UNIT 14 : LINEAR PARTIAL DIFFERENTIAL EQUATION

UNIT STUCTURE

14.1 Learning Objectives

14.2 Introduction

14.3 Solution of first order and first degree partial differential equation

14.4 Let Us Sum Up

14.5 Further Readings

14.6 Answers To Check Your Progress

14.7 Model Questions

14.1 LEARNING OBJECTIVES

After going through this unit, you will able to find

- solution of first order and first degree partial differential equation.

14.2 INTRODUCTION

We have already discussed the formation of partial differential equations by eliminating either arbitrary constants or arbitary functions in previous unit. In this unit, we shall proceed to solution of first order and first degree partial differential equation.

14.3 SOLUTION OF FIRST ORDER AND FIRST DEGREE PARTIAL DIFFERENTIAL EQUATION

A differential equation involving $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$ only and no higher order is called of degree one.If the degree of p and q are unity, then it is called a linear partial differential equation of order one.

Consider $Pp + Qq = R$ where P, Q and R are functions of x, y and dependent variable z.

Working Rule for solving $Pp + Qq = R$:

Step 1 : We put the given linear partial differential equation of the first order in the standard form

$$Pp + Qq = R \qquad \text{... (1)}$$

Step 2 : We write down Lagrange's auxiliary equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \qquad \text{... (2)}$$

Step 3 : We solve (2) by using some well known method.Let $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ be two independent solutions of (2).

Step 4 : The general solution of (1) is then written in one of the following three equivalent forms :

$\phi(u, v) = 0, u = \phi(v)$ and $v = \phi(u), \phi$ being an arbitrary function.

Example 10 : Solve $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$

Solution : Here auxiliary equations are

$$\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy} \qquad \text{... (1)}$$

Each of these equations =

$$\frac{dx - dy}{x^2 - y^2 - z(y - x)} = \frac{dy - dz}{y^2 - z^2 - x(z - y)} = \frac{dz - dx}{z^2 - xy - x^2 + yz}$$

so, $\frac{d(x - y)}{(x - y)(x + y + z)} = \frac{d(y - z)}{(y - z)(y + z + x)} = \frac{d(z - x)}{(z - x)(x + y + z)}$

or $\frac{d(x - y)}{(x - y)} = \frac{d(y - z)}{(y - z)} = \frac{d(z - x)}{(z - x)}$

Now $\frac{d(x - y)}{(x - y)} = \frac{d(y - z)}{(y - z)}$

Integrating, $\log(x - y) = \log(y - z) + \log C_1$

$$\text{or } \frac{x - y}{y - z} = C_1 \qquad \text{... (2)}$$

Again from $\frac{d(y - z)}{y - z} = \frac{d(z - x)}{z - x}$

Integrating we get $\log(y - z) - \log(z - x) = \log C_2$

$$\text{or } \frac{y - z}{z - x} = C_2 \quad \text{... (3)}$$

$\therefore$ the solution is $f\left(\frac{x-y}{y-z}, \frac{y-z}{z-x}\right) = 0$.

Example 11 : Solve $y^2 p - xyq = x(z - 2y)$

Solution : The auxiliary equations are

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

Taking the first two ratios, we get

$$\frac{dx}{y} = \frac{dy}{-x}$$

or $xdx + ydy = 0$

Integrating, we get $\frac{x^2}{2} + \frac{y^2}{2} = \frac{C_1}{2}$

or $x^2 + y^2 = C_1$

Again taking $\frac{dy}{-y} = \frac{dz}{z-2y}$

or $-2ydy + zdy + ydz = 0$

or $-y^2 + yz = C_2$

Then, $yz - y^2 = f\left(x^2 + y^2\right)$ is the required solution.

Example 12 : Solve $xzp + yzq = xy$

Solution : Auxiliary equations are $\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{xy}$

From first two $\frac{dx}{xz} = \frac{dy}{yz} \Rightarrow \frac{dx}{x} = \frac{dy}{y}$

Integrating $\log x = \log y + \log C_1$

or $\frac{x}{y} = C_1$

Similarly from the last two $\frac{y}{z} = C_2$

Hence the general solution is $\phi\left(\frac{x}{y}, \frac{y}{z}\right) = 0$.

Example 13 : Solve $\frac{y^2 z}{x} p + xzq = y^2$

Solution : The given equation can be written as

$$y^2zp + x^2zq = xy^2$$

Auxiliary equations are

$$\frac{dx}{y^2z} = \frac{dy}{x^2z} = \frac{dz}{xy^2}$$

From first two, $x^2dx = y^2dy$,i.e. $x^3 - y^3 = C_1$

From first and third , $xdx = zdz$,i.e. $x^2 - z^2 = C_2$

Hence $\phi\left(x^3 - y^3, x^2 - z^2\right) = 0$ is the required solution.

Example 14 : Solve $yzp + xzq = xy$

Solution : The auxiliary equations are

$$\frac{dx}{yz} = \frac{dy}{xz} = \frac{dz}{xy} \quad \text{... (1)}$$

From (1), $\frac{dx}{y} = \frac{dy}{x}$

$$\Rightarrow xdx = ydy$$

$$\Rightarrow x^2 - y^2 = C_1$$

Similarly, From (1), $\frac{dy}{z} = \frac{dz}{y}$

$$\Rightarrow ydy = zdz$$

$$\Rightarrow y^2 - z^2 = C_2$$

Hence, $\phi\left(x^2 - y^2, y^2 - z^2\right) = 0$ is the required solution.

Method of Multipliers :

Let the auxiliary equations of the given equation be

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Let l, m, n be constants or functions of x, y, z then we have

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{ldx + mdy + ndz}{lP + mQ + nR}$$

l, m, n are chosen in such a way that $lP + mQ + nR = 0$

Thus we get $ldx + mdy + ndz = 0$

Solving this differential equation, we get $u = C_1$

Similarly, we choose another set of multipliers (l_1, m_1, n_1), we get $v = C_2$.

Hence, the required solution is $f(u, v) = 0$.

Example 15 : Solve : $\frac{y-z}{yz}p + \frac{z-x}{zx}q = \frac{x-y}{xy}$

Solution : The given equation can be written as

$(xy - zx)p + (yz - xy)q = xz - yz$

The auxiliary equations are

$$\frac{dx}{xy-zx} = \frac{dy}{xy-zx} = \frac{dz}{yz-xy} = \frac{dx+dy+dz}{0}$$

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

Now, $dx + dy + dz = 0 \Rightarrow x + y + z = C_1$

and $\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0 \Rightarrow \log x + \log y + \log z = \log C_2$

$$\Rightarrow xyz = C_2$$

Hence the general solution is $\phi(x + y + z, xyz) = 0$

Example 16 : Solve $(mz - ny)\frac{\partial z}{\partial x} + (nx - lz)\frac{\partial z}{\partial y} = (ly - mx)$

Solution : Given, $(mz - ny)\frac{\partial z}{\partial x} + (nx - lz)\frac{\partial z}{\partial y} = (ly - mx)$

Here, the auxiliary equations are

$$\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx}$$

using multipliers x, y, z we get

each fraction $= \frac{xdx + ydy + zdz}{x(mz-ny) + y(nx-lz) + z(ly-mx)} = \frac{xdx + ydy + zdz}{0}$

$$\therefore xdx + ydy + zdz = 0 \quad \text{... (1)}$$

Integrating (1), we get $x^2 + y^2 + z^2 = C_1$ ... (2)

Again using multipliers l, m, n we get each fraction

$$= \frac{ldx + mdy + ndz}{l(mz - ny) + m(nx - lz) + n(ly - mx)} == \frac{ldx + mdy + ndz}{0}$$

$\therefore ldx + mdy + ndz = 0$ ... (3)

Integrating (2), we get $lx + my + nz = C_2$ ... (4)

Hence from (2) and (4), the required solution is

$x^2 + y^2 + z^2 = f(lx + my + nz)$.



CHECK YOUR PROGRESS

Q 1: Solve the following partial differential equations :

(a) $yq - xp = z$

(b) $p + q = \frac{z}{a}$

(c) $x^2(y - z)p + (z - x)y^2q = z^2(x - y)$

(d) $z(xp - yq) = y^2 - x^2$

(e) $x(y^2 + z)p - y(x^2 + z)q = z(x^2 - y^2)$



14.4 LET US SUM UP

- The partial differential equation of the form $Pp + Qq = R$, where P, Q and R are functions of x, y, z is called the standard form of the linear partial differential equation of the first order and is called Lagrange's linear partial differential equation of first order.
- To solve the equation $Pp + Qq = R$, we apply following steps

  (i) We form the auxiliary equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

  (ii) We solve the auxiliary equations by the method of grouping or the method of multipliers or both to get two independent solutions $u = a$ and $v = b$, where a, b are arbitrary constants.

(iii) Then $\phi(u,v)=0$ or $u=f(v)$ is the general solution of the equation $Pp+Qq=R$.



14.5 FURTHER READINGS

1) Bhamra. K. S. Partial Differential Equations : An Introductory Treatment with Applications. PHL Learning Private Limited, New Delhi - 110001.

2) Amaranasth, T. Partial Differential Equations, 2nd edition, Jones and Bartlett Publishers.

3) C. Evans, Lawrence. Partial Differential Equations, 2nd edition, American Mathematical Society.

4) Ioannis P Stavroulakis and Stepan A Tersia. Partial differential Equations : An Introduction with Mathematica and Maple, 2nd edition, World scientific Publishing Co. Pt. Ltd, 2004.

5) Donald Greenspan. Introduction to Partial Differential Equation, Mc Graw Hill, Inc.



14.6 ANSWERS TO CHECK YOUR PROGRESS

Check your progress

Ans to Q No 1 : (a) Here the auxiliary equations are

$$\frac{dx}{-x}=\frac{dy}{y}=\frac{dz}{z} \quad \ldots \ (1)$$

From the first two of (1), $\frac{dx}{-x}=\frac{dy}{y}$

or $-\log x=\log y-\log C_1$

or $\log x+\log y=\log C_1$

or $\log xy = \log C_1$

or $xy = C_1$ ... (2)

From the last two of (1), $\frac{dy}{y} = \frac{dz}{z}$

or $\log y = \log z + \log C_2$

or $\log y - \log z = \log C_2$

or $\log\left(\frac{y}{z}\right) = \log C_2$

or $\frac{y}{z} = C_2$ ... (3)

From (2) and (3), we get $f\left(xy, \frac{y}{z}\right) = 0$ which is required solution.

(b) The auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{z/a}$$

Taking the first two terms we have $dx = dy$

Integrating both sides, we get

$$x - y = c_1$$

Again taking the last two terms we have $\frac{dy}{a} = \frac{dz}{z}$

Integrating both sides, we get

$\frac{y}{a} = \log z + \log c_2$

or $\frac{y}{a} = \log(zc_2)$

or $e^{\frac{y}{a}} = zc_2$

or $z = ce^{\frac{y}{a}}$

$\therefore$ The general solution is $z = e^{\frac{y}{a}} f(y - x)$

(c) The auxiliary equations are

$$\frac{dx}{x^2(y-z)} = \frac{dy}{(z-x)y^2} = \frac{dz}{z^2(x-y)}$$

Using $\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{z^2}$ as multipliers

$$\text{Each fraction} = \frac{\frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2}}{0}$$

$$\therefore \frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2} = 0$$

Integrating , $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = C_1$ ... (1)

Again, using $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers

$$\text{Each fraction} = \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{0}$$

$$\therefore \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

Integrating, $\log x + \log y + \log z = \log C_2$

or $\log xyz = \log C_2$

or $xyz = C_2$ ... (2)

From (1) and (2), we get

$f\left(xyz, \frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = 0$ which is required solution.

(d) The given equation can be written as

$$zxp - zyq = y^2 - x^2$$

The auxiliary equations are $\frac{dx}{zx} = \frac{dy}{-zy} = \frac{dz}{y^2 - x^2}$ ... (1)

Using x, y, z as multipliers, we have

$$\text{Each fraction} = \frac{xdx + ydy + zdz}{0}$$

$$\therefore xdx + ydy + zdz = 0$$

Integrating, $x^2 + y^2 + z^2 = C_1$ ... (2)

From the first two members, we have

$ydx + xdy = 0$

or $d(xy) = 0$

or $xy = C_2$ ... (3)

From (2) and (3), we get

$f\left(xy, x^2 + y^2 + z^2\right) = 0$ which is required solution.

(e) The auxiliary equations are

$$\frac{dx}{x\left(y^2 + z\right)} = \frac{dy}{-y\left(x^2 + z\right)} = \frac{dz}{z\left(x^2 - y^2\right)}$$

using $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers,

$$\text{Each fraction} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$$\therefore \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

Integrating, $\log x + \log y + \log z = \log c_1$

or $\log xyz = \log c_1$

or $xyz = c_1$

Again using $x, y, -1$ as multipliers,

$$\text{Each fraction} = \frac{xdx + ydy - dz}{0}$$

$$\therefore 2xdx + 2ydy - 2dz = 0$$

Integrating, we get $x^2 + y^2 - 2z = c_2$

Hence the general solution is

$$f\left(x^2 + y^2 - 2z, xyz\right) = 0 .$$



14.7 MODEL QUESTIONS

Q 1 : Solve the following differential equations :

(a) $p\tan x + q\tan y = \tan z$

(b) $px^2 + qy^2 = z^2$

(c) $(x+2z)p + (4zx - y)q = 2x^2 + y$

(d) $p + q = 1$

(e) $(y^2 + z^2)p - xyq = -3x$

(f) $y^2 p - xyq = x(z - 2y)$

(g) $p + 3q = 5z + \tan(y - 3x)$

(h) $x^2 p + y^2 q = nxy$

(i) $(x^2 - y^2 - z^2)p + 2xyq = 2xz$

(j) $(z^2 - 2yz - y^2)p + (xy + zx)q = xy - zx$.

*** ***** ***



REFERENCES

- *Integral Calculus,* B.C. Das and B.N. Mukherjee, U. N. Dhur & Sons Pvt. Ltd., Kolkata.
- *Integral Calculus*, K. C. Maity & Ram Krishna Ghosh, New Central Book Agency (P) Ltd., Kolkata.
- *Integral Calculus*, P. N. Chatterji, Rajhans Prakashan Mandir, Meerut.
- Ordinary and partial differential equation, Raisinghania, M. D. S.Chand & company Ltd, New Delhi -110055
- Advanced differential Equations, Sharma, A. K. Discovery Publishing House, New Delhi - 110002
- Ramana, B. V. Higher Engineering Mathematics,11th edition, Tata McGraw Hill Education private Ltd, New Delhi.