UNIT 8 : DIFFERENTIAL EQUATION OF FIRST ORDER AND FIRST DEGREE

UNIT STUCTURE

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8.1 LEARNING OBJECTIVES

After going through this unit, you will able to learn

- differential equation of first order and first degree
- variable separable method
- homogeneous differential equation
- linear differential equation
- Bernouli's equation.

8.2 INTRODUCTION

In this unit,we shall discuss the different methods of solving differential equations which are of the first order and first degree.These equations contain only the first order derivatives of the depedent variable w.r.t to the independent variable.The general solution will contain only one arbitrary constant.There are different methods of solving differential equations of this type depending on the form of the equation.

8.3 DIFFERENTIAL EQUATION OF FIRST ORDER AND FIRST DEGREE

Such equations are classified into four types :

1. equations where variables can be separated directly or on special substitution;
2. homogeneous differential equations and equations reducible to such form;
3. linear differential equations and equations reducible to this form;
4. exact differential equations and equations reducible to this form.

8.4 VARIABLE SEPARABLE METHOD

In a differential equation variables are said to be separated if functions of x and dx be on one side and the functions of y and dy be on the other side. Such type of equation can be reduced to the form $f(x)dx = f(y)dy$. Then the solution of the equation can be obtained on integrating both sides. The required solution is

$$\int f(x)dx = \int f(y)y + C$$

Note :

(i) When y is absent, the equation reduces to $\frac{dy}{dx} = f(x)$ and therefore the solution is

$$y = \int f(x)dx + C .$$

(ii) When x is absent , it reduces to $\frac{dy}{dx} = g(y)$ and in this case, the solution is $\int \frac{dy}{g(y)} = \int dx + C$.

Example 1 : Solve $\frac{dy}{dx} = e^{3x-2y} + x^2 e^{-2y}$

Solution : The given equation can be written as

$$\frac{dy}{dx} = e^{-2y}\left(e^{3x} + x^2\right)$$

or $$e^{2y}dy = \left(e^{3x} + x^2\right)dx$$

Integrating both sides, we get $\int e^{2y}dy = \int \left(e^{3x} + x^2\right)dx$

or, $\frac{1}{2}e^{2y} = \frac{1}{3}e^{3x} + \frac{1}{3}x^3 + C$

or $3e^{2y} = 2\left(e^{3x} + x^3\right) + K$, which is the requird solution.

Example 2 : Solve $\left(e^y + 1\right)\cos x dx + e^y \sin x dy = 0$

Solution : Dividing both sides by $\left(e^y + 1\right)\sin x$, we separate the variables in the form

$$\frac{\cos x dx}{\sin x} + \frac{e^y dy}{e^y + 1} = 0$$

Integrating both sides, we get $\log \sin x + \log\left(e^y + 1\right) = \log C$

or $\log\left(\sin x\left(e^y + 1\right)\right) = \log C$

or $\sin x\left(e^y + 1\right) = C$,the required solution.

Note : If sometimes variables could not be separated then by combing some terms as total differentials it can be done.

Example 3 : Solve $y \sin 2x dx - \left(1 + y + \cos^2 x\right)dy = 0$

Solution : Here variables cannot be separated easily. We rearrange as

$$\left(y \sin 2x dx - \cos^2 x dy\right) - \left(1 + y\right)dy = 0$$

Integrating, we have

$$-y\cos^2 x - \left(y + \frac{y^2}{2}\right) = -C$$

or $y\cos^2 x + \left(y + \frac{y^2}{2}\right) = -C$, which is the required solution.

Example 4 : Solve : $x\frac{dy}{dx} = y + \left(\log x\right)\frac{dy}{dx}$

Solution : This is an equation where variables are separable but integration is not possible.

The given equation can be written as $\left(x - \log x\right)\frac{dy}{dx} = y$

$$\text{or } \frac{dy}{y} = \frac{dx}{x - \log x}$$

Integrating, $\int \frac{dy}{y} = \int \frac{dx}{x - \log x}$

or $\log y + \log C = \int \frac{dx}{x - \log x}$

or $\log(yC) = \int \frac{dx}{x - \log x}$

or $yC = e^{\int \frac{dx}{x - \log x}}$ which is the required solution.

Initial value problem : A differential equation together with an initial condition is called a initial value problem.Using the initial condition, the particular value of the arbitrary constant can be determined.

Example 5 : Find the particular solution of

$\cos y dx + (1 + 2e^{-x})\sin y dy = 0$ when $x = 0$, $y = \frac{\pi}{4}$.

Solution : Separating variables the given equation becomes

$$\frac{dx}{1 + 2e^{-x}} + \frac{\sin y dy}{\cos y} = 0$$

or $\frac{e^x dx}{e^x + 2} + \frac{\sin y dy}{\cos y} = 0$

Integrating, we have

$$\log(e^x + 2) - \log(\cos y) = \log c$$

or $e^x + 2 = c\cos y$ ... (1)

Using initial condition, $e^0 + 2 = c\cos\frac{\pi}{4}$

or $3 = c\frac{1}{\sqrt{2}}$

or $c = 3\sqrt{2}$

putting the value of c in (1), we get $e^x + 2 = 3\sqrt{2}\cos y$

or $\sec y(e^x + 2) = 3\sqrt{2}$ which is the required solution.

Note : Equations of the form $\frac{dy}{dx} = f(ax + by + c)$ can be transfomed by substitution $z = ax + by + c$.

Example 6 : Solve $\cos(x + y)dy = dx$

Let $z = x + y$ so that $\frac{dz}{dx} = 1 + \frac{dy}{dx}$.

The given ode becomes $\cos z\left(\frac{dz}{dx} - 1\right) = 1$

or $\frac{dz}{dx}.\cos z = 1 + \cos z$

or $\frac{dz}{dx} = \frac{1+\cos z}{\cos z}$

or $dx = \frac{\cos z}{1+\cos z}dz$

$$= \left(1 - \frac{1}{1+\cos z}\right)dz$$

$$= \left(1 - \frac{1}{2}\sec^2\frac{z}{2}\right)dz$$

Integrating, we have

$$x = z - \tan\frac{z}{2} + C$$

or $0 = y - \tan\frac{(x+y)}{2} + C$

or $y = -C + \tan\frac{(x+y)}{2}$ is the required solution.

Example 7 : Solve $\log\left(\frac{dy}{dx}\right) = ax + by$

Solution : Substituting $z = ax + by$ and hence $\frac{dz}{dx} = a + b\frac{dy}{dx}$

we have $\frac{1}{b}\left(\frac{dz}{dx} - a\right) = e^z$

or $\frac{dz}{a+be^z} = dx$

or $\frac{e^{-z}dz}{ae^{-z}+b} = dx$

or $\frac{du}{-au} = dx$, where $u = ae^{-z} + b$ and $du = -ae^{-z}dz$

or $\log u = -ax + \log C$

or $\log\left(ae^{-z} + b\right) = -ax + \log C$

or $ae^{-(ax+by)} + b = Ce^{-ax}$

or $ae^{-ax}e^{-by} - Ce^{-ax} + b = 0$

or $ae^{-by} - C + be^{ax} = 0$

or $ae^{-by} + be^{ax} = C$, which is the required solution.

Another substitution : If $x = r\cos\theta, y = r\sin\theta$, then from $x^2 + y^2 = r^2$. we have $xdx + ydy = rdr$.

From $\tan\theta = \frac{y}{x}$, we find $\frac{xdy - ydx}{x^2} = \sec^2\theta d\theta$

or $xdy - ydx = r^2 d\theta$

This shows that an ode containing terms $x^2 + y^2$, $xdx + ydy$ and $xdy - ydx$ can be reduced to a simple form. Then method of variable separation can be applied to solve the equation.

Example 8 : Solve $\frac{xdx + ydy}{xdy - ydx} = \sqrt{\frac{a^2 - x^2 - y^2}{x^2 + y^2}}$

Solution : Substituting $x = r\cos\theta, y = r\sin\theta$, we have $xdx + ydy = rdr$, $xdy - ydx = r^2 d\theta$ and $x^2 + y^2 = r^2$.

So, the given equation reduces to

$$\frac{rdr}{r^2 d\theta} = \sqrt{\frac{a^2 - r^2}{r^2}}$$

or $\frac{dr}{\sqrt{a^2 - r^2}} = d\theta$

Integrating $\int \frac{dr}{\sqrt{a^2 - r^2}} = \int d\theta$

or $\sin^{-1}\frac{r}{a} = \theta + C$

or $\frac{r}{a} = \sin(\theta + C)$

or $r = a(\sin\theta\cos C + \cos\theta\sin C)$

or $r^2 = a(r\sin\theta\cos C + r\cos\theta\sin C)$

or $x^2 + y^2 = a(y\cos C + x\sin C)$, which is the required solution.

Equation of the form $yf_1(xy) + xf_2(xy)dy = 0$:

Substituting $z = xy$, $dz = ydx + xdy$ and $dy = \frac{xdz - zdx}{x^2}$

or $xdz - zdx = x^2 dy$

or $dz - \frac{z}{x}dx = xdy$

The variables z and x can be separated. We illustrate with the help of the following example.

Example 8 : Solve $y(1+2xy)dx + x(1-xy)dy = 0$

Solution : Substituting $z = xy$, $xdy + ydx = z$ and $xdy = dz - \frac{z}{x}dx$,

we have, $\frac{z}{x}(1+2z)dx + (1-z)\left(dz - \frac{z}{x}dx\right) = 0$

or $\left\{\frac{z}{x}(1+2z) - (1-z)\frac{z}{x}\right\}dx + (1-z)dz = 0$

or $\left\{z + 2z^2 - z + z^2\right\}\frac{dx}{x} + (1-z)dz = 0$

or $3z^2\frac{dx}{x} + (1-z)dz = 0$

or $\frac{dx}{x} + \frac{1-z}{3z^2}dz = 0$

or $\frac{dx}{x} + \frac{1}{3}\left(\frac{1}{z^2} - \frac{1}{z}\right)dz = 0$

Integrating, we have

$\log x + \frac{1}{3}\left(-\frac{1}{z} - \log z\right) = \frac{C}{3}$

or $\log x - \frac{1}{3}\left(\frac{1}{xy} + \log xy\right) = \frac{C}{3}$

or $\log\frac{x^3}{xy} - \frac{1}{xy} = C = \log k$,(say)

or $\log\frac{x^2}{ky} = \frac{1}{xy}$

or $x^2 = kye^{\frac{1}{xy}}$, which is required solution.



CHECK YOUR PROGRESS - 1

Q 1 : Solve the following differential equations :

(i) $3e^x \tan y dx + \left(1 - e^x\right)\sec^2 y dy = 0$

(ii) $\frac{dy}{dx} = e^{x+y} + x^2 e^{x^3+y}$

(iii) $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$

Q 2 : Solve the following differential equations :

(i) $\frac{dy}{dx} = (4x + y + 1)^2$

(ii) $\frac{dy}{dx} = \sin(x+y) + \cos(x+y)$

8.5 HOMOGENEOUS DIFFERENTIAL EQUATION

A function $f(x,y)$ is called a homogeneous function equation of degree

n if $f(x,y)$ can be expressed as $x^n \phi\left(\frac{y}{x}\right)$ or $y^n \phi\left(\frac{x}{y}\right)$.

For example $f(x,y) = x^2 + xy + y^2 = x^2\left(1 + \frac{y}{x} + \frac{y^2}{x^2}\right)$

or $y^2\left(1 + \frac{x}{y} + \frac{x^2}{y^2}\right)$ is a homogeneous function of degree 2.

A differential equation of the form $M(x,y)dx + N(x,y)dy = 0$ is called a linear homogeneous equation if $M(x,y)$ and $N(x,y)$ are homogeneous function of the same order.This equation can be reduced to another form

$$\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)} = -\frac{\phi\left(\frac{y}{x}\right)}{\varphi\left(\frac{y}{x}\right)} = f\left(\frac{y}{x}\right).$$

Method of Solving a linear homogeneous equation $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$:

Let $z = \frac{y}{x}$ or $y = zx$ so that $\frac{dy}{dx} = \frac{dz}{dx}x + z$.

so, $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$ reduces to the form $x\frac{dz}{dx} + z = f(z)$

$$\text{or } \frac{dz}{f(z) - z} = \frac{dx}{x}$$

The solution of the equation is obtained on integration.

Example 1 : Solve the following differential equations :

(i) $(x^2 - y^2)dx - xydy = 0$

(ii) $\frac{dy}{dx} = \frac{y + \sqrt{x^2 + y^2}}{x}$

(iii) $\left(1 + e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy = 0$

(iv) $y^2 + x^2\frac{dy}{dx} = xy\frac{dy}{dx}$

Solution : (i) The given equation is $\frac{dy}{dx} = \frac{x^2 - y^2}{xy} = \frac{1 - \left(\frac{y}{x}\right)^2}{\frac{y}{x}} = f\left(\frac{y}{x}\right)$

which is a linear homogeneous equation.

Let $z = \frac{y}{x}$ or $y = zx$ so that $\frac{dy}{dx} = \frac{dz}{dx}x + z = \frac{1 - z^2}{z}$

$$\text{or } x\frac{dz}{dx} = \frac{1 - 2z^2}{z}$$

$$\text{or } \frac{zdz}{1 - 2z^2} = \frac{dx}{x}$$

Integrating both sides, we have $-\frac{1}{4}\log(1 - 2z^2) = \log x + \log C$

$$\text{or } -\frac{1}{4}\log(1 - 2z^2) = \log Cx$$

or $\left(1-2z^2\right)^{-\frac{1}{4}} = Cx$

or $1-2\frac{y^2}{x^2} = C^{-4}x^{-4}$

or $C^4\left(x^4-2x^2y^2\right)=1$, which is the required solution.

(ii) $\frac{dy}{dx} = \frac{y+\sqrt{x^2+y^2}}{x} = \frac{y}{x}+\sqrt{1+\frac{y^2}{x^2}} = f\left(\frac{y}{x}\right)$ is a homogeneous equation.

Let $z = \frac{y}{x}$ or $y = zx$ so that $\frac{dy}{dx} = \frac{dz}{dx}x + z$

So, the given equation reduces to $x\frac{dz}{dx} + z = z + \sqrt{1+z^2}$

or $\frac{dz}{\sqrt{1+z^2}} = \frac{dx}{x}$

Integrating both sides, we have

$\log\left(z+\sqrt{1+z^2}\right) = \log x + \log C = \log(Cx)$

or $\frac{y}{x}+\sqrt{1+\left(\frac{y}{x}\right)^2} = Cx$

or $y+\sqrt{x^2+y^2} = Cx^2$, which is the required solution.

(iii) $\left(1+e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy = 0$ can be written as

$$\frac{dx}{dy} = -\frac{e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)}{1+e^{\frac{x}{y}}}$$

Let $z = \frac{x}{y}$, $x = yz$ and $\frac{dx}{dy} = z + y\frac{dz}{dy}$.

So the equation becomes

$$z + y\frac{dz}{dy} = -\frac{e^z(1-z)}{1+e^z}$$

or $y\frac{dz}{dy} = -\frac{e^z(1-z)}{1+e^z} - z$

or $y\frac{dz}{dy}=\frac{-e^z+ze^z-z-ze^z}{1+e^z}$

or $\frac{(1+e^z)dz}{z+e^z}=-\frac{dy}{y}$

Integrating both sides, $\log(z+e^z)=-\log y+\log C$

or $\log(z+e^z)=\log\left(\frac{C}{y}\right)$

or $z+e^z=\frac{C}{y}$

or $\frac{x}{y}+e^{\frac{x}{y}}=\frac{C}{y}$, which is the required solution.

(iv) The given equation $y^2+x^2\frac{dy}{dx}=xy\frac{dy}{dx}$ can be written in standard form

as $\frac{dy}{dx}=\frac{y^2}{xy-x^2}=\frac{\left(\frac{y}{x}\right)^2}{\frac{y}{x}-1}$. ... (1)

Let $z=\frac{y}{x}$ or $y=zx$ so that $\frac{dy}{dx}=\frac{dz}{dx}x+z$

The equation (1) becomes $x\frac{dz}{dx}+z=\frac{z^2}{z-1}$

or $x\frac{dz}{dx}=\frac{z^2}{z-1}-z$

or $x\frac{dz}{dx}=\frac{z}{z-1}$

or $\frac{(z-1)dz}{z}=\frac{dx}{x}$

or $\left(1-\frac{1}{z}\right)dz=\frac{dx}{x}$

Integrating both sides, $\int\left(1-\frac{1}{z}\right)dz=\int\frac{dx}{x}$

or $z-\log z=\log x+\log C$

or $z = \log(Cxz)$

or $\frac{y}{x} = \log\left(Cx\frac{y}{x}\right)$

or $\frac{y}{x} = \log(Cy)$

or $Cy = e^{\frac{y}{x}}$,which is the required solution.

8.5.1 Equations reducible to Homogeneous form

The non homogeneous equation of the form $\frac{dy}{dx} = \frac{ax + by + c}{a'x + b'y + c'}$ $\left(c \neq 0, c' \neq 0\right)$ can be reduced to a homogeneous equations as follows

Case I : Suppose $\frac{a}{b} \neq \frac{a'}{b'}$ .Let $x = X + h$ and $y = Y + k$ where h and k are constants.Then $dx = dX$ and $dy = dY$ .Then the nonhomogeneous equation becomes

$$\frac{dY}{dX} = \frac{aX + bY + (ah + bk + c)}{a'X + b'Y + (a'h + b'k + c')} \quad \text{... (1)}$$

This becomes homogeneous equation choosing h and k such that $ah + bk + c = 0$ and $a'h + b'k + c' = 0$.

Here h and k are given by $\frac{h}{bc' - b'c} = \frac{k}{a'c - ac'} = \frac{1}{ab' - a'b}$

The equation (1) reduces to $\frac{dY}{dX} = \frac{aX + bY}{a'X + b'Y}$ which is homogeneous and can be solved by putting $Z = \frac{Y}{X}$.

Case II : If $\frac{a}{b} = \frac{a'}{b'} = m$,then the equation reduces to the form

$\frac{dy}{dx} = \frac{m(a'x + b'y) + c}{a'x + b'y + c'}$ .Now substituting $z = a'x + b'y$,it can be further reduced to form where variable separation is possible.In this case

$\frac{dz}{dx} = a' + b' \frac{dy}{dx}$ and the equation becomes $\frac{1}{b'}\left(\frac{dz}{dx} - a'\right) = \frac{mz + c}{z + c'}$

$$\text{or } \frac{dz}{dx} = a' + \frac{mb'z + b'c}{z + c'}$$

$$= \frac{\left(a' + mb'\right)z + \left(a'c' + b'c\right)}{z + c'}$$

Separating variables and integrating both sides,we get the solution

$$\int \frac{\left(z + c'\right)dz}{\left(a' + mb'\right)z + \left(a'c' + b'c\right)} = x + K .$$

Example 1 : Solve $(2x + y)dx - (4x + 2y - 1)dy = 0$

Solution : The given equation can be written as

$$\frac{dy}{dx} = \frac{2x + y}{4x + 2y - 1} \text{ and in this case } \frac{2}{4} = \frac{1}{2}.$$

We put $z = 2x + y$, the equation becomes

$$\frac{dz}{dx} - 2 = \frac{z}{2z - 1}$$

$$\text{or } \frac{dz}{dx} = \frac{5z - 2}{2z - 1}$$

$$\text{or } \frac{2z - 1}{5z - 2}dz = dx$$

Integrating both sides, we have $\int \frac{2z - 1}{5z - 2}dz = \int dx + C$

$$\text{or } \int \frac{\frac{2}{5}(5z - 2) - \frac{1}{5}}{5z - 2}dz = \int dx + C$$

$$\text{or } \frac{2z}{5} - \frac{1}{25}\log(5z - 2) = x + C$$

$$\text{or } \frac{2}{5}(2x + y) - \frac{1}{25}\log(10x + 5y - 2) = x + C$$

$$\text{or } -\frac{1}{5}x + \frac{2}{5}y - \frac{1}{25}\log(10x + 5y - 2) = C$$

or $10y - 10x - \log(10x + 5y - 2) = 25C$, which is the required solution.

Example 2 : Solve $\frac{dy}{dx}=\frac{y+x-2}{y-x-4}$

Solution : In this case $\frac{a}{a'}=1$ and $\frac{b}{b'}=-1$,so $\frac{a}{a'}\neq\frac{b}{b'}$.

Let $x=X-h$ and $y=Y-k$. where h and k are constants. Then $dx=dX$ and $dy=dY$. Then the given equation becomes

$$\frac{dY}{dX}=\frac{Y-k+X-h-2}{Y-k-X+h-4}=\frac{X+Y-(h+k+2)}{Y-X+(h-k-4)}$$

We choose h and k such that

$h+k+2=0$

$h-k-4=0$

Solving, we get $h=1$ and $k=-3$, the equation reduces to

$$\frac{dY}{dX}=\frac{X+Y}{Y-X}=\frac{1+\frac{Y}{X}}{\frac{Y}{X}-1}$$

Let $Y=ZX$ so that $\frac{dY}{dX}=Z+X\frac{dZ}{dX}$.

The equation becomes

$$Z+X\frac{dZ}{dX}=\frac{1+Z}{Z-1}$$

or $X\frac{dZ}{dX}=\frac{1+Z}{Z-1}-Z$

or $X\frac{dZ}{dX}=\frac{1+Z-Z^2+Z}{Z-1}$

or $\frac{Z-1}{1+Z-Z^2+Z}dZ=\frac{dX}{X}$

or $\frac{-\frac{1}{2}(2-2Z)}{1+Z-Z^2+Z}dZ=\frac{dX}{X}$

Integrating both sides, we have

$$\int\frac{-\frac{1}{2}(2-2Z)}{1+Z-Z^2+Z}dZ=\int\frac{dX}{X}$$

or $-\frac{1}{2}\log(1+2Z-Z^2)=\log X+\log C$

or $\log\left(1+2Z-Z^2\right)^{-\frac{1}{2}} = \log XC$

or $\frac{1}{\sqrt{1+2Z-Z^2}} = CX$

or $1 = CX\sqrt{1+2\frac{Y}{X}-\frac{Y^2}{X^2}} = C\sqrt{X^2+2XY-Y^2}$

or $\frac{1}{C^2} = X^2+2XY-Y^2$

or $\frac{1}{C^2} = (x+1)^2+2(x+1)(y-3)-(y-3)^2$, which is the required solution.

Example 3 : Solve $(2x+3y-6)\frac{dy}{dx} = 6x-2y-7$

Solution : The given equation can be written as

$$\frac{dy}{dx} = \frac{6x-2y-7}{2x+3y-6} \quad \text{... (1)}$$

In this case $\frac{a}{a'} = \frac{6}{2} = \frac{3}{1}$ and $\frac{b}{b'} = \frac{-2}{3}$, so $\frac{a}{a'} \neq \frac{b}{b'}$.

Let $x = X+h$ and $y = Y+k$ .where h and k are constants.Then $dx = dX$ and $dy = dY$. Then (1) becomes

$$\frac{dY}{dX} = \frac{6(X+h)-2(Y+k)-7}{2(X+h)+3(Y+k)-6} = \frac{6X-2Y+6h-2k-7}{2X+3Y+2h+3k-6}$$

We choose h and k such that

$6h-2k-7=0$

$2h+3k-6=0$

solving $h = \frac{3}{2}$ and $k = 1$ .The equation reduces to

$$\frac{dY}{dX} = \frac{6X-2Y}{2X+3Y}$$

We put $Y = ZX$,so that $\frac{dY}{dX} = Z+X\frac{dZ}{dX}$

$$Z+X\frac{dZ}{dX} = \frac{6-2Z}{2+3Z}$$

or $X\frac{dZ}{dX} = \frac{6-2Z}{2+3Z} - Z$

$$\text{or } X\frac{dZ}{dX} = \frac{6-4Z-3Z^2}{2+3Z}$$

$$\text{or } \int \frac{2+3Z}{6-4Z-3Z^2}dZ = \int \frac{dX}{X}$$

$$\text{or } -\frac{1}{2}\int \frac{-6Z-4}{6-4Z-3Z^2}dZ = \int \frac{dX}{X}$$

$$\text{or } -\frac{1}{2}\log\left(3Z^2+4Z-6\right) = \log X + \log C = \log CX$$

$$\text{or } \log\left(3Z^2+4Z-6\right) = \log\left(CX\right)^{-2}$$

$$\text{or } 3\left(\frac{Y}{X}\right)^2 + 4\frac{Y}{X} - 6 = C^{-2}X^{-2}$$

$$\text{or } 3\left(\frac{y-1}{x-\frac{3}{2}}\right)^2 + 4\frac{y-1}{x-\frac{3}{2}} - 6 = C^{-2}\left(x-\frac{3}{2}\right)^{-2}$$, which is the required solution.



CHECK YOUR PROGRESS - 2

Q 1 : Solve the following differential equations :

(a) $\left(x^3+3xy^2\right)dx + \left(y^3+3x^2y\right)dy = 0$

(b) $\frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x}$

(c) $\left(x^2-4xy-2y^2\right)dx + \left(y^2-4xy-2x^2\right)dy = 0$

(d) $\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3}$

8.6 LINEAR DIFFERENTIAL EQUATION

A differential equation of the form $\frac{dy}{dx} + P(x)y = Q(x)$ where $P(x)$ and $Q(x)$ are functions of x or constants is called a linear differential equation of first order.Here the dependent variable y and its derivative $\frac{dy}{dx}$ occur in the first degree.

Method of Solving a linear differential equation :

Let $R(x)$ be an integrating factor of the standard linear equation

$\frac{dy}{dx} + P(x)y = Q(x)$.Multiplying both sides by $R(x)$,we obtain

$R(x)\frac{dy}{dx} + P(x)R(x)y = Q(x)R(x)$.

It can be written as

$\frac{d}{dx}(yR(x)) + P(x)R(x)y - y\frac{dR(x)}{dx} = Q(x)R(x)$

Left side can be made total differential, choosing $R(x)$ such that

$P(x)R(x)y = y\frac{dR(x)}{dx}$

or $P(x)dx = \frac{dR(x)}{R(x)}$

On integration, we have $\log R = \int Pdx$ and $R = e^{\int Pdx}$.

Thus $e^{\int Pdx}$ is an integrating factor of the equation.

Then the linear equation becomes

$\frac{d}{dx}\left(y.e^{\int pdx}\right) = Qe^{\int pdx}$ and its solution is

$y.e^{\int pdx} = \int \left(Qe^{\int pdx}\right)dx + C$

or $y = e^{-\int pdx}\left[\int \left(Qe^{\int pdx}\right)dx + C\right]$

Remark : Interchanging x and y ,the solution of

$\frac{dx}{dy} + P(y)x = Q(y)$ is found to be

$x = e^{-\int pdy}\left[\int \left(Qe^{\int pdy}\right)dy + C\right]$

Note : We frequently use the following properties of logarithmic and exponential functions :

(i) $e^{\log x} = x$ (ii) $e^{m\log x} = x^m$ (iii) $e^{-m\log x} = \frac{1}{x^m}$

Example 1 : Solve $\frac{dy}{dx} + 2y\cot x = 3x^2 \cos ec^2 x$

Solution : Comparing with standard equation $\frac{dy}{dx} + Py = Q$,we have

$P = 2\cot x$, $Q = 3x^2 \cos ec^2 x$

Integrating factor (I.F) $= e^{\int p dx} = e^{\int 2\cot x dx} = e^{2\log \sin x} = \sin^2 x$

The solution is given by

$$y \sin^2 x = \int 3x^2 \cos ec^2 x \sin^2 x dx$$

$$= 3\int x^2 dx$$

$$= x^3 + C$$

or $y = \cos ec^2 x\left(x^3 + C\right)$.

Example 2 : Solve $\frac{dy}{dx} - y\tan x = e^x$

Solution : Comparing with standard equation $\frac{dy}{dx} + Py = Q$, we have

$P = -\tan x$, $Q = e^x$.

Integrating factor (I.F) $= e^{-\int \tan x dx} = e^{\log \cos x} = \cos x$

The solution of the given differential equation is given by

$$y.\cos x = \int \left(e^x.\cos x\right)dx + C$$

$$= \frac{1}{2}e^x\left(\sin x + \cos x\right) + C$$

or $y = \frac{1}{2}e^x \sec x\left(\sin x + \cos x + 2C\right)$

Example 3 : Solve $\cos^2 x\frac{dy}{dx} + y = \tan x$

Solution : The given equation can be written in standard form of linear equation as $\frac{dy}{dx} + \sec^2 xy = \tan x \sec^2 x$

Comparing with standard equation $\frac{dy}{dx} + Py = Q$, we have

$P = \sec^2 x$, $Q = \tan x \sec^2 x$.

Here I.F $= e^{\int \sec^2 x dx} = e^{\tan x}$

The solution of the given differential equation is given by

$$y.e^{\tan x} = \int e^{\tan x}.\tan x \sec^2 x dx + C$$

$$= e^{\tan x}\left(\tan x - 1\right) + C$$

or $y = \left(\tan x - 1\right) + Ce^{-\tan x}$.

Example 4 : Solve $\left(1 - x^2\right)\frac{dy}{dx} + 2xy = x\sqrt{1 - x^2}$

Solution : The equation can be written in the standard form as

$$\frac{dy}{dx}+\frac{2x}{\left(1-x^2\right)}y=\frac{x\sqrt{1-x^2}}{\left(1-x^2\right)}=\frac{x}{\sqrt{1-x^2}}$$

Comparing with standard equation $\frac{dy}{dx}+Py=Q$,we have

$P=\frac{2x}{\left(1-x^2\right)}, Q=\frac{x}{\sqrt{1-x^2}}$.

Here I.F $=e^{\int\frac{2x}{1-x^2}dx}=e^{-\log\left(1-x^2\right)}=\frac{1}{1-x^2}$

The solution is given by

$$y.\frac{1}{1-x^2}=\int\frac{1}{1-x^2}\frac{x}{\sqrt{1-x^2}}dx+C$$

$$=\int\frac{x}{\left(1-x^2\right)^{\frac{3}{2}}}dx+C$$

$$=\frac{1}{\sqrt{1-x^2}}+C$$

or $y=-\sqrt{1-x^2}+C\left(1-x^2\right)$.

Example 6 : Solve $\left(1+y^2\right)dx=\left(\tan^{-1}y-x\right)dy$

Solution : The given equation can be written as

$$\frac{dx}{dy}+\frac{x}{1+y^2}=\frac{\tan^{-1}y}{1+y^2}$$

Here I.F $=e^{\int\frac{dy}{1+y^2}}=e^{\tan^{-1}y}$

Thus the solution is $x.e^{\tan^{-1}y}=\int\frac{\tan^{-1}y}{1+y^2}.e^{\tan^{-1}y}dy$

$$=\int u.e^u du \text{ if } u=\tan^{-1}x.$$

$$=ue^u-e^u+C$$

$$=\left(\tan^{-1}y-1\right)e^{\tan^{-1}y}+C$$

or $x=\tan^{-1}y-1+Ce^{-\tan^{-1}y}$.

Example 7 : Solve $\left(x+\tan y\right)dy=\sin 2ydx$

Solution : The given equiation can be written as

$$\frac{dx}{dy} - \cos ec2yx = \frac{\tan y}{\sin 2y} = \frac{1}{2}\sec^2 y$$

Here I.F $= e^{\int pdy} = e^{-\int \cos ecydy} = e^{-\frac{1}{2}\log \tan y} = e^{\log \sqrt{\cot y}} = \sqrt{\cot y}$

So, the required solution is

$$x\sqrt{\cot y} = \frac{1}{2}\int \sec^2 y\sqrt{\cot y}dy + C$$

$$= \frac{1}{2}\int (\tan y)^{-\frac{1}{2}} d(\tan y) + C$$

$$= \frac{1}{2}\frac{(\tan y)^{\frac{1}{2}}}{\frac{1}{2}} + C = \sqrt{\tan y} + C$$

or $x = \tan y + C\sqrt{\tan y}$

Example 8 : Solve $(x + y + a)\frac{dy}{dx} = y^2 + b$

Solution : The given equiation can be written as

$$\frac{dx}{dy} = \frac{x + y + a}{y^2 + b}$$

or $\frac{dx}{dy} = \frac{x}{y^2 + b} + \frac{y + a}{y^2 + b}$

or $\frac{dx}{dy} - \frac{x}{y^2 + b} = \frac{y + a}{y^2 + b}$

Here I.F $= e^{-\int \frac{1}{y^2+b}dy} = e^{-\frac{1}{\sqrt{b}}\tan^{-1}\frac{y}{\sqrt{b}}}$

The required solution is

$$x.e^{-\frac{1}{\sqrt{b}}\tan^{-1}\frac{y}{\sqrt{b}}} = \int \frac{y + a}{y^2 + b}.e^{-\frac{1}{\sqrt{b}}\tan^{-1}\frac{y}{\sqrt{b}}}dy + C.$$

$$= -\int (y + a).e^{-\frac{1}{\sqrt{b}}\tan^{-1}\frac{y}{\sqrt{b}}} d\left(\frac{1}{\sqrt{b}}\tan^{-1}\frac{y}{\sqrt{b}}\right)y + C.$$

$$= -(y + a).e^{-\frac{1}{\sqrt{b}}\tan^{-1}\frac{y}{\sqrt{b}}} + \int e^{-\frac{1}{\sqrt{b}}\tan^{-1}\frac{y}{\sqrt{b}}} d\left(\frac{1}{\sqrt{b}}\tan^{-1}\frac{y}{\sqrt{b}}\right)y + C.$$

$$= -(y + a).e^{-\frac{1}{\sqrt{b}}\tan^{-1}\frac{y}{\sqrt{b}}} + e^{-\frac{1}{\sqrt{b}}\tan^{-1}\frac{y}{\sqrt{b}}} + C$$

or $x = -(y+a)+1+Ce^{\frac{1}{\sqrt{b}}\tan^{-1}\frac{y}{\sqrt{b}}}$

or $x+y+a-1 = Ce^{\frac{1}{\sqrt{b}}\tan^{-1}\frac{y}{\sqrt{b}}}$



CHECK YOUR PROGRESS - 3

Q 1 : Solve the following differential equations :

(a) $x\cos x\frac{dy}{dx}+y(x\sin x+\cos x)=1$

(b) $x(1-x^2)dy+(2x^2y-y-ax^3)dx=0$

(c) $(1+x+xy^2)dy+(y+y^3)dx=0$

(d) $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}}-\frac{y}{\sqrt{x}}\right)\frac{dx}{dy}=1$

(e) $(y\log y)dx+(x-\log y)dy=0$

8.7 BERNOULI'S EQUATION

The equation $\frac{dy}{dx}+Py=Qy^n$, where P, Q are functions of x is called Bernoulli's equation.

This equation is reducible to the linear differential equation $\frac{dz}{dx}+P(1-n)z=Q(1-n)$, by substitution $z=y^{1-n}$ which can be solved easily.

Differentiating $z=y^{1-n}$ w.r.t x, we have $\frac{dz}{dx}=(1-n)y^{-n}\frac{dy}{dx}$.

The Bernoull's equation then reduces to

$$\frac{1}{1-n}y^n\frac{dz}{dx}+P.y=Qy^n$$

or $\frac{dz}{dx}+Pz(1-n)=Q(1-n)$, which is linear.

I.F $=e^{\int(1-n)Pdx}$

The required solution is given by

$$z.e^{\int(1-n)Pdx}=\int e^{\int(1-n)Pdx}.Q(1-n)dx+C$$

Note : The equation is named after James Bernoulli (1655–1705).



James Bernoulli (1655–1705) was a Swiss Mathematician

Example 1 : Solve $\cos x \frac{dy}{dx} = y \sin x + y^3 \cos^2 x$

Solution : The given equation can be written as

$$\frac{dy}{dx} - y \frac{\sin x}{\cos x} = y^3 \cos x$$

Dividing by y^3, $y^{-3} \frac{dy}{dx} - y^{-2} \frac{\sin x}{\cos x} = \cos x$

We put $z = y^{-2}$ so that $\frac{dz}{dx} = -2y^{-3} \frac{dy}{dx}$

The equation becomes

$$-\frac{1}{2} \frac{dz}{dx} - z \frac{\sin x}{\cos x} = \cos x$$

or $\frac{dz}{dx} + 2z \frac{\sin x}{\cos x} = -2 \cos x$, which is linear.

$\therefore$ I. F $= e^{\int \frac{2 \sin x}{\cos x} dx} = \sec^2 x$

The required solution is

$z.\sec^2 x = \int \sec^2 x(-2 \cos x) dx = -2 \int \sec x dx$

or $z.\sec^2 x = -2 \log(\sec x + \tan x) + C$

or $y^{-2} \sec^2 x = -2 \log(\sec x + \tan x) + C$

or $\sec^2 x + 2y^2 \log(\sec x + \tan x) = Cy^2$

Example 2 : Solve $\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$

Solution : Dividing the given equation by $\cos^2 y$, we have

$$\sec^2 y \frac{dy}{dx} + 2x \sin y \sec y = x^3$$

or $\sec^2 y \frac{dy}{dx} + 2x \tan y = x^3$

we put $z = \tan y$ so that $\frac{dz}{dx} = \sec^2 y \frac{dy}{dx}$

The equation reduces to

$\frac{dz}{dx} + 2xz = x^3$, which is linear.

I.F $= e^{\int x dx} = e^{x^2}$

The required solution is

$$z.e^{x^2} = \int e^{x^2}.x^3 dx$$

$$= \int e^u.u\frac{du}{2} \qquad (u = x^2)$$

$$= \frac{1}{2}ue^u - \frac{1}{2}\int e^u du$$

$$= \frac{1}{2}\left(ue^u - e^u\right) + C$$

or $\tan y.e^{x^2} = \frac{1}{2}x^2e^{x^2} - \frac{1}{2}e^{x^2} + C$

or $\tan y = \frac{1}{2}x^2 - \frac{1}{2} + Ce^{-x^2} = \frac{1}{2}\left(x^2 - 1\right) + Ce^{-x^2}$.

Example 3 : Solve $\frac{dz}{dx} + \frac{z}{x}\log z = \frac{z}{x}(\log z)^2$

Solution : Dividing the given equation by z , we have

$$\frac{1}{z}\frac{dz}{dx} + \frac{1}{x}\log z = \frac{1}{x}(\log z)^2$$

we put $v = \log z$, $\frac{dv}{dx} = \frac{1}{z}\frac{dz}{dx}$

The equation becomes

$$\frac{dv}{dx} + \frac{1}{x}v = \frac{1}{x}v^2$$

or $\frac{1}{v^2}\frac{dv}{dx} + \frac{1}{x}\frac{1}{v} = \frac{1}{x}$

Again, we put $t = \frac{1}{v}$ so that $\frac{dt}{dx} = -\frac{1}{v^2}\frac{dv}{dx}$

Again the equation becomes

$$\frac{dt}{dx} - \frac{1}{x}t = -\frac{1}{x}$$

I.F $= e^{-\int \frac{1}{x}dx} = e^{-\log x} = \frac{1}{x}$

The required solution is

$$\frac{1}{x}.t = \int \frac{1}{x}.\left(-\frac{1}{x}\right)dx = \frac{1}{x} + C$$

or $t = 1 + Cx$

or $\frac{1}{\log z} = 1 + Cx$

or $\log z = \frac{1}{1+Cx}$

or $z = e^{\frac{1}{1+Cx}}$

Example 4 : Solve $x\frac{dy}{dx} + y = xy^3$

Solution : Dividing the given equation by xy^3, we have

$$\frac{1}{y^3}\frac{dy}{dx} + \frac{1}{xy^2} = 1$$

we put $z = \frac{1}{y^2}$, so that $\frac{dz}{dx} = -2y^3\frac{dy}{dx}$.

The equation becomes

$$-\frac{1}{2}\frac{dz}{dx} + \frac{1}{x}z = 1$$

$$\text{or } \frac{dz}{dx} + \left(\frac{-2}{x}\right)z = -2$$

$\text{I.F} = e^{-\int\frac{2}{x}dx} = e^{-2\log x} = \frac{1}{x^2}$

The required solution is

$$z.\frac{1}{x^2} = \int\frac{1}{x}.(-2)dx$$

$$= \frac{2}{x} + C$$

$$\text{or } \frac{1}{x^2y^2} = \frac{2}{x} + C$$

Example 5 : Solve $(x+a)\frac{dy}{dx} - 3y = (x+a)^5$

Solution : We put $z = x + a$ so that $\frac{dz}{dx} = 1$ and $\frac{dy}{dx} = \frac{dy}{dz}.\frac{dz}{dx} = \frac{dy}{dz}$

The given equation becomes

$$z\frac{dy}{dz} - 3y = z^5$$

or $\frac{dy}{dz}+\left(\frac{-3}{z}\right)y=z^4$

I.F $=e^{\int\left(\frac{-3}{z}\right)dz}=e^{-3\log z}=\frac{1}{z^3}$

The required solution is

$$y.\frac{1}{z^3}=\int\frac{1}{z^3}.z^4dz$$

$$=\int zdz$$

$$=\frac{z^2}{2}+C$$

or $\frac{y}{(x+a)^3}=\frac{1}{2}(x+a)^2+C$

or $y=\frac{1}{2}(x+a)^5+C(x+a)^3$.



CHECK YOUR PROGRESS - 4

Q 1 : Solve the following differential equations :

(a) $x\frac{dy}{dx}+y\log y=xye^x$

(b) $\left(x^2y^3+xy\right)\frac{dy}{dx}=1$

(c) $\frac{dy}{dx}+\frac{x}{1-x^2}y=x\sqrt{y}$

(d) $xy\left(1+xy^2\right)\frac{dy}{dx}=1$



8.8 LET US SUM UP

- The solution of the differential equation of the type $\frac{dy}{dx}=f(x)g(y)$ is obtained after separating the variables and integrating both sides.
- A differential equation of the form $M(x,y)dx+N(x,y)dy=0$ is called a linear homogeneous equation if $M(x,y)$ and $N(x,y)$ are

homogeneous functions of the same order. This equation can be reduced to another form $\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)} = -\frac{\phi\left(\frac{y}{x}\right)}{\varphi\left(\frac{y}{x}\right)} = f\left(\frac{y}{x}\right)$.

The solution of a homogeneous differential equation is obtained by substituting $y = vx$ or $x = vy$ and then separating the variables.

- The solution of the first order linear equation $\frac{dy}{dx} + Py = Q$ is

$$y.e^{\int pdx} = \int \left(Qe^{\int pdx}\right)dx + C$$

$e^{\int Pdx}$ is called an integrating factor of the equation. Similarly, the solution of the first order linear equation

$\frac{dx}{dy} + P(y)x = Q(y)$ is $xe^{\int Pdy} = \int \left(Qe^{\int Pdy}\right)dy + C$

$e^{\int Pdy}$ is called an integrating factor of the equation.

- The equation $\frac{dy}{dx} + Py = Qy^n$, where P, Q are functions of x is called the Bernoull's equation.

This equation is reducible to the linear differential equation $\frac{dz}{dx} + P(1-n)z = Q(1-n)$, by substitution $z = y^{1-n}$ which can be solved easily.



8.9 FURTHER READINGS

1) Tenenbaum, Morris and Pollard, Harry. Ordinary Differential Equations, Dover Publications, Inc, N.Y 11501.

2) Differential Equations : An Introduction with Mathematica. Clay C. Ross, second Edition, Springer.

3) Differential Equations : Linear, Nonlinear, ordinary, partial. A.C. King, J. Billinggham and S.R. Otto, Cambridge-2003.

4) Differential equations with Application and Programs. S. Balachandra Rao and H.R. Anuradha, University Press(India) Limited 1996.

8.10 ANSWERS TO CHECK YOUR PROGRESS

Check your progress -1

Ans to Q No 1: (i) Separating the variables, we get $\frac{3e^x}{1-e^x}dx+\frac{\sec^2 y}{\tan y}dy=0$

Integrating both sides,we get $\int\frac{3e^x}{1-e^x}dx+\int\frac{\sec^2 y}{\tan y}dy=\log C$

or $-3\log(1-e^x)+\log(\tan y)=\log C$

or $\log(\tan y)=\log(1-e^x)^3+\log C$

or $\tan y=C(1-e^x)^3$.

(ii) Given, $\frac{dy}{dx}=e^{x+y}+x^2e^{x^3+y}$

or $\frac{dy}{dx}=e^y\left(e^x+x^2e^{x^3}\right)$

Separating the variables, we get $e^{-y}dy=\left(e^x+x^2e^{x^3}\right)dx$ ... (1)

Integrating (1), $\int e^{-y}dy=\int e^x dx+\int x^2e^{x^3}dx$

or $-e^{-y}=e^x+\frac{1}{3}\int e^t dt+C$,putting $x^3=t$

or $-e^{-y}=e^x+\frac{1}{3}e^t+C$

or $-e^{-y}=e^x+\frac{1}{3}e^{x^3}+C$.

(iii) Separating the variables,the given differential equation can be written

as $\frac{dy}{1+y^2}=\frac{dx}{1+x^2}$ ... (1)

Integrating both sides of (1),we get

$$\int\frac{dy}{1+y^2}=\int\frac{dx}{1+x^2}$$

or $\tan^{-1}y=\tan^{-1}x+C$

which is the general solution of the equation (1).

Ans to Q No 2 : (i) Let $4x + y + 1 = v$ ... (1)

Differentiating (1) with respect to x, we get $4 + \frac{dy}{dx} = \frac{dv}{dx}$

or $\frac{dy}{dx} = \frac{dv}{dx} - 4$ ... (2)

using (1) and (2), the given equation becomes $\frac{dv}{dx} - 4 = v^2$

or $\frac{dv}{dx} = 4 + v^2$

Now, separating the variables x and v, $dx = \frac{dv}{4 + v^2}$

Integrating both sides, we get $x + C' = \frac{1}{2} \times \tan^{-1}\left(\frac{v}{2}\right)$

or $2x + 2C' = \tan^{-1}\left(\frac{v}{2}\right)$

or $2x + C = \tan^{-1}\left(\frac{v}{2}\right)$, where $C = 2C'$

or $v = 2\tan(2x + C)$

or $4x + y + 1 = 2\tan(2x + C)$.

(ii) Let $x + y = v$ ... (1)

Differentiating (1) w.r.t x, $1 + \frac{dy}{dx} = \frac{dv}{dx}$

or $\frac{dy}{dx} = \frac{dv}{dx} - 1$ ... (2)

From (1) and (2), the given equation becomes

$\frac{dv}{dx} - 1 = \sin v + \cos v$

or $\frac{dv}{dx} = 1 + \sin v + \cos v$

or $\frac{dv}{dx} = 1 + 2\sin\frac{v}{2}\cos\frac{v}{2} + 2\cos^2\frac{v}{2} - 1$

or $\frac{dv}{dx} = 2\cos^2\frac{v}{2}\left[1 + \tan\frac{v}{2}\right]$

Separating the variables, we get $dx = \dfrac{dv}{2\cos^2\dfrac{v}{2}\left[1+\tan\dfrac{v}{2}\right]}$

$$\text{or } dx = \frac{\frac{1}{2}\sec^2\frac{v}{2}dv}{1+\tan\frac{v}{2}}$$

Integrating both sides,we get $x + C = \log\left[1+\tan\dfrac{v}{2}\right]$

$$\text{or } x + C = \log\left[1+\tan\left(\frac{x+y}{2}\right)\right]$$

Check your progress - 2

Ans to Q No 1: (a) Given, $\dfrac{dy}{dx} = -\dfrac{x^3+3xy^2}{y^3+3x^2y} = -\dfrac{1+3\left(\frac{y}{x}\right)^2}{\left(\frac{y}{x}\right)^3+3\left(\frac{y}{x}\right)}$ ... (1)

Let $\dfrac{y}{x} = v$ i.e., $y = vx$ so that $\dfrac{dy}{dx} = v + x\left(\dfrac{dv}{dx}\right)$ ... (2)

From (1) and (2), $v + x\dfrac{dv}{dx} = -\dfrac{1+3v^2}{v^3+3v}$

or $x\dfrac{dv}{dx} = -\dfrac{1+3v^2}{v^3+3v} - v$

or $x\dfrac{dv}{dx} = -\dfrac{v^4+6v^2+1}{v^3+3v}$

or $4\dfrac{dx}{x} = -\dfrac{4v^3+12v}{v^4+6v^2+1}dv$

Integrating both sides, we get

$$4\log x = -\log\left(v^4+6v^2+1\right) + \log C$$

or $\log x^4 = \log\left[\dfrac{C}{v^4+6v^2+1}\right]$

or $x^4\left(v^4+6v^2+1\right) = C$

or $y^4 + 6x^2y^2 + x^4 = C$, which is the required solution.

(b) Let $\frac{y}{x} = v$, i.e., $y = vx$ so that $\frac{dy}{dx} = v + x\left(\frac{dv}{dx}\right)$ ... (1)

Using (1), the given equation becomes

$$v + x\frac{dv}{dx} = v + \tan v$$

or $\frac{dx}{x} = \frac{\cos v}{\sin v}dv$

Integrating both sides, we get

$$\log x + \log C = \log \sin v$$

or $Cx = \sin v$

or $Cx = \sin\left(\frac{y}{x}\right)$.

(c) The given equation can be written as

$$\frac{dy}{dx} = \frac{x^2 - 4xy - 2y^2}{2x^2 + 4xy - y^2}$$

Let $\frac{y}{x} = v$, i.e., $y = vx$ so that $\frac{dy}{dx} = v + x\left(\frac{dv}{dx}\right)$ ... (1)

Using (1), the given equation becomes

$$v + x\frac{dv}{dx} = \frac{1 - 4v - 2v^2}{2 + 4v - v^2}$$

or $x\frac{dv}{dx} = \frac{1 - 4v - 2v^2}{2 + 4v - v^2} - v$

or $x\frac{dv}{dx} = \frac{1 - 6v - 6v^2 + v^3}{2 + 4v - v^2}$

or $3\frac{dx}{x} + \frac{3\left(v^2 - 4v - 2\right)}{v^3 - 6v^2 - 6v + 1}dv = 0$

Integrating, we get

$$3\log x + \log\left(v^3 - 6v^2 - 6v + 1\right) = \log C$$

or $x^3\left(v^3 - 6v^2 - 6v + 1\right) = C$

or $x^3\left\{\left(\frac{y}{x}\right)^3 - 6\left(\frac{y}{x}\right)^2 - 6\left(\frac{y}{x}\right) + 1\right\} = C$

or $y^3 - 6xy^2 - 6x^2y + x^3 = C$.

(d) In this case $\frac{a}{a'} = \frac{1}{2}$ and $\frac{b}{b'} = \frac{2}{1}$, so $\frac{a}{a'} \neq \frac{b}{b'}$.

Let $x = X + h$ and $y = Y + k$, where h and k are constants. Then $dx = dX$ and $dy = dY$. Then the given equation becomes

$$\frac{dY}{dX} = \frac{X + 2Y + (h + 2k - 3)}{2X + Y + (2h + k - 3)} \quad \text{... (1)}$$

We choose h, k so that $h + 2k - 3 = 0$ and $2h + k - 3 = 0$ ... (2)

Solving (2), we get $h = 1, k = 1$

From (1), $\frac{dY}{dX} = \frac{X + 2Y}{2X + Y} = \frac{1 + \left(\frac{2Y}{X}\right)}{2 + \left(\frac{Y}{X}\right)}$ ... (3)

We take $\frac{Y}{X} = v$, i.e., $Y = vX$

Therefore, $\frac{dY}{dX} = v + X\frac{dv}{dX}$ ... (4)

From (3) and (4), we have

$$v + X\frac{dv}{dX} = \frac{1 + 2v}{2 + v}$$

or $X\frac{dv}{dX} = \frac{1 + 2v}{2 + v} - v = \frac{1 - v^2}{2 + v}$

or $\frac{dX}{X} = \frac{(2 + v)dv}{(1 - v)(1 + v)} = \left[\frac{1}{2}\left(\frac{1}{1 + v}\right) + \frac{3}{2}\left(\frac{1}{1 - v}\right)\right]dv$, resolving into partial fractions.

Integrating, we have $\log X + \log C = \frac{1}{2}[\log(1 + v) - 3\log(1 - v)]$

or $2\log CX = \log\frac{1 + v}{(1 - v)^3}$

or $X^2C^2 = \frac{1 + v}{(1 - v)^3}$

or $X^2C^2\left(1 - \frac{Y}{X}\right)^3 = 1 + \frac{Y}{X}$

or $C^2(X - Y)^3 = X + Y$

or $C^2\{x - 1 - (y - 1)\}^3 = x - 1 + y - 1$

or $C^{/}\{x-y\}^3 = x+y-2$,taking $C^{/} = C^2$.

Check your progress - 3

Ans to Q No 1: (a) Re-writing the given equation, we have

$$\frac{dy}{dx}+\left(\tan x+\frac{1}{x}\right)y=\frac{\sec x}{x} \quad \text{... (1)}$$

I.F $= e^{\int\left(\tan x+\frac{1}{x}\right)dx} = e^{\log\sec x+\log x} = e^{\log(x\sec x)} = x\sec x$

Hence the required solution is

$$yx\sec x = \int \sec^2 x dx + C$$

or $yx\sec x = \tan x + C$.

(b) Re-writing the given equation, we have

$$x\left(1-x^2\right)\frac{dy}{dx}+y\left(2x^2-1\right)=ax^3$$

or $\frac{dy}{dx}+\frac{\left(2x^2-1\right)}{x\left(1-x^2\right)}y=\frac{ax^2}{1-x^2}$ ... (1)

Here $P=\frac{\left(2x^2-1\right)}{x\left(1-x^2\right)}=-\frac{1}{x}-\frac{1}{2(x+1)}-\frac{1}{2(x-1)}$ and $Q=\frac{ax^2}{1-x^2}$ ... (2)

$$\int Pdx = -\int\left[\frac{1}{x}+\frac{1}{2(x+1)}+\frac{1}{2(x-1)}\right]dx$$

$$= -\left[\log x+\frac{1}{2}\times\log(x+1)+\frac{1}{2}\times\log(x-1)\right]$$

$$= -\left[\log x+\frac{1}{2}\times\log\left(x^2-1\right)\right]$$

$$= -\log\left[x\left(x^2-1\right)^{\frac{1}{2}}\right]$$

$$= \log\left[x\left(x^2-1\right)^{\frac{1}{2}}\right]^{-1}$$

I.F $= e^{\int Pdx} = e^{\log\left[x\left(x^2-1\right)^{\frac{1}{2}}\right]^{-1}} = \left\{x\left(x^2-1\right)^{\frac{1}{2}}\right\}^{-1} = \frac{1}{\left\{x\left(x^2-1\right)^{\frac{1}{2}}\right\}}$

Hence required solution is

$$\frac{y}{x\left(x^2-1\right)^{\frac{1}{2}}}=\int\frac{ax^2}{1-x^2}\times\frac{1}{x\left(x^2-1\right)^{\frac{1}{2}}}dx+C$$

or $$\frac{y}{x\left(x^2-1\right)^{\frac{1}{2}}}=C-a\int\frac{x}{\left(x^2-1\right)^{\frac{3}{2}}}dx$$

or $$\frac{y}{x\left(x^2-1\right)^{\frac{1}{2}}}=C-\frac{a}{2}\int\frac{dt}{t^{\frac{3}{2}}}$$, Putting $x^2-1=t$ and $2xdx=dt$

or $$\frac{y}{x\left(x^2-1\right)^{\frac{1}{2}}}=C-\frac{a}{2}\left[\frac{t^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)}\right]$$

or $$\frac{y}{x\left(x^2-1\right)^{\frac{1}{2}}}=C+\frac{a}{\sqrt{t}}$$

or $$\frac{y}{x\left(x^2-1\right)^{\frac{1}{2}}}=C+\frac{a}{\left(x^2-1\right)^{\frac{1}{2}}}$$

or $y=ax+Cx\left(x^2-1\right)^{\frac{1}{2}}$

(c) Re-writing the given equation, we have

$$\frac{dx}{dy}+\frac{1+x\left(1+y^2\right)}{y\left(1+y^2\right)}=0$$

or $$\frac{dx}{dy}+\frac{x}{y}=-\frac{1}{y\left(1+y^2\right)}$$

Here $P=\frac{1}{y}$ and $Q=-\frac{1}{y\left(1+y^2\right)}$

I.F $=e^{\int\frac{1}{y}dy}=e^{\log y}=y$

Hence required solution is

$$xy=-\int\frac{1}{y\left(1+y^2\right)}.ydy+C$$

or $xy=-\tan^{-1}y+C$.

(d) Re-writing the given equation, we have

$$\frac{dy}{dx}+\frac{1}{\sqrt{x}}y=\frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$

Comparing with $\frac{dy}{dx}+Py=Q$, we get

$$P=\frac{1}{\sqrt{x}}, Q=\frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$

Then I.F $=e^{\int Pdx}=e^{\int\frac{1}{\sqrt{x}}dx}=e^{2\sqrt{x}}$
Hence required solution is

$$y.e^{2\sqrt{x}}=\int\frac{e^{-2\sqrt{x}}}{\sqrt{x}}.e^{2\sqrt{x}}dx+C$$

or $y.e^{2\sqrt{x}}=\int\frac{1}{\sqrt{x}}dx+C$

or $y.e^{2\sqrt{x}}=2\sqrt{x}+C$.

(e) Re-writing the given equation, we have

$$\frac{dx}{dy}+\frac{x}{y\log y}=\frac{1}{y}$$

Comparing with $\frac{dx}{dy}+Px=Q$, we get

$$P=\frac{1}{y\log y}\ ,\quad Q=\frac{1}{y}$$

Then I.F $=e^{\int Pdy}=e^{\int\frac{1}{y\log y}dy}=e^{\log(\log y)}=\log y$
Hence, the solution is

$$x\log y=\int\frac{1}{y}.\log ydy+C$$

or $x\log y=\frac{1}{2}(\log y)^2+C$

Check your progress - 4

Ans to Q No 1: (a) Dividing by xy, the given becomes $\frac{1}{y}\frac{dy}{dx}+\frac{1}{x}\log y=e^x$ ... (1)

Let $\log y = v$ so that $\frac{1}{y}\frac{dy}{dx} = \frac{dv}{dx}$ ... (2)

Using (2), (1) reduces to $\frac{dv}{dx} + \frac{1}{x}v = e^x$, which is linear differential equation.

Here $P = \frac{1}{x}$ and $Q = e^x$

I.F $= e^{\int \frac{1}{x}dx} = e^{\log x} = x$

Hence the required solution is

$v.x = \int xe^x dx + C$

or $v.x = xe^x - \int e^x dx + C$

or $v.x = xe^x - e^x + C$

or $x\log y = e^x(x-1) + C$

(b) The given equation can be written as

$\frac{dx}{dy} = x^2y^3 + xy$

or $\frac{dx}{dy} - yx = y^3x^2$

or $x^{-2}\left(\frac{dx}{dy}\right) - x^{-1}y = y^3$ ... (1)

We put $x^{-1} = v$ and $-x^{-2}\left(\frac{dx}{dy}\right) = \frac{dv}{dy}$

(1) becomes $-\frac{dv}{dy} - yv = y^3$

or $\frac{dv}{dy} + yv = -y^3$, which is linear differential equation.

I.F $= e^{\int ydy} = e^{\frac{y^2}{2}}$

Hence the required solution is

$v.e^{\frac{y^2}{2}} = -\int y^3 e^{\frac{y^2}{2}} dy + C = -2\int te^t dt + C$, putting $\frac{y^2}{2} = t$ and $ydy = dt$

or $v.e^{\frac{y^2}{2}} = -2\left[t.e^t - \int 1.e^t dt\right] + C$

or $v.e^{\frac{y^2}{2}} = -2\left(te^t - e^t\right) + C$

or $x^{-1}.e^{\frac{y^2}{2}} = -2e^{\frac{y^2}{2}}\left(\frac{y^2}{2} - 1\right) + C$ as $x^{-1} = v$

or $\frac{1}{x} = -2\left(\frac{y^2}{2} - 1\right) + Ce^{-\frac{y^2}{2}}$

(c) Dividing by $\sqrt{y}$, the given equation becomes

$$y^{-\frac{1}{2}}\frac{dy}{dx} + \frac{x}{1-x^2}y^{\frac{1}{2}} = x \quad \text{... (1)}$$

Putting $y^{\frac{1}{2}} = z$ so that $\frac{1}{2}y^{-\frac{1}{2}}\frac{dy}{dx} = \frac{dz}{dx}$ or $y^{-\frac{1}{2}}\frac{dy}{dx} = 2\frac{dz}{dx}$

Equation (1) becomes

$$2\frac{dz}{dx} + \frac{x}{1-x^2}z = x$$

or $\frac{dz}{dx} + \frac{x}{2\left(1-x^2\right)}z = \frac{x}{2}$ which is linear in z.

I.F $= e^{\int \frac{x}{2\left(1-x^2\right)}dx} = e^{-\frac{1}{4}\int \frac{-2x}{1-x^2}dx} = e^{-\frac{1}{4}\log\left(1-x^2\right)} = \left(1-x^2\right)^{-\frac{1}{4}}$

Hence the required solution is

$$z.\left(1-x^2\right)^{-\frac{1}{4}} = \int \frac{x}{2}\left(1-x^2\right)^{-\frac{1}{4}}dx + C$$

or $z.\left(1-x^2\right)^{-\frac{1}{4}} = -\frac{1}{4}\int \left(1-x^2\right)^{-\frac{1}{4}}\left(-2x\right)dx + C$

or $z.\left(1-x^2\right)^{-\frac{1}{4}} = -\frac{1}{4}\frac{\left(1-x^2\right)^{\frac{3}{4}}}{\frac{3}{4}} + C$

or $z = -\frac{1}{3}\left(1-x^2\right) + C\left(1-x^2\right)^{\frac{1}{4}}$

or $\sqrt{y} = -\frac{1}{3}\left(1-x^2\right) + C\left(1-x^2\right)^{\frac{1}{4}}$.

(d) The given equation can be written as $\frac{dx}{dy} - yx = y^3x^2$

Dividing by x^2, we have $x^{-2}\frac{dx}{dy} - yx^{-1} = y^3$ ... (1)

Putting $x^{-1} = z$ so that $-x^{-2}\frac{dx}{dy} = \frac{dz}{dx}$ or $x^{-2}\frac{dx}{dy} = -\frac{dz}{dx}$

Equation (1) becomes

$$-\frac{dz}{dx} - yz = y^3$$

or $\frac{dz}{dx} + yz = -y^3$ which is linear in z.

I.F $= e^{\int ydx} = e^{\frac{1}{2}y^2}$

Hence the required solution is

$$z.e^{\frac{1}{2}y^2} = \int -y^3 e^{\frac{1}{2}y^2} dy + C$$

or $z.e^{\frac{1}{2}y^2} = \int -y^2 e^{\frac{1}{2}y^2}.ydy + C$

or $z.e^{\frac{1}{2}y^2} = -\int 2te^t dt + C$ where $t = \frac{y^2}{2}$

or $z.e^{\frac{1}{2}y^2} = -2\left[te^t - \int e^t dt\right] + C$

or $z.e^{\frac{1}{2}y^2} = -2\left(te^t - e^t\right) + C$

or $z.e^{\frac{1}{2}y^2} = -2e^{\frac{1}{2}y^2}\left(\frac{1}{2}y^2 - 1\right) + C$

or $z = -2\left(\frac{1}{2}y^2 - 1\right) + Ce^{-\frac{1}{2}y^2}$

or $z = 2 - y^2 + Ce^{-\frac{1}{2}y^2}$.



8.11 MODEL QUESTIONS

Q 1 : Solve the following differential equations :

(a) $\frac{y}{x}.\frac{dy}{dx}=\sqrt{1+x^2+y^2+x^2y^2}$

(b) $e^y\left(1+x^2\right)\frac{dy}{dx}-2x\left(1+e^y\right)=0$

(c) $\frac{dy}{dx}-x\tan\left(y-x\right)=1$

(d) $\sin\left(x+y\right)dy=dx$

Q 2 : Solve the following differential equations :

(a) $x\left(x-y\right)\frac{dy}{dx}=y\left(x+y\right)$

(b) $x\frac{dy}{dx}=y+x\cos^2\frac{y}{x}$

Q 3 : Solve the following differential equations :

(a) $\left(2x-2y+5\right)\frac{dy}{dx}=x-y+3$

(b) $\left(6x+2y-10\right)\frac{dy}{dx}-2x-9y+29=0$

(c) $\left(3y-7x+7\right)dx+\left(7y-3x+3\right)dy=0$

(d) $\left(2x^2+3y^2-7\right)xdx-\left(3x^2+2y^2-8\right)ydy=0$

Q 4 : Solve the following differential equations :

(a) $\left(1+x^3\right)\frac{dy}{dx}+6x^2y=1+x^2$

(b) $\left(x+2y^3\right)\frac{dy}{dx}=y$

(c) $\frac{dy}{dx}=\frac{y}{2y\log y+y-x}$

(d) $\sqrt{1-y^2}\,dx=\left(\sin^{-1}y-x\right)dy$

Q 5 : Solve the following differential equations :

(a) $2\frac{dy}{dx} = \frac{y}{x} + \frac{y^2}{x^2}$

(b) $(x+1)\frac{dy}{dx} + 1 = 2e^{-y}$

(c) $\frac{dy}{dx} + \frac{y \log y}{x} = \frac{y(\log y)^2}{x^2}$

(d) $y(2xy + e^x)dx = e^x dy$

Q 6 : Solve the following differential equations :

(a) $\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$

(b) $xdy + ydx + \frac{xdy - ydx}{x^2 + y^2} = 0$

(c) $(a^2 - 2xy - y^2)dx - (x+y)^2 dy = 0$

(d) $(2ax + by)ydx + (ax + 2by)xdy = 0$

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