{bmatrix}+\begin{bmatrix}1 & 2 & 4\\ 6 & 8 & 1\\ 3 & 5 & 7\end{bmatrix}\right)$$

$$=\frac{1}{2}\begin{bmatrix}1+1 & 6+2 & 3+4\\ 2+6 & 8+8 & 5+1\\ 4+3 & 1+5 & 7+7\end{bmatrix}$$

$$=\frac{1}{2}\begin{bmatrix}2 & 8 & 7\\ 8 & 16 & 6\\ 7 & 6 & 14\end{bmatrix}=\begin{bmatrix}1 & 4 & \frac{7}{2}\\ 4 & 8 & 3\\ \frac{7}{2} & 3 & 7\end{bmatrix}, \text{ which is symmetric.}$$

$$\frac{1}{2}(A-A')=\frac{1}{2}\left(\begin{bmatrix}1 & 6 & 3\\ 2 & 8 & 5\\ 4 & 1 & 7\end{bmatrix}-\begin{bmatrix}1 & 2 & 4\\ 6 & 8 & 1\\ 3 & 5 & 7\end{bmatrix}\right)$$

$$=\frac{1}{2}\begin{bmatrix}1-1 & 6-2 & 3-4\\ 2-6 & 8-8 & 5-1\\ 4-3 & 1-5 & 7-7\end{bmatrix}$$

$$=\frac{1}{2}\begin{bmatrix}0 & 4 & -1\\ -4 & 0 & 4\\ 1 & -4 & 0\end{bmatrix}=\begin{bmatrix}0 & 2 & -\frac{1}{2}\\ -2 & 0 & 2\\ \frac{1}{2} & -2 & 0\end{bmatrix},$$

which is skew-symmetric.

Ans. to Q. No. 6: Let $A=\begin{bmatrix}0 & -3i & -4\\ 3i & 0 & 8\\ 4 & -8 & 0\end{bmatrix}$

$$A'=\begin{bmatrix}0 & 3i & 4\\ -3i & 0 & -8\\ -4 & 8 & 0\end{bmatrix}=-\begin{bmatrix}0 & -3i & -4\\ 3i & 0 & 8\\ 4 & -8 & 0\end{bmatrix}=-A$$

$\therefore$ A is Skew Symmetric

Ans. to Q. No. 7: $|A| = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{vmatrix}$

Co-factors of the elements of 1st row are–

$$\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix}, -\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}, \text{ i.e.} -1, 8, -5$$

Co-factors of the elements of 2nd row are–

$$-\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix}, -\begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix}, \text{ i.e.1, - 6, 3}$$

Co-factors of the elements of 3rd row are–

$$\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}, -\begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix}, \text{ i.e.} -1, 2, -1$$

$$\therefore \text{ adj } A = \begin{vmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{vmatrix}$$

Ans. to Q. No. 8: $|A| = \begin{vmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{vmatrix} \quad \therefore |A| = \cos^2\alpha + \sin^2\alpha = 1 \neq 0$

$\therefore$ Co-factor of the elements of 1st row are–

$\cos\alpha, -\sin\alpha$

Co-factor of the element of 2nd row are–

$-(-\sin\alpha), \cos\alpha$ i.e.$\sin\alpha, \cos\alpha$

$$\therefore \text{adj } A = \begin{vmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{vmatrix}$$

$$\therefore A^{-1} = \frac{\text{adj}A}{|A|} = \begin{vmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{vmatrix}$$

Ans. to Q. No. 9: $|A| \neq 0 \quad \therefore A^{-1}$ exists

Now, $AA^{-1} = I$

$$\Rightarrow |AA^{-1}| = |I|$$

$$\Rightarrow |A||A^{-1}| = 1$$

$$\Rightarrow |A^{-1}| = \frac{1}{|A|} = |A|^{-1}$$

Ans. to Q. No. 10: Given that AB = BA

$$\therefore (AB)^{-1} = (BA)^{-1}$$

$$\Rightarrow B^{-1}A^{-1} = A^{-1}B^{-1}$$

Ans. to Q. No. 11: $|A| \neq 0 \quad \therefore A^{-1}$ exists

also $A^{/}=A$

Now, $(A^{-1})^{/} = (A^{/})^{-1} = A^{-1}$

$\therefore$ A^{-1} is symmetric.

Ans. to Q. No. 12: $A\,adjA = |A|I_n$

$\therefore \; |A\,adjA| = \|A\|I_n\|$

$\Rightarrow |A||adj\,A| = |A|^n \; |I_n| \quad \because |kA| = k^n|A|$

$= |A|^n \qquad \because |I_n| = 1$

$\Rightarrow |adj\,A| = |A|^{n-1}$ if $|A| \neq 0$

If $|A| = 0$, then $|adj\ A| = 0 = |A|^{n-1}$

Hence the result.

Ans. to Q. No. 13: $|A| \neq 0$

We have $A(adj\ A) = |A|I_n$

Replacing A by adj A, we get

$(adj\,A)(adj\,adj\ A) = |adj\,A|I_n$

$= |A|^{n-1}.I_n \qquad -(i)$

Premultiplying both side of (i) by A

$A\ (adj\,A)(adj\,adj\ A) = A\{|A|^{n-1} I_n\}$

$= |A|^{n-1}(AI_n)$

$= |A|^{n-1} A$

$\Rightarrow \{A(adj\ A)\}(adj\,adj\ A) = |A|^{n-1} A$

$\Rightarrow |A|I_n\ (adj\ adj\ A) = |A|^{n-1} A$

$\Rightarrow |A|(adj\ adj\ A) = |A|^{n-1} A$

$\Rightarrow adj\ adj\ A = |A|^{n-2} A \qquad [\text{Dividing both sides by } |A| \; \because |A| \neq 0]$

6.9 FURTHER READING

- University Algebra, N S Gopala Krishnan, New Age International (P) Limited.
- Theory of Matrices, B S Vatsa, New Age International Publishers..
- Linear Algebra: Schum Solved Problems Series; Seymour Lipschitz: Tata McGraw Hill.
- Text Book of Algebra, A.K Sharma, DPH, Mathematics Series.
- A Text Book of Matrices, Shanti Narayan and P K Mittal.

6.10 MODEL QUESTIONS

Q.1: If $A = \begin{pmatrix} 4 & 3 & 7 \\ 2 & 5 & 6 \end{pmatrix}$ show that $(A')' = A$

Q.2: If $A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix}$, then show that $(AB)' = B'A'$

Q.3: If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, then show that $A^4 = \begin{bmatrix} 1+2\times 4 & -4\times 4 \\ 4 & 1-2\times 4 \end{bmatrix}$

Q.4: Find A^{-1} where $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{pmatrix}$

Q.5: Find the inverse of the matrix $\begin{pmatrix} 4 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{pmatrix}$

Q.6: If $A = \begin{bmatrix} 2 & -4 & -2 \\ 4 & 6 & 2 \\ 0 & 10 & -4 \end{bmatrix}$, find adj A

Q.7: If $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$, find A^{-1}

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UNIT 7: RANK AND ELEMENTARY OPERATIONS

UNIT STRUCTURE

7.1 Learning Objectives
7.2 Introduction
7.3 Rank of a Matrix
7.4 Elementary Row Operations
7.5 Inverse Using Elementary Operations
7.6 Let Us Sum Up
7.7 Answers to Check Your Progress
7.8 Further Reading
7.9 Model Questions

7.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- define rank of matrices
- learn elementary operations of a matriix
- find inverse using of a matrix elementary operations
- find rank using definition and using elementary operations.

7.2 INTRODUCTION

In this unit, we shall define rank of a matrix. We discuss elementary operations of a matrix. We shall apply elementary operations in finding inverse of a matrix.

7.3 RANK OF A MATRIX

Submatrix of a Matrix: Let A be any matrix of order m×n. Then a matrix obtained by leaving some rows and columns from A is called a submatrix of A.

Note: A matrix A itself is a submatrix of A.

Minors of a Matrix: Let A be any matrix of order m×n. By leaving (m – r) rows and (n – r) columns of A, we get a square submatrix of order r. The determinant of the square submatrix of order r is called a minor of A of order r.

Example: $A = \begin{bmatrix} 3 & -1 & -2 \\ 1 & 3 & 2 \end{bmatrix}_{2\times 3}$ is of order 2 × 3.

So we get minors of order 1 and 2

Minor of order 2 are $\begin{vmatrix} 3 & -1 \\ 1 & 3 \end{vmatrix}, \begin{vmatrix} 3 & -2 \\ 1 & 2 \end{vmatrix}, \begin{vmatrix} -1 & -2 \\ 3 & 2 \end{vmatrix}$

Now, $\begin{vmatrix} 3 & -1 \\ 1 & 3 \end{vmatrix} = 9+1=10, \begin{vmatrix} 3 & -2 \\ 1 & 2 \end{vmatrix} = 6+2=8, \begin{vmatrix} -1 & -2 \\ 3 & 2 \end{vmatrix} = -2+6=4$

Rank of a Matrix: The rank of a matrix A is r if

a) Every minor of A of order r + 1 is zero.

b) There is atleast one minor of A of order r which does not vanish. The rank of a matrix A is denoted by ρ (A)

Note: i) The rank of a null matrix is zero.

ii) Rank of a sub-matrix $\leq$ rank of the matrix.

Working Rule:

1) Calculate the minors of highest possible order of the matrix.
2) If it is not zero, then the order of the minor is the rank of the matrix.
3) If it is zero and all other minors of same order are also zero, then calculate the minor of next lower order.
4) If it is not zero, then follow (2)
5) If it is zero, then follow (3) and so on.

Example: Find the rank of the matrices

(i) $A = \begin{bmatrix} 2 & 6 & -3 \\ 1 & 4 & -2 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(iii) $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (iv) $A = \begin{bmatrix} 4 & 5 & 6 \\ 4 & 2 & 3 \\ 3 & 4 & 4 \end{bmatrix}$

Solution:

i) As the matrix A is of order 2 × 3, the highest possible order of minors will be 2

Minors of order 2 : $\begin{vmatrix} 6 & -6 \\ 4 & -3 \end{vmatrix} = 6(-2) - 4(-3) = -12 + 12 = 0$

Another minor of order 2 is: $\begin{vmatrix} 2 & 6 \\ 1 & 4 \end{vmatrix} = 2 \times 4 - 1 \times 6 = 8 - 6 = 2 \neq 0$

Hence, there is atleast one minor of order 2 is non–zero.

$\therefore \rho$ (A) = 2.

ii) As the matrix A is of order 3×3, the highest possible order of minor will be 3.

Minor of order 3 is: $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$

$\therefore \rho(A) = 3$

iii) We know that the rank of a null matrix is defined to be zero.

$\therefore \rho(A) = 0$

iv) As the matrix A is of order 3×3, the highest possible order of minor will be 3.

Minor of order 3 is:

$$\begin{vmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 3 & 4 & 4 \end{vmatrix} = 4(2\times 4 - 4\times 3) - 5(1\times 4 - 3\times 3) + 6(1\times 4 - 3\times 2)$$

$= 4(8 - 12) - 5(4 - 9) + 6(4 - 6)$

$= 4(-4) - 5(-5) + 6(-2)$

$= -16 + 25 - 12$

$= 25 - 28$

$= -3 \neq 0 \qquad \therefore \rho(A) = 3$

CHECK YOUR PROGRESS

Q.1: Find the rank of the matrices:

i) $A = \begin{bmatrix} 0 & 1 & 3 \\ 2 & 1 & -4 \end{bmatrix}$ ii) $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -4 & -5 \\ 0 & -2 & 1 \\ 0 & 16 & 3 \end{bmatrix}$

iii) $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 7 & 13 \\ 4 & -3 & -4 \end{bmatrix}$ iv) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

7.4 ELEMENTARY ROW OPERATIONS

The following are called elementary row operations on a matrix $A = [a_{ij}]_{m\times n}$ with respective symbols:

i) Interchanging the i^{th} row and the j^{th} row: $R_i \leftrightarrow R_j$.

ii) Multiplying the i^{th} row by a non-zero scalar k: $R_i \rightarrow kR_i$.

iii) Replacing the i^{th} row by adding to this row k times the j^{th} row: $R_i \rightarrow R_i + kR_j$.

Definition 1:

Echelon Matrix: A matrix A is called an **echelon matrix**, or is said to be in **echelon form** if:

i) any zero rows, that is rows with all elements zero, are on the bottom of the matrix.

ii) the first non-zero entry, called the **leading entry** in a row, occurs to the right of the leading non-zero entry in the preceding row.

Example: The matrices $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & -20 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 & -4 & 3 \\ 0 & -1 & 3 & -2 \\ 0 & 0 & 4 & 5 \end{pmatrix}$

are echelon matrices.

Definition 2:

Row Canonical Form of Matrix: A matrix A is said to be in **row canonical form** if:

i) A is an echelon matrix,

ii) each leading non-zero entry in a row is 1,

iii) each leading non-zero entry is the only non-zero entry in its column.

A matrix in row canonical form is also called **Reduced Echelon Matrix**.

Example: The matrix $\begin{pmatrix} 1 & 0 & 5 & 0 & 2 \\ 0 & 1 & 3 & 0 & 5 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ is in row canonical form

whereas the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is not in row canonical form.

Any given matrix can be reduced to row canonical form with the help of some operations performed in its rows.

Row Equivalence: Two matrices are row equivalent if one can be changed to the other by a sequnce of elementary row operations.It is generally denoted by ~.

Example 1: Using elementary row operations, show that

$B = \begin{pmatrix} 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & 2 \\ 0 & 3 & 0 & 5 \end{pmatrix}$ is row equivalent to $A = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & 1 \\ -1 & 1 & 0 & 2 \end{pmatrix}$.

Solution: $A = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & 1 \\ -1 & 1 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 3 & 0 & 5 \\ -1 & 1 & 0 & 2 \end{pmatrix}$,

by $R_2 \to R_2 + 2R_3$

$\sim \begin{pmatrix} 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & 2 \\ 0 & 3 & 0 & 5 \end{pmatrix} = B$, by $R_2 \leftrightarrow R_3$

Thus A ~ B.

Example 2: Reduce the matrix $A = \begin{pmatrix} 1 & -2 & 3 & -1 \\ 2 & -1 & 2 & 2 \\ 3 & 1 & 2 & 3 \end{pmatrix}$ to echelon form.

Solution: $A = \begin{pmatrix} 1 & -2 & 3 & -1 \\ 2 & -1 & 2 & 2 \\ 3 & 1 & 2 & 3 \end{pmatrix}$

$$\sim \begin{pmatrix} 1 & -2 & 3 & -1 \\ 0 & 3 & -4 & 4 \\ 0 & 7 & -7 & 6 \end{pmatrix}, \text{ by } R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 3R_1$$

$$\sim \begin{pmatrix} 1 & -2 & 3 & -1 \\ 0 & 3 & -4 & 4 \\ 0 & 0 & 7 & -10 \end{pmatrix}, \text{ by } R_3 \to 3R_3 - 7R_2$$

Thus $A \sim \begin{pmatrix} 1 & -2 & 3 & -1 \\ 0 & 3 & -4 & 4 \\ 0 & 0 & 7 & -10 \end{pmatrix}$ which is in echelon form.

Example 3: Reduce $A = \begin{pmatrix} 0 & 1 & 3 & -2 \\ 2 & 1 & -4 & 3 \\ 2 & 3 & 2 & -1 \end{pmatrix}$ to echelon form.

Solution: $A = \begin{pmatrix} 0 & 1 & 3 & -2 \\ 2 & 1 & -4 & 3 \\ 2 & 3 & 2 & -1 \end{pmatrix}$

$$\sim \begin{pmatrix} 2 & 1 & -4 & 3 \\ 0 & 1 & 3 & -2 \\ 2 & 3 & 2 & -1 \end{pmatrix}, \text{ by } R_1 \leftrightarrow R_2$$

$$\sim \begin{pmatrix} 2 & 1 & -4 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 2 & 6 & -4 \end{pmatrix}, \text{ by } R_3 \to R_3 - R_{1,}$$

$$\sim \begin{pmatrix} 2 & 1 & -4 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{ by } R_3 \to R_3 - 2R_2$$

Thus $A \sim \begin{pmatrix} 2 & 1 & -4 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ which is in echelon form.

Example 4: Reduce the matrix $A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 6 & 7 & 4 \\ 4 & 7 & 10 & 13 & 16 \end{pmatrix}$

to row canonical form.

Solution: $A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 6 & 7 & 4 \\ 4 & 7 & 10 & 13 & 16 \end{pmatrix}$

$\sim \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & -1 & -2 & -3 & -4 \\ 0 & -1 & -3 & -5 & -11 \\ 0 & -1 & -2 & -3 & -4 \end{pmatrix}$, by $R_2 \to R_2 - 2R_1$,

$R_3 \to R_3 - 3R_1$, $R_4 \to R_4 - 4R_1$

$\sim \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & -1 & -3 & -5 & -11 \\ 0 & -1 & -2 & -3 & -4 \end{pmatrix}$, by $R_2 \to (-1).R_2$

$\sim \begin{pmatrix} 1 & 0 & -1 & -2 & -3 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & -1 & -2 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, by $R_3 \to R_3 + R_2$, $R_4 \to R_4 + R_2$

$\sim \begin{pmatrix} 1 & 0 & -1 & -2 & -3 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, by $R_3 \to (-1).R_3$

$\sim \begin{pmatrix} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 & -10 \\ 0 & 0 & 1 & 2 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, by $R_2 \to R_2 - 2R_3$

which is in row canonical form.

Example 5: Reduce to row canonical form: $A = \begin{pmatrix} 1 & -2 & 3 & 1 & 2 \\ 1 & 1 & 4 & -1 & 3 \\ 2 & 5 & 9 & -2 & 8 \end{pmatrix}$

Solution: $A = \begin{pmatrix} 1 & -2 & 3 & 1 & 2 \\ 1 & 1 & 4 & -1 & 3 \\ 2 & 5 & 9 & -2 & 8 \end{pmatrix}$

$\sim \begin{pmatrix} 1 & -2 & 3 & 1 & 2 \\ 0 & 3 & 1 & -2 & 1 \\ 0 & 9 & 3 & -4 & 4 \end{pmatrix}$, by $R_2 \to R_2 - R_1$, $R_3 \to R_3 - 2R_1$

$$\sim \begin{pmatrix} 1 & -2 & 3 & 1 & 2 \\ 0 & 3 & 1 & -2 & 1 \\ 0 & 0 & 0 & 2 & 1 \end{pmatrix}, \text{ by } R_3 \to R_3 - 3R_2$$

$$\sim \begin{pmatrix} 1 & -2 & 3 & 1 & 2 \\ 0 & 3 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{pmatrix}, \text{by } R_3 \to \tfrac{1}{2}R_3$$

$$\sim \begin{pmatrix} 1 & -2 & 3 & 0 & \frac{3}{2} \\ 0 & 3 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{pmatrix}, \text{ by } R_1 \to R_1 - R_3,\ R_2 \to R_2 + 2R_3$$

$$\sim \begin{pmatrix} 1 & -2 & 3 & 0 & \frac{3}{2} \\ 0 & 1 & \frac{1}{3} & 0 & 2 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{pmatrix}, \text{ by } R_2 \to \frac{1}{3}R_2$$

$$\sim \begin{pmatrix} 1 & 0 & \frac{11}{3} & 0 & \frac{17}{6} \\ 0 & 1 & \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{pmatrix}, \text{ by } R_1 \to R_1 + 2R_2$$

which is the row canonical form of A.

7.5 INVERSE USING ELEMENTARY ROW OPERATIONS

Finding inverse of a square matrix by elementary row operations is based on a theorem stated below (without proof).

Theorem: If a square matrix A of order n is reduced to I_n by successively performing elementary row operations, then the same successive elementary row operations reduces I_n to A^{-1}.

Thus, if we perform successive elementary row operations on $(A \mid I_n)$ which reduces A to I_n, then the resulting matrix is $(I_n \mid A^{-1})$. In symbol, $(A \mid I_n) \sim (I_n \mid A^{-1})$.

It should be noted that in performing elementary row operations on $(A \mid I_n)$, if we get a zero-row in the A-part, then A does not reduce I_n and in such case A^{-1} does not exist.

Example 6: Find inverse matrix if possible using elementary row operations :

i) $A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 1 & 2 & 5 \end{pmatrix}$ ii) $B = \begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{pmatrix}$

iii) $C = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 2 & -1 & 1 & 2 \\ -1 & 2 & 1 & -2 \\ 1 & -1 & 1 & -1 \end{pmatrix}$ iv) $D = \begin{pmatrix} 1 & 3 & -4 \\ 1 & 5 & -1 \\ 3 & 13 & -6 \end{pmatrix}$

Solution:

i) $(A \mid I) = \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 2 & 1 & -1 & 0 & 1 & 0 \\ 1 & 2 & 5 & 0 & 0 & 1 \end{array}\right)$

$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 2 & 6 & -1 & 0 & 1 \end{array}\right)$, by $R_2 \to R_2 - 2R_1$; $R_3 \to R_3 - R_1$

$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 4 & 3 & -2 & 1 \end{array}\right)$, by $R_3 \to R_3 - 2R_2$

$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & \frac{3}{4} & -\frac{1}{2} & \frac{1}{4} \end{array}\right)$, by $R_3 \to \frac{1}{4}R_3$

$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{4} & -\frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & -\frac{11}{4} & \frac{3}{2} & -\frac{1}{4} \\ 0 & 0 & 1 & \frac{3}{4} & -\frac{1}{2} & \frac{1}{4} \end{array}\right)$ by $R_1 \to R_1 + R_3$, $R_2 \to R_2 - R_3$

$\sim (I \mid A^{-1})$

and hence, $A^{-1} = \begin{pmatrix} \frac{7}{4} & -\frac{1}{2} & \frac{1}{4} \\ -\frac{11}{4} & \frac{3}{2} & -\frac{1}{4} \\ \frac{3}{4} & -\frac{1}{2} & \frac{1}{4} \end{pmatrix}$

ii) $(B \mid I) = \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & -1 & 3 & 0 & 1 & 0 \\ 4 & 1 & 8 & 0 & 0 & 1 \end{array}\right)$

$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \end{array}\right)$, by $R_2 \to R_2 - 2R_1$; $R_3 \to R_3 - 4R_1$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -1 & -6 & 1 & 1 \end{array}\right), \text{ by } R_3 \leftrightarrow R_3+R_2$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -11 & 2 & 2 \\ 0 & -1 & 0 & 4 & 0 & -1 \\ 0 & 0 & -1 & -6 & 1 & 1 \end{array}\right), \text{ by } R_1 \to R_1+2R_3;\ R_2 \to R_2-R_3$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -11 & 2 & 2 \\ 0 & 1 & 0 & -4 & 0 & 1 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{array}\right), \text{ by } R_2 \to (-1)R_2;\ R_3 \to (-1)R_3$$

$\sim (I \mid B^{-1})$

and hence, $B^{-1} = \left(\begin{array}{ccc} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{array}\right)$

iii) $(C \mid I) = \left(\begin{array}{cccc|cccc} 1 & 1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 2 & -1 & 1 & 2 & 0 & 1 & 0 & 0 \\ -1 & 2 & 1 & -2 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right)$

$$\sim \left(\begin{array}{cccc|cccc} 1 & 1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & -3 & 3 & 0 & -2 & 1 & 0 & 0 \\ 0 & 3 & 3 & -1 & 1 & 0 & 1 & 0 \\ 0 & -2 & -1 & -2 & -1 & 0 & 0 & 1 \end{array}\right),$$

$$R_2 \to R_2-2R_1,\ R_3 \to R_3+R_1,\ R_4 \to R_4-R_1$$

$$\sim \left(\begin{array}{cccc|cccc} 1 & 1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & \frac{2}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & 3 & 3 & -1 & 1 & 0 & 1 & 0 \\ 0 & -2 & -1 & -2 & -1 & 0 & 0 & 1 \end{array}\right), \ R_2 \to -\frac{1}{3}R_2$$

$$\sim \left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 1 & 0 & \frac{2}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & -2 & \frac{1}{3} & -\frac{2}{3} & 0 & 1 \end{array}\right),$$

$$R_1 \to R_1-R_2,\ R_3 \to R_3-3R_2,\ R_4 \to R_4+2R_2$$

$$\sim \left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 1 & 0 & \frac{2}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & -2 & \frac{1}{3} & -\frac{2}{3} & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 & -1 & 1 & 0 \end{array}\right), \; R_3 \leftrightarrow R_4$$

$$\sim \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 3 & \frac{1}{3} & \frac{1}{3} & 0 & -1 \\ 0 & 1 & 0 & 2 & \frac{1}{3} & -\frac{1}{3} & 0 & -1 \\ 0 & 0 & 1 & -2 & \frac{1}{3} & -\frac{2}{3} & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 & -1 & -1 & 0 \end{array}\right),$$

$$R_1 \to R_1 - R_3,\; R_2 \to R_2 - R_3,\; R_4 \to (-1)R_4$$

$$\sim \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -3 & 4 & 3 & -1 \\ 0 & 1 & 0 & 0 & -\frac{5}{3} & \frac{7}{3} & 2 & -1 \\ 0 & 0 & 1 & 0 & \frac{7}{3} & -\frac{8}{3} & -2 & 1 \\ 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 \end{array}\right),$$

$$R_1 \to R_1 - 3R_4,\; R_2 \to R_2 - 2R_4,\; R_3 \to R_3 + 2R_4$$

$$\sim (I \mid C^{-1})$$

and hence, $C^{-1} = \begin{pmatrix} -3 & 4 & 3 & -1 \\ -\frac{5}{3} & \frac{7}{3} & 2 & -1 \\ \frac{7}{3} & -\frac{8}{3} & -2 & 1 \\ 1 & -1 & -1 & 0 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -9 & 12 & 9 & -3 \\ -5 & 7 & 6 & -3 \\ 7 & -8 & -6 & 3 \\ 3 & -3 & -3 & 0 \end{pmatrix}$

iv) $(D \mid I) = \left(\begin{array}{ccc|ccc} 1 & 3 & -4 & 1 & 0 & 0 \\ 1 & 5 & -1 & 0 & 1 & 0 \\ 3 & 13 & -6 & 0 & 0 & 1 \end{array}\right)$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 3 & -4 & 1 & 0 & 0 \\ 0 & 2 & 3 & -1 & 1 & 0 \\ 0 & 4 & 6 & -3 & 0 & 1 \end{array}\right), \; R_2 \to R_2 - R_1,\; R_3 \to R_3 - 3R_1$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 3 & -4 & 1 & 0 & 0 \\ 0 & 2 & 3 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{array}\right), \; R_3 \to R_3 - 2R_2$$

Since the D-part on the r.h.s. has a zero-row, the given matrix is not invertible, i.e., D^{-1} does not exist.

CHECK YOUR PROGRESS

Q.2: Find A^{-1} using elementary row operations:

i) $A = \begin{pmatrix} -2 & 3 & 2 \\ 6 & 0 & 3 \\ 4 & 1 & -1 \end{pmatrix}$ ii) $A = \begin{pmatrix} 1 & -2 & 2 \\ 2 & -3 & 6 \\ 1 & 1 & 7 \end{pmatrix}$

To find Rank Using Elementary Row Operations: We have seen that to find the rank of a matrix we have to calculate values of a number of determinants, which is a tedious process. We can avoid it by the application of elementary row operations on the matrix reducing it to reduced echelon matrix. This method is based on the following theorem, stated without proof.

Theorem 7:

a) The rank of a matrix remains invariant under elementary row operations.
b) The rank of a matrix is equal to the rank of its reduced echelon matrix.
c) If a reduced echelon matrix has r non-zero rows, then its rank is r.

From this theorem, it is obvious that if A ~ R where R is the reduced echelon matrix, and if R has r non-zero rows, then $\rho(A) = \rho(R) = r$.

Note: We can apply elementary operations to columns also.In that case they are called elementary column operations We get analogores results.

Example 10: Find the rank of $A = \begin{pmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{pmatrix}$

Solution: We find the reduced echelon matrix of A performing elementary row operations :

$$A = \begin{pmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & \frac{1}{8} & \frac{3}{8} & \frac{3}{4} \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{pmatrix}, \; R_1 \to \frac{1}{8}R_1$$

$$\sim \begin{pmatrix} 1 & \frac{1}{8} & \frac{3}{8} & \frac{3}{4} \\ 0 & 1 & \frac{2}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 10 \end{pmatrix}, \; R_2 \to \frac{1}{3}R_2, \; R_3 \to R_3 + 8R_1$$

$$\sim \begin{pmatrix} 1 & \frac{1}{8} & \frac{3}{8} & \frac{3}{4} \\ 0 & 1 & \frac{2}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; R_1 \to R_1 - \frac{1}{8}R_2, \; R_3 \to \frac{1}{10}R_3$$

$$\sim \begin{pmatrix} 1 & 0 & \frac{7}{24} & 0 \\ 0 & 1 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; R_1 \to R_1 - \frac{2}{3}R_3, \; R_2 \to R_2 - \frac{2}{3}R_3$$

The right hand side matrix is in reduced echelon form having 3 non-zero rows. So, $\rho(A) = 3$.

Example 8: Find the rank of $A = \begin{pmatrix} 2 & 3 & 1 & 0 & 4 \\ 3 & 1 & 2 & -1 & 1 \\ 4 & -1 & 3 & -2 & -2 \\ 5 & 4 & 3 & -1 & 6 \end{pmatrix}$

Solution:

$$A = \begin{pmatrix} 2 & 3 & 1 & 0 & 4 \\ 3 & 1 & 2 & -1 & 1 \\ 4 & -1 & 3 & -2 & -2 \\ 5 & 4 & 3 & -1 & 6 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & \frac{3}{2} & \frac{1}{2} & 0 & 2 \\ 3 & 1 & 2 & -1 & 1 \\ 4 & -1 & 3 & -2 & -2 \\ 5 & 4 & 3 & -1 & 6 \end{pmatrix}, \; R_1 \to ½R_1$$

$$\sim \begin{pmatrix} 1 & \frac{3}{2} & \frac{1}{2} & 0 & 2 \\ 0 & -\frac{7}{2} & \frac{1}{2} & -1 & -5 \\ 0 & -7 & 1 & -2 & -10 \\ 0 & -\frac{7}{2} & \frac{1}{2} & -1 & -4 \end{pmatrix},$$

$$R_2 \to R_2 - 3R_1, \; R_3 \to R_3 - 4R_1, \; R_4 \to R_4 - 5R_1$$

$$\sim \begin{pmatrix} 1 & \frac{3}{2} & \frac{1}{2} & 0 & 2 \\ 0 & 1 & -\frac{1}{7} & \frac{2}{7} & \frac{10}{7} \\ 0 & -7 & 1 & -2 & -10 \\ 0 & -\frac{7}{2} & \frac{1}{2} & -1 & -4 \end{pmatrix}, \; R_2 \to -\frac{2}{7}R_2,$$

$$\sim \begin{pmatrix} 1 & 0 & \frac{5}{7} & -\frac{3}{7} & -\frac{1}{7} \\ 0 & 1 & -\frac{1}{7} & \frac{2}{7} & \frac{10}{7} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$R_1 \to R_1 - \frac{3}{2}R_2,\; R_3 \to R_3 + 7R_2,\; R_4 \to R_4 + \frac{7}{2}R_2$$

$$\sim \begin{pmatrix} 1 & 0 & \frac{5}{7} & -\frac{3}{7} & 0 \\ 0 & 1 & -\frac{1}{7} & \frac{2}{7} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \; R_1 \to R_1 + \frac{1}{7}R_4,\; R_2 \to R_2 - \frac{10}{7}R_4$$

$$\sim \begin{pmatrix} 1 & 0 & \frac{5}{7} & -\frac{3}{7} & 0 \\ 0 & 1 & -\frac{1}{7} & \frac{2}{7} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} R_3 \leftrightarrow R_4$$

The right hand side matrix is in reduced echelon form having 3 non-zero rows.

So, $\rho(A) = 3$.

CHECK YOUR PROGRESS

Q.3: Find rank of the following matrices using elementary row operations :

i) $\begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 1 \\ 1 & -1 & 4 & 0 \\ -2 & 2 & 8 & 0 \end{pmatrix}$ ii) $\begin{pmatrix} 1 & -1 & 3 & 6 \\ 1 & 3 & -3 & 4 \\ 5 & 3 & 3 & 11 \end{pmatrix}$

7.6 LET US SUM UP

- If a matrix A has atleast one non-zero minor of order r and all other higher order minors are zero, then r is called the rank of A, denoted by $\rho(A)$.
- Rank of a sub matrix $\leq$ Rank of the matrix.
- Elementary row operations are:
  i) interchanging any two rows: $R_i \leftrightarrow R_j$
  ii) multiplying a row by a non-zero scalar: $R_i \rightarrow kR_j$
  iii) replacing the i^{th} row by adding to this row k times the j^{th} row: $R_i \rightarrow R_i + kR_j$.
- A matrix A is in echelon form if:
  i) any zero rows are on the bottom of the matrix.
  ii) The first non-zero entry, called the leading entry in a row occurs to the right of the leading non-zero entry in the preceding row.
- A matrix A is said to be in row canonical form if:
  i) A is an echelon matrix,
  ii) each leading non-zero entry in a row is 1,
  iii) each leading non-zero entry is the only non-zero entry in its column.

7.7 ANSWER TO CHECK YOUR PROGRESS

Ans. to Q. No. 1:

i) As the matix A is of order 2 x 3, the highest possible order of minor is 2.

A minor of order $2 = \begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} = 0x1 - 2x1 = -2 \neq 0$

$\therefore \rho(A) = 2$

ii) As the matix A is of order 4 x 3, the highest possible order of minor is 3.

A minor of order $3 = \begin{vmatrix} 1 & 1 & 1 \\ 0 & -3 & -5 \\ 0 & -8 & -1 \end{vmatrix}$

$= \begin{vmatrix} -3 & -5 \\ -8 & -1 \end{vmatrix} = 3 - 40 = -37 \neq 0$

$\therefore \rho(A) = 3$

ii) As the matix A is of order 3 x 3, the highest possible order of minor is 3.

The minor of order $3 = \begin{vmatrix} 2 & 1 & 3 \\ 4 & 3 & 13 \\ 4 & -3 & -1 \end{vmatrix}$

$= 2x(-3+39)-(-4-52)+3(-12-12)$

$= 2x36+56+3(-24)$

$= 72+56-72$

$= 56 \neq 0$

$\therefore \rho(A) = 3$

iv) As the matix A is of order 2 x 2, the highest possible order of minor is 2.

The minor of order 2

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

$\therefore \rho(A) = 2$

Ans. to Q. No. 2:

i) $(A \mid I) = \left(\begin{array}{ccc|ccc} -2 & 3 & 2 & 1 & 0 & 0 \\ 6 & 0 & 3 & 0 & 1 & 0 \\ 4 & 1 & -1 & 0 & 0 & 1 \end{array}\right)$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -\frac{3}{2} & -1 & -\frac{1}{2} & 0 & 0 \\ 6 & 0 & 3 & 0 & 1 & 0 \\ 4 & 1 & -1 & 0 & 0 & 1 \end{array}\right), R_1 \to -½R_1$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -\frac{3}{2} & -1 & -\frac{1}{2} & 0 & 0 \\ 0 & 9 & 9 & 3 & 1 & 0 \\ 0 & 7 & 3 & 2 & 0 & 1 \end{array}\right), R_2 \to R_2 - 6R_1, R_3 \to R_3 - 4R_1$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -\frac{3}{2} & -1 & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{3} & \frac{1}{9} & 0 \\ 0 & 7 & 3 & 2 & 0 & 1 \end{array}\right), R_2 \to \frac{1}{9}R_2$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & 0 & \frac{1}{6} & 0 \\ 0 & 1 & 1 & \frac{1}{3} & \frac{1}{9} & 0 \\ 0 & 0 & -4 & -\frac{1}{3} & -\frac{7}{9} & 1 \end{array}\right), R_1 \to R_1 + \frac{3}{2}R_2, R_3 \to R_3 - 7R_2$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & 0 & \frac{1}{6} & 0 \\ 0 & 1 & 1 & \frac{1}{3} & \frac{1}{9} & 0 \\ 0 & 0 & 1 & -\frac{1}{12} & -\frac{7}{36} & -\frac{1}{4} \end{array}\right), R_3 \to --\frac{1}{4}R_3$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{24} & \frac{5}{72} & \frac{1}{8} \\ 0 & 1 & 0 & \frac{1}{4} & -\frac{1}{12} & \frac{1}{4} \\ 0 & 0 & 1 & -\frac{1}{12} & -\frac{7}{36} & -\frac{1}{4} \end{array}\right), \; R_2 \to R_1 - \tfrac{1}{2}R_3, \; R_2 \to R_2 - R_3$$

$$\sim (I \mid A^{-1})$$

$$\text{and so, } A^{-1} = \begin{pmatrix} -\frac{1}{24} & \frac{5}{72} & \frac{1}{8} \\ \frac{1}{4} & -\frac{1}{12} & \frac{1}{4} \\ \frac{1}{12} & \frac{7}{36} & -\frac{1}{4} \end{pmatrix}$$

$$= \frac{1}{72}\begin{pmatrix} -3 & 5 & 9 \\ 18 & -6 & 18 \\ 6 & 14 & -18 \end{pmatrix}$$

ii) $$(A \mid I) = \left(\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 2 & -3 & 6 & 0 & 1 & 0 \\ 1 & 1 & 7 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 3 & 5 & -1 & 0 & 1 \end{array}\right), \; R_2 \to R_2 - 2R_1, \; R_3 \to R_3 - R_1$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 6 & -3 & 2 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & -1 & 5 & -3 & 1 \end{array}\right), \; R_1 \to R_1 + 2R_2, \; R_3 \to R_3 - 3R_2$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 6 & -3 & 2 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -5 & 3 & -1 \end{array}\right), \; R_3 \to (-1)R_3$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 27 & -16 & 6 \\ 0 & 1 & 0 & 8 & -5 & 2 \\ 0 & 0 & 1 & -5 & 3 & -1 \end{array}\right), \; R_1 \to R_1 - 6R_3, \; R_2 \to R_2 - 2R_3$$

$$\sim (I \mid A^{-1})$$

$$\text{So, } A^{-1} = \begin{pmatrix} 27 & -16 & 6 \\ 8 & -5 & 2 \\ -5 & 3 & -1 \end{pmatrix}$$

Ans. to Q. No. 3:

i) $A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 1 \\ 1 & -1 & 4 & 0 \\ -2 & 2 & 8 & 0 \end{pmatrix}$

$\sim \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & -1 & 2 & -1 \\ 0 & 2 & 12 & 2 \end{pmatrix}$, $R_3 \to R_3 - R_1$, $R_4 \to R_4 + 2R_1$

$\sim \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 16 & 0 \end{pmatrix}$, $R_3 \to R_3 + R_2$, $R_4 \to R_4 - 2R_2$

$\sim \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $R_3 \leftrightarrow R_4$

$\sim \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $R_3 \to \frac{1}{16} R_3$

$\sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $R_1 \to R_1 - 2R_3$, $R_2 \to R_2 + 2R_3$

The right hand side matrix is reduced echelon matrix with three non-zero rows. Hence, $\rho(A) = 3$.

ii) Try yourselelf.

7.8 FURTHER READING

- University Algebra, N S Gopala Krishnan, New Age International (P) Limited.
- Theory of Matrices, B S Vatsa, New Age International Publishers..

- Linear Algebra: Schum Solved Problems Series; Seymour Lipschitz: Tata McGraw Hill.
- Text Book of Algebra, A.K Sharma, DPH, Mathematics Series.
- A Text Book of Matrices, Shanti Narayan and P K Mittal.

7.9 MODEL QUESTIONS

Q.1: Find inverse matrix using elementary row operations :

i) $A = \begin{pmatrix} 1 & 2 & -4 \\ -1 & -1 & 5 \\ 2 & 7 & -3 \end{pmatrix}$ ii) $A = \begin{pmatrix} 2 & 4 & 3 \\ 0 & 1 & 1 \\ 2 & 2 & -1 \end{pmatrix}$

iii) $A = \begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix}$ iv) $A = \begin{pmatrix} 3 & -2 & 0 & -1 \\ 0 & 2 & 2 & 1 \\ 1 & -2 & -3 & -2 \\ 0 & 1 & 2 & 1 \end{pmatrix}$

Q.2: Find rank of the following matrices using definition :

i) $\begin{pmatrix} 1 & 2 & 3 \\ -2 & 4 & 1 \\ -3 & -1 & 2 \end{pmatrix}$ ii) $\begin{pmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{pmatrix}$

iii) $\begin{pmatrix} 1-\sqrt{6} & \sqrt{3} & \sqrt{2} \\ 2 & \sqrt{6} & -\sqrt{2} \\ 1 & -\sqrt{3} & -\sqrt{6} \end{pmatrix}$

Q 3: Find rank using elementary row operations :

i) $\begin{pmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{pmatrix}$ ii) $\begin{pmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{pmatrix}$

iii) $\begin{pmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{pmatrix}$

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