UNIT 11: EXPANSION OF TRIGONOMETRIC FUNCTIONS

UNIT STRUCTURE

11.1 Learning Objectives

11.2 Introduction

11.3 Expansion of $\cos n\theta$ and $\sin n\theta$

11.4 Expansion of $\tan n\theta$

11.5 Expansion of $\sin\alpha$ in Terms of α

11.6 Expansion $\cos\alpha$ in Terms of α

11.7 Expansion of $\tan\alpha$ in Terms of α

11.8 Expansion of $\sin\alpha^0$

11.9 Expansion of $\cos\alpha^0$

11.10 Let Us Sum Up

11.11 Answers to check Your Progress

11.12 Further Reading

11.13 Model Questions

11.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- derive of $\cos n\theta$, $\sin n\theta$ and $\tan n\theta$
- derive of $\sin\alpha$, $\cos\alpha$ and $\tan\alpha$ in terms of α
- derive of $\sin\alpha^0$ and $\cos\alpha^0$.

11.2 INTRODUCTION

In this unit, We will obtain the expansion of some trigonometric functions.

11.3 EXPANSION OF $\cos n\theta$ AND $\sin n\theta$.(nBEING A +VE INTEGER)

We know from De moivre's Theorem, that

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

Expanding the left hand side by Binomial theorem, we have

$\cos n\theta + i\sin n\theta$

$= (\cos\theta)^n + {}^nC_1(\cos\theta)^{n-1}(i\sin\theta) + {}^nC_2(\cos\theta)^{n-2}(i\sin\theta)^2 + {}^nC_3(\cos\theta)^{n-3}(i\sin\theta)^3$

$+ {}^nC_4(\cos\theta)^{n-4}(i\sin\theta)^4 + ... + {}^nC_{n-1}(\cos\theta)\ (i\sin\theta)^{n-1} + {}^nC_n(i\sin\theta)^n$.

Now, $i^2=-1, i^3=i^2.i=-i, i^4=(i^2)^2=1$ and so on.

and ${}^nC_{n-1}={}^nC_1=n, {}^nC_n={}^nC_0=1$.

$\therefore \cos n\theta+i\sin n\theta$

$=(\cos\theta)^n+i^nC_1\cos^{n-1}\theta\sin\theta-{}^nC_2\cos^{n-2}\theta\sin^2\theta-i^nC_3\cos^{n-3}\theta\sin^3\theta$

$+{}^nC_4\cos^{n-4}\theta\sin^4\theta+\ldots+i^{n-1}n\cos\theta\sin^{n-1}\theta+i^n\sin^n\theta.$

There arise two cases, according as n is odd or even

Case I: If n is odd, then (n–1) is even $\cos n\theta+i\sin n\theta$

$=(\cos\theta)^n+i^nC_1\cos^{n-1}\theta\sin\theta-{}^nC_2\cos^{n-2}\theta\sin^2\theta-i^nC_3\cos^{n-3}\theta\sin^3\theta$

$+{}^nC_4\cos^{n-4}\theta\sin^4\theta+\ldots+n(-1)^{\frac{n-1}{2}}\cos\theta\sin^{n-1}\theta+i(-1)^{\frac{n-1}{2}}\sin^n\theta.$

$\left[\because i^{n-1}=(i^2)^{\frac{n-1}{2}}=(-1)^{\frac{n-1}{2}}; i^n=i.i^{n-1}=i.(-1)^{\frac{n-1}{2}}\right]$

Equating real and imaginary parts, we get

$\cos n\theta=(\cos\theta)^n-{}^nC_2\cos^{n-2}\theta\sin^2\theta+\ldots+n(-1)^{\frac{n-1}{2}}\cos\theta\sin^{n-1}\theta$ .... (i)

and

$\sin n\theta={}^nC_1\cos^{n-1}\theta\sin\theta-{}^nC_3\cos^{n-3}\theta\sin^3\theta+\ldots+(-1)^{\frac{n-1}{2}}\sin^n\theta.$ .... (ii)

Case II: If n is even, then(n–1) is odd, (n–2) is even $\cos n\theta+i\sin n\theta$

$+{}^nC_4\cos^{n-4}\theta\sin^4\theta+\ldots..+i(-1)^{\frac{n-2}{2}}n\cos\theta\sin^{n-1}\theta+(-1)^{\frac{n}{2}}\sin^n\theta\cdot$

$\left[\because i^{n-1}=i.i^{n-2}=i.(i^2)^{\frac{n-1}{2}}=i.(-1)^{\frac{n-1}{2}}; i^n=(i^2)^{\frac{n}{2}}=(-1)^{\frac{n}{2}}\right]$

Equating real and imaginary parts,we get

$\cos n\theta=\cos^n\theta-{}^nC_2\cos^{n-2}\theta\sin^2\theta+{}^nC_4\cos^{n-4}\theta\sin^4\theta-\cdots\cdots+(-1)^{\frac{n}{2}}\sin^n\theta$ ....(iii)

and

$\sin n\theta={}^nC_1\cos^{n-1}\theta\sin\theta-{}^nC_3\cos^{n-3}\theta\sin^3\theta+\ldots..+(-1)^{\frac{n-2}{2}}n\cos\theta\sin^{n-1}\theta.$ ....(iv)

Example 1: a) Expand $\cos 7\theta$ in descending powers of $\cos\theta$.

b) Expand $\sin 7\theta$ in ascending powers of $\sin\theta$.

Solution: We have $(\cos 7\theta+i\sin 7\theta)=(\cos\theta+i\sin\theta)^7$

Expanding the R.H.S. by Bionomial Theorem, we have

$\cos 7\theta+i\sin 7\theta$

$=\cos^7\theta+{}^7C_1\cos^6\theta\,(i\sin\theta)+{}^7C_2\cos^6\theta(i\sin\theta)^2+{}^7C_3\cos^5\theta(i\sin\theta)^3$

$+{}^7C_4(\cos\theta)^3(i\sin\theta)^4+{}^7C_5(\cos\theta)^2(i\sin\theta)^5+{}^7C_6(\cos\theta)(i\sin\theta)^6+{}^7C_7(i\sin\theta)^7$

Now, $i^2=-1$, $i^3=-i$, $i^4=1, i^5=i, i^6=-1, i^7=i^3.i^4=-i$.

Also, ${}^7C_7=1$, ${}^7C_6={}^7C_1=7$, ${}^7C_5={}^7C_2=\frac{7.6}{1.2}=21$, ${}^7C_4={}^7C_3=\frac{7.6.5}{1.2.3}=35$.

$\therefore \cos 7\theta+i\sin 7\theta=\cos^7\theta+7i\cos^6\theta\sin\theta$

$-21\cos^5\theta\sin^2\theta-35i\cos^4\theta\sin^3\theta+35\cos^3\theta\sin^4\theta$

$+21i\cos^2\theta\ \sin^5\theta\ -7\cos\theta\sin^6\theta\ -i\sin^7\theta\cdot$

Equating real and imaginary parts,we get

a) $\cos 7\theta = \cos^7\theta - 21\cos^5\theta\sin^2\theta + 35\cos^3\theta\sin^4\theta - 7\cos\theta\sin^6\theta$

$= \cos^7\theta - 21\cos^5\theta(1-\cos^2\theta) + 35\cos^3\theta(1-\cos^2\theta)^2 - 7\cos\theta(1-\cos^2\theta)^3$

$= \cos^7\theta - 21\cos^5\theta + 21\cos^7\theta + 35\cos^3\theta(1-2\cos^2\theta+\cos^4\theta)$

$-7\cos\theta(1-3\cos^2\theta+3\cos^4\theta-\cos^6\theta)$

$= 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$

b) $\sin 7\theta = 7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta$

$= 7(1-\sin^2\theta)^3\sin\theta - 35(1-\sin^2\theta)^2\sin^3\theta + 21(1-\sin^2\theta)\sin^5\theta - \sin^7\theta$

$= 7(1-3\sin^2\theta+3\sin^4\theta-\sin^6\theta)\sin\theta - 35(1-2\sin^2\theta+\sin^4\theta)\sin^3\theta$

$+21\sin^5\theta - 21\sin^7\theta - \sin^7\theta\cdot$

$= 7\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta.$

11.4 EXPANSION OF $\tan n\theta$

$$\tan n\theta = \frac{\sin n\theta}{\cos n\theta} = \frac{{}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + {}^nC_5\cos^{n-5}\theta\sin^5\theta.....}{\cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + {}^nC_4\cos^{n-4}\theta\sin^4\theta.....}$$

Dividing the numerator and denominator by $\cos^n\theta$,we get

$$\tan n\theta = \frac{{}^nC_1\tan\theta - {}^nC_3\tan^3\theta + {}^nC_5\tan^5\theta - .....}{1 - {}^nC_2\tan^2\theta + {}^nC_4\tan^4\theta - .....}.$$

Last terms:

i) When n is odd: The last terms of the numerator is $(-1)^{\frac{n-1}{2}}\tan^n\theta$ and that of the denominator is $n(-1)^{\frac{n-2}{2}}\tan^{n-1}\theta$.

ii) When n is even: The last term of the denominator is $n(-1)^{\frac{n-2}{2}}\tan^{n-1}\theta$ and that of the denominator is $(-1)^{\frac{n}{2}}\tan^n\theta$.

Example 2: Expand $\tan 5\theta$ in powers of $\tan\theta$.

Solution: We know that

$$\tan n\theta = \frac{{}^nC_1\tan\theta - {}^nC_3\tan^3\theta + {}^nC_5\tan^5\theta - .....}{1 - {}^nC_2\tan^2\theta + {}^nC_4\tan^4\theta - .....}$$

Putting n = 5, we have

$$\tan 5\theta = \frac{{}^5C_1\tan\theta - {}^5C_3\tan^3\theta + {}^5C_5\tan^5\theta}{1 - {}^5C_2\tan^2\theta + {}^5C_4\tan^4\theta}$$

$$= \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$$

Example 3: Write down the value of $\tan 9\theta$ in terms of $\tan\theta$.

Solution: We know that

$$\tan n\theta = \frac{{}^nC_1 \tan\theta - {}^nC_3 \tan^3\theta + {}^nC_5 \tan^5\theta - \cdots\cdots}{1 - {}^nC_2 \tan^2\theta + {}^nC_4 \tan^4\theta - {}^nC_6 \tan^6\theta + \cdots\cdots}$$

Putting n = 9, we get

$$\tan 9\theta = \frac{{}^9C_1 \tan\theta - {}^9C_3 \tan^3\theta + {}^9C_5 \tan^5\theta - \cdots\cdots}{1 - {}^9C_2 \tan^2\theta + {}^9C_4 \tan^4\theta - {}^9C_6 \tan^6\theta + \cdots\cdots}$$

$$= \frac{9\tan\theta - \frac{9.8.7}{3!}\tan^3\theta + \frac{9.8.7.6}{4!}\tan^5\theta - \frac{9.8}{2!}\tan^7\theta + \tan^9\theta}{1 - \frac{9.8}{4!}\tan^2\theta + \frac{9.8.7.6}{4!}\tan^4\theta - \frac{9.8.7}{3!}\tan^6\theta + 9\tan^8\theta}$$

$$= \frac{9\tan\theta - 84\tan^3\theta + 126\tan^5\theta - 36\tan^7\theta + \tan^9\theta}{1 - 36\tan^2\theta + 126\tan^4\theta - 84\tan^6\theta + 9\tan^8\theta}$$

CHECK YOUR PROGRESS

Q.1: Expand $\cos 8\theta$ in powers of $\cos\theta$ and $\sin\theta$.

Q.2: Prove that, $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

Q.3: Prove that,

$$\tan 8\theta = \frac{8\tan\theta - 56\tan^3\theta + 56\tan^5\theta - 8\tan^7\theta}{1 - 28\tan^2\theta + 70\tan^4\theta - 28\tan^6\theta + \tan^8\theta}$$

Q.4: Prove that:

$$\frac{\sin 8\theta}{\sin\theta} = 128\cos^7\theta - 192\cos^5\theta + 80\cos^3\theta - 8\cos\theta$$

11.5 EXPANSION OF $\sin\alpha$ IN TERMS OF α (RADIANS)

We know that if n is a positive integer, then

$$\sin n\theta = n\theta\cos^{n-1}\theta\left(\frac{\sin\theta}{\theta}\right) - \frac{n\theta(n\theta-\theta)(n\theta-2\theta)}{2!}\cos^{n-3}\theta\left(\frac{\sin\theta}{\theta}\right)^3 .....$$

Let $n\theta = \alpha$ and suppose that n increses without limit and θ decreases in such a way that $n\theta = \alpha$ remains constant, then

$$\sin\alpha = \alpha\cos^{n-1}\theta\left(\frac{\sin\theta}{\theta}\right) - \frac{\alpha(\alpha-\theta)(\alpha-2\theta)}{2!}\cos^{n-3}\theta\left(\frac{\sin\theta}{\theta}\right)^3 .....$$

Now, when $\theta \to 0, \frac{\sin\theta}{\theta} \to 1$ and $\cos\theta \to 1$.

$$\therefore \sin\alpha = \alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} .....\infty .$$

11.6 EXPANSION $\cos\alpha$ IN TERMS OF α (RADIANS)

We know that if n is a positive integer, then

$$\cos n\theta = \cos^n\theta - \frac{n(n-1)}{2!}\cos^{n-2}\theta\sin^2\theta + \frac{n(n-1)(n-2)(n-3)}{4!}\cos^{n-4}\theta\sin^4\theta - \cdots\cdots$$

$$= \cos^n\theta - \frac{n\theta(n\theta-\theta)}{2!}\cos^{n-2}\theta\left(\frac{\sin\theta}{\theta}\right)^2 + \frac{n\theta\,(n\theta-\theta)(n\theta-2\theta)(n\theta-3\theta)}{4!}\cos^{n-4}\theta\left(\frac{\sin\theta}{\theta}\right)^4 - \cdots\cdots$$

Let $n\theta = \alpha$ and supose that n increases without limit and θ decreases in such a way that $n\theta = \alpha$ remains constant, then

$$= \cos^n\theta - \frac{\alpha(\alpha-\theta)}{2!}\cos^{n-2}\theta\left(\frac{\sin\theta}{\theta}\right)^2 + \frac{\alpha(\alpha-\theta)(\alpha-2\theta)(\alpha-3\theta)}{4!}\cos^{n-4}\theta\left(\frac{\sin\theta}{\theta}\right)^4 - \cdots\cdots$$

Now when $\theta \to 0, \frac{\sin\theta}{\theta} \to 1$ and $\cos\theta \to 1$

$$\therefore \cos\alpha = 1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \cdots\cdots\infty$$

11.7 EXPANSION OF $\tan\alpha$ IN TERMS OF α (RADIANS), α BEING SMALL

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{(\alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} - \cdots)}{(1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \cdots)}$$

$$= (\alpha - \frac{\alpha^3}{6} + \frac{\alpha^5}{120} - \cdots)(1 - \frac{\alpha^2}{2} + \frac{\alpha^4}{24} - \cdots)^{-1}$$

$$= (\alpha - \frac{\alpha^3}{6} + \frac{\alpha^5}{120} - \cdots)\left[1 - (\frac{\alpha^2}{2} - \frac{\alpha^4}{24} + \cdots)\right]^{-1}$$

$$= (\alpha - \frac{\alpha^3}{6} + \frac{\alpha^5}{120} - \cdots)\left\{1 + (\frac{\alpha^2}{2} - \frac{\alpha^4}{24}) + (\frac{\alpha^2}{2} - \frac{\alpha^4}{24})^2 + \cdots\right\}$$

$$= (\alpha - \frac{\alpha^3}{6} + \frac{\alpha^5}{120} - \cdots)\left(1 + \frac{\alpha^2}{2} + \frac{5\alpha^4}{24} + \cdots\right)$$

$$= \alpha + \frac{\alpha^3}{6} + \frac{2\alpha^5}{15} + \cdots$$

11.8 EXPANSION OF $\sin\alpha^0$

We know that $1^0 = \frac{\pi}{180}$ radians

$$\therefore \quad \alpha^0 = \frac{\pi\alpha}{180} \text{ radians}$$

Therefore $\sin\alpha^0 = \sin\frac{\pi\alpha}{180}$

$$= \frac{\pi\alpha}{180} - \frac{1}{3!}\left(\frac{\pi\alpha}{180}\right)^3 + \cdots$$

11.9 EXPANSION OF $\cos\alpha^0$

$$\cos\alpha^0 = \cos\frac{\pi\alpha}{180}$$

$$= 1 - \frac{1}{2!}\left(\frac{\pi\alpha}{180}\right)^2 + \frac{1}{4!}\left(\frac{\pi\alpha}{180}\right)^4 - \cdots$$

Example 1: If $\frac{\sin\theta}{\theta} = \frac{5765}{5766}$, Show that $\theta = 2^0$ approximately.

Solution: Here $\frac{5765}{5766}$ is nearly equal to 1.

$\frac{\sin\theta}{\theta} = 1$ (nearly), and so θ must be very small.

We know that, $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$

$$\frac{\sin\theta}{\theta} = \frac{\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots}{\theta} = \frac{5765}{5766}$$

or $1 - \frac{\theta^2}{6} + \frac{\theta^4}{120} - \cdots = \frac{5765}{5766}$.

Neglecting powers of θ higher than θ^2, we have

$$1 - \frac{\theta^2}{6} = \frac{5765}{5766}$$

or $\frac{\theta^2}{6} = 1 - \frac{5765}{5766} = \frac{1}{5766}$.

or $\theta^2 = \frac{1}{961}$

or $\theta = \frac{1}{31}$ radians

or $\theta = \frac{1}{31} \times \frac{180}{\pi}$ degrees

or $\theta = \frac{180 \times 7}{31 \times 22}$ degrees

$= 1.84$ degrees

$\theta = 2^0$ nearly.

Example 2: Show that,

$$\sin^3\theta = \frac{3}{4}\left\{\frac{(3^2-1)}{3!}\theta^3 - \frac{(3^4-1)}{5!}\theta^5 + \cdots + \frac{(-1)^{n-1}(3^{2n}-1)}{(2n+1)!}\theta^{2n+1} + \cdots\right\}$$

Solution: We know that, $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$$\therefore \sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta)$$

$$= \frac{1}{4}\left[3(\theta - \frac{\theta^3}{3} + \frac{\theta^5}{5!} - \cdots) - \left\{(3\theta) - \frac{(3\theta)^3}{3!} + \frac{(3\theta)^5}{5!} - \cdots\right\}\right]$$

$$= \frac{1}{4}\left[(3\theta - 3\theta) - \frac{\theta^3}{3!}(3 - 3^3) + \frac{\theta^5}{5!}(3 - 3^5) - \cdots)\right]$$

$$= \frac{1}{4}\left[\frac{\theta^3}{3!}(3^3 - 3) - \frac{\theta^5}{5!}(3^5 - 3) + \frac{\theta^7}{7!}(3^7 - 3) - \cdots)\right]$$

$$= \frac{3}{4}\left[\frac{\theta^3}{3!}(3^2 - 1) - \frac{\theta^5}{5!}(3^4 - 1) + \frac{\theta^7}{7!}(3^7 - 1) - \cdots)\right].$$

Example 2: Prove that,

$$\sin^2\theta\cos\theta = \theta^2 - \frac{5}{6}\theta^4 + \cdots + (-1)^{n+1}\frac{3^{2n} - 1}{4}\frac{\theta^{2n}}{(2n)!} + \cdots\infty$$

Solution: We have

$$\sin^2\theta\cos\theta = \frac{1}{2}\sin\theta(2\sin\theta\cos\theta) = \frac{1}{2}\sin\theta\sin 2\theta$$

$$= \frac{1}{4}(2\sin\theta\sin 2\theta) = \frac{1}{4}(\cos\theta - \cos 3\theta)$$

$$= \frac{1}{4}\left[\left\{1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots + (-1)^n\frac{\theta^{2n}}{(2n)!} + \cdots\infty\right\} - \left\{1 - \frac{(3\theta)^2}{2!} + \frac{(3\theta)^4}{4!} - \cdots + (-1)^{2n}\frac{(3\theta)^{2n}}{(2n)!} + \cdots\infty\right\}\right]$$

$$= \frac{1}{4}\left[-\frac{\theta^2}{2!}(1 - 3^2) + \frac{\theta^4}{4!}(1 - 3^4) - \cdots\infty + (-1)^n\frac{\theta^{2n}}{(2n)!}(1 - 3^{2n}) + \cdots\infty\right]$$

$$= \frac{1}{4}\left[4\theta^2 - \frac{10}{3}\theta^4 + \cdots + \frac{(-1)^n(-1)(3^{2n} - 1)\theta^{2n}}{(2n)!} + \cdots\infty\right]$$

$$= \theta^2 - \frac{5}{6}\theta^4 + \cdots + \frac{(-1)^{n+1}(3^{2n} - 1)\theta^{2n}}{4(2n)!} + \cdots\cdots\infty$$

Example 3: If $x = \frac{2}{1!} - \frac{4}{3!} + \frac{6}{5!} - \frac{8}{7!} + \cdots\cdots\infty$

and $y = 1 + \frac{2}{1!} - \frac{2^3}{3!} + \frac{2^5}{5!} - \cdots\cdots\infty$, then show that $x^2 = y$.

Solution: Given, $x = \frac{2}{1!} - \frac{4}{3!} + \frac{6}{5!} - \frac{8}{7!} + \cdots\cdots\infty$

$$= \frac{1+1}{1!} - \frac{3+1}{3!} + \frac{5+1}{5!} - \frac{7+1}{7!} + \cdots\cdots\infty$$

$$= \left(\frac{1}{1!} - \frac{3}{3!} + \frac{5}{5!} - \frac{7}{7!} + \cdots\cdots\right) + \left(\frac{1}{1!} - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!}\cdots\right)$$

$$= \left(1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \cdots\right) + \left(\frac{1}{1!} - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} \cdots\right)$$

$$= \cos 1 + \sin 1$$

and $y = 1 + \frac{2}{1!} - \frac{2^3}{3!} - \frac{2^5}{5!} - \cdots = 1 + \sin 2$

$\therefore\ x^2 = (\cos 1 + \sin 1)^2 = \cos^2 1 + \sin^2 1 + 2\sin 1 \cos 1$

$= 1 + \sin 2 = y$

Thus, $x^2 = y$

Example 4: Find the value of the series: $1 - \frac{2}{3!} + \frac{3}{5!} - \frac{4}{7!} \cdots\cdots \infty$

Solution: The given series,

$$1 - \frac{2}{3!} + \frac{3}{5!} - \frac{4}{7!} \cdots$$

$$= \frac{1}{2}\left[2 - \frac{4}{3!} + \frac{6}{5!} - \frac{8}{7!} \cdots\right]$$

$$= \frac{1}{2}\left[1 + 1 - \frac{3+1}{3!} + \frac{5+1}{5!} - \frac{7+1}{7!} \cdots\right]$$

$$= \frac{1}{2}\left[(1 - \frac{3}{3!} + \frac{5}{5!} - \frac{7}{7!} \cdots) + (1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots)\right]$$

$$= \frac{1}{2}\left[(1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \cdots) + (1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots)\right]$$

$$= \frac{1}{2}(\cos 1 + \sin 1)$$

$$= \frac{1}{\sqrt{2}}(\frac{1}{\sqrt{2}}\cos 1 + \frac{1}{\sqrt{2}}\sin 1)$$

$$= \frac{1}{\sqrt{2}}(\sin\frac{\pi}{4}\cos 1 + \cos\frac{\pi}{4}\sin 1)$$

$$= \frac{1}{\sqrt{2}}\sin(\frac{\pi}{4} + 1).$$

CHECK YOUR PROGRESS

Q.5: If $\frac{\sin\theta}{\theta} = \frac{2165}{2166}$, Show that θ is nearly equal to 3^0.

Q.6: If $\cos\theta = \frac{1681}{1682}$, show that θ is nearly equal to $1^0.58^/$.

Q.7: Prove that,

$$\frac{1}{3}\sin^3\theta = \frac{\theta^3}{3!} - (1 + 3^2)\frac{\theta^5}{5!} + (1 + 3^2 + 3^4)\frac{\theta^7}{7!} - \cdots\cdots \infty.$$

Q.8: Prove that,

$$\frac{\pi^2}{2.4}-\frac{\pi^4}{2.4.6.8}+\frac{\pi^6}{2.4.6.8.10.12}-\cdots=1$$

11.10 LET US SUM UP

- We derive some important deductions from De-Moivre's theorem. These deductions are
  a) Expansions of $\cos n\theta$ and $\sin n\theta$ in powers of $\cos\theta$ and $\sin\theta$ (θ being a positive integer).
  b) Expansion of $\tan n\theta$ in powers of $\tan\theta$.
  c) Expansions of $\sin\theta$ and $\cos\theta$ in series of powers of θ.
  d) Expansions of $\sin\theta^0$ and $\cos\theta^0$.
  e) Expansion of $\tan\theta$ in powers of θ.

11.11 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1: We have, $\cos 8\theta+i\sin 8\theta=(\cos\theta+i\sin\theta)^8$

(By DeMovire's theorem)

$$=\cos^8\theta+i\,{}^8C_1\cos^7\theta\sin\theta+i^2\,{}^8C_2\cos^6\theta\sin^2\theta+i^3\,{}^8C_3\cos^5\theta\sin^3\theta+\cdots+i^8\sin^8\theta$$

Equating the real parts, we have

$$\cos 8\theta=\cos^8\theta-{}^8C_2\cos^6\theta\sin^2\theta+{}^8C_4\cos^4\theta\sin^4\theta-{}^8C_6\cos^2\theta\sin^6\theta+\sin^8\theta$$

Ans. to Q. No. 2: We have $\cos 5\theta+i\sin 5\theta=(\cos\theta+i\sin\theta)^5$

$$=\cos^5\theta+i5\cos^4\theta\sin\theta+i^2 10\cos^3\theta\sin^2\theta+i^3 10\cos^2\theta\sin^3\theta+i^4.5\cos\theta\sin^4\theta+i^5\sin^5\theta$$

Equating real parts, we get

$$\cos 5\theta=\cos^5\theta-10\cos^3\theta\sin^2\theta+5\cos\theta\sin^4\theta$$
$$=\cos^5\theta-10\cos^3\theta(1-\cos^2\theta)+5\cos\theta(1-\cos^2\theta)^2$$
$$=\cos^5\theta-10\cos^3\theta+10\cos^5\theta+5\cos\theta-10\cos^3\theta+5\cos^5\theta$$
$$=16\cos^5\theta-20\cos^3\theta+5\cos\theta$$

Ans. to Q. No. 3: We know that

$$\tan n\theta=\frac{{}^nC_1\tan\theta-{}^nC_3\tan^3\theta+{}^nC_5\tan^5\theta-.....}{1-{}^nC_2\tan^2\theta+{}^nC_4\tan^4\theta-.....}$$

Now putting $n=8$, we get the required result.

Ans. to Q. No. 4: First we find expansion of $\sin 8\theta$ in terms of $\sin\theta$ and $\cos\theta$ then dividing both sides by $\sin\theta$. We get the required result.

Ans. to Q. No. 5: Given $\frac{\sin\theta}{\theta}=\frac{2165}{2166}$ which is nearly equal to 1. So that we take θ to be very small. Then in the series for $\sin\theta$, we neglect higher powers of θ.

$$\therefore \sin\theta = \theta - \frac{\theta^3}{3!}$$

$$\text{or } \frac{\sin\theta}{\theta} = 1 - \frac{\theta^2}{6}$$

$$\text{or } \frac{\theta^2}{6} = 1 - \frac{2165}{2166}$$

$$\text{or } \theta^2 = \frac{6}{2166} = \frac{1}{361}$$

$$\text{or } \theta = \frac{1}{19}\text{radian} = 3^0\text{nearly}$$

Ans. to Q. No. 6: Try to solve yourself.

Ans. to Q. No. 7: We know that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$$\therefore \sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$$

$$= \frac{3}{4}\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right) - \frac{1}{4}\left(3\theta - \frac{3^3\theta^3}{3!} + \frac{3^5\theta^5}{5!} - \frac{3^7\theta^7}{7!} + \cdots\right)$$

$$= \frac{3^3 - 3}{4.3!}.\theta^3 - \frac{3^5 - 3}{4.5!}.\theta^5 + \frac{3^7 - 3}{4.7!}.\theta^7 - \cdots$$

$$= \frac{3}{4}\left[\frac{\left(3^2 - 1\right)}{3!}.\theta^3 - \frac{\left(3^4 - 1\right)}{5!}.\theta^5 + \frac{\left(3^6 - 1\right)}{7!}.\theta^7 - \cdots\right]$$

$$= \frac{3\left(3^2 - 1\right)}{4}\left[\frac{\theta^3}{3!} - \left(3^2 + 1\right)\frac{.\theta^5}{5!} + \left(3^4 + 3^2 + 1\right)\frac{\theta^7}{7!} - \cdots\right]$$

Hence the result.

Ans. to Q. No. 8: We know that $\cos\alpha = 1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \frac{\alpha^6}{6!} + \cdots$

If $\alpha = \frac{\pi}{2}$, we have $\cos\frac{\pi}{2} = 1 - \frac{\pi^2}{2^2 2!} + \frac{\pi^4}{2^4 4!} - \frac{\pi^6}{2^6 6!} + \cdots$

$$\text{or } 0 = 1 - \frac{\pi^2}{2.4} + \frac{\pi^4}{2.4.6.8} - \frac{\pi^6}{2.4.6.8.10.12} + \cdots$$

$$\because 2^2(2!) = (2.2)(1.2) = 2.4$$

$$2^4 4! = (2.2.2.2)(4.3.2.1) = 2.4.6.8$$

$$\text{or } \frac{\pi^2}{2.4} - \frac{\pi^4}{2.4.6.8} + \frac{\pi^6}{2.4.6.8.10.12} - \cdots = 1$$

Hence the result.

11.12 FURTHER READING

- University Algebra, N S Gopala Krishnan, New Age International (P) Limited.
- Trigonometry, for B. A/B.Sc. Classes (1st Edition), N.P. Bali: Laxmi Publication (P) Ltd.
- Trigonometry; (1st Edition), Hari Kishan: Atlantic Publishers.
- Linear Algebra: Schaum's Solved Problems Series; Seymour Lipschitz: Tata McGraw Hill.
- Text Book of Algebra, A.K Sharma, DPH, Mathematics Series.
- Simplified Course in Trigonometry, M D Raisinghania,HK Das,H C Sexana, S Chand & Company Ltd.

11.13 MODEL QUESTIONS

Q.1: Prove that,

$$\sin 6\theta = 6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta$$

Q.2: Prove that,

$$\tan 8\theta = \frac{8\tan\theta - 56\tan^3\theta + 56\tan^5\theta - 8\tan^7\theta}{1 - 28\tan^2\theta + 70\tan^4\theta - 28\tan^6\theta + \tan^8\theta}.$$

Q.3: If $\frac{\sin\theta}{\theta} = \frac{863}{864}$, Show that θ is nearly $4^0 47'$.

Q.4: If $\tan\theta = \frac{1}{15}$, Show that θ is nearly $\frac{42}{11}$ degrees.

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