UNIT 8: SYSTEM OF LINEAR EQUATIONS

UNIT STRUCTURE

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8.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- solve system of linear equations by matrix method
- solve system of linear equations by Cramer's rule.

8.2 INTRODUCTION

We are familiar with the method of solution of two linear equations in two variables. The equations ax + by = c, px + qy = d can easily be solved by the method of elimination of one of the two variables. But the method is difficult to work with when the number of variables and number of equations are more than two. It becomes more difficult when the number of equations and the number of variables are not equal. In this unit we shall discuss method of solution of such a system of linear equations, converting the system of equations to matrix form and then solving by applying Cramer's rule.

8.3 LINEAR EQUATIONS

Linear equations are divided into two types–

i) Homogeneous linear equations
ii) Non-homogeneous linear equations.

The nature of solutions of a system of linear equations can be studied with the help of matrices. We shall first consider a system of homogeneous linear equations. There are two types of equations

i) Consistent equation
ii) Inconsistent equation

i) **Consistent equation:** A system of equations is said to be consistent if they have one or more solutions.

Example: i) $x + 2y = 0$, $3x + y = 2$ ii) $x + 2y = 4$, $4x + 8y = 16$

ii) **Inconsistent Equations:** A system of equation is said to be inconsistent if they have no solution.

Example: $x + 2y = 4$

$$3x + 6y = 6$$

Homogeneous Linear Equations: A system of n-linear equations in n-unknown say, $x_1, x_{2,} \cdots, x_n$ is–

given as follows:

$$\left.\begin{aligned} &a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots\ldots + a_{1n}x_n = 0 \\ &a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots\ldots + a_{2n}x_n = 0 \\ &\text{- -} \\ &a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots\ldots\ldots + a_{mn}x_n = 0 \end{aligned}\right\} \quad (1)$$

$$\text{Let } A = \begin{bmatrix} a_{11} & a_{12}.....a_{1n} \\ a_{21} & a_{22}.....a_{2n} \\ a_{m1} & a_{m2} ..... a_{mn} \end{bmatrix}_{m\times n}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix}_{n\times 1}, 0 = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ 0 \end{bmatrix}_{m\times 1}$$

Then the above system of equations can be written in the matrix form as–

$$\begin{bmatrix} a_{11} & a_{12}.....a_{1n} \\ a_{21} & a_{22}.....a_{2n} \\ a_{m1} & a_{m2} ..... a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix}, = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ 0 \end{bmatrix}$$

i.e. $AX = 0$ (2)

The matrix A is called the co-efficient matrix of the system of equation (1)

Obviously, $x_1 = x_2 = \ldots \ldots \ldots \ldots \ldots = x_n = 0$

i.e. $X = 0$ is a solution of (1) called the trivial solution.

Again, let X_1 and X_2 be two solutions of (2). Then their linear combination $K_1X_1 + K_2X_2$ where K_1 and K_2 are any arbitrary numbers is also a solution of (2).

Now, $AX_1 = 0$ and $AX_2 = 0$ and hence

$$\begin{aligned} A(K_1X_1 + K_2X_2) &= K_1(AX_1) + K_2(AX_2) \\ &= K_1 0 + K_2 0 \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

Hence, $K_1X_1 + K_2X_2$ is also a solution of (2)

8.4 SOLUTION OF NON-HOMOGENEOUS SIMULTANEOUS EQUATIONS

Let a system of n simultaneous equation in n-unknown

$x_1, x_2, \ldots \ldots \ldots x_n$ be–

$a_{11} x_1 + a_{12} x_2 + \ldots\ldots\ldots\ldots + a_{1n} x_n = b_1$

$a_{21} x_1 + a_{22} x_2 + \ldots\ldots\ldots\ldots + a_{2n} x_n = b_2$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$a_{n1} x_1 + a_{n2} x_2 + \ldots\ldots\ldots\ldots + a_{nn} x_n = b_n$

This system can be written in the matrix form as

$$AX = B \qquad (1)$$

where $A = \begin{bmatrix} a_{11} & a_{12} \ldots\ldots\ldots a_{1n} \\ a_{21} & a_{22} \ldots\ldots\ldots a_{2n} \\ a_{n1} & a_{n2} \ldots\ldots\ldots a_{nn} \end{bmatrix}_{n\times n}$ $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$ $B = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_n \end{bmatrix}_{nx1}$

If $|A| \neq 0$, then A^{-1} exists.

Pre-multiplying both sides of (1) by A^{-1}

$A^{-1}(AX) = A^{-1} B$

$\Rightarrow (A^{-1}A) X = A^{-1}B$

$\Rightarrow IX = A^{-1} B$

$\Rightarrow X = A^{-1}B$ which is the solution of the given system of equations.

Example: Solve: $x + y + z = 1$, $x + 2y + z = 2$, $x + y + 2z = 0$

Solution: Given system of equations is–

$x + y + z = 1$

$x + 2y + z = 2$

$x + y + 2z = 0$

It can be written as–

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

i.e. $AX = B$ where $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

Now, $|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} = 1(4-1) - 1(2-1) + 1(1-2)$

$= 3 - 1 - 1 = 1 \neq 0$

Co-factors of the elements of 1st row are–

$$\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}, -\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} \quad \text{i.e.3,–1,-1}$$

Co-factors of the elements of 2nd row are –

$$-\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}, \quad \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} \text{ i.e. } -1, 1, 0$$

Co-factors of the elements of 3rd row are –

$$\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}, -\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \text{ i.e. } -1, 0, 1$$

$$\text{adj } A = \begin{vmatrix} 3 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix}$$

$$\therefore\ A^{-1} = \frac{\text{adj } A}{|A|} = \text{adj } A \quad \because |A| = 1$$

$$\therefore\ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}B = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -2 \\ -1+2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

i.e. x = 1, y = 1, z = –1

CHECK YOUR PROGRESS

Q.1: Solve: x + y + z = 6, x – y + z = 2,

2x + y + z = 1

Q.2: Solve: x – 3y + z = -1, 2x + y – 4z = –1, 6x – 7y + 8z = 7

8.5 SOLUTION OF A SYSTEM OF n NON-HOMOGENOUS LINEAR EQUATIONS IN n-UNKNOWNS BY CRAMER'S RULE

Let the system of n non-homogenous linear equations in n-unknowns be–

$a_{11} x_1 + a_{12} x_2 + ... + a_{1n} x_n = b_1$ (1)

$a_{21} x_1 + a_{22} x_2 + ... + a_{2n} x_n = b_2$ (2)

… … … … … … … … … … … … …

$a_{n1} x_1 + a_{n2} x_2 + ... + a_{nn} x_n = b_n$ (n)

The system can be written as–

$$\begin{bmatrix} a_{11} & a_{12} & .... & a_{1n} \\ a_{21} & a_{22} & .... & a_{2n} \\ ..... & ..... & .. & \\ a_{n1} & a_{n2} & ..... & a_{nn} \end{bmatrix}_{n\times n} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix}_{n\times 1} = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_n \end{bmatrix}_{n\times 1}$$

i.e. AX = B

Determinant of the co-efficient matrix A =|A| = D (say)

Multiplying the equations (1), (2), …, …, (n) respectively by the co-factors of $a_{11}, a_{21}, \cdots\cdots$ i.e. $A_{11}, A_{21}, \cdots, A_{n1}$ and adding we get

$$A_{11}(a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n) + A_{21}(a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n) + \cdots$$
$$+ A_{n1}(a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n) = b_1A_{11} + b_2A_{21} + \cdots + b_nA_{n1}$$
$$\Rightarrow (a_{11}A_1 + a_{21}A_{21} + \cdots + a_{n1}A_{n1})x_1 = b_1A_{11} + b_2A_{21} + \cdots + b_nA_{n1}$$
$$\Rightarrow Dx_1 = D_1$$

Where $D = \begin{vmatrix} a_{11} & a_{12} ..... & a_{1n} \\ a_{21} & a_{22} ..... & a_{2n} \\ ..... & & \\ a_{n1} & a_{n2} ..... & a_{nn} \end{vmatrix}$ and $D_1 = \begin{vmatrix} b_1 & a_{12} ..... & a_{1n} \\ b_2 & a_{22} ..... & a_{2n} \\ ..... & & \\ b_n & a_{n2} ..... & a_{nn} \end{vmatrix}$

D_1 is the determinant obtained from D by replacing the elements of 1st column by corresponding b's–

$$\Rightarrow \frac{x_1}{D_1} = \frac{1}{D} \quad \text{Provided } D \neq 0$$

Similarly multiplying the equations (1), (2), …… (n) by co-factors of the elements of 2nd column of |A| and adding, we get–

$$Dx_2 = D_2 \quad \text{where } D_2 = \begin{vmatrix} a_{11} & b_1 ..... & a_{1n} \\ a_{21} & b_2 .....a_{2n} \\ & & \\ a_{n1} & b_n .....a_{nn} \end{vmatrix}$$

$$\Rightarrow \frac{x_2}{D_2} = \frac{1}{D}, \; D \neq 0$$

As above we will get–

$$\frac{x_1}{D_1} = \frac{x_2}{D_2} = ... = \frac{x_n}{D_n} = \frac{1}{D}, D \neq 0$$

as the unique solution of the given system of equations provided

$D \neq 0$ i.e. the coefficient matrix is non-singular.

i.e. the rank of the co-efficient matrix is n = number of variables.

Note: For a system of n non-homogeneous linear equations with n-unknowns

$$\frac{x_1}{D_1} = \frac{x_2}{D_2} = \cdots = \frac{x_n}{D_n} = \frac{1}{D}, D \neq 0$$

Example: Solve the equations: $3x + y + 2z = 3$

$2x - 3y - z = -3$

$x + 2y + z = 4$

using Cramer's rule.

Solution: We have $D = |A| = \begin{vmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{vmatrix}$

$= 3(-3 + 2) - 1(2+1) + 2(4+3)$

$= -3 - 3 + 14$

$= 8 \neq 0$

$$D_1 = \begin{vmatrix} 3 & 1 & 2 \\ -3 & -3 & -1 \\ 4 & 2 & 1 \end{vmatrix}$$

$= 3(-3 + 2) - 1(-3 + 4) + 2(-6 + 12)$

$= -3 - 1 + 12$

$= 8$

$$D_2 = \begin{vmatrix} 3 & 3 & 2 \\ 2 & -3 & -1 \\ 1 & 4 & 1 \end{vmatrix}$$

$= 3(-3 + 4) - 3(2 + 1) + 2(8 + 3)$

$= 3 - 9 + 22 = 16$

$$D_3 = \begin{vmatrix} 3 & 1 & 3 \\ 2 & -3 & -3 \\ 1 & 2 & 4 \end{vmatrix}$$

$= 3(-12 + 6) - 1(8 + 3) + 3(4 + 3)$

$= -18 - 11 + 21$

$= -8$

$\therefore x = \frac{D_1}{D} = \frac{8}{8} = 1$

$y = \frac{D_2}{D} = \frac{16}{8} = 2$

$z = \frac{D_3}{D} = \frac{-8}{8} = -1$

Example: Solve completely the equations by Cramer's rule–

$x + 2y + 3z = 6, \quad 3x - 2y + z = 2, \quad 4x + 2y + z = 7$

Solution: $D = \begin{vmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{vmatrix} = 1(-2-2) - 2(3-4) + 3(6+8)$

$= -4 + 2 + 42 = 40 \neq 0$

$D_1 = \begin{vmatrix} 6 & 2 & 3 \\ 2 & -2 & 1 \\ 7 & 2 & 1 \end{vmatrix} = 6(-2-2) - 2(2-7) + 3(4+14)$

$= -24 + 10 + 54 = 40$

$D_2 = \begin{vmatrix} 1 & 6 & 3 \\ 3 & 2 & 1 \\ 4 & 7 & 1 \end{vmatrix}$

$= 1(2 - 7) - 6(3 - 4) + 3(21 - 8)$

$= -5 + 6 + 39$

$= 40$

$D_3 = \begin{vmatrix} 1 & 2 & 6 \\ 3 & -2 & 2 \\ 4 & 2 & 7 \end{vmatrix}$

$= 1(-14 - 4) - 2(21 - 8) + 6(6 + 8)$

$= -18 - 26 + 84$

$= 84 - 44$

$= 40$

$\therefore x = \frac{D_1}{D} = \frac{40}{40} = 1$

$y = \frac{D_2}{D} = \frac{40}{40} = 1$

$z = \frac{D_3}{D} = \frac{40}{40} = 1$

CHECK YOUR PROGRESS

Q.3: Solve completely the system of equations by using Cramer's rule:

i) $3x + 5y = 8, \ -x + 2y - z = 0, \ 3x - 6y + 4z = 1$

ii) $x_1 + x_2 + x_3 = 7, \ x_1 - x_2 + x_3 = 2, \ 2x_1 - x_2 + 3x_3 = 9$

8.6 LET US SUM UP

- If x_1 and x_2 be two solutions of AX = 0, the $k_1x_1 + k_2x_2$ is also a solution of AX = 0
- For system of n simultaneous non-homogenous equations with n-unknown written as AX = B, the solution matrix is $X = A^{-1}B$
- provided $|A| \neq 0$.
- For a system of n-simultaneous linear non-homogenous equation with n-unknowns

$$\frac{x_1}{D_1} = \frac{x_2}{D_2} = \cdots = \frac{x_n}{D_n} = \frac{1}{D}, D \neq 0$$

8.7 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1: $x + y + z = 6$

$x - y + z = 2$

$2x + y - z = 1$

This system of equations can be written in matix form as–

$Ax = B$

where $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix}$, $x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}$$

$$= 1 - 1 - (-1 - 2) + (1 + 2)$$
$$= 0 + 3 + 3 = 6 \neq 0$$

Now, Co-factors of the elements of 1st row are–

$$\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix}, -\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix}, \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}, \text{i.e. } 0,\ 3,\ 3$$

Co-factors of the elements of 2nd row are–

$$-\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix}, -\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}, \text{i.e. } 2, -3,\ 1$$

Co-factors of the elements of 3rd row are–

$$\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix}, -\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}, \text{i.e. } 2,\ 0,\ -2$$

$$\therefore \text{adj } A = \begin{bmatrix} 0 & 2 & 2 \\ 3 & -3 & 0 \\ 3 & 1 & -2 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}.\text{adj}.A = \frac{1}{6}\begin{bmatrix} 0 & 2 & 2 \\ 3 & -3 & 0 \\ 3 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{6} & -\frac{1}{3} \end{bmatrix}$$

$$\therefore X = A^{-1}B = \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{6} & -\frac{1}{3} \end{bmatrix}\begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{2}{3}+\frac{1}{3} \\ 3-1 \\ 3+\frac{1}{3}-\frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\therefore x = 1, \quad y = 2, \quad z = 3.$

Ans. to Q. No. 2: $x - 3y + z = -1$

$2x + y - 4z = -1$

$6x - 7y + 8z = 7$

The above system of equations can be written in matrix from as

$Ax = B.$

When $A = \begin{bmatrix} 1 & -3 & 1 \\ 2 & 1 & -4 \\ 6 & -7 & 8 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad = \begin{bmatrix} -1 \\ -1 \\ 7 \end{bmatrix}$

Now, $|A| = \begin{vmatrix} 1 & -3 & 1 \\ 2 & 1 & -4 \\ 6 & -7 & 8 \end{vmatrix}$

$$= \begin{vmatrix} 1 & -4 \\ -7 & 8 \end{vmatrix} + 3\begin{vmatrix} 2 & -4 \\ 6 & 8 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 6 & -7 \end{vmatrix}$$

$= 8 - 28 + 3(16 + 24) + (-14 - 6)$

$= 8 - 28 + 120 - 20$

$= 80 \neq 0$

Now, Co-factor of the elements of 1st row are–

$$\begin{vmatrix} 1 & -4 \\ -7 & 8 \end{vmatrix}, -\begin{vmatrix} 2 & -4 \\ 6 & 8 \end{vmatrix}, \begin{vmatrix} 2 & 1 \\ 6 & -7 \end{vmatrix}, \text{ i.e. } -20, \ -40, \ -20$$

Co-factor of the elements of 2nd row are–

$$-\begin{vmatrix}-3 & 1\\ -7 & 8\end{vmatrix}, \begin{vmatrix}1 & 1\\ 6 & 8\end{vmatrix}, -\begin{vmatrix}1 & -3\\ 6 & -7\end{vmatrix}, \text{i.e. } 17,\ 2,\ -11$$

Co-factor of the elements of 3rd row are–

$$-\begin{vmatrix}-3 & 1\\ 1 & -4\end{vmatrix}, -\begin{vmatrix}1 & 1\\ 2 & -4\end{vmatrix}, \begin{vmatrix}1 & -3\\ 2 & 1\end{vmatrix}, \text{i.e. } 11,\ 6,\ 7$$

$$\therefore \text{adj } A = \begin{bmatrix}-20 & 17 & 11\\ -40 & 2 & 6\\ -20 & -11 & 7\end{bmatrix} = -20\begin{bmatrix}1 & 17 & 11\\ 2 & 2 & 6\\ 1 & -11 & 7\end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}\text{adj}\,A = \frac{1}{80}\begin{bmatrix}20 & 17 & 11\\ 40 & 2 & 6\\ 20 & 11 & 7\end{bmatrix}$$

$$\therefore X = A^{-1}\ B = \frac{1}{80}\begin{bmatrix}-20 & 17 & 11\\ -40 & 2 & 6\\ -20 & -11 & 7\end{bmatrix}\begin{bmatrix}-1\\ -1\\ 7\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}x\\ y\\ z\end{bmatrix} = \frac{1}{80}\begin{bmatrix}20 - 17 + 77\\ 40 - 2 + 42\\ 20 + 11 + 49\end{bmatrix} = \frac{1}{80}\begin{bmatrix}80\\ 80\\ 80\end{bmatrix} = \begin{bmatrix}1\\ 1\\ 1\end{bmatrix}$$

$$\therefore x = 1,\quad y = 1,\quad z = 1$$

Ans. to Q. No. 3: i) $3x + 5y + 0z = 8$

$-x + 2y - z = 0$

$3x - 6y + 4z = 1$

$$D = \begin{vmatrix}3 & 5 & 0\\ -1 & 2 & -1\\ 3 & -6 & 4\end{vmatrix}$$

$$= 3\begin{vmatrix}2 & -1\\ -6 & 4\end{vmatrix} - 5\begin{vmatrix}-1 & -1\\ 3 & 4\end{vmatrix}$$

$= 3\,(8 - 6) - 5(-4 + 3)$

$= 6 + 5 = 11 \neq 0$

$$D_1 = \begin{vmatrix}8 & 5 & 0\\ 0 & 2 & -1\\ 1 & -6 & 4\end{vmatrix} = 8\begin{vmatrix}2 & -1\\ -6 & 4\end{vmatrix} - 5\begin{vmatrix}0 & -1\\ 1 & 4\end{vmatrix}$$

$= 8(8 - 6) - 5 \times 1 = 16 - 5 = 11$

$$D_2 = \begin{vmatrix}3 & 8 & 0\\ -1 & 0 & -1\\ 3 & 1 & 4\end{vmatrix}$$

$$= 3\begin{vmatrix} 0 & -1 \\ 1 & 4 \end{vmatrix} - 8\begin{vmatrix} -1 & -1 \\ 3 & 4 \end{vmatrix}$$

$= 3(0 + 1) – 8(–4 + 3)$

$= 3 + 8 = 11$

$$D_3 = \begin{vmatrix} 3 & 5 & 8 \\ -1 & 2 & 0 \\ 3 & -6 & 1 \end{vmatrix}$$

$$= 3\begin{vmatrix} 2 & 0 \\ -6 & 1 \end{vmatrix} - 5\begin{vmatrix} -1 & 0 \\ 3 & 1 \end{vmatrix} + 8\begin{vmatrix} -1 & 2 \\ 3 & -6 \end{vmatrix}$$

$= 3(2 – 0) – 5(–1 –0) + 8(6 – 6)$

$= 6 + 5 = 11$

$$\therefore x = \frac{D_1}{D} = \frac{11}{11} = 1$$

$$y = \frac{D_2}{D} = \frac{11}{11} = 1$$

$$y = \frac{D_3}{D} = \frac{11}{11} = 1$$

i.e. $x = 1$, $y = 1$, $z = 1$.

ii) $x_1 + x_2 + x_3 = 6$

$x_1 – x_2 + x_3 = 2$

$2x_1 – x_2 + 3x_3 = 9$

$$\therefore D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & -1 & 3 \end{vmatrix} = 1x\begin{vmatrix} -1 & 1 \\ -1 & 3 \end{vmatrix} - 1x\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} + a\,x\begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix}$$

$= (–3 + 1) – (3 – 2) + (–1 + 2)$

$= –2 – 1 + 1 = -2 \neq 0$

$$\therefore D_1 = \begin{vmatrix} 6 & 1 & 1 \\ 2 & -1 & 1 \\ 9 & -1 & 3 \end{vmatrix} = 6\begin{vmatrix} -1 & 1 \\ -1 & 3 \end{vmatrix} - 1x\begin{vmatrix} 2 & 1 \\ 9 & 3 \end{vmatrix} + 1x\begin{vmatrix} 2 & -1 \\ 9 & -1 \end{vmatrix}$$

$= 6(–3 + 1) – (6 – 9) + (–2 + 9)$

$= –12 + 3 + 7 = – 2$

$$D_2 = \begin{vmatrix} 1 & 6 & 1 \\ 1 & 2 & 1 \\ 2 & 9 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 1 \\ 9 & 3 \end{vmatrix} - 6\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 2 & 9 \end{vmatrix}$$

$= (6 – 9) – 6(3 – 2) +(9 – 4)$

$= –3 – 6 + 5 = –9 + 5 = – 4$

$$D_3 = \begin{vmatrix} 1 & 1 & 6 \\ 1 & -1 & 2 \\ 2 & -1 & 9 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 2 \\ -1 & 9 \end{vmatrix} - \begin{vmatrix} 1 & 2 \\ 2 & 9 \end{vmatrix} + 6\begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix}$$

$= (–9 + 2) – (9 – 4) + 6x(–1 + 2)$

$= –7 – 5 + 6 = –12 + 6 = –6$

$$x_1 = \frac{D_1}{D} = \frac{-2}{-2} = 1$$

$$x_2 = \frac{D_2}{D} = \frac{-4}{-2} = 2$$

$$x_3 = \frac{D_3}{D} = \frac{-6}{-2} = 3$$

8.8 FURTHER READING

- University Algebra, N S Gopala Krishnan, New Age International (P) Limited.
- Theory of Matrices, B S Vatsa, New Age International Publishers..
- Linear Algebra: Schum Solved Problems Series; Seymour Lipschitz: Tata McGraw Hill.
- Text Book of Algebra, A.K Sharma, DPH, Mathematics Series.
- A Text Book of Matrices, Shanti Narayan and P K Mittal.

8.9 MODEL QUESTIONS

Q.1: Solve by matrix method:

a) $3x + y + 2z = 3$, $2x – 3y – z = –3$, $x + 2y + z = 4$

b) $x + y + z = 0$, $2x + 3y – 5z = 7$, $3x – 4y – 2z = –1$

c) $x + y + z = 6$, $x + 2y – 3z = –4$, $–x – 4y + 9z = 18$

Q.2: Solve completely the system of equations by Cramer's rule.

a) $2x + 3y + z = 9$, $x + 2y + 3z = 6$, $3x + y + 2x = 8$

b) $4x + y + z = 4$, $x + 4y – 2z = 4$, $–x + 2y – 4z = 2$

c) $x + y + z + t = 4$, $x + y + z – t = 2$, $x – y + z – t = 0$

Q.3: Solve by Cramer's rule:

i) $x – y + 2z = 1$ ii) $x + y – 2z = 1$

$2x + y + z = 2$ $2x - 7z = 3$

$x - 3y + z = 1$ $x + y - z = 5$

iii) $x + 2y - z = 3$ iv) $3x + 2y - 2z = 1$

$3x - y + 2z = 1$ $-x + y + 4z = 13$

$2x - 2y + 3z = 2$ $2x - 3y + 4z = 8$

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UNIT 9: EIGENVALUES AND EIGENVECTORS

UNIT STRUCTURE

9.1 Learning Objectives
9.2 Introduction
9.3 Characteristic Equation of a Matrix
9.4 EigenValues and Eigen Vectors of a Matrix
9.5 Properties of Eigenvalues
9.6 Cayley Hamilton theorem
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9.8 Answers to Check Your Progress
9.9 Further Reading
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9.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- know characterstic equation of a matrix
- understand the concept of eigenValues and eigen vectors of a matrix
- state Cayley Hamilton theorem.

9.2 INTRODUCTION

In this unit, we will first introduce charcteristic equation of a matrix. In this unit, we shall study eigenvalues and eigen vectors of a matrix. We also state Cayley Hamilton theorem and apply this theorem in finding inverse of a matrix.

9.3 CHARACTERISTIC EQUATION OF A MATRIX

We consider a matrix. Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & a_{nn} \end{bmatrix} = \left[a_{ij}\right]_{n\times n}$$

be an $n \times n$ square matrix. Then we find

$$A - \lambda I = \begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & a_{nn} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & \cdot & \cdot & 0 \\ 0 & 1 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}-\lambda & a_{12} & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & a_{nn}-\lambda \end{bmatrix}$$

where I is the identity matrix of order n.

$A-\lambda I$ is called the characterstic matrix of order n.

The determinant $|A-\lambda I|$ of the characterstic matrix of A can be expanded in the form of $b_0+b_1\lambda+b_2\lambda^2+\cdots+b_{n-1}\lambda^{n-1}+b_n\lambda^n$ where b_i's are scalars.

The expression $b_0+b_1\lambda+b_2\lambda^2+\cdots+b_{n-1}\lambda^{n-1}+b_n\lambda^n$ is non zero polynomial of degree n and is denoted by $P(\lambda)$, called characterstic polynomial of A.

The equation $|A-\lambda I|=0$ i .e, $b_0+b_1\lambda+b_2\lambda^2+\cdots+b_{n-1}\lambda^{n-1}+b_n\lambda^n=0$ is called the characterstic equation of the matrix A.

9.4 EIGENVALUES AND EIGEN VECTORS OF A MATRIX

The roots of the characterstic equation are called the eigenvalues of the matrix A.

A non null vector X satisfying $AX=\lambda X$ is called eigen vector.

Note: Eigenvalues and eigenvectors are sometimes also called charactertic roots and characteristic vectors respectively.

Working rule to find eigen values and eigen vectors:

1) Find the characteristic equation $|A-\lambda I|=0$
2) Solve the characteristic equation to get characteristic roots. They are called eigen values.
3) To find the eigen vectors, solve $AX=\lambda X$ for different values of λ.

Note: Eigen vector corresponding to an eigen value of a matrixis not unique.

Example 1: Find the (i) the characterstic matrix (ii) characterstic polynomial (iii) characterstic equation (iv) characterstic roots (or eigen values) and (v) characterstic vectors(or eigen vectors) of the matrix $\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$.

Solution: Let $A=\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$

i) Characterstic matrix is $A-\lambda I=\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}-\lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$=\begin{pmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{pmatrix}$$

ii) Characterstic equation is $\begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix}=0$

$$\Rightarrow (1-\lambda)(3-\lambda)-8=0$$
$$\Rightarrow 3-3\lambda-\lambda+\lambda^2-8=0$$
$$\Rightarrow \lambda^2-4\lambda-5=0$$

iii) Characterstic roots are given by $\lambda^2-4\lambda-5=0$

$$\Rightarrow (\lambda-5)(\lambda+1)=0$$
$$\Rightarrow \lambda=-1,5$$

iv) For $\lambda=-1$,we find eigenvector $X=\begin{pmatrix}x\\y\end{pmatrix}$ such that $AX=\lambda X$

i.e. $\begin{pmatrix}1&4\\2&3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=(-1)\begin{pmatrix}x\\y\end{pmatrix}$

$$\Rightarrow \begin{pmatrix}x+4y\\2x+3y\end{pmatrix}=\begin{pmatrix}-x\\-y\end{pmatrix}$$

$$\Rightarrow \begin{matrix}x+4y=-x\\2x+3y=-y\end{matrix}$$

$$\Rightarrow 2x=-4y$$

i.e $x=-2y$

$$\therefore \frac{x}{-2}=\frac{y}{1}$$

$\therefore$ the solution of the system is $k(-2,1)$, where k is a real numbers.

$\therefore$ the eigen vectors are $c\begin{pmatrix}-2\\1\end{pmatrix}$, where c is a non-zero real number.

For $\lambda=5$, we find eigenvector $X=\begin{pmatrix}x\\y\end{pmatrix}$ such that $AX=\lambda X$

i.e. $\begin{pmatrix}1&4\\2&3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=(5)\begin{pmatrix}x\\y\end{pmatrix}$

$$\Rightarrow \begin{pmatrix}x+4y\\2x+3y\end{pmatrix}=\begin{pmatrix}5x\\5y\end{pmatrix}$$

$$\Rightarrow \begin{matrix}x+4y=5x\\2x+3y=5y\end{matrix}$$

$$\Rightarrow x=y$$

i.e $x=y$

$$\therefore \frac{x}{1}=\frac{y}{1}$$

$\therefore$ the solution of the system is $k(1,1)$,where k is a real numbers.

$\therefore$ the eigen vectors are $c\begin{pmatrix}1\\1\end{pmatrix}$, where c is a non-zero real number.

Example 2: Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$$

Solution: Characterstic equation of matrix is $|A - \lambda I| = 0$

$$\text{i.e., } \begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

Expanding we get $\lambda^3 - 18\lambda^2 + 45\lambda = 0$

i.e., $\lambda(\lambda^2 - 18\lambda + 45) = 0$

i.e, $\lambda(\lambda - 3)(\lambda - 15) = 0$

i.e. $\lambda = 0, 3, 15$

$\therefore$ Eigenvalues are 0, 3, 15.

For $\lambda = 0$,we find eigenvector $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ such that $AX = \lambda X$

$$\text{i.e.} \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\text{i.e.} \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

i.e. $8x - 6y + 2z = 0$

$-6x + 7y - 4z = 0$

$2x - 4y + 3z = 0$

Solving any two equations by cross multiplication method, we get

$$\frac{x}{24-14} = \frac{y}{32-12} = \frac{z}{56-36}$$

i.e., $\frac{x}{1} = \frac{y}{2} = \frac{z}{2}$

$\therefore$ the solution of the system is $k(1, 2, 2)$,where k is a real numbers.

$\therefore$ the eigen vectors are $c\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$,where c is a non-zero real number.

For $\lambda = 3$, we find eigenvector $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ such that $AX = \lambda X$

$$\text{i.e.} \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\text{i.e.} \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \\ 3z \end{pmatrix}$$

i.e. $5x - 6y + 2z = 0$

$-6x + 4y - 4z = 0$

$2x - 4y + 0z = 0$

Solving any two equations by cross multiplication method, we get

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{-2}$$

$\therefore$ the solution of the system is $k(2,1,-2)$, where k is a real numbers.

$\therefore$ the eigen vectors are $c \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$, where c is a non-zero real number.

For $\lambda = 15$, we find eigenvector $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ such that $AX = \lambda X$

$$\text{i.e.} \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 15 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\text{i.e.} \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 15x \\ 15y \\ 15z \end{pmatrix}$$

i.e. $-7x - 6y + 2z = 0$

$-6x - 8y - 4z = 0$

$2x - 4y - 12z = 0$

Solving any two equations by cross multiplication method, we get

$$\frac{x}{2} = \frac{y}{-2} = \frac{z}{1}$$

$\therefore$ Eigen vectors corresponding to eigenvalue $\lambda = 15$ are

$k \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$, where k is a non-zero real number.

CHECK YOUR PROGRESS

Q.1: Find the eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$.

Q.2: Find the eigenvalues and the corresponding eigenvectors for the following matrices.

i) $\begin{pmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{pmatrix}$ ii) $\begin{pmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{pmatrix}$ iii) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 3 & 2 & 3 \end{pmatrix}$

9.5 PROPERTIES OF EIGENVALUES

Now, we state the various properties of eigenvalues of a matrix without proof.

The properties are as follows:

a) The sum of the eigen values of a matrix is equal to the sum of the elements of the principal diagonal (trace of the matrix).

If $\lambda_1, \lambda_2, \lambda_3$ are eigenvalues of $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, then

Trace of the matrix $= \lambda_1 + \lambda_2 + \lambda_3 = a_{11} + a_{22} + a_{33}$.

b) The product of the eigen values of a matrix is equal to the determinant of the matrix.

If $\lambda_1, \lambda_2, \lambda_3$ are eigenvalues of $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$,

then $\lambda_1\lambda_2\lambda_3 = |A|$

c) The eigenvalues of a diagonal matrix are the same as its diagonal elements.

d) The eigenvalues of a triangular matrix are the same as its diagonal elements.

e) A square matrix A and its transpose A' have the same eigen values.

f) Two similar matrices have same eigen values.

Example 1: Find the sum and the product of the eigen values of

$A = \begin{pmatrix} 1 & 1 & 5 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$.

Solution: F

Sum of the Eigen values $= \lambda_1 + \lambda_2 + \lambda_3 = a_{11} + a_{22} + a_{33} = 1 + 5 + 1 = 7$

Product of the Eigen values $= \lambda_1\lambda_2\lambda_3 = |A|$

$$= 1(5-1) - 1(1-3) + 5(1-15)$$

$$= 4 + 2 - 70 = -64 .$$

Example 2: The product of two eigen values of the matrix

$A = \begin{pmatrix} 6 & -2 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$ is 14. Find the third eigen value.

Solution: Let the eigenvalues of the matrix be $\lambda_1, \lambda_2, \lambda_3$.

Given $\lambda_1\lambda_2 = 14$

We know that product of the eigen values $= \lambda_1\lambda_2\lambda_3 = |A|$

$\Rightarrow \lambda_1\lambda_2\lambda_3 = 6(9-1) + 2(-9+2) + 2(3-6)$

$\Rightarrow \lambda_1\lambda_2\lambda_3 = 48 - 14 - 6$

$\Rightarrow 14\lambda_3 = 28$

$\Rightarrow \lambda_3 = 2$

CHECK YOUR PROGRESS

Q.3: Find the sum and the product of the eigen values of–

i) $A = \begin{pmatrix} 1 & 3 & 4 \\ 6 & 2 & 1 \\ 1 & 3 & 1 \end{pmatrix}$ ii) $A = \begin{pmatrix} 1 & 1 & 5 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$

Q.4: If $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$, Write down the sum and product of the Eigenvalues of A.

Q.5: If 2,2,3 are the eigenvalues of $A = \begin{pmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{pmatrix}$, find the Eigenvalues of $A^{/}$.

Q.6: If 3 and 15 are the eigenvalues of $A = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$,what is the third eigenvalue.

9.6 CAYLEY-HAMILTON THEOREM

Statement: Every square matrix satisfies its own characteristic equation.

Uses of Cayley-Hamilton Theorem:

1) To calculate the positive integral powers of A
2) To calculate the inverse of a square matrix A

Example 1: Find the characteristic equation of the matrix

$A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$ and then verify Cayley Hamilton theorem.

Solution: The characteristic equation is given by $|A - \lambda I| = 0$

i.e $\begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0$

or $(1-\lambda)(3-\lambda) - 8 = 0$

or $\lambda^2 - 4\lambda - 5 = 0$ (1)

To verify the Cayley Hamilton theorem for the matrix A, we have to verify that A satisfies the characteristic equation given in (1).

Now, $A^2 = A.A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1+8 & 4+12 \\ 2+4 & 8+9 \end{pmatrix} = \begin{pmatrix} 9 & 16 \\ 8 & 17 \end{pmatrix}$

Hence, $A^2 - 4A - 5I = \begin{pmatrix} 9 & 16 \\ 8 & 17 \end{pmatrix} - 4\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} - 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 9 & 16 \\ 8 & 17 \end{pmatrix} - \begin{pmatrix} 4 & 16 \\ 8 & 12 \end{pmatrix} - \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$

$= \begin{pmatrix} 9-4-5 & 16-16-0 \\ 8-8-0 & 17-12-5 \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

= 0

i.e., $A^2 - 4A - 5I = 0$

Hence Cayley Hamilton theorem is verified for the given matrix *A*.

Example 2: Show that the matrix $A = \begin{pmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{pmatrix}$

satisfies Cayley-Hamilton theorem.

Solution: The characteristic equation of matrix A is $|A - \lambda I| = 0$

i.e., $\begin{vmatrix} -\lambda & c & -b \\ -c & -\lambda & a \\ b & -a & -\lambda \end{vmatrix} = 0$

i.e., $-\lambda(\lambda^2 + a^2) - c(c\lambda - ab) - b(ac + b\lambda) = 0$

i.e., $-\lambda^3 - \lambda(a^2 + b^2 + c^2) = 0$

i.e., $\lambda^3 + \lambda(a^2 + b^2 + c^2) = 0$

Now to prove that matrix A satisfies Cayley-Hamilton theorem we have to show that $A^3 + A(a^2 + b^2 + c^2) = 0$

$$\text{Now, } A^2 = A.A = \begin{pmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{pmatrix}\begin{pmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -c^2 - b^2 & ab & ac \\ ab & -c^2 - a^2 & bc \\ ac & bc & -b^2 - a^2 \end{pmatrix}$$

$$A^3 = A.A^2 = \begin{pmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{pmatrix}\begin{pmatrix} -c^2 - b^2 & ab & ac \\ ab & -c^2 - a^2 & bc \\ ac & bc & -b^2 - a^2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -c^3 - b^2c - a^2c & bc^2 + b^3 + a^2b \\ c^3 + b^2c + a^2c & 0 & -ac^2 - ab^2 - a^3 \\ -bc^2 - b^3 - a^2b & ab^2 + ac^2 + a^3 & 0 \end{pmatrix}$$

$$\text{Also, } (a^2 + b^2 + c^2)A = (a^2 + b^2 + c^2)\begin{pmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -c^3 - b^2c - a^2c & bc^2 + b^3 + a^2b \\ c^3 + b^2c + a^2c & 0 & -ac^2 - ab^2 - a^3 \\ -bc^2 - b^3 - a^2b & ab^2 + ac^2 + a^3 & 0 \end{pmatrix}$$

$$\therefore A^3 + (a^2 + b^2 + c^2)A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

Hence Cayley Hamilton theorem is verified for the given matrix *A*.

Example 3: Verify that the matrix A satisfies its characterstic equation and also compute A^{-1} where $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{pmatrix}$

Solution: The characteristic equation of matrix A is $|A - \lambda I| = 0$

$$\text{i.e., } \begin{vmatrix} 1-\lambda & 2 & 1 \\ 0 & 1-\lambda & -1 \\ 3 & -1 & 1-\lambda \end{vmatrix} = 0$$

or $(1-\lambda)(-2\lambda + \lambda^2) + 3(-3 + \lambda) = 0$

or $-\lambda^3+3\lambda^2+\lambda-9=0$

or $\lambda^3-3\lambda^2-\lambda+9=0$

By Cayley Hamilton Theorem $A^3-3A^2-A+9I=0$ Verification.

$$A^2=A.A=\begin{pmatrix}1&2&1\\0&1&-1\\3&-1&1\end{pmatrix}\begin{pmatrix}1&2&1\\0&1&-1\\3&-1&1\end{pmatrix}=\begin{pmatrix}4&3&0\\-3&2&-2\\6&4&5\end{pmatrix}$$

$$A^3=A^2.A=\begin{pmatrix}4&3&0\\-3&2&-2\\6&4&5\end{pmatrix}\begin{pmatrix}1&2&1\\0&1&-1\\3&-1&1\end{pmatrix}=\begin{pmatrix}4&11&1\\-9&-2&-7\\21&11&7\end{pmatrix}$$

Now, A^3-3A^2-A+9I

$$=\begin{pmatrix}4&11&1\\-9&-2&-7\\21&11&7\end{pmatrix}-3\begin{pmatrix}4&3&0\\-3&2&-2\\6&4&5\end{pmatrix}-\begin{pmatrix}1&2&1\\0&1&-1\\3&-1&1\end{pmatrix}+9\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$=\begin{pmatrix}4&11&1\\-9&-2&-7\\21&11&7\end{pmatrix}-\begin{pmatrix}12&9&0\\-9&6&-6\\18&12&15\end{pmatrix}-\begin{pmatrix}1&2&1\\0&1&-1\\3&-1&1\end{pmatrix}+\begin{pmatrix}9&0&0\\0&9&0\\0&0&9\end{pmatrix}$$

$$=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}=0.$$

Hence Cayley Hamilton theorem is verified for the given matrix *A*.

We know that $A^3-3A^2-A+9I=0$

Premultiplying both sides by A^{-1}, we get

$$A^{-1}\left(A^3-3A^2-A+9I\right)=A^{-1}0$$

or $A^{-1}\left(A^3-3A^2-A+9I\right)=A^{-1}0$

or $A^2-3A-I+9A^{-1}=0$

$\therefore 9A^{-1}=I+3A-A^2$

$$=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}+3\begin{pmatrix}1&2&1\\0&1&-1\\3&-1&1\end{pmatrix}-\begin{pmatrix}4&3&0\\-3&2&-2\\6&4&5\end{pmatrix}$$

$$=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}+\begin{pmatrix}3&6&1\\0&3&-3\\9&-3&3\end{pmatrix}-\begin{pmatrix}4&3&0\\-3&2&-2\\6&4&5\end{pmatrix}$$

$$=\begin{pmatrix}0&3&3\\3&2&-1\\3&-7&-1\end{pmatrix}$$

$$\therefore A^{-1} = \frac{1}{9}\begin{pmatrix} 0 & 3 & 3 \\ 3 & 2 & -1 \\ 3 & -7 & -1 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{9} & -\frac{1}{9} \\ \frac{1}{3} & -\frac{7}{9} & -\frac{1}{9} \end{pmatrix}$$

Example 4: Find the characterstic equation of the matrix

$$A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}.$$

Verify Cayley Hamilton theorem and hence obtain A^{-1}.

Solution: We have $|A - \lambda I| = \begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix}$

$= (2-\lambda)\{(2-\lambda)2-1\} + 1\{-1(2-\lambda)+1\} + 1\{1-(2-\lambda)\}$

$= (2-\lambda)(3-4\lambda+\lambda^2) + (\lambda-1) + (\lambda-1)$

$= -\lambda^3 + 6\lambda^2 - 9\lambda + 4$

$\therefore$ the characterstic equation of the matrix A is $\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$

Now, we are to verify that $A^3 - 6A^2 + 9A - 4I = 0$

$$A^2 = A.A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix}$$

$$A^3 = A^2.A = \begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix}\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{pmatrix}$$

Now, $A^3 - 6A^2 + 9A - 4I$

$$= \begin{pmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{pmatrix} - 6\begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix} + 9\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} - 4\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

= 0

We know that $A^3 - 6A^2 + 9A - 4I = 0$

Premultiplying both sides by A^{-1}, we get $A^2 - 6A + 9I - 4A^{-1} = 0$

$$\therefore A^{-1} = \frac{1}{4}\left(A^2 - 6A + 9I\right)$$

Now, $A^2 - 6A + 9I$

$$= \begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix} - 6\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} + 9\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{pmatrix}$$

$$\therefore A^{-1} = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{pmatrix}$$

CHECK YOUR PROGRESS

Q.7: Verify Cayley-Hamilton theorem for the matrix:

$A = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{pmatrix}$ and then using the theorem obtain A^{-1}.

Q.8: Find the characteristic equation of the matrix $A = \begin{pmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 0 & 2 & 1 \end{pmatrix}$

and show that it is satisfied by A.Hence obtain the inverse of the given matrix A.

Q.9: Verify that the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$ satisfies its characteristic equation and

compute A^{-1}.

9.7 LET US SUM UP

- University Algebra, N S Gopala Krishnan, New Age International (P) Limited.
- Theory of Matrices, B S Vatsa, New Age International Publishers..
- Linear Algebra: Schum Solved Problems Series; Seymour Lipschitz: Tata McGraw Hill.
- Text Book of Algebra, A.K Sharma, DPH, Mathematics Series.
- A Text Book of Matrices, Shanti Narayan and P K Mittal.

✓ 9.8 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1: The characteristic equation is $|A - \lambda I| = 0$

i.e., $\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$

$\Rightarrow (5-\lambda)(2-\lambda) - 4 = 0$

$\Rightarrow \lambda^2 - 7\lambda + 6 = 0$

$\Rightarrow \lambda = 6, 1$

$\therefore$ Eigenvalues are 6,1.

For $\lambda = 6$, we find eigenvector $X = \begin{pmatrix} x \\ y \end{pmatrix}$ such that $AX = \lambda X$

i.e. $\begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 6\begin{pmatrix} x \\ y \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 5x+4y \\ x+2y \end{pmatrix} = \begin{pmatrix} 6x \\ 6y \end{pmatrix}$

$\Rightarrow \begin{matrix} 5x+4y = 6x \\ x+2y = 6y \end{matrix}$

$\Rightarrow x = 4y$

i.e $x = 4y$

$\therefore \frac{x}{4} = \frac{y}{1}$

$\therefore$ Eigen vector corresponding to eigen value 6 are $k\begin{pmatrix} 4 \\ 1 \end{pmatrix}$,

where k is non-zero real number.

For $\lambda = 1$, we find eigenvector $X = \begin{pmatrix} x \\ y \end{pmatrix}$ such that $AX = \lambda X$

i.e. $\begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 1\begin{pmatrix} x \\ y \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 5x+4y \\ x+2y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\Rightarrow \begin{matrix} 5x+4y = x \\ x+2y = y \end{matrix}$

$\Rightarrow x = -y$

i.e $x = -y$

$\therefore \frac{x}{1} = \frac{y}{-1}$

$\therefore$ Eigen vectors corresponding to eigen value 1 are $k\begin{pmatrix} 1 \\ -1 \end{pmatrix}$,

where k is non-zero real number.

Ans. to Q. No. 2: i) Eigenvalues are : –1, 1, 2

Eigenvectors are : $k\begin{bmatrix}1\\0\\1\end{bmatrix},\ k\begin{bmatrix}3\\2\\1\end{bmatrix},\ k\begin{bmatrix}1\\3\\1\end{bmatrix},\ k \neq 0$

ii) Eigenvalues are: –2, 1, 3

Eigenvectors are : $k\begin{bmatrix}11\\1\\-14\end{bmatrix},\ k\begin{bmatrix}-1\\1\\1\end{bmatrix},\ k\begin{bmatrix}1\\1\\1\end{bmatrix},\ k \neq 0$

iii) Eigenvalues are : 0, 1, 5

Eigenvectors are : $k\begin{bmatrix}0\\-1\\1\end{bmatrix},\ k\begin{bmatrix}-1\\0\\1\end{bmatrix},\ k\begin{bmatrix}4\\5\\11\end{bmatrix},\ k \neq 0$

Ans. to Q. No. 3: i) Sum of the Eigen values

$= \lambda_1 + \lambda_2 + \lambda_3 = a_{11} + a_{22} + a_{33} = 1 + 2 + 1 = 4$.

Product of the Eigen values $= \lambda_1\lambda_2\lambda_3 = |A| = 48$.

ii) Sum of the Eigen values: $= \lambda_1 + \lambda_2 + \lambda_3 = a_{11} + a_{22} + a_{33}$

$= 1 + 5 + 1$

$= 7$

Product of the Eigen values: $= \lambda_1\lambda_2\lambda_3 = |A|$

$= 1(5-1) - 1(1-3) + 5(1-15)$

$= 4 + 2 - 70 = -64$.

Ans. to Q. No. 4: Sum of the Eigen values:

$$= \lambda_1 + \lambda_2 + \lambda_3 = a_{11} + a_{22} + a_{33} = 1 + 2 + 3 = 6$$

Product of the Eigen values: $= \lambda_1\lambda_2\lambda_3 = |A|$

$= 1(6-4) - 1(3-2) + 1(2-2)$

$= 2 - 1 = 1$.

Ans. to Q. No. 5: We know that a square matrix A and its transpose A' have the same eigenvalues.

$\therefore$ Eigenvalues of A' are 2, 2, 3.

Ans. to Q. No. 6: Let λ be third eigen value.

Sum of eigenvalues $= \lambda + 3 + 15 = \lambda + 18$

Sum of the diagonal elements $= 8 + 7 + 3 = 18$

$\therefore \lambda + 18 = 18$

i.e., $\lambda = 0$.

Ans. to Q. No. 7: The characteristic equation is given by $|A - \lambda I| = 0$

i.e., $\begin{vmatrix} 1-\lambda & 1 & 2 \\ 3 & 1-\lambda & 1 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0$

or $\lambda^3 - 3\lambda^2 - 7\lambda - 11 = 0$

Now, $A^2 = A.A = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 2 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 8 & 5 \\ 8 & 7 & 8 \\ 13 & 8 & 8 \end{pmatrix}$

$$A^3 = A^2.A = \begin{pmatrix} 8 & 8 & 5 \\ 8 & 7 & 8 \\ 13 & 8 & 8 \end{pmatrix}.\begin{pmatrix} 1 & 1 & 2 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 42 & 37 & 29 \\ 45 & 39 & 31 \\ 53 & 45 & 42 \end{pmatrix}$$

Verification: $A^3 - 3A^2 - 7A - 11I$

$$= \begin{pmatrix} 42 & 37 & 29 \\ 45 & 39 & 31 \\ 53 & 45 & 42 \end{pmatrix} - 3\begin{pmatrix} 8 & 8 & 5 \\ 8 & 7 & 8 \\ 13 & 8 & 8 \end{pmatrix} - 7\begin{pmatrix} 1 & 1 & 2 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{pmatrix} - 11\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 42 & 37 & 29 \\ 45 & 39 & 31 \\ 53 & 45 & 42 \end{pmatrix} - \begin{pmatrix} 24 & 24 & 15 \\ 24 & 21 & 24 \\ 39 & 24 & 24 \end{pmatrix} - \begin{pmatrix} 7 & 7 & 14 \\ 21 & 7 & 7 \\ 14 & 21 & 7 \end{pmatrix} - \begin{pmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0.$$

Hence Cayley Hamilton theorem is verified for the given matrix *A*.

We know that $A^3 - 3A^2 - 7A - 11I = 0$

Premultiplying both sides by A^{-1}, we get

$$A^{-1}\left(A^3 - 3A^2 - 7A - 11I\right) = A^{-1}0$$

or $A^2 - 3A - 7I - 11A^{-1} = 0$

or $11A^{-1} = A^2 - 3A - 7I$

$$= \begin{pmatrix} 8 & 8 & 5 \\ 8 & 7 & 8 \\ 13 & 8 & 8 \end{pmatrix} - 3\begin{pmatrix} 1 & 1 & 2 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{pmatrix} - 7\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 8 & 5 \\ 8 & 7 & 8 \\ 13 & 8 & 8 \end{pmatrix} - \begin{pmatrix} 3 & 3 & 6 \\ 9 & 3 & 3 \\ 6 & 9 & 3 \end{pmatrix} - \begin{pmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 5 & -1 \\ -1 & -3 & 5 \\ 7 & -1 & -2 \end{pmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{11}\begin{pmatrix} -2 & 5 & -1 \\ -1 & -3 & 5 \\ 7 & -1 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-2}{11} & \frac{5}{11} & \frac{-1}{11} \\ \frac{-1}{11} & \frac{-3}{11} & \frac{5}{11} \\ \frac{7}{11} & \frac{-1}{11} & \frac{-2}{11} \end{pmatrix}$$

Ans. to Q. No. 8: Characterstic equation is $\lambda^3 - 4\lambda^2 - 13\lambda - 40 = 0$

Inverse of the matrix $A^{-1} = \frac{1}{40}\begin{pmatrix} -4 & 11 & -5 \\ -4 & 1 & 25 \\ 8 & -2 & -10 \end{pmatrix}$

Ans. to Q. No. 9: Characteristic equation is $\lambda^3 - 3\lambda^2 - \lambda + 9 = 0$.

Inverse of A is given by $A^{-1} = \frac{1}{9}\begin{bmatrix} 0 & 3 & 3 \\ 3 & 2 & -1 \\ 3 & -7 & -1 \end{bmatrix}$

9.9 FURTHER READING

- University Algebra, N S Gopala Krishnan, New Age International (P) Limited.
- Theory of Matrices, B S Vatsa, New Age International Publishers..
- Linear Algebra: Schum Solved Problems Series; Seymour Lipschitz: Tata McGraw Hill.
- Text Book of Algebra, A.K Sharma, DPH, Mathematics Series.
- A Text Book of Matrices, Shanti Narayan and P K Mittal.

9.10 MODEL QUESTIONS

Q.1: Find eigenvalues and eigenvectors for the following matrices.

i) $\begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{bmatrix}$ ii) $\begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix}$ iii) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 3 & 2 & 3 \end{bmatrix}$

iv) $\begin{bmatrix} 1 & 3 & 0 \\ 3 & -2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$ v) $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix}$ vi) $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Q.2: Find the characteristic equation of the matrix $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$ and then verify Cayley Hamilton theorem.

Q.3: Find the characteristic equation of the matrix $A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$ and then verify Cayley Hamilton theorem.

Q.4: Verify Cayley-Hamilton theorem for the following matrices:

i) $\begin{pmatrix} 2 & 0 \\ \sqrt{3} & 1 \end{pmatrix}$ ii) $\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$ iii) $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}$

Q.5: Verify that the matrix A satisfies its characteristic equation and also compute A^{-1} for the following matrices:

i) $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{pmatrix}$ ii) $A = \begin{pmatrix} 2 & -1 & 1 \\ -2 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$

iii) $A = \begin{pmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 0 & 2 & 1 \end{pmatrix}$ iv) $A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix}$

Q.6: Verify Cayley-Hamilton theorem for the matrix $A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ and hence or otherwise obtain the value of A^{-1}.

Q.7: Show that the matrix $A = \begin{pmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{pmatrix}$ satisfies Cayley-Hamilton theorem.Find A^{-1} also.

Q.8: Determine the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

Q.9: Show that the matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ satifies Cayley-Hamilton theorem. Also determine the characteristic roots and the corresponding characteristic vectors of

the matrix A.

Q.10: If $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -8 \\ 2 & -4 & 3 \end{bmatrix}$ then find the characteristic equation of matrix A and hence verify Cayley Hamilton theorem and determine A^{-1}.

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UNIT 10: DE MOVIRE'S THEOREM AND ITS APPLICATION

UNIT STRUCTURE

10.1 Learning Objectives
10.2 Introduction
10.3 The Complex Number Plane
10.4 De Moivre's Theorem
10.5 Determination of roots using De Moivre's Theorem
10.6 Let Us Sum Up
10.7 Answers To Check Your Progress
10.8 Further Reading
10.9 Model Questions

10.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- know about complex numbers
- state De Moivre Theorem
- employ De Moivre's theorem in a number of applications
- solve equations using De moivre's theorem.

10.2 INTRODUCTION

In this unit,we will discuss one of the great theorems in mathematics. *French mathematician* Abraham De Moivre (1667-1754),established this theorem, known with his name as De Moivre's theorem. In this unit ,we introduce De Moivre's theorem and examine some of its consequences.

Another important aspect of De Moivre's theorem lies in its use in obtaining complex roots of polynomial equations.

10.3 THE COMPLEX NUMBER PLANE

A complex number, z, is a number of the form x + iy, where x (the real part) and y (the imaginary part) represent real numbers, and $i = \sqrt{-1}$ is the imaginary unit.

For examples:

i) $z = x + iy$ ii) $z = -6 + i3$

iii) $z = -\frac{1}{3} - i\frac{5}{7}$ iv) $z = 9 - i\frac{4}{7}$ etc.

Geometrical Representation of Complex Numbers: *Argand Diagram*: We can interpret a point (x, y) in the rectangular system as a complex number, z with real part x and imaginary part y. In this interpretation, we call the x-axis as the real axis and the y-axis as the imaginary axis.

A complex number z = x + iy can be represented by a point P in the cartesian plane whose coordinates are (x, y) referred to rectangular axes OX and OY,usually called the real and imaginary axes respectively. Corresponding to each complex number x+iy, we have a unique point (x, y) in the xy-plane and corresponding to each point (x, y) in the xy-plane, we have a unique complex number x + iy. The complex number 0+i0 corresponds to the origin, the real number x = x + i0 correspond to the points on the x-axis and the purely imaginary numbers iy = 0 + iy correspond to the points on the y-axis.

Modulus (or *Magnitude*) and Argument (or *Amplitude*) of a Complex Number: Let z = x + iy be a complex number. The magnitude or modulus of z is the distance from the origin (0, 0) to the point (x, y). The modulus of z, denoted by $|z|$, is $|z| = \sqrt{x^2 + y^2}$.

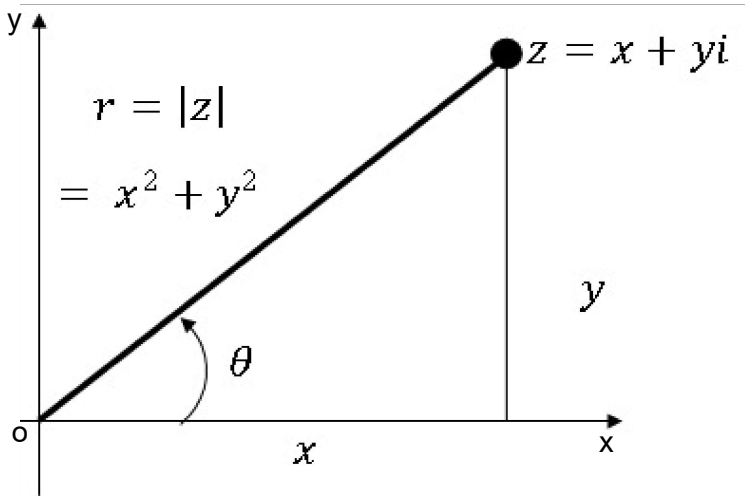


Fig. 9.1

Argument of a complex number (z = x + iy) is the angle made by the line representing z with the positive *x*-axis.It is denoted by arg(z) and is given by $\arg(z) = \theta = \tan^{-1}\frac{y}{x}$.

Let z = x + iy be a complex number. Then the complex conjugate of z, denoted by $\bar{z}$ is $\bar{z} = x - iy$.

Polar form of a complex number:

Let $r \geq 0$ and $0 \leq \theta \leq 2\pi$. Then the polar form of the complex number z = x + iy is $z = x + iy = r(\cos\theta + \sin\theta)$

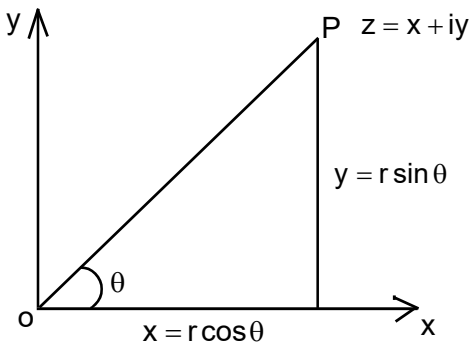


Fig. 9.2

To convert from the rectangular form to the polar form:

From the triangle OBP, we have, $\sin\theta = \frac{BP}{OP} = \frac{y}{r}$

$$\Rightarrow y = r\sin\theta$$

$$\cos\theta = \frac{OB}{OP} = \frac{x}{r}$$

$$\Rightarrow x = r\cos\theta$$

Therefore,we can rewrite the complex number z as follows:

$z = x + iy = r\cos\theta + ir\sin\theta$

$= r(\cos\theta + i\sin\theta)$

$z = x + iy$ can be expressed in the form $z = r(\cos\theta + i\sin\theta)$

Note: Polar form of $z = x + iy$ is also expressed as $z = rcis(\theta)$

Example: Find the modulus and amplitude of $(2+i3)^2$. Also,write it in polar form.

Solution: We have $(2+i3)^2 = 4 + 9i^2 + 12i = 4 - 9 + 12i = -5 + 12i$

Now, let $-5 + i12 = r(\cos\theta + i\sin\theta)$

Equating real and imaginary parts, we get

$r\cos\theta = -5$ (1)

and $r\sin\theta = 12$ (2)

Squaring and adding (1) and (2),we get

$$r^2 = (-5)^2 + (12)^2 = 25 + 144 = 169$$

Hence, $r = 13$

Therefore, Modulus of $(2+i3)^2$ = 13

Dividing (2) by (1), we get $\tan\theta = \frac{12}{-5}$ and

hence, $\theta = \tan^{-1}\left(\frac{12}{-5}\right)$ where $-\pi < \theta < \pi$.

Therefore, Argument $= \tan^{-1}\left(\frac{12}{-5}\right)$

Hence, $(2+i3)^2 = 13\left[\cos\left(\tan^{-1}\frac{12}{-5}\right) + i\sin\left(\tan^{-1}\frac{12}{-5}\right)\right]$

where $-\pi < \theta < \pi$.

which is the required polar form.

10.4 DE MOIVRE'S THEOREM

De Moivre's theorem enables us to find all *n*th roots of any complex number. Below we discuss the statement and the proof of De Moivre's theorem.

Statement: The value of $(\cos\theta + i\sin\theta)^n$ is $\cos n\theta + i\sin n\theta$,where n is an integer and one of the values of $(\cos\theta + i\sin\theta)^n$ is $\cos n\theta + i\sin n\theta$, when n is a fraction.

Proof: ***Case I: Let n be a positive integer:***

We have $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$

$= \cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + i^2\sin\theta_1\sin\theta_2$

$= (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1)$

where $i^2 = -1$

$= \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)$ (1)

Multiplying both sides of (1) by $(\cos\theta_3 + i\sin\theta_3)$, we have

$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3)$

$= \{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\}(\cos\theta_3 + i\sin\theta_3)$

$= \cos(\theta_1 + \theta_2 + \theta_3) + i\sin(\theta_1 + \theta_2 + \theta_3)$

Proceeding in this way,we obtain the product of n number of factors of the form $(\cos\theta + i\sin\theta)$. we obtain

$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\ldots(\cos\theta_n + i\sin\theta_n)$

$= \cos(\theta_1 + \theta_2 + \cdots + \theta_n) + \sin(\theta_1 + \theta_2 + \cdots + \theta_n)$ (2)

Putting $\theta_1 = \theta_2 = \cdots = \theta_n = \theta$ on both sides of (2), we get

$(\cos\theta + i\sin\theta)\ (\cos\theta + i\sin\theta)$ ...upto n factors

$= \cos(\theta + \theta + \cdots + \theta) + i\sin(\theta + \theta + \cdots + \theta)$

i.e $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

This proves De Moivre's Theorem when n is a positive integer.

Case II: Let n be a negative integer:

Let us suppose n = –m, m being a positive integer.

Then $(\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-m}$

$$= \frac{1}{(\cos\theta + i\sin\theta)^m}$$

$$= \frac{1}{(\cos m\theta + i\sin m\theta)}$$ (Since m is a positive integer)

$$= \frac{(\cos m\theta - i\sin m\theta)}{(\cos m\theta + i\sin m\theta)(\cos m\theta - i\sin m\theta)}$$

(On multiplying numerator and denominator by $(\cos m\theta - i\sin m\theta)$)

$$= \frac{(\cos m\theta - i\sin m\theta)}{(\cos^2 m\theta + \sin^2 m\theta)}$$

$$= \frac{(\cos m\theta - i\sin m\theta)}{1}$$ Since $\cos^2 m\theta + \sin^2 m\theta = 1$

$= (\cos m\theta - i\sin m\theta)$

$= \cos(-m\theta) + i\sin(-m\theta)$

$(\cos(-m\theta) = \cos m\theta$ and $\sin(-m\theta) = -\sin m\theta)$

$= \cos n\theta + i\sin n\theta$ for –m = n.

This proves the De Moivre's theorem when n is a negative integer.

Case III: Let n be a fraction i.e., rational number, positive or negative:

Suppose $n = \frac{p}{q}$, where q is a positive integer and p is an integer, positive or negative.

It follows from case I, that

$$\left(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right)^q = \cos(q.\frac{\theta}{q}) + i\sin(q.\frac{\theta}{q}) = \cos\theta + i\sin\theta$$

Taking qth root of both sides, $\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}$ is one of the values of

$(\cos\theta + i\sin\theta)^{\frac{1}{q}}$

Raising to pth power, $\left(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right)^p$ is one of the values of

$(\cos\theta + i\sin\theta)^{p/q}$

or $\cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta$ is one of the values of $(\cos\theta + i\sin\theta)^{p/q}$.

Since $\frac{p}{q} = n$, $\cos n\theta + i\sin n\theta$ is one of the values of $(\cos\theta + i\sin\theta)^n$.

This proves De-Moivre's Theorem when n is a fraction,positive or negative.

Hence, De-Moivre's theorem is completely established.

Corollary: $(\cos\theta - i\sin\theta)^n = (\cos n\theta - i\sin n\theta)$

Proof: We have $(\cos\theta - i\sin\theta)^n = [\cos(-\theta) + i\sin(-\theta)]^n$

= $\cos(-n\theta) + i\sin(-n\theta)$ (By DeMoivre's theorem)

= $\cos n\theta - i\sin n\theta$ (since $\cos(-n\theta) = \cos n\theta$ and $\sin(-n\theta) = -\sin n\theta$)

Note: The learner's should note the following points very carefully:

i) $(\sin\theta + i\cos\theta)^n \neq \sin n\theta + i\cos n\theta$

But $(\sin\theta + i\cos\theta)^n = \left[\cos(\frac{\pi}{2} - \theta) + i\sin(\frac{\pi}{2} - \theta)\right]^n$

= $\cos n(\frac{\pi}{2} - \theta) + i\sin n(\frac{\pi}{2} - \theta)$ (By De Moivre's theorem)

ii) $(\cos\theta + i\sin\phi)^n \neq \cos n\theta + i\sin n\phi$

i.e., De Moivre's theorem is applied only when the real and imaginary parts are cosine and sine of the same angle.

iii) We can use the notation $\text{cis}\,\theta$ to denote $\cos\theta + i\sin\theta$.

Using this notation, De Moivre's theorem can be written as

$(\text{cis}\theta)^n = \text{cis}n\theta$

iv) $(\cos\theta + i\sin\theta)^{-n} = \cos(-n\theta) + i\sin(-n\theta)$

$$= \cos n\theta - i\sin n\theta \quad (\text{Since } \cos(-n\theta) = \cos n\theta \text{ and } \sin(-n\theta) = -\sin n\theta)$$

v) $(\cos\theta - i\sin\theta)^{-n} = \cos(-n\theta) - i\sin(-n\theta)$

$$= \cos n\theta - i(-\sin n\theta)$$

$$= \cos n\theta + i\sin n\theta$$

vi) $\dfrac{1}{\cos\theta + i\sin\theta} = (\cos\theta + i\sin\theta)^{-1}$

$$= \cos(-\theta) + i\sin(-\theta)$$

$$= \cos\theta - i\sin\theta$$

vii) $\dfrac{1}{\cos\theta - i\sin\theta} = (\cos\theta - i\sin\theta)^{-1}$

$$= \cos(-\theta) - i\sin(-\theta)$$

$$= \cos\theta + i\sin\theta$$

Example 1: Simplify: $\dfrac{(\cos\theta + i\sin\theta)^4}{(\cos\theta - i\sin\theta)^3}$

Solution: Since $\cos\theta - i\sin\theta = (\cos\theta + i\sin\theta)^{-1}$

We write $\dfrac{(\cos\theta + i\sin\theta)^4}{(\cos\theta - i\sin\theta)^3} = \dfrac{(\cos\theta + i\sin\theta)^4}{(\cos\theta + i\sin\theta)^{-3}}$

$$= (\cos\theta + i\sin\theta)^7$$

$$= \cos 7\theta + i\sin 7\theta$$

Example 2: Prove that:

$$\frac{(\cos\theta + i\sin\theta)^8(\cos 2\theta - i\sin 2\theta)^9}{(\cos 2\theta + i\sin 2\theta)^9(\cos\theta - i\sin\theta)^8} = \cos 20\theta - i\sin 20\theta$$

Solution: L.H.S $= \dfrac{(\cos\theta + i\sin\theta)^8(\cos 2\theta - i\sin 2\theta)^9}{(\cos 2\theta + i\sin 2\theta)^9(\cos\theta - i\sin\theta)^8}$

$$= \frac{(\cos\theta + i\sin\theta)^8\{\cos(-2)\theta + i\sin(-2\theta\}^9}{(\cos\theta + i\sin\theta)^{2\times 9}(\cos\theta + i\sin\theta)^{-8}}$$

$$= \frac{(\cos\theta + i\sin\theta)^8\{\cos\theta + i\sin\theta\}^{-2\times 9}}{(\cos\theta + i\sin\theta)^{18}(\cos\theta + i\sin\theta)^{-8}}$$

$$= \frac{(\cos\theta + i\sin\theta)^8\{\cos\theta + i\sin\theta\}^{-18}}{(\cos\theta + i\sin\theta)^{18}(\cos\theta + i\sin\theta)^{-8}}$$

$$= \frac{(\cos\theta + i\sin\theta)^{8-18}}{(\cos\theta + i\sin\theta)^{18-8}}$$

$$= (\cos\theta + i\sin\theta)^{8-18-18+8}$$

$$= (\cos\theta + i\sin\theta)^{-20}$$

$$= \cos(-20)\theta + i\sin(-20)\theta$$

(By De-Moivre's theorem)

$= \cos 20\theta - i\sin 20\theta$ Hence proved.

Example 3: Prove that $\left(\frac{\cos\theta + i\sin\theta}{\sin\theta + i\cos\theta}\right)^4 = \cos 8\theta + i\sin 8\theta$

Solution: L.H.S = $\left(\frac{\cos\theta + i\sin\theta}{\sin\theta + i\cos\theta}\right)^4$

$$= \frac{(\cos\theta + i\sin\theta)^4}{(\sin\theta + i\cos\theta)^4}$$

$$= \frac{(\cos\theta + i\sin\theta)^4}{\left\{i\left(\frac{1}{i}\sin\theta + \cos\theta\right)\right\}^4}$$

$$= \frac{(\cos\theta + i\sin\theta)^4}{i^4(\cos\theta + \frac{1}{i}\sin\theta)^4}$$

$$= \frac{(\cos\theta + i\sin\theta)^4}{(\cos\theta - i\sin\theta)^4}$$

$$= \frac{(\cos\theta + i\sin\theta)^4}{(\cos\theta + i\sin\theta)^{-4}}$$

$$= (\cos\theta + i\sin\theta)^{4+4}$$

$$= (\cos\theta + i\sin\theta)^8$$

$$= (\cos 8\theta + i\sin 8\theta)$$ Hence proved.

Example 4: If $x + \frac{1}{x} = 2\cos\theta$ and $y + \frac{1}{y} = 2\cos\phi$ prove that: θ

i) One of the values of $x^m y^m + \frac{1}{x^m y^m}$ is $2\cos(m\theta + n\phi)$

ii) One of the values of $\frac{x^m}{y^n} + \frac{y^n}{x^m}$ is $2\cos(m\theta - n\phi)$

Solution: Given $x + \frac{1}{x} = 2\cos\theta$

Hence, $x^2 + 1 = 2x\cos\theta$ be $x^2 - 2x\cos\theta + 1 = 0$

$$x = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$$

Hence, $x = \frac{2\cos\theta \pm 2\sqrt{\cos^2\theta - 1}}{2}$

so, $x = \cos\theta \pm \sqrt{\cos^2\theta - 1}$

i.e., $x = \cos\theta \pm \sqrt{-\sin^2\theta}$

i.e., $x = \cos\theta \pm \sin\theta\sqrt{-1}$

i.e., $x = \cos\theta \pm i\sin\theta$

Let $x = \cos\theta + i\sin\theta$

Similarly, $y = \cos\phi + i\sin\phi$

i) Then, $x^m y^n = (\cos\theta + i\sin\theta)^m (\cos\phi + i\sin\phi)^n$

$= (\cos m\theta + i\sin m\theta)\ (\cos n\phi + i\sin n\phi)$

(By DeMoivre's theorem)

$= \cos(m\theta + n\phi) + i\sin(m\theta + n\phi)$ (1)

Also, $\frac{1}{x^m y^n} = x^{-m} y^{-n}$

$= [\cos(m\theta + n\phi) + i\sin(m\theta + n\phi)]^{-1}$

$= [\cos(m\theta + n\phi) - i\sin(m\theta + n\phi)]$ (2)

Adding (1) and (2), we have $x^m y^n + \frac{1}{x^m y^n}$

$= [\cos(m\theta + n\phi) + i\sin(m\theta + n\phi)] + [\cos(m\theta + n\phi) - i\sin(m\theta + n\phi)]$

$= 2\cos(m\theta + n\phi)$ Hence proved.

ii) We take, $x = \cos\theta + i\sin\theta$ and $y = \cos\phi + i\sin\phi$

Then, $x^m = (\cos\theta + i\sin\theta)^m$ and $y^n = (\cos\phi + i\sin\phi)^n$

$\frac{x^m}{y^n} = \frac{(\cos\theta + i\sin\theta)^m}{(\cos\phi + i\sin\phi)^n}$

$= \frac{(\cos m\theta + i\sin m\theta)}{(\cos n\phi + i\sin n\phi)}$ (By DeMoivre's theorem)

$= (\cos m\theta + i\sin m\theta)\ (\cos n\phi + i\sin n\phi)^{-1}$

$= (\cos m\theta + i\sin m\theta)\ [\cos(-n\phi) + i\sin(-n\phi)]$

(By DeMoivre's theorem)

$= \cos(m\theta - n\phi) + i\sin(m\theta - n\phi)$ (3)

$\frac{y^n}{x^m} = \left(\frac{x^m}{y^n}\right)^{-1} = [\cos(m\theta - n\phi) + i\sin(m\theta - n\phi)]^{-1}$

$= [\cos(m\theta - n\phi) - i\sin(m\theta - n\phi)]$ (4)

Adding (3) and (4), we have $\frac{x^m}{y^n} + \frac{y^n}{x^m}$

$= [\cos(m\theta - n\phi) + i\sin(m\theta - n\phi)] + [\cos(m\theta - n\phi) - i\sin(m\theta - n\phi)]$

$= 2\cos(m\theta - n\phi)$ Hence proved.

Example 5: Prove that:

$$(a + ib)^{\frac{m}{n}} + (a - ib)^{\frac{m}{n}} = 2(a^2 + b^2)^{\frac{m}{2n}} \cos\left\{\frac{m}{n}\tan^{-1}\frac{b}{a}\right\}$$

Solution: Let $(a + ib) = r(\cos\theta + i\sin\theta)$

$(a - ib) = r(\cos\theta - i\sin\theta)$

Equating real and imaginary parts, we have

$a = r\cos\theta$ and $b = r\sin\theta$

Hence $r = \sqrt{a^2 + b^2}$ and $\tan\theta = \frac{b}{a}$ or $\theta = \tan^{-1}\left(\frac{b}{a}\right)$

Now, $L.H.S = (a+ib)^{\frac{m}{n}} + (a-ib)^{\frac{m}{n}}$

$$= r^{\frac{m}{n}}\left[(\cos\theta + i\sin\theta)^{\frac{m}{n}} + (\cos\theta - i\sin\theta)^{\frac{m}{n}}\right]$$

$$= r^{\frac{m}{n}}\left[(\cos\frac{m}{n}\theta + i\sin\frac{m}{n}\theta) + (\cos\frac{m}{n}\theta - i\sin\frac{m}{n}\theta)\right]$$

$$= 2r^{\frac{m}{n}}\cos 2\frac{m}{n}\theta$$

$$= 2(a^2+b^2)^{\frac{m}{2n}}\cos\left\{\frac{m}{n}\tan^{-1}\frac{b}{a}\right\}$$, putting for r and θ.

Example 6: If $x = \cos\alpha + i\sin\alpha$, $y = \cos\beta + i\sin\beta$, $z = \cos\gamma + i\sin\gamma$ and $x + y + z = xyz$ then prove that, $\cos(\beta-\gamma) + \cos(\gamma-\alpha) + \cos(\alpha-\beta) + 1 = 0$

Solution: Since $x + y + z = xyz$, we have

$$(\cos\alpha + i\sin\alpha) + (\cos\beta + i\sin\beta) + (\cos\gamma + i\sin\gamma)$$

$$= (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma)$$

$$= \cos(\alpha+\beta+\gamma) + i\sin(\alpha+\beta+\gamma)$$

Hence, $(\cos\alpha + \cos\beta + \cos\gamma) + i(\sin\alpha + \sin\beta + \sin\gamma)$

$$= \cos(\alpha+\beta+\gamma) + i\sin(\alpha+\beta+\gamma)$$

Equating real and imaginary parts, we have

$\cos\alpha + \cos\beta + \cos\gamma = \cos(\alpha+\beta+\gamma)$ (1)

and $\sin\alpha + \sin\beta + \sin\gamma = \sin(\alpha+\beta+\gamma)$ (2)

Squaring and adding (1) and (2),we get

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + 2\cos\alpha\cos\beta + 2\cos\beta\cos\gamma + 2\cos\gamma\cos\alpha$$

$$+ \sin^2\alpha + \sin^2\beta + \sin^2\gamma + 2\sin\alpha\sin\beta + 2\sin\beta\sin\gamma + 2\sin\gamma\sin\alpha$$

$= \cos^2(\alpha+\beta+\gamma) + \sin^2(\alpha+\beta+\gamma)$.

$$\text{or}(\cos^2\alpha + \sin^2\alpha) + (\cos^2\gamma + \sin^2\gamma)$$

$$+ \cos^2\beta + \sin^2\beta + 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta)$$

$$+ 2(\cos\beta\cos\gamma + \sin\beta\sin\gamma) + 2(\cos\gamma\cos\alpha + \sin\gamma\sin\alpha) = 1$$

$$\text{or } 1 + 1 + 1 + 2\cos(\alpha-\beta) + 2\cos(\beta-\gamma) + 2\cos(\gamma-\alpha) = 1$$

$$\text{or } 3 + 2\cos(\alpha-\beta) + 2\cos(\beta-\gamma) + 2\cos(\gamma-\alpha) = 1$$

$$\text{or } 2\cos(\alpha-\beta) + 2\cos(\beta-\gamma) + 2\cos(\gamma-\alpha) = 1 - 3$$

$$\text{or } 2\cos(\alpha-\beta) + 2\cos(\beta-\gamma) + 2\cos(\gamma-\alpha) = -2$$

So, $\cos(\beta-\gamma)+\cos(\gamma-\alpha)+\cos(\alpha-\beta)+1=0$ Hence proved.

CHECK YOUR PROGRESS

Q.1: Simpify $\dfrac{(\cos\theta-i\sin\theta)^{10}}{(\cos\theta+i\sin\theta)^{12}}$

Q.2: If $a=\cos2\alpha+i\sin2\alpha$, $b=\cos2\beta+i\sin2\beta$ Prove that,

i) $a+b=2\cos2(\alpha-\beta)\{\cos(\alpha+\beta)+i\sin(\alpha+\beta)\}$

ii) $a-b=2i\sin2(\alpha-\beta)\{\cos(\alpha+\beta)+i\sin(\alpha+\beta)\}$

and iii) $\dfrac{a+b}{a-b}=i\tan(\alpha-\beta)$

Q.3: If n be a positive integer,prove that

$(\sqrt{3}+i)^n+(\sqrt{3}-i)^n=2^{n+1}\cos\left(\dfrac{n\pi}{6}\right)$.

Q.4: If $x=\cos\alpha+i\sin\alpha$, $y=\cos\beta+i\sin\beta$, $z=\cos\gamma+i\sin\gamma$

and if $x+y+z=0$, then prove that $\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=0$.

Q.5: Show that $(1+i)^n=2^{\frac{n}{2}}\left[\cos\dfrac{1}{4}n\pi+i\sin\dfrac{1}{4}n\pi\right]$.

Hence prove that if $(1+x)^n=c_0+c_1x+c_2x^2+.....$, where n is a positive integer, then

i) $c_0-c_2+c_4-.....=2^{\frac{n}{2}}\cos\dfrac{1}{4}n\pi$.

ii) $c_1-c_3+c_5-.....=2^{\frac{n}{2}}\sin\dfrac{1}{4}n\pi$.

10.5 DETERMINATION OF ROOTS USING DEMOIVRE'S THEOREM

The complex number $z=x+iy$ is an nth root of the complex number z if $w=z^n=(x+iy)^n$.

Working rule for finding the qth roots of complex number:

$z=x+iy$

Let $z=x+iy=r(\cos\theta+i\sin\theta)$

$\therefore z^{\frac{1}{q}}=(x+iy)^{\frac{1}{q}}=r^{\frac{1}{q}}(\cos\theta+i\sin\theta)^{\frac{1}{q}}$

$=r^{\frac{1}{q}}[\cos(2n\pi+\theta)+i\sin(2n\pi+\theta)]^{\frac{1}{q}}$ (since $\cos\theta=\cos(2n\pi+\theta)$ & $\sin\theta=\sin(2n\pi+\theta)$)

$$= r^{\frac{1}{q}}\left[\cos\frac{2n\pi+\theta}{q} + i\sin\frac{2n\pi+\theta}{q}\right]$$

Putting $n = 0, 1, 2, 3, ..., ..., q-1$ the q values of $(x+iy)^{\frac{1}{q}}$ are obtained.

Note: Some standard results of frequent use

a) $1 = \cos 0 + i\sin 0$ b) $-1 = \cos\pi + i\sin\pi$

c) $i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$ d) $-i = \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)$

Example 1: Find the cube roots of unity.

Soution: We have to evaluate $(1)^{\frac{1}{3}}$

$$(1)^{\frac{1}{3}} = (\cos 0 + i\sin 0)^{\frac{1}{3}}$$

$$= [\cos(2n\pi + 0) + i\sin(2n\pi + 0)]^{\frac{1}{3}}$$

$$= (\cos 2n\pi + i\sin 2n\pi)^{\frac{1}{3}} = \cos\frac{2n\pi}{3} + i\sin\frac{2n\pi}{3} \qquad (1)$$

Putting $n = 0, 1, 2$ the three cube roots of unity are

For $n = 0, \cos 0 + i\sin 0 = 1 + i.0 = 1$

For $n = 1, \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = \cos\left(\pi - \frac{\pi}{3}\right) + i\sin\left(\pi - \frac{\pi}{3}\right)$

$$= -\cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

For $n = 2, \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = \cos\left(\pi + \frac{\pi}{3}\right) + i\sin\left(\pi + \frac{\pi}{3}\right)$

$$= -\cos\frac{\pi}{3} - i\sin\frac{\pi}{3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

Hence cube roots of 1 are $1, -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$.

Example 2: Find the values of $(-1)^{\frac{1}{6}}$.

Solution: We know that $-1 = \cos\pi + i\sin\pi$

$$= \cos(2m\pi + \pi) + i\sin(2m\pi + \pi)$$

$$= \cos(2m+1)\pi + i\sin(2m+1)\pi$$

Now, $(-1)^{\frac{1}{6}} = [\cos(2m+1)\pi + i\sin(2m+1)\pi]^{\frac{1}{6}}$

$$= \cos\frac{(2m+1)\pi}{6} + i\sin\frac{(2m+1)\pi}{6}$$ (By DeMovire's Theory)

Putting $m = 0, 1, 2, 3, 4, 5$, We get the required values

$$\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}, \cos\frac{3\pi}{6} + i\sin\frac{3\pi}{6}, \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}, \cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6},$$

$\cos\frac{9\pi}{6}+i\sin\frac{9\pi}{6}$ and $\cos\frac{11\pi}{6}+i\sin\frac{11\pi}{6}$.

or $\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}, \cos\frac{\pi}{2}+i\sin\frac{\pi}{2}, \cos\left(\pi-\frac{\pi}{6}\right)+i\sin\left(\pi-\frac{\pi}{6}\right)$,

$\cos\left(\pi+\frac{\pi}{6}\right)+i\sin\left(\pi+\frac{\pi}{6}\right), \cos\left(\pi+\frac{3\pi}{6}\right)+i\sin\left(\pi+\frac{3\pi}{6}\right)$ and

$\cos\left(2\pi-\frac{\pi}{6}\right)+i\sin\left(2\pi-\frac{\pi}{6}\right)$.

i.e. $\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}, \cos\frac{\pi}{2}+i\sin\frac{\pi}{2}, -\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}, -\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}$,

$-\cos\frac{\pi}{2}-i\sin\frac{\pi}{2}$ and $\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}$.

i.e. $\frac{\sqrt{3}}{2}+i\frac{1}{2}, i, -\frac{1}{2}+i\frac{\sqrt{3}}{2}, -\frac{1}{2}-i\frac{\sqrt{3}}{2}, -i$ and $\frac{\sqrt{3}}{2}-i\frac{1}{2}$.

Example 3: Find all the values of $\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^{3/4}$ and show that their continued product is 1.

Solution : Let $\frac{1}{2}+\frac{\sqrt{3}}{2}i = r(\cos\theta+i\sin\theta)$

Equating real and imaginary parts $r\cos\theta=\frac{1}{2}$, $r\sin\theta=\frac{\sqrt{3}}{2}$

Squaring and adding, $r^2(\cos^2\theta+\sin^2\theta)=\frac{1}{4}+\frac{3}{4}=1$. $\therefore\ r=1$

Also, $\frac{r\sin\theta}{r\cos\theta}=\frac{\sqrt{3}/2}{1/2}=\sqrt{3}$

or $\tan\theta=\sqrt{3}=\tan\frac{\pi}{3}$

so, $\theta=\frac{\pi}{3}$

$\therefore\ \frac{1}{2}+i\frac{\sqrt{3}}{2}=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$

$\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^{3/4}=\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^{3/4}=\left[\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^3\right]^{\frac{1}{4}}$

$=(\cos\pi+i\sin\pi)^{\frac{1}{4}}$

$=\left[\cos(2n\pi+\pi)+i\sin(2n\pi+\pi)\right]^{\frac{1}{4}}$

$= \cos\frac{(2n+1)\pi}{4} + i\sin\frac{(2n+1)\pi}{4}$ (1)

where n = 0, 1, 2, 3.

Required values are:

for $n = 0$, $\cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} = \frac{1+i}{\sqrt{2}}$;

for $n = 1$, $\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = \cos(\pi - \frac{\pi}{4}) + i\sin(\pi - \frac{\pi}{4}) = -\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$

$= -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} = \frac{-1+i}{\sqrt{2}}$;

for $n = 2$, $\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} = \cos(\pi + \frac{\pi}{4}) + i\sin(\pi + \frac{\pi}{4}) = -\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}$

$= -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}} = \frac{-1-i}{\sqrt{2}}$

for $n = 3$, $\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4} = \cos(2\pi - \frac{\pi}{4}) + i\sin(2\pi - \frac{\pi}{4}) = \cos\frac{\pi}{4} - i\sin\frac{\pi}{4}$

$= \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}} = \frac{1-i}{\sqrt{2}}$

The required product:

$= (\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4})(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4})$

$= (\cos(\frac{\pi}{4} + \frac{3\pi}{4} + \frac{5\pi}{4} + \frac{7\pi}{4}) + i\sin(\frac{\pi}{4} + \frac{3\pi}{4} + \frac{5\pi}{4} + \frac{7\pi}{4})$

$= \cos 4\pi + i\sin 4\pi = 1$.

Example 4: Use De Moivre's theorem to solve the following equation: $x^7 + x^4 + x^3 + 1 = 0$

Solution: Given equation is $x^7 + x^4 + x^3 + 1 = 0$

or $x^4(x^3 + 1) + 1(x^3 + 1) = 0$

or $(x^4 + 1)(x^3 + 1) = 0$ (1)

either $x^4 + 1 = 0$ or $x^3 + 1 = 0$

First we consider $x^4 + 1 = 0$

or $x^4 = -1$

or $x = (-1)^{1/4}$

$\therefore\ x = (\cos\pi + i\sin\pi)^{1/4}$

$= [\cos(2n\pi + \pi) + i\sin(2n\pi + \pi)]^{1/4}$

$= \cos\frac{(2n+1)\pi}{4} + i\sin\frac{(2n+1)\pi}{4}$ (2)

Putting $n = 0, 1, 2, 3$ the four roots of $x^4 + 1 = 0$ are for $n = 0$,

$\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}=\frac{1+i}{\sqrt{2}}$;

for $n=1,\ \cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}=\cos(\pi-\frac{\pi}{4})+i\sin(\pi-\frac{\pi}{4})=-\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}$

$=-\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}=\frac{-1+i}{\sqrt{2}}$

for $n=2,\ \cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}=\cos(\pi+\frac{\pi}{4})+i\sin(\pi+\frac{\pi}{4})=-\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}$

$=-\frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}=\frac{-1-i}{\sqrt{2}}$

for $n=3,\ \cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}=\cos(2\pi-\frac{\pi}{4})+i\sin(2\pi-\frac{\pi}{4})=\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}$

$=\frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}=\frac{1-i}{\sqrt{2}}$.

Now, $x^3+1=0$

or $x^3=-1$

or $x=(-1)^{1/3}$

$=(\cos\pi+i\sin\pi)^{\frac{1}{3}}$

$=\left[\cos(2n\pi+\pi)+i\sin(2n\pi+\pi\right]^{\frac{1}{3}}$

$=\cos\frac{(2n+1)\pi}{3}+i\sin\frac{(2n+1)\pi}{3}$ (2)

Putting $n=0,1,2$ the three roots of $x^3+1=0$ are

for $n=0,\ \cos\frac{\pi}{3}+i\sin\frac{\pi}{3}=\frac{1}{2}+i\frac{\sqrt{3}}{2}=\frac{1+i\sqrt{3}}{2}$;

for $n=1,\ \cos\pi+i\sin\pi=-1+i0=-1$;

For $n=2,\ \cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}=\cos(2\pi-\frac{\pi}{3})+i\sin(2\pi-\frac{\pi}{3})=\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}$

$=\frac{1}{2}-i\frac{\sqrt{3}}{2}=\frac{1-i\sqrt{3}}{2}$.

Hence roots of (1) are: $-1,\frac{1+i}{\sqrt{2}},\frac{1-i}{\sqrt{2}},\frac{-1+i}{\sqrt{2}},\frac{-1-i}{\sqrt{2}},\frac{1+i\sqrt{3}}{\sqrt{2}},\frac{1-i\sqrt{3}}{\sqrt{2}}$.

CHECK YOUR PROGRESS

Q.6: Find all the values of $(-1)^{1/3}$

Q.7: Find all the values of

i) $(1-i\sqrt{3})^{1/3}$ ii) $(1+i)^{1/4}$

iii) $(1-i)^{\frac{1}{3}}$ iv) $(1+i)^{\frac{1}{7}}$

Q.8: Find all the values of $(1+i\sqrt{3})^{1/3}+(1-i\sqrt{3})^{1/3}$.

Q.9: Use De Moivre's theorem to solve the following equations:

i) $x^4+x^3+x^2+x+1=0$ ii) $x^4-x^3+x^2-x+1=0$

iii) $x^8+x^5-x^3-1=0$ iii) $x^9-x^5+x^4-1=0$

10.6 LET US SUM UP

- A complex number $x+iy$ where x and y are real numbers and $i=\sqrt{-1}$, is called a complex number and is denoted by z i.e $z=x+iy$. The modulus and argument of a complex $z=x+iy$ are defined by $|z|=\sqrt{x^2+y^2}$ and $\theta=\tan^{-1}\left(\frac{y}{x}\right)$ respectively.Also,we define polar form of $z=x+iy$ as $z=r(\cos\theta+i\sin\theta)$ where $r=|z|=\sqrt{x^2+y^2}$ and $\theta=\tan^{-1}\left(\frac{y}{x}\right)$.

- According to De-Moivre's theorem :

  a) If n is any integer (positive, negativeor zero), then $(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta$.

  b) If n is a fraction (positive or negative), then one of the values of $(\cos\theta+i\sin\theta)^n$ is $\cos n\theta+i\sin n\theta$.

  Also, we derive the follwing results from De-Moivre's theorem.

  i) $(\cos\theta-i\sin\theta)^n=\cos n\theta-i\sin n\theta$

  ii) $(\sin\theta+i\cos\theta)^n\neq\sin n\theta+i\cos n\theta$

  iii) $(\cos\theta-i\sin\theta)^{-n}=\cos(-n\theta)-i\sin(-n\theta)=\cos n\theta+i\sin n\theta$

  iv) $\frac{1}{\cos\theta+i\sin\theta}=\cos\theta-i\sin\theta$

  v) $\frac{1}{\cos\theta-i\sin\theta}=\cos\theta+i\sin\theta$

- We can find roots of a complex number using De-Moivre's theorem. For any positive integer n,the complex number $z=r(\cos\theta+i\sin\theta)$ has exactly n distinct roots.then n roots are given by–

$\sqrt[n]{r}\left[\cos\left(\frac{\theta+2\pi k}{n}\right)+i\sin\left(\frac{\theta+2\pi k}{n}\right)\right]$ where $k=0,1,2,.....,n-1.$

10.7 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1: $\dfrac{(\cos\theta - i\sin\theta)^{10}}{(\cos\theta + i\sin\theta)^{12}} = \dfrac{(\cos\theta + i\sin\theta)^{-10}}{(\cos\theta + i\sin\theta)^{12}}$

$= (\cos\theta + i\sin\theta)^{-22} = \cos 22\theta - i\sin 22\theta$

Ans. to Q. No. 2: i) we have

$a + b = (\cos 2\alpha + i\sin 2\alpha) + (\cos 2\beta + i\sin 2\beta)$

$= (\cos 2\alpha + \cos 2\beta) + i(\sin 2\alpha + \sin 2\beta)$

$= 2\cos(\alpha + \beta)\cos(\alpha - \beta) + 2i\sin(\alpha + \beta)\cos(\alpha - \beta)$

$= 2\cos(\alpha - \beta)[\cos(\alpha + \beta) + i\sin(\alpha + \beta)]$

ii) we have

$a - b = (\cos 2\alpha + i\sin 2\alpha) - (\cos 2\beta + i\sin 2\beta)$

$= (\cos 2\alpha - \cos 2\beta) + i(\sin 2\alpha - \sin 2\beta)$

$= 2\sin(\alpha + \beta)\sin(\beta - \alpha) + 2i\cos(\alpha + \beta)\sin(\alpha - \beta)$

$= 2\sin(\alpha - \beta)[i\cos(\alpha + \beta) - \sin(\alpha + \beta)]$

$= 2\sin(\alpha - \beta)[i\cos(\alpha + \beta) + i^2\sin(\alpha + \beta)]$

$= 2i\sin(\alpha - \beta)[\cos(\alpha + \beta) + i\sin(\alpha + \beta)]$

iii) Dividing (ii) by (i), we get

$$\frac{a - b}{a + b} = \frac{2i\sin(\alpha - \beta)[\cos(\alpha + \beta) + i\sin(\alpha + \beta)]}{2\cos(\alpha - \beta)[\cos(\alpha + \beta) + i\sin(\alpha + \beta)]} = i\tan(\alpha - \beta)$$

Ans. to Q. No. 3: Let $r\cos\theta = \sqrt{3}$ and $r\sin\theta = 1$ so that $r^2 = 4$

i.e $r = 2$ and $\tan\theta = \dfrac{1}{\sqrt{3}}$ i.e., $\theta = \dfrac{\pi}{6}$

Hence, $\left(\sqrt{3} + i\right)^n + \left(\sqrt{3} - i\right)^n$

$= r^n\left[(\cos\theta + i\sin\theta)^n + (\cos\theta - i\sin\theta)^n\right]$

$= r^n\left(\cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta\right)$

$= 2r^n\cos n\theta = 2.2^n\cos\left(\dfrac{n\pi}{6}\right)$

$= 2^{n+1}\cos\left(\dfrac{n\pi}{6}\right)$

Ans. to Q. No. 4: We have $x + y + z = 0$

So, $(\cos\alpha + i\sin\alpha) + (\cos\beta + i\sin\beta) + (\cos\gamma + i\sin\gamma) = 0$

or $(\cos\alpha + \cos\beta + \cos\gamma) + i(\sin\alpha + \sin\beta + \sin\gamma) = 0 + i0$

Equating real and imaginary parts,we have

$\cos\alpha + \cos\beta + \cos\gamma = 0$ (1)

and $\sin\alpha + \sin\beta + \sin\gamma = 0$ (2)

Now, $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=x^{-1}+y^{-1}+z^{-1}$

$$=(\cos\alpha+i\sin\alpha)^{-1}+(\cos\beta+i\sin\beta)^{-1}+(\cos\gamma+i\sin\gamma)^{-1}$$
$$=(\cos\alpha-i\sin\alpha)+(\cos\beta-i\sin\beta)+(\cos\gamma-i\sin\gamma)$$
$$=(\cos\alpha+\cos\beta+\cos)-i(\sin\alpha+\sin\beta+\sin\gamma)$$
$$=0-i.0=0$$

Ans. to Q. No. 5: Let $1+i=r(\cos\theta+i\sin\theta)$

Equating real and imaginary parts, we have $r\cos\theta=1$ and $r\sin\theta=1$

So that $r=\sqrt{1+1}=\sqrt{2}$ and $\tan\theta=1$ i.e., $\theta=\frac{\pi}{4}$

$$(1+i)^n=r^n(\cos\theta+i\sin\theta)^n=r^n(\cos n\theta+i\sin n\theta)$$
$$=\left(\sqrt{2}\right)^n\left(\cos\frac{n\pi}{4}+i\sin\frac{n\pi}{4}\right) \quad (1)$$

Putting for r and θ.

Also, we have

$$(1+x)^n=C_0+C_1x+C_2x^2+C_3x^3+\cdots\cdots \quad (2)$$

Putting $x=i$ in (2),we have

$$(1+i)^n=C_0+C_1i-C_2-C_3i+C_4+C_5i-C_6-C_7i+\cdots\cdot$$
$$=(C_0-C_2+C_4-C_6+\cdots\cdot)+i(C_1-C_3+C_5-C_7+\cdots\cdot)$$
$$\left[\because i^2=-1,i^3=-i,i^4=-1\right] \quad (3)$$

Equating the values of $(1+i)^n$ from (1) and (3),we get

$$(C_0-C_2+C_4-C_6+\cdots\cdot)+i(C_1-C_3+C_5-C_7+\cdots\cdot)=\left(\sqrt{2}\right)^n\left[\cos\left(\frac{n\pi}{4}\right)+i\sin\left(\frac{n\pi}{4}\right)\right]$$

Equating real and imaginary parts on both sides,we have

$$C_0-C_2+C_4-C_6+\cdots\cdot=\left(\sqrt{2}\right)^n\cos\left(\frac{n\pi}{4}\right) \quad (4)$$

and $C_1-C_3+C_5-C_7+\cdots\cdot=\left(\sqrt{2}\right)^n\sin\left(\frac{n\pi}{4}\right)$ (5)

Ans. to Q. No. 6: Since $\cos\pi=-1$ and $\sin\pi=0$, we have $-1=\cos\pi+i\sin\pi$

$$\therefore(-1)^{\frac{1}{3}}=(\cos\pi+i\sin\pi)^{\frac{1}{3}}$$
$$=\left[\cos(2n\pi+\pi)+i\sin(2n\pi+\pi)\right]^{\frac{1}{3}}=\cos(2n+1)\frac{\pi}{3}+i\sin(2n+1)\frac{\pi}{3}$$

Now, putting $r=0,1,2$,we obtain the required values as

$\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$, $\cos\pi+i\sin\pi$ and $\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}$

Now, $\cos\pi+i\sin\pi=-1+i.0=-1$,

Also, $\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}=\cos\left(2\pi-\frac{\pi}{3}\right)+i\sin\left(2\pi-\frac{\pi}{3}\right)=\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}$

i.e., the three values are –1, $\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$ and $\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}$.

i.e, $-1,\frac{1}{2}+i\frac{\sqrt{3}}{2},\frac{1}{2}-i\frac{\sqrt{3}}{2}$

Ans. to Q. No. 7: i) we have $\left(1-i\sqrt{3}\right)=2\left(\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}\right)$

$$\therefore\left(1-i\sqrt{3}\right)^{\frac{1}{3}}=2^{\frac{1}{3}}\left[\cos\left(2n\pi+\frac{1}{3}\pi\right)-i\sin\left(2n\pi+\frac{1}{3}\pi\right)\right]^{\frac{1}{3}}$$

$$=2^{\frac{1}{3}}\left[\cos\frac{1}{9}(6n+1)\pi-i\sin\frac{1}{9}(6n+1)\pi\right]$$

$$=2^{\frac{1}{3}}\left[\cos\frac{1}{3}\left(2n\pi+\frac{1}{3}\pi\right)-i\sin\frac{1}{3}\left(2n\pi+\frac{1}{3}\pi\right)\right]$$

Putting $n=0,1,2$,we get the three values of $\left(1-i\sqrt{3}\right)^{\frac{1}{3}}$.

ii) Try to solve yourself.

iii) As $\cos\frac{\pi}{4}=\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}$, we have

$$1-i=\sqrt{2}\left(\cos\frac{\pi}{4}-\sin\frac{\pi}{4}\right)=\sqrt{2}\left\{\cos\left(2n\pi+\frac{\pi}{4}\right)-i\sin\left(2n\pi+\frac{\pi}{4}\right)\right\},$$

n is an integer

$$(1-i)^{\frac{1}{3}}=2^{\frac{1}{6}}\left\{\cos\left(\frac{2n\pi+\frac{\pi}{4}}{3}\right)-i\sin\left(\frac{2n\pi+\frac{\pi}{4}}{3}\right)\right\}$$

Putting $n=0,1,2$, we get the three values of

$(1-i)^{\frac{1}{3}}$, i.e., $2^{\frac{1}{6}}\left(\cos\frac{\pi}{12}-i\sin\frac{\pi}{12}\right)$,

$2^{\frac{1}{6}}\left(\cos\frac{3\pi}{4}-i\sin\frac{3\pi}{4}\right)$ and $2^{\frac{1}{6}}\left(\cos\frac{17\pi}{12}-i\sin\frac{17\pi}{12}\right)$.

iv) Since, $1+i=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)=\sqrt{2}\left\{\cos\left(2n\pi+\frac{\pi}{4}\right)+i\sin\left(2n\pi+\frac{\pi}{4}\right)\right\}$

$$\therefore(1+i)^{\frac{1}{7}}=2^{\frac{1}{14}}\left\{\cos\left(2n\pi+\frac{\pi}{4}\right)+i\sin\left(2n\pi+\frac{\pi}{4}\right)\right\}^{\frac{1}{7}}=2^{\frac{1}{14}}\left\{\cos\frac{2n\pi+\frac{\pi}{4}}{7}+i\sin\frac{2n\pi+\frac{\pi}{4}}{7}\right\}$$

Now, putting $n=0, 1, 2, 3, 4, 5$ and 6, we obtain the seven values of the expression.

Ans. to Q. No. 8: We have to find the values of $(1+i\sqrt{3})^{1/3}+(1-i\sqrt{3})^{1/3}$

First we find the three values of $(1+i\sqrt{3})^{1/3}$. Also, we find the three values of $(1-i\sqrt{3})^{1/3}$ (by Q2.(i)). Then adding corresponding values of $(1+i\sqrt{3})^{1/3}$ and $(1-i\sqrt{3})^{1/3}$, we get the required values of $(1+i\sqrt{3})^{1/3}+(1-i\sqrt{3})^{1/3}$.

Ans. to Q. No. 9: i) We know that $(x-1)(x^4+x^3+x^2+x+1)=x^5-1$

Therefore, the roots of $(x^4+x^3+x^2+x+1)=0$ are those of $x^5-1=0$ excluding that of $x-1=0$ which is 1.

Now, $x^5-1=0$ gives $x^5=1$ or $x=1^{1/5}$

$=(\cos 0+i\sin 0)^{1/5}=(\cos 2n\pi+i\sin 2n\pi)^{1/5}$

$=\cos\frac{2n\pi}{5}+i\sin\frac{2n\pi}{5}$

Putting *n* = 0, 1, 2, 3, 4, the roots of $x^5-1=0$ are

$1,\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5},\cos\frac{4\pi}{5}+i\sin\frac{4\pi}{5}$,

$\cos\frac{6\pi}{5}+i\sin\frac{6\pi}{5}$ and $\cos\frac{8\pi}{5}+i\sin\frac{8\pi}{5}$.

But, $\cos\frac{8\pi}{5}+i\sin\frac{8\pi}{5}=\cos\left(2\pi-\frac{2\pi}{5}\right)+i\sin\left(2\pi-\frac{2\pi}{5}\right)=\cos\frac{2\pi}{5}-i\sin\frac{2\pi}{5}$

Also, $\cos\frac{6\pi}{5}+i\sin\frac{6\pi}{5}=\cos\left(2\pi-\frac{4\pi}{5}\right)+i\sin\left(2\pi-\frac{4\pi}{5}\right)=\cos\frac{4\pi}{5}-i\sin\frac{4\pi}{5}$

Hence, the root of $x^5-1=0$ are

$1,\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5},\cos\frac{4\pi}{5}+i\sin\frac{4\pi}{5},\cos\frac{2\pi}{5}-i\sin\frac{2\pi}{5}$ and $\cos\frac{4\pi}{5}-i\sin\frac{4\pi}{5}$.

Here, 1 is the root of $x-1=0$ and the remaining four are the roots of $(x^4+x^3+x^2+x+1)=0$.

ii) Try to Solve Yourself.

iii) We have $x^8+x^5-x^3-1=0$

Factorising the left hand side, we get $(x^5-1)(x^3+1)=0$

From the first factor we get $x^5=1=\cos 2n\pi+i\sin 2n\pi$,

where n is any integer.

$\therefore x=(\cos 2n\pi+i\sin 2n\pi)^{\frac{1}{5}}=\cos\frac{2n\pi}{5}+i\sin\frac{2n\pi}{5}$

Putting $n=0, 1, 2, 3$ and 4 we get five roots from the first equation.

From the second factor,we have

$x^3=-1=\cos\pi+i\sin\pi=\cos(2n\pi+\pi)+i\sin(2n\pi+\pi)$

n is any integer.

$\therefore x = \{\cos(2n\pi+\pi)+i\sin(2n\pi+\pi)\}^{\frac{1}{3}} = \cos\frac{2n\pi+\pi}{3}+i\sin\frac{2n\pi+\pi}{3}$

Putting $n = 0, 1, 2.$ we get three roots from the second equation.

Thus all the eight roots of the equation are obtained.

iv) Try to Solve Yourself.

10.8 FURTHER READING

- University Algebra, N S Gopala Krishnan, New Age International (P) Limited.
- Trigonometry, for B. A/B.Sc. Classes (1st Edition), N.P. Bali: Laxmi Publication (P) Ltd.
- Trigonometry; (1st Edition), Hari Kishan: Atlantic Publishers.
- Linear Algebra: Schaum's Solved Problems Series; Seymour Lipschitz: Tata McGraw Hill.
- Text Book of Algebra, A.K Sharma, DPH, Mathematics Series.
- Simplified Course in Trigonometry, M D Raisinghania,HK Das,H C Sexana, S Chand & Company Ltd.

10.9 MODEL QUESTIONS

Q.1: Prove that:

$$\frac{(1+\cos\varphi+i\sin\varphi)^n}{(1+\cos\varphi-i\sin\varphi)^n} = \cos n\varphi + i\sin n\varphi.$$

Q.2: If $x_r = \cos\frac{\pi}{2^r}+i\sin\frac{\pi}{2^r}$, then prove that $x_1.x_2.x_3.....\infty = -1$.

Q.3: If $\cos\alpha+\cos\beta+\cos\gamma = \sin\alpha+\sin\beta+\sin\gamma = 0$, then prove that

$\cos 3\alpha+\cos 3\beta+\cos 3\gamma = 3\cos(\alpha+\beta+\gamma)$ and

$\sin 3\alpha+\sin 3\beta+\sin 3\gamma = 3\sin(\alpha+\beta+\gamma)$

Q.4: Simplify: a) $\frac{(\cos\alpha+i\sin\alpha)^4}{(\sin\beta+i\cos\beta)^5}$

b) $\frac{(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3})^5(\cos\frac{\pi}{3}-i\sin\frac{\pi}{3})^2}{(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3})^{-3}(\cos\frac{\pi}{3}-i\sin\frac{\pi}{3})^{-4}}$

c) $\frac{(\cos\frac{\pi}{15}+i\sin\frac{\pi}{15})^{10}+(\cos\frac{\pi}{15}-i\sin\frac{\pi}{15})^{10}}{(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3})^3}$

Q.5: If $x = \cos\alpha+i\sin\alpha$, $y = \cos\beta+i\sin\beta$,

Prove that $\frac{x-y}{x+y} = i\tan\frac{\alpha-\beta}{2}$.

Q.6: Evaluate: a) $(8i)^{\frac{1}{3}}$ b) $(-i)^{\frac{1}{6}}$

Q.7: Find all the values of

a) $(1+i)^{\frac{1}{3}}$ b) $(1+i)^{\frac{2}{3}}$

c) $(\sqrt{3}+i)^{\frac{1}{3}}$ d) $(1-i\sqrt{3})^{\frac{1}{3}}$

Q.8: Solve the following equations:

a) $x^4 - x^3 + x^3 - x + 1 = 0$

b) $x^7 + x^4 + x^3 + 1 = 0$

c) $x^9 - x^5 + x^4 - 1 = 0$

d) $x^4 + x^3 + x^2 + x + 1 = 0$

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