UNIT 8: SUCCESSIVE DIFFERENTIATIONS

UNIT STRUCTURE

8.1 Learning Objectives

8.2 Introduction

8.3 The n^{th} Derivatives of Some Standard Functions

8.4 Leibnitz's Theorem for the n^{th} Derivative of the Product of two Functions

8.5 Let Us Sum Up

8.6 Answers to Check Your Progress

8.7 Further Reading

8.8 Model Questions

8.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- understand that the functions and their derivatives of different orders exist on their respective domains
- find n^{th} derivative of some standard functions
- differentiate product to two functions upto desired order.

8.2 INTRODUCTION

The derived function of a differentiable function f(x) itself may be a differentiable function. In that case, we can find the derivative of $f'(x)$ at any variable point x and this derivative, denoted by $f''(x)$, is called the second derivativeof f(x). So, $f''(x)$ is the derivative of the derivative of f(x). If f(x) is denoted by y then $f''(x)$ is denoted by $\frac{d^2y}{dx^2}$ or y_2.

If this second derivative of f(x) is also differentiable function, we can again differentiate $f''(x)$ to get the third derivative denoted by $f'''(x)$ or $\frac{d^3y}{dx^3}$ or y_3 or $f^{(3)}(x)$.

This process can be continued provided the derivative is again differentiable each time. In general, we have n^{th} derivative of f(x) denoted by $\frac{d^ny}{dx^n}$ or $f^n(x)$ or y_n.

A function f(x) is said to be differentiable n times in an interval if the successive derivatives $f'(x)$, $f''(x)$, ..., $f^n(x)$ exist at every point of the interval.

Polynomial functions are differentiable any number of times.

8.3 THE NTH DERIVATIVE OF SOME STANDARD FUNCTIONS

1) If $y = (ax+b)^m$, then $y_1 = ma(ax+b)^{m-1}$

$y_2 = m(m-1)a^2(ax+b)^{m-2}$

$y_3 = m(m-1)(m-2)a^3(ax+b)^{m-3}$

In general,

$y_n = m(m-1)(m-2)....\{m-(n-1)\}a^n\ (ax+b)^{m-n}$

$y = (ax+b)^m \Rightarrow y_n = m(m-1)(m-2).....\{m-(n-1)\}a^n\ (ax+b)^{m-n}$

Note : Putting m = –1, We get,

$y_n = (-1)(-1-1)(-1-1) \ldots (-1-(n-1))a^n(ax+b)^{-1-n}$

$= (-1)^n n!a^n(ax+b)^{-(n+1)}$

$$= \frac{(-1)^n n!a^n}{(ax+b)^{n+1}}$$

$$y = \frac{1}{ax+b} \Rightarrow y_n = \frac{(-1)^n n!a^n}{(ax+b)^{n+1}}$$

2) If $y = x^m$, then $y_1 = mx^{m-1}$

$y_2 = m(m-1)x^{m-2}$

In general, $y_n = m(m-1)(m-2) \ldots \ldots (m-n+1)x^{m-n}$

$y = x^m \Rightarrow y_n = m\ (m-1)(m-2) \ldots \ldots (m-n+1)x^{m-n}$

Note: a) If n > m, then $y_n = 0$

b) If n = m, then $y_n = n!$

3) If $y = e^{ax}$, then $y_1 = ae^{ax}$

$y_2 = a^2e^{ax}$

In general, $y_n = a^ne^{ax}$

$y = e^{ax} \Rightarrow y_n = a^ne^{ax}$

Note: If $y = e^x$, then $y_n = e^x$.

4) If y = sin(ax+b), then,

$$y_1 = a\cos(ax+b) = a\sin\left\{\frac{\pi}{2}+(ax+b)\right\}$$

$$y_2 = a^2\cos\left\{\frac{\pi}{2}+(ax+b)\right\} = a^2\sin\left\{2.\frac{\pi}{2}+(ax+b)\right\}$$

$$y_3 = a^3\cos\left\{2.\frac{\pi}{2}+(ax+b)\right\} = a^3\sin\left\{3.\frac{\pi}{2}+(ax+b)\right\}$$

In general, $y_n = a^n \sin\left\{n\frac{\pi}{2}+(ax+b)\right\}$

$$y = \sin(ax+b) \Rightarrow y_n = a^n\sin\left\{n\frac{\pi}{2}+(ax+b)\right\}$$

Note: Putting a = 1, b = 0, We get, $y = \sin x \Rightarrow y_n = \sin\left(\frac{n\pi}{2}+x\right)$

5) If y = cos(ax+b), then,

$$y_1 = -a\sin(ax+b) = a\cos\left\{\frac{\pi}{2}+(ax+b)\right\}$$

$$y_2 = -a^2\sin\left\{\frac{\pi}{2}+(ax+b)\right\} = a^2\cos\left\{2.\frac{\pi}{2}+(ax+b)\right\}$$

$$y_3 = -a^3\sin\left\{2\frac{\pi}{2}+(ax+b)\right\} = a^3\cos\left\{3.\frac{\pi}{2}+(ax+b)\right\}$$

In general, $y_n = a^n\cos\left\{n\frac{\pi}{2}+(ax+b)\right\}$

$$y = \cos(ax+b) \Rightarrow y_n = a^n\cos\left\{n\frac{\pi}{2}+(ax+b)\right\}$$

Note: Putting a = 1, b = 0, We get, $y = \cos x \Rightarrow y_n = \cos\left(\frac{n\pi}{2}+x\right)$

6) If y = log(ax+b), then, $y_1 = \frac{a}{ax+b} = a(ax+b)^{-1}$

$$y_2 = (-1)a^2(ax+b)^{-2}$$

$$y_3 = (-1)(-2)\,a^3(ax+b)^{-3}$$

In general, $y_n = (-1)(-2)\ldots\ldots \{-(n-1)\}\, a^n(ax+b)^{-n}$

$$= (-1)^{n-1}\,\underline{\lfloor n-1}\; a^n\,(ax+b)^{-n}$$

$$y = \log(ax+b) \Rightarrow y_n = (-1)^{n-1}\,\underline{\lfloor n-1}\; a^n(ax+b)^{-n}$$

7) If $y = e^{ax} \sin bx$, then, $y_1 = ae^{ax} \sin bx + be^{ax} \cos bx$

Let $a = r\cos\theta$, $b = r\sin\theta$

$\therefore r = \sqrt{a^2 + b^2}$ and $\tan\theta = \frac{b}{a}$

$\therefore \quad y_1 = r\cos\theta . e^{ax} \sin bx + r\sin\theta . e^{ax} \cos bx$

$\Rightarrow y_1 = re^{ax}(\cos\theta \sin bx + \sin\theta \cos bx)$

$\Rightarrow y_1 = re^{ax}\sin(bx+\theta)$

$\therefore \quad y_2 = r[ae^{ax} \sin(bx+\theta) + be^{ax} \cos(bx+\theta)]$

$= re^{ax}[r\cos\theta\sin(bx+\theta) + r\sin\theta \cos(bx+\theta)]$

$= r^2e^{ax} \sin(bx+\theta+\theta)$

$= r^2e^{ax} \sin(bx+2\theta)$

In general, $y_n = r^n e^{ax} \sin(bx+n\theta)$

$$= \left(a^2 + b^2\right)^{\frac{n}{2}} e^{ax} \sin\left(bx + n \tan^{-1}\frac{b}{a}\right)$$

$y = e^{ax} \sin bx$

$$\Rightarrow y_n = \left(a^2 + b^2\right)^{\frac{n}{2}} e^{ax} \sin\left(bx + n \tan^{-1}\frac{b}{a}\right)$$



CHECK YOUR PROGRESS

Q.1: Find the n^{th} differential coefficients of the following with respect to x:

i) $y = \sin x$ ii) $y = \cos x$ iii) $y = e^{ax} \cos bx$

iv) $y = \frac{1}{ax+b}$ v) $y = e^x \sin x$ vi) $y = \log x$.

ILLUSTRATIVE EXAMPLES:

Example 1: Find the n^{th} differential coefficients of the following with respect to x.

a) $y = \frac{1}{x^2 - 16}$ b) $y = \frac{1-x}{1+x}$

Solution: a) $y = \frac{1}{x^2 - 16}$

$$\Rightarrow y = \frac{1}{(x-4)(x+4)} = \frac{1}{8}\left[\frac{1}{x-4} - \frac{1}{x+4}\right]$$

$$\therefore y_n = \frac{1}{8}\left[\frac{(-1)^n n!.1^n}{(x-4)^{n+1}} - \frac{(-1)^n n!.1^n}{(x+4)^{n+1}}\right]$$

$$\left[\because y = \frac{1}{ax+b} \Rightarrow y_n = \frac{(-1)^n n!.a^n}{(ax+b)^{n+1}}\right]$$

$$\Rightarrow y_n = \frac{1}{8}\left[\frac{(-1)^n n!}{(x-4)^{n+1}} - \frac{(-1)^n n!}{(x+4)^{n+1}}\right] = \frac{(-1)^n n!}{8}\left[\frac{1}{(x-4)^{n+1}} - \frac{1}{(x+4)^{n+1}}\right]$$

b) $y = \dfrac{1-x}{1+x} = \dfrac{2-(1+x)}{1+x}$

$$= \frac{2}{1+x} - \frac{1+x}{1+x}$$

$$= 2.\frac{1}{1+x} - 1$$

$$\therefore y_n = 2.\frac{(-1)^n n!.1^n}{(1+x)^{n+1}} - 0 = \frac{2.(-1)^n n!}{(1+x)^{n+1}}$$

Example 2: Find the n^{th} derivative of

a) $y = \sin 3x \cos 5x$ b) $y = \tan^{-1}x$

c) $y = \cos^2 x$ d) $y = \cos^4 x$

Solution: a) $y = \sin 3x \cos 5x$

$$\Rightarrow y = \frac{1}{2}(2\sin 3x \cos 5x)$$

$$\Rightarrow y = \frac{1}{2}[\sin 8x - \sin 2x]$$

$$\therefore y_n = \frac{1}{2}\left[8^n \sin\left(n\frac{\pi}{2} + 8x\right) - 2^n \sin\left(n\frac{\pi}{2} + 2x\right)\right]$$

$$= 2^{n-1}\left[2^{2n}\sin\left(n\frac{\pi}{2} + 8x\right) - \sin\left(n\frac{\pi}{2} + 2x\right)\right]$$

b) $y = \tan^{-1}x \Rightarrow y_1 = \dfrac{1}{1+x^2}$

$$\Rightarrow y_1 = \frac{1}{(x-i)(x+i)}$$

$$\Rightarrow y_1 = \frac{1}{2i}\left(\frac{1}{x-i} - \frac{1}{x+i}\right)$$

$$\Rightarrow y_n = \frac{1}{2i}\left[\frac{(-1)^{n-1}(n-1)!}{(x-i)^n} - \frac{(-1)^{n-1}(n-1)!}{(x+i)^n}\right]$$

$$\Rightarrow y_n = \frac{(-1)^{n-1}(n-1)!}{2i}\left[\frac{1}{(x-i)^n} - \frac{1}{(x+i)^n}\right]$$

Let $x = r\cos\theta$ and $1 = r\sin\theta$

$\therefore r = \sqrt{x^2+1}$ and $\tan\theta = \frac{1}{x}$

Then, $\frac{1}{(x-i)^n} = (r\cos\theta - ir\sin\theta)^{-n}$

$= r^{-n}(\cos\theta - i\sin\theta)^{-n}$

$= r^{-n}(\cos n\theta + i\sin n\theta)$, by De-Moivre's theorem

And, $\frac{1}{(x+i)^n} = (r\cos\theta + ir\sin\theta)^{-n}$

$= r^{-n}(\cos\theta + i\sin\theta)^{-n}$

$= r^{-n}(\cos n\theta - i\sin n\theta)$

$$\therefore y_n = \frac{(-1)^{n-1}(n-1)!}{2i}\left[r^{-n}(\cos n\theta + i\sin n\theta) - r^{-n}(\cos n\theta - i\sin n\theta s\right]$$

$$= \frac{(-1)^{n-1}(n-1)!}{2i}\left[2ir^n \sin n\theta\right]$$

$$= (-1)^n(n-1)!\ r^{-n}\sin n\theta$$

$$= \frac{(-1)^n(n-1)!}{(x^2+1)^{\frac{n}{2}}}.\sin\left(n\tan^{-1}\frac{1}{x}\right)$$

c) $y = \cos^2 x \Rightarrow y = \frac{1}{2}(1+\cos 2x)$

$$\therefore y_n = \frac{1}{2}\left[0 + 2^n\cos\left(n\frac{\pi}{2} + 2x\right)\right]$$

$$= 2^{n-1}\cos\left(n\frac{\pi}{2} + 2x\right)$$

d) $y = \cos^4 x = (\cos^2 x)^2$

$$= \left(\frac{1+\cos 2x}{2}\right)^2$$

$$= \frac{1}{4}\left[1 + 2\cos 2x + \cos^2 2x\right]$$

$$= \frac{1}{4}\left[1 + 2\cos 2x + \frac{1+\cos 4x}{2}\right]$$

$$= \frac{1}{8}\left[3 + 4\cos 2x + \cos 4x\right]$$

$$\therefore y_n = \frac{1}{8}\left[0 + 4.2^n \cos\left(n\frac{\pi}{2} + 2x\right) + 4^n \cos\left(n\frac{\pi}{2} + 4x\right)\right]$$

$$= \frac{1}{2^3}\left[2^{n+2}\cos\left(n\frac{\pi}{2} + 2x\right) + 2^{2n}\cos\left(n\frac{\pi}{2} + 4x\right)\right]$$

$$= 2^{n-1}\cos\left(n\frac{\pi}{2} + 2x\right) + 2^{2n-3}\cos\left(n\frac{\pi}{2} + 4x\right)$$

Example 3: If y = sinnx + cosnx, show that $y_r = n^r\left\{1 + (-1)^r \sin 2nx\right\}^{\frac{1}{2}}$

Solution: Given, y = sin nx + cosnx

$$\therefore y_r = n^r \sin\left(r\frac{\pi}{2} + nx\right) + n^r \cos\left(r\frac{\pi}{2} + nx\right)$$

$$\Rightarrow y_r = n^r\left[\left\{\sin\left(r\frac{\pi}{2} + nx\right) + \cos\left(r\frac{\pi}{2} + nx\right)\right\}^2\right]^{\frac{1}{2}}$$

$$= n^r\left[\sin^2\left(r\frac{\pi}{2} + nx\right) + \cos^2\left(r\frac{\pi}{2} + nx\right) + 2\sin\left(r\frac{\pi}{2} + nx\right)\cos\left(r\frac{\pi}{2} + nx\right)\right]^{\frac{1}{2}}$$

$$= n^r\left[1 + \sin\left\{2\left(r\frac{\pi}{2} + nx\right)\right\}\right]^{\frac{1}{2}}$$

$$= n^r\left[1 + \sin(r\pi + 2nx)\right]^{\frac{1}{2}}$$

$$= n^r\left[1 + (-1)^r \sin 2nx\right]^{\frac{1}{2}}$$

Example 4: Show that the value of the n^{th} differential coefficient of $\frac{x^3}{x^2-1}$ for x = 0 is 0 or –n! according as n is even or odd and n > 1.

Solution: Let $y = \frac{x^3}{x^2-1} = x + \frac{x}{x^2-1}$

$$= x + \frac{x}{(x-1)(x+1)}$$

$$= x + \frac{1}{2(x-1)} + \frac{1}{2(x+1)}$$

$\therefore$ When n > 1, we have $y_n = 0 + \frac{1}{2}.\frac{(-1)^n n!}{(x-1)^{n+1}} + \frac{1}{2}.\frac{(-1)^n n!}{(x+1)^{n+1}}$

$$\Rightarrow y_n = \frac{(-1)^n n!}{2}\left[\frac{1}{(x-1)^{n+1}} + \frac{1}{(x+1)^{n+1}}\right]$$

$$\therefore (y_n)_0 = \frac{(-1)^n n!}{2!}\left[\frac{1}{(-1)^{n+1}} + \frac{1}{1}\right]$$

$$= \frac{(-1)^n n!}{2}\left[(-1)^{n+1} + 1\right]$$

$$= \begin{cases} \frac{n!}{2}(-1+1), \text{if n is even} \\ \frac{-n!}{2}(1+1), \text{if n is odd} \end{cases}$$

$$= \begin{cases} 0, \text{ if n is even} \\ -n!, \text{ if n is odd} \end{cases}$$



CHECK YOUR PROGRESS

Q.2: Find the nth differential coefficient of:

a) $\frac{1}{x+2}$ b) $\frac{1}{x^2-4}$ c) $\frac{1}{x^2-1}$

Q.3: Find y_n if:

a) $y = \cos 3x \cos 7x$ b) $y = \sin^2 x$

c) $y = \tan^{-1}\frac{1-x}{1+x}$ d) $y = \log(2x+5)$

e) $y = e^x \cos 3x$

Q.4: If $y = (ax+b)^m$ and n is a positive interger less than m, then prove that $y_n = \frac{m!}{(m-n)!}a^n(ax+b)^{m-n}$

Also state the condition for which:

a) y_n vanishes b) y_n becomes a constant.

8.4 LEIBNITZ'S THEOREM

Let u and v be any two functions of x derivable n times, then

$(uv)_n = u_n v + {}^nc_1\, u_{n-1}v_1 + {}^nc_2\, u_{n-2}v_2 + \ldots + {}^nc_r\, u_{n-r}v_r + \ldots + uv_n$

Proof: We shall prove the theorem by mathematical induction.

By actual differentiation, we have

$(uv)_1 = u_1v + uv_1$... ... ... (1)

From (1) we see that the theorem holds for n = 1.

Let us assume that the theorem holds for n = m; m > 1, $m \in N$.

Then,

$$(uv)_m = u_mv + {}^mc_1u_{m-1}v_1 + {}^mc_2u_{m-2}v_2 + ... + {}^mc_{r-1}u_{m-r+1}v_{r-1} + {}^mc_ru_{m-r}v_r + ... + uv_m \quad ... ... (2)$$

Differentiavting both sides of (2) with respect to x, we get,

$$(uv)_{m+1} = (u_{m+1}v+u_mv_1) + {}^mc_1(u_mv_1 + u_{m-1}v_2) + {}^mc_2(u_{m-1}v_2 + u_{m-2}v_3) + ... + {}^mc_{r-1}(u_{m-r+2}v_{r-1} + u_{m-r+1}v_r) + {}^mc_r(u_{m-r+1}v_r + u_{m-r}v_n) + ... + (u_1v_m + uv_{m+1})$$

$$= u_{m+1}v + (1+{}^mc_1)u_mv_1 + ({}^mc_1+{}^mc_2)u_{m-1}v_2 + ... + ({}^mc_{r-1}+{}^mc_r)u_{m-r+1}v_r + ... + uv_{m+1}$$

$$= u_{m+1}v + {}^{m+1}c_1u_mv_1 + {}^{m+1}c_2u_{m-1}v_2 + ... + {}^{m+1}c_ru_{m-r+1}v_r + ... + uv_{m+1}$$

$[\because {}^mc_0 = 1,\ {}^mc_{r-1} + {}^mc_r = {}^{m+1}c_r]$

$\therefore$ The theorem holds for n = m + 1 if it is true for n = m.

But the theorem is proved to be true for n = 1. Hence, it must be true for n = 2 and so for n = 3 and so on.

Hence the theorem is true for all positive integral values of n.

Note: Gottfried Wilhelm Leibnitz (1646-1716) was a celebrated mathematician.



Gottfried W
Leibnitz (1
was a Ger
mathemati
philosophe

ILLUSTRATIVE EXAMPLES:

Example 1: If $y = x^n\log x$, show that $xy_{n+1} = n!$

Solution: Given, $y = x^n\log x$

$$\therefore\ y_1 = x^n\frac{1}{x} + nx^{n-1}\log x$$

$$\Rightarrow xy_1 = x^n + nx^n\log x$$

$$\Rightarrow xy_1 = x^n + ny$$

Differentiating n times using Leibnitz's theorem, we get

$$xy_{n+1} + {}^nc_1.1.y_n = n! + ny_n$$

$$\Rightarrow xy_{n+1} + ny_n = n! + ny_n$$

$$\Rightarrow xy_{n+1} = n!$$



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Example 2: If y = a cos(logx) + b sin (logx), show that

i) $x^2y_2 + xy_1 + y = 0$

ii) $x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$

Solution: Given, y = a cos(logx) + b sin (logx) ... ... (1)

$$\therefore y_1 = -a\frac{\sin(\log x)}{x} + \frac{b\cos(\log x)}{x}$$

$\Rightarrow xy_1 = -a\sin(\log x) + b\cos(\log x)$ ... ... (2)

Differentiating (2) w.r. to x, we get,

$$xy_2 + 1.y_1 = -\frac{a\cos(\log x)}{x} - \frac{b\sin(\log x)}{x}$$

$\Rightarrow x^2y_2 + xy_1 = -[a\cos(\log x) + b\sin(\log x)]$

$\Rightarrow x^2y_2 + xy_1 = -y$, by (1)

$\Rightarrow x^2y_2 + xy_1 + y = 0$, which gives (i).

Differentating (1) n times using Leibnitz's theorem, we get,

$$(x^2y_{n+2} + {}^nc_1.2xy_{n+1} + {}^nc_2.2.y_n) + (xy_{n+1} + {}^nc_1.1.y_n) + y_n = 0$$

$$\Rightarrow x^2y_{n+2} + 2nxy_{n+1} + \frac{n(n-1)}{2!}.2y_n + xy_{n+1} + ny_n + y_n = 0$$

$\Rightarrow x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2-n+n+1)y_n = 0$

$\Rightarrow x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$

Example 3: If $y = e^{a\sin^{-1}x}$, Prove that,

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0.$$

Solution: Given $y = e^{a\sin^{-1}x} \Rightarrow \log y = a\sin^{-1}x$

Differentiating w.r.to x, we get, $\frac{1}{y}.y_1 = a.\frac{1}{\sqrt{1-x^2}}$

$$\Rightarrow \frac{y_1^2}{y^2} = \frac{a^2}{1-x^2}$$

$$\Rightarrow (1-x^2)y_1^2 = a^2y^2$$

Differentiating again, w.r. to x we get,

$(1-x^2)2\,y_1y_2 + (-2x)y_1^2 = a^2.2yy_1 \Rightarrow (1-x^2)y_2 - xy_1 = a^2y$

Differentiating n times using Leibnitz's theorem, we get,

$$\{(1-x^2)y_{n+2} + {}^n c_1(-2x)y_{n+1} + {}^n c_2(-2)y_n\} - \{xy_{n+1} + {}^n c_1.1.y_n\} = a^2y_n$$

$$\Rightarrow (1-x^2)y_{n+2} - 2nxy_{n+1} - 2.\frac{n(n-1)}{2!}y_n - xy_{n+1} - ny_n = a^2y_n$$

$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-n+n+a^2)y_n = 0$

$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0$

Example 4: If $y = \sin(m \sin^{-1}x)$, then prove that

i) $(1-x^2)y_2 - xy_1 + m^2y = 0$

ii) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-m^2)y_n = 0$

Also find $y_n(0)$

Solution: Given, $y = \sin(m \sin^{-1}x)$ ... ... (1)

$\Rightarrow \sin^{-1}y = m\sin^{-1}x$

$\Rightarrow \frac{1}{\sqrt{1-y^2}}.y_1 = m.\frac{1}{\sqrt{1-x^2}}$ (on differentiation)

$\Rightarrow (1-x^2)y_1^2 = m^2(1-y^2)$ ... ... (2)

Differentiating again, w.r. to x we get,

$(1-x^2)2y_1y_2 + (-2x)y_1^2 = m^2(0-2y\,y_1)$

$\Rightarrow (1-x^2)y_2 - xy_1 = -m^2y$

$\Rightarrow (1-x^2)y_2 - xy_1 + m^2y = 0$ ... ... (3)

This gives (i)

Differentiating (3) n times using Leibnitz's theorem we get,

$$\left\{(1-x^2)y_{n+2} + {}^n c_1(-2x)y_{n+1} + {}^n c_2(-2)y_n\right\} - \left\{xy_{n+1} + {}^n c_1.1.y_n\right\} + m^2y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - 2nxy_{n+1} - 2.\frac{n(n-1)}{2!}y_n - xy_{n+1} - ny_n + m^2y_n = 0$$

$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-n+n-m^2)y_n = 0$

$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-m^2)y_n = 0$ ... ... (4)

This gives (ii),

Putting x = 0, we get, 1) $\Rightarrow y(0) = 0$

2) $\Rightarrow y_1(0) = m$

3) $\Rightarrow y_2(0) = 0$

4) $\Rightarrow y_{n+2}(0) = (n^2-m^2)y_n(0)$ ... ... (5)

Putting n = 1, 2, 3, ... in (5), we get,

$y_3(0) = (1^2-m^2)y_1(0) = m(1^2-m^2)$

$y_4(0) = (2^2-m^2)y_2(0) = 0$

$y_5(0) = (3^2-m^2)y_3(0) = m(1^2-m^2)(3^2-m^2)$

$y_6(0) = (4^2-m^2)\,y_4(0) = 0$

$y_7(0) = (5^2-m^2)y_5(0) = m(1^2-m^2)\,(3^2-m^2)\,(5^2-m^2)$ and so on

Thus, we get, $y_n(0)=0$, if n is even and

$$y_n(0) = m(1^2-m^2)(3^2-m^2)(5^2-m^2) \ldots \ldots\{(n-2)^2-m^2\}$$

Example 5: If $y = (\sin^{-1}x)^2$, Prove that

$$(1-x^2)y_{n+2} + (2n+1)xy_{n+1} - n^2y_n = 0$$

Solution: Given, $y = (\sin^{-1}x)^2$ ... ... (1)

$$\Rightarrow y_1 = 2\sin^{-1}x.\frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \sqrt{1-x^2}.y_1 = 2\sin^{-1}x$$

$$\Rightarrow (1-x^2)y_1^2 = 4(\sin^{-1}x)^2$$

$$\Rightarrow (1-x^2)y_1^2 = 4y, \text{ by (1)}$$

Differentiating w.r. to x, we get, $(1-x^2)2y_1y_2 + (-2x)y_1^2 = 4y_1$

$$\Rightarrow (1-x^2)y_2 - xy_1 = 2$$

Differentiating n times using Leibritz's theorem, we get,

$$\left\{(1-x^2)y_{n+2} + {}^nc_1(-2x)y_{n+1} + {}^nc_2(-2)y_n\right\} - \left\{xy_{n+1} + {}^nc_1.1.y_n\right\} = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - 2nxy_{n+1} - 2.\frac{n(n-1)}{2!}.y_n - xy_{n+1} - ny_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-n+n)y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0$$

Example 6: If $I_n = \frac{d^n}{dx^n}(x^n \log x)$, show that $I_n = n I_{n-1} + (n-1)!$

Hence show that $I_n = n!\left(\log x + 1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots + \frac{1}{n}\right)$

Solution: $I_n = \frac{d^n}{dx^n}\left(x^n \log x\right)$

$$= \frac{d^{n-1}}{dx^{n-1}}\left\{\frac{d}{dx}\left(x^n \log x\right)\right\}$$

$$= \frac{d^{n-1}}{dx^{n-1}}\left(x^n.\frac{1}{x} + nx^{n-1}\log x\right)$$

$$= \frac{d^{n-1}}{dx^{n-1}}\left(x^{n-1}\right) + n\frac{d^{n-1}}{dx^{n-1}}\left(x^{n-1}\log x\right)$$

$$= (n-1)! + nI_{n-1}$$

$$\Rightarrow I_n = n I_{n-1} + (n-1)! \quad \ldots \ldots \ldots (1)$$

Second Part:

1) $\Rightarrow \frac{I_n}{n!} = \frac{nI_{n-1}}{n!} + \frac{(n-1)!}{n!}$

$\Rightarrow \frac{I_n}{n!} = \frac{I_{n-1}}{(n-1)!} + \frac{1}{n}$ ... ... (2)

Replacing n by n–1 in (2), we get, $\frac{I_{n-1}}{(n-1)!} = \frac{I_{n-2}}{(n-2)!} + \frac{1}{(n-1)}$

$\therefore$ (2) $\Rightarrow \frac{I_n}{n!} = \frac{I_{n-2}}{(n-2)!} + \frac{1}{n-1} + \frac{1}{n}$

Proceeding in this way and making use of (2), we get,

$\frac{I_n}{n!} = \frac{I_1}{1!} + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}$

$\Rightarrow I_n = n!\left(I_1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}\right)$ ... ... (3)

Now $I_1 = \frac{d}{dx}(x\log x)$

$= x.\frac{1}{x} + 1.\log x$

$= 1 + \log x$

$\therefore (3) \Rightarrow I_n = n!\left(\log x + 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}\right)$

Example 7: If $y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$, Prove that

$(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$

Solution: Given, $y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$ ... ... (1)

$\Rightarrow y^{\frac{2}{m}} + 1 = 2xy^{\frac{1}{m}}$

$\Rightarrow y^{\frac{2}{m}} - 2xy^{\frac{1}{m}} + 1 = 0$

$\Rightarrow y^{\frac{1}{m}} = \frac{2x \pm \sqrt{4x^2 - 4}}{2}$

$\Rightarrow y^{\frac{1}{m}} = \frac{2x \pm 2\sqrt{x^2 - 1}}{2}$

$\Rightarrow y = \left(x \pm \sqrt{x^2 - 1}\right)^m$ ... ... (2)

$$\therefore y_1 = m\left(x \pm \sqrt{x^2-1}\right)^{m-1}\left(1 \pm \frac{1}{2\sqrt{x^2-1}}(2x)\right)$$

$$\Rightarrow y_1 = \frac{m\left(x \pm \sqrt{x^2-1}\right)^m}{x \pm \sqrt{x^2-1}} \cdot \frac{\sqrt{x^2-1} \pm x}{\sqrt{x^2-1}}$$

$$\Rightarrow y_1 = \pm \frac{m\,y}{\sqrt{x^2-1}}, \text{ by(2)}$$

$\Rightarrow (x^2-1)y_1^2 = m^2y^2$

Differentiating w.r. to x, we get, $(x^2-1)2y_1y_2 + 2xy_1^2 = m^2.2y\,y_1$

$$\Rightarrow (x^2-1)y_2 + xy_1 = m^2y$$

Differentiating n times using Leibnitz's theorem, we get,

$\{(x^2-1)y_{n+2} + {}^nc_1.2x.y_{n+1} + {}^nc_2.2.y_n\} + (xy_{n+1} + {}^nc_1.1.y_n\} = m^2y_n$

$\Rightarrow (x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-n+n-m^2)y_n = 0$

$\Rightarrow (x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$

Example 8: If $\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{n}\right)^n$, Prove that

$$x^2y_{n+2} + (2n+1)xy_{n+1} + 2n^2y_n = 0$$

Solution: Given, $\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{n}\right)^n$ ... ... (1)

$$\Rightarrow \cos^{-1}\left(\frac{y}{b}\right) = n\left(\log x - \log n\right)$$

Differentiating w.r. to x, we get, $\frac{-1}{\sqrt{1-\frac{y^2}{b^2}}} \cdot \frac{y_1}{b} = n\left(\frac{1}{x} - 0\right)$

$$\Rightarrow \frac{-y_1}{\sqrt{b^2-y^2}} = \frac{n}{x}$$

$$\Rightarrow x^2y_1^2 = n^2(b^2-y^2)$$

Differentiating w.r. to x, we get, $x^2.2y_1y_2 + 2xy_1^2 = n^2(0-2y\,y_1)$

$$\Rightarrow x^2y_2 + xy_1 = -n^2y$$

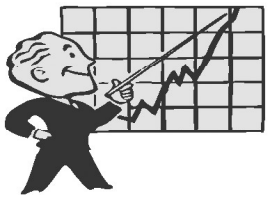
Differentiating n times using Leibnitz's theorem, we get,

$(x^2y_{n+2} + {}^nc_1.2xy_{n+1} + {}^nc_2.2.y_n) + (xy_{n+1} + {}^nc_1.1.y_n) = -n^2y_n$

$$\Rightarrow x^2y_{n+2} + 2nxy_{n+1} + 2.\frac{n(n-1)}{L2}y_n + xy_{n+1} + ny_n + n^2y_n = 0$$

$\Rightarrow x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2–n+n+n^2)y_n = 0$

$\Rightarrow x^2y_{n+2} + (2n+1)xy_{n+1} + 2n^2y_n = 0$



CHECK YOUR PROGRESS

Q.5: Find y_{n+1} using Leibnitz's theorem if $y = x^n\log x$.

Q.6: If $y = \cos(\log x)$, Prove that

$x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$

Q.7: If $y = (x^2–1)^n$, Prove that $(x^2–1)y_{n+2} + 2xy_{n+1}–n(n+1)y_n = 0$

Q.8: If $y = e^{\tan^{-1}x}$, Prove that

$(1+x^2)y_{n+2} + [2(n+1)x–1]y_{n+1} + n(n+1)y_n = 0$

Q.9: If $y = \tan^{-1}x$, Prove that

$(1+x^2)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n = 0$

Hence evaluate $y_n(0)$.

Q.10: If $y = \left(\log\left(x + \sqrt{1+x^2}\right)\right)^2$, Prove that

$(1+x^2)y_{n+2} + 2(n+1)xy_{n+1} + n^2y_n = 0$

Show that $(y_{n+2})_0 = –n^2(y_n)_0$

Q.11: If $y = e^x\log x$, Prove that

i) $xy_2 – (2x–1)y_1 + (x–1)y = 0$

ii) $xy_{n+2} + (n–2x+1)y_{n+1} – (2n–x+1)y_n + ny_{n–1} = 0$

Q.12: If $y = \dfrac{\sin^{-1}x}{\sqrt{1-x^2}}$, Prove that

$(1–x^2)y_{n+2} – (2n+3)xy_{n+1} – (n+1)^2y_n = 0$

Q.13: If $x = \tan(\log y)$, Prove that

$(1+x^2)y_{n+1} + (2nx–1)y_n + n(n–1)y_{n–1} = 0$



8.5 LET US SUM UP

- The nth derivative of y is obtained by differentiating a function $y = f(x)$, n times successively. The nth derivative of y is denoted by

$y^{(n)}(x)$, $f^{(n)}(x)$, $D^{(n)}y$, $\dfrac{d^ny}{dx^n}$ or $y_n(x)$.

- The nth derivatives of some standard functions are given below.

  a) For $y = (ax+b)^m$, $y_n = m(m-1)(m-2)\ldots\ldots\{m-(n-1)\}a^n (ax+b)^{m-n.}$

  b) For $y = x^m$, $y_n = m(m-1)(m-2)\ldots\ldots(m-n+1)x^{m-n}$

  d) For $y = \sin(ax+b)$, $y_n = a^n \sin\left\{n\frac{\pi}{2} + (ax+b)\right\}$

  e) For $y = \cos(ax+b)$, $y_n = a^n \cos\left\{n\frac{\pi}{2} + (ax+b)\right\}$

  f) For $y = \log(ax+b)$, $y_n = (-1)^{n-1} \lfloor n-1 \; a^n (ax+b)^{-n}$

  g) For $y = e^{ax} \sin bx$, $y_n = \left(a^2+b^2\right)^{\frac{n}{2}} e^{ax} \sin\left(bx + n\tan^{-1}\frac{b}{a}\right)$.

- Leibnitz's Theorem for the n^{th} derivatives of the product of two functions is as follows.

  Let u and v be any two functions of x derivable n times, then $(uv)_n =$ $u_n v + {}^nc_1 u_{n-1} v_1 + {}^nc_2 u_{n-2} v_2 + \ldots\ldots + {}^nc_r u_{n-r} v_r + \ldots\ldots + uv_n$



8.6 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1: i) Given, $y = \sin x$

$$y_1 = \cos x = \sin\left(\frac{\pi}{2} + x\right)$$

$$y_2 = \cos\left(\frac{\pi}{2} + x\right) = \sin\left(\frac{\pi}{2} + \frac{\pi}{2} + x\right) = \sin\left(2.\frac{\pi}{2} + x\right)$$

$$y_3 = \cos\left(2.\frac{\pi}{2} + x\right) = \sin\left(\frac{\pi}{2} + 2.\frac{\pi}{2} + x\right) = \sin\left(3.\frac{\pi}{2} + x\right)$$

In general, $y_n = \sin\left(n.\frac{\pi}{2} + x\right)$

ii) Given, $y = \cos x$

$$y_1 = -\sin x = \cos\left(\frac{\pi}{2} + x\right)$$

$$y_2 = -\sin\left(\frac{\pi}{2} + x\right) = \cos\left(\frac{\pi}{2} + \frac{\pi}{2} + x\right)$$

$$y_3 = -\sin\left(2.\frac{\pi}{2} + x\right) = \cos\left(\frac{\pi}{2} + 2.\frac{\pi}{2} + x\right) = \cos\left(3.\frac{\pi}{2} + x\right)$$

In general, $y_n = \cos\left(n.\frac{\pi}{2} + x\right)$

iii) Given, $y = e^{ax}\cos bx$

$y_1 = ae^{ax}\cos bx - be^{ax}\sin bx$ (Putting $a = r\cos\theta$ and $b = r\sin\theta$

$\therefore r = \sqrt{a^2 + b^2}$ and $\tan\theta = \frac{b}{a}$)

$= r\cos\theta\, e^{ax}\cos bx - r\sin\theta e^{ax}\sin bx$

$= re^{qx}\cos(bx+\theta)$

$y_2 = r[ae^{ax}\cos(bx+\theta) - be^{ax}\sin(bx+\theta)]$

$= re^{ax}[r\cos\theta\cos(bx+\theta) - r\sin\theta\sin(bx+\theta)]$

$= r^2e^{ax}[\cos\theta\cos(bx+\theta) - \sin\theta\sin(bx+\theta)]$

$= r^2e^{ax}\cos(bx+2\theta)$

In general, $y_n = r^n e^{ax}\cos(bx+n\theta)$

$$= \left(a^2 + b^2\right)^{\frac{n}{2}} e^{ax}\cos\left(bx + n\tan^{-1}\frac{b}{a}\right).$$

iv) Given, $y = \frac{1}{ax+b}$

$$y_1 = -\frac{1}{(ax+b)^2}$$

$$y_2 = \frac{(-1)(-2)}{(ax+b)^3}a^2 = \frac{(-1)^2 2!}{(ax+b)^3}a^2$$

$$y_3 = \frac{(-1)^2 2!(-3)}{(ax+b)^4}a^3 = \frac{(-1)^2 3!}{(ax+b)^4}a^3$$

In general, $y_n = \frac{(-1)^2 n!}{(ax+b)^{n+1}}a^n$.

v) Given, $y = e^x \sin x$

$y = e^x\sin x + e^x\cos x$

$$y_1 = \sqrt{2}e^x\left(\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right)$$

$$= \sqrt{2}e^x\sin\left(\frac{\pi}{4} + x\right)$$

$$= 2^{\frac{1}{2}}e^x\sqrt{2}\left[\frac{1}{\sqrt{2}}\sin\left(\frac{\pi}{4}+x\right) + \frac{1}{\sqrt{2}}\cos\left(\frac{\pi}{4}+x\right)\right]$$

$$= 2^{\frac{2}{2}}\left[e^x \cos\frac{\pi}{4}\sin\left(\frac{\pi}{4}+x\right) + e^x \sin\frac{\pi}{4}\cos\left(\frac{\pi}{4}+x\right)\right]$$

$$= 2^{\frac{2}{2}} e^x \sin\left(\frac{\pi}{4}+\frac{\pi}{4}+x\right)$$

$$= 2^{\frac{2}{2}} e^x \sin\left(2.\frac{\pi}{4}+x\right)$$

In general, $y_n = 2^{\frac{n}{2}} e^x \sin\left(n.\frac{\pi}{4}+x\right)$.

vi) Given, $y = \log x$

$$y_1 = \frac{1}{x}$$

$$y_2 = \frac{-1}{x^2}$$

$$y_3 = \frac{(-1)(-2)}{x^3} = \frac{(-1)^2 2!}{x^3}$$

$$y_4 = \frac{(-1)(-2)(-3)}{x^4} = \frac{(-1)^3 3!}{x^4}$$

In general, $y_n = \frac{(-1)^{n-1}(n-1)!}{x^n}$.

Ans. to Q. No. 2: a) Let $y = \frac{1}{x+2}$

$$\Rightarrow y_1 = \frac{-1}{(x+2)^2}$$

$$\Rightarrow y_2 = \frac{(-1)(-2)}{(x+2)^3} = \frac{(-1)^2 2!}{(x+2)^3}$$

Similarly, $y_3 = \frac{(-1)(-2)(-3)}{(x+2)^4} = \frac{(-1)^3 3!}{(x+2)^4}$

In general, $y_n = \frac{(-1)^n n!}{(x+2)^{n+1}}$.

b) Let $y = \frac{1}{x^2-4} \Rightarrow y = \frac{1}{(x+2)(x-2)}$

$$\Rightarrow y = \frac{1}{4}\left[\frac{1}{x-2} - \frac{1}{x+2}\right]$$

$$y_n = \frac{1}{4}\left[\frac{(-1)^n n!.1^n}{(x-2)^{n+1}} - \frac{(-1)^n n!.1^n}{(x+2)^{n+1}}\right]$$

$$\Rightarrow y_n = \frac{1}{4}\left[\frac{(-1)^n n!}{(x-2)^{n+1}} - \frac{(-1)^n n!}{(x+2)^{n+1}}\right]$$

$$= \frac{(-1)^n n!}{4}\left[\frac{1}{(x-2)^{n+1}} - \frac{1}{(x+2)^{n+1}}\right]$$

c) Let $y = \frac{1}{x^2-1}$

$$\Rightarrow y = \frac{1}{(x-1)(x+1)}$$

$$\Rightarrow y = \frac{1}{2}\left[\frac{1}{x-1} - \frac{1}{x+1}\right]$$

$$\therefore\ y_n = \frac{1}{2}\left[\frac{(-1)^n n!.1^n}{(x-1)^{n+1}} - \frac{(-1)^n n!.1^n}{(x+1)^{n+1}}\right]$$

$$= \frac{1}{2}\left[\frac{(-1)^n n!}{(x-1)^{n+1}} - \frac{(-1)^n n!}{(x+1)^{n+1}}\right]$$

$$= \frac{(-1)^n n!}{2}\left[\frac{1}{(x-1)^{n+1}} - \frac{1}{(x+1)^{n+1}}\right]$$

Ans. to Q. No. 3: a) Given, y = cos3x cos7x

$$= \frac{1}{2}(2\cos 3x\cos 7x)$$

$$= \frac{1}{2}[\cos 4x + \cos 10x]$$

$$y_n = \frac{1}{2}\left[4^n \cos\left(n\frac{\pi}{2} + 4x\right) + 10^n \cos\left(n\frac{\pi}{2} + 10x\right)\right]$$

b) Given, $y = \sin^2 x$

$$= \frac{1}{2}(2\sin^2 x)$$

$$= \frac{1}{2}(1 - \cos 2x)$$

$$= \frac{1}{2} - \frac{1}{2}\cos 2x$$

$$y_n = 0 - \frac{1}{2}.2^n \cos\left(n\frac{\pi}{2} + 2x\right)$$

$$= -2^{n-1}\cos\left(n\frac{\pi}{2} + 2x\right).$$

c) Given, $y = \tan^{-1}\frac{1-x}{1+x}$

$= \tan^{-1} 1 - \tan^{-1} x$

$$= \frac{\pi}{4} - \tan^{-1} x$$

$y_n = 0 - (-1)^{n-1}(n-1)!\frac{\sin n\theta}{r^n}$, where, $r = \sqrt{x^2+1}$ and $\tan\theta = \frac{1}{x}$.

$$\Rightarrow y_n = \frac{(-1)^n(n-1)!}{(x^2+1)^{\frac{n}{2}}}\sin\left(n\tan^{-1}\frac{1}{x}\right)$$

d) Given, y = log(2x+5)

$$y_1 = \frac{2}{2x+5}$$

$$y_2 = (-1)\frac{2^2}{(2x+5)^2}$$

$$y_3 = (-1)(-2)\frac{2^3}{(2x+5)^3} = (-1)^2 2!\frac{2^3}{(2x+5)^3}$$

In general, $y_n = (-1)^{n-1}(n-1)!\frac{2^n}{(2x+5)^n}$.

e) Given, $y = e^x \cos 3x \Rightarrow y_1 = e^x\cos 3x - 3e^x \sin 3x$

$= e^x(\cos 3x - 3\sin 3x)$

$= e^x(r\cos\theta\cos 3x - r\sin\theta\sin 3x)$

where, $r\cos\theta = 1$ and $r\sin\theta = 3$

$r = \sqrt{10}$ and $\tan^{-1} 3 = \theta$

$= e^x r\cos(3x+\theta)$

$= re^x\cos(3x+\theta)$

$y_2 = r[e^x\cos(3x+\theta) - 3e^x\sin(3x+\theta)]$

$= re^x[r\cos\theta\cos(3x+\theta) - r\sin\theta\sin(3x+\theta)]$

$= r^2e^x\cos(\theta+3x+\theta)$

$= r^2e^x\cos(3x+2\theta)$

In general, $y_n = r^n e^x \cos(3x + n\theta)$

$= (10)^{\frac{n}{2}} e^x \cos(3x + n\tan^{-1} 3)$.

Ans. to Q. No. 4: a) We know that if $y = (ax+b)^m$, then

$y_n = m(m-1)(m-2), ..., [m-(n-1)]a^n(ax+b)^{m-n}$, $n < m$

$= \frac{m!}{(m-n)!} a^n (ax+b)^{m-n}$

Second part: a) If $n > m$, $y_n = 0$

b) If $n = m$, $y_n = n!$, a constant.

Ans. to Q. No. 5: Given, $y = x^n \log x$

$\therefore y_1 = nx^{n-1} \log x + x^n . \frac{1}{x}$

$\Rightarrow xy_1 = nx^n \log x + x^n$

$\Rightarrow xy_1 = ny + x^n$

Differentiating n times using Leibnitz's theorem, we get,

$xy_{n+1} + n.1.y_n = n.y_n + n!$

$\Rightarrow xy_{n+1} = n!$

$\Rightarrow y_{n+1} = \frac{n!}{x}$.

Ans. to Q. No. 6: Given, $y = \cos(\log x)$

$\Rightarrow \cos^{-1} y = \log x$

$\therefore \frac{d}{dx}(\cos^{-1} y) = \frac{d}{dx}(\log x)$

$\Rightarrow \frac{-1}{\sqrt{1-y^2}} y_1 = \frac{1}{x}$

$\Rightarrow -xy_1 = \sqrt{1-y^2}$

$\Rightarrow x^2 y_1^2 = 1 - y^2$

Differentiating w.r.t. x, we get, $x^2 2y_1 y_2 + 2x.y_1^2 = -2y.y_1$

$\Rightarrow x^2 y_2 + x.y_1 = -y$

$\Rightarrow x^2 y_2 + x.y_1 + y = 0$

Differentiating n times using Leibnitz's theorem, we get,

$\left\{x^2 y_{n+2} + n.2x.y_{n+1} + \frac{n(n-1)}{2!}.2y_n\right\} + \{xy_{n+1} + n.1.y_n\} + y_n = 0$

$\Rightarrow x^2 y_{n+1} + (2n+1)y_{n+1} + (n^2+1)y_n = 0$

Ans. to Q. No. 7: Given, $y = (x^2-1)^n$ ... ... (i)

$y_1 = n(x^2-1)^{n-1}.2x$

$\Rightarrow y_1 = \dfrac{n(x^2-1)^n.2x}{(x^2-1)}$

$\Rightarrow (x^2-1)y_1 = ny.2x$

$\Rightarrow (x^2-1)y_1 = 2nxy$ using (i)

Differentiating w.r.t. x, we get, $(x^2-1)y_2 + 2xy_1 = 2n(1.y + xy_1)$

$\Rightarrow (x^2-1)y_2 + 2(1-n)xy_1 - 2ny = 0$

Differentiating n times using Leibnitz's theorem, we get,

$$\left\{(x^2-1)y_{n+2} + n.2xy_{n+1} + \frac{n(n-1)}{2!}.2.y_n\right\} + 2(1-n)\{xy_{n+1} + n.1.y_n\} - 2ny_n = 0$$

$\Rightarrow (x^2-1)y_{n+2} + 2nxy_{n+1} + (n^2-n)y_n + 2(1-n)xy_{n+1} + 2(1-n)ny_n - 2ny_n = 0$

$\Rightarrow (x^2-1)y_{n+2} + 2xy_{n+1} - n(n+1)y_n = 0$

Ans. to Q. No. 8: Given $y = e^{\tan^{-1}x}$

$\Rightarrow \log y = \tan^{-1}x$

$\therefore \dfrac{d}{dx}(\log y) = \dfrac{d}{dx}(\tan^{-1}x)$

$\Rightarrow \dfrac{1}{y}.y_1 = \dfrac{1}{1+x^2}$

$\Rightarrow (1+x^2)y_1 = y$

Differentiating (n+1) times using Leibnitz's theorem, we get,

$$(1+x^2)y_{n+2} + (n+1).2x.y_{n+1} + \frac{(n+1).n}{2!}.2.y_n = y_{n+1}$$

$\Rightarrow (1+x^2)y_{n+2} + \{2(n+1)x - 1\}y_{n+1} + n(n+1)y_n = 0$

Ans. to Q. No. 9: Given, $y = \tan^{-1}x$ ... ... (1)

$\therefore y_1 = \dfrac{1}{1+x^2}$ ... ... (2)

$\Rightarrow (1+x^2)y_1 = 1$

Differentiating w.r.t. x, we get,

$(1+x^2)y_2 + 2xy_1 = 0$ ... ... (3)

Differentiating (3) n times using Leibnitz's theorem, we get,

$$\left\{(1+x^2)y_{n+2} + n.2x.y_{n+1} + \frac{n(n-1)}{2!}.2y_n\right\} + 2\{xy_{n+1} + n.1.y_n\} = 0$$

$\Rightarrow (1+x^2)y_{n+2} + 2nxy_{n+1} + n(n-1)y_n + 2xy_{n+1} + 2ny_n = 0$

$\Rightarrow (1+x^2)y_{n+2} + 2(n+1)xy_{n+1} + \{n(n-1) + 2n\}n_n = 0$

$\Rightarrow (1+x^2)y_{n+2} + 2(n+1)xy_{n+1} + (n^2+n)y_n = 0$

$\Rightarrow (1+x^2)y_{n+2} + 2(n+1)xy_{n+1} + (n^2+1)y_n = 0$ ... ... (4)

Putting x = 0, we get, (1) $\Rightarrow y(0) = 0$

(2) $\Rightarrow y_1(0) = 1$

(3) $\Rightarrow y_2(0) = 0$

(4) $\Rightarrow y_{n+2}(0) = -n(n+1)y_n(0)$ ... ... (5)

Putting n = 1, 2, 3, ... in (5), we get,

$y_3(0) = -1.2.y_1(0) = -2!$

$y_4(0) = -2.3.y_2(0) = 0$

$y_5(0) = -3.4.y_3(0) = -3.4.(-2!) = 4!$

$y_6(0) = -4.5.y_4(0) = 0$

$y_7(0) = -5.6.y_5(0) = -5.6.(4!) = -6!$

and so on.

In general, $y_n(0) = \begin{cases} 0, \text{if } n \text{ is even} \\ (-1)^{\frac{n-1}{2}}(n-1)!, \text{if } n \text{ is odd} \end{cases}$

Ans. to Q. No. 10: Given, $y = \left\{\log\left(x + \sqrt{1+x^2}\right)\right\}^2$ ... ... (1)

$$y_1 = 2\log\left(x + \sqrt{1+x^2}\right).\frac{1}{x + \sqrt{1+x^2}}\left(1 + \frac{1}{2\sqrt{1+x^2}}.2x\right)$$

$$\Rightarrow y_1 = \frac{2\sqrt{y}}{x + \sqrt{1+x^2}}.\frac{\sqrt{1+x^2} + x}{\sqrt{1+x^2}}$$

$\Rightarrow \sqrt{1+x^2}.y_1 = 2\sqrt{y}$

$\Rightarrow (1+x^2)y_1^2 = 4y$ ... ... (2)

Differentiating w.r.t. x, we get,

$(1+x^2)2y_1y_2 + 2xy_1^2 = 4y_1$

$\Rightarrow (1+x^2)y_2 + xy_1 - 4 = 0$ ... ... (3)

Differentiating (3) n times using Leibnitz's theorem, we get,

$$\left\{(1+x^2)y_{n+2} + n.2x.y_{n+1} + \frac{n(n-1)}{2!}.2.y_n\right\} + \left\{xy_{n+1} + n.1.y_n\right\} = 0$$

$\Rightarrow (1+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2y_n = 0$ ... ... (4)

Putting x = 0, in (4), we get, $(y_{n+2})_0 + 0 + n^2(y_n)_0 = 0$

$\Rightarrow (y_{n+2})_0 = -n^2(y_n)_0$

Ans. to Q. No. 11: Given, $y = e^x \log x$ ... ... (1)

$$y_1 = e^x \log x + e^x.\frac{1}{x}$$

$\Rightarrow y_1 = y + e^x.\frac{1}{x}$, using (1)

$\Rightarrow xy_1 = xy + e^x$

$\Rightarrow xy_1 - xy = e^x$... ... ... (2)

Differentiating (2) w.r.t. x, we get,

$(xy_2+1.y_1) - (1.y+xy_1) = e^x$

$\Rightarrow xy_2 + y_1 - y - xy_1 = xy_1 - xy$, using (2)

$\Rightarrow xy_2 + y_1 - 2xy_1 - y + xy = 0$

$\Rightarrow xy_2 - (2x-1)y_1 + (x-1)y = 0$, which gives the 1st part.

Differentiating n times using Leibnitz's theorem, we get,

$\{xy_{n+2} + n.1.y_{n+1}\} - \{(2x-1)y_{n+1} + n.2.y_n\} + \{(x-1)y_n + n.1.y_{n-1}\} = 0$

$\Rightarrow xy_{n+2} + ny_{n+1} - (2x-1)y_{n+1} - 2ny_n(x-1)y_n + ny_{n-1} = 0$

$\Rightarrow xy_{n+2} + (n-2x+1)y_{n+1} - (2n-x+1)y_{n+1} + ny_{n-1} = 0$

Ans. to Q. No. 12: Given, $y = \frac{\sin^{-1}x}{\sqrt{1-x^2}} \Rightarrow y\sqrt{1-x^2} = \sin^{-1}x$

Differentiating w.r.t. x, we get,

$$y_1\sqrt{1-x^2} + y.\frac{1}{2\sqrt{1-x^2}}(-2x) = \frac{1}{\sqrt{1-x^2}}$$

$\Rightarrow (1-x^2)y_1 - xy = 1$

Differentiating again w.r.t. x, we get,

$\{(1-x^2)y_2 + (-2x)y_1\} - \{xy_1 + 1.y\} = 0$

$\Rightarrow (1-x^2)y_2 - 3xy_1 - y = 0$

Differentiating n times using Leibnitz's theorem, we get,

$$\left\{(1-x^2)y_{n+2} + n.(-2x)y_{n+1} + \frac{n(n-1)}{2!}(-2)y_n\right\} - 3\{xy_{n+1} + n.1.y_n) - y_n = 0$$

$\Rightarrow (1-x^2)y_{n+2} - 2nxy_{n+1} - (n^2-n)y_n - 3xy_{n+1} - 2ny_n - y_n = 0$

$\Rightarrow (1-x^2)y_{n+2} - (2n+3)xy_{n+1} - (n^2+2n+1)y_n = 0$

$\Rightarrow (1-x^2)y_{n+2} - (2n+3)xy_{n+1} - (n+1)^2y_n = 0$

Ans. to Q. No. 13: Given, x = tan(log y)

$\Rightarrow \tan^{-1} x = \log y$

$\therefore \frac{d}{dx}\left(\tan^{-1} x\right) = \frac{d}{dx}\left(\log y\right)$

$\Rightarrow \frac{1}{1+x^2} = \frac{1}{y}.y_1$

$\Rightarrow (1+x^2)y_1 = y$

Differentiating n times using Leibnitz's theorem, we get,

$$\left(1+x^2\right)y_{n+1} + n.2x.y_n + \frac{n(n-1)}{2!}.2y_{n-1} = y_n$$

$\Rightarrow (1+x^2)y_{n+1} + 2nxy_n + n(n-1)y_{n-1} - y_n = 0$

$\Rightarrow (1+x^2)y_{n+1} + (2nx - 1)y_n + n(n-1)y_{n-1} = 0$

$\Rightarrow \left(1+x^2\right)y_{n+1} + \left(2nx-1\right)y_n + n(n-1)y_{n-1} = 0$



8.7 FURTHER READINGS

1) Avez, A.: *An Elementary Treatise on Differential Calculus*; Princeton: Princeton University Press, 2001.

2) Rao, S. (2004): *Differential Calculus*; Berkeley: University of California Press.



8.8 MODEL QUESTIONS

Q.1: Find the n^{th} derivatives of:

a) $y = e^{ax} x^2$ b) $y = \log(1+x)$ c) $y = \frac{1-x}{1+x}$

d) $y = \sin x$ e) $y = e^x(2x+3)^3$

Q.2: If $y = \tan^{-1}\left(\frac{1+x}{1-x}\right)$, show that the n^{th} derivative is

$y_n = (-1)^{n-1}(n-1)!\sin^n\theta \sin n\theta$ where $\theta = \cot^{-1}x$.

Q.3: If $\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{n}\right)^n$, Prove that $x^2y_{n+2} + (2n+1)xy_{n+1} + 2n^2y_n = 0$.

Q.4: Find $y_n(0)$ if $y = (\sin^{-1} x)^2$.

Q.5: Determine $y_n(0)$, if $y = e^{m\sin^{-1}x}$.

Q.6: If $y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$, Prove that $(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$.

Q.7: Determine $y_n(0)$, when $y = \cos(m \sin^{-1}x)$.

Q.8: If $\sin^{-1}y = 2\log(x+1)$,

show that $(x+1)^2y_{n+2} + (2n+1)(x+1)y_{n+1} + (n^2+4)y_n = 0$.

Q.9: Find the n^{th} derivatives of the followings:

a) $\dfrac{x+1}{x^2-4}$ b) $\dfrac{x^4}{(x-1)(x-2)}$

c) $\log x^2$ d) $e^x \sin4x \cos6x$

Q.10: If $y = (x^2-1)^n$, show that $(x^2-1)y_{n+2} + 2xy - n(n+1)y_n = 0$.

Q.11: If $y = \left[\log_e\left(x + \sqrt{a^2+x^2}\right)\right]^2$,

Prove that $(a^2+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2y = 0$.

Q.12: If $y = e^{\tan^{-1}x}$, Prove that $(1+x^2)y_{n+2} + (2nx-1)xy_{n+1} + n(n-1)y_n = 0$.

Q.13: If $y = \dfrac{\sin^{-1}x}{\sqrt{1-x^2}}$, find the value of $y_{n+1}(0)$ in terms of $y_{n-1}(0)$.

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UNIT 9: APPLICATION OF DERIVATIVES

UNIT STRUCTURE

9.1 Learning objectives
9.2 Introduction
9.3 Increasing and Decreasing Function
9.4 Derivative as a Rate Measurer
9.5 Maximum and Minimum of a Function
    9.5.1 Methods for finding Maximum and Minimum
9.6 Application of Maxima and Minima
9.7 Curvature
    9.7.1 Definition
    9.7.2 Radius of Curvature
    9.7.3 Radius of Curvature for Cartesian Curves
    9.7.4 Radius of Curvature in Polar Form
9.8 Tangents and Normals
    9.8.1 Equations of Tangent and Normal
9.9 Let Us sum Up
9.10 Answers to Check Your Progress
9.11 Further Reading
9.12 Model Questions

9.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- obtain intervals in which a given function increases or decreases
- determine the rate of increase or decrease of length, area, volume etc
- determine the points on a given curve where extremum exist, if any
- obtain sharpness of a curve at any point of it
- find equations of tangents and normals at a given point on a particular curve.

9.2 INTRODUCTION

In this unit, we shall discuss the nature of a function which is governed by the sign of its derivative. If the graph of a function is in upward going direction or in downward coming direction, it is called as monotonic function. Also, we shall study those points of the domain of a function where its graph changes its direction from upwards to downwards or from downwards to upwards. The derivative of the function vanishes at those points, if exists. Also, we also discuss the way to find equations of tangents and normals to a curve at any given point on it by using derivatives. Lastly, we discuss the bendness of a curve with the help of derivatives.

9.3 INCREASING AND DECREASING FUNCTIONS

Functions are said to be monotonic if they are either increasing or decreasing in their entire domain. $f(x) = e^x$, $f(x) = 2x + 5$ are some of the examples of functions which are increasing whereas $f(x) = -x^5$, $f(x) = e^{-x}$ are some of the examples of functions which are decreasing.

A function is said to be monotonically increasing at x = a if,

f(a+h) > f(a) and f(a–h) < f(a) for a small positive h. See Figure 9.1.

A function is said to be monotonically decreasing at x = a if

f(a+h) < f(a) and f(a–h) > f(a) for a small positive h. See Figure 9.2.

In Figure 9.3, f(x) is neither monotonically increasing nor decreasing.

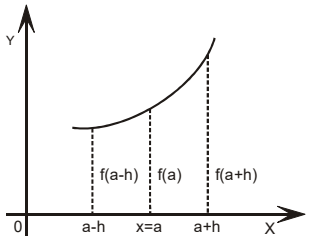


Fig. 9.1

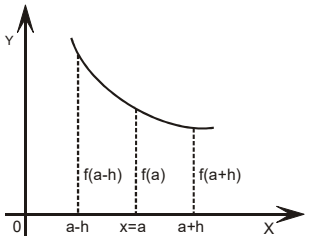


Fig. 9.2

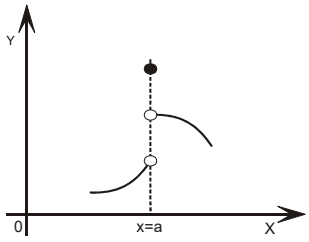


Fig. 9.3

Condition for f(x) to be increasing or decreasing:

Let f(x) be an increasing function

Then f(x+h) > f(x) for h>0 and $h \to 0$

$$\Rightarrow \frac{f(x+h)-f(x)}{h} > 0 \text{ for } h > 0 \text{ and } h \to 0$$

Hence, $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h} \geq 0$

$\Rightarrow f'(x) \geq 0$ and conversely

Hence f(x) is increasing at x if and only if $f'(x) \geq 0$. Similarly f(x) is decreasing at x if and only if $f'(x) \leq 0$.

Note: A monotonic function is invertible and vice-versa.

Illustrative Examples:

Example 1: Show that $f(x) = \frac{x}{\sin x}$ is increasing for $0 < x < \frac{\pi}{2}$

Solution: We have $f(x) = \frac{x}{\sin x}$

$$\Rightarrow f'(x) = \frac{\sin x - x\cos x}{\sin^2 x}$$

$$= \frac{\cos x(\tan x - x)}{\sin^2 x} > 0 \left[\begin{array}{l} \because 0 < x < \frac{\pi}{2} \\ \therefore \cos x > 0, \sin^2 x > 0 \text{ and } \tan x > x \end{array}\right]$$

= f' is strictly increasing for $0 < x < \frac{\pi}{2}$

Example 2: Show that f(x) = x – sin x is an increasing function.

Solution: f(x) = x – sin x

$\Rightarrow$ $f'(x) = 1 - \cos x$

$\Rightarrow$ $f'(x) > 0$ everywhere except at x = 0, $\pm 2\pi, \pm 4\pi$ etc.

$\Rightarrow$ f(x) is monotonically increasing for $x \in R$ as the points 0, $\pm 2\pi, \pm 4\pi$ etc. are discrete and do not form an interval.

Example 3: Show that $x - \frac{x^2}{2} < \log(1+x) < x - \frac{x^2}{2(1+x)}$ for x > 0

Solution: Let f(x) = log (1+x) – $\left(x - \frac{x^2}{2}\right)$

$$\Rightarrow f'(x) = \frac{1}{1+x} - (1-x)$$

$$= \frac{1-(1-x^2)}{1+x}$$

$$= \frac{x^2}{1+x} > 0 \text{ for } x > 0$$

$\Rightarrow$ f(x) is an increasing function for x > 0 .

Also f(0) = 0

$\therefore$ f(x) > 0 for x > 0

$\Rightarrow \log(1+x) > x - \frac{x^2}{2}$ for x > 0 ... ... (1)

Let g(x) = $\left\{x - \frac{x^2}{2(1+x)}\right\} - \log(1+x)$

$\Rightarrow g'(x) = 1 - \frac{2(1+x).2x - x^2.2}{\{2(1+x)\}^2} - \frac{1}{1+x}$

$= 1 - \frac{2x+x^2}{2(1+x)^2} - \frac{1}{1+x}$

$= \frac{2(1+x)^2 - (2x+x^2) - 2(1+x)}{2(1+x)^2}$

$= \frac{2+4x+2x^2-2x-x^2-2-2x}{2(1+x)^2}$

$= \frac{x^2}{2(1+x)^2} > 0$ for x > 0

$\Rightarrow$ g(x) is an increasing function for x > 0. But g(0) = 0

$\therefore$ g(x) > 0 for x > 0

$\Rightarrow x - \frac{x^2}{2(1+x)} > \log(1+x)$ for x > 0 ... ... (2)

Combining (1) and (2), we get,

$$x - \frac{x^2}{2} < \log(1+x) < x - \frac{x^2}{2(1+x)} \text{ for } x > 0$$

Example 4: Find the intervals of increase and/or decrease for the function $f(x) = 2x^3 - 15x^2 + 36x + 1$.

Solution: Given, $f(x) = 2x^3 - 15x^2 + 36x + 1$

$\Rightarrow f'(x) = 6x^2 - 30x + 36$

$= 6(x^2 - 5x + 6)$

$= 6(x-2)(x-3)$

Now, $f'(x) > 0 \Rightarrow 6(x-2)(x-3) > 0$

$\Rightarrow (x-2 > 0 \text{ and } x-3 > 0) \text{ or } (x-2 < 0 \text{ and } x-3 < 0)$

$\Rightarrow (x > 2 \text{ and } x > 3) \text{ or } (x < 2 \text{ and } x < 3)$

$\Rightarrow x > 3 \text{ or } x < 2$

$\therefore$ f(x) strictly increases in $]-\infty, 2[\cup]3, \infty[$

Also, $f'(x) < 0 \Rightarrow 6(x-2)(x-3) < 0$

$\Rightarrow (x-2 > 0 \text{ and } x-3 < 0) \text{ or } (x-2 < 0 \text{ and } x-3 > 0)$

$\Rightarrow (x > 2 \text{ and } x < 3) \text{ or } (x < 2 \text{ and } x > 3)$

$\Rightarrow x > 2 \text{ and } x < 3$

$\Rightarrow 2 < x < 3$

$\therefore f(x)$ strictly decreases in $]2, 3[$



CHECK YOUR PROGRESS

Q.1: Show that $f(x) = x^3 - 3x^2 + 3x + 2$ increases for all x.

Q.2: Show that if $x > 0$

a) $\frac{x}{1+x} < \log(1+x) < x$ b) $x - \frac{x^2}{2(1+x)} > \log(1+x)$

Q.3: Find the interval of increase or decrease for the following functions:

a) $f(x) = 2x^3 - 9x^2 - 168x + 36$

b) $f(x) = \frac{\log x}{x}$ for $x > 0$

c) $f(x) = 2x^3 - 15x^2 - 84x + 7$

9.4 DERIVATIVE AS A RATE-MEASURER

The derivative of a function $y = f(x)$ with respect to the independent variable x is the rate measure of change of y with respect to change in x. Let us consider a small change Δx in the value of x and let Δy be the corresponding small change in the value of y, Then,

$y + \Delta y = f(x+\Delta x)$

$\Rightarrow \Delta y = f(x+\Delta x) - y$

$\Rightarrow \Delta y = f(x+\Delta x) - f(x)$

$\Rightarrow \frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$,

which gives average rate of change of y w.r. to x.

When Δx is very small, $\frac{\Delta y}{\Delta x}$ is very nearly equal to the instantaneous rate of change of y w.r. to x when $\Delta x \to 0$, the limiting value of this average rate of change becomes the instantaneous rate of change of y.

Thus, instantaneous rate of change of y w.r. to change in x

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = f'(x)$$

$\therefore$ $f'(x)$ measures the rate of change of f(x) w.r. to x.

Illustrative Examples:

Example 1: The volume of a spherical balloon is increasing at the rate of 25 cm^3/sec. Find the rate at which the surface area of the balloon is increasing when its radius is 5 cm.

Solution: Let at any instant of time t,

Radius of the balloon = r

Surface area of the balloon = S

Volume of the balloon = V

Then, $S = 4\pi r^2$

$$V = \frac{4}{3}\pi r^3$$

Now, $\frac{dV}{dt} = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)$

$$\Rightarrow \frac{dV}{dt} = \frac{4}{3}\left(\pi.3r^2\right)\frac{dr}{dt}$$

$$\Rightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \frac{1}{4\pi r^2}.\frac{dV}{dt} \qquad \text{... ... ... (1)}$$

Also, $\frac{dS}{dt} = \frac{d}{dt}\left(4\pi r^2\right)$

$$= 4\pi(2r)\frac{dr}{dt}$$

$$= 8\pi r.\frac{1}{4\pi r^2}\frac{dV}{dt}, \text{ by (1)}$$

$$= \frac{2}{r}\frac{dV}{dt}$$

Given, $\frac{dV}{dt} = 25 cm^3/s$ and r = 5 cm

$\therefore \frac{dS}{dt} = \frac{2}{5} \times 25 \, cm^2/s$

= 10 cm^2/sec

Example 2: Let x and y be the sides of two squares such that $y = x - x^2$. Find the rate of change of area of second square with respect to the area of the first square.

Solution: Let A and S be the area of the two squares of sides x and y respectively. Then $A = x^2$ and $S = y^2$.

To find $\frac{dS}{dA}$

Now, $\frac{dS}{dx} = \frac{d}{dx}(y^2)$

$= \frac{d}{dx}(x - x^2)^2$ [$\because$ $y = x - x^2$, given]

$= \frac{d}{dx}(x^2 - 2x^3 + x^4)$

$= 2x - 6x^2 + 4x^3$ ... . (1)

Also, $\frac{dA}{dx} = \frac{d}{dx}(x^2) = 2x$ ... ... (2)

$\therefore \frac{dS}{dA} = \frac{\frac{dS}{dx}}{\frac{dA}{dx}} = \frac{2x - 6x^2 + 4x^3}{2x}$

= $1 - 3x + 2x^2$

Example 3: A conical vessel whose height is 10 metres and radius of whose base is half that of the height is being filled with a liquid at a uniform rate of 1.5 m^3/min. Find the rate at which the level of the water in the vessel is rising when it is 3m below the top of the vessel.

Solution: Let OAB be the cone of height OC = 10 meters.

Radius of the base, AC $= \frac{1}{2}$OC

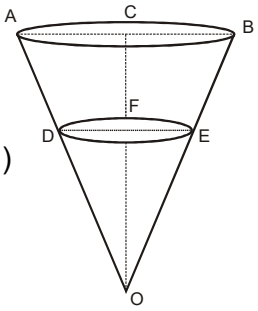
= 5 meters.

At only instant of time t, the liquid in the vessel form a cone, say ODE. Let at time t, the base radius and height of this cone be respectively r and h.

Then, $\frac{DF}{AC} = \frac{OF}{OC}$ [$\because \Delta OAC$ and ΔODF are similar]

$\Rightarrow \frac{r}{5} = \frac{h}{10}$

$\Rightarrow r = \frac{h}{2}$ ... ... (1)



Let volume of the liquid cone at time t be V.

Then, $V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$, using (1)

$= \frac{1}{12}\pi h^3$

$\Rightarrow \frac{dV}{dt} = \frac{d}{dt}\left(\frac{\pi h^3}{12}\right)$

$= \frac{\pi}{12}.3h^2\frac{dh}{dt}$

$= \frac{\pi h^2}{4}\frac{dh}{dt}$

$\Rightarrow \frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt}$ ... ... (2)

Given, $\frac{dV}{dt} = 1.5\ m^3/min$

h = (10 – 3)m = 7m

$\therefore (2) \Rightarrow \frac{dh}{dt} = \frac{4}{\pi(7)^2} \times 1.5\ m/min$

$= \frac{6}{49\pi} m/min$

Example 4: The length of a rectangle is increasing at the rate of 3.5 cm/sec and its breadth is decreasing at the rate of 3 cm/sec. Find the rate of change of the area of the rectangle when length is 12 cm and breadth is 8 cm.

Solution: Let at instant of time t,

length of rectangle = x

breadth of rectangle = y

Area of rectangle = A

$\therefore$ A = xy

$$\Rightarrow \frac{d}{dt}(A) = \frac{d}{dt}(xy)$$

$$\Rightarrow \frac{dA}{dt} = x\frac{dy}{dt} + y\frac{dx}{dt} \qquad \text{... ... ... (1)}$$

Given, $\frac{dx}{dt} = 3.5 \text{ cm/sec}$

$$\frac{dy}{dt} = -3 \text{ cm/sec}$$

$$\therefore (1) \Rightarrow \frac{dA}{dt} = (-3x + 3.5y) \text{ cm/sec}$$

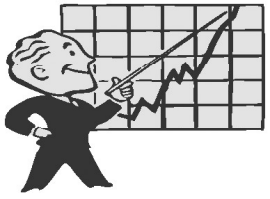
When x = 12 cm, y = 8 cm, we get,

$$\frac{dA}{dt} = (-3 \times 12 + 3.5 \times 8) \text{ cm}^2/\text{sec}$$

$= (-36+28)$ cm^2/sec

$= -8$ cm^2/sec

Hence area of the rectangle decreases at the rate of 8 cm^2/sec.



CHECK YOUR PROGRESS

Q.4: Find the rate of change of the volume V of a sphere of radius r with respect to change in the radius.

Q.5: The volume of a spherical ballon is increasing at the rate of 20 cm^3/sec. Find the rate at which its surface area is increasing at the instant when its radius is 4 cm.

Q.6: A stone is dropped into a quiet lake and waves move in circles at a speed of 3.5 cm/sec. At the instant when the radius of the circular wave is 7.5 cm, how fast is the enclosed area increasing?

Q.7: Sand is pouring from a pipe at the rate of 9 cm^3/sec. The falling sand forms a cone on the ground in such a way that the height of the cone is always one sixth of the radius of the

base. How fast is the height of the sand cone increasing when the height is 3 cm?

Q.8: Find the point on the curve $y^2 = 8x$ for which the abscissa and ordinate change at the same rate.

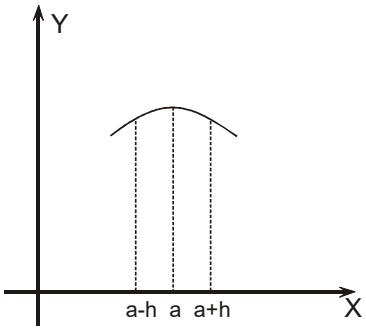
Q.9: A man 2 metre high walks at a uniform speed of 6 km/h away from a lamp post 6 metre high. Find the rate at which the length of his shadow increases. Also find the rate at which the tip of the shadow is moving away from the lamp post.

Q.10: An edge of a variable cube is increasing at the rate of 3 cm/sec. How fast is the volume of the cube increasing when the edge is 10 cm long?

9.5 MAXIMUM AND MINIMUM OF A FUNCTION

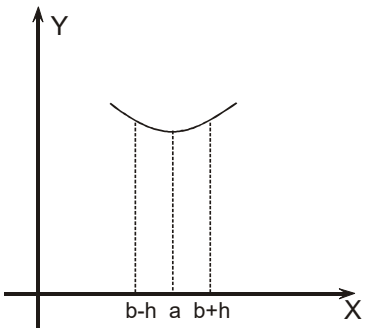
A function f(x) is said to have a maximum at x = a if f(a) is greater than every other value assumed by f(x) in the immediate neighbourhood of x = a.

This means that f(a) > f(a+h) and f(a) > f(a–h) for a sufficiently small positive h. (See Figure 9.4)



Local Maxima

Fig. 9.4



Local Minumum

Fig. 9.5

Similarly, a function f(x) is said to have a minimum at x = b if f(b) is smaller than every other value assumed by f(x) in the immediate neighbourhood of x = b.

This means that f(b) < f(b+h)

and f(b) < f(b–h)

for a sufficiently small positive h (See Figure 9.5).

Geometrically, a function f continuous at a point has a maximum or a minimum at that point according as the graph of f has a peak or a cavity at that point.

Note: i) The maximum and minimum values of a function are also known as local (relative) maxima or local (relative) minima as these are the greatest and least values of the function relative to some neighbourhood of the point in the question. The term extremum is also used for both a maximum and minimum value.

ii) A local maximum value at a point may be less than a local minimum value at another point.

iii) A function can have several maximum and minimum values. The maximum and minimum values of a continuous function occur alternatively.

iv) At maximum, the function f(x) changes from increasing to decreasing state and at minimum the function f(x)changes from decreasing to increasing state.

9.5.1 Methods of Finding Maximum or Minimum

Method I:

1) Find $\frac{dy}{dx}$.

2) Solve the equation $\frac{dy}{dx} = 0$ for real values of x. Let these values be a, b, c etc.

3) Consider x = a. Study the sign of $\frac{dy}{dx}$ for values of a slightly less than a and slightly greater than a.

4) If $\frac{dy}{dx}$ changes sign from positive to negative, then f(x) is maximum at x = a.

5) If $\frac{dy}{dx}$ changes sign from negative to positive, then f(x) is minimum at x = a.

6) If $\frac{dy}{dx}$ does not change sign, then x=a is a point neither of maximum nor of minimum.

Method II:

Steps 1 and 2 are same as in method I.

3) Find $\frac{d^2y}{dx^2}$ and put values. Consider x = a.

i) If $\frac{d^2y}{dx^2} < 0$ at x = a, then f(x) is maximum at x = a.

ii) If $\frac{d^2y}{dx^2} > 0$ at x = a, then f(x) is minimum at x = a.

iii) If $\frac{d^2y}{dx^2} = 0$ but $\frac{d^3y}{dx^3} \neq 0$ at x = a, then there is neither maximum nor minimum at x = a.

Illustrative Examples:

Example 1: Find the maximum and minimum of the function

$x^5 - 5x^4 + 5x^3 - 1$.

Solution: Let $y = x^5 - 5x^4 + 5x^3 - 1$

$$\therefore \frac{dy}{dx} = 5x^4 - 20x^3 + 15x^2$$

For a maximum or a minimum $\frac{dy}{dx} = 0$

$$\Rightarrow 5x^4 - 20x^3 + 15x^2 = 0$$

$$\Rightarrow 5x^2(x^2 - 4x + 3) = 0$$

$$\Rightarrow 5x^2(x-1)(x-3) = 0$$

$$\Rightarrow x = 0, x = 1, x = 3$$

Consider x = 0

Let δ be a very small positive number.

For very small positive δ

When $x = 0 - \delta$, we get,

$$\frac{dy}{dx} = 5(0-\delta)^2(0-\delta-1)(0-\delta-3)$$

$$= 5\delta^2(\delta+1)(\delta+3) > 0$$

When $x = 0+\delta$, we get,

$$\frac{dy}{dx} = 5(0+\delta)^2 (0+\delta-1) (0+\delta-3)$$

$$= 5\delta^2(\delta-1)(\delta-3)$$

>0

$\therefore$ There is no change in sign of $\frac{dy}{dx}$.

Hence at x = 0, there is neither maximum nor minimum.

Consider x = 1:

When $x = 1-\delta$, we get,

$$\frac{dy}{dx} = 5(1-\delta)^2(1-\delta-1)(1-\delta-3)$$

$$= 5(1-\delta)^2(-\delta)(-\delta-2)$$

> 0

When $x = 1+\delta$, we get,

$$\frac{dy}{dx} = 5(1+\delta)^2(1+\delta-1)(1+\delta-3)$$

$$= 5(1+\delta)^2\delta(\delta-2)$$

< 0

Thus the sign of $\frac{dy}{dx}$ changes from positive to negative. So, there is maximum at x = 1.

And the maximum value $= 1^5-5(1)^4 + 5(1)^3 - 1$

$= 0$

Consider x = 3:

When $x = 3-\delta$, we have

$$\frac{dy}{dx} = 5(3-\delta)^2(3-\delta-3)(3-\delta-1)$$

$$= 5(3-\delta)^2(-\delta)(-\delta+2)$$

< 0

When $x = 3+\delta$, we have

$$\frac{dy}{dx} = 5(3+\delta)^2 (3+\delta-3) (3+\delta-1)$$

$$= 5(3+\delta)^2 (\delta) (\delta+2)$$

> 0

$\therefore$ The value of $\frac{dy}{dx}$ changes from negative to positive.

$\therefore$ There is a minimum at x = 3 and the minimum value is:

$= 3^5 - 5(3)^4 + 5(3)^3 - 1$

$= 243 - 405 + 135 - 1$

$= -28$

Example 2: Find x for which f(x) = x(5–x) is the maximum. What is the maximum value of f(x)?

Solution: Given, f(x) = x(5–x)

$\Rightarrow f(x) = 5x - x^2$

$\therefore f'(x) = 5 - 2x$

For on extremum, we have $f'(x) = 0$

$\Rightarrow 5 - 2x = 0$

$\Rightarrow x = \frac{5}{2}$

Also, $f''(x) = \frac{d}{dx}(5 - 2x)$

$= -2$

$f''\left(\frac{5}{2}\right) = -2 < 0$

$\therefore f(x)$ is maximum at $x = \frac{5}{2}$ and the maximum value

$= f\left(\frac{5}{2}\right)$

$= \frac{5}{2}\left(5 - \frac{5}{2}\right)$

$= \frac{25}{4}$

Example 3: If $y = a\log x + bx^2 + x$ has its extreme values at x = –1 and x = 2, find a and b.

Solution: Given $y = a\log x + bx^2 + x$

$\Rightarrow \frac{dy}{dx} = \frac{a}{x} + 2bx + 1$

For extreme points, we have $\frac{dy}{dx} = 0$

$$\Rightarrow \frac{a}{x} + 2bx + 1 = 0$$

At x = –1, we get, –a – 2b + 1 = 0

$$\Rightarrow a = 1 - 2b \quad \text{... ... ... (1)}$$

At x = 2, we get, $\frac{a}{2} + 4b + 1 = 0$

$$\Rightarrow \frac{1-2b}{2} + 4b + 1 = 0, \text{ by (1)}$$

$$\Rightarrow 1 - 2b + 8b + 2 = 0$$

$$\Rightarrow 6b = -3$$

$$\Rightarrow b = -\frac{1}{2}$$

$$\therefore a = 1 - 2\left(-\frac{1}{2}\right) = 2$$

Example 4: Prove that the maximum value of the function sinx + cosx is $\sqrt{2}$.

Solution: Let f(x) = sinx + cosx

$$\therefore f'(x) = \cos x - \sin x$$

For extremum points, we have f′(x) = 0

$$\Rightarrow \cos x - \sin x = 0$$

$$\Rightarrow \cos x = \sin x$$

$$\Rightarrow \tan x = 1$$

$$\Rightarrow x = \frac{\pi}{4}$$

Now, $f''(x) = \frac{d}{dx}(\cos x - \sin x)$

$$= -\sin x - \cos x$$

$$\therefore f''\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\frac{2}{\sqrt{2}} = -\sqrt{2} < 0$$

$\therefore$ f(x) has maximum at $x = \frac{\pi}{4}$ and

$$\text{maximum value} = f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

Example 5: Prove that the maximum value of $\left(\frac{1}{x}\right)^x$ is $e^{\frac{1}{e}}$

Solution: Let $y = \left(\frac{1}{x}\right)^x$

Taking logarithm of both sides, we get, $\log y = x\log\frac{1}{x}$

$= x\log x^{-1}$

$= -x\log x.$

Differentiating bothsides w.r. to x, we get,

$$\frac{1}{y}\frac{dy}{dx} = -\left[x.\frac{1}{x} + 1.\log x\right]$$

$$\Rightarrow \frac{dy}{dx} = -y(1+\log x) \qquad \text{... ... ... (1)}$$

For maximum or minimum value of y, we have

$$\frac{dy}{dx} = 0$$

$\Rightarrow -y(1+\log x) = 0$

$\Rightarrow \log x = -1$

$\Rightarrow \log x = -\log e$

$\Rightarrow \log x = \log e^{-1}$

$$\therefore x = \frac{1}{e}$$

Differentiating (1) w.r. to x, we get,

$$\frac{d^2y}{dx^2} = -\left[y.\frac{1}{x} + (1+\log x).\frac{dy}{dx}\right]$$

$$= -\left[\frac{y}{x} - y(1+\log x)^2\right], \text{ using (1)}$$

At $x = \frac{1}{e}$, we have

$$\frac{d^2y}{dx^2} = -\left[\frac{e^{\frac{1}{e}}}{\frac{1}{e}} - e^{\frac{1}{e}}\left(1+\log\frac{1}{e}\right)^2\right]$$

$$= -e.e^{\frac{1}{e}} < 0$$

$\therefore$ At $x = \frac{1}{e}$ the function is maximum. The maximum value of the function.

$$= \left(\frac{1}{\frac{1}{e}}\right)^{\frac{1}{e}}$$

$$= e^{\frac{1}{e}}$$



CHECK YOUR PROGRESS

Q.11: Find the maximum or minimum values of the following functions.

a) $2x^2 - 6x + 3$ b) $x + \frac{1}{x}$ c) $4x^3 - 15x^2 + 12x - 1$

Q.12: Show that the function x^x is the minimum, when $x = \frac{1}{e}$ and find the minimum value.

Q.13: Prove that $\sin x(1+\cos x)$ is maximum when $x = \frac{\pi}{3}$.

9.6 APPLICATION OF MAXIMA AND MINIMA TO PROBLEMS

The theory of maxima and minima finds applications in practical life. For example, one has to find out the relation between the radius and height of a conical tent which may have a definite capacity and at the same time use minimum amount of cloth. In solving such type of problems we find out the function whose maximum or minimum has to be found out. If it involves two variables, then by eliminating one variable from the given data, the function is reduced to one having only one variable.

Example 1: The sum of two numbers is constant. Prove that their product is maximum when each number is half of their sum.

Solution: Let the two numbers be x and y.

Then x + y = constant = c, say,

$\Rightarrow y = c - x$ ... . (1)

Let P be the product of the numbers. Then,

$P = xy$

$\Rightarrow P = x(c-x)$, by (1)

$\Rightarrow P = cx - x^2$

$\therefore \frac{dP}{dx} = c - 2x$

For maximum or minimum value of P, we have,

$\frac{dP}{dx} = 0$

$\Rightarrow c - 2x = 0$

$\Rightarrow x = \frac{c}{2}$

Now, $\frac{d^2P}{dx^2} = \frac{d}{dx}(c - 2x)$

$= -2$

< 0

$\therefore$ P is maximum when $x = \frac{c}{2}$

Also, when $x = \frac{c}{2}$, we get from (1) $y = \frac{c}{2}$

Hence P is maximum when $x = y = \frac{c}{2}$

i.e., the product is maximum when each number is half of their sum.

Example 2: Prove that the rectangle of maximum area inscribed in a circle is a square.

Solution: Let ABCD be the rectangle inscribed in a circle of radius a. Let x any y be the length and breadth of the rectangle respectively.

Then, $AB^2 + BC^2 = AC^2$

$\Rightarrow x^2 + y^2 = (2a)^2$

$\Rightarrow x^2 + y^2 = 4a^2$ ... ... (1)

Let area of the rectangle ABCD be A . Then $A = xy$

$\Rightarrow A = x\sqrt{4a^2 - x^2}$, using (1)

$\because A > 0$

$\therefore$ A is maximum when A^2 is maximum.

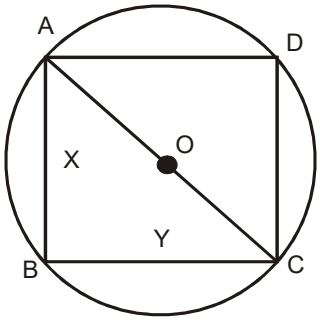
Let $u = A^2$

$\Rightarrow u = x^2(4a^2 - x^2)$

$\Rightarrow u = 4a^2x^2 - x^4$

$\therefore \frac{du}{dx} = 8a^2x - 4x^3$ ... ... (2)

For maximum or minimum of u, we have



$$\frac{du}{dx} = 0$$

$\Rightarrow 8a^2x - 4x^3 = 0$

$\Rightarrow 4x(2a^2 - x^2) = 0$

$\Rightarrow x = \sqrt{2}a \qquad [\because x > 0]$

Differentiating (2) w.r. to x, we get,

$$\frac{d^2u}{dx^2} = \frac{d}{dx}\left(8a^2x - 4x^3\right)$$

$$= 8a^2 - 12x^2$$

When $x = \sqrt{2}a$, we get,

$$\frac{d^2u}{dx^2} = 8a^2 - 12\left(\sqrt{2}a\right)^2$$

$= 8a^2 - 24a^2$

$= -16a^2$

< 0

$\therefore$ u is maximum when $x = \sqrt{2}a$

$\Rightarrow$ A is maximum when $x = \sqrt{2}a$

And $y = \sqrt{4a^2 - x^2}$

$= \sqrt{4a^2 - 2a^2}$ (when $x = \sqrt{2}a$)

$= \sqrt{2a^2}$

$= \sqrt{2}a$

$\therefore$ A is maximum when $x = y = \sqrt{2}a$.

Hence rectangle has maximum area when both adjacent sides are equal and so it is a square.

Example 3: Inside a cone of height h and semi vertical angle α, a cylinder is inscribed. Show that its maximum volume is $\frac{4}{27}\pi h^2 \tan^2\alpha$.

Solution: Let ABC be the cone whose semi vertical angle is α and height OA = h. Let PQRS be the cylinder inscribed in it whose radius is r and height h'. Then, $h' = OO' = AO - AO' = h - r\cot\alpha$

Let the volume of the cylinder be V. Then $V = \pi r^2 h'$

$$\Rightarrow V = \pi r^2(h - r\cot\alpha)$$

$$\therefore \frac{dV}{dr} = 2\pi r(h - r\cot\alpha) + \pi r^2(-\cot\alpha)$$

$$= 2\pi rh - 3\pi r^2\cot\alpha$$

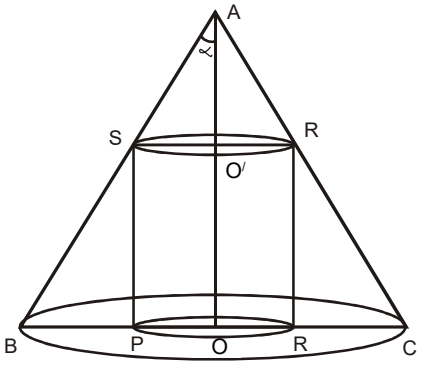
$$\therefore \frac{dV}{dr} = \pi r(2h - 3r\cot\alpha) \quad \text{... ... ... (1)}$$

For maximum or minimum value of V, we have $\frac{dV}{dr} = 0$

$$\Rightarrow \pi r(2h - 3r\cot\alpha) = 0$$

$$\Rightarrow r = 0,\ r = \frac{2h}{3\cot\alpha}$$

$$\Rightarrow r = 0,\ r = \frac{2}{3}h\tan\alpha$$



But r > 0 (radius of cone)

$$\therefore r = \frac{2}{3}h\tan\alpha$$

Differentiating (1) w.r. to r, we get,

$$\frac{d^2V}{dr^2} = 2\pi h - 6\pi r\cot\alpha$$

When $r = \frac{2}{3}h\tan\alpha$, we get,

$$\frac{d^2V}{dr^2} = 2\pi h - 6\pi\left(\frac{2}{3}h\tan\alpha\right)\cot\alpha$$

$$= 2\pi h - 4\pi h$$

$$= -2\pi h < 0$$

$\therefore$ V is maximum when $r = \frac{2}{3}h\tan\alpha$, and the maximum volume is:

$$= \pi.\frac{4}{9}h^2\tan^2\alpha\left(h - \frac{2}{3}h\tan\alpha\cot\alpha\right)$$

$$= \frac{4\pi}{9}h^2\tan^2\alpha\left(h - \frac{2}{3}h\right)$$

$$= \frac{4\pi}{9} h^2 \tan^2 \alpha . \frac{h}{3}$$

$$= \frac{4}{27} \pi h^3 \tan^2 \alpha$$

Example 4: Show that that cone inscribed in a given sphere has maximum volume when the height of the cone bears to the diameter of the sphere a ratio 2 : 3.

Solution: Let the diameter of the sphere be d.

$\therefore$ AE = d

Let ABC be the cone inscribed in the sphere. Let D be the centre of the base of the cone. Let the height AD of the cone be x and radius BD be r.

By plane geometry, $AD.DE = BD^2$

$\Rightarrow x(d-x) = r^2$

$\Rightarrow r^2 = xd - x^2$ ... ... (1)

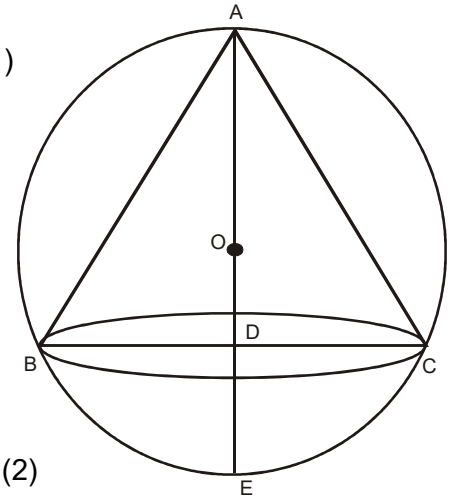
$\therefore$ The volume of the cone

$$V = \frac{1}{3}\pi r^2 x$$

$$= \frac{1}{3}\pi\left(xd - x^2\right)x \text{, by (1)}$$

$$= \frac{1}{3}\pi\left(x^2 d - x^3\right)$$

$$\therefore \frac{dV}{dx} = \frac{1}{3}\pi\left(2dx - 3x^2\right) \quad \text{... ... ... (2)}$$



For maximum or minimum value of V,

$$\frac{dV}{dx} = 0 \Rightarrow \frac{1}{3}\pi\left(2dx - 3x^2\right) = 0$$

$$\Rightarrow x(2d-3x) = 0$$

$$\Rightarrow x = \frac{2d}{3} \quad \left[\because x > 0\right]$$

Differentiating (2) w.r. to x, we get,

$$\frac{d^2V}{dx^2} = \frac{1}{3}\pi(2d - 6x)$$

When $x = \frac{2d}{3}$, we get,

$$\frac{d^2V}{dx^2} = \frac{1}{3}\pi\left(2d - 6.\frac{2d}{3}\right)$$

$$= -\frac{2\pi d}{3} < 0$$

$\therefore$ The volume of the cone is maximum when

$$x = \frac{2}{3}d \text{ i.e., } x : d = 2 : 3.$$



CHECK YOUR PROGRESS

Q.14: Find two numbers whose sum is 20 and product is maximum.

Q.15: A figure consists of a rectangle and a semicircle, the semicircle having diameter as the side of the rectangle. The perimeter of the figure is 20m. Find the dimensions of the rectangle for which the area of the figure is maximum.

Q.16: Prove that the semi vertical angle of the cone of given slant height and maximum volume is $\tan^{-1}\sqrt{2}$.

Q.17: Show that of all rectangles of given perimeter, the square has the largest area.

Q.18: A square piece of tin of side 10 cm is to be made into a box without top cutting a square from each corner and folding up the flaps to form the box. What should be the sides of the square to be cut off from each corner in order that the volume of the box is maximum?

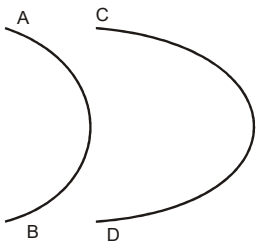
Q.19: Prove that the volume of the largest cone that can be inscribed in a sphere of radius R is $\frac{8}{27}$ of the volume of the sphere.

Q.20: Find the area of the largest rectangle inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

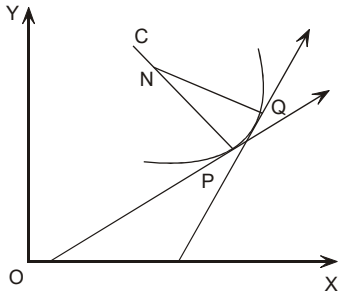
9.7 CURVATURE

Let us consider two curves AB and CD. In figure, CD bends more sharply than AB. In other words we can say CD has greater curvature than AB.

9.7.1 Definitions

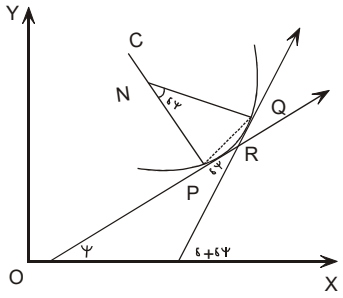


Let P and Q be two neghibouring points on a given curves such that the arc PQ is concave towards its chord. Let N be the point of intersection of of normals at P and Q to the curve. If Q approaches P, then N approaches a definite position C, called the centre of curvature of the curve at P. CP is called the radius of curvature and is usually denoted by ρ.



$\frac{1}{CP}$ is called the curvature of the curve at P. The circle with its centre at C and radius CP is called the **circle of curvature** of the curve at P.

9.7.2 Radius of Curvature



Let A be the fixed point on the curve. Let tangents at two neighbouring points P and Q make angles ψ and $\psi + \delta\psi$ respectively with x-axis.

Let arc AP = s.

arc PQ = δs.

Let N be the point of intersection of normals at P and Q. Also, let R be the point of intersection of tangents at P and Q.

We have $\angle PNQ = \angle TRT' = \delta\psi$

The radius of curvature at P = $\rho = \lim_{\delta s \to 0} PN$

In Δ PNQ, we have, $\frac{PN}{\sin NQP} = \frac{\text{chord } PQ}{\sin PNQ}$

$$\Rightarrow \frac{PN}{\sin NQP} = \frac{\text{chord } PQ}{\sin \delta\psi}$$

$$\Rightarrow PN = \frac{\text{chord } PQ . \sin NQP}{\sin \delta\psi}$$

$$\Rightarrow PN = \frac{\text{chord } PQ}{\delta s} . \frac{\delta s}{\delta\psi} . \frac{\delta\psi}{\sin \delta\psi} . \sin NQP$$

Hence, $\rho = \lim_{\substack{\partial s \to 0 \\ \partial\psi \to 0}} \left\{ \frac{\text{chord} PQ}{\delta s} . \frac{\delta s}{\delta\psi} . \frac{\delta\psi}{\sin \delta\psi} . \sin NQP \right\}$

$$= 1 . \frac{ds}{d\psi} . 1 . \sin\frac{\pi}{2}$$

$$= \frac{ds}{d\psi}$$

9.7.3 Radius of Curvature for Cartesian Curves

Let y = f(x) be the curve.

We know $\frac{dy}{dx} = \tan\psi$, where ψ is the inclination of the tangent at any point on the curve with x-axis.

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}(\tan\psi)$$

$$= \sec^2\psi . \frac{d\psi}{dx}$$

$$= \sec^2\psi . \frac{d\psi}{ds} . \frac{ds}{dx}$$

$$= (1 + \tan^2\psi) . \frac{1}{\rho} . \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\Rightarrow \rho \frac{d^2y}{dx^2} = \left\{ 1 + \left(\frac{dy}{dx}\right)^2 \right\} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\Rightarrow \rho = \frac{\left\{ 1 + \left(\frac{dy}{dx}\right)^2 \right\}^{3/2}}{\frac{d^2y}{dx^2}}$$

Note: The value of ρ is independent of the choice of axis, so we may take

$$\rho = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}}$$

Note: ρ denotes length, so it can not be negative.

9.7.4 Radius of Curvature in Polar Form

We know that $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$

Differentiating w.r. to r, we get,

$$-\frac{2}{p^3}\frac{dp}{dr} = -\frac{2}{r^3} - \frac{4}{r^5}\left(\frac{dr}{d\theta}\right)^2 + \frac{1}{r^4}.2\frac{dr}{d\theta}.\frac{d}{dr}\left(\frac{dr}{d\theta}\right)$$

$$\Rightarrow \frac{1}{p^3}\frac{dp}{dr} = \frac{1}{r^3} + \frac{2}{r^5}\left(\frac{dr}{d\theta}\right)^2 - \frac{1}{r^4}\frac{dr}{d\theta}.\frac{d}{d\theta}\left(\frac{dr}{d\theta}\right).\frac{d\theta}{dr}$$

$$\Rightarrow \frac{1}{p^3}\frac{dp}{dr} = \frac{1}{r^3} + \frac{2}{r^5}\left(\frac{dr}{d\theta}\right)^2 - \frac{1}{r^4}\frac{d^2r}{d\theta^2}$$

$$\Rightarrow \frac{1}{p^3}\frac{dp}{dr} = \frac{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}{r^5}$$

$$\Rightarrow r\frac{dr}{dp} = \frac{r^6}{p^3\left[r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}\right]} \quad \text{... ... ... (1)}$$

But $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$

$$\Rightarrow \frac{r^4}{p^2} = r^2 + \left(\frac{dr}{d\theta}\right)^2$$

$$\Rightarrow \left(\frac{r^4}{p^2}\right)^{3/2} = \left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}^{3/2}$$

$$\Rightarrow \frac{r^6}{p^3} = \left\{ r^2 + \left(\frac{dr}{d\theta} \right)^2 \right\}^{3/2} \quad \text{... ... ... (2)}$$

$$\therefore \rho = r \frac{dr}{dp}$$

$$= \frac{\left\{ r^2 + \left(\frac{dr}{d\theta} \right)^2 \right\}^{3/2}}{r^2 - r \frac{d^2 r}{d\theta^2} + 2\left(\frac{dr}{d\theta} \right)^2}$$

Illustrative Examples:

Example 1: Find the radius of curvature of the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at the point $\left(\frac{a}{4}, \frac{a}{4} \right)$.

Solution: Given curve

$$\sqrt{x} + \sqrt{y} = \sqrt{a} \quad \text{... ... ... (1)}$$

$$\therefore \frac{d}{dx}\left(\sqrt{x} + \sqrt{y} \right) = \frac{d}{dx}\left(\sqrt{a} \right)$$

$$\Rightarrow \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}} \quad \text{... ... ... (2)}$$

Differentiating (2) w.r. to x we get,

$$\frac{d^2 y}{dx^2} = \frac{-\sqrt{x} \cdot \frac{1}{2\sqrt{y}} \frac{dy}{dx} + \sqrt{y} \cdot \frac{1}{2\sqrt{x}}}{x}$$

$$= -\frac{-\frac{\sqrt{x}}{2\sqrt{y}} \cdot \left(\frac{-\sqrt{y}}{\sqrt{x}} \right) - \frac{\sqrt{y}}{2\sqrt{x}}}{x}$$

$$= \frac{\sqrt{x} + \sqrt{y}}{2x\sqrt{x}}$$

$$= \frac{\sqrt{a}}{2x\sqrt{x}}, \text{ using (1)}$$

At $\left(\frac{a}{4},\frac{a}{4}\right)$, we get,

$$\frac{dy}{dx}=-1 \text{ and } \frac{d^2y}{dx^2}=\frac{\sqrt{a}}{2.\frac{a}{4}.\frac{\sqrt{a}}{2}}=\frac{4}{a}$$

$$\therefore \text{Radius of curvature } \rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$= \frac{(1+1)^{3/2}}{\frac{4}{a}}=\frac{a\sqrt{2}}{2}$$

Example 2: Find the radius of curvature of the curve $y = e^x$ at the point where it crosses the y-axis.

Solution: Given curve $y = e^x$ ... ... (1)

$$\therefore \frac{dy}{dx}=e^x \text{ and } \frac{d^2y}{dx^2}=e^x$$

When x = 0, we get from (1)

y = 1

$\therefore$ The (1) crosses curve y-axis at the point (0, 1).

At (0,1), we get, $\frac{dy}{dx}=1, \quad \frac{d^2y}{dx^2}=1$

$$\therefore \text{Radius of curvature } \rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$= \frac{(1+1)^{3/2}}{1} = 2\sqrt{2}$$

Example 3: Find the radius of curvature at any point θ of the curve $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$.

Solution: Given, $x = a(\theta - \sin\theta)$ ... ... (1)

$y = a(1-\cos\theta)$ ... ... (2)

$$\therefore \frac{dx}{d\theta} = a(1-\cos\theta) \text{ and } \frac{dy}{d\theta} = a\sin\theta$$

$$\therefore \frac{dy}{dx} = \frac{a\sin\theta}{a(1-\cos\theta)} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = \cot\frac{\theta}{2}$$

Also, $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\cot\frac{\theta}{2}\right) = -\operatorname{cosec}^2\frac{\theta}{2}.\frac{1}{2}.\frac{d\theta}{dx}$

$$= -\frac{1}{2}\operatorname{cosec}^2\frac{\theta}{2}.\frac{1}{a(1-\cos\theta)}$$

$$= -\frac{1}{2}\operatorname{cosec}^2\frac{\theta}{2}.\frac{1}{a.2\sin^2\frac{\theta}{2}}$$

$$= -\frac{1}{4a}\operatorname{cosec}^4\frac{\theta}{2}$$

$$\therefore \rho = \frac{\left(1+\cot^2\frac{\theta}{2}\right)^{3/2}}{-\frac{1}{4a}\operatorname{cosec}^4\frac{\theta}{2}} = \frac{-4a\operatorname{cosec}^3\frac{\theta}{2}}{\operatorname{cosec}^4\frac{\theta}{2}} = -4a\sin\frac{\theta}{2}$$

But radius of curvature cannot be negative.

$$\therefore \rho = 4a\sin\frac{\theta}{2}$$

Example 4: Find the radius of curvature at the point (r, θ) of the curve r = a(1–cos θ).

Solution : The given curve is r = a(1–cos θ)

$$\therefore \frac{dr}{d\theta} = a\sin\theta \text{ and } \frac{d^2r}{d\theta^2} = a\cos\theta$$

We know that radius of curvature

$$\rho = \frac{\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]^{3/2}}{r^2 - r\frac{d^2r}{d\theta^2} + 2\left(\frac{dr}{d\theta}\right)^2}$$

$$= \frac{\left[a^2(1-\cos\theta)^2 + a^2\sin^2\theta\right]^{3/2}}{a^2(1-\cos\theta)^2 - a(1-\cos\theta).a\cos\theta + 2a^2\sin^2\theta}$$

$$= \frac{a^3\left[1+\cos^2\theta-2\cos\theta+\sin^2\theta\right]^{3/2}}{a^2\left[1+\cos^2\theta-2\cos\theta+\cos^2\theta-\cos\theta+2\sin^2\theta\right]}$$

$$= \frac{a(2-2\cos\theta)^{3/2}}{(3-3\cos\theta)}$$

$$= \frac{a.2^{3/2}(1-\cos\theta)^{3/2}}{3(1-\cos\theta)}$$

$$= \frac{2\sqrt{2}a}{3}\sqrt{1-\cos\theta}$$

$$= \frac{2\sqrt{2}\,a}{3}\sqrt{\frac{r}{a}}$$

$$= \frac{2}{3}\sqrt{2ar}$$

Example 5: In the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, show that the radius of curvature at the end of the major axis is equal to the semi latus rectum of the ellipse.

Solution: The given ellipse is:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \text{... ... ... (1)}$$

Differentiating w.r. t x we get,

$$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=-\frac{b^2x}{a^2y} \qquad \text{... ... ... (2)}$$

$$\therefore \frac{d^2y}{dx^2}=-\frac{b^2}{a^2}\frac{d}{dx}\left(\frac{x}{y}\right)$$

$$= -\frac{b^2}{a^2}.\frac{y.1-x.\frac{dy}{dx}}{y^2}$$

$$= -\frac{b^2}{a^2y^2}\left[y+\frac{b^2x^2}{a^2y}\right], \text{ using (2)}$$

$$= -\frac{b^2}{a^4y^3}\left[a^2y^2+b^2x^2\right]$$

$$= -\frac{b^2}{a^4y^3}\left[a^2b^2\right], \text{ using (1)}$$

$$= -\frac{b^4}{a^2y^3}$$

$\therefore$ Radius of currature at (x, y)

$$\rho = \frac{\left\{1+\left(-\frac{b^2x}{a^2y}\right)^2\right\}^{3/2}}{-\frac{b^4}{a^2y^3}}$$

$$= \frac{\left(a^4y^2+b^4x^2\right)^{3/2}}{a^4b^4}, \text{ neglecting –ive signs}$$

$\because$ The co-ordinates of one end of major axis is (a, 0)

$\therefore$ Radius of curvature at (a, 0)

$$\rho = \frac{\left(0+b^4a^2\right)^{3/2}}{a^4b^4}$$

$$= \frac{b^2}{a}$$

= Semi-latus rectum of the ellipse.



CHECK YOUR PROGRESS

Q.21: Find the radius of currature at (x, y) on the curve $a^2y = x^3 - a^3$.

Q.22: Find the radius of curvature at any point (s, ψ) on the following curves.

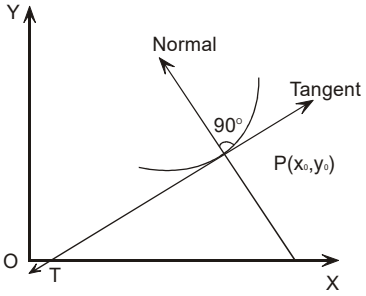
a) $s = 4a \sin\psi$ b) $s = c\log\sec\psi$

Q.23: Find the curvature at the point $\left(3\frac{a}{2}, 3\frac{a}{2}\right)$ of the curve $x^3 + y^3 = 3axy$.

Q.24: Find the radius of curvature at any point of the cycloid $x = a(t+\sin t)$, $y = a(1-\cos t)$.

Q.25: Find the curvature at any point θ of the curve $x = a\cos^3\theta$, $y = a\sin^3\theta$.

9.8 TANGENTS AND NORMALS



We know that the gradient of the tangent to a curve y = f(x) at a point (x, y) on the curve is given by the derivative $\frac{dy}{dx}$. The gradient is $\tan\theta$, where θ is the inclination made by the tangent with x–axis. In case $\frac{dy}{dx} = 0$, the tangent is parallel to x-axis. If $\frac{dx}{dy} = 0$, the tangent is parallel to y-axis.

9.8.1 Equations of Tangent and Normal

Let the given curve be y = f(x) and let $P(x_0, y_0)$ be a point on it.

Then, the slope of the tangent to the curve is $\left[\frac{dy}{dx}\right]_{(x_0, y_0)}$

$\therefore$ By co-ordinate geometry, the equation of the tangent at $P(x_0, y_0)$ is: $y - y_0 = \left[\frac{dy}{dx}\right]_{(x_0, y_0)} (x - x_0)$

Also, the equation of normal at $P(x_0, y_0)$ is:

$$y - y_0 = \frac{-1}{\left[\frac{dy}{dx}\right]_{(x_0, y_0)}} (x - x_0)$$

Remark 1: If $\left[\frac{dx}{dy}\right]_{(x_0, y_0)} = 0$, then the tangent is parallel to y-axis and its equation is $x = x_0$.

Remark 2: If $\left[\frac{dy}{dx}\right]_{(x_0, y_0)} = 0$, then the tangent is parallel to x-axis and its equation is $y = y_0$.

Illustration Examples:

Example 1: Find the equations of the tangent and the normal to the curve $y = 2x^2 + 3\sin x$ at x = 0.

Solution: Given curve is $y = 2x^2 + 3\sin x$ ... ... (1)

At x = 0, we get from (1), y = 0

$\therefore$ The point is (0, 0).

Differentiating (1) w.r. t. x, we get,

$$\frac{dy}{dx} = 4x + 3\cos x$$

$$\Rightarrow \left[\frac{dy}{dx}\right]_{(0,0)} = 0 + 3 = 3$$

$\therefore$ The equation of tangent at (0, 0) is: $y - 0 = 3(x-0)$

$$\Rightarrow 3x - y = 0$$

And, the equation of normal at (0, 0) is: $y - 0 = \frac{-1}{3}(x-0)$

$$\Rightarrow x + 3y = 0$$

Example 2: Find the points on the curve $y = 2x^3 - 3x^2 - 12x + 20$ at which the tangents are parallel to x-axis.

Solution: Given curve is $y = 2x^3 - 3x^2 - 12x + 20$ ... ... (1)

$$\therefore \frac{dy}{dx} = 6x^2 - 6x - 12$$

Let (α,β) be the point on (1) at which the tangent is parallel to x-axis.

$$\therefore \left[\frac{dy}{dx}\right]_{(\alpha,\beta)} = 0$$

$\Rightarrow 6\alpha^2 - 6\alpha - 12 = 0$

$\Rightarrow \alpha^2 - \alpha - 2 = 0$

$\Rightarrow (\alpha-2)(\alpha+1) = 0$

$\Rightarrow \alpha = 2, -1$

$\because$ (α,β) lies on (1)

$\therefore$ (1) $\Rightarrow \beta = 2\alpha^3 - 3\alpha^2 - 12\alpha + 20$

When $\alpha = 2$, we get, $\beta = 16 - 12 - 24 + 20 = 0$

When $\alpha = -1$, we get, $\beta = -2 - 3 + 12 + 20 = 27$

$\therefore$ The required points are (2, 0) and (–1, 27).

Example 3: From the origin tangents are drawn to the curve $y = \sin x$. Show that their points of contact lie on $x^2y^2 = x^2 - y^2$.

Solution: Given curve is $y = \sin x$ ... ... (1)

Let (x_1, y_1) be any point on (1)

$\therefore$ $y_1 = \sin x_1$ ... ... (2)

Differentiating (1) w.r.t. x, we get,

$$\frac{dy}{dx} = \cos x$$

$$\therefore \left[\frac{dy}{dx}\right]_{(x_1, y_1)} = \cos x_1$$

$\therefore$ The equation of tangent at the point (x_1, y_1) is:

$$y - y_1 = \cos x_1 . (x - x_1)$$

$\because$ This tangent passes through origin

$$\therefore 0 - y_1 = \cos x_1 . (0 - x_1)$$

$$\Rightarrow y_1 = x_1 \cos x_1 \quad \text{... ... ... (3)}$$

From (2) and (3), we get,

$$y_1^2 + \left(\frac{y_1}{x_1}\right)^2 = \sin^2 x_1 + \cos^2 x_1$$

$$\Rightarrow x_1^2 y_1^2 + y_1^2 = x_1^2$$

$$\Rightarrow x_1^2 - y_1^2 = x_1^2 y_1^2$$

$\therefore$ The locus of the point (x_1, y_1) is the curve $x^2 - y^2 = x^2 y^2$

Example 4: If the normal to the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ makes an angle Φ with the axis of x, show that its equation is:

$$y \cos\Phi - x \sin\Phi = a \cos 2\Phi .$$

Solution: The given curve is $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ ... ... (1)

Differentiating (1) with respect to x, we get,

$$\frac{2}{3} x^{-\frac{1}{3}} + \frac{2}{3} y^{-\frac{1}{3}} \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\left(\frac{y}{x}\right)^{\frac{1}{3}}$$

$\therefore$ Gradient of the normal to (1) at (x, y)

$$\Rightarrow \frac{-1}{\frac{dy}{dx}} = \left(\frac{x}{y}\right)^{\frac{1}{3}} = \tan\Phi \text{ (given)}$$

$$\Rightarrow \frac{x^{\frac{1}{3}}}{y^{\frac{1}{3}}} = \frac{\sin\Phi}{\cos\Phi}$$

$$\Rightarrow \frac{x^{\frac{1}{3}}}{\sin\Phi} = \frac{y^{\frac{1}{3}}}{\cos\Phi} = \frac{\sqrt{x^{\frac{2}{3}} + y^{\frac{2}{3}}}}{\sqrt{\sin^2\Phi + \cos^2\Phi}}$$

$$= \frac{\sqrt{a^{\frac{2}{3}}}}{1}, \text{ by (1)}$$

$\therefore x^{\frac{1}{3}} = a^{\frac{1}{3}}\sin\Phi$ and $y^{\frac{1}{3}} = a^{\frac{1}{3}}\cos\Phi$

i.e, $x = a\sin^3\Phi$ and $y = a\cos^3\Phi$

$\therefore$ $(a\sin^3\Phi, a\cos^3\Phi)$ is the point on (1) the normal at which makes an angle Φ with x-axis.

Hence, the equation of normal is:

$$y - a\cos^3\Phi = \tan\Phi(x - a\sin^3\Phi)$$

$$\Rightarrow y\cos\Phi - a\cos^4\Phi = x\sin\Phi - a\sin^4\Phi$$

$$\Rightarrow y\cos\Phi - x\sin\Phi = a(\cos^4\Phi - \sin^4\Phi)$$

$$= a(\cos^2\Phi - \sin^2\Phi)(\cos^2\Phi + \sin^2\Phi)$$

$$= a\cos 2\Phi$$

Example 5: The line $2y - x = 2$ touches the curve $y^2 = 2x$. Find the cordinates of the point of contact.

Solution: Let the point of contact be (x_1, y_1).

It lies on the given curve $y^2 = 2x$ ... ... (1)

$\therefore$ $y_1^2 = 2x_1$ ... ... (2)

Differentiating (1) w.r.t. x, we get,

$$2y\frac{dy}{dx} = 2$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{y}$$

$$\therefore \left[\frac{dy}{dx}\right]_{(x_1,y_1)} = \frac{1}{y_1}$$

Given line is $2y - x = 2$

$$\Rightarrow y = \frac{1}{2}x + 1$$

Its slope is $\frac{1}{2}$.

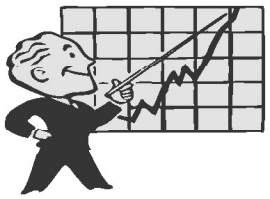
$$\therefore \frac{1}{2} = \frac{1}{y_1}$$

$$\Rightarrow y_1 = 2$$

Putting $y_1 = 2$ in (2) we get, $4 = 2x$

$$\Rightarrow x_1 = 2$$

$\therefore$ The point of contact is (2, 2).



CHECK YOUR PROGRESS

Q.26: Find the equations of tanget and normal to the curve $x^2 - 4y^2 = 9$ at the point (5, –2).

Q.27: Find the points on the curve $y = x^3 - 3x^2 - 9x + 5$ at which the tangent are parallel to x-axis.

Q.28: Find the equation of normal to the curve $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$ at the point (a,b).

Q.29: Find the equation of tangent and normal to the curve y(x–2)(x–3) – x + 7 = 0 at the point where it crosses the x-axis.

Q.30: Prove that all points of the curve $y^2 = 4a\left\{x + a\sin\left(\frac{x}{a}\right)\right\}$ at which the tangent is parallel to x–axis lie on a parabola.



9.9 LET US SUM UP

- f(x) is increasing in [a, b] if $f'(x) \geq 0, \forall x \in [a,b]$.
- f(x) is strictly increasing in [a, b] if $f'(x) > 0, \forall x \in [a,b]$.
- f(x) is decreasing in [a, b] if $f'(x) \leq 0, \forall x \in [a,b]$
- f(x) is strictly decreasing in [a, b] if $f'(x) < 0, \forall x \in [a,b]$.
- A function may be increasing in some interval and decreasing in some other interval.
- $\frac{dy}{dx}$ represents the rate of change of y with respect to x.
- The values of x obtained on solving $f'(x) = 0$ are called stationary or critical points. A function have local maximum or minimum at these points according at $f''(x) < 0$ or $f''(x) > 0$ respectively.

- The radius of curvature ρ for cartesion curve is:

$$\rho = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

- The radius of curvature ρ in polar form is:

$$\rho = \frac{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{3}{2}}}{r^2 - r\frac{d^2r}{d\theta^2} + 2\left(\frac{dr}{d\theta}\right)^2}$$

- Curvature is reciprocal of radius of curvature.
- $\frac{dy}{dx}$ at a point on the curve y = f(x) represents the gradient of the tangent line at that point.
- If $\frac{dy}{dx} > 0$, the tangent makes an acute angle with the x–axis.
- If $\frac{dy}{dx} < 0$, the tangent makes an obtuse angle with the x-axis.
- If $\frac{dy}{dx} = 0$, the tangent is parallel to x–axis.
- If $\frac{dy}{dx} = 0$, the tangent is parallel to y-axis.
- The equation of tangent to a curve y = f(x) at the point (x_0, y_0)

$$y - y_0 = \left[\frac{dy}{dx}\right]_{(x_0,y_0)}(x - x_0)$$

- The equation of normal to a curve y = f(x) at the point (x_0, y_0)

$$y - y_0 = \frac{-1}{\left[\frac{dy}{dx}\right]_{(x_0,y_0)}}(x - x_0)$$

9.10 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1: $f(x) = x^3 - 3x^2 + 3x + 2$

$\Rightarrow f'(x) = 3x^2 - 6x + 3$

$= 3(x^2 - 2x + 1)$

$= 3(x-1)^2$

$\geq 0, \ \forall x \in R$

$\therefore$ f(x) increases for all x.

Ans. to Q. No. 2: a) Let $f(x) = \log(1+x) - \frac{x}{1+x}$

$$\Rightarrow f'(x) = \frac{1}{1+x} - \frac{(1+x).1 - x.1}{(1+x)^2}$$

$$= \frac{1+x-1}{(1+x)^2}$$

$$= \frac{x}{(1+x)^2} > 0 \text{ for } x > 0$$

$\Rightarrow$ f(x) is an increasing function for x > 0 Also f(0) = 0

$\therefore$ f(x) > 0 for x > 0

$$\therefore \log(1+x) - \frac{x}{1+x} > 0$$

$$\Rightarrow \frac{x}{1+x} < \log(1+x) \qquad \text{... ... ... (1)}$$

Let g(x) = x – log(1+x)

$$\Rightarrow g'(x) = 1 - \frac{1}{1+x}$$

$$= \frac{1+x-1}{1+x}$$

$$= \frac{x}{1+x} > 0 \text{ for } x > 0$$

$\Rightarrow$ g(x) is an increasing function for x > 0 Also g(0) = 0

$\Rightarrow$ g(x) > 0 for x > 0

$\Rightarrow$ log (1+x) <x ... ... (2)

Combining (1) and (2), we get, $\frac{x}{1+x} < \log(1+x) < x$; if $x > 0$

b) Let $f(x) = \left\{x - \frac{x^2}{2(1+x)}\right\} - \log(1+x)$

$$\therefore f'(x) = 1 - \frac{(1+x).2x - x^2.1}{2(1+x)^2} - \frac{1}{1+x}$$

$$= 1 - \frac{2x + x^2}{2(1+x)^2} - \frac{1}{1+x}$$

$$= \frac{2(1+x)^2 - 2x - x^2 - 2(1+x)}{2(1+x)^2}$$

$$= \frac{2 + 4x + 2x^2 - 2x - x^2 - 2 - 2x}{2(1+x)^2}$$

$$= \frac{x^2}{2(1+x)^2} > 0 \text{ for } x > 0$$

$\Rightarrow$ f(x) is an increasing function for x > 0

Also, f(0) = 0

$\therefore$ f(x) > 0 for x > 0

$$\Rightarrow x - \frac{x^2}{2(1+x)} > \log(1+x)$$

Ans. to Q. No. 3: a) $f(x) = 2x^3 - 9x^2 - 168x + 36$

$\Rightarrow f'(x) = 6x^2 - 18x - 168$

$= 6(x^2 - 3x - 28)$

$= 6(x-7)(x+4)$

Now, $f'(x) > 0$

$\Rightarrow 6(x-7)(x+4) > 0$

$\Rightarrow$ (x – 7 > 0 and x + 4 > 0) or (x – 7 < 0 and x + 4 < 0)

$\Rightarrow$ (x > 7 and x > –4) or (x < 7 and x < –4)

$\Rightarrow$ x > 7 or x < – 4

$\therefore$ f(x) strictly increases in $]-\infty,-4[\cup]7,\infty[$

Also, $f'(x) < 0$

$\Rightarrow 6\,(x-7)(x+4) < 0$

$\Rightarrow$ (x – 7 > 0 and x + 4 < 0) or (x –7 < 0 and x + 4 > 0)

$\Rightarrow$ (x > 7 and x < –4) or (x < 7 and x > –4)

$\Rightarrow$ x < 7 and x > –4

$\Rightarrow -4 < x < 7$

$\therefore f(x)$ strictly decreases in $]-4,7[$

b) $f(x) = \frac{\log x}{x}; x > 0$

$$\Rightarrow f'(x) = \frac{\frac{1}{x}.x - \log x.1}{x^2}$$

$$= \frac{1-\log x}{x^2}$$

Now, $f'(x) \geq 0 \Rightarrow \frac{1-\log x}{x^2} \geq 0$

$\Rightarrow 1 \geq \log x$

$\Rightarrow 0 < x \leq e$

$\therefore f(x)$ is increasing in $]0,e[$

Also, $f'(x) \leq 0 \Rightarrow \log x \geq 1; x > 0$

$\Rightarrow x \geq e ; x > 0$

$\therefore f(x)$ is decreasing in $[e,\infty[$

c) $f(x) = 2x^3 - 15x^2 - 84x + 7$

$\therefore f'(x) = 6x^2 - 30x - 84$

$= 6(x^2-5x-14)$

$= 6(x-7)(x+2)$

Now, $f'(x) > 0$

$\Rightarrow (x-7)(x+2) > 0$

$\Rightarrow x < -2$ or $x > 7$

$\therefore f(x)$ is an increasing function in $]-\infty,-2[\cup]7,\infty[$

Also, $f'(x) < 0 \Rightarrow (x-7)(x-2) < 0$

$\Rightarrow -2 < x < 7$

$\therefore f(x)$ is decreasing function in $]-2,7[$

Ans. to Q. No. 4: Given, V = volume of the sphere.

r = radius of the sphere.

$\therefore V = \frac{4}{3}\pi r^3$, To find $\frac{dV}{dr}$

$$\frac{dV}{dr} = \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right)$$

$= \frac{4}{3}\pi.3r^2$

$= 4\pi r^1$

Ans. to Q. No. 5: Let at instant of time t, radius of spherical balloon = r,

volume of spherical balloon = V

surface area of spherical balloon = S

$\therefore V = \frac{4}{3}\pi r^3$ ... ... (1)

$S = 4\pi r^2$ ... ... (2)

According to question, $\frac{dV}{dt} = 20\,cm^3/sec$

$\therefore \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = 20$, by (1)

$\Rightarrow \frac{4}{3}\pi.3r^2\frac{dr}{dt} = 20$

$\Rightarrow \frac{dr}{dt} = \frac{5}{\pi r^2}$

From (2), $\frac{dS}{dt} = \frac{d}{dt}\left(4\pi r^2\right) = 4\pi.2r\frac{dr}{dt} = 8\pi r.\frac{5}{\pi r^2} = \frac{40}{r}$

When radius is 4 cm, we get, $\frac{dS}{dt} = 10cm^2/sec$

Ans. to Q. No. 6: Let at instant of time t, radius of circular wave = r

area of the enclosed portion = A

$\therefore A = \pi r^2$ ... ... (i)

Differentiating (i) w.r. to t, we get, $\frac{dA}{dt} = \pi.2r\frac{dr}{dt}$

$= 2\pi r(3.5)cm^2/sec\left[\because \frac{dr}{dt} = 3.5\,cm/sec\right]$

$= 7\pi r\ cm^2/sec$

When r = 7.5 cm, we get, $\frac{dA}{dt} = 7\pi(7.5)\,cm^2/sec = 52.5\,\pi\ cm^2/sec$

Ans. to Q. No. 7: Let at any instant of time t,

Radius of right circular cone = r

Height of right circular cone = h

Volume of right circular cone = V

$\therefore V = \frac{1}{3}\pi r^2 h$ ... ... (i)

Now, $h = \frac{1}{6}r$ (given)

$\therefore$(i) $\Rightarrow V = \frac{1}{3}\pi(6h)^2 h \Rightarrow V = 12\pi h^3$ ... ... (ii)

Differentiating (ii) w.r. to t, we get, $\frac{dV}{dt} = 12\pi.3h^2\frac{dh}{dt}$

$\Rightarrow 9 = 36\pi h^2\frac{dh}{dt}$ $\left[\because \frac{dV}{dt} = 9 \text{ cm}^3/\text{sec (given)}\right]$

$\Rightarrow \frac{dh}{dt} = \frac{1}{4\pi h^2}$

When h = 3, we get, $\frac{dh}{dt} = \frac{1}{4\pi.3^2} = \frac{1}{36\pi}$

Thus, the height of the sand cone is increasing at the rate of $\frac{1}{36\pi}$ cm/sec, when its height is 3 cm.

Ans. to Q. No. 8: Given $y^2 = 8x$ ... ... (1)

According to question $\frac{dx}{dt} = \frac{dy}{dt}$ ... ... (2)

(1) $\Rightarrow \frac{d}{dt}(y^2) = \frac{d}{dt}(8x)$

$\Rightarrow 2y\frac{dy}{dt} = 8\frac{dx}{dt}$

$\Rightarrow 2y\frac{dx}{dt} = 8\frac{dx}{dt}$, using (2)

$\Rightarrow$ 2y = 8

$\Rightarrow$ y = 4

Putting y = 4 in (1) we get, 16 = 8x $\Rightarrow$ x = 2

$\therefore$ The required point on the curve is (2, 4).

Ans. to Q. No. 9: Let AB be the lamp post and B be the position of light. At any instant of time t, let CD be the position of man.In this position, let x be the distance between man and the pole and y be the length of the shadow.

$\therefore$ AB = 6, CD = 2, AC = x, CL = y

From similar triangles ABL and CDL, we get,

$$\frac{AB}{CD} = \frac{AL}{CL}$$

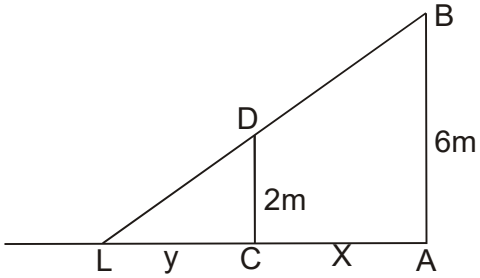
$$\Rightarrow \frac{6}{2} = \frac{x+y}{y}$$

$$\Rightarrow 3y = x + y$$

$$\Rightarrow 2y = x$$

$$\therefore \frac{d}{dt}(2y) = \frac{d}{dt}(x)$$

$$\Rightarrow 2\frac{dy}{dt} = \frac{dx}{dt}$$



Now, $\frac{dx}{dt} = 6\,km/h$ (given)

$$\therefore 2\frac{dy}{dt} = 6 \text{ km/h}$$

$$\Rightarrow \frac{dy}{dt} = 3 \text{ km/h}$$

$\therefore$ Length of shadow increases at the rate of 3 km/h. Also, distance tip of the shadow from lamp post = x + y

$\therefore$ Rate at which the tip of the shadow is moving away from the post:

$$= \frac{d}{dt}(x+y)$$

$$= \frac{dx}{dt} + \frac{dy}{dt}$$

= 6 km/h + 3 km/h

= 9 km/h

Ans. to Q. No. 10: Let at any instant of time t,

Length of edge of the cube = x

Volume of the cube = V

$\therefore V = x^3$ ... ... (i)

Now, $\frac{dx}{dt} = 3\,cm/sec$ (given) (ii)

Differentiating (i) w.r. to t, we get, $\frac{dV}{dt} = 3x^2\frac{dx}{dt}$

$$\Rightarrow \frac{dV}{dt} = 9x^2, \text{ using (ii)}$$

When x = 10cm, we get, $\frac{dV}{dt} = 9 \times 10^2$ cm^3/sec = 900 cm^2/sec

Ans. to Q. No. 11: a) Let $f(x) = 2x^2 - 6x + 3$

$\therefore f'(x) = 4x - 6$

For maximum or minimum, we have $f'(x) = 0$

$\Rightarrow 4x - 6 = 0$

$\Rightarrow x = \frac{3}{2}$

Now, $f''(x) = \frac{d}{dx}(4x-6)$

$\Rightarrow f''(x) = 4$

$\Rightarrow f''\left(\frac{3}{2}\right) = 4 > 0$

$\therefore$ f(x) is minimum when $x = \frac{3}{2}$ and the minimum value is

$= f\left(\frac{3}{2}\right)$

$= 2 \times \frac{9}{4} - 6 \times \frac{3}{2} + 3$

$= \frac{9 - 18 + 6}{2}$

$= -\frac{3}{2}$

b) Let $f(x) = x + \frac{1}{x}$

$\therefore f'(x) = 1 - \frac{1}{x^2}$

For maximum or minimum, we have $f'(x) = 0$

$\Rightarrow 1 - \frac{1}{x^2} = 0$

$\Rightarrow x^2 - 1 = 0$

$\Rightarrow (x+1)(x-1) = 0$

$\Rightarrow x = -1, x = 1$

Now, $f''(x) = \frac{d}{dx}\left(1 - \frac{1}{x^2}\right)$

$= 0 - \frac{-2}{x^3}$

$= \frac{2}{x^3}$

$\therefore$ $f''(-1) = -2 < 0$

$\Rightarrow$ f(x) is maximum at x = –1, and the maximum value is

$f(-1) = -1 + \frac{1}{-1} = -2.$

Again, $f''(1) = 2 > 0$

$\therefore$ f(x) is minimum at x = 1 and the minimum value is

f(1) = 1 + 1 = 2

c) Let $f(x) = 4x^3 - 15x^2 + 12x - 1$

$\Rightarrow f'(x) = 12x^2 - 30x + 12$

For maximum or minimum, we have $f'(x) = 0$

$\Rightarrow 12x^2 - 30x + 12 = 0$

$\Rightarrow 6(2x^2 - 5x + 2) = 0$

$\Rightarrow (2x-1)(x-2) = 0$

$\Rightarrow x = \frac{1}{2}, x = 2$

Now, $f''(x) = \frac{d}{dx}(12x^2 - 30x + 12)$

$= 24x - 30$

$\therefore f''\left(\frac{1}{2}\right) = 12 - 30$

$= -18 < 0$

$\Rightarrow$ f(x) is maximum when $x = \frac{1}{2}$ and the maximum value is:

$= f\left(\frac{1}{2}\right)$

$= 4 \times \frac{1}{8} - 15 \times \frac{1}{4} + 12 \times \frac{1}{2} - 1$

$= \frac{1}{2} - \frac{15}{4} + 5$

$= \frac{2 - 15 + 20}{4}$

$= \frac{7}{4}$

Also, $f''(2) = 24 \times 2 - 30$

$= 18 > 0$

$\Rightarrow$ f(x) is minimum when x = 2 and the minimum value is

$= f(2)$

$= 4 \times 8 - 15 \times 4 + 12 \times 2 - 1$

$= 32 - 60 + 24 - 1$

$= 56 - 61$

$= -5$

Ans. to Q. No. 12: Let $y = x^x$

$\therefore \log y = \log x^x$

$\Rightarrow \log y = x \log x$

$\therefore \frac{d}{dx}(\log y) = \frac{d}{dx}(x \log x)$

$\Rightarrow \frac{1}{y}.\frac{dy}{dx} = x.\frac{1}{x} + 1.\log x$

$\Rightarrow \frac{dy}{dx} = y(1 + \log x)$ ... ... (1)

For maximum or minimum, we have $\frac{dy}{dx} = 0$

$\Rightarrow y(1+\log x) = 0$

$\Rightarrow \log x = -1$

$\Rightarrow x = e^{-1} = \frac{1}{e}$

Now, $\frac{d^2y}{dx^2} = \frac{d}{dx}\{y(1 + \log x)\}$

$= \frac{dy}{dx}.(1 + \log x) + y.\frac{1}{x}$

$= y(1 + \log x)^2 + \frac{y}{x}$, using (1)

At $x = \frac{1}{e}$, we get, $\frac{d^2y}{dx^2} = \left(\frac{1}{e}\right)^{\frac{1}{e}}(1-1)^2 + \left(\frac{1}{e}\right)^{\frac{1}{e}}.e$

$= e\left(\frac{1}{e}\right)^{\frac{1}{e}} > 0$

$\therefore$ At $x = \frac{1}{e}$ the function is minimum. And the minimum value of the function is $\left(\frac{1}{e}\right)^{\frac{1}{e}}$.

Ans. to Q. No. 13: Let f(x) = sin x(1+cosx)

$\therefore f'(x) = \cos x(1+\cos x) + \sin x(-\sin x)$

$= \cos x + \cos^2 x - \sin^2 x$

$= \cos x + \cos 2x$

For maximum or minimum, we have $f'(x) = 0$

$\Rightarrow \cos x + \cos 2x = 0$

$\Rightarrow \cos 2x = -\cos x$

$\Rightarrow \cos 2x = \cos(\pi - x)$

$\Rightarrow 2x = \pi - x$

$\Rightarrow x = \frac{\pi}{3}$

$f''(x) = -\sin x - 2\sin 2x$

$\therefore f''\left(\frac{\pi}{3}\right) = -\sin\frac{\pi}{3} - 2\sin\frac{2\pi}{3} < 0$

$\therefore$ the function is maximum at $x = \frac{\pi}{3}$.

Ans. to Q. No. 14: Let the two numbers be x and y.

According to question, x + y = 20

$\Rightarrow y = 20 - x$ ... ... (1)

Let P be the product of the numbers

$\therefore P = xy$

$\Rightarrow P = x(20-x)$, using (1)

$\Rightarrow P = 20x - x^2$

$\therefore \frac{dP}{dx} = 20 - 2x$

For maximum or minimum of P, we have $\frac{dP}{dx} = 0$

$\Rightarrow 20 - 2x = 0$

$\Rightarrow x = 10$

Now, $\frac{d^2P}{dx^2} = \frac{d}{dx}(20 - 2x)$

$= -2 < 0$

$\therefore$ P is maximum when x = 10

Also, x = 10, gives y = 10

$\therefore$ The numbers are 10 and 10.

Ans. to Q. No. 15: Let ABCD be the rectangle and DEC the semicircle having diameter DC.

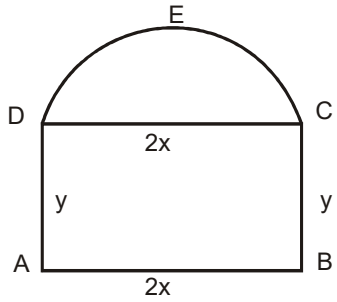
Let DC = 2x

AD = y

$\therefore$ Perimeter of the figure = $2x + 2y + \pi x$

$\therefore$ $2x + 2y + \pi x = 20$

$\Rightarrow y = 10 - x - \frac{\pi}{2}x$ ... ... (1)



Let area of the figure be A.

$\therefore A = 2xy + \frac{1}{2}\pi x^2$

$= 2x\left[10 - x - \frac{\pi}{2}x\right] + \frac{1}{2}\pi x^2$, using (1)

$= 20x - 2x^2 - 2.\frac{\pi}{2}x^2 + \frac{1}{2}\pi x^2$

$= 20x - 2x^2 - \frac{\pi}{2}x^2$

$\therefore \frac{dA}{dx} = 20 - 4x - \pi x$ ... ... (2)

For maximum or minimum of A, we have $\frac{dA}{dx} = 0$

$\Rightarrow 20 - 4x - \pi x = 0$

$\Rightarrow (4+\pi)x = 20$

$\Rightarrow x = \frac{20}{4+\pi}$

Differentiating (2) w.r.t x, we get, $\frac{d^2A}{dx^2} = -4 - \pi < 0$

$\therefore$ The area is maximum when $x = \frac{20}{4+\pi}$

$\therefore$ One side of the rectangle $2x = \frac{40}{4+\pi}$ metre

The other side of the rectangle = y

$= 10 - \frac{20}{4+\pi} - \frac{\pi}{2}.\frac{20}{4+\pi}$

$= 10 - \frac{20}{4+\pi} - \frac{10\pi}{4+\pi}$

$$= \frac{40 + 10\pi - 20 - 10\pi}{4 + \pi}$$

$$= \frac{20}{4 + \pi} \text{ metre.}$$

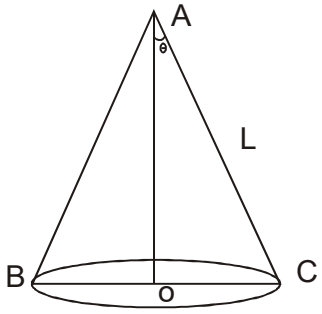
Ans. to Q. No. 16: Let ABC be the cone whose base is BC and vertex is A.

Let its height be AO. Let $\angle OAC = \theta$

Let l be the slant height AC.

The radius of the cone = OC = $l \sin\theta$

The height of the cone = OA = $l \cos\theta$



$$\therefore \text{Volume } V = \frac{1}{3}\pi(l\sin\theta)^2 . l\cos\theta$$

$$= \frac{\pi l^3}{3}\sin^2\theta\cos\theta$$

$$\therefore \frac{dV}{d\theta} = \frac{\pi l^3}{3}\left[2\sin\theta\cos^2\theta - \sin^3\theta\right]$$

$$= \frac{\pi l^3}{3}\sin\theta\left(2\cos^2\theta - \sin^2\theta\right) \quad \text{... ... ... (2)}$$

For the volume to be maximum,

$$\frac{dV}{d\theta} = 0$$

$$\Rightarrow \frac{\pi l^3}{3}\sin\theta\left(2\cos^2\theta - \sin^2\theta\right) = 0$$

$$\Rightarrow 2\cos^2\theta - \sin^2\theta = 0 \qquad \left[\because \sin\theta \neq 0 \text{ as } \theta > 0\right]$$

$$\Rightarrow 2 - \tan^2\theta = 0$$

$$\Rightarrow \tan^2\theta = 2$$

$$\Rightarrow \tan\theta = \sqrt{2} \quad \left[\because \theta > 0 \quad \therefore \tan\theta > 0\right]$$

$$\Rightarrow \theta = \tan^{-1}\sqrt{2}$$

Differentiating (2) w.r. to θ, we get,

$$\frac{d^2V}{d\theta^2} = \frac{\pi l^3}{3}\left[\sin\theta\left(-4\cos\theta\sin\theta - 2\sin\theta\cos\theta\right) + \cos\theta\left(2\cos^2\theta - \sin^2\theta\right)\right]$$

$$= \frac{\pi l^3}{3}\left[\cos\theta\left\{-4\sin^2\theta - 2\sin^2\theta + 2\cos^2\theta - \sin^2\theta\right\}\right]$$

$$= \frac{\pi l^3}{3}\cos\theta\left(-7\sin^2\theta + 2 - 2\sin^2\theta\right)$$

$$= \frac{\pi l^3}{3}\cos\theta\left(2 - 9\sin^2\theta\right)$$

When $\theta = \tan^{-1}\sqrt{2}$, we get,

$$\frac{d^2V}{d\theta^2} = \frac{\pi l^3}{3}.\frac{1}{\sqrt{3}}\left(2 - 9.\frac{2}{3}\right) \qquad \left[\because \tan\theta = \sqrt{2},\ \therefore \sin\theta = \sqrt{\frac{2}{3}}, \cos\theta = \frac{1}{\sqrt{3}}\right]$$

$$= -\frac{4\pi l^3}{3\sqrt{3}} < 0$$

$\therefore$ V is maximum when $\theta = \tan^{-1}\sqrt{2}$.

Ans. to Q. No. 17: Let ABCD be a rectangle of given perimeter 'a'.

Let AB = x

AD = y

$\therefore$ 2x + 2y = a

$$\Rightarrow y = \frac{1}{2}(a - 2x) \qquad \text{... ... ... (i)}$$

Let A be the area of the rectangle

Then, A = xy

$$\Rightarrow A = \frac{1}{2}\left(ax - 2x^2\right) \text{ using (i)}$$

$$\therefore \frac{dA}{dx} = \frac{1}{2}(a - 4x) \qquad \text{... ... ... (ii)}$$

For maximum or minimum value of A, we have

$$\frac{dA}{dx} = 0$$

$$\Rightarrow \frac{1}{2}(a - 4x) = 0$$

$$\Rightarrow x = \frac{a}{4}$$

Differentiating (ii) w.r. to x, we get

$$\frac{d^2A}{dx^2} = \frac{1}{2}(0 - 4)$$

$$= -2 < 0$$

$\therefore$ A is maximum when x = $\frac{a}{4}$

Also, $x = \frac{a}{4}$ gives $y = \frac{1}{2}\left(a - 2.\frac{a}{4}\right) = \frac{a}{4}$

$\therefore$ A is maximum when $x = y = \frac{a}{4}$

Hence, all rectangles of given perimeter, the square has the largest area.

Ans. to Q. No. 18: Let the portion of the side of the square to be cut from each of the four corners be x cm.

$\therefore$ Dimension of base of the box is of (18–2x)cm. and (18–2x)cm.
Height of the box = x cm.

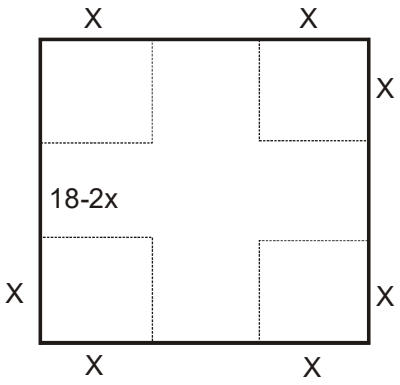
Let V be the volume of the box.

Then, V = x(18–2x)(18–2x); x > 0, x < 9.

$\Rightarrow$ $V = 324x - 7x - 72x^2 + 4x^3$

$\therefore \frac{dV}{dx} = 324 - 144x + 12x^2$

For maximum or minimum, we have



$\frac{dV}{dx} = 0$

$\Rightarrow 324 - 144x + 12x^2 = 0$

$\Rightarrow 12(x^2 - 12x + 27) = 0$

$\Rightarrow x^2 - 12x + 27 = 0$

$\Rightarrow (x-3)(x-9) = 0$

$\Rightarrow x = 3 \quad [\because x < 9]$

Now, $\frac{d^2V}{dx^2} = \frac{d}{dx}(324 - 144x + 12x^2)$

$= -144 + 24x$

When x = 3, we get $\frac{d^2V}{dx^2} = -144 + 72$

$= -72 < 0$

$\therefore$ V is maximum when x = 3

$\therefore$ A square of side 3cm. should be cut from each corner for the box to have a maximum volume.

Ans. to Q. No. 19: Let O be the centre of the sphere of radius R.

Let ABC be the cone inscribed in the sphere.

Let the height of the cone be R+x and radius be y.

Then, $x^2 + y^2 = R^2$

$\Rightarrow y^2 = R^2 - x^2$ ... ... (1)

Let V and V_1 denote the volume of the cone and sphere respectively.

Then $V = \frac{1}{3}\pi y^2(R + x)$ ... ... (2)

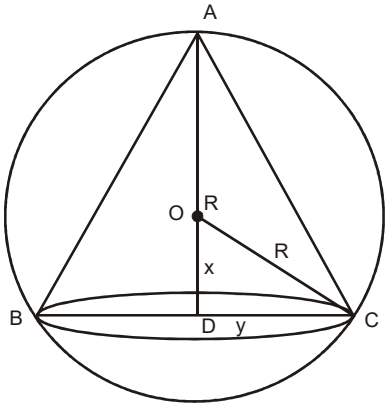
$V_1 = \frac{4}{3}\pi R^3$ ... ... (3)

Now, (2) $\Rightarrow V = \frac{1}{3}\pi(R^2 - x^2)(R + x)$, using (1)

$\Rightarrow V = \frac{1}{3}\pi(R^3 + R^2x - Rx^2 - x^3)$

$\therefore \frac{dV}{dx} = \frac{1}{3}\pi(R^2 - 2Rx - 3x^2)$

For maximum or minimum value of V, we get,



$\frac{dV}{dx} = 0$

$\Rightarrow R^2 - 2Rx - 3x^2 = 0$

$\Rightarrow 3x^2 + 2Rx - R^2 = 0$

$\Rightarrow (3x - R)(x + R) = 0$

$\Rightarrow 3x - R = 0 \quad [\because x + R \neq 0]$

$\Rightarrow x = \frac{R}{3}$

Now, $\frac{d^2V}{dx^2} = \frac{d}{dx}\left[\frac{1}{3}\pi(R^2 - 2Rx - 3x^2)\right]$

$= \frac{1}{3}\pi(-2R - 6x)$

At $x = \frac{R}{3}$, we get $\frac{d^2V}{dx^2} = \frac{1}{3}\pi\left(-2R - 6.\frac{R}{3}\right)$

$= -\frac{4}{3}\pi R < 0$

$\therefore$ V is maximum when $x = \frac{R}{3}$ and

Volume of the largest cone $= \frac{1}{3}\pi\left(R^2 - \frac{R^2}{9}\right)\left(R + \frac{R}{3}\right)$

$= \frac{1}{3}\pi.\frac{8R^2}{9}.\frac{4R}{3}$

$= \frac{8}{27}\left(\frac{4}{3}\pi R^3\right)$

$= \frac{8}{27}V_1 = \frac{8}{27}$ (Volume of the sphere)

Ans. to Q. No. 20: Given ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ... ... (1)

Let ABCD be the rectangle inscribed in (1).

Let AB = 2x, BC = 2y.

Let A be the area of the rectangle

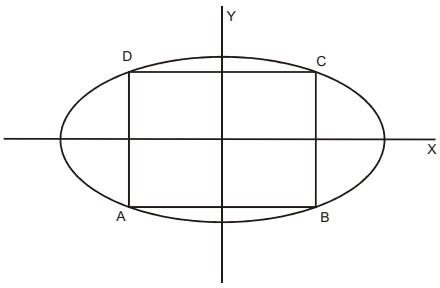
Then, A = (2x)(2y)

$= 4xy$

$= 4\frac{b}{a}x\sqrt{a^2 - x^2}$ $\left[\because \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} \quad \therefore y^2 = \frac{b^2}{a^2}\left(a^2 - x^2\right)\right]$

$\because A > 0 \ \therefore$ A is maximum $\Rightarrow A^2$ is maximum.

Let $u = A^2$



$\Rightarrow u = \frac{16b^2}{a^2}x^2\left(a^2 - x^2\right)$

$\Rightarrow u = \frac{16b^2}{a^2}\left(a^2x^2 - x^4\right)$

$\therefore \frac{du}{dx} = \frac{16b^2}{a^2}\left(2a^2x - 4x^3\right)$

For maximum or minimum value of u, we have

$\frac{du}{dx} = 0$

$\Rightarrow 2a^2x - 4x^3 = 0$

$\Rightarrow 2x(a^2 - 2x^2) = 0$

$\Rightarrow 2x^2 = a^2 \ [\because x > 0]$

$\Rightarrow x = \frac{a}{\sqrt{2}}$

Now, $\frac{d^2u}{dx^2} = \frac{16b^2}{a^2}\left(2a^2 - 12x^2\right)$

When $x = \frac{a}{\sqrt{2}}$, we get, $\frac{d^2u}{dx^2} = \frac{16b^2}{a^2}\left(2a^2 - 12.\frac{a^2}{2}\right)$

$= \frac{16b^2}{a^2}\left(-4a^2\right)$

$= -64b^2 < 0$

$\therefore$ u is maximum when $x = \frac{a}{\sqrt{2}}$

i.e. A is maximum when $x = \frac{a}{\sqrt{2}}$

$x = \frac{a}{\sqrt{2}} \Rightarrow y = \frac{b}{a}\sqrt{a^2 - \frac{a^2}{2}} = \frac{b}{\sqrt{2}}$

$\therefore$ Area of the largest rectangle = $4xy = 4\frac{a}{\sqrt{2}}.\frac{b}{\sqrt{2}} = 2ab$

Ans. to Q. No. 21: Given curve is $a^2y = x^3 - a^3$ ... ... (1)

Differentiating w.r. to x, we get, $a^2\frac{dy}{dx} = 3x^2$

$\Rightarrow \frac{dy}{dx} = \frac{3x^2}{a^2}$

Differentiating again, w.r. to x, we get, $\frac{d^2y}{dx^2} = \frac{6x}{a^2}$

$$\because \rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$= \frac{\left[1 + \left(\frac{3x^2}{a^2}\right)^2\right]^{3/2}}{\frac{6x}{a^2}}$$

$$= \frac{\left(a^4 + 9x^4\right)^{3/2}}{6a^4x}$$

Ans. to Q. No. 22: a) $s = 4a \sin\psi$

$$\therefore \frac{ds}{d\psi} = 4a\cos\psi$$

$$\Rightarrow \rho = 4a\cos\psi$$

b) $s = c \log \sec\psi$

$$\Rightarrow \frac{ds}{d\psi} = c.\frac{1}{\sec\psi}.\sec\psi\tan\psi$$

$$\Rightarrow \rho = c\tan\psi$$

Ans. to Q. No. 23: Given curve is $x^3 + y^3 = 3axy$ ... ... (1)

Differentiating w.r. to x, we get

$$3x^2 + 3y^2\frac{dy}{dx} = 3a\left(x\frac{dy}{dx} + y\right)$$

$$\Rightarrow x^2 + y^2\frac{dy}{dx} = ax\frac{dy}{dx} + ay \qquad \text{... ... ... (2)}$$

$$\Rightarrow \left(y^2 - ax\right)\frac{dy}{dx} = ay - x^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}$$

At $\left(\frac{3a}{2}, \frac{3a}{2}\right)$, we get, $\frac{dy}{dx} = \frac{\frac{3a^2}{2} - \frac{9a^2}{4}}{\frac{9a^2}{4} - \frac{3a^2}{2}} = -1$

Differentiating (2) w.r. to x, we get,

$$2x + 2y\left(\frac{dy}{dx}\right)^2 + y^2\frac{d^2y}{dx^2} = a\frac{dy}{dx} + ax\frac{d^2y}{dx^2} + a\frac{dy}{dx}$$

$$\Rightarrow \left(y^2 - ax\right)\frac{d^2y}{dx^2} = 2a\frac{dy}{dx} - 2x - 2y\left(\frac{dy}{dx}\right)^2$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{2a\frac{dy}{dx} - 2x - 2y\left(\frac{dy}{dx}\right)^2}{y^2 - ax}$$

At $\left(\frac{3a}{2}, \frac{3a}{2}\right)$, we get, $\frac{d^2y}{dx^2} = \frac{2a(-1) - 3a - 3a(1)}{\left(\frac{3a}{2}\right)^2 - \frac{3a^2}{2}}$

$$= \frac{-8a}{9a^2 - 6a^2} \times 4$$

$$= \frac{-32}{3a}$$

Hence, the radius of curvature ρ at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$

$$= \left[\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}\right]_{\left(\frac{3a}{2},\frac{3a}{2}\right)}$$

$$= \frac{(1+1)^{3/2}}{\frac{-32}{3a}}$$

$$= \frac{-2\sqrt{2}.3a}{32}$$

$$= -\frac{3a}{8\sqrt{2}}$$

$$= \frac{3a}{8\sqrt{2}}, \text{ neglecting –ive sign}$$

$\therefore$ Curvature at $\left(\frac{3a}{2}, \frac{3a}{2}\right) = \frac{1}{p} = \frac{8\sqrt{2}}{3a}$

Ans. to Q. No. 24: Given cycloid is:

$x = a(t+\sin t)$ ... ... (1)

$y = a(1-\cos t)$ ... ... (2)

$$\therefore \frac{dx}{dt} = a(1+\cos t)$$

and $$\frac{dy}{dt} = a\sin t$$

$$\therefore \frac{dy}{dx} = \frac{\sin t}{1+\cos t} = \frac{2\sin\frac{t}{2}\cos\frac{t}{2}}{2\cos^2\frac{t}{2}} = \tan\frac{t}{2}$$

Also, $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\tan\frac{t}{2}\right) = \sec^2\frac{t}{2}.\frac{1}{2}.\frac{dt}{dx}$

$$= \frac{1}{2}\sec^2\frac{t}{2}.\frac{1}{a(1+\cos t)}$$

$$= \frac{1}{2a}\sec^2\frac{t}{2}.\frac{1}{2\cos^2\frac{t}{2}}$$

$$= \frac{1}{4a}\sec^4\frac{t}{2}$$

Radius of curvature $\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$

$$= \frac{\left[1+\tan^2\frac{t}{2}\right]^{3/2}}{\frac{1}{4a}\sec^4\frac{t}{2}}$$

$$= 4a\sec^3\frac{t}{2}.\cos^4\frac{t}{2}$$

$$= 4a\cos\frac{t}{2}$$

Ans. to Q. No. 25: Given curve is

$x = a\cos^2\theta$ ... ... (1)

$y = a\sin^3\theta$ ... ... (2)

$\therefore \frac{dx}{d\theta} = a.3\cos^2\theta(-\sin\theta)$

$= -3a\cos^2\theta.\sin\theta$

and $\frac{dy}{d\theta} = a.3\sin^2\theta.\cos\theta$

$= 3a\sin^2\theta.\cos\theta$

$\therefore \frac{dy}{dx} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = -\tan\theta$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}(-\tan\theta)$$

$$= -\sec^2\theta.\frac{d\theta}{dx}$$

$$= \frac{-\sec^2\theta}{-3a\cos^2\theta\sin\theta}$$

$$= \frac{\sec^4\theta\,\mathrm{cosec}\theta}{3a}$$

Radius of curvature $\rho = \dfrac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}{\dfrac{d^2y}{dx^2}}$

$$\Rightarrow \rho = \frac{\left(1+\tan^2\theta\right)^{\frac{3}{2}}}{\dfrac{\sec^4\theta\,\mathrm{cosec}\theta}{3a}}$$

$$\Rightarrow \rho = \frac{3a\sec^3\theta}{\sec^4\theta\,\mathrm{cosec}\theta}$$

$$\Rightarrow \rho = 3a\sin\theta\cos\theta$$

$$\Rightarrow \rho = \frac{3}{2}a\sin 2\theta$$

$$\therefore \text{ Curvature } = \frac{1}{\rho} = \frac{2}{3a}\mathrm{cosec}2\theta$$

Ans. to Q. No. 26: Given equation of the curve is:

$$x^2 - 4y^2 = 9 \qquad \text{... ... ... (1)}$$

$$\therefore \frac{d}{dx}\left(x^2-4y^2\right) = \frac{d}{dx}(9)$$

$$\Rightarrow 2x - 8y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{x}{4y}$$

$$\therefore \left[\frac{dy}{dx}\right]_{(5,-2)} = -\frac{5}{8}$$

$\therefore$ The equation of tangent at (5, –2) is:

$$y + 2 = -\frac{5}{8}(x - 5)$$

or $8y + 16 = -5x + 25$

or $5x + 8x - 9 = 0$

The equation of normal at (5, –2) is:

$$y + 2 = \frac{-1}{-\frac{5}{8}}(x - 5)$$

or $y + 2 = \frac{8}{5}(x - 5)$

or $5y + 10 = 8x - 40$

or $8x - 5y - 50 = 0$

Ans. to Q. No. 27: Given curve is

$$y = x^3 - 3x^2 - 9x + 5 \qquad \text{... ... ... (1)}$$

$$\therefore \frac{d}{dx}(y) = \frac{d}{dx}(x^3 - 3x^2 - 9x + 5)$$

$$\Rightarrow \frac{dy}{dx} = 3x^2 - 6x - 9$$

For tangents parallel to x-axis, we have

$$\frac{dy}{dx} = 0$$

$\Rightarrow 3x^2 - 6x - 9 = 0$

$\Rightarrow 3(x^2 - 2x - 3) = 0$

$\Rightarrow (x-3)(x+1) = 0$

$\Rightarrow x = 3, -1$

When x = 3, we get from (1),

$y = 27 - 27 - 27 + 5 = -22$

When x = –1, we get from (1), y = –1

$y = -1 - 3 + 9 + 5 = 10$

$\therefore$ The required points are:

(3, –22) and (–1, 10).

Ans. to Q. No. 28: Given equation is

$$\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2 \qquad \text{... ... ... (1)}$$

Differentiating w.r. t. x, we get,

$$\frac{nx^{n-1}}{a^n}+\frac{ny^{n-1}}{b^n}\frac{dy}{dx}=0$$

$$\Rightarrow \frac{ny^{n-1}}{b^n}\frac{dy}{dx}=-\frac{nx^{n-1}}{a^n}$$

$$\Rightarrow \frac{dy}{dx}=-\frac{b^n}{a^n}.\frac{x^{n-1}}{y^{n-1}}$$

$$\therefore \left[\frac{dy}{dx}\right]_{(a,b)}=-\frac{b^n}{a^n}.\frac{a^{n-1}}{b^{n-1}}=-\frac{b}{a}$$

Equation of normal at (a, b) is

or

or $by - b^2 = ax - a^2$

or $ax - by = a^2 - b^2$

Ans. to Q. No. 29: Given curve is

$y(x-2)(x-3) - x + 7 = 0$ ... ... (1)

(1) Crosses x-axis at the point where ordinate is 0, i.e., y = 0

$x = 7$

(1) crosses x-axis at (7, 0).

Differentiating (1) w.r.t. x, we get,

The equation of tangent at (7, 0) is:

$y - 0 =$

or $20y = x - 7$

or $x - 20y - 7 = 0$

The equation of normal at (7, 0) is:

or $y = -20(x-7)$

or $y = -20x + 140$

or $20x + y = 140$

Ans. to Q. No. 30: The given curve is:

... ... (1)

Differentiating (1) w.r. to x, we get,

Let the tangent to (1) at (α, β) be parallel to x-axis.

Then

... ... (2)

Also, (α, β) lies as (1)

=

=

Hence, the locus of $^2 = 4ax$, which is a parabola.



9.12 FURTHER READING

1) Gupta, Sudhir (2011) (1st ed.): *Basics of Differentential Calculus*; New Delhi: Anmol publication Pvt. Ltd.
2) Habib, Amber. (2011) (1st ed.): *The Calculus of Finance*; Hyderabad: Universities Press India Pvt. Ltd.
3) Maron, I .A. (1993) (1st ed.): *Problems in Calculus of One Variable*; New Delhi: CBS Publishers & Distributors Pvt Ltd.
4) Sharma, A. K. (2004) (1st ed.): *Text Book of Differential Calculus for B.A, B.Sc, B.Com, I.A.S., P.C.S.*; New Delhi: Discovery Publishing House.



9.13 MODEL QUESTIONS

Q.1: Find the intervals in which the following functions are increasing or decreasing:

a) $x^3 - 9x^2 + 15x + 11$

b) $2x^3 - 12x^2 + 18x + 15$

c) $x^4 - 8x^3 + 22x^2 - 24x + 21$

d) $(x+3)^3(x-5)^3$

Q.2: Show that [redacted] for all [redacted]

Q.3: Find the points at which the following functions have maximum and minimum values :

a) $y = 9x^3 - 45x^2 + 48x + 11$ b) $y = 11 - 12x + 6x^2 - x^3$

Q.4: Show that the function [redacted] extremum at the point x = 0. Identify whether this extremum is a maximum or a minimum.

Q.5: Show that sin x(1+cos x) is a maximum when [redacted].

Q.6: For what value of x, sinx increases half as fast as x?

Q.7: The radius of a cylinder is increasing at the rate of 2 m/sec and its height is decreasing at the rate of 3 m/sec. Find the rate at which the volume of the cylinder is changing when the radius is 3m and height is 5m.

Q.8: Find the equation of the tangent to the curve  point $(a\cos\theta, b\sin\theta)$.

Q 9: Find the tangent to the curve $xy^2=4(4-x)$ at the point where it is cut by the line $y = x$.

Q.10: Find the radius of curvature at (r, 0) on the curve $r = 3(1+\cos\theta)$.

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UNIT 10: MEAN VALUE THEOREMS

UNIT STRUCTURE

10.1 Learning Objectives
10.2 Introduction
10.3 Rolle's Theorem
10.3.1 Geometrical Interpretation of Rolle's Theorem
10.4 Lagrange's Mean Value Theorem
10.4.1 Geometrical Meaning of Lagrange's Mean Value Theorem
10.4.2 Another form of Lagrange's Mean Value Theorem
10.5 Cauchy's Mean Value Theorem
10.5.1 Geometrical Meaning of Cauchy's Mean Value Theorem
10.6 Let Us Sum Up
10.7 Answers to Check Your Progress
10.8 Further Reading
10.9 Model Questions

10.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- understand that in a given section of a smooth curve, there is a point at which the slope of the curve is same as the average derivative of the section
- derive geometrical significance of various tangents to a curve
- know about Rolle's theorem
- know about Lagrange's mean value theorem
- know about Cauchy's mean value theorem.

10.2 INTRODUCTION

In this unit, we discuss some useful theorems which deal with tangents at interior points of a continuous curve in a closed interval. The most fundamental theorem is Rolle's theorem which is pivotal to the mean value theorems. Michel Rolle (1652-1719) was a French mathematician.



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theorem.

We also discuss Lagrange's mean value theorem (due to Joseph Louis Lagrange, 1736-1813) and Cauchy's mean value theorem (due to Augustin Louis Cauchy, 1789-1857).

10.3 ROLLE'S THEOREM

Let f be a function defined an [a, b] such that–

i) f is continuous in [a, b],

ii) f is derivable in]a, b[,

iii) f(a) = f(b).

Then there exists at least one value. say c $f'(c) = 0$.

10.3.1 Geometrical Interpretation of Rolle's Theorem



Fig. 10.1



Fig. 10.2

If the graph of a function y = f(x) is continuous at each point from the point A[a, f(a)] to the point B[b, f(b)] and tangent at each point between A and B exists and y co-ordinates of points A and B are equal, then there will be at least one point P on the curve between A and B at which tangent will be parallel to x-axis.

In Figure 10.1 there is only one such point where tangent is parallel to x-axis but in Figure 10.2 there are more than one such points.

Converse of Rolle's Theorem is Not True:

Consider $f(x) = x^3 - x^2 - x + 1$ in [–1, 2].

Here $f'(x) = 3x^2 - 2x - 1$

$f'(1) = 3 - 2 - 1 = 0$ and 1

But f(–1)



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Proof of Rolle's Theorem:

Since f is continuous in [a, b]

f is bounded in [a, b]

Let M and m be supremum and infimum respectively of f in [a, b].

Here there are two possibilities

M = m or M

Case I: Let M = m

In this case, f(x) = f(a) = f(b) = M = m,

f(x) = constant;

$f'(x) = 0$,

$f'(c) = 0$, for c

Case II: Let M

In this case, at least one of the bounds M or m must be different from f(a) = f(b).

Let M

f(a) < M

Let c

Then f(c+h) f(c) = M for h

f(c+h) – f(c) 0 for h

$f'(c) = 0$ [$'(c)$ exists $'(c) = Lf'(c) = f'(c)$]

Hence, in both cases $f'(c) = 0$; c

Illustrative Example:

Example 1: Verify Rolle's theorem for $f(x) = x^3 - 6x^2 + 11x - 6$, .

Solution: Given $f(x) = x^3 - 6x^2 + 11x - 6$,

Clearly, f(x) is continuous on R, being a polynomial function.

Also, $f'(x) = 3x^2 - 12x + 11$;

And $f(1) = 0 = f(3)$

f satisfies all the conditions of Rolle's theorem .

c $f'(c) = 0$, which is to be verified.

Now, $f'(x) = 3x^2 - 12x + 11$

$f'(c) = 3c^2 - 12c + 11$

$f'(c) = 0$ $3c^2 - 12c + 11 = 0$

and

Hence Rolle's theorem is verified.

Example 2: Are the conditions of Rolle's theorem satisfied for the following functions?

a) $f(x) = x^2$ in [2,3]

b) $f(x) = \tan x$ in $[0,$

Solution:

a) f is continuous as well as differentiable in [2, 3].

But $f(2) = 4$, $f(3) = 9$.

$f(2)$

The third condition of Rolle's theorem is not satisfied.

b) Here $f(0) = 0$ and $f(\pi) = 0$

$f(0) = f(\pi)$

Third condition of Rolle's theorem is satisfied.

But f is not continuous at

First two conditions of Rolle's theorem are not satisfied.

Example 3: If the function $f(x) = x^3 - 6x^2 + ax + b$ defined on [1,3] satisfies the Rolle's theorem for [redacted] and b.

Solution:

Since f(x) satisfies conditions of Rolle's theorem on [1, 3]

[redacted]

[redacted] $1 - 6 + a + b = 27 - 54 + 3a + b$

[redacted] $-5 + a = -27 + 3a$

[redacted] $22 = 2a$

[redacted] $a = 11$

[redacted] f(1) = f(3) is independent of b

[redacted] b[redacted]

Hence a = 11, b[redacted]

Example 4: If $2a + 3b + 6c = 0$, then prove that at least one root of the equation $ax^2 + bx + c = 0$, lies in the interval]0, 1[.

Solution: Consider the function [redacted]

We have f(0) = 0

And [redacted]

[redacted]

= 0 [redacted]

[redacted] f(0) = f(1) =0

Also f is continuous on [0, 1] and derivable in]0, 1[.

By Rolle's Theorem

[redacted] There exists at least one number, say α, in]0, 1[such that $f'(\alpha) = 0$ i.e.there exists at least one root of the equation $ax^2 + bx + c = 0$ in]0, 1[.



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CHECK YOUR PROGRESS

Q.1: Verify Rolle's theorem for the function $f(x) = x^3 - 4x$ in the interval $[-2, 2]$.

Q.2: Discuss the applicability of Rolle's theorem for the following functions in the indicated intervals:

a) $y = x(x-4)$; $x \in [0, 4]$ b) $f(x) = x$ in $[1, 2]$

c) $f(x) =$ in $[-1, 1]$.

Q.3: It is given that for the function $f(x) = x^3 + bx^2 + ax + 5$ on $[1, 3]$, Rolle's theorem holds with . Find values of a and b.

Q.4: Apply Rolle's theorem to find point (or points) on the following curves where the tangant is parallel to x-axis.

a) $f(x) = 16 - x^2$; $x \in [-1, 1]$ b) $f(x) = \cos x - 1$; $x \in [0, 2\pi]$

10.4 LAGRANGE'S MEAN VALUE THEOREM



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Let f be a function defined on [a, b] such that–

i) f is continuous in [a, b],

ii) f is derivable in]a, b[.

Then there exists at least one value ,say c

10.4.1 Geometrical Meaning of Lagrange's Mean Value Theorem



Fig. 10.3



Fig. 10.4

Slope of Chord 

If the graph of a function y = f(x) is continuous at each point from the point A[a, f(a)] to the point B[b, f(b] and tangent at each point between A and B exists and y co-ordinates of points A and B are not equal, then there will be at least one point P on the curve between A and B at which tangent will be parallel to the chord AB.

In Figure 10.3 there is only one such point where tangent is parallel to chord AB but in Figure 10.4 there are more than one such points.

Proof of Lagrange's Mean Value Theorem:

Let us consider the function g defined by

$g(x) = f(x) + Ax$ ... ... (1)

where A is a constant (real number)

Such that g(a) = g(b)

$f(a) + Aa = f(b) + Ab$

$Aa - Ab = f(b) + Ab$

$-A(b-a) = f(b) - f(a)$

$A = -$ 

With the value of A found above–

i) g is continuous in [a, b] as both f and Ax are continuous in [a, b]

ii) g is derivable in]a, b[as both f and Ax are derivable in]a, b[.

iii) g(a) = g(b)

So, g satisfies all the conditions of Rolle's theorem.

at least one c $g'(c) = 0$ ... ... (3)

Differentiating (1) with respect to x, we get,

$g'(x) = f'(x) + A$

$g'(c) = f'(c) + A$

$0 = f'(c) + A$, using (3)

$f'(c) = -A$

$f'(c) =$ 

10.4.2 Another form of Lagrange's M.V. Theorem

Let f be a function defined on [a, a+h] such that–

i) f is continuous in [a, a+h],

ii) f is derivable in]a, a+h[,

Then there exists t

$$f(a+h) = f(a) + h f'(a+th)$$

Proof: Let us consider the function g defined by

$$g(x) = f(x) + Ax \quad \text{... ... ...(1)}$$

where A is a constant such that–

$g(a) = g(a+h)$

$f(a) + Aa = f(a+h) + A(a+h)$

$f(a) + Aa = f(a+h) + Aa + Ah$

$f(a) - f(a+h) = Ah$

$A =$

Now, i) g is continuous in [a, a+h]

ii) g is derivable in]a, a+h[

iii) g(a) = g(a+h)

So, g satisfies all the conditions of Rolle's theorem

c $\quad f'(c) = 0 \quad$ ... ... (3)

$0 < t < 1$ (given)

a+th

Thus we can take c = a + th

(3) $f'(a+th) = 0 \quad$ ... ... (4)

Differentiating (1) with respect to x, we get,

$g'(x) = f'(x) + A$

$g'(a+th) = f'(a+th) + A$

$0 = f'(a+th) + A$, by (4)

$0 = f'(a+th) +$

$0 = hf'(a+th) + f(a) - f(a+h)$

$f(a+h) = f(a) + hf'(a+th)$

Illustrative Examples:

Example 1: Verify Lagrange mean value theorem for the function $f(x) = 2x^2 – 10x + 29$ in the interval [2, 7].

Solution: Given $f(x) = 2x^2 – 10x + 29$

Clearly, f(x) is continuous as well as differentiable for all x

a polynomial.

Therefore, i) f(x) is continuous in [2, 7]

ii) f(x) is differentiable in]2, 7[

So, f(x) satisfies conditions of Lagrange's theorem on [2, 7]

at least one real number c

Here,

$= 4x–10$

$f'(c) = 4c – 10$

$= 8$

$4c – 10 = 8$

$4c = 18$

$2 < c < 7$

Lagrange's mean value theorem is verified.

Example 2: Find the point on the curve $y = (x–3)^2$ where the tangent is parallel to the chord joining (3, 0) and (4, 1).

Solution: We have $y = f(x) = (x–3)^2$

$f(x) = x^2 – 6x + 9$, which is a polynomial.

f(x) is continuous in [3, 4] and differentiable in]3, 4[.

f(x) satisfies conditions of Lagrange's theorem.

Hence, c

2c – 6 = 1

Hence, the required point on the curve is

Example 3: If 0 < a < b, prove that

Solution: Let f(x) = $\tan^{-1}x$

Lagrange's mean value theorem in [a, b] is:

... ... (1)

0 < a < c, $^2 < c^2$

$1 + a^2 < 1 + c^2$

, by (1)

i.e.,

Again, 0 < c < b, $^2 < b^2$

$1 + c^2 < 1 + b^2$





, by (1)

i.e., 

Combining (2) and (3), we get,



Example 4: Using Lagrange's theorem find an approximate value of 

Solution: Let us consider the function f(x) = ; x

Since is continuous in [0,[and derivable in]0,

 f is continuous in [64, 66] and derivable in]64, 66[

 By Lagrange's theorem, 

Such that,





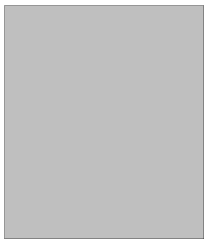








... ... (1)



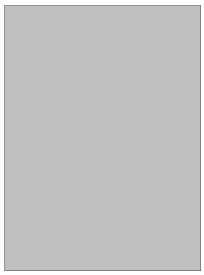






... ... (2)

Again,

(1)  



 ... ... (3)

From (2) and (3), we get,





Example 5: If a function f satisfies Lagrang's mean value theorem and $f'(x) = 0$, 

Solution: Let $a < x_1 < x_2 < b$

 f satisfies Lagrange's theorem in [a, b] and , $[x_1, x_2]$

 f also satisfies Lagrange's theorem in $[x_1, x_2]$

$]x_1, x_2[$ such that

 ... ... (1)

Given, $f'(x) = 0$, 

 $f'(c) = 0$ (c 

, by (1)

 $f(x_2) - f(x_1) = 0$

 $f(x_1) = f(x_2)$ for all $a < x_1 < x_2 < b$

 f(x) is a constant in [a, b]



CHECK YOUR PROGRESS

Q.5: Verify Lagrange's mean value theorem for the following functions in the given intervals.

a) $f(x) = x^3 - 2x^2 - x + 3$; $x \in [0, 1]$

b) $f(x) =$; $x \in [1, 5]$

c) $f(x) = \log_e x$; $x \in [1, e]$

Q.6: If $f(x) = (x-1)(x-2)(x-3)$ and $a = 0$, $b = 4$; find the value of c satisfying .

Q.7: Find the points on the curve $y = x^3 - 3x$ whose tangent to the curve is parallel to the chord joining (1, –2) and (2, 2).

Q.8: Compute the value of t in the Lagrange's mean value theorem $f(x+h) = f(x) + hf'(x+th)$ if $f(x) = ax^2 + bx + c$

10.5 CAUCHY'S MEAN VALUE THEOREM

Let f and g be two functions defined on [a, b] such that–

i) f and g are continuous in [a, b],

ii) f and g are derivable in]a, b[,

iii) $g'(x) \neq 0$

Then there exists at least one value, say c



10.5.1 Geometrical Meaning of Cauchy's Mean Value Theorem





Augustin L
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was a Fre
mathemat

Also,

and

Thus, if the graph of y = f(x) and y = g(x) are continuous at each point from

A[a, f(a)] to B[b, f(b)] and A/[a, g(a)] to B/[b, g(b)] respectively, tangents at each point between A, B and A/, B/ exist,tangent at no point between A/ and B/ is parallel to the x-axis,then there will be at least one point c

chords AB and A/B/ is equal to the ratio of the slopes of the tangents to the curves at which x = c cuts the curves.

Proof of Cauchy's Mean Value Theorem:

Let us consider the function

(x) = f(x) + Ag(x) ... ... (1)

where A is a constant (real number) such that–

(a) =

f(a) + Ag(a) = f(b) + Ag(b)

A[g(a)–g(b)] = f(b) – f(a)

A = –

With the value of A found above

i) is continuous in [a, b] as both of and g are continuous in [a, b]

ii) is derivable in]a, b[as both f and g are derivable in]a, b[.

iii) (a) = (b)

So,

at least one c]a, b[such that / (c) = 0... ... ... (3)

Differentiating (1) with respect to x, we get,

uis
39-1857)
h
an.

$\phi'(x) = f'(x) + Ag'(x)$

$\phi'(c) = f'(c) + Ag'(c)$

$0 = f'(c) + Ag'(c)$, by (3)

$-Ag'(c) = f'(c)$

$-A =$

Corollary: Another from of Cauchy's mean value theorem.

Let f and g be two functions defined on [a, a+h] such that.

i) f and g are continuous in [a, a+h],

ii) f and g are derivable in]a, a+h[,

iii) $g'(x) \neq 0$,

Then there exists t

Illustrative Examples:

Example 1: Verify Cauchy's mean value theorem for the functions

$f(x) = x^2$ and $g(x) = x^3$ over [1, 2].

Solution: Given, $f(x) = x^2$

$g(x) = x^3$

f and g are both continuous as well as differentiable in [1, 2] as both of them are polynomial functions.

Also, $f'(x) = 2x$

$g'(x) = 3x^2$

f and g satisfy all the conditions of Cauchy's M.V. theorem

such that–



$9c^2 = 14c$

$9c = 14$ 





Cauchy's mean value theorem is verified.

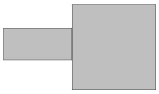
Example 2: Let a, b and θ satisfy $0 < a < \theta < b <$ 



Solution: Let $f(x) = \sin x$, $g(x) = \cos x$.

Clearly, f(x) and g(x) are continuous and derivable.

Also, $f'(x) = \cos x$, $g'(x) = -\sin x$

and $g'(x)$  0; 

]a, b[such that–









Example 3: If the function f is continuous in [a, b] and differentiable in]a, b[and $a < c < b$, deduce from Cauchy's M.V. theorem that $f(b) - f(a) = cf'(c)$ , $b > a > 0$

Solution: Let $g(x) = \log x$, $x > 0$

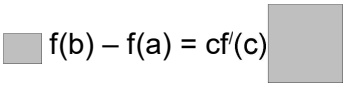
 for $x > 0$

Cauchy's mean value theorem is:







 f(b) – f(a) = cf′(c)

Example 4: In Cauchy's mean value theorem,let f(x) = 

g(x) = 

Solution: Cauchy M.V. theorem in [a, b] is:

 ... ... (1)

Here, f(x) =  

and g(x) =  









c is the geometric mean between a and b.



CHECK YOUR PROGRESS

Q.9: Calculate c in Cauchy's mean value theorem for each of the following pairs of functions.

i) f(x) = logx, g(x) = [ ] on [1, e]

ii) f(x) = [ ], g(x) = [ ] on [1, 3]

Q.10: In Cauchy's M.V. theorem if f(x) = [ ], g(x) = [ ], prove that c is harmonic mean between a and b.

Q.11: In Cauchy's M.V. theorem if f(x) = e^{-x}, g(x) = e^{x}, prove that c is arithmetic mean between a and b.



10.6 LET US SUM UP

- Rolle's theorem is as follows:

  Let f be a function defined an [a, b] such that–

  i) f is continuous in [a, b],

  ii) f is derivable in]a, b[,

  iii) f(a) = f(b).

  Then there exists at least one value. say c[ ] $f'(c) = 0$

- Lagrange's Mean value theorem is as follows:

  Let f be a function defined on [a, b] such that

  i) f is continuous in [a, b],

  ii) f is derivable in]a, b[.

  then there exists at least one value, say c[ ]

  [ ].

  Another form of Lagrange's Mean value theorem is as follows:

  Let f be a function defined on [a, a+h] such that–

  i) f is continuous in [a, a+h],

  ii) f is derivable in]a, a+h[.

then there exists t

$f(a+h) = f(a) + h\, f'(a+th)$

- Cauchy's Mean value theorem is as follows:

  Let f and g be two functions defined on [a, b] such that–

  i) f and g are continuous in [a, b],

  ii) f and g are derivable in]a, b[,

  iii) $g'(x)$ 0

  Then there exists at least one value, say c

  Another from of Cauchy's mean value theorem is as follows:

  Let f and g be two functions defined on [a, a+h] such that–

  i) f and g are continuous in [a, a+h],

  ii) f and g are derivable in]a, a+h[,

  iii) $g'(x)$ 0,

  Then there exists t



10.7 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1: Since f(x) is a polynomial function,so it is continuous in [–2, 2].

Also, $f'(x) = 3x^2 - 4$ exists in]–2, 2[.

And f(–2) = –8 + 8 = 0, f(2) = 8 – 8 = 0

f(–2) = f(2)

f(x) satisfies all the conditions of Rolle's Theorem in [–2, 2].

$'(c) = 0$.

$f'(c) = 0$ $^2 - 4 = 0$

Both values of c lies between –2 and 2.

Hence, Rolle's theorem is verified.

Ans. to Q. No. 2: a) Given $y = f(x) = x(x–4)$;

$= x^2 – 4x$

y is a polynomial function,so it is contiuous in [0, 4].

Also,

Again, $f(0) = 0$

$f(4) = 0$

$f(0) = f(4)$

y satisfies all the conditions of Rolle's theorem.

Therefore, such that

$2c – 4 = 0$

$c = 2$

Rolle's theorem is applicable.

b) Given $f(x) = x$,

Being a polynomial function f(x) is both continuous and differentiable on R.

$f(x) = x$ is continuous in [1, 2] and $f'(x) = 1$ exists in]1, 2[.

But

Hence,Rolle's Theorem is not applicable.

c) ;

f(x) is contnuous in [–1, 1]

f(x) is not differentable in

x = 0.

Hence, Rolle's Theorem is not applicable.

Ans. to Q. No. 3: Given, $f(x) = x^3 + bx^2 + ax + 5$

$f'(x) = 3x^2 + 2bx + a$

Rolle's theorem holds with

f(1) = f(3) and

Now, f(1) = f(3)

$1 + b + a + 5 = 27 + 9b + 3a + 5$

$2a + 8b + 26 = 0$

$a + 4b + 13 = 0$ ... ... (1)

Also, $f'(c) = 0$

$3c^2 + 2bc + a = 0$

given

, using (1)

$b = -6$

(1) $a - 24 + 13 = 0$

$a = 11.$

Ans. to Q. No. 4 : a) Given, $f(x) = 16 - x^2$

$f'(x) = -2x$

Now, $f'(c) = 0$ gives $-2c = 0$

$f(0) = 16 - 0 = 16$

The required point is (0, 16).

b) Given, $f(x) = \cos x - 1$

$f'(x) = -\sin x$

Now, $f'(c) = 0$ gives $-\sin c = 0$

$\sin c = 0$

$c = \pi$

$f(p) = \cos\pi - 1 = -1 - 1 = -2$

The required point is $(\pi, -2)$.

Ans. to Q. No. 5: a) Given $f(x) = x^3 - 2x^2 - x + 3$

$f'(x) = 3x^2 - 4x - 1$

f(x) is a polynomial

f(x)is continuous and derivable for all real values of x.

f(x) is continuous in [0, 1] and differentiable in]0, 1[.

Thus f(x) satisfies conditions of Lagrange's Mean value theorem in [0, 1].

$3c^2 - 4c - 1 = (1-2-2+3) - (3)$

$3c^2 - 4c - 1 = -2$

$3c^2 - 4c + 1 = 0$

$(3c-1)(c-1) = 0$

.

But $0 < c < 1$

Therefore, Lagrange's Mean value Theorem is verified for the given function.

b) Given,

Clearly, f(x) is continuous on [1, 5].

Also,

f(x) is differentiable on]1, 5[.

Therefore, f(x) satisfies conditions of Lagrange's Theorem on [1, 5].

such that





$2c^2 = 75 - 3c^2$

$5c^2 = 75$

$c^2 = 15$



But $1 < c < 5$



This verifies Lagrange's Mean value Theorem.

c) Given, $f(x) = \log_e x$

$f(x)$ is continuous and derivable for all real values of x.

Therefore, $f(x)$ is continuous in $[1, e]$ and differentiable in $]1, e[$.

 such that 







Hence, Lagrange's Mean Value Theorem is verified for the given function.

Ans. to Q. No. 6: Given, $f(x) = (x-1)(x-2)(x-3)$, $a = 0$, $b = 4$

Now, $f(0) = (-1)(-2)(-3) = -6$

$f(4) = (3)(2)(1) = 6$

$f(x) = x^3 - 6x^2 + 11x - 6$

$f'(x) = 3x^2 - 12x + 11$

$f'(c) = 3c^2 - 12c + 11$





$3 = 3c^2 - 12c + 11$

$3c^2 - 12c + 8 = 0$

Ans. to Q. No. 7: Let $f(x) = y$

$f(x) = x^3 - 3x$

$f(x)$ is a polynomial.

$f(x)$ is continuous and differentiable for all real values of x.

Lagrange's mean value theorem is appicable in [1, 2].

such that

$3c^2 - 3 = 4$

$f(c) = c^3 - 3c$

and

The points on the curve are and

Ans. to Q. No. 8: Given, $f(x) = ax^2 + bx + c$

$f(x+h) = a(x+h)^2 + b(x+h) + c$

and $f'(x) = 2ax + b$

Now, $f(x+h) = f(x) + hf'(x+th)$

$a(x+h)^2 + b(x+h) + c = ax^2 + bx + c + h[2a(x+th) + b]$

$ax^2 + 2ahx + ah^2 + bx + bh + c = ax^2 + bx + c + 2ahx + 2ah^2t + bh$

$ah^2 = 2ah^2t$

.

Ans. to Q. No. 9: i) Given, f(x) = log x,

Both f(x) and g(x) are continuous on [1, e].

Also, and

And

Therefore, f(x) and g(x) satisfy the conditions of Cauchy's Mean value theorem.

such that

.

ii) Given, ,

Both f(x) and g(x) are continuous on [1, 3].

Also, and

Also, ,

Therefore, f(x) and g(x) satisfy the conditions of cauchy's Mean value theorem.

such that

Ans. to Q. No. 10: Given, [redacted] and [redacted].

[redacted] and [redacted]

From Cauchy's Mean Value Theorem, we have

[redacted]

[redacted] c is the harmonic mean between a and b.

Ans. to Q. No. 11: Given $f(x) = e^{-x}$ and $g(x) = e^{x}$

$f'(x) = -e^{-x}$ and $g'(x) = e^{x}$

From Cauchy's Mean Value Theorem, we have

[redacted]

$e^{2c} = e^{a+b}$

$2c = a + b$

c is arithmetic mean between a and b.



10.8 FURTHER READINGS

1) Gupta, Sudhir (2011) (1st ed.): *Basics of Differentential Calculus*; New Delhi: Anmol publication Pvt. Ltd.
2) Habib, Amber. (2011) (1st ed.): *The Calculus of Finance*; Hyderabad: Universities Press India Pvt. Ltd.
3) Maron, I .A. (1993) (1st ed.): *Problems in Calculus of One Variable*; New Delhi: CBS Publishers & Distributors Pvt Ltd.
4) Sharma, A. K. (2004) (1st ed.): *Text Book of Differential Calculus for B.A, B.Sc, B.Com, I.A.S., P.C.S.*; New Delhi: Discovery Publishing House.



10.9 MODEL QUESTIONS

Q.1: Verify Rolle's theorem for the functions given below.

i) ;

ii) $f(x) = (x-a)^m(x-b)^n$;

iii) ;

iv) $f(x) = e^x(\sin x - \cos x)$;

v) $f(x) = (x-1)(x-2)^2$;

Q.2: Verify Lagrange's Mean value theorem for the functions given below.

i) $f(x) = x^3 - 6x$ in the interval

ii) $f(x) = x^3 - 3x - 1$ in

iii) $f(x) = \log x$ in .

Q.3: Verify the truth of Rolle's Theorem for the following function:
$f(x) = x^3 + 5x^2 - 6x$ on the interval [0, 1].

Q.4: Test whether the Lagrange's Mean value theorem of differential calculus holds for the function $f(x) = x - x^3$ on the interval [–2, 1] and find appropriate intermediate value of c.

Q.5: Show that  for all real a and b.

Q.6: Prove that if a rational integral function f(x) has n zeros between a and b, then $f'(x)$ has (n–1) zeros in (a, b).

Q.7: Use Lagrange's Mean value theorem to prove that if $0 < u < v < 1$

ss

Deduce that 

Q.8: Verify Cauchy's Mean value theorem for the following pair of functions in the indicated interval:

i) x^2, x in [a, b] ii) e^x, e^{-x} in [a, b]

iii) ,  in [a, b] iv) ,  in [a, b]

Q.9: If 

Q.10: Show that , if 

*** ***** ***

UNIT 11: PARTIAL DIFFERENTIATION

UNIT STRUCTURE

11.1 Learning Objectives
11.2 Introduction
11.3 Function of Several Variables
11.4 Partial Derivatives
11.5 Chain Rule for Partial Differentiation
11.6 Homogeneous Function
11.7 Euler's Theorem on Homogeneous Function
11.8 Let Us Sum Up
11.9 Answers to Check Your Progress
11.10 Further Reading
11.11 Model Questions

11.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- know about functions having more than one variable
- understand the concept partial differentiation
- evaluate partial derivatives using Chain Rule
- learn about homogeneous functions
- describe Euler's theorem on homogeneous functions.

11.2 INTRODUCTION

In this unit, we introduce functions of several variables. In Unit 1 and 5, we have already discussed about function of single variable and derivative of function of single variable. Here, we discuss partial derivatives of a functions of several variables. We also discuss about homogeneous functions. Finally, we introduce Euler's theorem on homogeneous functions.

11.3 FUNCTION OF SEVERAL VARIABLES

We are familiar with function of a single variable. But in nature there arise situations where we have to deal with functions of more than one variable.

Suppose we want to forecast the weather this weekend in your town. We construct a formula for the temperature as a function of several environmental variables, each of which is not predictable. Now, we would like to see how our weather forecast would change as one particular environmental factor changes, holding all the other factor constant. To do this investigation, we would use the concept of partial derivatives.

For example, the area of a rectangle A = xy which is a function of the variable x(length) and variable y (breadth). Also, the volume of cuboid V = xyz which is a function of the variables x(length), y(breadth) and z(height). If three variables u, y, z are so related that the value of u depends upon the values of x and y,then u is called a function of two variables x and y and this is denoted by u = f(x, y). Here u is called the dependent variable while x and y are called independent variables.

Hence, $u = f(x_1, x_2)$ is a function of two variables where u is dependent variable and x_1, x_2 are indepedent variables.

Also, $u = f(x_1, x_2, x_3)$ is a function of three variables where u is dependent variable and x_1, x_2, x_3 are independent variables and $u = f(x_1, x_2, x_3, x_4)$ is a function of four variables where u is dependent variable and x_1, x_2, x_3, x_4 are indepedent variables.

In general, a function of the form $u = f(x_1, x_2, x_3, ..., ..., x_n)$ is a function of n variables where u is dependent variable and $x_1, x_2, x_3, ..., ..., x_n$ are independent variables.

Examples:

i) $u = f(x, y) = x^2 + 2xy + y^2$ is a function of two variables x, y.

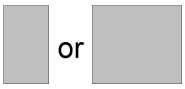
ii) $u = f(x, y, z) = x^2 + 3xy + yz^2 + z^3$ is a function of three variables x,y,z.

11.4 PARTIAL DERIVATIVES

If a derivative of a function of several independent variables is found with respect to any of them, keeping the others as constants, it is said to be a **Partial Derivative**. The process of finding the partial derivatives of a function of more than one independent variables is called **Partial differentiation.**

Let z = f(x, y) be a function of two variables.

The partial derivative of z with respect to x , denoted by

 or $_x$, is defined by

, provided the limit exists.

Also,the partial derivative of z with respect to y, denoted by

 or  $_y$ is defined by

,provided the limit exists.

 and  **first order partial derivatives of z.**

Example 1: Let $f(x, y) = xy^2$

Then , $f_x(x, y)$ =  $_y(x, y)$ = 



$= y^2$





$= 2xy$

Functions of more than two variables: Partial derivatives can also be defined for functions of three or more variables.

Thus if u = f(x, y, z), then the partial differential coefficient of u w.r.t x i.e. 

i.e. 

provided the limit exists.

Similarly,the partial differential coefficient of u w.r.t y i.e.  obtained by differentiating u w.r.t y keeping x and z as constants

i.e., 

provided the limit exists.

Again, the partial differential coefficient of u w.r.t z i.e. 

by differentiating u w.r.t z keeping x and y as constants.

i.e., 

In general, if u is a function of n variables, $u = f(x_1, x_2, ..., ..., x_n)$, Its partial derivative w.r.t the variable 



,

provided the limit exists.

Second order partial differential coefficients: If z = f(x, y), then  or f_x and  $_y$ are themselves functions of x and y and can be again differentiated partially.

We define , ,  and  as the second order partial derivatives of u and these are as denoted by

, ,   respectively**.**

Higher order partial derivatives: Partial derivatives of higher orderof a function are calculated by successive differentiation.

We denote the third order partial derivatives by

, ,

, etc.

and, in general, the n-th partial derivatives are



 etc.

Note: 1) All the rules of differentiation applicable to function of a single independent variable are also applicable in partial differentiation with the only difference that while differentiating (partially) with respect to one variable, all the other variables are treated (temporarily) as constants.

2) If $z = f(x)$ is a function of single variable x, we get .

3) If $z = f(x, y)$, a function of two independent variables xand y, we get  and 

independent variables $x_1, x_2, ..., ..., x_n$,we get , , ..., ..., .

4) If $u = f(x, y)$ and its partial derivatives are continuous i.e., 

 are continuous, the order of differentiation is immaterial i.e., .

5) If $z = u + v$, where $u = f(x, y)$, $v = g(x, y)$ then z is a function of x andy.

 and 

6) If $z = uv$, where $u = f(x, y)$, $v = g(x, y)$, then

; 

7) If , where $u = f(x, y)$, $v = g(x, y)$, then

; 

8) In repeated differentiation of a function of two variables,the order of differentiation can be changed arbitarily,provided only the derivatives in question are continuous functions.

9) A function of two variables has two first order partial derivatives, four second order partial derivatives and 2^n of nth order partial

derivatives. A function of m independent variables will have m^n partial derivatives of order n.

Illustative Examples:

Example 1: If $z = x^2y + xy^3$, find out the first order partial derivatives.

Solution: We have $z = x^2y + xy^3$





$= 2xy + y^3$

Also, 



$= x^2 + 3xy^2$

Example 2: If , then find out  and 

Solution: Given 







Also, 



.

Example 3: If $f(x, y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2, 1)$ and $f_y(2, 1)$.

Solution: Holding y constant and differentiating w.r.t x, we get

$$f_x(x, y) = 3x^2 + 2xy^3$$

and so $f_x(2, 1) = 3.2^2 + 2.2.1^3 = 16$

Similarly, holding x constant and differentiating w.r.t y, we get

$f_y(x, y) = 3x^2y^2 - 4y$

$f_y(2, 1) = 3.2^2.1^2 - 4.1 = 8$

Example 4: If

Solution: We have $u = \log(x^2 + y^2 + z^2)$

... ... (1)

Also,

... ... (2)

and

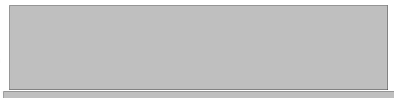
... ... (3)

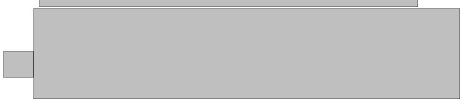
Adding (1), (2) and (3), we get

Example 5: If $z = f(x+ay) + \phi(x-ay)$, prove that

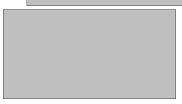
Solution: We have $z = f(x+ay) + \phi(x-ay)$

... ...(1)

Also, 

 ... ... (2)

From (1) and (2), we get .

Example 6: Verify that 

i) z = x log(y) ii) 

Solution: (i) We have z = x log(y)

Differentiating z partially w.r.t x taking y as constant, we have



Now, 

Again differentiating z partially w.r.t y taking x as constant, we have



Now, 

 Verified.

(ii) We have 

Differentiating z partially w.r.t x taking y as constant, we have



Now, 





and







Hence,  verified.

Example 7: If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, show that:

i) 

ii) 

Solution: we have $u = \log(x^3 + y^3 + z^3 - 3xyz)$







Adding, 

[$^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$]

i) 



ii)

Example 8: If $x = r\cos\theta$, $y = r\sin\theta$, prove that

a)

b) c)

Solution: (a) We have $x = r\cos\theta$, $y = r\sin\theta$

Therefore, $r^2 = x^2 + y^2$ ... ... (1)

Now,

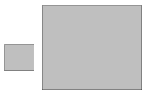


using (2)



... ... (3)

Again differentiating (1) partially w.r.t y, we get 



Differentiating (4) partially w.r.t y, we get

,

from (4)



... ... (5)

Adding (3) and (5), we get 

Also, 

Hence, 

(b) Differentiating (4) partially w.r.t x, we get



From (4), 



From (2), 

Now, 



(c) From (2) and (4), on squaring and adding, we get

Example 9 : If $u = e^{xyz}$, show that

Solution: Here, $u = e^{xyz}$

Differentiating partially w.r.t z, we get

Now, differentiating partially w.r.t y, we get

$= e^{xyz}(x^2yz + x)$

Again, differentiating partially w.r.t x, we get

$= e^{xyz}(2xyz + 1) + yze^{xyz}(x^2yz + x)$

$= e^{xyz}(2xyz + 1 + x^2y^2z^2 + xyz)\backslash$

$= e^{xyz}(1 + 3xyz + x^2y^2z^2)$



CHECK YOUR PROGRESS

Q.1: Find and for each of the following:

a) $z = x^2y^4$ b) $z = (x^4 + x^2)y^3$

c) $z = \sin x \cos xy$ d) $z = \sin(x^2 + y)$

Q.2: Find if z =

Q.3: If $f = x^3 + y^3 + z^3 + 3xyz$, find (a) f_x, (b) f_{xx}, (c) f_{xy}, (d) f_{xyz}.

Q.4: If , for what value of n, .

Q.5: Verify that  where u = x log y.

Q.6: If $u = \log(x^2 + y^2 + z^2)$, show that .

Q.7: If $z = xf(x + y) + y\phi(x + y)$, prove that



11.5 CHAIN RULE FOR PARTIAL DIFFERENTIATION

Total Derivatives: If $u = f(x, y)$, where $x = \phi_1(t)$ and $y = \phi_2(t)$, then



 is called the total differential coefficient of u w.r.t 't'.

while  and  are partial derivatives of u w.r.t x and y respectively.

Proof: Let there be a small increment δt in t and let the corresponding increments in u, x and y be δu, δx and δy respectively. Then we have

$u = f(x, y)$ ... ... (1)

and $u + \delta u = f(x + \delta x, y + \delta y)$ ... ... (2)

 $\delta u = f(x + \delta x, y + \delta y) - f(x, y)$

$= [f(x + \delta x, y + \delta y) - f(x, y + \delta y)] + [f(x, y + \delta y) - f(x, y)]$

Dividing by δt, we get

 ... ... (3)

Let δt  δx  δy 

Now, from (3), we get



 ... ... (4)

Since δx and δy tend to zero with δt, therefore (4) becomes



Note: 1) If $u = f(x, y, z)$, where $x = \phi_1(t)$, $y = \phi_2(t)$, $z = \phi_3(t)$, then



2) The above result can be extended to any finite number of variables. If $u = f(x_1, x_2, ..., x_n)$,
where $x_1 = \phi_1(t)$, $x_2 = \phi_2(t)$, ..., $x_n = \phi_n(t)$. then



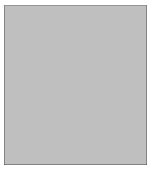
3) If u is a function of x and y, i.e., $u = f(x, y)$, where y is a function of x, then 

Deductions: 1) If $u = f(x, y)$, where $x = f_1(t_1, t_2)$ and $y = f_2(t_1, t_2)$, then we have 

and 

2) Differentiation of implicit functions: If we are given an implicit relation between x and y of the form,
i.e., $u = f(x, y) = c$,
where c is a constant then 

i.e. 

or 

ILLUSTRATIVE EXAMPLES:

Example 1: If $u = x^5 y^4$, where $x = t^2$ and $y = t^3$, find 

Solution: Here u is a function of x and y where x and y are the functions of t. u is also a function of single variable t.



$= (5x^4 y^4).2t + (4x^5 y^3).3t^2$

$= 10.x^4 y^4 t + 12x^5 y^3 t^2$

$= 10.t^8 t^{12} t + 12t^{10} t^9 t^2$

$= 22t^{21}$

Example 2: If $u = f(x – y, y – z, z – x)$, prove that 

Solution: Let $x – y = X$, $y – z = y$, $z – x = Z$.

Then $u = f(X, Y, Z)$, where X, Y, Z are functions of x, y, z.







... ... (1)

Similarly, 





... ... (2)

and 

Adding (1), (2) and (3), we get 

Example 3: Find  when $z = xy^2 + x^2y$, $x = at^2$, $y = 2at$. Verify by direct substitution.

Solution: , 

, 

Now, we have 

$= (y^2 + 2xy)2at + (2xy + x^2)2a$

$= (4a^2t^2)2at + (4a^2t^3 + a^2t^4)2a$

$= a^3(16t^3 + 10t^4)$

Second Part:

$z = xy^2 + x^2y = 2a^3t^5 + 4a^3t^4$



$= a^3(10t^4 + 16t^3)$

$= a^3(16t^3 + 10t^4)$ Hence verified.

Example 4: If z is a function of x and y, prove that if

. if $x = e^u + e^{-v}$, $y = e^{-u} - e^v$.

Solution: Here z is a function of x and y, while x and y are functions of u and v.

We have 

 ... ... (1)

Also, 

 ... ... (2)

Subtracting (2) from (1), we get





Example 5: If $w = f(x, y)$, $x = r\cos\theta$, $y = r\sin\theta$, show that



Solution: Here w is a function of x and y. Also, x and y are functions of r and θ.



 ... ... (1) [

Also,

... ... (2)

Squaring and adding (1) and (2), we get

Hence proved.

Example 6: Find $y + y^x = c$.

Solution: Let $f(x, y) = x^y + y^x$.

Example 7: Prove that

where $x = u\cos\alpha - v\sin\alpha$, $y = u\sin\alpha + v\cos\alpha$.

Solution: Here z is a function of u and v.

... ... (1)

Also,

... ... (2)

Now,

... ... (3)

... ... (4)

Adding (3) and (4), we get

Example 8: The altitude of a right circular cone is 15 cm and is increasing at 0.2 cm/sec.The radius of the base is 10 cm and is decreasing at 0.3 cm/sec. How fast is the volume changing?

Solution: Let x be the radius and y be the altitude of the cone. So, the volume v of the right circular cone is

Since x and y are changing w.r.t time t, differentiate v w.r.t 't'.

... ... (1)

Now, it is given that at x = 10, y = 15, and

From (1), we get

=

i.e., Volume is decreasing at the rate of ^{3}/sec.



CHECK YOUR PROGRESS

Q.8: Find as a total derivative and verify the result by direct substitution if $u = x^2 + y^2 + z^2$ and $x = e^{2t}$, $y = e^{2t}\cos 3t$, $z = e^{2t}\sin 3t$.

Q.9: If $u = x^2 - y^2$, $x = 2r - 3s + 4$, $y = -r + 8s - 5$, find and .

Q.10: Find from the given implicit function f connecting x and y:

a) $f(x, y) = x \sin(x-y) - (x+y)$ b) $x^y = y^x$

Q.11: If $u = xe^yz$, where , $z = \sin^2 x$. find .

Q.12: If $u = \sin^{-1}(x-y)$, $x = 3t$, $y = 4t^3$, show that .

11.6 HOMOGENEOUS FUNCTION

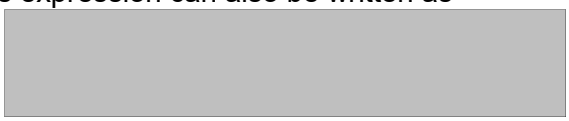
Consider the polynomial $f(x, y) = a_o x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \ldots\ldots\ldots + a_n y^n$.

Here the degree of each term is n. We say that f is a homogeneous function of degree n.

For example $f(x.y) = x^3 + y^3 + 3x^2y$ is a homogeneous function of degree 3.

Definition 1: A function f(x, y) is said to homogeneous function of degree n if each term in x and y is of the same degree n. Thus the expression $a_0x^n + a_0x^{n-1}y + a_2x^{n-2}y^2 + \ldots + a_ny^n$ is a homogeneous function of degree n.

This expression can also be written as

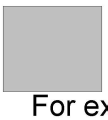


=  , where 

The expression can also be written as

Also 

 , where 

Definition 2: A function of two variables x, y is said to be a homogeneous function of degree n in x and y if it can be expressed in the form  or 

For example, the expression

i) $ax^2 + by^2 + 2hxy$ i.e. 

function of degree 2,

ii)  is a homogeneous function

of degree 

Whether a given function f(x, y) is homogeneous or not: We put tx for x and ty for y. f(fx, fy) = f(x y) then the function f(x y) is homogeneous of degree n, otherwise f(x y) is not homogeneous.

Examples: 1) $f(x.y) = x^2+y^2$ is homogeneous of degree 2, since

$f(tx, ty) = (tx)^2 + (ty)^2 = t^2(x^2+y^2) = t^2f(x, y)$.

2) f(x. y) =

3) f(x, y) =

4) f(x, y) =

5) $f(x, y) = x^2 + 5xy + y^2$ is homogeneous of degree 2, since

$f(tx, ty) = (tx)^2 + 5(tx)(ty) + (th)^2$

$= t^2x^2 + t^25xy + t^2y^2$

$= t^2(x^2+5xy+y^2)$

$= t^2 f(x, y)$

6) $f(x, y) = x^3 + y^3 + 3x^2y$ is a homogeneous of degree 3. Since

$f(tx, ty) = (tx)^3 + (ty)^3 + 3(tx)^2(ty)$

$= t^3(x^3 + y^3 + 3x^2y)$

$= t^3f(x.y)$

7) f(x, y) =

$= t^{-1} f(x. y)$

8) $f(x, y) = x^2 + x + y$ is not homogeneous.



Leonhard
1783) was
swiss mat

Note: The function f(x, y, z) of three variables x, y, z is homogeneous of degree n if $f(tx, ty, tz) = t^n f(x, y, z)$.



CHECK YOUR PROGRESS

Q.13: Determine whether or not each of the following functions is homogeneous and if so of what degree.

a) $3x + 4y$ b) $2x^2 + xy$ c) $x^2 + x^3$ d)

Q.14: Is the function homogeneous of any degree? If so, what degree?

Q.15: Is the function homogeneous of any degree?

Q.16: Is the function $x^3y^2 + x^{1/2}$ homogeneous of any degree?

Q.17: The function g(x, y) is homogeneous of degree r. Is f defined by f(x, y) = homogeneous of any degree?

11.7 EULER'S THEOREM ON HOMOGENEOUS FUNCTION



ler (1707-
famous
matician.

In previous section. we have already discussed about homogeneous functions. In this section, we will discuss Euler's theorem on homogeneous functions.

Statement: If z = f(x, y) be a homogeneous function of x, y of degree n, then

Proof: Since z = f(x, y) be a homogeneous function of x, y of degree n, therefore we can write z = f(x, y) as

... ... (1)

Differentaiting (1) portially w. r + x we get











... ... (2)

Again, differentiating (1) partially w. r .t y we get









... ... (3)

Now, we have





[from (2) & (3)]



= nz. Heree the result.

Note: Leonhard Euler (1707-1783) was a celebrated Swiss mathematician.

Corollory 1: If z = f(x y) is a homogeneous function of x, y of degree n then 

Proof: We have  ]

Differentiating (1) partially w. r .t 'x', we get

 ... ...(2)

Again differentiating (1) partially w. r .t 'y', we get

 ... ... (3)

Multiplying (2). (3) by x. y respectively and adding we obtain

 

Corollory 2: If z is a homogeneous function of x and y of degree n and z = f(v), then .

Proof : By Euler's theorem

 ... ... (1)

Now, given that z = f(v)

 and 

Putting these values in eq^{n} (1) we get

or 

Corollory 3: If z is a homogeneous function of x and y of degree n and z = f(v) then

where 

Proof: By Corollory 2, we have 

Given that 

 ... ... (1)

Differentiating (1) parlially w. r .t 'x', we get



or 

Againg, differentiating (1) parlially w. r .t 'y', we get



or  

or 

Multiplying (2) by x and (3) by y and adding we get



or  $'(v) - 1]\ g(v)$ [from (1)]

or  $'(v) - 1]$

Note: Euler's theorem can be generalized to homogeneous functions of any finite number of variables.

i) If u = f(x. y. z) is a homogeneous function of degree n of variables x, y, z, then 

ii) If $u = f(x_1, x_2, ..., ..., ..., x_n)$ be a function of degree n of variables x_1, x_2, ..., x_n, then 

ILLUSTRATIVE EXAMPLES:

Example 1: Verify Euler's theorem for the function

$f(x, y) = ax^2 + 2hxy + by^2$

Solution: Here $f(x, y) = ax^2 + 2hxy + by^2$ ... ... (1)

Clearly f(x, y) is a homogeneoue function of degree 2.

We verify that 

Dfferentiating (1) partially w. r .t 'x' we get





... ... (2)

Again dfferentiating (1) partially w. r .t 'y' we get





... ... (3)

Adding (2) and (3) we get



$= 2(ax^2 + 2hxy + by^2)$

Thus, Euler's theorem is verified.

Example 2: Verify Euler's theorem for the function

$f = x^3 – 2x^2y + 3xy^2 + y^3$

Solution: Here, f is a homogeneous function of degree 3.

$f_x = 3x^2 – 4xy + 3y^2$

$f_y = –2x^2 + 6xy + 3y^2$

$xf_x + yf_y = x(3x^2 – 4xy + 3y^2) + y(–2x^2 + 6xy + 3y^{2)}$

$= 3(x^3 – 2x^2y + 3xy^2 + y^3)$

$= 3f$

Here, Euler's theorem is verified.

Example 3: If $u = \sin^{-1}$ , prove that $xu_x + yu_y =$ 

Solution: Here

is a homogeneous function of degree i.e.

Example 4: If

Solution: We have

Here sinv is homogeneous function of degree 1 – 4 = –3

Then by Euler's theorem

or

Example 5: i) If

ii) If

Solution: i) Here

Thus, v is a homogeneous function in x and y of degree

Apply Euler's Theorem, we have

ii) Here

= u + v ... ... (1)

Where

and

Now,

Therefore u is a homogeneous function in x and y of degree 6.

Also,

Therefore, v is homogeneous in x and y of degree 0.

By apply Euler's theorem, we have

... ... (2)

and

Adding (2) and (3), we get

[from (1)]

Hence Proved

Example 6: If

Solution:

, corollary 3 (11.7)

Proved

Example 7: If

Solution: Let $u = z_1 + z_2$ ... ... (1)

where [ ] and [ ]

Obviously, z_1 is a homogeneous function of x and y of degree 1 and z_2 is a homogeneous function of x and y of degree o.

Now, [ ]

[ ]

$= 1.z_1 + 0.z_2$ (By Euler's theorem)

Thus [ ]

Differentiating (2) partially w. r. t x and y respectively,

we get [ ]

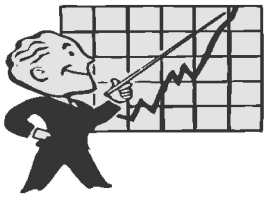
and [ ] ... ... (4)

Multiplying (3) by x and (4) by y and adding we get

[ ]

or [ ]

or [ ]



CHECK YOUR PROGRESS

Q.18: Verify Euler's theorem for [ ].

Q.19: If [ ] find $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy}$.

Q.20: If , find the value of .

Q.21: If , prove that

Q.22: If , prove that



11.8 LET US SUM UP

- A function of the form $u = f(x_1, x_2, x_3, ..., x_n)$ is n variables where u is dependent variable and $x_1, x_2, x_3, ..., x_n$ are independent variables is called function of several variables.
- If u is a function of n variables i.e $u = f(x_1, x_2, ..., ..., ..., x_n)$, its partial derivative w. r. t the variable x_i is given by

  provided the limit exists.

  If u is a function of two or more independent variables then partial derivative of u w. r. t any one of the independent vatiables is the ordinary derivative of z w. r. t that variable treating all other variables as constant.
- Partial derivatives of higher order of a function are calculated by successive differentiation. We denote the nth partial derivatives by

  etc.

- If u = f(x, y) where x = $\phi_1(t)$ and y = $\phi_2(t)$, then

   is called the total differentiation co-efficient of u. w. r.+ 't'.

  The above result can be extended to any finite number of variables.

  i.e. if u = f(x_1, x_2, ..., ..., ..., x_n) where $x_1 = \phi_1(t)$, $x_2 = \phi_2(t)$, ..., $x_n = \phi_n t(t)$

  then 

  Also, if u = f(x, y) where x = $f_1(t_1, t_2)$ and y = $f_2(t_1, t_2)$, then

  we have 

  and 

- A function f(x, y) is said to be homogeneous function of degrees if each of them in x and y is of the same degree n.

  Also, a function f(x, y) of two variables x, y is said to be a homogeneous function of degree n in x and y if it can be expressed

  in the form x^nf  or 

- If z = f(x, y) be a homogeneous function of x, y of degree n then

  Euler's theorem states that 

  Deductions from Euler's Theorem:

  a) If z = f(x, y) is a homogeneous function of x, y of degree n, then

  b) If z = f(x, y) is a homogeneous function of x and y of degree n

     and z = f(), then 

     where 

- Euler's theorem can be extended to homogeneous functions of any finite number of vatiables.



11.9 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1: a)

b)

c) = cox x cox xy – y sin y sin xy

= –x sinx sinxy

d) = 2x cos(x^2+y)

Ans. to Q. No. 2: Given

Ans. to Q. No. 3: f = $x^3 + y^3 + z^3 + 3xyz$

a)

b)

c)

d)

Ans. to Q. No. 4: Given

Also,

Since

Ans. to Q. No. 5: Given u(x.y) = x logy

Now

Also

Now 

Hence 

Ans. to Q. No. 6: We have $u = \log(x^2 + y^2 + z^2)$

 [treating x and y as constants]

and 



Now 

By symmetry, 

Hence 

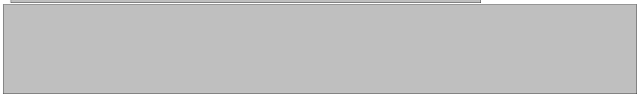
Ans. to Q. No. 7: We have $z = xf(x+y) + y\phi(x+y)$



and 



Also 

and 



Again 





Now, 

 Proved

Ans. to Q. No. 8: Solution: Here u is a function of x, y, z and x, y, z are in turn functions of t. Thus u is a function of 't' via the intermediate variables x, y, z. Then



$= 2x.2e^{2t} + 2y(2e^{2t}\cos 3t - 3e^{2t}\sin 3t) + 2z(2e^{2t}\sin 3t + 3e^{2t}\cos 3t)$

$= 2e^{2t}.2e^{2t} + 2e^{2t}\cos 3t(2e^{2t}\cos t - 3e^{2t}\sin 3t) +$

$2e^{2t}\sin 3 + (2z(2e^{2t}\sin 3t + 3e^{2t}\cos 3t)$

$= 4e^{4t} + 4e^{4t}\cos 3t - 6e^{4t}\cos 3t \sin 3t + 4e^{41}\sin^2 3t + 6e^{4t}\sin 3t \cos 3t$

$= 4e^{4t} + 4e^{4t}(\cos^2 3t + \sin^2 3t)$

$= 4e^{4t} + 4e^{4t}$

$= 8e^{4t}$

Now, verification by direct substitution

$u = x^2 + y^2 + z^2$

$= e^{4t} + e^{4t}\cos^2 3t + e^{4t}\sin^2 3t$

$= e^{4t} + e^{4t}(\cos^2 3t + \sin^2 3t)$

$= 2e^{4t}$



Ans. to Q. No. 9: Solution: Here, u is a function of x, y which are functions of r,s. So by chain rule



$= 2x.2 + (-2y)(-1)$

$= 2(2x+y)$



$= 2x(-3) + (-2y).8$

$= -6x - 16y$

Ans. to Q. No. 10: Solution: a) Let f(x, y) = xsin(x–y) – (x+y)





b) Given $x^y = y^x$

Taking log ylogx = xlogy

Let f(x, y) = ylogx – xlogy = 0



Ans. to Q. No. 11: Solution: Here u is afunction of x, y and z while y and z are functions of x.







Ans. to Q. No. 12: By Total derivative,











Ans. to Q. No. 13: a) Homogeneous of degree 1 : $3(tx) + 4(ty) = t(3x+4y)$

b) Homogeneous of degree 2 : $2(tx)^2 + (tx)(ty) = t^2(2x^2+xy)$

c) Not Homogeneous

d) Homogeneous of degree :

Ans. to Q. No. 14: We have

So the function is homogeneous of degree 2.

Ans. to Q. No. 15: Yes,

the function is homogeneous of degree

Ans. to Q. No. 16: No. it is not homogeneous of any degree.

Ans. to Q. No. 17: We have

Thus, f is homogeneous of degree (r–2)

Ans. to Q. No. 18: More u is a homogeneous function of degree 0.

Now,

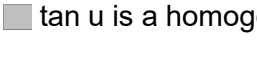
= 0



Henee Euler's theorem is verified.

Ans. to Q. No. 19: 



 tan u is a homogeneous function of degree (3-1) i.e., 2.

by Euler's theorem, 











 $xu_x + yu_y = \sin 2u$ ... ... (1)

Differentiating (1) w.r.t. x and y, we get

$xu_{xn} + u_x + yu_{yx} = 2u_x \cos 2u$ ... ... (2)

$xu_{xy} + u_y + yu_{yy} = 2u_y \cos 2u$ ... ... (3)

Multiplying (2) by x and (3) by 'y' and adding, we get

$x^{2u}{}_{xx} + 2xyu_{xy} + y^2u_{xy} = (2\cos 2u - 1)(xu_{x+} yu_y)$

$= (2\cos 2u - 1)\sin 2u$ By (1)

$= \sin 4u - \sin 2u$

Ans. to Q. No. 20: Given 





 U is a homogeneous function of degree n = 1.

By using Euler's theorem 

Differentiating (1) partially w. r. t 'x', we get







... ... (2)

Again, differentiating (1) Partially w. r. t 'y', we get







... ... (3)

Adding (2) and (3), 

Ans to Q No 21: We have 



Here sin u is a homogeneous function of degree (3–1) i.e 2.

Hence by Euler's theorem



or 

or 

or 

Ans. to Q. No. 22: Here

is a homogeneous function of degree

in x and y.

By Euler's theorem, we have

or

or

Differentiating (1) partially w. r. t 'x', we get

or

Differentiating (1) partially w. r. + 'y', we get

... ...(3)

Multiplying (2) by x, (3) by y and adding, we get

[using (1)]

[$^2u - 1 = \cos^2 u$]



11.10 FURTHER READING

1) Dhami, H. S., *Differential Calculus*; New Delhi: New Age International (P) Limited Publishers.
2) Ginzburg, A.: *Calculus Problems and Solutions*.
3) Kishan, Hari: *Differential Calculus*; New Delhi: Atlantic Publishers Distributors (P) Ltd.
4) Naryan, Shanti: Differential Calculus; S. Chand Limited.
5) Rao, S. Balachandra & Shantha, C. K.: *Differential Calculus*; New Delhi: New Age International (P) Limited Publishers.
6) Sharma, A. K.: *Text Book of Differential Calculus*; New Delhi: Discovery Publishing House.



11.11 MODEL QUESTIONS

Q.1: Find and

i) $u = x \sin y + y \sin x$ ii)

iii) $u = x^y$ iv)

Q.2: If , Prove that

Q.3: If , Prove that

Q.4: If , Show that

Q.5: Verify Euler's theorem for the following functions :

i) $u = 3x^2yz + 4xy^2z + 5y^4$ ii)

iii) 

iv) u = axy + byz + czx

v) 

Q.6: If , find 

Q.7: If z = z(x, y), $x = e^{u} + e^{-v}$, $y = e^{-u} - e^{v,}$

Prove that 

Q.8: If z = f(x, y) , where $x = e^{u}\cos v$, $y = e^{u}\sin v$

Prove that 

Q.9: If u is a homogeneous function of degree n, Prove that

a) 

b) 

Q.10: If $u = e^{ax+by} f(ax–by)$, Prove that 

Q.11: If u = cos xyz, find 

Q.12: If , Prove that $U_{xx} + U_{yy} + U_{zz} = 0$

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UNIT 12: EXTREME VALUE OF A FUNCTION OF SEVERAL VARIABLES

UNIT STRUCTURE

12.1 Learning Objective
12.2 Introduction
12.3 Taylor's Theorem for a Function of Two Variables
12.4 Maximum and Minimum of a Function of two Variables
12.4.1 Necessary Conditions for the Existence of a Maximum or a Minimum of a Function of two Variables
12.4.2 Stationary and Extreme Points
12.4.3 Sufficient Condition for Maxima or Minima of a Function of two Variables
12.4.4 Working rule for Finding Maxima or Minima of a Function of two Variables
12.5 Lagrange's Method of Undetermined Multipliers
12.6 Let Us Sum Up
12.7 Answer to Check Your Progress
12.8 Further Reading
12.9 Model Questions

12.1 LEARNING OBJECTIVS

After going through this unit, you will able to:

- Taylor's theorem for a function of two variables
- Maximum and minimum of a function of two variables.

12.2 INTRODUCTION

In unit 11, we have discussed function of several variables partial derivatives, homogeneous functions and Euler's theorem on homogeneous functions. In this unit, we will discuss Taylor's servive in two variables. We will also study maximum and minimum of a function of two variables.

12.3 TAYLOR'S SERIES IN TWO VARIABLES

Let f(x, y) be a function of two variables having partial derivatives of all orders.

Then,

(1)

The above series is known as Taylor's series after the name of Brook Taylor(1685-1731).



Brook Tayl
1731) was
mathemat

Note: 1) The above result can be extended to any finite number of independent vatiables.

2) The above result can be written in the following form.

3) Putting x = a, y = b in (1), we get,

f(a+h, y+k) = f(a, b) + [hfx(a, b) + kfy(a, b)]

Another Form: Taylor's series about the point (a, b)

... ... (4)

Putting a = 0, b = 0 in (4), we get

... ... (5)

The above result (4) is called Maclaurin's series for two variables after the name of Colin Maclaurin (1698-1746).



Colin Mac
1746) was
mathemat



(1685-
British
an.

Example 1: Expand e^{xy} about the point (1, 1)

Solution: Let $f(x, y) = e^{xy}$

$f(1, 1) = e^{1.1} = e$

By Taylor's series for function of two variables, we have

Example 2: Expand

Solution: Here ,

,

,



urin (1698-
Scottish
an.



Example 3: Expand $f(x, y) = e^y \log(1+x)$ in powers of x and y.

Solution: Here $f(x, y) = e^y \log(1+x)$, $f(0, 0) = 0$



$f_y(x, y) = e^y \log(1+x)$, $f_y(0, 0) = 0$



$_{xy}(0, 0) = 1$



$_{xx}(0, 0) = -1$

$f_{yy}(x, y) = e^y \log(1+x)$, $f_{yy}(0, 0) = 0$

By Taylor series, we have

$f(x, y) = e^y \log(1+x) = f(0, 0) + [xf_x(0, 0) + yf_y(0, 0)] +$







Example 4: Expand $x^2y + 3y - 2$ in powers of $(x-1)$ and $(y+2)$ using Taylor's series.

Solution: Here $f(x,y) = x^2y + 3y - 2$, $a = 1$, $b = -2$

$f(1, -2) = 1^2 \times (-2) + 3(-2) - 2 = -10$

$f_x = 2xy$, $f_x(1, -2) = 2(1)(-2) = -4$

$f_y = x^2+3$, $f_y(1, -2) = 1^2 + 3 = 4$

$f_{xx} = 2y$, $f_{xx}(1, -2) = 2(-2) = -4$

$f_{xy} = 2x$, $f_{xy}(1, -2) = 2(1) = 2$

$f_{yy} = 0$, $f_{yy}(1, -2) = 0$

Expansion of $f(x, y)$ in powers of $(x-a)$ and $(y-b)$ is given by

$f(x, y) = f(a, b) + [(x-a)f_x(a.b) + (y-b)f_y(a.b)] +$

$x^2y + 3y - 1 = f(x, y)$

$= -10 + [(x-1)(-4) + (y+2)(4)] + \frac{1}{2}[(x-1)^2(-4) + 2(x-1)(y+2)$

$+ (y+2)^2 0] + \ldots$

$= -10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + 2(x-1)(y+2) + \ldots$



CHECK YOUR PROGRESS

Q.1: Find the first six terms of the function $e^x \log(1+y)$ in a Taylor series in the neighbourhood of the point $(0, 0)$.

Q.2: Expand of $f(x, y) = x^3 + y^3 + xy^2$ in powers of $(x-1)$ and $(y-2)$ using Taylor's series.

Q.3: Obtain the expansion of e^{xy} in powers of $(x-1)$ and $(y-1)$.

Q.4: Expand e^{ax} sin by about origin upto 3rd degree terms.

12.4 MAXIMUM AND MINIMUM OF FUNCTIONS OF TWO VARIABLES

In unit 9, we have already discussed the method of obtaining the maxima and minima of a function of one variable.

We may recall that the function $f(x)$ has a maximum at a if $f'(a) = 0$ and $f''(a) < 0$ and has a minimum at a if $f'(a) = 0$ and $f''(a) > 0$.

In this section, we discuss the method of finding the maximum and minimum of a function of two variables.

Definition: A function f(x, y) is said to have a maximum value at x = a, y = b if f(a, b) > f(a+h, b+k) for small and independent values of h and k, positive or negative.

A function f(x. y) is said to have a minimum value at x = a, y = b if f(a, b) < f(a+h, b+k) for small and independent values of h and k, positive or negative.

Therefore, f(x, y) has a maximum or minimum value at a point (a, b) according as

$\Delta f = f(a+h, b+k) - f(a, b) < 0$ or > 0.

A maximum or a minimum value of a function is called its extreme value. The value of a function at an extremum (maximum or minimum) point is known as the extremum (maximum or minimum) value of the function.

12.4.1 Conditions for f(x y) to be Maximum or Minimum

By Taylor's series $f(a+h, b+k) = f(a, b) + [hf_x(a, b) + kf_y(a, b)]$

... ... (1)

Since h, k are small,we neglect terms h^2, hk, k^2 etc. the above expansion (1) becomes

$\Delta f = [f(a+h, b+k) - f(a, b)]$

$= hf_x(a, b) + kf_y(a, b)$ ... ... (2)

Thus sign of Δf $_x(a, b) + kf_y(a, b)]$

Taking h = 0, the sign of Δf ,

takng k = 0, the sign of Δf changes with the sign of h. Since Δf changes sign with h and k, f(x, y) cannot have a maximum or a minimum value at (a, b) unless $f_x(a, b) = 0 = f_y(a, b)$.

Therefore, the necessary conditions for f(x, y) to have a maximum or a minimum value at (a, b) are $f_x(a, b) = 0$, $f_y(a, b) = 0$

If these conditions are satisfied, then for small values of h and k, we have from (1),





[where $r = f_{xx}(a, b)$, $s = f_{xy}(a, b)$, $t = f_{yy}(a, b)$]

or 



 ... ... (3)

Now, $(hr + ks)^2$ is always positive and $k^2 (rt – s^2)$ will be positive if $rt – s^2 > 0$. In this case, 

all values of h and k.

Hence,

i) If $rt – s^2 > 0$, then f(x, y) has a maximum or a minimum at (a, b) according as $r < 0$ or $r > 0$.

ii) If $rt – s^2 < 0$, then 

neither a maximum nor a minimum value at (a, b).

iii) If $rt – s^2 = 0$, no conclusion can be drawn about a maximum or a minimum value at (a, b) and hence further investigation is required.

Stationary Point: The point (a, b) is called a stationary point if, $f_x(a. b) = 0$, $f_y(a. b) = 0$.

The value f(a, b) is called a stationary value.

12.4.2 Rule to find the Extreme Values of a Function z = f(x, y)

i) First find  and 

ii) Solve  and 

Let (a, b) ; (c, d), ... be the solutions of these equations.

iii) For each solution in step (ii) find ,  and .

iv) a) If $rt - s^2 > 0$ and $r < 0$ for a particular solution (a, b) of step (ii), then z has a maximum value at (a, b).

b) If $rt - s^2 > 0$ and $r > 0$ for a particular solution of (a, b) of step (ii), then z has a minimum value at (a, b).

c) If $rt - s^2 < 0$ for a particular solution (a, b) of step (ii), then z has no extreme value at (a, b).

d) If $rt - s^2 = 0$, the case is doubtful and requires further investigation.

Example 1: Find the maximum or minimum values of

$$u = x^3y^2(1-x-y)$$

Solution: Given, $u = x^3y^2(1 - x - y)$

Now, 

$= x^2y^2[3(1-x-y) - x]$

$= x^2y^2[3 - 4x - 3y]$



$= 6xy^2(1-x-y) - 6x^2y^2$







For a maximum or minimum of u, we must have

 and 

 $3x^2y^2(1-x-y) - x^2y^2 = 0$

 $x^2y^23-4x-3y = 0$

 $3 - 4x - 3y = 0$

 $4x + 3y = 3$ ... (1)

 $2x^3y(1-x-y) - x^3y^2 = 0$

 $x^3y\,[2(1-x-y) - y] = 0$

 $2 - 2x - 3y = 0$

 $2x + 3y = 2$ ... (2)

Solving equations (1) and (2), we get , 

Now, we find at the point

$= 6xy^2(1–x–y) – 6x^2y^2$ at

at

$= 6x^2y(1–x–y) – 2x^3y – 3x^2y^2$ at

and at

$= 2x^3(1–x–y) – 4x^3y$ at

Now

Also, r is negative.

The function has a maximum at

The maximum value of u is given by

Example 2: Examine for maximum and minimum values of the function $z = x^2 – 3xy + y^2 + 2x$.

Solution: Given, $z = x^2 - 3xy + y^2 + 2x$.

Now, ... (1)

... ... (2)

Also,

and

For a maximum or minimum of z, we must have

and

$2x - 3y + 2 = 0$ ... (1) and $-3x + 2y = 0$ ... (2)

Solving (1) and (2), we get ,

Now,

$rt - s^2 = 2 \times 2 - (-3)^2 = -5 < 0$

Hence, z is neither maximum nor minimum at ,

Example 3: Discuss the maxima and minma of the function $u = x^2y^2 - 5x^2 - 8xy - 5y^2$

Solution: The given function is $u = x^2y^2 - 5x^2 - 8xy - 5y^2$

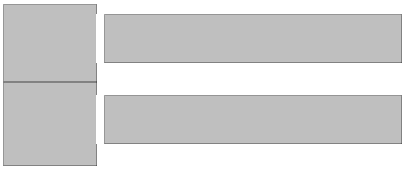
Now,

Also





For a maximum or minimum of u, we must have



and ... ... (2)

From (2) we have $y(x^2–5) = 4x$

Putting  in (1), we get



or 

or $16x^3 – 5x(x^2 – 5)^2 – 16x(x^2 – 5) = 0$

or $x[16x^2 – 5x^4 + 50x^2 –125 – 16x^2 + 80] = 0$

or $x[-5x^4 + 50x^2 – 45] = 0$

or $x[x^4 – 10x^2 + 9] = 0$

or $x(x^2 – 1)(x^2 – 9) = 0$



From (2), when x = 0, we have y = 0,

When x = 1 y = –1

When x = –1 y = 1

When x = 3 y = 3

When x = –3 y = –3

The function u is stationary at the points (0, 0), (1, –1), (–1, 1), (3, 3) and (–3, –3).

At (0, 0) r = – 10, s = – 8, t = – 10.

So that $rt – s^2 = 100 – 64 = 36 > 0$

Since r < 0, so u is maximum at (0, 0).

At (1, –1), r = – 8, s = – 12, t = – 8

So that $rt – s^2 = 64 – 144 < 0$.

Therefore, the stationary value of u at (1, –1) is neither maximum nor minimum.

Similarly, we can show that the stationary value of u at each of the point (–1, 1), (3, 3) and (–3, –3) is neither maximum nor minimum.

Hence the only point at which u has an extreme value at (0, 0), and u has maximum at this point.

Example 4: Examine the function $x^3 + y^3 - 3axy$ for maxima and minima.

Solution: Here $f(x, y) = x^3 + y^3 - 3axy$

$f_x = 3x^2 - 3ay$

$f_y = 3y^2 - 3ax$

$r = f_{xx} = 6x$

$s = f_{xy} = -3a$

$t = f_{yy} = 6y$

Now, $f_x = 0$ and $f_y = 0$

$x^2 - ay = 0$ ... (1) and $y^2 - ax = 0$ ... (2)

From (1)

From (2), $^3 - a^3) = 0$ or $x = 0, a$

When x = 0, y = 0. When x = a, y = a.

There are two stationary points (0, 0) and (a, a),

Now, $rt - s^2 = 36xy - 9a^2$

At (0, 0), $rt - s^2 = -9a^2 < 0$

There is no extreme value at (0, 0)

At (a, a), $rt - s^2 = 36a^2 - 9a^2 = 27a^2 > 0$

f(x, y) has extreme value at (a, a)

Now, r = 6a

If a > 0, r > 0 so that f(x, y) has a minimum value at (a, a).

If a < 0, r < 0 so that f(x, y) has maximum value at (a, a).

Example 5: Examine for minimum and maximum values of sinx + siny + sin(x+y)

Solution: Here f(x, y) = sinx + siny + sin(x + y)

$f_x = \cos x + \cos(x+y)$, $f_y = \cos y + \cos(x+y)$

$r = f_{xx} = -\sin x - \sin(x+y)$, $s = f_{xy} = -\sin(x+y)$

$t = f_{yy} = -\sin y - \sin(x+y)$

Now, $f_x = 0$ and $f_y = 0$

$\cos x + \cos(x+y) = 0$ ... (1) $\cos y + \cos(x+y) = 0$ ... (2)

Subtracting (2) from (1), $\cos x - \cos y = 0$ or $\cos x = \cos y$

From (1), $\cos x + \cos 2x = 0$ or $\cos 2x = -\cos x =$

or $2x = \pi - x$

is a stationary point.

At

Again,

Also, $r < 0$

f(x, y) has a maximum value at

Maximum value =

Example 6: Find the maximum and minimum values of $f(x.y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$.

Solution: Here $f(x, y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$ ... ... (1)

Differentating (1) partially w.r.t x and y, we get

The stationary points are given by $f_x = 0$ and $f_y = 0$.

$3x^2 + 3y^2 - 30x + 72 = 0$ ... (2) and $6xy - 30y = 0$

$6y(x-5) = 0$ i.e. either $y = 0$ or $x = 5$

for $y = 0$, (2) $^2 - 30x\ 72 = 0$

for $x = 5$, (2) $^2 - 150 + 72 = 0$

$3y^2 - 3 = 0$

$y =$

Thus the four stationary points are given by

(6, 0), (4, 0), (5, 1) (5, –1).

To detemine the nature of these points, calculate f_{xx}, f_{xy} and f_{xy}.

$f_{xx} = r = 6x - 30$, $f_{xy} = s = 6y$, $t = f_{yy} = 6x - 30$.

So $AC = B^2 = (6x - 30)^2 - 36y^2 = 36\{(x-5)^2 - y^2\}$ ... ... (3)

Now (i) At (6, 0) we have $r = 36 - 30 = 6 > 0$ and $rt - s^2 = 36 > 0$.

So (6, 0) is a minimum point of the given function f and the minimum value of f at (6, 0) is $6^3 + 0 - 15.36 + 72.6 = 108$

(ii) At (4, 0), $Ar = 24 - 30 = -6 < 0$ and $rt - s^2 = 36 > 0$.

So a maximum occurs at the point (4, 0) and the maximum value of f at (4, 0) is 112.

(iii) At (5, 1) $r = 0$, $rt - s^2 = -36 < 0$

(5, 1) is a saddle point (it is neither maximum nor minimum).

(iv) At (5, -1), $r = 0$, $rt - s^2 = -36 < 0$

So (5, –1) is neither a maximum nor a minimum (it is a saddle print).



CHECK YOUR PROGRESS

Q.5: Examine $f(x, y) = x^3 + y^3 - 3xy$ for maximum and minimum values.

Q.6: Discuss the maximum and minimum of $x^2 + y^2 + 6x + 12$.

Q.7: Discuss the maximum and minimum of the function

.

Q.8: Examine for maximum and minimum values of the function

$Z = x^2 - 3xy + y^2 + 2x = 50$



12.5 LET US SUM UP

- Let f(x, y) be a function of two variables having partial derivatives of all orders.Then,Taylor's series in two variables is given by

Another Form: Taylor's sevies about the point (a, b)

(A)

Putting a = 0, b = 0, we get

The above result (A) is called Maclaurin's series for two variables.

- A function f(x. y) is said to have a maximum value at x = a, y = b if f(a, b) > f(a+h, b+k) for small and independent values of h and k, positive or negative.

  A function f(x. y) is said to have a minimum value at x = a, y = b if f(a, b) < f(a+h, b+k) for small and independent values of hand k, positive or negative.

- The necessary conditions for f(x, y) to have a maximum or a minimum value at (a, b) are $f_x(a, b) = 0$, $f_y(a, b) = 0$



12.6 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1: Here $f(x, y) = e^x \log(1+y)$, $f(0, 0) = 0$

$f_x(x, y) = e^x \log(1+y)$, $f_x(0, 0) = 0$

, $f_y(0, 0) = 1$

$f_{xx}(x, y) = e^x \log(1+y)$, $f_{xx}(0, 0) = 0$

, $f_{xy}(0, 0) = 1$

, $f_{yy}(0, 0) = -1$

$f_{xxx}(x, y) = e^x \log(1+y)$, $f_{xxx}(0, 0) = 0$

, $f_{xxy}(0, 0) = 1$,

, $f_{xyy}(0, 0) = -1$

, $f_{yyy}(0, 0) = 2$

.....

$e^x \log(1+y) = f(x, y)$

$= f(0\ 0) + [xf_x(0, 0) + yf_y(0, 0)] +$

Ans. to Q. No. 2: Do yourself $13 + 7(x–1) + 16(y–2) + 3(x–1)^2 +$

$4(x–1)(y–2) + 7(y–2)^2 + (x–1)^3 + (x–1)(y–2)^2 + (y–2)^3 + ...$

Ans. to Q. No. 3: Do yourself



Ans. to Q. No. 4: Do yourself 

Ans. to Q. No. 5: Given $f(x, y) = x^3 + y^3 – 3xy$

, 

, , 

For maxima and minima,

 and 

 $3x^2 – 3y = 0$ and $3y^2 – 3x = 0$

or $x^2 = y$ ... (1) and $y^2 = x$ ... (2)

Putting the value of y in (2), we get

$x^4 = x$ or $x(x^3–1) = 0$

or $x(x–1)(x^2+x+1) = 0$

or $x = 0, 1$ [$x^2 + x + 1 = 0$ gives complex roots]

Putting $x = 0$ in (1), we get $y = 0$

Putting $x = 1$ in (1), we get $y = 1$

 (0, 0) and (1, 1) are the stationary points.

At (0, 0) $r = 0, s = –3, t = 0$

$rt – s^2 = 0 – 9 < 0$

At (0, 0) there is no extreme value.

At (1, 1) $r = 6, s = –3, t = 6$

$rt – s^2 = 36 – 9 = 27 > 0$

Also, $r = 6 > 0$

Therefore, (1, 1) is a point of minimum value.

The minimum value = $1 + 1 – 3 = –1$.

Ans. to Q. No. 6: Let $f(x, y) = x^2 + y^2 + 6x + 12$

For maxima and minima,

$2x + 6 = 0$

and

Now,

At $(-3, 0)$ $rt - s^2 = 2 \times 2 - 0 = 4 > 0$

$r = 2 > 0$

Hence, $f(x, y)$ is minimum at $(-3, 0)$

Minimum value $= f(-3, 0) = 9 + 0 - 18 + 12 = 3$.

Ans. to Q. No. 7: We have

For maxima and minima,

and

These relation give $a^3 = x^2y = xy^2$

Thus there can be maxima or minima at (a, a).

Now,

and at (a, a).

At (a, a), $rt - s^2 = 2 \times 2 - 1 = 3 > 0$

Also, $r = 2 > 0$. Hence the given function has a minimum value at (a, a). Also, the minimum value is $u = a^2 + a^2 + a^2 = 3a^2$.

Ans. to Q. No. 8: We have $z = x^2 - 3xy + y^2 + 2x$

For maximum or minimum of z,we must have

... ... (1)

... ...(2)

Solving (1) and (2), we get ,

Thus, Z

Also,

rt – s^2 = 2 × 2 – (–3)^2 = –5 which is < 0.

The function has no extreme value at



12.7 FURTHER READING

1) Apostol Tom M.: *Calculus*; New Delhi: Wiley India (P.) Ltd..
2) Dhami, H. S.: *Differential Calculus*; New Delhi: New Age International (P) Limited Publishers.
3) Ginzburg, A.: *Calculus Problems and Solutions*.
4) Kishan, Hari: *Differential Calculus*; New Delhi: Atlantic Publishers Distributors (P) Ltd.
5) Naryan, Shanti: *Differential Calculus*; S. Chand Limited.
6) Rao, S. Balachandra & Shantha, C. K: *Differential Calculus*; New Delhi: New Age International (P) Limited Publishers.
7) Sharma, A. K.: *Text Book of Differential Calculus*; New Delhi: Discovery Publishing House.
8) Thomas, George B. & Finney, Ross L: *Calculus and Analytic Geometry*; New Delhi: Narosa Publishing House.



12.8 MODEL QUESTIONS

Q.1: Expand y^x as a Taylor series at (1,1).

Q.2: Obtain the Taylor's expansion of $x^2y + 2y$ about (1,2).

Q.3: Expand $2x^2y + x + y$ using Taylor's theorem about (1,-2) upto terms of third order.

Q.4: Expand [redacted]

degree terms.

Q.5: Find the maxima and minima of the following functions.

a) $2a^2xy - 3ax^2y - ay^3 + x^3y + xy^3$

b) $3x^4y^2 - 6x^2y^2 + 3x^4 + 2y^3 - 6x^2 - 3x + 1$

c) $\sin x \sin y \sin z$

d) [redacted]

Q.6: Show that the function $u(x, y) = 2x^2y - x^2 - y^2 + 2y$ has no extreme value.

Q.7: Prove that $x^4 + 2x^2y - x^2 + 3y^2$ is minimum when [redacted]

Q.8: Prove that $u = x^2y^2 - 5x^2 - 8xy - 5y^2$ is maximum at $x = y = 0$.

Q.9: Show that the function $u(x, y) = x^2 - 2xy + y^2 + x^3 - y^3 + x^5$ has neither a maximum nor a minimum at (0, 0).

*** ***** ***



Carl Gusta
(1804-185
german
mathemati

UNIT 13: JACOBIANS

UNIT STRUCTURE

13.1 Learning Objectives
13.2 Introduction
13.3 Jacobians
13.3.1 Definitions
13.3.2 Properties of Jacobians
13.4 Let Us Sum Up
13.5 Answer to Check Your Progress
13.6 Further Reading
13.7 Model Questions

13.1 LEARNING OBJECTIVES

After going through this unit, you will able to:

- Jocabians and its properties.

13.2 INTRODUCTION

In unit 11, we have discussed function of several variables partial derivatives, homogeneous functions and Euler's theorem on homogeneous functions. In this unit, we will discuss about Jacobians and its properties.

13.3 JACOBIANS



Jacobi
was a

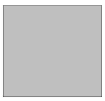
an.

For function of several variables, we find a special function of their partial derivatives known as functional determinant of the given functions. The functional determinants were first studied by the German mathematician Carl Gustav Jacobi (1804-1851). These functional determinants are called Jacobians after his name.

13.3.1 Definition

Jacobian is a functional determinant whose elements are functions.

Let u (x, y) and v(x, y) be two given functions of two independent variables x and y.

The Jocabian of u, v with respect to x, y denoted by  or  is a second order functional determinant defined as:



Similarly, the Jacobian of three functions u, v, w of three independent variables x, y, z, defined as:



In a similar way, Jacobian of n functions in n variables can be defined.

If $u_1, u_2 ..., u_n$ are functions of n independent variables $x_1, x_2, ..., x_n$, then the determinant:



is called the Jacobian of $u_1, u_2, ..., ..., ..., u_n$ with respect to $x_1, x_2, ..., ..., ..., x_n$ and is denoted either by  .

Note: 1) 



2) If the functions u_1, u_2, ..., ..., ..., u_n of the variables x_1, x_2, ..., x_n are defined by the relation.

$u_1 = f_1(x_1)$, $u_2 = f_2(x_1, x_2)$, ..., ..., ..., $u_n = f_n(x_1, ..., x_n)$

then, 

i.e. the Jacobian reduces to the product of the leading diagonal elements of the determinants.

13.3.2 Properties of Jacobians

I) If u, v are functions of r, s where r, s are functions of x, y then



Proof: 

= 

On interchanging the columns and rows

=

=

=

Similarly,

Note: The above result can be extended to any finite number of variables.

II) If J_1 is the Jacobian of u, v, with respect to x, y and J_2 is the Jacobian of x, y with respect to u, v, then $J_1 J_2 = 1$

i. e.,

Proof: Let $u = f(x, y)$ ... ... (1)

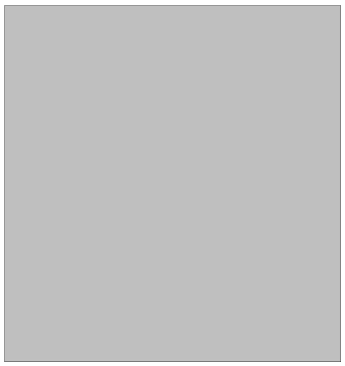
$v = \phi(x, y)$ ... ... (2)

On interchanging the rows and columns of second determinant,

=

= 

On differentiating (1) and (2) w.r.t, u and v. We get



... ... (4)

On making substitutions from (4) in (3), we get,



Note: The above result can be extended to any finite number of variables.

III) If functions u, v, w of three independent variables x, y, z are not independent, then 

Proof: As u, v, w are not independent,

f(u, v, w) = 0 ... ... (1)

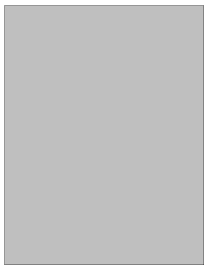
Differentiating (1) w.r.t x, y, z, we get

 ... ... (2)

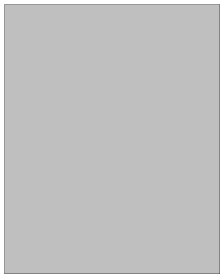
 ... ... (3)

 ... ... (4)

Eliminating 



On interchanging rows and columns, we get,



or 

Note: Two functions u and v are functionally dependent if their Jacobian



If Jacobian is not equal to zero, then u and v are said to be functionally independent.

This concept can be extended to more than two functions.

Example 1: If , Find 

Solution: We have  and 



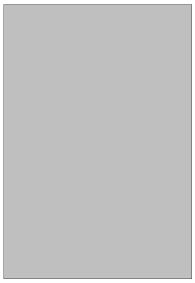




Example 2: If 

Prove that $J(u_1, u_2, u_3) = 4$

Solution: We have

$J(u_1, u_2, u_3) =$ 





,

adding R_2 to R_1 and then R_3 to R_2.

.

Example 3: If $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r \cos\theta$

Show that 

Solution: We have





$= \cos\theta(r^2\sin\theta\cos\theta\cos^2\phi + r^2\sin\theta\cos\theta\sin^2\phi) +$

$r\sin\theta(r\sin^2\theta\cos^2\phi + r\sin^2\theta\sin^2\phi)$

(expanding the determinant along the third row)

$= r^2\sin\theta\cos^2\theta + r^2\sin^3\theta$

$= r^2\sin\theta(\cos^2\theta + \sin^2\theta)$

$= r^2\sin\theta$

Example 4: If $y_1 = \cos x_1$, $y_2 = \sin x_1 \cos x_2$ and $y_3 = \sin x_1 \sin x_2 \cos x_3$,

Prove that 

Solution: We have $y_1 = \cos x_1$, $y_2 = \sin x_1 \cos x_2$, $y_3 = \sin x_1 \sin x_2 \cos x_3$

Now, 



$= -\sin x_1 \sin x_2 \sin x_3(\sin^2 x_1 \sin x_2)$

$= -\sin^3 x_1 \sin^2 x_2 \sin x_3$

Example 5: If $x = u\cos v$, $y = u\sin v$, Prove that 

Solution: Given, $x = u \cos v$ ... ... (1)

$y = u \sin v$ ... ... (2)



$= u \cos^2 v + u \sin^2 v$

$= u(\cos^2 v + \sin^2 v)$

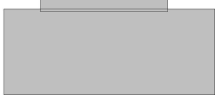
$= u$

Now, squaring (1) and (2) and then adding, we get,

$x^2 + y^2 = u^2(\cos^2 v + \sin^2 v) = u^2 \times 1\ u^2$



Dividing (2) by (1), we get 

















Example 6: Show that the functions

, $v = \tan^{-1}x + \tan^{-1}y$ are functionally dependent and find the relation between them.

Solution: We know that







Hence u, v are functionally dependent.

Now we know that 

 $v = \tan^{-1}u$

 $u = \tan v$

Example 7: Compute the Jacobian 

where $u = 2xy$, $v = x^2 - y^2$, $x = r\cos\theta$, $y = r\sin\theta$

Solution: We know that

 ... ... (1)

Now, 

Also, 

$= r(\cos^2\theta + \sin^2\theta)$ [From $x = r\cos\theta c$, $y = r\sin\theta s$, we have $x^2 + y^2 = r^2$]

$= r$

Hence from (1), $^2 + y^2)r = -4r^2.r = -4r^3$

Example 8: Calculate and

given ,

Verify that $J.J^* = 1$

given ,

Solution:

Solving for x, y in terms of u and v, we have

$y^2 = vx$

$y^2 = v(u-v)$

Now,

Verification:

Example 9: Verify whether the following functions are functionally related $u = e^x \sin y$, $v = e^x \cos y$

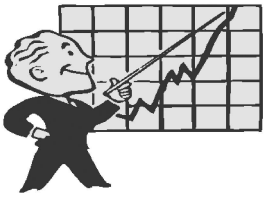
Solution: Jacobian =

$= e^{2x}(-\sin^2 y - \cos^2 y)$

$= -e^{2x}(\sin^2 y + \cos^2 y)$

$= -e^x$

u. v are functionally related.



CHECK YOUR PROGRESS

Q.1: a) If $u\ x^2$, $v = y^2$ find .

b) If $x = ar\cos\theta$, $y = br\sin\theta$ find .

Q.2: If $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, find .

Q.3: Fill in the blanks:

i) If $x = r\cos\theta$, $y = r\sin\theta$ then the value of Jacobian is .....

ii) If $u = x(1-y)$, $v = xy$ then the value of the Jacobian = ...

iii) [ ] = ...

Q.4: If u = xyz, v = $x^2 + y^2 + z^2$, w = x + y + z find [ ]

Q.5: Verify whether the following functions are related and if so find the relation between them.

[ ]



13.4 LET US SUM UP

- Jacobian is a functional determinant whose elements are functions. The Jacobian of three functions u, v, w of three independent variables x, y, z is defined as

[ ]

- Properties of Jacobians:
  - i) If u, v are functions of r, s where r, s are functions of x, y; then

    [ ]
  - ii) If J_1 is the Jacobian of u, v, with respect to x, y and J_2 is the Jacobian of x, y with respect to u, v, then $J_1 J_2 = 1$,

    i. e., [ ]
  - iii) If functions u, v, w of three independent variables x, y, z are not independent, then [ ]

13.5 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1: (a) 

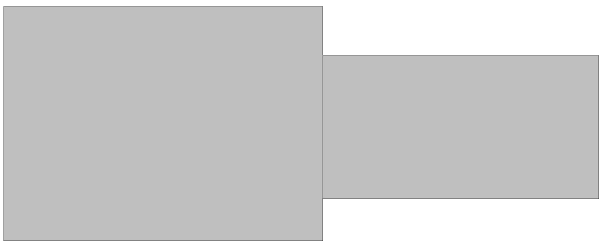
$= 4xy$

(b) 

$= abr\cos^2\theta + abr\sin^2\theta$

$= abr(\cos^2\theta + \sin^2\theta)$

$= abr$

Ans. to Q. No. 2: 

$= \cos\theta(r\cos\theta - 0) - (-r\sin\theta)(\sin\theta - 0) + 0$

$= r\cos^2\theta + r\sin^2\theta$

$= r$

Ans. to Q. No. 3: i) 

$= r\cos^2\theta + r\sin^2\theta$

$= r$

ii) 

$= x(1–y) + xy$

$= x – xy + xy$

$= x$

iii) 

Ans. to Q. No. 4: Since u, v, w are explicitly given, so first we evaluate













$= 2(x – y)(y – z)(–z + x)$

$= –2(x – y)(y – z)(z – x)$

Hence, by $JJ' = 1$, we have

Ans. to Q. No. 5:

u, v are functionally dependent.

is the functional relation between u and v.



13.6 FURTHER READING

1) Apostol Tom M.: *Calculus*; New Delhi: Wiley India (P.) Ltd..
2) Dhami, H. S.: *Differential Calculus*; New Delhi: New Age International (P) Limited Publishers.
3) Ginzburg, A.: *Calculus Problems and Solutions*.
4) Kishan, Hari: *Differential Calculus*; New Delhi: Atlantic Publishers Distributors (P) Ltd.
5) Naryan, Shanti: *Differential Calculus*; S. Chand Limited.
6) Rao, S. Balachandra & Shantha, C. K: *Differential Calculus*; New Delhi: New Age International (P) Limited Publishers.
7) Sharma, A. K.: *Text Book of Differential Calculus*; New Delhi: Discovery Publishing House.
8) Thomas, George B. & Finney, Ross L: *Calculus and Analytic Geometry*; New Delhi: Narosa Publishing House.



13.7 MODEL QUESTIONS

Q.1: Find the Jacobian of following transformations

a) $x + y = u,\ x - y = v$

b) $x = u\sin\alpha + v\cos\alpha;\ y = u\cos\alpha - v\sin\alpha$ (α constant)

c) $x = r\cos\theta\cos\phi;\ y = r\cos\theta\sin\phi;\ z = r\sin\theta$

d) .

Q.2: If $x = e^u \cos v$ and $y = e^u \sin v$, show that 

Q.3: If $u = x + y$, $v = x - y$ and $x = r\sin\theta$, $y = r\sin\theta$, Prove that

.

Q.4: If  and 

$r^2 = x^2 + y^2 + z^2$, Prove that 

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UNIT 14: LAGRANGE METHOD OF UNDETERMINED MULTIPLIERS

UNIT STRUCTURE

14.1 Learning Objectives

14.2 Introduction

14.3 Lagrange's Method of Undetermined Multipliers

14.4 Let Us Sum Up

14.5 Answer to Check Your Progress.

14.6 Further Reading

14.7 Model Questions

14.1 LEARNING OBJECTIVES

After going through this unit, you will able to:

- Lagrange's method of undertermined multipliers.

14.2 INTRODUCTION

In unit 12, we have discussed maximum and minimum of a function of two variables. In this unit, we will discuss Lagrange's Method of undetermined multipliers.

14.3 LAGRANGE'S METHOD OF UNDETERMINED MULTIPLIERS

The maximum or minimum of a function of several variables, where the variables are connected by some given relation or condition known as constraint can be found by using Lagrange's Method of Undetermined Multipliers.

Now, let f(x, y, z) be a function of three independent variables where x, y, z are related by a known constraint $\phi(x, y, z) = 0$.We are to find the exterme values of f(x, y, z)

Subject to $\phi(x. y. z) = 0$ ... ... (1)

f(x, y, z) will have maximum or minimum values if





... ... (2)

By total differentiation of (1)



... ... (3)

Multiplying (3) by λ and adding to (2), we get,







... ... (4)



... ... (5)



... ... (6)

On solving (1), (4), (5), (6), we can find the values of x, y, z and λ for which f(x, y, z) has either maximum or minimum value.

Note: Lagrange's Method does not enable us to find whether there is a maximum or minimum.

Example 1: Find the maximum value of $u = x^p y^q z^r$ when the variables x, y, z are subjected to the condition $ax + by + cz = p + q + r$.

Solution: Given $u = x^p y^q z^r$ ... ... (1)

if $ax + by + cz = p + q + r$ ... ... (2)

(1) 

Now differentiating (3) partially w.r.t x, we get,





Similarly differentiating (3) partially w.r.t y and z, we get,





and 



Also, $\phi(x, y, z) = ax + by + cz - p - q - r$



Now, Lagrange's equations are







Putting in (2), we have







$\Rightarrow \lambda = -u$







Putting in (1) we have 

Which is the required minimum value.

Example 2: The temperature T at any point (x, y, z) in space is $T = 400xyz^2$. Find the highest temperature at the surface of a unit sphere $x^2 + y^2 + z^2 = 1$.

Solution: Given $T = 400\ xyz^2$

$x^2 + y^2 + z^2 = 1$

$\phi(x, y, z) = x^2 + y^2 + z^2 - 1$

Now Lagrange's equations are

$\Rightarrow 400z^2 + \lambda(2x) = 0$ ... ... (1)

$\Rightarrow 400xz^2 + \lambda(2y) = 0$ ... ... (2)

and $\Rightarrow 800xyz + \lambda(2z) = 0$ ... ... (3)

Multiplying (1) by x, (2) by y and (3) z and adding, we get,

$1600xyz^2 + 2\lambda(x^2 + y^2 + z^2) = 0$

$\Rightarrow 1600xyz^2 + 2\lambda(1) = 0$

$\Rightarrow \lambda = -800xyz^2$

Putting the value of λ in (1), we get,

$400yz^2 + 2x(-800xyz^2) = 0$

$1 - 4x^2 = 0$

Putting the value of λ in (2), we get

On putting the value of λ in (3), we get,

$800xyz - 1600xyz^3 = 0$

or $1 - 2z^2 = 0$

or

On putting the value of x, y, z in T, we get,

$= 50$

Example 3: Find the entreme value of $x^2 + y^2 + z^2$ subject to the condition $xy + yz + zx = p$.

Solution: Given function $f = x^2 + y^2 + z^2$

Condition is $xy + yz + zx = p$

$\phi = xy + yz + zx - p$

By Lagrange's Method

$\Rightarrow 2x + \lambda(y+z) = 0$ ... ... (1)

$\Rightarrow 2y + \lambda(x+z) = 0$ ... ... (2)

and $\Rightarrow 2z + \lambda(x+y) = 0$ ... ... (3)

Now (1) – (2) gives $2(x-y) + \lambda(y-x) = 0$

$\Rightarrow 2(x-y) - \lambda(x-y) = 0$

$\Rightarrow (x-y)(2-\lambda) = 0$

$x - y = 0$

$\Rightarrow x = y$

(2) – (3) gives $2(y-z) + \lambda(z-y) = 0$

$\Rightarrow 2(y-z) - \lambda(y-z) = 0$

$\Rightarrow (y-z)(2-\lambda) = 0$

$y - z = 0$

$\Rightarrow y = z$

Thus, $x = y = z$.

$xy + yz + zx = p$

$\Rightarrow 3x^2 = p$

Extreme value of $f = x^2 + y^2 + z^2$

Example 4: Find the minimum distance from the point (1, 2, 0) to the cone $z^2 = x^2 + y^2$

Solution: Equation of given cone is $z^2 = x^2 + y^2$

$x^2 + y^2 - z^2 = 0$ ... ... (1)

Let P(x, y, z) be the co-ordinates of desired point. Then distance of P from (1, 2, 0).

 [By distance formule]

$\Rightarrow d^2 = (x-1)^2 + (y-2)^2 + z^2$

Let $f = (x-1)^2 + (y-2)^2 + z^2$

and $\phi = x^2 + y^2 - z^2$

By Lagrange's Method,

 $\Rightarrow 2(x-1) + \lambda 2x = 0$ ... ... (2)

 $\Rightarrow 2(y-2) + \lambda 2y = 0$ ... ... (3)

 $\Rightarrow 2z + \lambda 2z = 0$ ... ... (4)

From (4), $\lambda = 1$

Putting $\lambda = 1$ in (2), we get 

Putting $\lambda = 1$ in (3), we get $y = 1$

Putting , $y = 1$ in (1), we get,





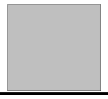




 the point is 

 Minimum distance 







CHECK YOUR PROGRESS

Q.1: Use the method of the Lagrange's multipliers to find the volume of the largest rectangular parallelopiped that can be inscribed in the ellipsoid



Q.2: Find the maximum and minimum distances of the point (3, 4, 12) from the sphere $x^2 + y^2 + z^2 = 1$.

Q.3: Using the Lagrange's Method of undetermined multipliers find shortest distances from the point (1, 2, 2) to the sphere $x^2 + y^2 + z^2 = 36$.

Q.4: The temperature T at any point (x, y, z) in space is $T = kxyz^2$ where k is constant. Find the highest temperature at the surface of.



14.4 LET US SUM UP

- The maximum or minimum of a function of several variables, where the variables are connected by some given relation or condition known as constraint can be found by using Lagrange's Method of Undetermined Multipliers.



14.5 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1: Let (x, y, z) be the coordinates of one vertex of the parallelopiped, the lengths of three coterminous edges are 2x, 2y, 2z, so that volume is given by

$$V = (2x)(2y)(2z) = 8xyz \quad \text{... ... ... (1)}$$

Subject to the condition  ... ... (2)

Taking log on both sides of (1), we get,

$$\log V = \log 8 + \log x + \log y + \log z$$



For maximum V, 

Also, 

i.e., 

Multiplying (4) by λ and adding to (3) and then equating the co-efficients of dx, dy, dz to zero, we get,



Multiplying these equation by x, y, z respectively and adding, we have



$\Rightarrow 4 + \lambda(1) = 0$ [from (2)]

$\Rightarrow \lambda = -3$

Also, since 





Similarly  and 

Thus, 

Ans. to Q. No. 2: Let (x, y, z) be any point on the sphere. Distance of the point A(3, 4, 12) from (x, y, z) is given by .

If the distance is maximum or minimum, so will be the square of the distance.

Let $f(x, y, z) = (x-3)^2 + (y-4)^2 + (z-12)^2$ ... ... (1)

Subject to the condition that

$\phi(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$ ... ... (2)

Consider Lagrange's function

$f(x, y, z) = f(x, y, z) + \lambda\phi(x, y, z)$

$= (x-3)^2 + (y-4)^2 + (z-12)^2 + \lambda(x^2 + y^2 + z^2 - 1)$

For stationary values, $dF = 0$

$\Rightarrow [2(x-3) + 2\lambda x]dx + [2(y-4) + 2\lambda y]dy + [2(z-12) + 2\lambda z]dz = 0$

$\Rightarrow (x-3) + \lambda x = 0$ ... ... (3)

$(y-4) + \lambda y = 0$ ... ... (4)

$(z-12) + \lambda z = 0$ ... ... (5)

Multipling (3) by x, (4) by y , (5) by z and adding, we get,

$(x^2 + y^2 + z^2) - 3x - 4y - 12z + \lambda(x^2 + y^2 + z^2) = 0$

or $1 - 3x - 4y - 12z + 2\lambda = 0$ [using (2)]

or $3x + 4y + 12z = 1 + \lambda$ ... ... (6)

From (3), (4) and (5),

Putting these values of x, y, z in (6), we have

or $(1+\lambda)^2 = 169$

or $1 + \lambda =$

$\lambda = 12$ or $\lambda = -14$

When

When

Thus, we get two points and

Now,

 is at a minimum distance from A and the minimum distance = 12.

 is at a maximum distance from A and the maximum distance =14.

Ans. to Q. No. 3: Let (x, y, z) be a point on the sphere. Then its distance from (1, 2, 2) is 

Consider, $f(x, y, z) = \{(x-1)^2 + (y-2)^2\} + \lambda(x^2 + y^2 + z^2 - 16)$

where λ is a multiplier.

For the stiationary values of F, 







From these we get , 

Putting these in equation $x^2 + y^2 + z^2 = 36$,

We obtain 



giving . Thus, we get two points, viz. (2, 4, 4), (–2, –4, –4)

Thus, the minimum distance 

.

and the maximum distance 



.

Ans. to Q. No. 4: Consider the Lagrange's function

$$f(x, y, z) = kxyz^2 + \lambda(x^2 + y^2 + z^2 - a^2) \quad \ldots\ \ldots\ \ldots\ (1)$$

For extreme values,

$(kyz^2 + \lambda 2x)dx + (kxz^2 + \lambda 2y)dy + (2kxyz + \lambda 2z)dz = 0$

$\Rightarrow kyz^2 + 2\lambda x = 0$ ... ... (2)

$kxz^2 + 2\lambda y = 0$ ... ... (3)

$2kxyz + 2\lambda z = 0$ ... ... (4)

Multiplying (2) by x, (3) by y and (4) by z and adding, we get,

$(4kxyz^2) + 2\lambda(x^2 + y^2 + z^2) = 0$ ... ... (5)

$x^2 + y^2 + z^2 = a^2$

From (2),

Similarly,

From (4),

Substituting value of x, y and z in T, we get,



14.5 FURTHER READING

1) Apostol Tom M.: *Calculus*; New Delhi: Wiley India (P.) Ltd..

2) Dhami, H. S.: *Differential Calculus*; New Delhi: New Age International (P) Limited Publishers.

3) Ginzburg, A.: *Calculus Problems and Solutions*.
4) Kishan, Hari: *Differential Calculus*; New Delhi: Atlantic Publishers Distributors (P) Ltd.
5) Naryan, Shanti: *Differential Calculus*; S. Chand Limited.
6) Rao, S. Balachandra & Shantha, C. K: *Differential Calculus*; New Delhi: New Age International (P) Limited Publishers.
7) Sharma, A. K.: *Text Book of Differential Calculus*; New Delhi: Discovery Publishing House.
8) Thomas, George B. & Finney, Ross L: *Calculus and Analytic Geometry*; New Delhi: Narosa Publishing House.



14.6 MODEL QUESTIONS

Q.1: Find the minimum distance from the point (2, 1, –3) to the plane $2x + y – 2z = 4$.

Q.2: Find the maximum and minimum value of $x^2 + y^2$ subject to the condition $3x^2 + 4xy + 16y^2 = 140$.

Q.3: Find the shortest and longest distances from the point (1, 2, –1) to the sphere $x^2 + y^2 + z^2 = 24$.

Q.4: Find the maximum value of $x^m y^n z^p$ when $x + y + z = a$.

Q.5: Find the minimum value of $x + y + z$ subject to the condition .

Q.6: Find the minimum value of the function $x^2 + y^2 + z^2$ subject to the condition $ax + by + cz = d$.

Q.7: Find the shortest and longest distances from the origin to the curve $5x^2 + 6xy + 5y^2 – 8 = 0$.

Q 8: Find the maximum and minimum values of $x^2 + y^2 + z^2$ subject to the condition $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 1$.

*** ***** ***



REFERENCES

1) Apostol Tom M.: *Calculus*; New Delhi: Wiley India (P.) Ltd..
2) Avez, A.: *An Elementary Treatise on Differential Calculus*; Princeton: Princeton University Press, 2001.
3) Dhami, H. S., *Differential Calculus*; New Delhi: New Age International (P) Limited Publishers.
4) Ginzburg, A.: *Calculus Problems and Solutions*.
5) Gupta, Sudhir (2011) (1st edition): *Basics of Differentential Calculus*; New Delhi: Anmol publication Pvt. Ltd.
6) Habib, Amber (2011) (1st edition): *The Calculus of Finance*; Hyderabad: Universities Press India Pvt. Ltd.
7) Kishan, Hari: *Differential Calculus*; New Delhi: Atlantic Publishers Distributors (P) Ltd.
8) Maron, I. A. (1993) (1st edition): *Problems in Calculus of One Variable*; New Delhi: CBS Publishers & Distributors Pvt. Ltd.
9) Naryan, Shanti: Differential Calculus; S. Chand Limited.
10) Rao, S. (2004): *Differential Calculus*; Berkeley: University of California Press.
11) Rao, S. Balachandra & Shantha, C. K.: *Differential Calculus*; New Delhi: New Age International (P) Limited Publishers.
12) Sharma, A. K. (2004) (1st edition): *Text Book of Differential Calculus for B.A., B.Sc., B.Com., I.A.S., P.C.S.*; New Delhi: Discovery Publishing House.
13) Thomas, George B. & Finney, Ross L: *Calculus and Analytic Geometry*; New Delhi: Narosa Publishing House.