UNIT 11 : LINEAR DIFFERENTIAL EQUATION OF HIGHER ORDER

UNIT STUCTURE

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11.1 LEARNING OBJECTIVES

After going through this unit, you will able to know

- linear differential equation of higher order with constant coefficients,

11.2 INTRODUCTION

In unit 10, we discussed different types of first order and higher degree nonlinear differential equations and their solutions. In this unit, we shall discuss about solution of linear differential equation of higher order with constant coefficients.

11.3 LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

An ordinary linear differential equation of nth order with constant coefficients is of the form

$$\frac{d^n y}{dx^n}+P_1\frac{d^{n-1}y}{dx^{n-1}}+P_2\frac{d^{n-2}y}{dx^{n-2}}+\cdots+P_{n-1}\frac{dy}{dx}+P_n y=X$$

where $P_1, P_2, \cdots, P_n$ are constants and X is a function of x or a constant.

If the coefficient of $\frac{d^n y}{dx^n}$ is not unity,we divide each term by that coefficient.

We consider two forms of the equation,viz,(i) $X=0$, (ii) X is a function of x only or a constant.

Theorem 11.3.1 : If $y=f(x)$ is the general solution of the equation

$$\frac{d^n y}{dx^n}+P_1\frac{d^{n-1}y}{dx^{n-1}}+P_2\frac{d^{n-2}y}{dx^{n-2}}+\cdots+P_{n-1}\frac{dy}{dx}+P_n y=0 \quad \text{... (1)}$$

and $y=\phi(x)$ be a solution of the equation

$$\frac{d^n y}{dx^n}+P_1\frac{d^{n-1}y}{dx^{n-1}}+P_2\frac{d^{n-2}y}{dx^{n-2}}+\cdots+P_{n-1}\frac{dy}{dx}+P_n y=X \quad \text{... (2)}$$

then $y=f(x)+\phi(x)$ is the general solution of the equation (2).

Proof : Since $y=f(x)$ is the general solution of (1),

$$\therefore \frac{d^n f}{dx^n}+P_1\frac{d^{n-1}f}{dx^{n-1}}+P_2\frac{d^{n-2}f}{dx^{n-2}}+\cdots+P_{n-1}\frac{df}{dx}+P_n f=0$$

Again, since $y=\phi(x)$ be a solution of (2),

$$\therefore \frac{d^n \phi}{dx^n}+P_1\frac{d^{n-1}\phi}{dx^{n-1}}+P_2\frac{d^{n-2}\phi}{dx^{n-2}}+\cdots+P_{n-1}\frac{d\phi}{dx}+P_n\phi=X$$

Now, $\frac{d^n(f+\phi)}{dx^n}+P_1\frac{d^{n-1}(f+\phi)}{dx^{n-1}}+P_2\frac{d^{n-2}(f+\phi)}{dx^{n-2}}+\cdots+P_{n-1}\frac{d(f+\phi)}{dx}+P_n(f+\phi)=0+X=X$

$\therefore y=f(x)+\phi(x)$ is a solution of the equation (2).

Theorem 11.3.2 : If $y=y_1$, $y=y_2$,..., $y=y_n$ are solutions of the equation

$$\frac{d^n y}{dx^n}+P_1\frac{d^{n-1}y}{dx^{n-1}}+P_2\frac{d^{n-2}y}{dx^{n-2}}+\cdots+P_{n-1}\frac{dy}{dx}+P_n y=0 \quad \text{... (1)}$$

then $y=C_1y_1+C_2y_2+\cdots+C_ny_n$ is a solution of the equation (1).

Here $C_1, C_2, \cdots, C_n$ are arbitrary constants.

Proof : We have

$$C_1\left(\frac{d^n y_1}{dx^n}+P_1\frac{d^{n-1}y_1}{dx^{n-1}}+\cdots+P_{n-1}\frac{dy_1}{dx}+p_n y_1\right)+$$

$$C_2\left(\frac{d^n y_2}{dx^n}+P_1\frac{d^{n-1}y_2}{dx^{n-1}}+\cdots+P_{n-1}\frac{dy_2}{dx}+p_n y_2\right) \quad +,\cdots,+$$

$$\left(\frac{d^n y_n}{dx^n}+P_1\frac{d^{n-1}y_n}{dx^{n-1}}+\cdots+P_{n-1}\frac{dy_n}{dx}+p_n y_n\right)$$

$= C_1\times 0 + C_2\times 0 +,\cdots,+ C_n\times 0$

$= 0$

$\therefore y = C_1y_1 + C_2y_2 + \cdots + C_ny_n$ is a solution of (1).

This solution contains n arbitrary constants.So,it is the general or complete solution of the equation (1).

Note (a) From the above two results, it is clear that the complete solution of (2) consists of two parts.The first part is the general solution of (1), called the complementary function (C.F). It contains n arbitrary constants. The second part is a solution of (2), called particular integral (P.I). It does not contain an arbitrary constant.

Thus,the complete or general solution of (2) is

$y = \text{C.F} + \text{P.I}$

(b) If $y_1(x)$ is a solution of a linear differential equation,then $Cy_1(x)$ is also a solution of the equation.

If $y_1(x)$ and $y_2(x)$ are two solutions of a linear differential equation, then $C_1y_1(x) + C_2y_2(x)$ is also a solution of the equation.

Operator D and its properties :

Let $D = \frac{d}{dx}$ and $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(Dy) = D(Dy) = D^2y$ and in general $\frac{d^n y}{dx^n} = D^n y$. So the equation (1) can be written as

$$\left(D^n + P_1D^{n-1} + \cdots + P_n\right)y = X .$$

If m and n are positive integer,then

$$D^m D^n = D^n D^m = D^{m+n} .$$

Result : If α and β are scalars then

$$(D-\alpha)(D-\beta)y = (D-\beta)(D-\alpha)y = \left(D^2 - (\alpha+\beta)D + \alpha\beta\right)y$$

Proof : $(D-\alpha)(D-\beta)y = (D-\alpha)(Dy-\beta y)$

$$= D(Dy - \beta y) - \alpha(Dy - \beta y)$$
$$= D^2 y - \beta Dy - \alpha Dy + \alpha\beta y$$
$$= D^2 y - (\alpha + \beta)Dy + \alpha\beta y$$

Similarly, $(D-\beta)(D-\alpha)y = D^2 y - (\alpha+\beta)Dy + \alpha\beta y$

$$= (D^2 - (\alpha+\beta)D + \alpha\beta)y$$

11.3.1 Rules of Finding Complementary Function

Consider the linear equation

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + k_n y = 0$$

where $k_1, k_2, \cdots, k_n$ are constants and right hand expression $X = 0$ and it is called the auxiliary equiation of

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + k_n y = X$$

In terms of operator D, it can be written as

$(D^n + k_1 D^{n-1} + \cdots + k_n)y = X$

Let $y = e^{mx}$ be a solution of $(D^n + k_1 D^{n-1} + \cdots + k_n)y = 0$ ... (3)

Then $Dy = me^{mx}$, $D^2 y = m^2 e^{mx}$,..., $D^n y = m^n e^{mx}$ and equation (3) becomes $(m^n + k_1 m^{n-1} + k_2 m^{n-2} + \cdots + k_n)e^{mx} = 0$

$\Rightarrow m^n + k_1 m^{n-1} + k_2 m^{n-2} + \cdots + k_n = 0 \quad (\because e^{mx} \neq 0)$ ... (4)

case I : If $m_1, m_2, \cdots, m_n$ be real and distinct roots of (4), then (3) can be written as

$(D-m_1)(D-m_2)\cdots(D-m_n)y = 0$ ... (5)

(3) will be satisfied by $(D-m_n)y = 0$. Solving it

$$\frac{dy}{dx} = m_n y \text{ ,we get}$$

or $$\frac{dy}{y} = m_n dx$$

or $$\log y = m_n x + \log c$$

or $$\log \frac{y}{c} = m_n x$$

or $$y = e^{m_n x}$$

Since the order of the factors in (5) are independent of their position (3) will be satisfied by the solutions of $(D-m_i)y=0, 1\le i\le n$.

i.e., $y=c_ie^{m_ix}\ (1\le i\le n)$

So, the general solution of (3) i.e., the complementary function is

$$y=c_1e^{m_1x}+c_2e^{m_2x}+\cdots+c_ne^{m_nx}$$

Csae II : If two roots are equal,say, $m_1=m_2$ .then complete solution of (3) is $y=(c_1+c_2x)e^{m_1x}+c_3e^{m_3x}+\cdots+c_ne^{m_nx}$

Similarly, If $m_1=m_2=m_3, m_4, m_5, \cdots, m_n$ are roots, then complete solution is $y=(c_1+c_2x+c_3x^2)e^{m_1x}+c_4e^{m_4x}+c_5e^{m_5x}+\cdots+c_ne^{m_nx}$

Case III : If one pair of roots be imaginary i.e $m_1=\alpha+i\beta$ and $m_2=\alpha-i\beta$ then the complete solution is

$$y=c_1e^{(\alpha+i\beta)x}+c_2e^{(\alpha-i\beta)x}+c_3e^{m_3x}+\cdots+c_ne^{m_nx}$$

$$y=e^{\alpha x}\left[c_1(\cos\beta x+i\sin\beta x)+c_2(\cos\beta x-i\sin\beta x)\right]+c_3e^{m_3x}+\cdots+c_ne^{m_nx}$$

$$=e^{\alpha x}\left[c_1\cos\beta x+c_2\sin\beta x\right]+c_3e^{m_3x}+\cdots+c_ne^{m_nx}$$

where $C_1=c_1+c_2$ and $C_2=i(c_1-c_2)$

Case IV : If two pairs of imaginary roots are equal, i.e., $m_1=m_2=\alpha+i\beta$, $m_3=m_4=\alpha-i\beta$, then as in caseII, the complete solution is

$$y=e^{\alpha x}\left[(c_1+c_2x)\cos\beta x+(c_3+c_4x)\sin\beta x\right]+c_5e^{m_5x}+\cdots+c_ne^{m_nx}$$

Example 1: Solve (i) $2D^2y+9Dy-18y=0$

(ii) $\dfrac{d^3y}{dx^3}-\dfrac{dy}{dx}=0$

(iii) $(D^4-7D^2+6D)y=0$

Solution : (i) Auxiliary equation is $2m^2+9m-18=0$

or $2m^2+12m-3m-18=0$

or $2m(m+6)-3(m+6)=0$

or $(2m-3)(m+6)=0$

or $m=\dfrac{3}{2},-6$

So, $\dfrac{3}{2}$ and - 6 are distinct realroots. Hence, the solution is

$y = Ae^{\frac{3}{2}x} + Be^{-6x}$.

(ii) Auxiliary equation is $m^3 - m = 0$

or $m(m+1)(m-1) = 0$

or $m = 0,-1,1$

Hence the complete solution is $y = A + Be^{-x} + Ce^{x}$.

(iii) Auxiliary equation is $m^4 - 7m^2 + 6m = 0$

By inspection $m = 0$ and 1 are two roots.

So, the equation can be written as

$$m(m^3 - 7m + 6) = 0$$

or $m(m-1)(m^2 + m - 6) = 0$

or $m(m-1)(m+3)(m-2) = 0$

So, m = 0, 1, 2, –3 are roots of the equation.

Hence its solution is $y = A + Be^{x} + Ce^{2x} + De^{-3x}$.

Example 2 : Solve (i) $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 9x = 0$

(ii) $(D^3 - 3mD^2 + 3m^2D - m^3)y = 0$

(iii) $\frac{d^4y}{dx^4} - 4\frac{d^3y}{dx^3} + 4\frac{d^2y}{dx^2} = 0$

(iv) $(D^5 - D^4 - 2D^3 + 2D^2 + D - 1)y = 0$

Solution : (i) Auxiliary equation is $m^2 + 6m + 9 = 0$

or $(m+3)^2 = 0$

$\therefore \quad m = 3,-3$ are repeated roots.

So, the complete solution is $y = (Ax + B)e^{-3x}$.

(ii) Auxiliary equation is $u^3 - 3mu^2 + 3m^2u - u^3 = 0$

or $(u - m)^3 = 0$

So, $u = m, m, m$ are repeated roots.

The general solution is $y = (Ax^2 + Bx + C)e^{mx}$.

(iii) Auxiliary equation is $m^4 - 4m^3 + 4m^2 = 0$

or $m^2(m^2 - 4m + 4) = 0$

or $m^2(m-2)^2 = 0$

So, $m = 0,0$ and 2, 2 are roots of the equation.

Hence the general solution is

$y = (Ax + B)e^{0.x} + (Cx + D)e^{2x}$

$= Ax + B + (Cx + D)e^{2x}$

(iv) Auxiliary equation is $m^5 - m^4 - 2m^3 + 2m^2 + m - 1 = 0$

By inspective $m = \pm 1$ are two roots so that $(m^2 - 1)$ is a factor.

$$m^5 - m^4 - 2m^3 + 2m^2 + m - 1 = (m^2 - 1)(m^3 - m^2 - m + 1)$$
$$= (m^2 - 1)\{m^2(m-1) - (m-1)\}$$
$$= (m^2 - 1)(m-1)(m^2 - 1)$$
$$= (m+1)(m-1)(m-1)(m+1)(m-1)$$
$$= (m+1)^3(m-1)^2$$

$\therefore$ m = 1,1,1,-1,-1 are five roots of the auxiliary equation.

So, the complete solution of the equation is

$$y = (Ax^2 + Bx + C)e^x + (Dx + E)e^{-x}.$$

Example 3 : Solve (i) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 10y = 0$

(ii) $(D^2 + n^2)y = 0$

(iii) $D^3y + D^2y + Dy + y = 0$

(iv) $(D^4 + 8D^2 + 16)y = 0$

(v) $(D^3 - 8)y = 0$

(vi) $D^2 - 2mD + (m^2 + n^2)y = 0$

(ii) $\frac{d^5y}{dx^5} + 4\frac{d^2y}{dx^2} = 0$

Solution : (i) Auxiliary equation is $m^2 + 2m + 10 = 0$

So, $m = \frac{-2 \pm i6}{2} = -1 \pm 3i$

are two complex roots. So the solution of the equation is

$$y = (A\cos 3x + B\sin 3x)e^{-x}$$

(ii) Auxiliary equation is $m^2 + n^2 = 0$

or $m = 0 \pm in$ are complex conjugate roots.

Hence the complete solution is $y = A\cos nx + B\sin nx$.

(iii) Auxiliary equation is $m^3 + m^2 + m + 1 = 0$

or $m^2(m+1) + (m+1) = 0$

or $(m^2+1)(m+1)=0$

or $m=-1, m=0\pm i$ are roots.

So, the solution is $y=Ae^{-x}+B\cos x+C\sin x$.

(iv) Auxiliary equation is $m^4+8m^2+16=0$

or $(m^2+4)^2=0$

or $m=\pm 2i,\pm 2i$ are repeated complex roots.

so, the solution is $y=(Ax+B)\cos 2x+(Cx+D)\sin 2x$.

(v) Auxiliary equation is $m^3-8=0$

or $(m-2)(m^2+2m+4)=0$

or $m=2,\dfrac{-2\pm i2\sqrt{3}}{2}$

or $m=2,-1\pm i\sqrt{3}$ are roots of the equation.

Hence the solution is $y=Ae^{2x}+(B\cos\sqrt{3}x+C\sin\sqrt{3}x)e^{-x}$.

(vi) Auxiliary equation is $t^2-2mt+(m^2+n^2)=0$

so, $t=\dfrac{2m\pm\sqrt{4m^2-4m^2-4n^2}}{2}=m\pm in$ are complex conjugate roots.

So, the complete solution is $y=e^{mx}(A\cos nx+B\sin nx)$.

(vii) Auxiliary equation is $m^5+4m^2=0$

or $m^2\left(m^3+\left(4^{\frac{1}{3}}\right)^3\right)=0$

Roots of the equation are $m=0,0,-4^{\frac{1}{3}},\dfrac{1}{2}\left(4^{\frac{1}{3}}\pm\sqrt{4^{\frac{2}{3}}-4.4^{\frac{2}{3}}}\right)$

i.e., $m=0,0,-4^{\frac{1}{3}},2^{-\frac{1}{3}}(1\pm i\sqrt{3})$

So, the complete solution is

$$y=(Ax+b)e^{o.x}+Ce^{-4^{\frac{1}{3}}x}+e^{2^{-\frac{1}{3}}x}\left(D\cos(2^{-\frac{1}{3}}\sqrt{3})x+E\sin(2^{-\frac{1}{3}}\sqrt{3})x\right)$$

$$=Ax+B+Ce^{-\sqrt[3]{4}x}+e^{-\sqrt[3]{2}x}\left(D\cos\frac{\sqrt{3}}{\sqrt[3]{2}}x+E\sin\frac{\sqrt{3}}{\sqrt[3]{2}}x\right).$$



CHECK YOUR PROGRESS - 1

Q 1 : Solve the following differential equations :

(i) $\frac{d^3y}{dx^3}-7\frac{dy}{dx}-6y=0$

(ii) $\frac{d^4y}{dx^4}+13\frac{d^2y}{dx^2}+36y=0$

(iii) $\left(D^2+1\right)^2\left(D-1\right)y=0$

(iv) $\left(D^4-2D^3+2D^2-2D+1\right)y=0$

(v) $\frac{d^3y}{dx^3}+6\frac{d^2y}{dx^2}+11\frac{dy}{dx}+6y=0$

(vi) $\left(D^3+3D^2+3D+1\right)y=0$

(vii) $\left(D^4-5D^2+4\right)y=0$

(viii) $\left(D^4-81\right)y=0$

(ix) $\left(D^6-1\right)y=0$

(x) $\left(D^4+2D^3+3D^2+2D+1\right)y=0$

11.3.2 Methods of finding particular integral

We have already discussed that the complete integral or general solution of $\left(D^n+k_1D^{n-1}+k_2D^{n-2}+\cdots+k_n\right)y=X$ ... (1) has two parts i.e., complementary function (C.F) and particular integral (P.I).

General solution (G.S) = C.F +P.I.

(I) can be written as $f\left(D\right)y=X$ ... (2)

Particular integral of (2) $=\frac{1}{f\left(D\right)}X$

Result I $\frac{1}{D}X=\int Xdx$

suppose $\frac{1}{D}X=v$ so, $D\left(\frac{1}{D}\right)X=Dv$

or $X=Dv=\frac{dv}{dx}$

Hence, $\int Xdx=v=\frac{1}{D}X$

Note : $\frac{1}{D^2}X = \frac{1}{D}\left(\frac{1}{D}X\right) = \frac{1}{D}\int X dx = \int\left(\int X dx\right)dx$

For example, $\frac{1}{D^2}(\sin x) = \frac{1}{D}\left(\int \sin x dx\right) = \frac{1}{D}(-\cos x) = -\sin x$.

Result II $\frac{1}{D-a}(X) = e^{ax}\int Xe^{-ax}dx$

Let $\frac{1}{D-a}(X) = y$

So, $(D-a)\frac{1}{D-a}(X) = (D-a)y$

or $(D-a)y = X$

or $\frac{dy}{dx} - ay = X$, which is a linear equation with I.F $= e^{-ax}$.

So, its solution is $y = e^{ax}\int Xe^{-ax}dx$.

Hence, $\frac{1}{D-a}(X) = e^{ax}\int Xe^{-ax}dx$.

Similarly, $\frac{1}{D+a}(X) = e^{-ax}\int Xe^{ax}dx$

We discuss the methods of finding P.I for $X(x) = e^{ax}, \sin(ax+b), \cos(ax+b), x^m, e^{ax}v(x)$. Finally we discuss the general method to find P.I for any function of x.

Casel (a) when $X(x) = e^{ax}$, $\frac{1}{f(D)}e^{ax} = \frac{e^{ax}}{f(a)}$ if $f(a) \neq 0$.

$$f(D) = \left(D^n + k_1 D^{n-1} + k_2 D^{n-2} + \cdots + k_{n-1} + k_n\right)e^{ax}$$

$$= a^n e^{ax} + k_1 a^{n-1}e^{ax} + k_2 a^{n-2}e^{ax} + \cdots + k_{n-1}ae^{ax} + k_n e^{ax}$$

$$= f(a)e^{ax}$$

So, $\frac{1}{f(D)}\left[f(D)e^{ax}\right] = \frac{1}{f(D)}\left[f(a)e^{ax}\right] = f(a)\frac{1}{f(D)}e^{ax}$

or $e^{ax} = f(a)\frac{1}{f(D)}e^{ax}$

so, $\frac{1}{f(D)}e^{ax} = \frac{e^{ax}}{f(a)}$ if $f(a) \neq 0$.

(b) If $f(a)=0$, $\frac{1}{f(D)}e^{ax}=x\frac{1}{f'(a)}e^{ax}$

$f(a)=0$ implies $D-a$ is a factor of $f(D)=0$.

so, $f(D)=(D-a)\phi(D)$ and $\phi(a)\neq 0$.

$$\frac{1}{f(D)}e^{ax}=\frac{1}{D-a}.\frac{1}{\phi(D)}e^{ax}=\frac{1}{D-a}.\frac{1}{\phi(a)}e^{ax}$$

$$=\frac{1}{\phi(a)}\frac{1}{D-a}e^{ax}$$

$$=\frac{1}{\phi(a)}e^{ax}\int e^{-ax}.e^{ax}dx \text{ (Result II)}$$

$$=x\frac{1}{\phi(a)}e^{ax}$$

$f(D)=(D-a)\phi(D)$

Differentiating, $f'(D)=\phi(D)+\phi'(D)(D-a)$

or $f'(a)=\phi(a)+\phi'(a)(a-a)$

so, $f'(a)=\phi(a)+\phi'(a)\times 0$

$f'(a)=\phi(a)$

Hence $\frac{1}{f(D)}e^{ax}=x\frac{1}{f'(a)}e^{ax}$ provided $f'(a)\neq 0$.

Note : If $f'(a)=0$, it can similarly be shown that $\frac{1}{f(D)}e^{ax}=x^2\frac{1}{f''(a)}e^{ax}$ provided $f''(a)\neq 0$.

Example 1 : Find $\frac{1}{(D-2)^3}e^{2x}$

Solution : We know that $\frac{1}{D-a}(X)=e^{ax}\int Xe^{-ax}dx$

$$\frac{1}{(D-2)}e^{2x}=e^{2x}\int e^{-2x}e^{2x}dx$$

$$=e^{2x}x$$

So, $\frac{1}{(D-2)^2}e^{2x}=\frac{1}{D-2}xe^{2x}$

$$= \frac{1}{2} e^{2x} \int x e^{2x} e^{-2x} dx$$

$$= \frac{e^{2x} x^2}{2}$$

and $\frac{1}{(D-2)^3} e^{2x} = \frac{1}{2} \frac{1}{D-2} \left(x^2 e^{2x} \right)$

$$= \frac{1}{2} e^{2x} \int \left(x^2 e^{2x} \right) e^{-2x} dx$$

$$= \frac{x^3 e^{2x}}{6}$$

Example 2 : Solve $\left(D^2 + 5D + 6 \right) y = e^x$

Solution : The auxiliary equation is $m^2 + 5m + 6 = 0$

or $(m+2)(m+3) = 0$

or $m = -2, -3$

The complementary function is $Ae^{-2x} + Be^{-3x}$

$$\text{P. I} = \frac{1}{D^2 + 5D + 6} e^x$$

$$= \frac{1}{1 + 5 + 6} e^x = \frac{e^x}{12}$$

G.S is $y = \text{C.F} + \text{P.I} = Ae^{-2x} + Be^{-3x} + \frac{e^x}{12}$.

Example 3 : Solve $(D+2)(D-1)^2 y = e^{-2x} + 2\sinh x$

Solution : The auxiliary equation is $(m+2)(m-1)^2 = 0$ and its roots are -2,1,1.

C.F $= Ae^{-2x} + (Bx + C)e^x$

$$\text{P.I} = \frac{1}{(D+2)(D-1)^2} \left(e^{-2x} + 2\sinh x \right)$$

$$= \frac{1}{(D+2)(D-1)^2} \left(e^{-2x} + e^x - e^{-x} \right)$$

$$= \frac{1}{(D+2)(D-1)^2} e^{-2x} + \frac{1}{(D+2)(D-1)^2} e^x - \frac{1}{(D+2)(D-1)^2} e^{-x}$$

$$= \frac{1}{9} \frac{1}{(D+2)} e^{-2x} + \frac{1}{3} \frac{1}{(D-1)^2} e^x - \frac{1}{4} e^{-x} \quad \left[\frac{1}{f(D)} e^{ax} = \frac{e^{ax}}{f(a)} \text{ if } f(a) \neq 0 \right]$$

$= \frac{1}{9}xe^{-2x} + \frac{1}{3}\frac{x^2}{2!}e^{x} - \frac{1}{4}e^{-x}$ $\left[\frac{1}{f(D)}e^{ax} = x\frac{1}{f'(a)}e^{ax}\right]$

$= \frac{x}{9}e^{-2x} + \frac{x^2}{6}e^{x} - \frac{1}{4}e^{-x}$

G.S is $y = Ae^{-2x} + (Bx + C)e^{x} + \frac{x}{9}e^{-2x} + \frac{x^2}{6}e^{x} - \frac{1}{4}e^{-x}$.

Example 4 : Solve $(D^3 - 2D^2 + D)y = e^{-x}$

The auxiliary equation is $m^3 - 2m^2 + m = 0$

or $m(m^2 - 2m + 1) = 0$

or $m(m-1)^2 = 0$

The roots of the equation are 0,1,1.

C.F $= Ae^{o.x} + (Bx + C)e^{x} = A + (Bx + C)e^{x}$

P.I $= \frac{1}{D(D-1)^2}e^{-x} = \frac{e^{-x}}{-1.4} = \frac{-1}{4}e^{-x}$

The general solution is $y = A + (Bx + C)e^{x} - \frac{1}{4}e^{-x}$.

Case II When $X = \sin(ax + b)$ or $\cos(ax + b)$

$\frac{1}{f(D^2)}\sin(ax + b) = \frac{1}{f(-a^2)}\sin(ax + b)$ if $f(-a^2) \neq 0$

Proof :

Let $f(D^2) = (D^2)^n + k_1(D^2)^{n-1} + k_2(D^2)^{n-2} + \cdots + k_{n-1}D^2 + k_n$

$D(\sin(ax + b)) = a\cos(ax + b)$ and

$D^2(\sin(ax + b)) = (-a^2)\sin(ax + b)$

$D^3(\sin(ax + b)) = (-a^3)\cos(ax + b)$ and

$D^4(\sin(ax + b)) = a^4\cos(ax + b) = (-a^2)^2\sin(ax + b)$

and in general $(D^2)^r(\sin(ax + b)) = (-a^2)^r\sin(ax + b)$

So, $f(D^2)\sin(ax + b) = f(-a^2)(\sin(ax + b))$

as $\sin(ax + b) = f(-a^2)\frac{1}{f(D^2)}\sin(ax + b)$ so that

$\frac{1}{f(D^2)}\sin(ax + b) = \frac{\sin(ax + b)}{f(-a^2)}$,if $f(-a^2) \neq 0$

Similarly, $\frac{1}{f(D^2)}\cos(ax+b)=\frac{\cos(ax+b)}{f(-a^2)}$, if $f(-a^2)\neq 0$

If $f(-a^2)=0$, $f(D^2)=(D^2+a^2)\phi(D^2)$ and

$$\frac{1}{f(D^2)}\sin(ax+b)=\frac{1}{D^2+a^2}\frac{1}{\phi(D^2)}\sin(ax+b)$$

$$=\frac{1}{\phi(-a^2)}\frac{1}{D^2+a^2}\sin(ax+b)$$

$$=\frac{1}{\phi(-a^2)}x\sin(ax+b)$$

$$=\frac{1}{f'(-a^2)}x\sin(ax+b)$$

and $\frac{1}{f(D^2)}\cos(ax+b)=\frac{1}{f'(-a^2)}x\cos(ax+b)$ if $f'(-a^2)\neq 0$.

Example 5 : Solve $(D^2+3D+2)y=4\cos^2 x$

Solution : The auxiliary equation is $m^2+3m+2=0$ and its roots are $m=-1,-2$.

$$\text{C.F} = Ae^{-x}+Be^{x}$$

$$\text{P.I} = \frac{1}{D^2+3D+2}4\cos^2 x$$

$$=\frac{1}{D^2+3D+2}2(1+\cos 2x)$$

$$=\frac{1}{D^2+3D+2}2+\frac{1}{D^2+3D+2}2\cos 2x$$

$$=\left(\frac{1}{D+1}-\frac{1}{D+2}\right)2+2.\frac{D^2+2-3D}{(D^2+2)^2-9D^2}\cos 2x$$

$$=e^{-x}\int e^{x}2dx-e^{-2x}\int e^{2x}.2dx+\frac{1}{20}(-4\cos 2x+6\sin 2x+2\cos 2x)$$

$$\left[\ \frac{1}{D-a}(X)=e^{ax}\int Xe^{-ax}dx\ .\right]$$

$$=2-1+\frac{1}{10}(3\sin 2x-\cos 2x)$$

$$= 1 + \frac{1}{10}(3\sin 2x - \cos 2x)$$

G.S is $y = Ae^{-x} + Be^{-2x} + 1 + \frac{1}{10}(3\sin 2x - \cos 2x)$

Example 6 : Solve $(D^3 + 2D^2 + D)y = e^{2x} + \sin 2x$

Solution : The auxiliary equation is $m^3 + 2m^2 + m = 0$

or $m(m^2 + 2m + 1) = 0$

or $m = 0, -1, -1$

C.F $= A + (Bx + C)e^{-x}$

P.I $= \frac{1}{D^3 + 2D^2 + D}(e^{2x} + \sin 2x)$

$$= \frac{1}{D^3 + 2D^2 + D}e^{2x} + \frac{1}{D^3 + 2D^2 + D}\sin 2x$$

$$= \frac{1}{18}e^{2x} + \frac{1}{-2^2.D + 2(-2^2) + D}\sin 2x$$

$$= \frac{1}{18}e^{2x} + \frac{1}{-8 - 3D}\sin 2x$$

$$= \frac{1}{18}e^{2x} - \frac{3D - 8}{9D^2 - 64}\sin 2x$$

$$= \frac{1}{18}e^{2x} + \frac{1}{100}(6\cos 2x - 8\sin 2x)$$

G.S is $y =$ C.F +P.I

$$= A + (Bx + C)e^{-x} + \frac{1}{18}e^{2x} + \frac{1}{100}(6\cos 2x - 8\sin 2x).$$

Example 7 : Solve $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 3x = \sin t$

Solution : The auxiliary equation is $m^2 + 2m + 3 = 0$

or $m = \frac{1}{2}(-2 \pm \sqrt{-8})$

or $m = -1 \pm i\sqrt{2}$

C.F $= e^{-t}(A\cos\sqrt{2}t + B\sin\sqrt{2}t)$

P.I $= \frac{1}{D^2 + 2D + 3}\sin t$

$$= \frac{1}{-1^2 + 2D + 3} \sin t$$

$$= \frac{2D - 2}{4D^2 - 4} \sin t$$

$$= -\frac{1}{8}(2\cos t - 2\sin t)$$

G. S is $x = e^{-t}\left(A\cos\sqrt{2}t + B\sin\sqrt{2}t\right) - \frac{1}{4}(\cos t - \sin t)$.

Case III : When $X = x^m$,then P.I $= \frac{1}{f(D)} x^m = \left[f(D)\right]^{-1} x^m$

$$= \left[1 + \phi(D)\right]^{-1} x^m$$

Expand $\left[1 + \phi(D)\right]^{-1}$ upto the term D^m and neglect the powers of D higher than D^m ,since $D^{m+i}\left(x^m\right) = 0$.

Example 8 : Solve (i) $\left(D^3 + 3D^2 + 2D\right)y = x^2$

(ii) $\frac{d^2y}{dx^2} + 4y = x^2 + \cos 2x$

Solution : (i) The auxiliary equation is $m^3 + 3m^2 + 2m = 0$

or $m\left(m^2 + 3m + 2\right) = 0$

or $m(m+1)(m+2) = 0$

or $m = 0, -1, -2$

C.F $= A + Be^{-x} + Ce^{-2x}$

P.I $= \frac{1}{D^3 + 3D^2 + 2D} x^2$

$$= \frac{1}{2D} \cdot \frac{1}{\left(\frac{1}{2}D^2 + \frac{3}{2}D + 1\right)} x^2$$

$$= \frac{1}{2D}\left[1 + \left(\frac{3}{2}D + \frac{1}{2}D^2\right)\right]^{-1} x^2$$

$$= \frac{1}{2D}\left[1 - \left(\frac{3}{2}D + \frac{1}{2}D^2\right) + \left(\frac{3}{2}D + \frac{1}{2}D^2\right)^2 - \cdots\right] x^2$$

$$= \frac{1}{2D}\left[1 - \frac{3}{2}D + \left(\frac{9}{4} - \frac{1}{2}\right)D^2\right]x^2$$

$$= \frac{1}{2D}\left[x^2 - 3x + \frac{7}{4} \times 2\right]$$

$$= \frac{1}{2}\int\left(x^2 - 3x + \frac{7}{2}\right)dx$$

$$= \frac{x^3}{6} - \frac{3x^2}{4} + \frac{7}{4}x$$

General solution is $y = A + Be^{-x} + Ce^{-2x} + \frac{x^3}{6} - \frac{3x^2}{4} + \frac{7}{4}x$.

(ii) The auxiliary equation is $m^2 + 4 = 0$

or $m = \pm 2i$

C.F $= A\cos 2x + B\sin 2x$

P.I $= \frac{1}{D^2 + 4}\left(x^2 + \cos 2x\right)$

$$= \frac{1}{D^2 + 4}x^2 + \frac{1}{D^2 + 4}\cos 2x \quad \text{... (i)}$$

Now, $\frac{1}{D^2 + 4}x^2 = \frac{1}{4}\left(1 + \frac{D^2}{4}\right)^{-1}x^2 = \frac{1}{4}\left(1 - \frac{D^2}{4}\right)x^2$

$$= \frac{1}{4}x^2 - \frac{1}{16} \times 2$$

$$= \frac{1}{4}x^2 - \frac{1}{8}$$

Let $Y = \frac{1}{D^2 + 4}\cos 2x$ and $Z = \frac{1}{D^2 + 4}\sin 2x$ so that

$$Y + iZ = \frac{1}{D^2 + 4}\left(\cos 2x + i\sin 2x\right)$$

$$= \frac{1}{D^2 + 4}e^{2xi}$$

$$= \frac{1}{4i}\left[\frac{1}{D - 2i} - \frac{1}{D + 2i}\right]e^{2xi}$$

$$= \frac{1}{4i}\left[\frac{1}{D-2i}e^{2xi} - \frac{1}{D+2i}e^{2xi}\right]$$

$$= \frac{1}{4i}\left[\frac{1}{D-2i}e^{2xi} - \frac{e^{2xi}}{4i}\right] \quad \left(\frac{1}{f(D)}e^{ax} = \frac{e^{ax}}{f(a)}\right)$$

$$= \frac{1}{4i}\left[e^{2xi}\int e^{-2xi}.e^{2xi}dx - \frac{e^{2xi}}{4i}\right] \quad \left(\frac{1}{D-a}(X) = e^{ax}\int Xe^{-ax}dx\right)$$

$$= \frac{1}{4i}\left[x(\cos 2x + i\sin 2x) - \frac{1}{4i}(\cos 2x + i\sin 2x)\right]$$

$$= -\frac{x}{4}i\cos 2x + \frac{1}{4}x\sin 2x + \frac{1}{16}\cos 2x - \frac{i}{16}\sin 2x$$

$$= \left(\frac{1}{4}x\sin 2x + \frac{1}{16}\cos 2x\right) + i\left(-\frac{1}{16}\sin 2x - \frac{x}{4}\cos 2x\right)$$

Equating real and imaginary parts, we get

$$Y = \frac{1}{D^2+4}\cos 2x = \frac{1}{4}x\sin 2x + \frac{1}{16}\cos 2x$$

The complete solution is

$$y = A\cos 2x + B\sin 2x + \frac{1}{4}x^2 - \frac{1}{8} + \frac{1}{4}x\sin 2x + \frac{1}{16}\cos 2x\,.$$

Case IV : when $X = e^{ax}V(x)$.If u be a function of x, then

$$D(e^{ax}u) = e^{ax}Du + ae^{ax}u = e^{ax}(D+a)u$$

$$D^2(e^{ax}u) = (ae^{ax}Du + e^{ax}D^2u) + (ae^{ax}Du + a^2e^{ax}u)$$

$$= e^{ax}(D^2 + 2aDu + a^2u) = e^{ax}(D+a)^2u$$

so, $f(D)(e^{ax}u) = \{D^n + k_1D^{n-1} + k_2D^{n-2} + \cdots + k_0\}(e^{ax}u)$

$$= e^{ax}(D+a)^n u + k_1e^{ax}(D+a)^{n-2}u + \cdots + e^{ax}k_0u$$

$$= e^{ax}f(D+a)u$$

$$= e^{ax}V \quad \text{where } V = f(D+a)u$$

and $\frac{1}{f(D)}e^{ax}V = e^{ax}u = e^{ax}\frac{1}{f(D+a)}V$

Thus $\frac{1}{f(D)}(e^{ax}V) = e^{ax}\frac{1}{f(D+a)}V$.

Example 9 : Solve $(D^2 - 2D + 4)y = e^x\cos x$

Solution : The auxiliary equation is $m^2 - 2m + 4 = 0$

$$\text{or } m = \frac{2 \pm \sqrt{-12}}{2} = 1 \pm i\sqrt{3}$$

$$\text{C.F } = e^x\left(A\cos\sqrt{3}x + B\sin\sqrt{3}x\right)$$

$$\text{P. I} = \frac{1}{D^2 - 2D + 4}\left(e^x \cos x\right)$$

$$= e^x \frac{1}{(D+1)^2 - 2(D+1) + 4}\cos x \quad \left(\frac{1}{f(D)}\left(e^{ax}V\right) = e^{ax}\frac{1}{f(D+a)}V\right)$$

$$= e^x \frac{1}{D^2 + 3}\cos x$$

$$= e^x \frac{1}{-1^2 + 3}\cos x \quad \left(\frac{1}{f(D^2)}\cos(ax+b) = \frac{\cos(ax+b)}{f(-a^2)}\right)$$

$$= \frac{e^x}{2}\cos x$$

General solution is $y = e^x\left(A\cos\sqrt{3}x + B\sin\sqrt{3}x\right) + \frac{e^x}{2}\cos x$.

Example 10 : Solve $(D^2 - 1)y = x\sin 3x + \cos x$

Solution : The auxiliary equation is $m^2 - 1 = 0$

$$\text{or } m = \pm 1$$

$$\text{C.F } = Ae^x + Be^{-x}$$

$$\text{P.I} = \frac{1}{D^2 - 1}\left(x\sin 3x + \cos x\right)$$

$$= \frac{1}{D^2 - 1}x\sin 3x + \frac{1}{D^2 - 1}\cos x$$

Let $Z = \frac{1}{D^2 - 1}x\sin 3x$ and $Y = \frac{1}{D^2 - 1}x\cos 3x$

so that $Y + iZ = \frac{1}{D^2 - 1}\left(xe^{i3x}\right)$

$$= e^{i3x}\frac{1}{(D + i3)^2 - 1}x$$

$$= e^{i3x}\frac{1}{D^2 + 6iD - 10}x$$

$$= \frac{e^{i3x}}{-10}\left(1 - \frac{D^2 + 6iD}{10}\right)^{-1} x$$

$$= \frac{e^{i3x}}{-10}\left[1 - \frac{1}{10}\left(D^2 - 6iD\right)\right]x$$

$$= \frac{e^{i3x}}{-10}\left(x + \frac{3}{5}i\right)$$

$$= \frac{1}{-10}\left(\cos 3x + i\sin 3x\right)\left(x + \frac{3}{5}i\right)$$

$$= \frac{1}{-10}\left(x\cos 3x + ix\sin 3x\right) + \frac{1}{-10}\left(\frac{3}{5}i\cos 3x + i^2\frac{3}{5}\sin 3x\right)$$

$$= \frac{1}{-10}(x\cos 3x - \frac{3}{5}\sin 3x) - i\frac{1}{10}(x\sin 3x + \frac{3}{5}\cos 3x)$$

so, equating real and imaginary parts, we get

$$Z = -\frac{1}{10}(x\sin 3x + \frac{3}{5}\cos 3x)$$

and $\frac{1}{D^2 - 1}\cos x = -\frac{1}{2}\cos x$

General solution is

$$y = Ae^x + Be^{-x} - \frac{1}{10}(x\sin 3x + \frac{3}{5}\cos 3x) - \frac{1}{2}\cos x\,.$$

Example 11 : Solve $\left(D^2 - 2D + 1\right)y = xe^x \sin x$

Solution : The auxiliary equation is $m^2 - 2m + 1 = 0$

or $\left(m - 1\right)^2 = 0$

or $m = 1,1$

C.F $= \left(Ax + B\right)e^x$

P.I $= \frac{1}{D^2 - 2D + 1}xe^x \sin x$

$$= \frac{1}{\left(D - 1\right)^2}xe^x \sin x$$

$$= e^x \frac{1}{\left(D + 1 - 1\right)^2}x\sin x$$

$$= e^x \frac{1}{D^2}x\sin x$$

$$= e^x \frac{1}{D} \int x \sin x dx$$

$$= e^x \frac{1}{D} \left[- x \cos x + \int \cos x dx \right]$$

$$= e^x \frac{1}{D} \left[- x \cos x + \sin x \right]$$

$$= e^x \int \left[- x \cos x + \sin x \right] dx$$

$$= -e^x \int x \cos x dx - \int \sin x dx$$

$$= -e^x \left[x \sin x - 2 \int \sin x dx \right]$$

$$= -e^x \left(x \sin x + 2 \cos x \right)$$

General solution is $y = \left(Ax + B \right) e^x - e^x \left(x \sin x + 2 \cos x \right)$.

Example 12 : Solve $\left(D^4 + 2D^2 + 1 \right) y = x^2 \cos x$

Solution : The auxiliary equation is $m^4 + 2m^2 + 1 = 0$

or $\left(m^2 + 1 \right)^2 = 0$

or $m = \pm i, \pm i$

C.F $= \left(A + Bx \right) \cos x + \left(C + Dx \right) \sin x$

P.I $= \dfrac{1}{\left(D^2 + 1 \right)^2} x^2 \cos x$

Let $Y = \dfrac{1}{\left(D^2 + 1 \right)^2} x^2 \cos x$ and $Z = \dfrac{1}{\left(D^2 + 1 \right)^2} x^2 \sin x$

Now, $Y + iZ = \dfrac{1}{\left(D^2 + 1 \right)^2} e^{ix} x^2$

$$= e^{ix} \frac{1}{\left\{ \left(D + i \right)^2 + 1 \right\}^2} x^2$$

$$= e^{ix} \frac{1}{\left(D^2 + 2iD \right)^2} x^2$$

$$= \frac{e^{ix}}{-4D^2} \left(1 + \frac{D}{2i} \right)^{-2} x^2$$

$$= e^{ix} \frac{1}{-4D^2} \left(1 + \frac{D}{i} + \frac{3D^2}{-4} \right) x^2$$

$$= \frac{e^{ix}}{-4}\frac{1}{D^2}\left(x^2 + \frac{2x}{i} - \frac{3}{2}\right)$$

$$= \frac{1}{-4}(\cos x + i\sin x)\left(\frac{x^4}{12} - i\frac{x^3}{3} - \frac{3}{4}x^2\right)$$

$$= \frac{1}{-4}\left(\frac{x^4}{12}\cos x - i\frac{x^3}{3}\cos x - \frac{3}{4}x^2\cos x + i\frac{x^4\sin x}{12} + \frac{x^3}{3}\sin x - \frac{3}{4}ix^2\sin x\right)$$

So, $Y = $ real part $= -\frac{x^4}{48}\cos x + \frac{3x^2}{16}\cos x - \frac{x^3}{12}\sin x$

$$= -\frac{1}{48}(x^4\cos x - 9x^2\cos x + 4x^3\sin x)$$

General solution is

$$y = (A + Bx)\cos x + (C + Dx)\sin x - \frac{1}{48}(x^4\cos x - 9x^2\cos x + 4x^3\sin x)$$

Case V : when X is any function of x.

$$\text{P.I} = \frac{1}{f(D)}X$$

Let $f(D) = (D - m_1)(D - m_2)(D - m_3)\cdots(D - m_n)$

and $\frac{1}{f(D)} = \frac{A_1}{(D - m_1)} + \frac{A_2}{(D - m_2)} + \frac{A_3}{(D - m_3)} + \cdots + \frac{A_n}{(D - m_n)}$ (say)

So,

$$\frac{1}{f(D)}X = A_1\frac{1}{(D - m_1)}X + A_2\frac{1}{(D - m_2)}X + \cdots + A_n\frac{1}{(D - m_n)}X$$

$$= A_1e^{m_1x}\int e^{-m_1x}Xdx + A_2e^{m_2x}\int e^{-m_2x}Xdx + \cdots + A_ne^{m_nx}\int e^{-m_nx}Xdx$$

Example 13 : Solve $(D^2 + a^2)y = \tan ax$

Solution : The auxiliary equation is $m^2 + a^2 = 0$

or $m = \pm ia$

$$\text{C.F} = A\cos ax + B\sin ax$$

$$\text{P.I} = \frac{1}{D^2 + a^2}\tan ax$$

$$= \frac{1}{2ia}\left[\frac{1}{D - ia} - \frac{1}{D + ia}\right]\tan ax$$

$$= \frac{1}{2ia}\left[e^{iax}\int e^{-iax}\tan axdx - e^{-iax}\int e^{iax}\tan axdx\right]$$

$$= \frac{1}{2ai}\left[e^{aix}\int(\cos ax - i\sin ax)\frac{\sin ax}{\cos ax}dx - e^{-aix}\int(\cos ax + i\sin ax)\frac{\sin ax}{\cos ax}dx \right]$$

$$= \frac{1}{2ia}\left[e^{iax}\left(\int \sin axdx - i\int \frac{\sin^2 ax}{\cos ax} \right) - e^{-iax}\left(\int \sin axdx + i\int \frac{\sin^2 ax}{\cos ax} \right) \right]$$

$$= \frac{1}{2ai}[e^{aix}\left(\frac{-\cos ax}{a} \right) - e^{aix} i$$

$$\int(\sec ax - \cos ax)dx - e^{-aix}\left(\frac{-\cos ax}{a} \right) - e^{-aix} i\int(\sec ax - \cos ax)dx]$$

$$= \frac{1}{2a^2 i}(-\cos ax)\times 2i\sin ax - \frac{2\cos ax}{2a^2}\left[\log(\sec ax + \sin ax) + \sin ax\right]$$

$$= -\frac{1}{a^2}\sin 2ax - \frac{\cos ax}{a^2}\log(\sec ax + \tan ax) - \frac{\sin 2ax}{2a^2}$$

$$= -\frac{3}{2a^2}\sin 2ax - \frac{\cos ax}{a^2}\log(\sec ax + \tan ax)$$

General solution is

$$y = A\cos ax + B\sin ax - \frac{3}{2a^2}\sin 2ax - \frac{\cos ax}{a^2}\log(\sec ax + \tan ax).$$

Case VI : When $X = x\phi(x)$

$$\frac{1}{f(D)}x\phi(x) = \left\{ x - \frac{1}{f(D)}f'(D) \right\}\frac{1}{f(D)}\varphi(x)$$

Proof : We have $D(x\phi) = xD\phi + \phi$

$$D^2(x\phi) = D\phi + xD^2\phi + D\phi = xD^2\phi + 2D\phi$$

Similarly $D^n(x\phi) = xD^n\phi + nD^{n-1}\phi = xD^n\phi + \left(\frac{dD^n}{dD} \right)\phi$

Hence $f(D)(x\phi) = xf(D)\phi + f'(D)\phi$

Substituting $f(D)\phi = \phi_1$ and hence $\phi = \frac{1}{f(D)}\phi_1$

we have from preceding step

$$f(D)\left\{ x\frac{1}{f(D)}\phi_1 \right\} = x\phi_1 + f'(D)\frac{1}{f(D)}\phi_1$$

and $x\frac{1}{f(D)}\phi_1 = \frac{1}{f(D)}(x\phi_1) + \frac{1}{f(D)}f'(D)\frac{1}{f(D)}\phi_1$

So, $\frac{1}{f(D)}(x\phi_1) = x\frac{1}{f(D)}\phi_1 - \frac{1}{f(D)}f'(D)\frac{1}{f(D)}\phi_1$

Replacing ϕ_1 by ϕ, we have

$$\frac{1}{f(D)}x\phi(x) = \left\{x - \frac{1}{f(D)}f'(D)\right\}\frac{1}{f(D)}\varphi(x).$$

Example 14 : Solve $(D^2-1)y = x\sin x + (1+x^2)e^x$

Solution : The auxiliary equation is $m^2 - 1 = 0$

or $m = \pm 1$

C.F $= Ae^x + Be^{-x}$

$$\text{P.I} = \frac{1}{D^2-1}\left[x\sin x + (1+x^2)e^x\right]$$

$$= \frac{1}{D^2-1}x\sin x + \frac{1}{D^2-1}(1+x^2)e^x$$

$$= \left[\left\{x - \frac{1}{D^2-1}.\frac{d}{dD}(D^2-1)\right\}\frac{1}{D^2-1}\sin x\right] + e^x\frac{1}{(D+1)^2-1}(1+x^2)$$

$$= \left\{x - \frac{1}{D^2-1}.2D\right\}\frac{1}{-1^2-1}\sin x + e^x\frac{1}{D^2+2D}(1+x^2)$$

$$= -\frac{x\sin x}{2} + \frac{1}{D^2-1}\cos x + e^x\frac{1}{2D}\left(1+\frac{D}{2}\right)^{-1}(1+x^2)$$

$$= -\frac{x\sin x}{2} + \frac{1}{D^2-1}\cos x + e^x\frac{1}{2D}\left(1-\frac{D}{2}+\frac{D^2}{4}\right)(1+x^2)$$

$$= -\frac{x\sin x}{2} - \frac{1}{2}\cos x + e^x\frac{1}{2D}\left(1+x^2-\frac{1}{2}.2x+\frac{1}{4}.2\right)(1+x^2)$$

$$= -\frac{1}{2}(x\sin x + \cos x) + \frac{e^x}{2}\frac{1}{D}\left(x^2 - x + \frac{3}{2}\right)$$

$$= -\frac{1}{2}(x\sin x + \cos x) + \frac{e^x}{2}\left(\frac{x^3}{3} - \frac{x^2}{2} + \frac{3}{2}x\right)$$

$$= -\frac{1}{2}(x\sin x + \cos x) + \frac{xe^x}{12}(2x^2 - 3x + 9)$$

The general solution is

$$y = Ae^x + Be^{-x} - \frac{1}{2}(x\sin x + \cos x) + \frac{xe^x}{12}(2x^2 - 3x + 9).$$



CHECK YOUR PROGRESS - 2

Q 1 : Solve the following differential equations :

(a) $\left(4D^2+12D+9\right)y=144e^{-x}$

(b) $\frac{d^2y}{dx^2}-6\frac{dy}{dx}+9y=6e^{3x}+7e^{-2x}-\log 2$

Q 2 : Solve the following differential equations :

(a) $\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=\cos 2x$

(b) $\left(D^2+4\right)y=\sin^2 x$

Q 3 : Solve the following differential equations :

(a) $\left(D^3+8\right)y=x^4+2x+1$

(b) $(D-2)^2y=8\left(e^{2x}+\sin 2x+x^2\right)$

Q 4 : Solve the following differential equations :

(a) $\left(D^2-2D+1\right)y=x^2e^{3x}$

(b) $\left(D^2-3D+2\right)y=xe^x+\sin 2x$

Q 5 : Solve the following differential equations :

(a) $\frac{d^2y}{dx^2}+4y=\tan 2x$

(b) $\frac{d^2y}{dx^2}+y=\sec^2 x$



11.4 LET US SUM UP

- The complete solution of the equation

$$\frac{d^ny}{dx^n}+P_1\frac{d^{n-1}y}{dx^{n-1}}+P_2\frac{d^{n-2}y}{dx^{n-2}}+\cdots+P_n\frac{dy}{dx}=X$$

or $\left(D^n+P_1D^{n-1}+P_2D^{n-2}+\cdots+P_n\right)y=X$ has two parts. The first part is complementary function (C.F) obtained in general observing the nature of roots of the equation :

$$m^n+P_1m^{n-1}+P_2m^{n-2}+\cdots+P_n=0$$

This equation is known as auxiliary equation (A.E).

- Methods of finding C.F :

(i) If the roots are real and distinct i.e., $m_1, m_2, \cdots, m_n$

then C.F $= C_1 e^{m_1 x} + C_2 e^{m_2 x} + \cdots + C_n e^{m_n x}$

(ii) If out of n roots r roots are real and repeated, say, m and the remaining are non-repeated ,say, $m_{r+1}, m_{r+2}, \cdots, m_n$ then C.F

$= \left(C_1 x^r + C_2 x^{r-1} + \cdots + C_{r-1} x + C_r\right) e^{mx} + C_{r+1} e^{m_{r+1} x}$

$+ C_{r+2} e^{m_{r+2} x} + \cdots + C_n e^{(m_r + n - r)x}$.

For example if the roots are $m, m, m, m_4, m_5, \cdots, m_{n-3}$,then

C.F $= e^{mx}\left(C_1 + C_2 x + C_3 x^2\right) + C_4 e^{m_4 x} + \cdots + C_{n-3} e^{m_{n-3}, x}$

(iii) If one pair of roots are imaginary and remaining are real and distinct,say, $\alpha + i\beta, \alpha - i\beta, m_3, m_4, \cdots, m_n$,then

C.F $= e^{\alpha x}\left(C_1 \cos \beta x + C_2 \sin \beta x\right) + C_3 e^{m_3 x} + \cdots + C_n e^{m_n x}$

(iv) If two pairs are complex and repeated and the remaining are different,say, $\alpha \pm i\beta, \alpha \pm i\beta, m_5, m_6, \cdots, m_n$,then C.F

$= e^{\alpha x}\left\{\left(C_1 + C_2 x\right)\cos \beta x + \left(C_3 + C_4 x\right)\sin \beta x\right\} + C_5 e^{m_5 x} + \cdots + C_n e^{m_n x}$

(v) One pair real and repeated, one pair complex and the remaining are real and distinct,say, $m_1, m_1, m_2 \pm i m_3, m_5, m_6, \cdots, m_n$ .then C.F

$= \left(C_1 + C_2 x\right) e^{m_1 x} + e^{m_2 x}\left(C_3 \cos m_3 x + C_4 \sin m_3 x\right) + C_5 e^{m_5 x} + \cdots + C_n e^{m_n x}$.

- The second part of the general solution of $\left(D^n + P_1 D^{n-1} + P_2 D^{n-2} + \cdots + P_n\right) y = X$ ... $(*)$ is a paticular solution which is called particular integral (P.I) of $(*)$.

The general solution of $(*)$ is given by $y =$ C.F +P.I.

If y is a P.I of $(*)$, then $f(D) y = X$ .It is written as $y = \frac{1}{f(D)} X$.

- Methods of finding P.I :

(i) If $f(D) = \frac{1}{D - \alpha}$,P.I $= \frac{1}{D - \alpha} X = e^{\alpha x} \int e^{-\alpha x} X dx$.

and $\frac{1}{D} X = \int X dx$

(ii) If $f(D) = (D - \alpha_1)(D - \alpha_2) \cdots (D - \alpha_n)$ and $\alpha_1, \alpha_2, \cdots, \alpha_n$ are distinct.

then $\frac{1}{f(D)}X = \frac{1}{(D-\alpha_1)(D-\alpha_2)\cdots(D-\alpha_n)}(X)$

$= \left(\frac{A_1}{D-\alpha_1} + \frac{A_2}{D-\alpha_2} + \cdots + \frac{A_n}{D-\alpha_n}\right)(X)$ (Resolving into partial fraction)

$= A_1 e^{\alpha_1 x}\int e^{-\alpha_1 x} X dx + \cdots + A_n e^{\alpha_n x}\int e^{-\alpha_n x} X dx$

(iii) If $X = e^{ax}$ then $\frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}$ if $f(a) \neq 0$

If $f(a) = 0$, $\frac{1}{f(D)}e^{ax} = x\frac{1}{f'(a)}e^{ax}$

If $f'(a) = 0$, it can similarly be shown that $\frac{1}{f(D)}e^{ax} = x^2\frac{1}{f''(a)}e^{ax}$

provided $f''(a) \neq 0$ and so on.

(iv) When $X = \sin(ax+b)$ or $\cos(ax+b)$

$\frac{1}{f(D^2)}\sin(ax+b) = \frac{1}{f(-a^2)}\sin(ax+b)$ if $f(-a^2) \neq 0$

Similarly, $\frac{1}{f(D^2)}\cos(ax+b) = \frac{\cos(ax+b)}{f(-a^2)}$, if $f(-a^2) \neq 0$

If $f(-a^2) = 0$, $\frac{1}{f(D^2)}\sin(ax+b) = \frac{1}{f'(-a^2)}x\sin(ax+b)$

and $\frac{1}{f(D^2)}\cos(ax+b) = \frac{1}{f'(-a^2)}x\cos(ax+b)$ if $f'(-a^2) \neq 0$.

(v) If $X = e^{ax}\phi(x)$, then

$\frac{1}{f(D)}e^{ax}\phi(x) = e^{ax}\frac{1}{f(D+a)}\phi(x)$

(vi) If $X = x\phi(x)$, then

$\frac{1}{f(D)}X = \left(x - \frac{1}{f(D)}.f'(D)\right)\frac{1}{f(D)}\phi(x)$

(vii) If $X = x^m$, then

$\frac{1}{f(D)}X = \frac{1}{D^n(1+P_1D+P_2D^2+\cdots)}X = \frac{1}{D^n}(1+P_1D+P_2D^2+\cdots)^{-1}X$.

where $u = \int \frac{-y_2 X}{W} dx$ and $v = \int \frac{y_1 X}{W} dx$, $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$.

General solution = C.F + P.I.



11.5 FURTHER READINGS

1) Tenenbaum, Morris and Pollard, Harry. Ordinary Differential Equations, Dover Publications, Inc, N.Y 11501.
2) Differential Equations: An Introduction with Mathematica. Clay C.Ross, second Edition, Springer.
3) Differential Equations: Linear, Nonlinear, ordinary, partial. A.C. King, J.Billinggham and S. R. Otto, Cambridge-2003.
4) Differential equations with Application and Programs. S. Balachandra Rao and H.R. Anuradha, University Press (India) Limited 1996.



11.6 ANSWERS TO CHECK YOUR PROGRESS

CHECK YOUR PROGRESS – 1

Ans to Q No 1: (i) The given equation is $\left(D^3 - 7D - 6\right)y = 0$

Auxiliary equation is $m^3 - 7m - 6 = 0$

or $(m+1)(m+2)(m-3) = 0$

or $m = -1, -2, 3$

Hence complete solution is $y = C_1 e^{-x} + C_2 e^{3x} + C_3 e^{-2x}$.

(ii) The given equation is $\left(D^4 + 13D^2 + 36\right)y = 0$

The auxiliary equation is $m^4 + 13m^2 + 36 = 0$

or $m^4 + 4m^2 + 9m^2 + 36 = 0$

or $\left(m^2 + 4\right)\left(m^2 + 9\right) = 0$

or $m^2+4=0, m^2+9=0$

or $m=\pm 2i, m=\pm 3i$

The complete solution is

$y=C_1\cos 2x+C_2\sin 2x+C_3\cos 3x+C_4\sin 3x$.

where C_1, C_2, C_3, C_4 and C_5 are arbitrary constants.

(iii) The auxiliary equation is $(m^2+1)^2(m-1)=0$

or $m=\pm i, \pm i, 1$

The complete solution is $y=$ C.F +P.I

$y=(C_1+xC_2)\cos x+(C_3+xC_4)\sin x+C_5e^x$

(iv) Here auxiliary equiation is $m^4-2m^3+2m^2-2m+1=0$

or $(m-1)^2(m^2+1)=0$

or $m=1,1,\pm i$

Hence complete solution is $y=(C_1+xC_2)+(C_3\cos x+C_4\sin x)$.

(v) The given equation can be re-written as

$(D^3+6D^2+11D+6)y=0,$ where $D=\frac{d}{dx}$.

The auxiliary equation is $m^3+6m^2+11m+6=0$

or $(m+1)(m+2)(m+3)=0$

or $m=-1,-2,-3$

Hence complete solution is $y=C_1e^{-x}+C_2e^{-2x}+C_3e^{-3x}$.

(vi) The auxiliary equation is $m^3+3m^2+3m+1=0$

or $(m+1)^3=0$

or $m=-1,-1,-1$

Hence complete solution is $y=(C_1+C_2x+C_3x^2)e^{-x}$.

(vii) The auxiliary equation is $m^4-5m^2+4=0$

or $(m^2-4)(m^2-1)=0$

or $m^2=4$ and $m^2=1$

or $m=2,-2,1-1$

Hence complete solution is $y=C_1e^{2x}+C_2e^{-2x}+C_3e^x+C_4e^{-x}$.

(viii) Here auxiliary equation is $m^4-81=0$

or $(m^2+9)(m^2-9)=0$

or $(m+3i)(m-3i)(m+3)(m-3)=0$

or $m=\pm 3i, 3, -3$

Hence complete solution is $y = C_1 e^{3x} + C_2 e^{-3x} + C_3 \cos 3x + C_4 \sin 3x$.

(ix) Here auxiliary equation is $m^6 - 1 = 0$

or $\left(m^2\right)^3 - (1)^3 = 0$

or $\left(m^2 - 1\right)\left(m^4 + m^2 + 1\right) = 0$

or $\left(m^2 - 1\right)\left\{\left(m^4 + 2m^2 + 1\right) - m^2\right\} = 0$

or $(m-1)(m+1)\left\{\left(m^2 + 1\right)^2 - m^2\right\} = 0$

or $(m-1)(m+1)\left(m^2 + 1 + m\right)\left(m^2 + 1 - m\right) = 0$

or $m = 1, -1, \dfrac{-1 \pm (1-4)^{\frac{1}{2}}}{2}, \dfrac{1 \pm (1-4)^{\frac{1}{2}}}{2}$

$$m = 1, -1, -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}, \frac{1}{2} \pm \frac{i\sqrt{3}}{2}$$

Hence complete solution is

$$y = C_1 e^x + C_2 e^{-x} + e^{-\frac{x}{2}}\left[C_3 \cos\frac{x\sqrt{3}}{2} + C_4 \sin\frac{x\sqrt{3}}{2}\right] + e^{\frac{x}{2}}\left[C_5 \cos\frac{x\sqrt{3}}{2} + C_6 \sin\frac{x\sqrt{3}}{2}\right].$$

(x) Here auxiliary equation is $m^4 + 2m^3 + 3m^2 + 2m + 1 = 0$

or $\left(m^4 + 2m^3 + m^2\right) + \left(2m^2 + 2m + 1\right) = 0$

or $\left(m^2 + m\right)^2 + 2\left(m^2 + m\right) + 1 = 0$

or $\left(m^2 + m + 1\right)^2 = 0$

or $m^2 + m + 1 = 0$ (twice)

or $\therefore m = \dfrac{\left\{-1 \pm (1-4)^{\frac{1}{2}}\right\}}{2} = -\left(\dfrac{1}{2}\right) \pm i\left(\dfrac{\sqrt{3}}{2}\right)$ (twice)

Hence complete solution is

$$y = e^{-\frac{x}{2}}\left[(C_1 + C_2 x)\cos\left(\frac{\sqrt{3}x}{2}\right) + (C_3 + C_4 x)\sin\left(\frac{\sqrt{3}x}{2}\right)\right]$$

CHECK YOUR PROGRESS – 2

Ans to Q No 1: (a) Here auxiliary equation is $(2m+3)^2 = 0$

or $m = -\dfrac{3}{2}, -\dfrac{3}{2}$

C.F $= (C_1 + C_2 x)e^{-\frac{3x}{2}}$

and P.I $= \dfrac{1}{4D^2+12D+9}144e^{-3x}$

$= 144\dfrac{1}{(2D+3)^2}e^{-3x}$

$= \dfrac{144}{(-6+3)^2}e^{-3x}$

$= 16e^{-3x}$

Hence the required solution is $y = (C_1 + C_2x)e^{-\frac{3x}{2}} + 16e^{-3x}$.

(b) The given equation can be written as

$(D^2 - 6D + 9)y = 6e^{3x} + 7e^{-2x} - \log 2$

A. E is $m^2 - 6m + 9 = 0$

or $(m-3)^2 = 0$

or $m = 3,3$

C.F $= (C_1 + C_2x)e^{3x}$

P.I $= \dfrac{1}{D^2-6D+9}(6e^{3x} + 7e^{-2x} - \log 2)$

$= \dfrac{1}{D^2-6D+9}6e^{3x} + \dfrac{1}{D^2-6D+9}7e^{-2x} - \dfrac{1}{D^2-6D+9}\log 2$

$= x\dfrac{1}{2D-6}6e^{3x} + \dfrac{1}{4+12+9}7e^{-2x} - \log 2\dfrac{1}{D^2-6D+9}e^{ox}$

$= x\dfrac{1}{2}6e^{3x} + \dfrac{7}{25}e^{-2x} - \log 2\left(\dfrac{1}{9}\right)$

$= 3x^2e^{3x} + \dfrac{7}{25}e^{-2x} - \dfrac{1}{9}\log 2$

Complete solution is

$y = (C_1 + C_2x)e^{3x} + 3x^2e^{3x} + \dfrac{7}{25}e^{-2x} - \dfrac{1}{9}\log 2\left(\dfrac{1}{9}\right)$

Ans to Q No 2 : (a) Given, $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y = \cos 2x$

The given equation can be written as $(D^2 + D + 1)y = \cos 2x$ ...(1)

Auxiliary equation is $m^2 + m + 1 = 0$

or $m = \dfrac{-1 \pm i\sqrt{3}}{2}$

$$\text{C.F} = e^{-\frac{x}{2}}\left[C_1 \cos\frac{\sqrt{3}}{2}x + C_2 \sin\frac{\sqrt{3}}{2}x\right]$$

$$\text{P.I} = \frac{1}{D^2 + D + 1}\cos 2x$$

$$= \frac{1}{-2^2 + D + 1}\cos 2x$$

$$= \frac{1}{D - 3}\cos 2x$$

$$= \frac{D + 3}{D^2 - 9}\cos 2x$$

$$= \frac{D + 3}{-2^2 - 9}\cos 2x$$

$$= \frac{1}{-13}(D + 3)\cos 2x$$

$$= \frac{1}{-13}\left[D(\cos 2x) + 3\cos 2x\right]$$

$$= \frac{1}{-13}\left[-2\sin 2x + 3\cos 2x\right]$$

Hence the required solution is

$$y = e^{-\frac{x}{2}}\left[C_1 \cos\frac{\sqrt{3}}{2}x + C_2 \sin\frac{\sqrt{3}}{2}x\right] + \frac{1}{13}\left[2\sin 2x - 3\cos 2x\right]$$

(b) Given, $(D^2 + 4)y = \sin^2 x$ ... (1)

Here auxiliary equation is $m^2 + 4 = 0$

or $m = \pm 2i$

Hence C.F $= C_1 \cos 2x + C_2 \sin 2x$

$$\text{P.I} = \frac{1}{D^2 + 4}\sin^2 x$$

$$= \frac{1}{2}\frac{1}{D^2 + 4}(1 - \cos 2x)$$

$$= \frac{1}{2}\left[\frac{1}{D^2 + 4}e^{0.x} - \frac{1}{D^2 + 4}\cos 2x\right]$$

$$= \frac{1}{2}\left[\frac{1}{0 + 4}e^{0.x} - \left(\frac{x}{2 \times 2}\right)\sin 2x\right]$$

$$= \frac{1}{8} - \frac{1}{8} x \sin 2x$$

Hence the required solution is $y = C_1 \cos 2x + C_2 \sin 2x + \frac{1}{8} - \frac{1}{8} x \sin 2x$

Ans to Q No 3 : (a) Given, $\left(D^3 + 8\right)y = x^4 + 2x + 1$

The auxiliary equation is $m^3 + 8 = 0$

or $\left(m + 2\right)\left(m^2 - 2m + 4\right) = 0$

or $m = -2, 1 \pm i\sqrt{3}$

C.F $= C_1 e^{-2x} + e^x \left(C_2 \cos x\sqrt{3} + C_3 \sin x\sqrt{3}\right)$

P.I $= \frac{1}{D^3 + 8}\left(x^4 + 2x + 1\right)$

$$= \frac{1}{8\left(1 + \frac{1}{8} D^3\right)}\left(x^4 + 2x + 1\right)$$

$$= \frac{1}{8}\left(1 + \frac{1}{8} D^3\right)^{-1}\left(x^4 + 2x + 1\right)$$

$$= \frac{1}{8}\left(1 - \frac{1}{8} D^3 + \right)\left(x^4 + 2x + 1\right)$$

$$= \frac{1}{8}\left[\left(x^4 + 2x + 1\right) - \frac{1}{8} D^3\left(x^4 + 2x + 1\right)\right]$$

$$= \frac{1}{8}\left[\left(x^4 + 2x + 1\right) - 3x\right]$$

(b) Given, $\left(D - 2\right)^2 y = 8\left(e^{2x} + \sin 2x + x^2\right)$ ... (1)

Auxiliary equation is $\left(m - 2\right)^2 = 0$

or $m = 2, 2$

C.F $= \left(C_1 + C_2 x\right)e^{2x}$

P.I $= \frac{1}{\left(D - 2\right)^2}\left(e^{2x} + \sin 2x + x^2\right)$

$$= \frac{1}{\left(D - 2\right)^2} e^{2x} + \frac{1}{\left(D - 2\right)^2} \sin 2x + \frac{1}{\left(D - 2\right)^2} x^2 \quad \text{... (2)}$$

Now, $\frac{1}{\left(D - 2\right)^2} e^{2x} = x.\frac{1}{2\left(D - 2\right)} e^{2x}$

$$= x^2 . \frac{1}{2} e^{2x}$$

$$= \frac{x^2}{2} e^{2x}$$

$$\frac{1}{(D-2)^2} \sin 2x = \frac{1}{D^2 - 2D + 4} \sin 2x$$

$$= \frac{1}{-2^2 - 2D + 4} \sin 2x$$

$$= -\frac{1}{2D} \sin 2x$$

$$= -\frac{1}{4} \int \sin 2x dx$$

$$= -\frac{1}{4} \left(-\frac{\cos 2x}{2} \right)$$

$$= \frac{1}{8} \cos 2x$$

$$\frac{1}{(D-2)^2} x^2 = \frac{1}{(2-D)^2} x^2 = \frac{1}{4\left(1 - \frac{D}{2}\right)^2} x^2 = \frac{1}{4} \left(1 - \frac{D}{2}\right)^{-2} x^2$$

$$= \frac{1}{4} \left[1 - 2\left(-\frac{D}{2} \right) + \frac{(-2)(-3)}{2} \left(\frac{D}{2} \right)^2 - \cdots \right] x^2$$

$$= \frac{1}{4} \left[1 + D + \frac{3}{4} D^2 + \cdots \right] x^2$$

$$= \frac{1}{4} \left(x^2 + 2x + \frac{3}{2} \right)$$

From (2), P.I $= \frac{x^2}{2} e^{2x} + \frac{1}{8} \cos 2x + \frac{1}{4} \left(x^2 + 2x + \frac{3}{2} \right)$

Hence the required solution is

$$y = (C_1 + C_2 x) e^{2x} + \frac{x^2}{2} e^{2x} + \frac{1}{8} \cos 2x + \frac{1}{4} \left(x^2 + 2x + \frac{3}{2} \right).$$

Ans to Q No 4 : (a). Given, $(D^2 - 2D + 1) y = x^2 e^{3x}$

Auxiliary equation is $m^2 - 2m + 1 = 0$

or $m = 1,1$

C.F $= (C_1 + C_2 x)e^x$

P.I $= \dfrac{1}{D^2 - 2D + 1} x^2 e^{3x}$

$= \dfrac{1}{(D-1)^2} x^2 e^{3x}$

$= e^{3x} \dfrac{1}{(D+3-1)^2} x^2$

$= \dfrac{1}{4} e^{3x} \dfrac{1}{\left(1 + \dfrac{D}{2}\right)^2} x^2$

$= \dfrac{1}{4} e^{3x} \left(1 + \dfrac{D}{2}\right)^{-2} x^2$

$= \dfrac{1}{4} e^{3x} \left[1 - 2\left(\dfrac{D}{2}\right) + 3\left(\dfrac{D}{2}\right)^2 + \cdots\right] x^2$

$= \dfrac{1}{4} e^{3x} \left[x^2 - Dx^2 + \dfrac{3}{4} D^2 x^2 + \cdots\right]$

$= \dfrac{1}{4} e^{3x} \left[x^2 - 2x + \dfrac{3}{2}\right]$

Hence the required solution is

$$y = (C_1 + C_2 x)e^x + \frac{1}{4} e^{3x} \left[x^2 - 2x + \frac{3}{2}\right]$$

(b) Given, $(D^2 - 3D + 2)y = xe^x + \sin 2x$ ... (1)

Auxiliary equation is $m^2 - 3m + 2 = 0$

or $m = 1, 2$

C.F $= C_1 e^x + C_2 e^{2x}$

P. I $= \dfrac{1}{D^2 - 3D + 2} (xe^x + \sin 2x)$

$= \dfrac{1}{D^2 - 3D + 2} xe^x + \dfrac{1}{D^2 - 3D + 2} \sin 2x$

$= e^x \dfrac{1}{(D+1)^2 - 3(D+1) + 2} x + \dfrac{1}{-2^2 - 3D + 2} \sin 2x$

$$= e^x \frac{1}{D^2 - D} x - \frac{1}{3D + 2} \sin 2x$$

$$= e^x \frac{1}{(-D)(1 - D)} x - \frac{(3D - 2)}{9D^2 - 4} \sin 2x$$

$$= -e^x \frac{1}{D} (1 - D)^{-1} x - \frac{(3D - 2)}{9(-2^2) - 4} \sin 2x$$

$$= -e^x \frac{1}{D} [1 + D + \cdots] x + \frac{1}{40} (3D \sin 2x - 2 \sin 2x)$$

$$= -e^x \frac{1}{D} [x + 1] + \frac{1}{40} (6 \cos 2x - 2 \sin 2x)$$

$$= -e^x \int (x + 1) dx + \frac{1}{20} (3 \cos 2x - \sin 2x)$$

$$= -e^x \left(\frac{x^2}{2} + x \right) + \frac{1}{20} (3 \cos 2x - \sin 2x)$$

Hence the complete solution is

$$y = C_1 e^x + C_2 e^{2x} - e^x \left(\frac{x^2}{2} + x \right) + \frac{1}{20} (3 \cos 2x - \sin 2x).$$

Ans to Q No 5: (a).Given, $\frac{d^2 y}{dx^2} + 4y = \tan 2x$

The auxiliary equation is $m^2 + 4 = 0$

or $m = \pm 2i$

C.F $= A \cos 2x + B \sin 2x$

P.I $= \frac{1}{D^2 + 4} \tan 2x$

$$= \frac{1}{4i} \left[\frac{1}{D - 2i} - \frac{1}{D + 2i} \right] \tan 2x$$

$$= \frac{1}{4i} \left[e^{2xi} \int e^{-2xi} \tan 2x dx - e^{-2xi} \int e^{2xi} \tan 2x dx \right]$$

$$= \frac{1}{4i} [(\cos 2x + i \sin 2x) \int (\cos 2x - i \sin 2x) \tan 2x dx -$$

$$(\cos 2x - i \sin 2x) \int (\cos 2x + i \sin 2x) \tan 2x dx]$$

$$= \frac{1}{4i} [(\cos 2x + i \sin 2x) \left\{ -\frac{1}{2} \cos 2x - \frac{1}{2} \log(\sec 2x + \tan 2x) + \frac{1}{2} \sin 2x \right\} -$$

$$(\cos 2x - i\sin 2x)\left\{-\frac{1}{2}\cos 2x + \frac{1}{2}\log\left(\sec 2x + \tan 2x - \frac{1}{2}\sin 2x\right)\right\}]$$

$$= \frac{1}{4i}\left[\frac{1}{2} - \frac{i}{2}e^{2ix}\log(\sec 2x + \tan 2x) - \frac{1}{2} - \frac{i}{2}e^{-2ix}\log(\sec 2x + \tan 2x)\right]$$

$$= \frac{1}{4i}\left[-\frac{i}{2}2\cos 2x \log(\sec 2x + \tan 2x)\right]$$

$$= -\frac{1}{4}\cos 2x \log(\sec 2x + \tan 2x)$$

The complete solution is

$$y = A\cos 2x + B\sin 2x - \frac{1}{4}\cos 2x \log(\sec 2x + \tan 2x).$$

(b) The given equation can be written as $(D^2 + 1)y = \sec^2 x$

Auxiliary equation is $m^2 + 1 = 0$

or $m = \pm i$

C.F $= C_1 \cos x + C_2 \sin x$

P.I $= \frac{1}{D^2 + 1}\sec^2 x$

$$= \frac{1}{(D - i)(D + i)}\sec^2 x$$

$$= \frac{1}{2i}\left[\frac{1}{D - i} - \frac{1}{D + i}\right]\sec^2 x \quad ...(1)$$

Now, $\frac{1}{D - i}\sec^2 x = e^{ix}\int e^{-ix}\sec^2 x dx$

$$= e^{ix}\int \frac{\cos x - i\sin x}{\cos^2 x}dx$$

$$= e^{ix}\int (\sec x - i\sec x \tan x)dx$$

$$= e^{ix}[\log(\sec x + \tan x) - i\sec x] \quad ... (2)$$

Replacing i by $-i$, we have

$$\frac{1}{D + i}\sec^2 x = e^{-ix}[\log(\sec x + \tan x) + i\sec x] \quad ... (3)$$

From (1), (2) and (3), we have

$$\text{P. I} = \frac{1}{2i}\left[e^{ix}\log(\sec x + \tan x) - ie^{ix}\sec x - e^{-ix}\log(\sec x + \tan x) - ie^{-ix}\sec x\right]$$

$$= \frac{e^{ix} - e^{-ix}}{2i}\log(\sec x + \tan x) - \frac{e^{ix} + e^{-ix}}{2}\sec x$$

$$= \sin x \log(\sec x + \tan x) - \cos x \sec x$$

$$= \sin x \log(\sec x + \tan x) - 1$$

The complete solution is

$$y = C_1 \cos x + C_2 \sin x + \sin x \log(\sec x + \tan x) - 1$$



11.7 MODEL QUESTION

Q 1 : Solve the following differential equations :

(a) $(D^4 - 2D^3 + 5D^2 - 8D + 4)y = 0$

(b) $(D^5 - 13D^3 + 26D^2 + 82D + 104)y = 0$

Q 2 : Solve the following differential equations :

(a) $(D^2 + 9)y = \sec 3x$

(b) $(D^2 - 9D + 18)y = e^{e^{-3x}}$

(c) $(D^3 - D^2 - 4D + 4)y = e^{3x}$

(d) $(D^2 - 2kD + k^2)y = e^{kx}$

(e) $(D^2 - a^2)y = \cos mx$

(f) $(D^2 + 1)y = \cos x \sin 3x$

(g) $(D^2 + 2D + 1)y = 2x + x^2$

(h) $(D^4 - 2D^3 + 5D^2 - 8D + 4)y = x^2$

(i) $(D^3 - 7D - 6)y = (x + 1)e^{2x}$

(j) $(D^2 - 1)y = e^x(1 + x^2)$

(k) $(D^2 - 5D + 6)y = x \sin 3x$

(l) $(D^2 + 5D + 6)y = e^{-2x} + 5\sin 4x$

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