UNIT 13 : PARTIAL DIFFERENTIAL EQUATION

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13.1 LEARNING OBJECTIVES

After going through this unit, you will able to learn

- how to form a partial differential equation
- solution of first order and first degree partial differential equation.

13.2 INTRODUCTION

Learners are already introduced to partial derivative. Here, we shall discuss the formation of partial differential equations by eliminating either arbitrary constants or arbitary functions.

A differential equation which involves partial derivatives is called a partial differential equation. So, in partial differential equations there are more than one independent variables.

Examples :

$$\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=1 \qquad \text{... (1)}$$

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0 \qquad \text{... (2)}$$

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z \qquad \text{... (3)}$$

$$\frac{\partial^2 z}{\partial x\partial y}=xy^2 \qquad \text{... (4)}$$

Order and degree of partial differential equation :

The **order** of a partial differential equation is the order of the highest ordered derivative present in the equation.

The **degree** of a partial differential equation is the degree of the highest order partial derivative occurring in the equation.

Example : (1) $\frac{\partial z}{\partial x}+\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial x\partial y}=xy^2$ is of order two and degree one.

(2) $\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=y^2$ is of order one and degree one.

Usually, x and y are treated as independent variables and z as dependent variable so that $z=f(x,y)$.

$\frac{\partial z}{\partial x}$ means derivative of z w.r.t x treating y as constant.

For example, if $z=ax^2+2hxy+by^2$, then $\frac{\partial z}{\partial x}=2ax+2hy$.

$\frac{\partial z}{\partial y}$ means derivative of z w.r.t y treating x as constant.

Note : Throughout we shall follow the following symbols

$$p=\frac{\partial z}{\partial x},\ q=\frac{\partial z}{\partial y},\ r=\frac{\partial^2 z}{\partial x^2},\ s=\frac{\partial^2 z}{\partial x\partial y},\ t=\frac{\partial^2 z}{\partial y^2}.$$

Linear and Nonlinear Partial Differential Equations :

A partial differential equation is called a linear partial differential equation if z (the dependent variable) and all its partial derivatives appear in the first degree and product of z and its derivatives donot appear in the equation.

A partial differential equation which is not linear is called a nonlinear partial differential equation.

8.3 FORMATION OF PARTIAL DIFFERENTIAL EQUATION

An ordinary differential equation can be formed by eliminating arbitrary constants.

A partial differential equation can be formed by the elimination of arbitrary constants or arbitrary functions.

13.3.1 Formation of partial differential equation by elimination of arbitrary constants

Let $f(x, y, z, a, b) = 0$ ... (1) be an equation involving two arbitrary constants a and b.

Differentiating (1) partially with respect to x and y ,we get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = 0$$

or $$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}p = 0 \qquad \text{... (2)}$$

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} = 0$$

or $$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}q = 0 \qquad \text{... (3)}$$

Eliminating a and b from (1), (2), (3), we get an equation of the form

$$F(x, y, z, p, q) = 0$$

which is a partial differential equation of first order.

Note :

(1) If the number of arbitrary constants to be eliminated is less than the number of independent variables, then the elimination of arbitrary constants give rise to more than one partial differential equation of order one.

(2) If the number of arbitrary constants to be eliminated is equal to the number of independent variables, the partial differential that forms is of first order.

(3) If the number of arbitrary constants to be eliminated is more than the number of independent variables, the partial differential equation so obtained is of second or higher order.

Example 1 : Form partial differential equations by eliminating arbitrary constants from the following equations :

(a) $z = a(x+y)+b$ (b) $z = ax + a^2y^2 + b$

Solution : (a) Given $z = a(x+y)+b$ ... (1)

Differentiating (1) partially with respect to x and y, we get

$$\frac{\partial z}{\partial x} = a \quad \text{... (2)} \quad \text{and} \quad \frac{\partial z}{\partial y} = a \quad \text{... (3)}$$

Eliminating a between these, we get

$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}$, which is the required partial differential equation.

(b) Given, $z = ax + a^2y^2 + b$ ... (1)

Differentiating (1) partially with respect to x and y, we get

$$\frac{\partial z}{\partial x} = a \quad \text{... (2)} \quad \text{and} \quad \frac{\partial z}{\partial y} = 2a^2y \quad \text{... (3)}$$

putting the value of a from (2) in (3), we get

$\frac{\partial z}{\partial y} = 2\left(\frac{\partial z}{\partial x}\right)^2 y$, which is the required differential equation.

Example 2 : Form the partial differential equation from

$(x-a)^2 + (y-b)^2 + z^2 = r^2$ where a and b are arbitrary constants

Solution : Differentiating the given equation partially w.r.t x and y respectively

we have $2(x-a)+2zp = 0$ ... (1)

and $2(y-b)+2zq = 0$ ... (2)

From (1), $x - a = -zp$

From (2), $y - b = -zq$

Putting the values of $(x-a)$ and $(y-b)$ in the given equation,we have

$$(-zp)^2 + (-zq)^2 + z^2 = r^2$$

or $z^2(p^2 + q^2 + 1) = r^2$

which is the required equation.

Example 3 : Eliminate a and b from $z = axe^y + \frac{1}{2}a^2e^{2y} + b$ and form a partial differential equation.

Solution : Given, $z = axe^y + \frac{1}{2}a^2e^{2y} + b$ ... (1)

Differentiating (1) partially with respect to x and y, we get

$$\frac{\partial z}{\partial x} = ae^y \quad \text{... (2)}$$

and $\frac{\partial z}{\partial y} = axe^y + a^2e^{2y} = x\left(ae^y\right) + \left(ae^y\right)^2$ ... (3)

substituting the value of ae^y from (2) in (3), we get

$\frac{\partial z}{\partial y} = x\left(\frac{\partial z}{\partial x}\right) + \left(\frac{\partial z}{\partial x}\right)^2$,which is the required equation.

Example 4 : Form a partial differential equation by eliminating the arbitrary constants a and b from $\log(az-1) = x + ay + b$.

Solution : Given, $\log(az-1) = x + ay + b$ ... (1)

Differentiating (1) partially w.r.t. 'x', we get

$$\frac{a}{az-1}\frac{\partial z}{\partial x} = 1 \quad \text{... (2)}$$

Differentiating (1) partially w.r.t. 'y', we get

$$\frac{a}{az-1}\frac{\partial z}{\partial y} = a \quad \text{... (3)}$$

From (3), $az - 1 = \frac{\partial z}{\partial y}$ so that $a = \frac{1 + \frac{\partial z}{\partial y}}{z}$ ... (4)

putting the above values of $az-1$ and a in (2), we have

$$\frac{1 + \left(\frac{\partial z}{\partial y}\right)}{z\left(\frac{\partial z}{\partial y}\right)} \cdot \frac{\partial z}{\partial x} = 1$$

or $1 + \left(\frac{\partial z}{\partial y}\right)\frac{\partial z}{\partial x} = z\left(\frac{\partial z}{\partial y}\right)$ which is the required equation.



CHECK YOUR PROGRESS - 1

Q 1 : Form partial differential equations from the following by eliminating arbitrary constants :

(i) $z = ax + by + a^2 + b^2$

(ii) $z = a(x+y) + b(x-y) + abt + c$

(iii) $z = (x+a)(y+b)$

(iv) $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

Q 2 : Find the partial differential equations of all spheres of fixed radius having their centres in the xy -plane.

Q 3 : Find the partial differential equations of all spheres whose centres lie on the z-axis.

13.3.2 Formation of partial differential equation by elimination by arbitrary functions

(a) **One arbitrary function**

Let $z = f(u)$ ... (1) where $f(u)$ is an arbitrary function of u and u is a given function of x, y, z i.e., $u = u(x, y, z)$.

Differentiating (1) partially w.r.t. x and y by chain rule

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial z}\frac{\partial z}{\partial x} \quad \text{... (2)}$$

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial z}\frac{\partial z}{\partial y} \quad \text{... (3)}$$

By eliminating the arbitrary function $f(u)$ from (1),(2),(3), we get a partial differential equation of first order.

(b) **Two arbitrary functions :**

Differentiating twice or more, the elimination process results in a partial differential equation of 2nd or higher order.

Note :

(1) If the partial differential equation is obtained by elimination of arbitrary functions, then the order of the partial differential equation, in general, is equal to the number of arbitrary functions eliminated.

(2) When n is the number of arbitrary functions, one may get several partial differential equations.

Example 5 : Find the partial differential equations by eliminating the arbitrary functions from the following :

(i) $z = f\left(x^2 - y^2\right)$

Solution : Differentiating z partially w.r.t. to x and y,

$$p = \frac{\partial z}{\partial x} = f'\left(x^2 - y^2\right).2x \quad \text{... (1)}$$

and $$q = \frac{\partial z}{\partial y} = f'\left(x^2 - y^2\right).(-2y) \quad \text{... (2)}$$

Dividing (1) by (2),

$$\frac{p}{q} = -\frac{x}{y}$$

or $py + qx = 0$ which is the required equation.

(ii) $z = yf(x) + xg(y)$

Differentiating z partially w.r.t. to x and y respectively, we get

$$p = \frac{\partial z}{\partial x} = yf'(x) + g(y)$$

$$q = \frac{\partial z}{\partial y} = f(x) + xg'(y)$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = f'(x) + g'(y)$$

$$\therefore \quad xys = xy\left(f'(x) + g'(y)\right) \quad \text{... (1)}$$

Also $px + qy - z = \left(yf'(x) + g(y)\right)x + \left(f(x) + xg'(y)\right)y - z$

$$= xyf'(x) + xg(y) + yf(x) + xyg'(y) - yf(x) - xg(y)$$

$$= xy\left(f'(x) + g'(y)\right) \quad \text{... (2)}$$

From (1) and (2)

which is the required differential equation.

Example 6 : Find partial differential equation from

$$z = f(x + at) + g(x - at) \quad \text{... (i)}$$

Solution : Differentiating (i) partially w.r.t. x and t,

$$\frac{\partial z}{\partial x} = f'(x + at) + g'(x - at) \quad \text{.... (ii)}$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x+at) + g''(x-at) \quad \text{... (iii)}$$

$$\frac{\partial z}{\partial t} = af'(x+at) - ag'(x-at)$$

$$\frac{\partial^2 z}{\partial t^2} = a^2\left\{f''(x+at) + g''(x-at)\right\} \quad \text{... (iv)}$$

$$= a^2 \frac{\partial^2 z}{\partial x^2}$$

The required differential equation is

$$\frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2}.$$

Example 7: Form the PDE by eliminating the arbitrary function from

$$z = f\left(x^2 + y^2 + z^2\right)$$

Solution : Differentiating partially w.r.t. x and y, we have

$$\frac{\partial z}{\partial x} = f'\left(x^2 + y^2 + z^2\right) \times \left(2x + 2z\frac{\partial z}{\partial x}\right) \quad \text{... (i)}$$

$$\frac{\partial z}{\partial y} = f'\left(x^2 + y^2 + z^2\right) \times \left(2y + 2z\frac{\partial z}{\partial y}\right) \quad \text{... (ii)}$$

Dividing (i) by (ii),

$$\frac{\frac{\partial z}{\partial x}}{\frac{\partial z}{\partial y}} = \frac{\left(2x + 2z\frac{\partial z}{\partial x}\right)}{\left(2y + 2z\frac{\partial z}{\partial y}\right)}$$

or $\frac{\partial z}{\partial x} 2\left(y + z\frac{\partial z}{\partial y}\right) = 2\left(x + z\frac{\partial z}{\partial x}\right)\frac{\partial z}{\partial y}$

or $\frac{\partial z}{\partial x}\left(y + z\frac{\partial z}{\partial y}\right) = \frac{\partial z}{\partial y}\left(x + z\frac{\partial z}{\partial x}\right)$

or $y\frac{\partial z}{\partial x} + z\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} = x\frac{\partial z}{\partial y} + z\frac{\partial z}{\partial y}\frac{\partial z}{\partial x}$

or $y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 0$, which is the required equation.

Elimination of Arbitrary Function F

Let $F(u,v) = 0$ ... (1)

where $u = u(x, y, z)$ and $v = v(x, y, z)$ are given functions of x, y, z.
Differentiating (1) partially w.r.t. x by chain rule, we get

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right)+\frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right)=0 \quad \text{... (2)}$$

Similarly, differentiating (1) partially w.r.t. y by chain rule, we get

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial y}\right)+\frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\frac{\partial z}{\partial y}\right)=0 \quad \text{... (3)}$$

Eliminating $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v}$ from (2) and (3), we get

$$\begin{vmatrix} \frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial x} & \frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\frac{\partial z}{\partial x} \\ \frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial y} & \frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\frac{\partial z}{\partial y} \end{vmatrix}=0 \quad \text{.... (4)}$$

Now, expanding the determinant (4) results in a P.D.E. which is free of the arbitrary function F as

$$\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right)\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\frac{\partial z}{\partial y}\right)-\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial y}\right)\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right)=0$$

or $\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}\right)p+\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}\right)q=\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}\right)$

which is a first order linear P.D.F.

Example 8: Obtain the partial differential equation by eliminating arbitrary function from

$$f\left(x^2+y^2, z-xy\right)=0$$

Solution : Let $u = x^2 + y^2$ and $v = z - xy$ so that $f(u, v) = 0$
Differentiating partially w.r.t. x and y, we have

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}p\right)+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p\right)=0$$

or $\frac{\partial f}{\partial u}(2x)+\frac{\partial f}{\partial v}(-y+p)=0$ ... (1)

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q\right)+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q\right)=0$$

or $\frac{\partial f}{\partial u}(2y)+\frac{\partial f}{\partial v}(-x+q)=0$ ....(2)

Eliminating $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ from (1) and (2), we get

$$\begin{vmatrix} 2x & -y+p \\ 2y & -x+q \end{vmatrix}=0$$

or $2x(-x+q)-2y(-y+p)=0$

or $xq-yp=x^2-y^2$, which is the required PDE.

Example 9 : Obtain the partial differential equation by eliminating arbitrary function from

$$F(xy+z^2, x+y+z)=0$$

Solution : Let $u=xy+z^2$ and $v=x+y+z$ so that $F(u,v)=0$.

Differentiating partially w.r.t. x and y, we have

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}p\right)+\frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p\right)=0$$

or $\frac{\partial F}{\partial u}(y+2zp)+\frac{\partial F}{\partial v}(1+p)=0$ ... (1)

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q\right)+\frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q\right)=0$$

or $\frac{\partial F}{\partial u}(x+2zq)+\frac{\partial F}{\partial v}(1+q)=0$ ....(2)

Eliminating $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v}$ from (1) and (2), we get

$$\begin{vmatrix} y+2zp & 1+p \\ x+2zq & 1+q \end{vmatrix}=0$$

or $(1+q)(y+2zp)-(1+p)(x+2zq)=0$

or $(2z-x)p+(y-2z)q=x-y$



CHECK YOUR PROGRESS - 2

Q 1 : Form partial differential equations by eliminating arbitrary functions from the following :

(i) $z = f\left(\frac{y}{x}\right)$ (ii) $z = f(xy)$

(iii) $e^x f(y-x) = z$ (iv) $z = f(x^2 - y) + g(x^2 + y)$

(v) $z = f_1(2x+y) + f_2(x-2y)$ (vi) $f(xyz, x+y+z) = 0$



13.4 LET US SUM UP

- An equation containing one or more partial derivatives of an unknown function of two or more independent variables is known as a partial differential equation.
- The order of a partial differential equation is defined as the order of the highest partial derivative occuring in the partial differential equation.
- The degree of a partial differential equation is the degree of the highest order derivative which occurs in it after the equation has been rationalised, i.e., made free from radicals and fractions so far as derivatives are concerned.
- A partial differential equation is said to be linear if the dependent variable z and its derivatives are of degree one and product of z and its derivatives donot appear in the equation.
- The partial differential equations may be formed
  (i) by the elimination of arbitrary constants from a given equation,
  (ii) by the elimination of arbitrary functions of the variables.
- If the number of arbitary constants to be eliminated is equal to the number of the independent variables, the partial differentials that arise are of the first order.
- If the number of arbitrary constants to be eliminated is more than the number of independent variables, the partial differential equations obtained are of second or higher order.

- If the partial differential equation is obtained by elimination of arbitrary functions, then the order of the partial differential equation is equal to the number of arbitrary functions eliminated.



13.5 FURTHER READINGS

1) Bhamra. K. S. Partial Differential Equations : An Introductory Treatment with Applications. PHL Learning Private Limited, New Delhi - 110001.
2) Amaranasth, T. Partial Differential Equations, 2nd edition, Jones and Bartlett Publishers.
3) C. Evans, Lawrence. Partial Differential Equations, 2nd edition, American Mathematical Society.
4) Ioannis P Stavroulakis and Stepan A Tersia. Partial differential Equations : An Introduction with Mathematica and Maple, 2nd edition, World scientific Publishing Co. Pt. Ltd, 2004.
5) Donald Greenspan. Introduction to Partial Differential Equation, Mc Graw Hill, Inc.



13.6 ANSWERS TO CHECK YOUR PROGRESS

Check your progress -1

Ans to Q No 1 : (i) Given, $z = ax + by + a^2 + b^2$ ... (1)

Differentiating (1) partially w.r.t. x , y we get

$\frac{\partial z}{\partial x} = a$ ... (2) and $\frac{\partial z}{\partial y} = b$ ... (3)

Putting the values of a and b from (2) , (3) in (1), we get

$z = px + qy + p^2 + q^2$ where $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$.

which is required partial differential equation.

(ii) Given, $z = a(x+y)+b(x-y)+abt+c$ ... (1)

Differentiating (1) partially w.r.t. x, y and t, we get

$$\frac{\partial z}{\partial x} = a+b, \frac{\partial z}{\partial y} = a-b, \frac{\partial z}{\partial t} = ab \quad \text{... (2)}$$

Since $(a+b)^2-(a-b)^2 = 4ab$ s

We get, $\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = 4\frac{\partial z}{\partial t}$ (using (2))

which is required differential equation.

(iii) we have $z = (x+a)(y+b)$ ... (1)

$$\frac{\partial z}{\partial x} = y+b \quad \text{and} \quad \frac{\partial z}{\partial y} = x+a$$

substituting in (1), we have $z = \frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}$ which is required equation.

(iv) we have $2z = \frac{x^2}{a^2}+\frac{y^2}{b^2}$ ... (1)

Differentiating (1) partially w.r.t. x and y, we get

$$2\frac{\partial z}{\partial x} = \frac{2x}{a^2}$$

or $\frac{1}{x}\frac{\partial z}{\partial x} = \frac{1}{a^2}$

or $\frac{1}{a^2} = \frac{1}{x}\frac{\partial z}{\partial x} = \frac{p}{x}$ ... (2)

and $2\frac{\partial z}{\partial y} = \frac{2y}{b^2}$

or $\frac{1}{y}\frac{\partial z}{\partial y} = \frac{1}{b^2}$

or $\frac{1}{b^2} = \frac{1}{y}\frac{\partial z}{\partial y} = \frac{q}{y}$ ... (3)

Substituting the values of $\frac{1}{a^2}$ and $\frac{1}{b^2}$ from (2) and (3) in (1), we get

$2z = px+qy$ which is required partial differential equation.

Ans to Q No 2 : The equation of any sphere of fixed radius 'λ' having centre $(h,k,0)$ in the xy-plane is given by

$(x-h)^2+(y-k)^2+(z-0)^2=\lambda^2$

or $(x-h)^2+(y-k)^2+z^2=\lambda^2$ ... (1) where h and k are arbitrary constants.

Differentiating (1) partially w.r.t. x and y, we get

$$2(x-h)+2z\frac{\partial z}{\partial x}=0$$

or $(x-h)=-z\frac{\partial z}{\partial x}$ ... (2)

and $2(y-k)+2z\frac{\partial z}{\partial y}=0$

or $(y-k)=-z\frac{\partial z}{\partial y}$ ... (3)

Substituting the values of $(x-h)$ and $(y-k)$ from (2) and (3) in (1),

we get, $z^2\left(\frac{\partial z}{\partial x}\right)^2+z^2\left(\frac{\partial z}{\partial y}\right)^2+z^2=\lambda^2$

or $z^2\left[\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1\right]=\lambda^2$, which is required partial differential equation.

Ans to Q No 3 : The equation of any sphere of fixed radius 'λ' having centre $(0,0,k)$ in the z-axis is given by

$(x-0)^2+(y-0)^2+(z-k)^2=\lambda^2$

or $x^2+y^2+(z-k)^2=\lambda^2$ ... (1) where k is an arbitrary constant.

Differentiating (1) partially w.r.t. x and y, we get

$$2x+2(z-k)\frac{\partial z}{\partial x}=0$$

or $x+(z-k)\frac{\partial z}{\partial x}=0$

or $x+(z-k)p=0$

or $z-k=-\frac{x}{p}$ ... (2)

and $2y + 2(z-k)\frac{\partial z}{\partial y} = 0$

or $y + (z-k)\frac{\partial z}{\partial y} = 0$

or $y + (z-k)q = 0$

or $z - k = -\frac{y}{q}$ ... (3)\

From (2) and (3), we get $-\frac{x}{p} = -\frac{y}{q}$

or $xq - yp = 0$, which is required partial differential equation.

Check your progress - 2

Ans to Q No 1 : (i) Given, $z = f\left(\frac{y}{x}\right)$ ... (1)

Differentiating (1) partially w.r.t. x and y, we get

$\frac{\partial z}{\partial x} = f'\left(\frac{y}{x}\right) \times \left(-\frac{y}{x^2}\right)$

or $f'\left(\frac{y}{x}\right) = -\left(\frac{x^2}{y}\right) \times \frac{\partial z}{\partial x}$ ... (2)

and $\frac{\partial z}{\partial y} = f'\left(\frac{y}{x}\right) \times \left(\frac{1}{x}\right)$

or $f'\left(\frac{y}{x}\right) = x\left(\frac{\partial z}{\partial y}\right)$ ... (3)

Eliminating $f'\left(\frac{y}{x}\right)$ between (2) and (3), we have

$-\frac{x^2}{y}\frac{\partial z}{\partial x} = x\frac{\partial z}{\partial y}$

or $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$, which is required partial differential equation.

(ii) Given, $z = f(xy)$ ... (1)

Differentiating (1) partially w.r.t. x and y, we get

$\frac{\partial z}{\partial x} = f'(xy)y$ ... (2)

$$\frac{\partial z}{\partial y} = f'(xy)x \quad \text{... (3)}$$

Dividing (2) by (3), we get

$$\frac{\frac{\partial z}{\partial x}}{\frac{\partial z}{\partial y}} = \frac{y}{x}$$

or $\frac{p}{q} = \frac{y}{x}$

or $xp - yq = 0$.

(iii) Given, $e^x f(y-x) = z$ ... (1)

Differentiating (1) partially w.r.t. x, we get

$$e^x f(y-x) - e^x f'(y-x) = \frac{\partial z}{\partial x}$$

or $z - e^x f'(y-x) = p$, using (1)

or $z - p = e^x f'(y-x)$ ... (2)

Again, differentiating (1) partially w.r.t. y, we get

$$e^x f(y-x) + e^x f'(y-x) = \frac{\partial z}{\partial y}$$

or $z + z - p = q$, using (1) and (2)

or $p + q = 2z$.

(iv) Given, $z = f(x^2 - y) + g(x^2 + y)$ ... (1)

Differentiating (1) partially w.r.t. x and y, we have

$$\frac{\partial z}{\partial x} = 2xf'(x^2 - y) + 2xg'(x^2 + y)$$

or, $\frac{\partial z}{\partial x} = 2x\{f'2(x^2 - y) + g'(x^2 + y)\}$ ... (2)

and $\frac{\partial z}{\partial y} = -f'(x^2 - y) + g'(x^2 + y)$ ... (3)

Differentiating (2) and (3) partially w.r.t. x and y respectively, we have

$$\frac{\partial^2 z}{\partial x^2} = 2\{f'(x^2 - y) + g'(x^2 + y)\} + 4x^2\{f''(x^2 - y) + g''(x^2 + y)\} \quad \text{...(4)}$$

and $\frac{\partial^2 z}{\partial y^2} = f''\left(x^2 - y\right) + g''\left(x^2 + y\right)$ ... (5)

Again (2) $\Rightarrow f'\left(x^2 - y\right) + g'\left(x^2 + y\right) = \frac{1}{2x} \times \frac{\partial z}{\partial x}$ ... (6)

Using (5) and (6) in (4), we get

$$\frac{\partial^2 z}{\partial x^2} = 2 \times \frac{1}{2x} \frac{\partial z}{\partial x} + 4x^2 \frac{\partial^2 z}{\partial y^2}$$

or $$x \frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial x} + 4x^3 \frac{\partial^2 z}{\partial y^2}$$

which is the required partial differential equation.

(v) Do Yourself.

(vi) Let $u = x + y + z$ and $v = xyz$ so that $f(u, v) = 0$.

Differentiating partially w.r.t. x and y, we have

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} p\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} p\right) = 0$$

or $\frac{\partial f}{\partial u}(1 + p) + \frac{\partial f}{\partial v}(yz + xyp) = 0$ ... (1)

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} q\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} q\right) = 0$$

$$\frac{\partial f}{\partial u}(1 + q) + \frac{\partial f}{\partial v}(xz + xyq) = 0 \qquad \text{....(2)}$$

Eliminating $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ from (1) and (2), we get

$$\begin{vmatrix} 1 + p & yz + xyp \\ 1 + q & xz + xyq \end{vmatrix} = 0$$

or $(xz + xyq)(1 + p) - (1 + q)(yz + xyp) = 0$

or $x(z - y)p + (x - z)yq + z(x - y) = 0$.



13.7 MODEL QUESTIONS

Q 1 : Eliminating the arbitrary constants from the following equations form partial differential equations.

(a) $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ (b) $(x-a)^2+(y-b)^2=z^2-c$

Q 2 : Form partial differential equations by eliminating the arbitrary functions from the following equations.

(a) $z=F(x^2+y^2)$ (b) $z=e^{my}\phi(x-y)$

(c) $lx+my+nz=\phi(x^2+y^2+z^2)$

Q 3 : Eliminate the arbitrary functions from

$$y=f(x-at)+xF(x-at)+x^2\phi(x-at)$$

Q 4 : Obtain the partial differential equation of all planes which are at a constant distance p from the origin.

Q 5 : Obtain a partial differential equation by eliminating arbitrary functions f from $z=x+y+f(xy)$.

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