UG MA S4-01 (DSE)

FOUR YEAR BACHELOR'S DEGREE COURSE

(CBCS in Accordance with NEP 2020)

SEMESTER IV

INTEGRAL CALCULUS AND DIFFERENTIAL EQUATIONS

MATHEMATICS

Indira Miri School of Education
KRISHNA KANTA HANDIQUI STATE OPEN UNIVERSITY

Subject Experts

Prof. Bipan Hazarika, Dept of Mathematics, G. U.

Dr. Dillip Kumar Sarma, Dept of Mathematics, Cotton College

Course Coordinator : Harekrishna Deka, K.K.H.S.O.U

SLM Preparation Team

Units	Contributor
1, 2, 3, 4, 5, 6	**Dr. Dibyajyoti Mahanta,** K.K.H.S.O.U
7, 8, 9, 10, 11, 12, 13, 14	**Harekrishna Deka,** K.K.H.S.O.U

Editorial Team

Content : **Dr. Dillip Kumar Sarma,** Dept of Mathematics, Cotton College

Structure & Format : **Harekrishna Deka,** K.K.H.S.O.U

March, 2025

© Krishna Kanta Handiqui State Open University.

This Self Learning Material (SLM) of the Krishna KantaHandiqui State Open University is made available under a Creative Commons Attribution-NonCommercial-ShareAlike4.0 License (international): http://creativecommons.org/licenses/by-nc-sa/4.0/

For the avoidance of doubt, by applying this license KKHSOU does not waive any privileges or immunities from claims that it may be entitled to assert, nor does KKHSOU submit to the jurisdiction, courts, legal processes or laws of any jurisdiction.

Printed and published by Registrar on behalf of the Krishna Kanta Handiqui State Open University.

The University acknowledges with thanks the financial support provided by the **Distance Education Bureau, UGC, New Delhi,** for the preparation of this study material.

Headquarters : Patgaon, Rani Gate, Guwahati - 781017

City Office : Resham Nagar, Khanapara, Guwahati-781 022; Web: www.kkhsou.in

COURSE INTRODUCTION

'Integral Calculus and Differential Equations' is the fourth discipline specific elective (DSE) course of the programme in Mathematics. This course consists of fourteen units.

In Unit 1, the basics of the Integral calculus are introduced. In this unit, we will discuss different methods of Integration.

In Unit 2, we will discuss about definite integrals and the properties of definite integrals.

In Unit 3, we will derive some reduction formulae of integrations.

In Unit 4, we will discuss quadrature of plane areas.

In Unit 5, we will discuss rectification of plane curves.

In Unit 6, we will discuss volume of revolution and surface of revolution.

In Unit 7 introduces fundamental concepts differential equation.

In Unit 8 deals with the solution of first order and first degree ordinary differential equation.In this unit, we will discuss different methods for solving first order and first degree ordinary differential equation.

In unit 9, we will introduce you to exact differential equation of first order and first degree.In this unit, we will discuss different methods for solving exact differential equation of first order and first degree.

In Unit 10 covers the solution of first order and higher degree differential equation.Clairaut's form is also explored in this unit.

In Unit 11 begins with introduction of linear differential equation of higher order.Solution procedure for linear differential equation of higher order is discussed in this unit.

In unit 12, we will linear homogeneous differential equation of higher order.Method of variation is also included in this unit.

Unit 13 deals with partial differential equation and its formation.

In unit 14, we will discuss solution of first order and first degree partial differential.

While going through a unit, you will notice some along-side boxes, which have been included to help you know some of the difficult, unseen terms. Some 'ACTIVITY' (s) has been included to help you apply your own thoughts. Again, we have included some relevant concepts in 'LET US KNOW' along with the text. And, at the end of each section, you will get 'CHECK YOUR PROGRESS' questions. These have been designed to self-check your progress of study. It will be better if you solve the

problems put in these boxes immediately after you go through the sections of the units and then match your answers with ‘ANSWERS TO CHECK YOUR PROGRESS’ given at the end of each unit.

CONTENTS

		Pages
UNIT 1 :	**Integration and Methods of Integration** Intregration of Simple Functions, Properties of Indefinite Integrals, Methods of Integration : Integration by Substitution, Integration by Parts, Integration by Partial Fractions.	**7 - 65**
UNIT 2 :	**Definite Integrals** Definite integral– Integration as the Limit of a Sum; Fundamental Theorem of Integral Calculus; Summation of Series with the help of Integration; Evaluation of Definite Integrals by Using Properties.	**66 - 102**
UNIT 3 :	**Reduction Formulae** Reduction Formulae : Reduction formulae for (i) $\int \sin^n x dx$, and (ii) $\int \cos^n x dx$, n being a positive integer, Reduction Formulae for (i) $\int \tan^n x dx$ and (ii) $\int \cot^n x dx$, n being a positive Integer, Reduction Formula for $\int x^n e^{ax} dx$, where n be a posititve Integer, Reduction Formula for $\int \cos^m x \sin x dx$.	**103 - 121**
UNIT 4 :	**Quadrature of Plane Areas** Quadrature of Plane Curves : Area Between Two Given Curves.	**122 - 140**
UNIT 5 :	**Rectification of Plane Curves** Rectification of Plane Curves.	**141 - 152**
UNIT 6 :	**Volume of Revolution and Surface of Revolution** Volume of Solids of Revolution; Surface of Solids of Revolution.	**153 - 170**
UNIT 7 :	**Dirrerential Equation : Fundamental Concepts** Definition; Order and Degree of Ordinary Differential Equation; Linear and Non-Linear Differential Equation; Formation of a Differential Equation; Solution of a Differential Equation.	**175-194**

UNIT 8 : Differential Equation of First Order and First Degree **195-233**

Differential Equation of First Order and First Degree; Variable Separable Method; Homogeneous Differential Equation : Equations Reducible to Homogeneous form; Linear Differential Equation; Bernouli's Equation.

UNIT 9 : Exact Differential Equation of First Order and First Degree **234-267**

Exact differential Equation; Equations reducible to Exact Equations.

UNIT 10 : First Order and Higher Degree Differential Equations **268-284**

Differential Equation of First Order and Higher Degree : Equation Solvable for p, Equation Solvable for x, Equation Solvable for y; Clairaut's form : Equations Reducible to Clairaut's form.

UNIT 11 : Linear Differential Equation of Higher Order **285-322**

Linear Differential Equations with Constant Coefficients : Rules of Finding Complementary Function, Methods of Finding Particular Integral.

UNIT 12 : Linear Homogeneous Differential Equation of Higher Order **323-338**

Linear Homogeneous Equation : Equation Reducible to Homogeneous form; Method of Variation of Parameters.

UNIT 13 : Partial Differential Equation **339-356**

Formation of Partial Differential Equation : Formation of Partial Differential Equation by Elimination of Arbitrary Constants, Formation of Partial Differential Equation by Elimination of Arbitrary Functions.

UNIT 14 : Linear Partial Differential Equation **357-367**

Solution of first order and first degree partial differential equation.

UNIT 1 : INTEGRATION AND METHODS OF INTEGRATION

UNIT STRUCTURE

1.1 Learning Objectives

1.2 Introduction

1.3 Intregration of Simple Functions

1.4 Properties of Indefinite Integrals

1.5 Methods of Integration

1.5.1 Integration by Substitution

1.5.2 Integration by Parts

1.5.3 Integration by Partial Fractions

1.6 Let Us Sum Up

1.7 Further Readings

1.8 Answers to Check Your Progress

1.9 Model Questions

1.1 LEARNING OBJECTIVES

After going through this unit, you will be able to

- understand about integration of simple functions from the view point of inverse of differentiation
- know the properties of indefinite integrals
- understand the various methods of integrations viz
  – method of substitution
  – integration by parts
  – integration by partial fractions.

1.2 INTRODUCTION

The development of integral calculus can be viewed from two different directions which are related to solving of problems of the following two types.

(i) to find the function when its derivative is known

(ii) to find the area bounded by the curve (or graph) of a function under certain given conditions.

The process related to the first category leads to the development of "indefinite integration" and the other leads to "definite integration".

The word "integration" literally means "summation". It is, in fact, the process of finding the "limit of a sum" of a certain number of elements, as the number of elements tends to infinity and each of the elements becomes infinitesimally small.

In this unit, we shall confine ourselves to the study of indefinite integral including some process of integration.

1.3 INTEGRATION OF SIMPLE FUNCTIONS

We have already learnt how to find the derivative *of* a functional of defined in an interval. If $y = f(x)$ is a function then its derivative is denoted by $f'(x)$ or $\frac{dy}{dx}$. Now, here in this unit, we will learn how to find f whenever f' is given. Integration is the inverse process of differentiation, i.e., to find a function when the differential coefficient is given.

Let us consider the following examples.

We know that (i) $\frac{d}{dx}\left(\frac{x^4}{4}\right) = x^3$

and (ii) $\frac{d}{dx}(\sin x) = \cos x$

Here we say that, in case of (i) $\frac{x^4}{4}$ is an antiderivative (or an integral or primitive) of x^3. Similarly, in (ii) $\sin x$ is an antiderivative of $\cos x$.

Let us take two other examples :

$$\text{(iii)}\ \frac{d}{dx}\left(\frac{x^4}{4} + C\right) = x^3$$

$$\text{and (iv)}\ \frac{d}{dx}(\sin x + C) = \cos x$$

Here C is any real number

It is evident from (iii), that $\frac{x^4}{4}+C$ is also an antiderivative of x^3 and in (iv) $\sin x + C$ is also an antiderivative of $\cos x$. Therefore, we have understood that the antiderivative of a function is not unique.

Definition : The set of functions that could possibly have the given function as a derivative are called "antiderivatives" (or integrals or primitives) of the function.

More generally, if there is a function $\phi(x)$ defined in an interval I such that $\frac{d}{dx}[\phi(x)]=f(x),\ x\in I$, then for any real number C $\frac{d}{dx}[\phi(x)+C]=f(x),\ x\in I$. Thus $\phi(x)+C$ denotes the family of anti-derivatives of $f(x)$. We introduce the symbol $\int f(x)dx$ which will represent the whole class of anti-derivatives, i.e. $\int f(x)dx=\phi(x)+C$.

The symbol $\int f(x)dx$ is read as the indefinite indegral of $f(x)$ with respect to *x*. The process of finding the anti-derivatives is known as "integration". The sign '$\int$' is called the integral sign. The symbol '$\int$' is an elongated S which is taken from the first letter of 'sum'.

Some Basic Results

Based upon the above definition and also on some standard derivatives, we obtain the following integration formulae.

	Derivatives	**Integrals (or anti derivatives)**
1.	$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right)=x^n$	$\int x^n dx=\frac{x^{n+1}}{n+1}+C,\ (n\neq -1)$
2.	$\frac{d}{dx}(\sin x)=\cos x$	$\int \cos x dx=\sin x+C$
3.	$\frac{d}{dx}(\cos x)=-\sin x$	$\int \sin x dx=-\cos x+C$
4.	$\frac{d}{dx}(\tan x)=\sec^2 x$	$\int \sec^2 x dx=\tan x+C$
5.	$\frac{d}{dx}(\sec x)=\sec x\tan x$	$\int \sec x\tan x dx=\sec x+C$
6.	$\frac{d}{dx}(\cot x)=-\text{cosec}^2 x$	$\int \text{cosec}^2 x dx=-\cot x+C$

7. $\frac{d}{dx}(\text{cosec}x) = -\text{cosec}x \cot x$ $\quad \int \text{cosec}x \cot x dx = -\text{cosec}x + C$

8. $\frac{d}{dx}(e^x) = e^x$ $\quad \int e^x dx = e^x + C$

9. $\frac{d}{dx}\left(\frac{a^x}{\log_e a}\right) = a^x$ $\quad \int a^x dx = \frac{a^x}{\log_e a} + C$

10. $\frac{d}{dx}(\log x) = \frac{1}{x}$ $\quad \int \frac{1}{x} dx = \log|x| + C$

11. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$ $\quad \int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$

$= -\cos^{-1} x + C$

12. $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$

13. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$ $\quad \int \frac{1}{1+x^2} dx = \tan^{-1} x + C$

$= -\cot^{-1} x + C$

14. $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$

15. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$ $\quad \int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + C$

$= -\text{cosec}^{-1} x + C$

16. $\frac{d}{dx}(\text{cosec}^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$

Example : Evaluate (i) $\int x^4 dx$ (ii) $\int \frac{1}{\sqrt{x}} dx$ (iii) $\int 3^x dx$

Solution : (i) $\int x^4 dx = \frac{x^{4+1}}{4+1} + C = \frac{x^5}{5} + C$ (using formula 1)

(ii) $\int \frac{1}{\sqrt{x}} dx = \int x^{-½} dx = \frac{x^{-½+1}}{-½+1} + C = 2\sqrt{x} + C$

(iii) $\int 3^x dx = \frac{3^x}{\log_e 3} + C$ (using formula 9)

1.4 PROPERTIES OF INDEFINITE INTEGRAL

P–1 Differentiation and integration are inverse processes of each other in the sense of the results given below–

$\frac{d}{dx}\int f(x)dx = f(x)$ and $\int f'(x)dx = f(x) + C$, C is a constant.

Proof : Let $\phi(x)$ be any antiderivative of $f(x)$ i.e. $\frac{d}{dx}[\phi(x)] = f(x)$

Then $\int f(x)dx = \phi(x) + C$

$\therefore \frac{d}{dx}\int f(x)dx = \frac{d}{dx}[\phi(x) + C] = \frac{d}{dx}[\phi(x)] + \frac{d}{dx}(C) = f(x)$

(first part is proved)

Similarly, we know that $f'(x) = \frac{d}{dx} f(x)$

Hence $\int f'(x)dx = f(x) + C$, C is any arbitrary constant.

P-2 Two identical integrals with the same derivative lead to the same family of curves and hence both are equivalent.

Proof : Let $\frac{d}{dx}\int f(x)dx = \frac{d}{dx}\int g(x)dx$

Then $\frac{d}{dx}\int f(x)dx - \frac{d}{dx}\int g(x)dx = 0$

or, $\frac{d}{dx}[\int f(x)dx - \int g(x)dx] = 0$

or, $\int f(x)dx - \int g(x)dx = C$

or, $\int f(x)dx = \int g(x)dx + C$

It is customarily expressed as $\int f(x)dx = \int g(x)dx$ without mentioning the constant of integration.

P-3 $\int[f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$

Proof : By property P-1, We have

$\frac{d}{dx}\int [f(x) + g(x)]dx = f(x) + g(x)$ ... (a)

Also we know that

$\frac{d}{dx}[\int f(x)dx + \int g(x)dx] = \frac{d}{dx}\int f(x)dx + \frac{d}{dx}\int g(x)dx$

$= f(x) + g(x)$ ... ...(b)

In view of property P-2, from (a) and (b) we can write

$$\int[f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$$

Corollary

(A) $\int[f(x) - g(x)]dx = \int f(x)dx - \int g(x)dx$

(B) We can generalize property P-3 as below -

$$\int [f(x)+g(x)+h(x)+....+\phi(x)]dx$$
$$=\int f(x)dx+\int g(x)dx+\int h(x)dx+...+\int \phi(x)dx$$

P-4 For any real number K,

$$\int K f(x)dx = K\int f(x)dx$$

Proof : We know that $\frac{d}{dx}\int K f(x)dx = K f(x)$, by property P-1

Also $\frac{d}{dx}\left[K\int f(x)dx\right]= K\frac{d}{dx}\int f(x)dx = K f(x)$

So by property P-2

$$\int K f(x)dx = K\int f(x)dx$$

Illustrative Examples

Example 1 : Evaluate (i) $\int\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2 dx$

(ii) $\int\left(x^{-3/2}-4e^x+\cos x\right)dx$ (iii) $\int\left(2x^2+4\sec^2 x-\sqrt[3]{x}\right)dx$

Solutions : (i) $\int\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2 dx$

$$=\int\left(x+2+\frac{1}{x}\right)dx$$

$$=\int x dx+\int 2dx+\int\frac{1}{x}dx \qquad \text{(by property P-3)}$$

$$=\frac{x^{1+1}}{1+1}+C_1+2\int dx+\log x+C_2$$

$$=\frac{x^2}{2}+C_1+2x+C_3+\log x+C_2$$

$$=\frac{x^2}{2}+2x+\log x+C \qquad (\text{Here } C_1+C_2+C_3=C)$$

(ii) $\int\left(x^{-3/2}-4e^x+\cos x\right)dx$

$$=\int x^{-3/2}-dx-\int 4e^x dx+\int\cos x\,dx$$

$$=\frac{x^{-\frac{3}{2}+1}}{\frac{-3}{2}+1}+C_1-4\int e^x+\sin x+C_2$$

$$= \frac{x^{-\frac{1}{2}}}{\frac{-1}{2}} + C_1 - 4e^x + C_3 + \sin x + C_2$$

$$= -2x^{-\frac{1}{2}} - 4e^x + \sin x + C$$

(iii) $\int \left(2x^2 + 4\sec^2 x - \sqrt[3]{x}\right)dx$

$$= \int 2x^2 dx + \int 4\sec^2 x\, dx - \int \sqrt[3]{x}\, dx$$

$$= 2\int x^2 dx + 4\int \sec^2 x\, dx - \int x^{\frac{1}{3}}\, dx$$

$$= \frac{2}{3}x^3 + C_1 + 4\tan x + C_2 - \frac{x^{\frac{4}{3}}}{\frac{4}{3}} + C_3$$

$$= \frac{2}{3}x^3 + 4\tan x - \frac{3}{4}x^{\frac{4}{3}} + C$$

Example 2 : Evaluate (i) $\int \frac{x^3 - x^2 + x - 1}{x - 1}\, dx$

(ii) $\int \frac{3x^4 + 7x + 5}{2\sqrt{x}}\, dx$ (iii) $\int \frac{1 + \sin x}{\cos^2 x}\, dx$

Solution : (i) $\int \frac{x^3 - x^2 + x - 1}{x - 1}\, dx$

$$= \int \frac{(x-1)(x^2+1)}{(x-1)}dx$$

$$= \int \left(x^2 + 1\right)dx$$

$$= \frac{x^3}{3} + x + C$$

(ii) $\int \frac{3x^4 + 7x + 5}{2\sqrt{x}}\, dx$

$$= \int \left(\frac{3}{2}x^{\frac{7}{2}}dx + \frac{7}{2}x^{\frac{1}{2}} + \frac{5}{2}x^{-\frac{1}{2}}\right)dx$$

$$= \frac{3}{2}.\frac{2}{9}x^{\frac{9}{2}} + \frac{7}{2}.\frac{2}{3}x^{\frac{3}{2}} + \frac{5}{2}.\frac{2}{1}x^{\frac{1}{2}} + C$$

$$= \frac{1}{3}x^{\frac{9}{2}} + \frac{7}{3}x^{\frac{3}{2}} + 5\sqrt{x} + C$$

(iii) $\int \frac{1 + \sin x}{\cos^2 x}\, dx$

$$= \int \frac{1}{\cos^2 x}\,dx + \int \frac{\sin x}{\cos^2 x}\,dx$$

$$= \int \sec^2 dx + \int \tan x \sec x\,dx$$

$$= \tan x + \sec x + C$$

Example 3 : Write an antiderivative of $\cos 2x$.

Solution : We know that $\frac{d}{dx}(\sin 2x) = 2\cos 2x$

$$\text{or } \cos 2x = \frac{1}{2}\frac{d}{dx}(\sin 2x) = \frac{d}{dx}\left(\frac{1}{2}\sin 2x\right)$$

$\therefore$ antiderivative of $\cos 2x$ is $\frac{1}{2}\sin 2x + C$.



CHECK YOUR PROGRESS - 1

Q.1 : Find an antiderivative of each of the followings–

(i) $\frac{1}{x}, (x \neq 0)$ (ii) $\sin 3x$ (iii) $(ax-b)^2$

Q.2 : Find the following integrals

(i) $\int x^3\left(x + \frac{1}{x}\right)^2 dx$ (ii) $\int (\cos x + \sin x)^2 dx$

(iii) $\int \left(\frac{2\cos x}{5\sin^2 x} + \frac{1}{3\cos^2 x}\right)dx$ (iv) $\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x}dx$

(v) $\int \left(e^{x\log a} + e^{a\log x} + e^{a\log a}\right)dx$ (vi) $\int \sqrt{1+\cos 2x}\,dx$

(vii) $\int \frac{(1+x)^2}{\sqrt{x}}\,dx$ (viii) $\int \left(\frac{x}{m} + \frac{m}{x} + x^m + m^x\right)dx$

Q.3 : Evaluate (i) $\int \left(x^4 + x^2 + 1\right) d\left(x^2\right)$ (ii) $\int \frac{x^4 + x^2 + 1}{x^2 - x + 1}\,dx$

(iii) $\int \frac{x^{-1/3} + \sqrt{x} + 2}{\sqrt[3]{x}}\,dx$

Q.4 : If the derivative of $f(x)$ w.r.t. x is $\frac{1}{x} + \frac{1}{1+x^2}$ and $f(1) = \pi/4$, then find $f(x)$.

Q.5 : The gradient at a point (x,y) of a curve is $3x^2 - 12$ and the curve passes through the point (2,–7). Find the equation of the curve.

1.5 METHODS OF INTEGRATION

In our earlier discussion, we have considered integrals of those functions which are readily obtainable from derivatives of some standard functions. This method is based on inspection, i.e. it is dependent on the search of a function F whose derivative is *f* which lead us to the integrals of *f.* But integrals of certain functions cannot be obtained directly if they are not in one of the standard forms. Hence we need to develop some other techniques or methods for finding the integrals by reducing the given functions to standard forms. Some of the methods are– (i) Integration by substitution, (ii) Integration by parts ,(iii) Integration by partial fractions.

1.5.1 Integration by Substitution

The method of evaluating an integral by reducing it to standard form by a proper substitution is called "integration by substitution".

Suppose $\phi(x)$ is a differentiable function. Then to evaluate integrals of the form $\int f(\phi(x))\phi'(x)dx$, we substitute $\phi(x) = t$ which then gives $\phi'(x)dx = dt$. This substitution reduces the above integral to the form $\int f(t)dt$.

If now $\int f(t)dt$ is the standard form then we can find the integrals. Finally we replace t by x with the help of the above substitution.

It is important to guess what will be the useful substitution. Usually we make a substitution for a function whose derivative also occurs in the integrand.

A few examples are given below to illustrate the method.

Illustrative Examples

Example 1 : Evaluate $\int (ax + b)^n dx$

Solution : Let $ax + b = t$, then a $\dfrac{dx}{dt} = 1$

or $dx = \dfrac{1}{a}dt$

$$\therefore \int (ax+b)^n dx = \int t^n \frac{1}{a} dt = \frac{1}{a}\int t^n dt = \frac{1}{a}\frac{1}{n+1}t^{n+1} + C$$

$$= \frac{1}{a(n+1)}(ax+b)^{n+1} + C$$

Example 2 : Evaluate $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\, dx$

Solution : Let $\sqrt{x} = t$, then $\frac{1}{2\sqrt{x}}dx = dt$, or $\frac{1}{\sqrt{x}}dx = 2dt$

Now $\int \frac{\sin\sqrt{x}}{\sqrt{x}} dx = \int 2\sin t\, dt = 2(-\cos t) + C = -2\cos\sqrt{x} + C$

Example 3 : Evaluate $\int \tan^4\theta\, \sec^2\theta\, d\theta$

Solution : Let $\tan\theta = t$, then $\sec^2\theta\, d\theta = dt$

Now $\int \tan^4\theta \sec^2\theta\, d\theta = \int t^4 dt = \frac{1}{5}t^5 + C = \frac{1}{5}\tan^5\theta + C$

Example 4 : Evaluate $\int \frac{10x^9 + 10^x \log_e^{10}}{10^x + x^{10}}\, dx$

Solution : Let $10^x + x^{10} = t$, then $(10^x \log_e 10 + 10x^9)dx = dt$

Now $\int \frac{10x^9 + 10^x \log_e 10}{10^x + x^{10}} dx = \int \frac{1}{t} dt = \log t + C = \log(10^x + x^{10}) + C$

A Few Special Results

1. $\int \frac{f'(x)}{f(x)}\, dx = \log|f(x)| + C$

Proof : Let $f(x) = z$, then $f'(x)dx = dz$

Therefore, $\int \frac{f'(x)}{f(x)} dx = \int \frac{dz}{z} = \log|z| + C = \log|f(x)| + C$

This result can be applied in case of the followings

(a) $\int \tan x\, x = \log|\sec x| + C$ (b) $\int \cot x\, dx = \log|\sin x| + C$

(c) $\int \sec x\, dx = \log|\sec x + \tan x| + C$

(d) $\int \operatorname{cosec} x\, dx = \log|\operatorname{cosec} x - \cot x| + C$

Proof (a) : $\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx = -\int \frac{-\sin x}{\cos x}\, dx$

$$= -\log|\cos x| + C = \log|\sec x| + C$$

Proof (c) : $\int \sec x dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} dx$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$$

$$= \log|\sec x + \tan x| + C$$

N.B. : (b) and (d) are left for the learners



CHECK YOUR PROGRESS - 2

Q. Find the integrals

1. $\int \frac{x}{9-4x^2} dx$
2. $\int (4x+2)\sqrt{x^2+x+1}\, dx$
3. $\int e^{\tan^{-1} x} \frac{1}{1+x^2} dx$
4. $\int \frac{dx}{x\sqrt{x^2-a^2}}$
5. $\int x\sqrt{x^2+5}\, dx$
6. $\int \frac{dx}{x^2\sqrt{1-x^2}}$
7. $\int \frac{\cos x\, dx}{(a+b\sin x)^2}$
8. $\int \frac{e^{2x}-1}{e^{2x}+1} dx$
9. $\int \frac{\sin(\tan^{-1} x)}{1+x^2} dx$
10. $\int \frac{(x+1)(x+\log x)^2}{x} dx$
11. $\int \sqrt{\frac{1+x}{1-x}}\, dx$
12. $\int \frac{3x-1}{\sqrt{3x^2-2x+7}} dx$
13. $\int \frac{e^x(1+x)}{\cos^2(xe^x)} dx$
14. $\int \frac{1}{x^2(x^4+1)^{3/4}} dx$
15. $\int \frac{x^2+5x+2}{x+2} dx$
16. $\int \frac{1}{\sqrt{x+a}+\sqrt{x+b}} dx$
17. $\int \cos^4 x\, dx$
18. $\int \sin^3 x\, dx$
19. $\int \cos^2 x\, dx$
20. $\int \frac{2x+5}{x^2+5x-7} dx$
21. $\int \frac{1}{\sin(x-a)\sin(x-b)} dx$
22. $\int \frac{dx}{\sin(x-a)\cos(x-b)}$
23. $\int \sin x \sin 2x \sin 3x\, dx$
24. $\int \frac{x+2}{(x+1)^2} dx$
25. $\int \frac{3x+1}{(3x^2+2x+1)^3} dx$

SOME STANDARD INTEGRALS

(A) $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C, \ (a \neq 0)$

(B) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C, \ (|x| > |a|)$

(C) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C, \ (|x| < |a|)$

(D) $\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left|x+\sqrt{x^2+a^2}\right| + C$

(E) $\int \frac{dx}{x^2-a^2} = \log\left|x+\sqrt{x^2-a^2}\right| + C$

(F) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$

Proof (A) : Let us put $x = a\tan\theta$, then $dx = a\sec^2\theta d\theta$

Now $\int \frac{dx}{x^2+a^2} = \int \frac{a\sec^2\theta d\theta}{a^2\tan^2\theta + a^2} = \int \frac{a\sec^2\theta}{a^2\sec^2\theta} d\theta = \frac{1}{a}\int d\theta$

$= \frac{1}{a}\theta + C = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

Proof (B) : $\int \frac{dx}{x^2-a^2} = \int \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right)dx$, by partial fraction

$= \frac{1}{2a}\left[\int \frac{1}{x-a}dx - \int \frac{1}{x+a}dx\right]$

$= \frac{1}{2a}\left[\log|x-a| - \log|x+a|\right] + C$

$= \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$

Proof (C) : First apply partial fraction and then proceed as in (B).

Proof (D) : Let $x = a\tan\theta$, then $dx = a\sec^2\theta d\theta$

Now $\int \frac{dx}{\sqrt{x^2+a^2}} = \int \frac{a\sec^2\theta d\theta}{\sqrt{a^2\tan^2\theta + a^2}} = \int \sec\theta d\theta$

$= \log|\sec\theta + \tan\theta| + C_1 = \log\left|\frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a}\right| + C_1$

$= \log\left|\sqrt{x^2+a^2} + x\right| - \log|a| + C_1$

$= \log\left|x+\sqrt{x^2+a^2}\right| + C$

Proof (E) : Here put $x = a\sec\theta$ and then proceed.

Proof (F) : Here put $x = a\sin\theta$ and then proceed.

SOME OTHER TYPES OF INTEGRALS

Here we will discuss the application of the above standard integrals for finding the integrals of the following types

(A) $\int \frac{dx}{ax^2+bx+c}$ (B) $\int \frac{dx}{\sqrt{ax^2+bx+c}}$ (C) $\int \frac{px+q}{ax^2+bx+c}$

(D) $\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx$ (E) $\int \frac{dx}{(ax+b)\sqrt{cx+d}}$

(F) $\int \frac{dx}{(px+q)\sqrt{ax^2+bx+c}}$

(Here a, b, c, d, p, q, r are constants)

Solution : (A) We can wirte ax^2+bx+c

$$= a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)$$

$$= a\left\{\left(x+\frac{b}{2a}\right)^2+\left(\frac{c}{a}-\frac{b^2}{4a^2}\right)\right\}$$

Now we put $x+\frac{b}{2a}=t$ and then $dx=dt$

$$\therefore \int \frac{dx}{ax^2+bx+c} = \frac{1}{a}\int \frac{dx}{\left(x+\frac{b}{2a}\right)^2+\left(\frac{4ac-b^2}{4a^2}\right)} = \frac{1}{a}\int \frac{dt}{t^2\pm \mathrm{K}^2}$$

where $\mathrm{K}^2 = \frac{4ac-b^2}{4a^2}$

Now this can be evaluated by using standard formula.

(B) Proceeding as in (A) we can write

$\int \frac{dx}{\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{a}}\int \frac{dt}{\sqrt{t^2\pm \mathrm{K}^2}}$ which can be evaluated by using standard formula.

(C) In this case we are to find real numbers A and B such that

$$px+q = \mathrm{A}\frac{d}{dx}\left(ax^2+bx+c\right)+\mathrm{B} = \mathrm{A}\left(2ax+b\right)+\mathrm{B}.$$

Now for finding A and B, we equate from both sides the coefficients of x and the constant terms. Then we can write

$$\int \frac{px+q}{ax^2+bx+c}\,dx = A\int \frac{f'(x)}{f(x)}dx + B\int \frac{1}{f(x)}dx \text{ where}$$

$$f(x) = ax^2 + bx + c .$$

Now the right hand side can be integrated by suitable methods discussed earlier.

(D) In this case we can write

$$\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx \text{ as } A\int \frac{f'(x)}{\sqrt{f(x)}}dx + B\int \frac{1}{\sqrt{f(x)}}\,dx$$

by the procedure discussed in (C).

(E) Let $cx + d = t^2$, then $c\,dx = 2t\,dt$

$$\text{Now } \int \frac{dx}{(ax+b)\sqrt{(cx+d)}} = \frac{2}{c}\int \frac{t\,dt}{\left(a\frac{t^2-d}{c}+b\right)t}$$

$$= 2\int \frac{dt}{at^2+(bc-ad)}$$

It can be now evaluated with the help of standard integrals.

(F) Here we take $px + q = \frac{1}{t}$, then $p\,dx = -\frac{1}{t^2}dt$. Then the given integral reduces to the form $-\int \frac{dt}{\sqrt{At^2+Bt+C}}$ which can be evaluated by the method discussed in (B).

ILLUSTRATIVE EXAMPLES

Example 1 : $\int \frac{\cos x + \sin x}{\cos x - \sin x}\,dx$

Solution : $\int \frac{\cos x + \sin x}{\cos x - \sin x}\,dx = \int \frac{1+\tan x}{1-\tan x}dx$, (Dividing both numerator and denominator by $\cos x$)

$$= \int \tan\left(\frac{\pi}{4}+x\right)dx = \log\left|\sec\left(\frac{\pi}{4}+x\right)\right| + C$$

Example 2 : $\int \frac{3x^2}{x^6+1}\,dx$

Solution : $\int \frac{3x^2}{x^6+1}\,dx = \int \frac{3x^2}{(x^3)^2+1}dx$

We take

$x^3 = t$

$\therefore 3x^2dx = dt$

$$= \int \frac{dt}{t^2+1}$$

$$= \tan^{-1}(t) + C = \tan^{-1}\left(x^3\right) + c$$

Examples 3 : $\int \frac{1}{2x^2+8x+20}\,dx$

Solution : $\int \frac{1}{2x^2+8x+20}\,dx$

$$= \frac{1}{2}\int \frac{1}{x^2+4x+10}\,dx$$

$$= \frac{1}{2}\int \frac{1}{\left(x^2+4x+4\right)+6}\,dx$$

$$= \frac{1}{2}\int \frac{dx}{(x+2)^2+\left(\sqrt{6}\right)^2}$$

$$= \frac{1}{2}\frac{1}{\sqrt{6}}\tan^{-1}\left(\frac{x+2}{\sqrt{6}}\right) + C$$

Example 4 : $\int \frac{dx}{\sqrt{2x^2+3x+4}}$

Solution : $\int \frac{dx}{\sqrt{2x^2+3x+4}} = \int \frac{dx}{\sqrt{2}\sqrt{x^2+\frac{3}{2}x+2}}$

$$= \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{x^2+2.\frac{3}{4}.x+\frac{9}{16}+\frac{23}{16}}} = \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{\left(x+\frac{3}{4}\right)^2+\left(\frac{\sqrt{23}}{4}\right)^2}}$$

$$= \frac{1}{\sqrt{2}}\log\left|\left(x+\frac{3}{4}\right)+\sqrt{x^2+\frac{3}{2}x+2}\right| + C$$

Example 5 : $\int \frac{4x+3}{3x^2+3x+1}\,dx$

Solution : We write

$$4x+3 = A\frac{d}{dx}\left(3x^2+3x+1\right)+B$$

$$\Rightarrow 4x+3 = A(6x+3)+B$$

$$\Rightarrow 4x+3 = 6Ax+3A+B$$

Equating the coefficients of x and the constant terms from both sides

$6A = 4$ and $3A + B = 3$

$$\therefore A = \frac{2}{3} \quad \text{and} \quad B = 3 - 3A = 3 - 3.\frac{2}{3} = 1$$

$\therefore$ The given integral is

$$\int \frac{\frac{2}{3}(6x+3)+1}{3x^2+3x+1}\,dx$$

$$= \frac{2}{3}\int \frac{6x+3}{3x^2+3x+1}\,dx + \int \frac{1}{3x^2+3x+1}\,dx$$

$$= \frac{2}{3}\log\left|3x^2+3x+1\right| + \frac{1}{3}\int \frac{1}{x^2+x+\frac{1}{3}}\,dx$$

$$= \frac{2}{3}\log\left|3x^2+3x+1\right| + \frac{1}{3}\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\left(\frac{1}{\sqrt{12}}\right)^2}\,dx$$

$$= \frac{2}{3}\log\left|3x^2+3x+1\right| + \frac{\sqrt{12}}{3}\tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{1}{\sqrt{12}}}\right) + C$$

$$= \frac{2}{3}\log\left|3x^2+3x+1\right| + \frac{2}{\sqrt{3}}\tan^{-1}\left[\sqrt{3}(2x+1)\right] + C$$

Example 6 : $\int \frac{dx}{(1+x)\sqrt{1+2x-x^2}}$

Solution : Let us take $1+x = \frac{1}{t}$, then $dx = -\frac{1}{t^2}dt$

$$\therefore \int \frac{dx}{(1+x)\sqrt{1+2x-x^2}} = \int \frac{-\frac{1}{t^2}dt}{\frac{1}{t}\sqrt{1+2\left(\frac{1}{t}-1\right)-\left(\frac{1}{t}-1\right)^2}}$$

$$= \int \frac{-\frac{1}{t}dt}{\sqrt{-2+\frac{4}{t}-\frac{1}{t^2}}} = -\int \frac{dt}{\sqrt{-2t^2+4t-1}}$$

$$= -\frac{1}{\sqrt{2}}\int \frac{dt}{\sqrt{-t^2+2t-\frac{1}{2}}} = -\frac{1}{\sqrt{2}}\int \frac{dt}{\sqrt{\left(\frac{1}{\sqrt{2}}\right)^2-(t-1)^2}}$$

$$= -\frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{t-1}{\frac{1}{\sqrt{2}}}\right) = -\frac{1}{\sqrt{2}}\sin^{-1}\left[\sqrt{2}\left(\frac{1}{1+x}-1\right)\right]$$

$$= -\frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{-x\sqrt{2}}{x+1}\right) = \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{x\sqrt{2}}{x+1}\right)$$



CHECK YOUR PROGRESS - 3

Q. Integrate the followings :

1. $\int \frac{1}{4x^2+9}\,dx$
2. $\int \frac{e^x dx}{e^{2x}+6e^x+5}$
3. $\int \frac{dx}{x\left(x^5+1\right)}$
4. $\int \frac{dx}{\sqrt{1-x-x^2}}$
5. $\int \frac{2x+3}{\sqrt{x^2+x+1}}\,dx$
6. $\int \frac{x+2}{\sqrt{x^2+5x+6}}\,dx$
7. $\int \frac{dx}{(x+2)\sqrt{x+1}}$
8. $\int \frac{dx}{(1+x)\sqrt{1-x^2}}$

1.5.2 INTEGRATION BY PARTS

An important integration formula can be derived from the derivative of the product of two functions.

We know from differential calculus that for two differentiable functions $f(x)$ and $q(x)$,

$$\frac{d}{dx}\{f(x).q(x)\} = f(x).\frac{d}{dx}\{q(x)\} + q(x).\frac{d}{dx}\{f(x)\}$$

Integrating both sides w.r.t. x we get

$$f(x).q(x) = \int \left[f(x).\frac{d}{dx}\{q(x)\} + q(x).\frac{d}{dx}\{f(x)\} \right] dx$$

$$= \int f(x)\frac{d}{dx}\{q(x)\}dx + \int q(x).\frac{d}{dx}\{f(x)\}dx$$

or $$\int f(x)\frac{d}{dx}\{q(x)\}dx = f(x).q(x) - \int q(x).\frac{d}{dx}\{f(x)\}dx \quad \ldots(1)$$

Now let us take $f(x) = u$ and $\frac{d}{dx}\{q(x)\} = v$.

Then $q(x) = \int v dx$

If we substitute there in (1) then we get

$$\int uvdx = u\int vdx - \int \left\{ \frac{d}{dx}(u).\int vdx \right\} dx$$

The above formula can be stated as below—

"The integral of the product of two functions = 1st function × integral of the 2nd function – integral of [differential coefficient of 1st function × integral of the 2nd function]"

Note : While applying the rule of integration by parts for the product of two functions, it is very important to choose the first and the second function very carefully. As an aid to the learners, we can choose the first function as the function that comes first in the word "ILATE". Here the letters stand for as given below—

I – Inverse trigonometric function

L – Logarithmic function

A – Algebraic function

T – Trigonometic function

E – Exponential function.

ILLUSTRATIVE EXAMPLES

Example 1 : Evaluate $\int x^2 e^x dx$

Solution : $\int x^2 e^x dx = x^2 \int e^x dx - \int \left\{ \frac{d}{dx}(x^2) \int e^x dx \right\} dx$

$= x^2 e^x - \int 2x e^x dx$ | Here we have taken x^2 = 1st function, e^x = 2nd function

$= x^2 e^x - 2\left[x \int e^x dx - \int \left\{ \frac{d}{dx}(x) . \int e^x dx \right\} dx \right]$

$= x^2 e^x - 2\{ x e^x - \int 1 . e^x dx \} = x^2 e^x - 2(x e^x - e^x) + C$

$= e^x (x^2 - 2x + 2) + C$

Example 2 : Evaluate $\int x \sin 3x \, dx$

Solution : $\int x \sin 3x \, dx = x \int \sin 3x \, dx - \int \left\{ \frac{d}{dx}(x) . \int \sin 3x \, dx \right\} dx$

$= x\left(-\frac{\cos 3x}{3} \right) - \int 1 . \left(-\frac{\cos 3x}{3} \right) dx$

$= -\frac{x}{3} \cos 3x + \frac{1}{3} \int \cos 3x \, dx$

$= -\frac{1}{3} x \cos 3x + \frac{1}{3} \frac{\sin 3x}{3} + C$

$= -\frac{1}{3} x \cos 3x + \frac{1}{9} \sin 3x + C$

Example 3 : Evaluate $\int x \tan^{-1} x \, dx$

Solution : $\int x \tan^{-1} x \, dx$ [Here 1st function = $\tan^{-1} x$, 2nd function = x]

$= \tan^{-1} x \int x \, dx - \int \left\{ \frac{d}{dx}(\tan^{-1} x) . \int x \, dx \right\} dx$

$= \frac{x^2}{2} \tan^{-1} x - \int \frac{1}{1+x^2} \frac{x^2}{2} dx = \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \frac{x^2}{1+x^2} dx$

$= \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} \int \left(1 - \frac{1}{1+x^2} \right) dx = \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} (x - \tan^{-1} x) + C$

$= \frac{1}{2} \{ (x^2 + 1) \tan^{-1} x - x \} + C$

Example 4 : $\int \log x \, dx$

Solution : $\int \log x\,dx = \int \log x.1\,dx$

Now take 1st function = log x

2nd function = 1 and proceed.

Similarly in case of $\int \sin^{-1} x\,dx$ we take $\sin^{-1} x$ as first function and 1 as second function.

STANDARD INTEGRALS

A. $\int e^{ax} \sin bx\,dx = \frac{1}{a^2+b^2} e^{ax}(a\sin bx - b\cos bx)$

B. $\int e^{ax} \cos bx\,dx = \frac{1}{a^2+b^2} e^{ax}(a\cos bx + b\sin bx)$

Proof A : Let $I = \int e^{ax} \sin bx\,dx$

Then $I = e^{ax}\int \sin bx\,dx - \int\left\{\frac{d}{dx}(e^{ax})\int \sin bx\,dx\right\}dx$

$$= -\frac{1}{b}e^{ax}\cos bx - \int ae^{ax}\left(-\frac{1}{b}\cos bx\right)dx$$

$$= -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b}\int e^{ax}\cos bx\,dx$$

$$= -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b}\left[e^{ax}\int \cos bx\,dx - \int\left\{\frac{d}{dx}(e^{ax})\int \cos bx\,dx\right\}dx\right]$$

$$= -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b}\left[e^{ax}\frac{\sin bx}{b} - \int ae^{ax}\frac{\sin bx}{b}\,dx\right]$$

$$= -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b}\left[\frac{1}{b}e^{ax}\sin bx - \frac{a}{b}\int e^{ax}\sin bx\,dx\right]$$

$$= -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b^2}e^{ax}\sin bx - \frac{a^2}{b^2}I$$

$$\therefore I + \frac{a^2}{b^2}I = -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b^2}e^{ax}\sin bx$$

$$\Rightarrow \frac{a^2+b^2}{b^2}I = \frac{1}{b^2}(ae^{ax}\sin bx - be^{ax}\cos bx)$$

$$\therefore I = \frac{1}{a^2+b^2}e^{ax}(a\sin bx - b\cos bx)$$

Proof B : It is left for the learners to verify the result.

SOME STANDARD FORMULAE

1. $\int \sqrt{x^2+a^2}\,dx = \frac{1}{2}x\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|$

2. $\int \sqrt{x^2-a^2}\,dx = \frac{1}{2}x\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|$

3. $\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}x\sqrt{a^2-x^2} - \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$

Proof 1 : Let $I = \int 1.\sqrt{x^2+a^2}\,dx$

$$\Rightarrow I = \sqrt{x^2+a^2}\int 1.dx - \int\left\{\frac{d}{dx}\left(\sqrt{x^2+a^2}\right).\int 1dx\right\}dx$$

$$= x\sqrt{x^2+a^2} - \int \frac{2x}{2\sqrt{x^2+a^2}}.x\,dx$$

$$= x\sqrt{x^2+a^2} - \int \frac{x^2}{\sqrt{x^2+a^2}}dx = x\sqrt{x^2+a^2} - \int \frac{x^2+a^2-a^2}{\sqrt{x^2+a^2}}dx$$

$$= x\sqrt{x^2+a^2} - \int\sqrt{x^2+a^2}\,dx + a^2\int \frac{1}{\sqrt{x^2+a^2}}dx$$

$$= x\sqrt{x^2+a^2} - I + a^2\log\left|x+\sqrt{x^2+a^2}\right|$$

$$\Rightarrow 2I = x\sqrt{x^2+a^2} + a^2\log\left|x+\sqrt{x^2+a^2}\right|$$

$$\therefore I = \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|$$

Note : (1) Proofs of (2) and (3) are left for the learners

(2) Though we have not given the constant of integration in each of the above results, the learners should not forget to put it in each cases.

SOME OTHER SPECIAL FORMS

(a) $\int \sqrt{ax^2+bx+c}\,dx$ where $a \neq 0$

In this case, ax^2+bx+c is to be expressed either in the form $a\left\{(x+\alpha)^2 \pm \beta^2\right\}$ or $b\left\{\beta^2-(x+\alpha)^2\right\}$.

Now if $x+\alpha$ is substituted by t, then the integral will reduce to a known form discussed earlier.

(b) $\int (px+q)\sqrt{ax^2+bx+c}\,dx$, $a \neq 0$

Here we take $px+q = \mathrm{A}\frac{d}{dx}\left(ax^2+bx+c\right)+\mathrm{B}$

Now by simple calculation we can write

$$px+q = \frac{p}{2q}(2ax+b)+\left(q-\frac{bp}{2a}\right)$$

Then the given integral can be expressed in two parts. For the first part we take $ax^2+bx+c=t$ and the second part is similar to any one form discussed earlier.

(c) $\int e^x\left\{f(x)+f'(x)\right\}dx$

Here taking $\mathrm{I}=\int e^x f(x)dx$ and applying the formula for "integration by parts" we can prove that $\int e^x\left\{f(x)+f'(x)\right\}dx = e^x f(x)$.

ILLUSTRATIVE EXAMPLES

Example 1 : Evaluate $\int e^x \cos x\, dx$

Solution : Let $\mathrm{I}=\int e^x \cos x\, dx$

$$= e^x\int \cos x\, dx - \int\left\{\frac{d}{dx}\left(e^x\right).\int \cos x\, dx\right\}dx$$

$$= e^x \sin x - \int e^x \sin x\, dx$$

$$= e^x \sin x - \left[e^x\int \sin x\, dx - \int\left\{\frac{d}{dx}\left(e^x\right).\int \sin x\, dx\right\}dx\right]$$

$$= e^x \sin x + e^x \cos x - \int e^x \cos x\, dx$$

$$= e^x(\sin x+\cos x)-\mathrm{I}$$

$$\Rightarrow 2\mathrm{I} = e^x(\sin x+\cos x)$$

$$\therefore\ \mathrm{I}=\frac{1}{2}e^x(\sin x+\cos x)+\mathrm{C}$$

Example 2 : $\int\sqrt{4-3x-2x^2}\,dx$

Solution : $\int\sqrt{4-3x-2x^2}\,dx = \sqrt{2}\int\sqrt{2-\frac{3}{2}x-x^2}\,dx$

$$= \sqrt{2}\int\sqrt{2+\frac{9}{16}+\left\{-\left(\frac{3}{4}\right)^2-2.\frac{3}{4}.x-x^2\right\}}\,dx$$

$$= \sqrt{2}\int \sqrt{\frac{41}{16} - \left(x + \frac{3}{4}\right)^2}\, dx \qquad \left|\begin{array}{l}\text{Let} \quad x + \frac{3}{4} = t \\ \text{then} \quad dx = dt\end{array}\right.$$

$$= \sqrt{2}\left[\frac{t\sqrt{\frac{41}{16} - t^2}}{2} + \frac{41/16}{2}\sin^{-1}\left(\frac{t}{\frac{\sqrt{41}}{4}}\right)\right] + C$$

$$= \sqrt{2}\left[\frac{1}{2}\left(x + \frac{3}{4}\right)\sqrt{\frac{41}{16} - \left(x + \frac{3}{4}\right)^2} + \frac{41}{32}\sin^{-1}\left(\frac{x + 3/4}{\sqrt{41/4}}\right)\right] + C$$

$$= \frac{\sqrt{2}}{8}(4x + 3)\frac{\sqrt{4 - 3x - 2x^2}}{\sqrt{2}} + \frac{41\sqrt{2}}{32}\sin^{-1}\left(\frac{4x + 3}{\sqrt{41}}\right) + C$$

$$= \frac{1}{8}(4x + 3)\sqrt{4 - 3x - 2x^2} + \frac{41\sqrt{2}}{32}\sin^{-1}\left(\frac{4x + 3}{\sqrt{41}}\right) + C$$

Example 3 : $\int (3x - 2)\sqrt{1 - x + x^2}\, dx$

Solution : We can write $3x - 2 = \frac{3}{2}(2x - 1) - \frac{1}{2}$

$$\therefore I = \frac{3}{2}\int (2x - 1)\sqrt{x^2 - x + 1}\, dx - \frac{1}{2}\int \sqrt{x^2 - x + 1}\, dx$$

For the first part of the R.H.S. we take $x^2 - x + 1 = t$

Then $\int (2x - 1)\sqrt{x^2 - x + 1}\, dx = \int \sqrt{t}\, dt = \frac{2}{3}t^{3/2} = \frac{2}{3}\left(x^2 - x + 1\right)^{3/2}$

For the 2nd part

$$\int \sqrt{x^2 - x + 1}\, dx = \int \sqrt{\left(x - \frac{1}{2}\right)^2 + \frac{3}{4}}\, dx$$

$$= \int \sqrt{z^2 + \left(\sqrt{3}/2\right)^2}\, dt \quad (\text{taking } x - \frac{1}{2} = z)$$

$$= \frac{z\sqrt{z^2 + 3/4}}{2} + \frac{3}{4}.\frac{1}{2}\log\left|z + \sqrt{z^2 + \frac{3}{4}}\right|$$

$$= \frac{1}{2}\left(x - \frac{1}{2}\right)\sqrt{x^2 - x + 1} + \frac{3}{8}\log\left|\left(x - \frac{1}{2}\right) + \sqrt{x^2 - x + 1}\right|$$

$$\therefore I = \left(x^2 - x + 1\right)^{3/2} - \frac{1}{8}(2x-1)\sqrt{x^2 - x + 1} - \frac{3}{16}\log\left|\left(x - \frac{1}{2}\right) + \sqrt{x^2 - x + 1}\right| + C$$

Example 4: Evaluate $\int e^x \left(\frac{1-\sin x}{1-\cos x}\right) dx$

Solution : $\int e^x \left(\frac{1-\sin x}{1-\cos x}\right) dx$

$$= \int e^x \frac{1 - 2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2 x/2}\, dx = \int e^x \left(\frac{1}{2}\operatorname{cosec}^2\frac{x}{2}\right) dx - \int e^x \cot\frac{x}{2}\, dx$$

$$= -\int e^x \cot\frac{x}{2}\, dx + \frac{1}{2}\int e^x \operatorname{cosec}^2\frac{x}{2}\, dx$$

Here $f(x) = -\cot\frac{x}{2}$

$$\therefore f'(x) = \frac{1}{2}\operatorname{cosec}^2\frac{x}{2}$$

$$= -e^x \cot\frac{x}{2} + c \qquad \left[\int e^x \left[f(x) + f'(x)\right] dx = e^x f(x)\right]$$



CHECK YOUR PROGRESS - 4

Q. Evaluate the followings :

1. $\int x^2 \cos x\, dx$
2. $\int x^3 \log 2x\, dx$
3. $\int x^3 \tan^{-1} x\, dx$
4. $\int x^2 e^{-x} dx$
5. $\int \sin^{-1} x\, dx$
6. $\int \cos\sqrt{x}\, dx$
7. $\int \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) dx$
8. $\int e^x \sin x\, dx$
9. $\int e^x \sin x \sin 2x\, dx$
10. $\int e^x (\cos x + \sin x) dx$

11. $\int e^x \sec x(1+\tan x)dx$ 12. $\int \sin(\log x)dx$

13. $\int (3x-2)\sqrt{x^2+x+1}\,dx$ 14. $\int \sqrt{7x-10-x^2}\,dx$

1.5.3 INTEGRATION BY PARTIAL FRACTIONS

From our earlier discussion we know the method of finding the integrals of the form $\int \frac{1}{ax^2+bx+c}\,dx$, while *a, b, c* are constants. But when we have to integrate a rational fraction, say of the form $\frac{f(x)}{\phi(x)}$, where the degree of the polynomial in the numerator is less than the degree of the polynomial in the denominator, the best way of effecting the integration is first to decompose the given fraction into a number of partial fractions whose sum gives the original fraction and then to integrate each fraction separately by suitable method. If in case, the degree of the polynomial in the numberator i.e. *f*(*x*), be not of a lower degree than $\phi(x)$, then we first divide $f(x)$ by $\phi(x)$ by method of ordinary division of a polynomial by another polynomial and express the given fraction to a form like $a_p x^p + a_{p-1}x^{p-1} + \ldots\ldots\ldots + a_0 + \frac{g(x)}{\phi(x)}$ where $g(x)$ is the remainder and hence of a lower degree than $\phi(x)$. After that $\frac{g(x)}{\phi(x)}$ is resolved into partial fractions and integrate the terms separately.

Here we shall avoid a detailed discussion of the various methods of partial fractions, since these topics are covered in Higher Secondary Algebra.

In our present discussion, we will study integration of rational fractions $\frac{f(x)}{\phi(x)}$ where the denominator contains

(i) real, linear and non-repeated factors

(ii) real, linear but some repeated factors

(iii) real, quadratic and non-repeated factors

(iv) real, quadratic and repeated factors

Integration of rational fractions when the denominator contains factors which are real, linear but non-repeated

The method can be understood from the following examples.

Example 1 : Integrate $\int \frac{x^2 dx}{(x-1)(x-2)(x-3)}$

Solution : Let us take

$\frac{x^2}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{(x-2)} + \frac{C}{x-3}$ where A, B, C are constants to be determined.

Then $x^2 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$

If $x = 1$,then $1 = A(-1)(-2) \Rightarrow A = \frac{1}{2}$

$x = 2$,then $4 = B(1)(-1) \Rightarrow B = -4$

and $x = 3$ then, $9 = C(2)(1) \Rightarrow C = \frac{9}{2}$

$$\therefore \frac{x^2}{(x-1)(x-2)(x-3)} = \frac{\frac{1}{2}}{x-1} + \frac{-4}{x-2} + \frac{\frac{9}{2}}{x-3}$$

$$\therefore \int \frac{x^2}{(x-1)(x-2)(x-3)}\, dx = \int \left\{ \frac{1}{2}\left(\frac{1}{x-1}\right) - 4\left(\frac{1}{x-2}\right) + \frac{9}{2}\left(\frac{1}{x-3}\right) \right\} dx$$

$$= \frac{1}{2}\log|x-1| - 4\log|x-2| + \frac{9}{2}\log|x-3| + c$$

Example 2: Integrate $\int \frac{2x+3}{x^3+x^2-2x} dx$

Solution : Here $x^3 + x^2 - 2x = x(x^2 + x - 2) = x(x+2)(x-1)$

Now $\frac{2x+3}{x^3+x^2-2x} = \frac{2x+3}{x(x+2)(x-1)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{(x-1)}$ (say)

$$\therefore 2x + 3 = A(x+2)(x-1) + Bx(x-1) + Cx(x+2)$$

Let $x = 0$, then $3 = A(2)(-1) \Rightarrow A = -\frac{3}{2}$

$x = -2$, then $-1 = B(-2)(-3) \Rightarrow B = -\frac{1}{6}$

and $x = 1$, then $5 = C(1)(3) \Rightarrow C = \frac{5}{3}$

$$\therefore \int \frac{2x+3}{x^3+x^2-2x}\,dx = \int\left(-\frac{3}{2}\frac{1}{x}-\frac{1}{6}\frac{1}{x+2}+\frac{5}{3}\frac{1}{x-1}\right)dx$$

$$= -\frac{3}{2}\log|x| - \frac{1}{6}\log|x+2| + \frac{5}{3}\log|x-1| + C$$

When the denominator contains factors which are real, linear but some are repeated

Example 3: Evaluate $\int \frac{x^2 dx}{(x+1)(x+2)^2}$

Solution : Let $\frac{x^2}{(x+1)(x+2)^2} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$

$\therefore x^2 = A(x+2)^2 + B(x+1)(x+2) + C(x+1)$

$\Rightarrow 1.x^2 + 0.x + 0 = (A+B)x^2 + (4A+3B+C)x + 4A + 2B + C$

Equating the coefficients of x^2, x and of the constant terms from both sides

$A + B = 1, \quad 4A + 3B + C = 0, \quad 4A + 2B + C = 0$

Now solving for A, B and C we get

$A = 1, B = 0, C = -4$

$$\therefore \int \frac{x^2}{(x+1)(x+2)^2}dx = \int\left(\frac{1}{x+1} + \frac{-4}{(x+2)^2}\right)dx$$

$$= \log|x+1| + \frac{4}{x+2} + c$$

Example 4 : Evaluate $\int \frac{x^2+1}{(x-1)^2(x+3)}\,dx$

Solution : Let $\frac{x^2+1}{(x-1)^2(x+3)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+3} \ldots\ldots\ldots$(i)

Then, $x^2 + 1 = A(x-1)(x+3) + B(x+3) + C(x-1)^2 \ldots\ldots\ldots$(ii)

If $x = 1$, then we get $B = \frac{1}{2}$

If $x = -3$, then $C = \frac{5}{8}$

Now equating the coefficients of x^2 from both sides of the identity (ii) we get

$1 = A + C \quad \Rightarrow A = 1 - \frac{5}{8} = \frac{3}{8}$

Then from (i), $\dfrac{x^2+1}{(x-1)^2(x+3)} = \dfrac{3/8}{x-1} + \dfrac{1/2}{(x-2)^2} + \dfrac{5/8}{x+3}$

$$\therefore \int \frac{x^2+1}{(x-1)^2(x+3)}\,dx = \frac{3}{8}\int \frac{1}{x-1}\,dx + \frac{1}{2}\int \frac{1}{(x-1)^2}\,dx + \frac{5}{8}\int \frac{1}{x+3}\,dx$$

$$= \frac{3}{8}\log|x-1| - \frac{1}{2(x-1)} + \frac{5}{8}\log|x+3| + C$$

When the denominator contains factors which are real, quadratic, but non-repeated

Example 5: $\int \dfrac{x}{(x-1)(x^2+4)}\,dx$

Solution : Let $\dfrac{x}{(x-1)(x^2+4)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+4}$

$$\Rightarrow x = A(x^2+4) + (Bx+C)(x-1)$$

If $x = 1$, then $1 = 5A \Rightarrow A = 1/5$

If $x = 0$, then $0 = 4A - C \Rightarrow C = 4/5$

If $x = -1$, then $-1 = 5A + 2B - 2C$ which gives $B = -\dfrac{1}{5}$

$$\therefore \int \frac{x}{(x-1)(x^2+4)}\,dx = \int \left(\frac{1/5}{x-1} + \frac{-\frac{1}{5}x + \frac{4}{5}}{x^2+4} \right) dx$$

$$= \frac{1}{5}\int \frac{1}{x-1}\,dx - \frac{1}{5}\int \frac{x-4}{x^2+4}\,dx$$

$$= \frac{1}{5}\log|x-1| - \frac{1}{10}\int \frac{2x}{x^2+4}\,dx + \frac{4}{5}\int \frac{1}{x^2+4}\,dx$$

$$= \frac{1}{5}\log|x-1| - \frac{1}{10}\log|x^2+4| + \frac{4}{5}\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$$

$$= \frac{1}{5}\log|x-1| - \frac{1}{10}\log|x^2+4| + \frac{2}{5}\tan^{-1}\left(\frac{x}{2}\right) + C$$

When the denominatior contains factors which are real, quadratic but some are repeated.

Example 6: Evaluate $\int \frac{x^3+x^2+2x+1}{(x^2+1)^2}\,dx$

Solution : Let $\frac{x^3+x^2+2x+1}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{(Cx+D)}{(x^2+1)^2}$

Then $x^3+x^2+2x+1 = (Ax+B)(x^2+1)+(Cx+D)$

$\Rightarrow x^3+x^2+2x+1 = Ax^3+Bx^2+(A+C)x+(B+D)$

Equating the coefficients of various powers of x A=1, B=1, C=1, D=0.

$$\therefore \int \frac{x^3+x^2+2x+1}{(x^2+1)^2}dx = \int \left\{\frac{x+1}{x^2+1} + \frac{x}{(x^2+1)^2}\right\}dx$$

$$= \int \frac{x}{x^2+1} + \int \frac{1}{x^2+1} + \int \frac{x}{(x^2+1)^2}dx$$

$$= \frac{1}{2}\log|x^2+1| + \tan^{-1}x - \frac{1}{2}\frac{1}{x^2+1} + C$$

Example 7 : Integrate $\int \frac{x^2+1}{(x^2+2)(2x^2+1)}\,dx$

Solution : Let $x^2 = y$, then

$$\frac{x^2+1}{(x^2+2)(2x^2+1)} = \frac{y+1}{(y+2)(2y+1)} = \frac{A}{y+2} + \frac{B}{2y+1} \quad \ldots\ \ldots\ \ldots \text{(i) (say)}$$

$$\therefore y+1 = A(2y+1)+B(y+2)$$

For $y=-2$, $\quad -1 = -3A \quad$ i.e. $A = 1/3$

For $y = -1/2$, $\quad \frac{1}{2} = B\left(\frac{3}{2}\right) \quad$ i.e. $B = 1/3$

Then by (i) $\quad \frac{y+1}{(y+2)(2y+1)} = \frac{1/3}{y+2} + \frac{1/3}{2y+1}$

Then replacing y by x^2 and integerting w.r.t. x

$$\int \frac{x^2+1}{(x^2+2)(2x^2+1)}dx = \frac{1}{3}\left[\int \frac{1}{x^2+2}dx + \int \frac{1}{2x^2+1}dx\right]$$

$$= \frac{1}{3}\left[\int \frac{1}{x^2 + \left(\sqrt{2}\right)^2} dx + \int \frac{1}{\left(\sqrt{2}x^2\right) + (1)^2} dx\right]$$

$$= \frac{1}{3}\left[\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{\sqrt{2}x}{1}\right)\right]$$

$$= \frac{1}{3\sqrt{2}}\left\{\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + \tan^{-1}\left(\sqrt{2}x\right)\right\} + C$$



CHECK YOUR PROGRESS - 5

Q. Evaluate the following

1. $\int \frac{x-1}{(x-2)(x-3)} dx$
2. $\int \frac{x\,dx}{x^2 - 12x + 35}$
3. $\int \frac{3x}{x^2 - x - 2} dx$
4. $\int \frac{1}{x(x+1)^2} dx$
5. $\int \frac{x^3 - 2x + 3}{x^2 + x - 2} dx$
6. $\int \frac{x^2}{(x+1)^2(x+2)} dx$
7. $\int \frac{(x-1)(x-5)}{(x-2)(x-4)} dx$
8. $\int \frac{x}{(x-2)^2} dx$
9. $\int \frac{1}{x^3 + x} dx$
10. $\int \frac{2x^3}{(x^2+1)^2} dx$
11. $\int \frac{x^2 + 3x - 4}{x^2 - 2x - 8} dx$
12. $\int \frac{x+5}{x^3(x+1)} dx$
13. $\int \frac{1}{1 - x^3} dx$
14. $\int \frac{x^2}{(x-1)^3(x+1)} dx$
15. $\int \frac{1}{(x^2-1)^2} dx$
16. $\int \frac{x dx}{(x^2 + a^2)(x^2 + b^2)}$
17. $\int \frac{dx}{x^3 - x^2 - x + 1}$
18. $\int \frac{x^2}{x^4 + x^2 - 2} dx$
19. $\int \frac{1 - 3x^2}{3x - x^3} dx$
20. $\int \frac{x^2 dx}{(x^2+1)(x^2+4)}$
21. $\int \frac{\sin 2x\, dx}{(1 + \sin x)(2 + \sin x)}$
22. $\int \frac{\cos x\, dx}{(2 + \sin x)(3 + 4\sin x)}$
23. $\int \frac{dx}{x(x^4 + 1)}$
24. $\int \frac{\sin 2x\, dx}{(1 + \sin x)(2 + \sin x)}$
25. $\int \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} dx$



1.6 LET US SUM UP

- Differentiation and integration are inverse processes of one another
- We may also define integration as a process of finding the limit of a sum [The method is discussed in the next unit].
- The set of functions that could possibly have in given functon as a derivative are called anti derivatives (or integrats or primitives) of a function.
- Based on the results of some standard derivatives, we obtain the formula of some standard simple Integration.
- There are different methods for determination of integrals.Some of the methods are (i) Integration by substitution,(ii) Integration by parts,(iii)Integration by partial fractions.
- A few important forms of integrals

(i) $\int \frac{f'(x)}{f(x)}dx = \log|f(x)| + C$

(ii) $\int \{f(x)\}^n f'(x)dx = \frac{1}{n+1}[f(x)]^{n+1} + C, \quad n \neq -1$

(iii) $\int f'(ax+b)dx = \frac{1}{a}f(ax+b) + C$

- An important formula for product of two functions is

$$\int uv\,dx = u\int v dx - \int \left\{\frac{d}{dx}(u).\int v dx\right\} dx$$

- When we have to integrate a rational fraction say $f(x)/\phi(x)$ where the degree of $f(x)$ is lower than that of $\phi(x)$, then we apply a suitable method to reduce $f(x)/\phi(x)$ into partial fractions and finally integrate each of the terms.
- When we have to integrate a rational fraction, say $f(x)/\phi(x)$, where the degree of $f(x)$ is not lower than that of $\phi(x)$, then we divide $f(x)$

by $\phi(x)$ by the method of ordinary division. The form $f(x)/\phi(x)$ then reduces to $a_p x^p + a_{p-1}x^{p-1} + \ldots\ldots\ldots + a_0 + \dfrac{g(x)}{\phi(x)}$ where the degree of $\phi(x)$ is greater than that of $g(x)$.



1.7 FURTHER READINGS

1) *Integral Calculus,* Kishan,Hari,Atlantic Publishers & Distributors.New Delhi -110027.
2) *Integral Calculus*, Dhami,H.S. New age International Publishers,New Delhi - 110002.
3) *Text book of Integral Calculus*,Sharma,A.K. Discovery Publishing House,New Delhi -110002.



1.8 ANSWERS TO CHECK YOUR PROGRESS

CHECK YOUR PROGRESS – 1

Ans to Q No 1 : (i) We know that $\dfrac{d}{dx}(\log x) = \dfrac{1}{x}, x > 0$

and $\dfrac{d}{dx}(\log(-x)) = \dfrac{1}{-x}(-1), x < 0$

Combining above ,we get $\dfrac{d}{dx}(\log|x|) = \dfrac{1}{x}, x \neq 0$

Therefore, $\int \dfrac{1}{x}dx = \log|x|$ is anti-derivative of $\dfrac{1}{x}$.

(ii) We know that $\dfrac{d}{dx}(\cos 3x) = -3\sin 3x$

$\Rightarrow \sin 3x = -\dfrac{1}{3}\dfrac{d}{dx}(\cos 3x)$

$$\therefore \sin 3x = \frac{d}{dx}\left(-\frac{1}{3}\cos 3x\right)$$

Therefore, anti -derivative of $\sin 3x$ is $-\frac{1}{3}\cos 3x$.

(iii) We know that $\frac{d}{dx}(ax-b)^3 = 3a(ax-b)^2$

$$\Rightarrow (ax-b)^2 = \frac{1}{3a}\frac{d}{dx}(ax-b)^3$$

$$\therefore (ax-b)^2 = \frac{d}{dx}\left\{\frac{1}{3a}(ax-b)^3\right\}$$

Therefore, anti -derivative of $(ax-b)^2$ is $\frac{1}{3a}(ax-b)^3$.

Ans to Q No 2:

(i) $\int x^3\left(x^2+\frac{1}{x^2}+2x.\frac{1}{x}\right)dx = \int\left(x^5+x+2x^3\right)dx$

$$= \frac{x^6}{6}+\frac{x^2}{2}+2\frac{x^4}{4}+C$$

$$= \frac{x^6}{6}+\frac{x^2}{2}+\frac{x^4}{2}+C.$$

(ii) $\int(\cos x+\sin x)^2 dx = \int\left(\cos^2 x+\sin^2 x+2\sin x\cos x\right)dx$

$$= \int(1+\sin 2x)dx$$

$$= x-\frac{\cos 2x}{2}+C.$$

(iii) $\int\left(\frac{2\cos x}{5\sin^2 x}+\frac{1}{3\cos^2 x}\right)dx = \int\left(\frac{2}{5}\cot x\,\mathrm{cosec}x+\frac{1}{3}\sec^2 x\right)dx$

$$= \frac{2}{5}\int\cot x\,\mathrm{cosec}x\,dx+\frac{1}{3}\int\sec^2 x\,dx$$

$$= \frac{2}{5}(-\mathrm{cosec}x)+\frac{1}{3}\tan x+C.$$

(iv) $\int\frac{\sin^2 x-\cos^2 x}{\sin^2 x\cos^2 x}dx = \int\left(\frac{\sin^2 x}{\sin^2 x\cos^2 x}-\frac{\cos^2 x}{\sin^2 x\cos^2 x}\right)dx$

$$= \int\left(\sec^2 x-\mathrm{cosec}^2 x\right)dx$$

$$= \tan x+\cot x+C$$

(v) $\int\left(e^{x\log a}+e^{a\log x}+e^{a\log a}\right)dx=\int\left(e^{\log a^x}+e^{\log x^a}+e^{\log a^a}\right)dx$

$=\int\left(a^x+x^a+a^a\right)dx$

$=\int a^x dx+\int x^a dx+\int a^a dx$

$=\frac{a^x}{\log a}+\frac{x^{a+1}}{a+1}+a^a.x+C$.

(vi) $\int\sqrt{1+\cos 2x}\,dx=\int\sqrt{2\cos^2 x}dx$, $\cos 2x=2\cos^2 x-1$

$=\sqrt{2}\int\cos x dx$

$=\sqrt{2}\sin x+C$.

(vii) $\int\frac{(1+x)^2}{\sqrt{x}}dx=\int\frac{1+2x+x^2}{\sqrt{x}}dx$

$=\int\left(\frac{1}{\sqrt{x}}+2\sqrt{x}+x^{\frac{3}{2}}\right)dx$

$=\int x^{-\frac{1}{2}}dx+2\int x^{\frac{1}{2}}dx+\int x^{\frac{3}{2}}dx$

$=\frac{x^{\frac{1}{2}}}{\frac{1}{2}}+2.\frac{x^{\frac{3}{2}}}{\frac{3}{2}}+\frac{2}{5}x^{\frac{5}{2}}+C$

$=2x^{\frac{1}{2}}+\frac{4}{3}x^{\frac{3}{2}}+\frac{2}{5}x^{\frac{5}{2}}+C$

(viii) $\int\left(\frac{x}{m}+\frac{m}{x}+x^m+m^x\right)dx=\frac{1}{m}\int xdx+m\int\frac{1}{x}dx+\int x^m dx+\int m^x dx$

$=\frac{1}{m}.\frac{x^2}{2}+m\log|x|+\frac{x^{m+1}}{m+1}+\frac{x^m}{\log m}+C$

Ans to Q No 3 : (i) Let $x^2=t$ .Then $\int\left(x^4+x^2+1\right)d\left(x^2\right)=\int\left(t^2+t+1\right)dt$

$=\frac{t^3}{3}+\frac{t^2}{2}+t+C$

$=\frac{x^6}{6}+\frac{x^4}{2}+x^2+C$

(ii) $\int \frac{x^4+x^2+1}{x^2-x+1}dx = \int \frac{\left(x^2+1\right)^2-x^2}{x^2-x+1}dx$

$$= \int \frac{\left(x^2+1+x\right)\left(x^2+1-x\right)}{x^2-x+1}dx$$

$$= \int \left(x^2+x+1\right)dx = \frac{x^3}{3}+\frac{x^2}{2}+x+C\,.$$

(iii) $\int \frac{x^{-1/3}+\sqrt{x}+2}{\sqrt[3]{x}}\,dx = \int \frac{x^{-\frac{1}{3}}+x^{\frac{1}{2}}+2}{x^{\frac{1}{3}}}dx$

$$= \int \left(\frac{x^{-\frac{1}{3}}}{x^{\frac{1}{3}}}+\frac{x^{\frac{1}{2}}}{x^{\frac{1}{3}}}+\frac{2}{x^{\frac{1}{3}}}\right)dx$$

$$= \int \left(x^{-\frac{2}{3}}+x^{\frac{1}{2}-\frac{1}{3}}+2x^{-\frac{1}{3}}\right)dx$$

$$= \int \left(x^{-\frac{2}{3}}+x^{\frac{1}{6}}+2x^{-\frac{1}{3}}\right)dx$$

$$= \frac{x^{-\frac{2}{3}+1}}{-\frac{2}{3}+1}+\frac{x^{\frac{1}{6}+1}}{\frac{1}{6}+1}+2\frac{x^{-\frac{1}{3}+1}}{-\frac{1}{3}+1}+C$$

$$= \frac{x^{\frac{1}{3}}}{\frac{1}{3}}+\frac{x^{\frac{7}{6}}}{\frac{7}{6}}+2\frac{x^{\frac{2}{3}}}{\frac{2}{3}}+C$$

$$= 3x^{\frac{1}{3}}+\frac{6}{7}x^{\frac{7}{6}}+3x^{\frac{2}{3}}+C\,.$$

Ans to Q No 4: We have $f(x) = \int f'(x)dx = \int \left(\frac{1}{x}+\frac{1}{1+x^2}\right)dx$

$$= \int \frac{1}{x}dx + \int \frac{1}{x^2+1}dx = \log|x| + \tan^{-1}x + C$$

When $x = 1$, we have $f(x) = \frac{\pi}{4}$

$$\therefore \frac{\pi}{4} = \log|1| + \tan^{-1} 1 + C$$

$$\Rightarrow \frac{\pi}{4} = \frac{\pi}{4} + C$$

$$\Rightarrow C = 0$$

$$\therefore f(x) = \log|x| + \tan^{-1} x .$$

Ans to Q No 5: Given, $\frac{dy}{dx} = 3x^2 - 12$

$$\therefore f'(x) = 3x^2 - 12$$

$$\therefore f(x) = \int f'(x)dx = \int (3x^2 - 12)dx$$

$$= 3.\frac{x^3}{3} - 12x + C$$

$$f(x) = x^3 - 12x + C \quad \ldots (1)$$

(1) passes through the point (2,-7).

$$\therefore -7 = 2^3 - 12.2 + C$$

$$\Rightarrow C = 9$$

Required equation of the curve is $f(x) = x^3 - 12x + 9$.

CHECK YOUR PROGRESS – 2

Ans to Q No 1: Let $9 - 4x^2 = t$

$$\therefore -8xdx = dt$$

$$\int \frac{x}{9 - 4x^2} dx = -\frac{1}{8}\int \frac{1}{t} dt$$

$$= -\frac{1}{8}\log|t| + C$$

$$= -\frac{1}{8}\log|9 - 4x^2| + C$$

Ans to Q No 2: Let $x^2 + x + 1 = t \quad \therefore (2x + 1)dx = dt$

$$\int (4x + 2)\sqrt{x^2 + x + 1}dx$$

$$= \int 2\sqrt{t}dt = 2\int \sqrt{t}dt = 2\frac{t^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{4}{3}(x^2 + x + 1)^{\frac{3}{2}} + C$$

Ans to Q No 3: Let $\tan^{-1} x = t$

$$\therefore \frac{1}{1+x^2}dx = dt$$

$$\int \frac{e^{\tan^{-1}x}dx}{1+x^2} = \int e^t dt$$

$$= e^t + C$$

$$= e^{\tan^{-1}x} + C$$

Ans to Q No 4: We put $x = a\sec\theta \Rightarrow \frac{x}{a} = \sec\theta \Rightarrow \theta = \sec^{-1}\frac{x}{a}$

$$dx = a\sec\theta\tan\theta d\theta$$

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \int \frac{a\sec\theta\tan\theta d\theta}{a\sec\theta\sqrt{a^2\sec^2\theta - a^2}}$$

$$= \int \frac{a\sec\theta\tan\theta d\theta}{a\sec\theta\sqrt{a^2\tan^2\theta}}$$

$$= \frac{1}{a}\int d\theta$$

$$= \frac{1}{a}\theta + C$$

$$= \frac{1}{a}\sec^{-1}\frac{x}{a} + C$$

Ans to Q No 5 : Let $x^2 + 5 = t$

$$\Rightarrow 2xdx = dt$$

$$\Rightarrow xdx = \frac{1}{2}dt$$

$$\int x\sqrt{x^2+5}\,dx = \frac{1}{2}\int t^{\frac{1}{2}}dt = \frac{1}{2}\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$$

$$= \frac{1}{2}\frac{t^{\frac{3}{2}}}{\frac{3}{2}} + C$$

$$= \frac{1}{3}t^{\frac{3}{2}} + C$$

$$= \frac{1}{3}\left(x^2+5\right)^{\frac{3}{2}} + C$$

Ans to Q No 6 : Let $x = \sin\theta \Rightarrow dx = \cos\theta d\theta$

$$\int \frac{dx}{x^2\sqrt{1-x^2}} = \int \frac{\cos\theta d\theta}{\sin^2\theta\sqrt{1-\sin^2\theta}}$$

$$= \int \frac{\cos\theta d\theta}{\sin^2\theta\cos\theta}$$

$$= \int \operatorname{cosec}^2\theta d\theta$$

$$= -\cot\theta + C$$

$$= -\cot\left(\sin^{-1}x\right) + C$$

Ans to Q No 7: Let $a + b\sin x = t \Rightarrow b\cos x dx = dt$

$$\cos x dx = \frac{1}{b}dt$$

$$\int \frac{\cos x\, dx}{(a+b\sin x)^2} = \int \frac{\frac{1}{b}dt}{t^2} = \frac{1}{b}\int \frac{dt}{t^2}$$

$$= \frac{1}{b}.\frac{t^{-2+1}}{-2+1} + C$$

$$= -\frac{1}{b}.\frac{1}{t} + C$$

$$= -\frac{1}{b}.\frac{1}{a+b\sin x} + C$$

Ans to Q No 8: Dividing numerator and denominator by e^x, we have

$$\frac{\frac{e^{2x}-1}{e^x}}{\frac{e^{2x}+1}{e^x}} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Let $e^x + e^{-x} = t$

$$\therefore \left(e^x - e^{-x}\right)dx = dt$$

$$\int \frac{e^{2x}-1}{e^{2x}+1}dx = \int \frac{e^x - e^{-x}}{e^x + e^{-x}}dx$$

$$= \int \frac{dt}{t}$$

$$= \log|t| + C$$

$$= \log\left|e^{x} + e^{-x}\right| + C$$

Ans to Q No 9: Let $\tan^{-1} x = t \Rightarrow \frac{1}{1+x^2}dx = dt$

$$\int \frac{\sin\left(\tan^{-1} x\right)}{1+x^2}dx = \int \sin t\,dt = -\cos t + C$$

$$= -\cos\left(\tan^{-1} x\right) + C$$

Ans to Q No 10 :

$$\frac{(x+1)(x+\log x)^2}{x} = \left(\frac{x+1}{x}\right)(x+\log x)^2 = \left(1+\frac{1}{x}\right)(x+\log x)^2$$

Let $x + \log x = t \qquad \therefore \left(1+\frac{1}{x}\right)dx = dt$

$$\int \left(1+\frac{1}{x}\right)(x+\log x)^2 dx = \int t^2 dt$$

$$= \frac{t^3}{3} + C$$

$$= \frac{1}{3}(x+\log x)^3 + C$$

Ans to Q No 11 : Let $I = \int \sqrt{\frac{1+x}{1-x}}dx = \int \frac{1+x}{\sqrt{1-x^2}}dx$

$$= \int \frac{dx}{\sqrt{1-x^2}} + \int \frac{x}{\sqrt{1-x^2}}dx$$

$$= \sin^{-1} x + I_1 + C$$

In I_1,we substitute $1 - x^2 = t^2 \Rightarrow -2xdx = 2tdt \Rightarrow xdx = -tdt$

$$I_1 = -\int \frac{tdt}{\sqrt{t^2}} = -\int dt = -t = -\sqrt{1-x^2}$$

$\therefore I = \sin^{-1} x - \sqrt{1-x^2} + C \cdot$

Ans to Q No 12 : Let $3x^2 - 2x + 7 = t \Rightarrow (6x-2)dx = dt$

$$\Rightarrow 2(3x-1)dx = dt$$

$$\Rightarrow (3x-1)dx = \frac{1}{2}dt$$

$$\int \frac{3x-1}{\sqrt{3x^2-2x+7}}\,dx = \int \frac{\frac{1}{2}}{\sqrt{t}}dt = \frac{1}{2}\int t^{-\frac{1}{2}}dt$$

$$= \frac{1}{2}\frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C$$

$$= \frac{1}{2}\frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$$= \sqrt{t} + C$$

$$= \sqrt{3x^2-2x+7} + C$$

Ans to Q No 13 : Let $e^x x = t \Rightarrow (e^x x + e^x.1)dx = dt$

$$\Rightarrow e^x(x+1)dx = dt$$

$$\therefore \int \frac{e^x(1+x)}{\cos^2(e^x x)}dx = \int \frac{dt}{\cos^2 t} = \int \sec^2 t\,dt$$

$$= \tan t + C$$

$$= \tan(e^x x) + C$$

Ans to Q No 14 : Multiplying and dividing by x^{-3}, we have

$$\frac{x^{-3}}{x^2.x^{-3}(x^4+1)^{\frac{3}{4}}} = \frac{x^{-3}(x^4+1)^{-\frac{3}{4}}}{x^2.x^{-3}} = \frac{(x^4+1)^{-\frac{3}{4}}}{x^5(.x^4)^{-\frac{3}{4}}}$$

$$= \frac{1}{x^5}\left(\frac{x^4+1}{x^4}\right)^{-\frac{3}{4}}$$

$$= \frac{1}{x^5}\left(1+\frac{1}{x^4}\right)^{-\frac{3}{4}}$$

Let $\frac{1}{x^4} = t \Rightarrow -\frac{4}{x^5}dx = dt \Rightarrow \frac{1}{x^5}dx = -\frac{dt}{4}$

$$\therefore \int \frac{1}{x^2\left(x^4+1\right)^{\frac{3}{4}}}dx = \int \frac{1}{x^5}\left(1+\frac{1}{x^4}\right)^{-\frac{3}{4}}dx$$

$$= -\frac{1}{4}\int (1+t)^{-\frac{3}{4}}dt$$

$$= -\frac{1}{4}\left[\frac{(1+t)^{\frac{1}{4}}}{\frac{1}{4}}\right]+C$$

$$= -\frac{1}{4}\frac{\left(1+\frac{1}{x^4}\right)^{\frac{1}{4}}}{\frac{1}{4}}+C$$

$$= -\left(1+\frac{1}{x^4}\right)^{\frac{1}{4}}+C$$

Ans to Q No 15: $\int \frac{x^2+5x+2}{x+2}\,dx = \int \frac{(x+3)(x+2)}{x+2}dx = \int (x+3)dx$

$$= \frac{x^2}{2}+3x+C$$

Ans to Q No 16: $\frac{1}{\sqrt{x+a}+\sqrt{x+b}} = \frac{\left(\sqrt{x+a}-\sqrt{x+b}\right)}{\left(\sqrt{x+a}+\sqrt{x+b}\right)\left(\sqrt{x+a}-\sqrt{x+b}\right)}$

$$= \frac{\left(\sqrt{x+a}-\sqrt{x+b}\right)}{\left(\sqrt{x+a}\right)^2-\left(\sqrt{x+b}\right)^2}$$

$$= \frac{\sqrt{x+a}-\sqrt{x+b}}{(x+a)-(x+b)} = \frac{1}{a-b}\left(\sqrt{x+a}-\sqrt{x+b}\right)$$

$$\therefore \int \frac{1}{\sqrt{x+a}+\sqrt{x+b}}dx = \frac{1}{a-b}\left\{\int \sqrt{x+a}dx - \int \sqrt{x+b}dx\right\}$$

$$= \frac{1}{a-b}\left\{\int \sqrt{u}du - \int \sqrt{v}dv\right\} \text{ where } x+a=u \text{ and } x+b=v$$

$$\therefore dx = du \text{ and } dx = dv$$

$$= \frac{1}{a-b}\left\{\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{v^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right\} + C$$

$$= \frac{1}{a-b}\left\{\frac{2}{3}u^{\frac{3}{2}} - \frac{2}{3}v^{\frac{3}{2}}\right\} + C$$

$$= \frac{2}{3(a-b)}\left\{(x+a)^{\frac{3}{2}} - (x+b)^{\frac{3}{2}}\right\} + C$$

Ans to Q No 17 : $\cos^4 x = (\cos^2 x)^2 = \left(\frac{1+\cos 2x}{2}\right)^2$

$$= \frac{1}{4}\left[1 + \cos^2 2x + 2\cos 2x\right]$$

$$= \frac{1}{4}\left[1 + \left(\frac{1+\cos 4x}{2}\right) + 2\cos 2x\right]$$

$$= \frac{1}{4}\left[1 + \frac{1}{2} + \frac{\cos 4x}{2} + 2\cos 2x\right]$$

$$= \frac{1}{4}\left[\frac{3}{2} + \frac{\cos 4x}{2} + 2\cos 2x\right]$$

$$\therefore \int \cos^4 x dx = \int\left(\frac{3}{8} + \frac{\cos 4x}{8} + \frac{\cos 2x}{2}\right)dx$$

$$= \frac{3}{8}x + \frac{\sin 4x}{32} + \frac{\sin 2x}{4} + C$$

Ans to Q No 18 : From the identity $\sin 3x = 3\sin x - 4\sin^3 x$

$$\sin^3 x = \frac{3\sin x - \sin 3x}{4}$$

$$\therefore \int \sin^3 x dx = \frac{3}{4}\int \sin x dx - \frac{1}{4}\int \sin 3x dx$$

$$= -\frac{3}{4}\cos x + \frac{1}{12}\cos 3x + C$$

Ans to Q No 19 : From the identity, $\cos 2x = 2\cos^2 x - 1$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$\int \cos^2 x dx = \frac{1}{2}\int (1+\cos 2x)dx$$

$$= \frac{1}{2}\int dx + \frac{1}{2}\int \cos 2x dx$$

$$= \frac{x}{2} + \frac{1}{4}\sin 2x + C$$

Ans to Q No 20 : Let $x^2 + 5x - 7 = t \Rightarrow (2x+5)dx = dt$

$$\int \frac{2x+5}{x^2+5x-7}\, dx = \int \frac{dt}{t} = \log t + C$$

$$= \log\left(x^2 + 5x - 7\right) + C$$

Ans to Q No 21 :

We have $\int \frac{1}{\sin(x-a)\sin(x-b)}dx = \frac{1}{\sin(a-b)}\int \frac{\sin\{(x-b)-(x-a)\}}{\sin(x-a)\sin(x-b)}dx$

$$= \frac{1}{\sin(a-b)}\int \{\cot(x-a) - \cot(x-b)\}dx$$

$$= \frac{1}{\sin(a-b)}\{\log|\sin(x-a)| - \log|\sin(x-b)|\} + C$$

$$= \cos ec(a-b)\left\{\log\left|\frac{\sin(x-a)}{\sin(x-b)}\right|\right\} + C\,.$$

Ans to Q No 22 :

$$\int \frac{1}{\sin(x-a)\cos(x-b)}dx = \frac{1}{\cos(a-b)}\int \frac{\cos(a-b)}{\sin(x-a)\cos(x-b)}dx$$

$$= \frac{1}{\cos(a-b)}\int \frac{\cos\{(x-b)-(x-a)\}}{\sin(x-a)\cos(x-b)}dx$$

$$= \frac{1}{\cos(a-b)}\int \frac{\cos(x-a)\cos(x-b)+\sin(x-a)\sin(x-b)}{\sin(x-a)\cos(x-b)}dx$$

$$= \frac{1}{\cos(a-b)}\int \{\cot(x-a) + \tan(x-b)\}dx$$

$$= \frac{1}{\cos(a-b)}[\log|\sin(x-a)| - \log|\cos(x-b)|] + C$$

$$= \frac{1}{\cos(a-b)}\log\left|\frac{\sin(x-a)}{\cos(x-b)}\right| + C\,.$$

Ans to Q No 23 : We have

$$\int \sin x \sin 2x \sin 3x dx = \frac{1}{2}\int (2\sin 2x \sin x)\sin 3x dx$$

$$= \frac{1}{2}\int (\cos x - \cos 3x)\sin 3x dx$$

$$= \frac{1}{4}\int (2\sin 3x \cos x - 2\sin 3x \cos 3x) dx$$

$$= \frac{1}{4}\int (\sin 4x + \sin 2x - \sin 6x) dx$$

$$= \frac{1}{4}\left[-\frac{\cos 4x}{4} - \frac{\cos 2x}{2} + \frac{\cos 6x}{6} \right] + C .$$

Ans to Q No 24 : $\int \frac{x+2}{(x+1)^2} dx = \int \frac{(x+1)+1}{(x+1)^2} dx$

$$= \int \frac{1}{x+1} dx + \int \frac{1}{(x+1)^2} dx$$

$$= \int \frac{1}{z} dz + \int \frac{1}{z^2} dz \quad \text{put} \quad x+1 = z \quad \therefore dx = dz$$

$$= \log|z| + \frac{z^{-2+1}}{-2+1} + C$$

$$= \log|z| - \frac{1}{z} + C$$

$$= \log|x+1| - \frac{1}{x+1} + C$$

Ans to Q No 25 : Let $3x^2 + 2x + 1 = t \Rightarrow (6x+2)dx = dt$

$$\Rightarrow (3x+1)dx = \frac{1}{2} dt$$

$$\int \frac{3x+1}{(3x^2+2x+1)^3} dx = \frac{1}{2}\int \frac{dt}{t^3} = \frac{1}{2}\int t^{-3} dt$$

$$= \frac{1}{2}\frac{t^{-3+1}}{-3+1} + C$$

$$= \frac{1}{2}.\frac{t^{-2}}{-2} + C$$

$$= -\frac{1}{4}.\frac{1}{t^2} + C$$

$$= -\frac{1}{4}.\frac{1}{\left(3x^2+2x+1\right)^2}+C\ .$$

CHECK YOUR PROGRESS – 3

Ans to Q No 1 : $I=\int\frac{dx}{4x^2+9}=\frac{1}{2}\int\frac{2dx}{(2x)^2+(3)^2}=\frac{1}{6}\tan^{-1}\frac{2x}{3}+C$

Ans to Q No 2 : We have $\int\frac{e^x}{e^{2x}+6e^x+5}dx=\int\frac{e^x}{\left(e^x\right)^2+6e^x+5}dx$

Let $e^x=t$,Then $e^x dx=dt$

$$\int\frac{1}{t^2+6t+5}dt=\int\frac{1}{(t+3)^2-2^2}dt$$

$$=\frac{1}{2\times 2}\log\left|\frac{t+3-2}{t+3+2}\right|+C$$

$$=\frac{1}{4}\log\left|\frac{e^x+1}{e^x+5}\right|+C\ .$$

Ans to Q No 3 : We have $\int\frac{1}{x\left(x^5+1\right)}dx=\int\frac{x^4}{x^5\left(x^5+1\right)}dx$ ...(1)

Let $x^5+1=t$

$$5x^4dx=dt$$

From (1), $\int\frac{1}{x\left(x^5+1\right)}dx=\frac{1}{5}\int\frac{1}{t(t-1)}dt$

$$=\frac{1}{5}\int\frac{1}{t^2-t}dt$$

$$=\frac{1}{5}\int\frac{1}{t^2-t+\frac{1}{4}-\frac{1}{4}}dt$$

$$=\frac{1}{5}\int\frac{1}{\left(t-\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}dt$$

$$= \frac{1}{5} \cdot \frac{1}{2\left(\frac{1}{2}\right)} \log \left| \frac{t - \frac{1}{2} - \frac{1}{2}}{t - \frac{1}{2} + \frac{1}{2}} \right| + C$$

$$= \frac{1}{5} \log \left| \frac{t-1}{t} \right| + C$$

$$= \frac{1}{5} \log \left| \frac{x^5}{x^5 + 1} \right| + C$$

Ans to Q No 4 : We have $\int \frac{1}{\sqrt{1 - x - x^2}} dx = \int \frac{dx}{\sqrt{\left\{1 - \left(x^2 + x\right)\right\}}}$

$$= \int \frac{dx}{\sqrt{\left\{1 - \left(x^2 + 2x\frac{1}{2} + \frac{1}{4} - \frac{1}{4} \right)\right\}}}$$

$$= \int \frac{dx}{\sqrt{\left\{1 - \left(x + \frac{1}{2} \right)^2 + \frac{1}{4}\right\}}}$$

$$= \int \frac{dx}{\sqrt{\left\{\frac{5}{4} - \left(x + \frac{1}{2} \right)^2 \right\}}}$$

$$= \sin^{-1} \left\{ \frac{x + \frac{1}{2}}{\frac{1}{2}\sqrt{5}} \right\} + C$$

$$= \sin^{-1} \left(\frac{2x+1}{\sqrt{5}} \right) + C .$$

Ans to Q No 5 : We have $\frac{d}{dx}\left(x^2 + x + 1\right) = 2x + 1$

Now $2x + 3 = \left(2x + 1\right) + 2$

$$\therefore \int \frac{2x+3}{\sqrt{x^2 + x + 1}} dx = \int \frac{\left(2x+1\right)}{\sqrt{x^2 + x + 1}} dx + \int \frac{2}{\sqrt{x^2 + x + 1}} dx$$

$$= 2\sqrt{x^2+x+1} + \int \frac{2}{\sqrt{\left\{\left(x+\frac{1}{2}\right)^2 + 1 - \frac{1}{4}\right\}}} dx$$

$$= 2\sqrt{x^2+x+1} + 2\int \frac{dx}{\sqrt{\left\{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2\right\}}}$$

$$= 2\sqrt{x^2+x+1} + 2\sinh^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C$$

$$= 2\sqrt{x^2+x+1} + 2\sinh^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C .$$

Ans to Q No 6 : Let $x+2 = A\frac{d}{dx}\left(x^2+5x+6\right)+B$

$\Rightarrow x+2 = A(2x+5)+B$

Comparing the coefficients of like powers of x, we get

$1 = 2A$ and $5A+B=2$

$\Rightarrow A = \frac{1}{2}$ and $B = -\frac{1}{2}$

$$\therefore \int \frac{x+2}{\sqrt{x^2+5x+6}} dx = \int \frac{\frac{1}{2}(2x+5)-\frac{1}{2}}{\sqrt{x^2+5x+6}} dx$$

$$= \frac{1}{2}\int \frac{(2x+5)}{\sqrt{x^2+5x+6}} dx - \frac{1}{2}\int \frac{1}{\sqrt{x^2+5x+6}} dx$$

$$= \frac{1}{2}\int \frac{dt}{\sqrt{t}} - \frac{1}{2}\int \frac{1}{\sqrt{\left(x+\frac{5}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} dx \quad \text{where } t = x^2+5x+6$$

$$= \sqrt{t} - \frac{1}{2}\log\left|\left(x+\frac{5}{2}\right) + \sqrt{x^2+5x+6}\right| + C$$

$$= \sqrt{x^2+5x+6} - \frac{1}{2}\log\left|\left(x+\frac{5}{2}\right)+\sqrt{x^2+5x+6}\right| + C$$

Ans to Q No 7 : We put $x+1=t^2$, so that $dx = 2tdt$

$$\therefore \int \frac{dx}{(x+2)\sqrt{x+1}} = \int \frac{2tdt}{(t^2+1)t}$$

$$= 2\int \frac{dt}{t^2+1}$$

$$= 2\tan^{-1} t + C$$

$$= 2\tan^{-1}\left[\sqrt{x+1}\right] + C$$

Ans to Q No 8 : We put $1+x=\frac{1}{t}$, so that $dx = -\frac{1}{t^2}dt$. Also, $x = \frac{1}{t} - 1$

$$\therefore \int \frac{dx}{(1+x)\sqrt{1-x^2}} = \int \frac{-\left(\frac{1}{t^2}\right)dt}{\frac{1}{t}\sqrt{\left[1-\left\{\left(\frac{1}{t}\right)-1\right\}^2\right]}}$$

$$= -\int \frac{dt}{\sqrt{\left[t^2-(1-t)^2\right]}}$$

$$= -\int \frac{dt}{\sqrt{(2t-1)}}$$

$$= -\frac{1}{2}\int (2t-1)^{-\frac{1}{2}}.2dt$$

$$= -\sqrt{2t-1} + C$$

$$= -\sqrt{\frac{2}{1+x}-1} + C$$

$$= -\sqrt{\frac{1-x}{1+x}} + C$$

CHECK YOUR PROGRESS – 4

Ans to Q No 1 : $\int x^2 \cos x dx = x^2 \sin x - \int 2x \sin x dx$

$$= x^2 \sin x - 2\left[x(-\cos x) - \int (-\cos x)dx\right]$$

$$= x^2 \sin x + 2x\cos x - 2\sin x$$
$$= (x^2 - 2)\sin x + 2x\cos x$$

Ans to Q No 2 : $\int x^3 \log 2x dx = \log 2x \int x^3 dx - \int \left(\frac{d}{dx} \log 2x . \int x^3 dx \right) dx$

$$= \log 2x \frac{x^4}{4} - \int \frac{1}{2x} . 2 . \frac{x^4}{4} dx$$

$$= \frac{x^4 \log 2x}{4} - \frac{1}{4} \int x^3 dx$$

$$= \frac{x^4 \log 2x}{4} - \frac{1}{4} . \frac{x^4}{4} + C$$

$$= \frac{x^4 \log 2x}{4} - \frac{x^4}{16} + C .$$

Ans to Q No 3 : $\int x^3 \tan^{-1} x dx = \frac{x^4}{4} \tan^{-1} x - \int \frac{x^4}{4} . \frac{1}{1+x^2} dx$

$$= \frac{x^4}{4} \tan^{-1} x - \frac{1}{4} \int \left[x^2 - 1 + \frac{1}{1+x^2} \right] dx$$

$$= \frac{1}{4} \left[x^4 \tan^{-1} x - \frac{x^3}{3} + x - \tan^{-1} x \right]$$

$$= \frac{1}{4} \left[(x^4 - 1)\tan^{-1} x - \frac{x^3}{3} + x \right]$$

Ans to Q No 4 : $\int x^2 e^{-x} dx = x^2 \int e^{-x} dx - \int \left(\frac{d}{dx} x^2 \int e^{-x} dx \right) dx$

$$= x^2 (-e^{-x}) - \int 2x (-e^{-x}) dx$$
$$= -x^2 e^{-x} + 2 \int x e^{-x} dx$$
$$= -x^2 e^{-x} + 2 \left[x(-e^{-x}) - \int 1 . (-e^{-x}) dx \right]$$
$$= -x^2 e^{-x} + 2(-xe^{-x} - e^{-x})$$
$$= -x^2 e^{-x} - 2xe^{-x} - 2e^{-x}$$

Ans to Q No 5 : Let $\sin^{-1} x = t$,Then $x = \sin t \Rightarrow dx = \cos t dt$

$$\therefore \int \sin^{-1} x dx = \int t \cos t dt$$
$$= t \sin t - \int 1 . (\sin t) dt$$
$$= t \sin t + \cos t + C$$

$$= x\sin^{-1}x + \sqrt{1-\sin^2 t} + C$$

$$= x\sin^{-1}x + \sqrt{1-x^2} + C$$

Ans to Q No 6 : We substitute $x = t^2$, We get $dx = 2tdt$

$$\therefore \int \cos\sqrt{x}dx = 2\int t\cos t dt$$

$$= 2\left[t\sin t - \int \sin t dt\right]$$

$$= 2\left[t\sin t + \cos t\right] + C$$

$$= 2\left[\sqrt{x}\sin\sqrt{x} + \cos\sqrt{x}\right] + C$$

Ans to Q No 7 : We put $x = \tan\theta$, so that $dx = \sec^2\theta d\theta$

$$\int \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)dx = \int \cos^{-1}\frac{1-\tan^2\theta}{1+\tan^2\theta}\sec^2\theta d\theta$$

$$= \int \cos^{-1}(\cos 2\theta)\sec^2\theta d\theta$$

$$= 2\int \theta\sec^2\theta d\theta$$

$$= 2\left[\theta\tan\theta - \int 1.\tan\theta d\theta\right]$$

$$= 2\left[\theta\tan\theta - \log\sec\theta\right] + C$$

$$= 2\left[x\tan^{-1}x - \log\sqrt{1+x^2}\right] + C.$$

Ans to Q No 8 : Let $I = \int e^x \sin x dx = e^x(-\cos x) + \int e^x \cos x dx$

$$= -e^x\cos x + e^x\sin x - \int e^x \sin x dx$$

$$\Rightarrow I = -e^x\cos x + e^x\sin x - I$$

$$\Rightarrow 2I = e^x(\sin x - \cos x)$$

$$\therefore I = \frac{e^x}{2}(\sin x - \cos x)$$

Ans to Q No 9 : Let $I = \int e^x \sin x \sin 2x dx$

$$= \frac{1}{2}\int e^x(\cos x - \cos 2x)dx$$

$$= \frac{1}{2}\int e^x \cos x dx - \int e^x \cos 2x dx$$

$$= \frac{1}{2}e^x\frac{(\cos x + \sin x)}{1^2+1^2} - \frac{1}{2}e^x\frac{(\cos 3x + 3\sin 3x)}{1^2+3^2} + C$$

$$= \frac{1}{4}e^x(\cos x + \sin x) - \frac{1}{20}e^x(\cos 3x + 3\sin 3x) + C$$

Ans to Q No 10 : We have $\int e^x(\sin x+\cos x)dx = \int e^x \sin x dx + \int e^x \cos x dx$

$$= \sin x e^x - \int \cos x e^x dx + \int \cos x e^x dx$$

$$= \sin x e^x + C$$

Ans to Q No 11 : Let $I = \int e^x \sec x(1+\tan x)dx = \int e^x(\sec x + \sec x \tan x)dx$

Also, let $\sec x = f(x) \Rightarrow \sec x \tan x = f'(x)$

We know that, $\int e^x\{f(x)+f'(x)\}dx = e^x f(x)+C$

$\therefore I = e^x \sec x + C$.

Ans to Q No 12 : Let $\log x = t$,then $x = e^t \Rightarrow dx = e^t dt$

$$\therefore I = \int(\sin t)e^t dt = -e^t \cos t - \int e^t(-\cos t)dt$$

$$= -e^t \cos t + \int e^t \cos t dt$$

$$= -e^t \cos t + \left[e^t \sin t - \int e^t \sin t dt\right]$$

$$\Rightarrow I = -e^t \cos t + e^t \sin t - I$$

$$\Rightarrow 2I = -e^t \cos t + e^t \sin t$$

$$\Rightarrow I = \frac{e^t}{2}(\sin t - \cos t) + C$$

Hence, $\int \sin(\log x)dx = \frac{x}{2}\{\sin(\log x) - \cos(\log x)\} + C$

Ans to Q No 13 : Let $3x-2 = A\frac{d}{dx}(x^2+x+1)+B$, A and B are constants.

$$\Rightarrow 3x-2 = A(2x+1)+B$$

On comparing the coefficients,we get $3 = 2A \Rightarrow A = \frac{3}{2}$

Also, $-2 = A+B \Rightarrow B = -2-\frac{3}{2}$

$$\Rightarrow B = -\frac{7}{2}$$

$$\therefore \int(3x-2)\sqrt{x^2+x+1}\,dx = \int\left[\frac{3}{2}(2x+1)-\frac{7}{2}\right]\sqrt{x^2+x+1}dx$$

$$= \frac{3}{2}\int(2x+1)\sqrt{x^2+x+1}dx - \frac{7}{2}\int(x^2+x+1)dx$$

$$= \frac{3}{2}\int(x^2+x+1)^{\frac{1}{2}}(2x+1)dx - \frac{7}{2}\int\sqrt{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}dx$$

$$= \frac{3}{2}.\frac{2}{3}\left(x^2+x+1\right)^{\frac{3}{2}} - \frac{7}{2}\left[\frac{1}{2}\left(x+\frac{1}{2}\right)\sqrt{\left\{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\right\}} + \frac{1}{2}.\frac{3}{4}\sinh^{-1}\left\{\frac{\left(x+\frac{1}{2}\right)}{\sqrt{\frac{3}{4}}}\right\}\right]$$

$$= \left(x^2+x+1\right)^{\frac{3}{2}} - \frac{7}{4}\left(x+\frac{1}{2}\right)\sqrt{\left\{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\right\}} - \frac{21}{16}\sinh^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+C$$

Ans to Q No 14 : Do yourself.

CHECK YOUR PROGRESS – 5

Ans to Q No 1 : Let $I = \int \frac{x-1}{(x-2)(x-3)}dx$

$$\frac{x-1}{(x-2)(x-3)} = \frac{A}{x-2}+\frac{B}{x-3} = \frac{A(x-3)+B(x-2)}{(x-2)(x-3)}$$

$$\therefore x-1 = A(x-3)+B(x-2)$$

Putting $x=3$ on both sides we get $B=2$. Again putting $x=2$ on both sides we get $1 = A(2-3) \therefore A=-1$.

$$\therefore I = -\int\frac{dx}{x-2}+2\int\frac{dx}{x-3}$$

$$= -\log(x-2)+2\log(x-3)+C$$

Ans to Q No 2 : Let $I = \int \frac{x}{x^2-12x+35}dx = \int \frac{x}{(x-7)(x-5)}dx$

Now, $\frac{x}{(x-7)(x-5)} = \frac{A}{x-7}+\frac{B}{x-5} = \frac{A(x-5)+B(x-7)}{(x-7)(x-5)}$

$$\therefore x = A(x-5)+B(x-7)$$

Putting $x=7$ and $x=5$ successively on both sides we get, $A=\frac{7}{2}$, $B=-\frac{5}{2}$.

$$\therefore I = \frac{7}{2}\int\frac{1}{x-7}dx - \frac{5}{2}\int\frac{1}{x-5}dx$$

$$= \frac{7}{2}\log(x-7)-\frac{5}{2}\log(x-5)+C.$$

Ans to Q No 3 : Let $I = \int \frac{3xdx}{x^2-x-2} = \int \frac{3xdx}{(x-2)(x+1)}$

$$\frac{3x}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1} = \frac{A(x+1)+B(x-2)}{(x-2)(x+1)}$$

$$\therefore 3x = A(x+1)+B(x-2)$$

Putting $x=2$ on both sides we get $6=3A \quad \therefore A=2$.

Again putting $x=-1$,we get $-3=-3B \quad \therefore B=1$

$$\therefore I = 2\int \frac{1}{x-2}dx + \int \frac{1}{x+1}dx$$

$$= 2\log(x-2)+\log(x+1)+C$$

Ans to Q No 4 : Let $\frac{1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$

$$= \frac{A(x+1)^2 + B(x^2+x)+Cx}{x(x+1)^2}$$

$$\therefore 1 = A(x+1)^2 + B(x^2+x)+Cx$$

Putting $x=0$ on both sides,we get $A=1$ and putting $x=-1$ on both sides we get, $1=-C$, or $C=-1$.

Equating coefficient of x^2 from both sides we get

$$A+B=0$$

or $$B=-A=-1$$

$$\therefore \int \frac{dx}{x(x+1)^2} = \int \frac{1}{x}dx - \int \frac{1}{x+1}dx - \int \frac{1}{(x+1)^2}dx$$

$$= \log x - \log(x+1) + \frac{1}{x+1} + C.$$

Ans to Q No 5 : Do yourself.

Ans to Q No 6 : Do yourself.

Ans to Q No 7 : Do yourself.

Ans to Q No 8 : Do yourself.

Ans to Q No 9 : Let $\frac{1}{x^3+x} = \frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$

Equating the co-efficients of x^2, x and constant terms,we get

$A+B=0, C=0, A=1$

On solving these equations, we obtain $A=1, B=-1, C=0$.

$$\therefore \frac{1}{x\left(x^2+1\right)} = \frac{1}{x} + \frac{-x}{x^2+1}$$

$$\Rightarrow \int \frac{1}{x\left(x^2+1\right)}dx = \int \left\{\frac{1}{x} - \frac{x}{x^2+1}\right\}dx = \log|x| - \frac{1}{2}\log\left|x^2+1\right| + C$$

Ans to Q No 10 : Do yourself.

Ans to Q No 11 : Do yourself.

Ans to Q No 12 : Do yourself.

Ans to Q No 13 : Let $I = \int \frac{1}{1-x^3}dx$

$$= \int \frac{1}{(1-x)\left(1+x+x^2\right)}dx$$

$$\frac{1}{(1-x)\left(1+x+x^2\right)} = \frac{A}{1-x} + \frac{Bx+C}{1+x+x^2}$$

$$\therefore 1 = A\left(1+x+x^2\right) + (Bx+C)(1-x) \quad \ldots (1)$$

Putting $x = 1$ in (1) we get $A = \frac{1}{3}$ and putting $x = 0$,we get

$$A + C = 1 \ \therefore C = 1 - \frac{1}{3} = \frac{2}{3}$$

Equating the co-efficients of x^2 on both sides we get

$$A - B = 0 \ \therefore B = \frac{1}{3}$$

$$\therefore I = \frac{1}{3}\int \frac{1}{1-x}dx + \frac{1}{3}\int \frac{x+2}{1+x+x^2}dx$$

$$= \frac{1}{3}\int \frac{1}{1-x}dx + \frac{1}{3}.\frac{1}{2}\int \frac{2x+1+3}{1+x+x^2}dx$$

$$= \frac{1}{3}\int \frac{1}{1-x}dx + \frac{1}{6}\int \frac{2x+1}{1+x+x^2}dx + \frac{1}{6}.3\int \frac{1}{1+x+x^2}dx$$

$$= -\frac{1}{3}\log(1-x) + \frac{1}{6}\log\left(1+x+x^2\right) + \frac{1}{2}\int \frac{1}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}dx$$

$$= -\frac{1}{3}\log(1-x) + \frac{1}{6}\log\left(1+x+x^2\right) + \frac{1}{2}.\frac{2}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}} + C$$

$$= -\frac{1}{3}\log(1-x) + \frac{1}{6}\log\left(1+x+x^2\right) + \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}} + C$$

Ans to Q No 14 : Do yourself.

Ans to Q No 15 : Do yourself.

Ans to Q No 16 : Let $I = \int \frac{xdx}{\left(x^2+a^2\right)\left(x^2+b^2\right)}$

Let $x^2 = z \quad \therefore 2xdx = dz$

$$\therefore I = \frac{1}{2}\int \frac{dz}{\left(z+a^2\right)\left(z+b^2\right)}$$

Let $\frac{1}{\left(z+a^2\right)\left(z+b^2\right)} = \frac{A}{z+a^2} + \frac{B}{z+b^2}$

$$\therefore 1 = A\left(z+a^2\right) + B\left(z+b^2\right) \quad \ldots (1)$$

Now, putting $z = -a^2$ and $z = -b^2$ successively in (1), we get

$$A = \frac{1}{a^2-b^2} \quad \text{and} \quad B = -\frac{1}{a^2-b^2}$$

$$\therefore I = \frac{1}{2\left(a^2-b^2\right)}\int \frac{1}{z+a^2}dz - \frac{1}{2\left(a^2-b^2\right)}\int \frac{dz}{z+b^2}$$

$$= \frac{1}{2\left(a^2-b^2\right)}\log\left(z+a^2\right) - \frac{1}{2\left(a^2-b^2\right)}\log\left(z+b^2\right) + C$$

$$= \frac{1}{2\left(a^2-b^2\right)}\log\frac{z+a^2}{z+b^2} + C$$

$$= \frac{1}{2\left(a^2-b^2\right)}\log\frac{x^2+a^2}{x^2+b^2} + C$$

Ans to Q No 17: Let $I = \int \frac{dx}{x^3-x^2-x+1} = \int \frac{dx}{(x+1)\left(x^2-1\right)}$

Now, $\frac{1}{(x+1)(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$

$$\therefore 1 = A(x-1)(x+1) + B(x+1) + C(x-1)^2 \quad \ldots (1)$$

Putting $x = 1$ in (1),we get $2B = 1$ or $B = \frac{1}{2}$

Again putting $x = -1$ in (1),we get $4C = 1$ or $C = \frac{1}{4}$

Equating coefficients of x^2 on both sides we get

$A+C=0$, $A=-C=-\frac{1}{4}$ or $A=-\frac{1}{4}$

$$\therefore I=-\frac{1}{4}\int\frac{dx}{x-1}+\frac{1}{2}\int\frac{dx}{(x-1)^2}+\frac{1}{4}\int\frac{dx}{x+1}$$

$$=-\frac{1}{4}\log(x-1)+\frac{1}{2}\frac{1}{x+1}+\frac{1}{4}\log(x+1)+C$$

Ans to Q No 18 : Do yourself.

Ans to Q No 19 : Do yourself.

Ans to Q No 20 : Let $x^2=y$, then $\frac{x^2}{(x^2+1)(x^2+4)}=\frac{y}{(y+1)(y+4)}$

Let $\frac{y}{(y+1)(y+4)}=\frac{A}{y+1}+\frac{B}{y+4}$ ... (1)

$\Rightarrow y=A(y+4)+B(y+1)$ ... (2)

Putting $y=-1$ and $y=-4$ successively in (2), we get $A=-\frac{1}{3}$ and $B=\frac{4}{3}$ Substituting the values of A and B in (1), we obtain

$$\frac{y}{(y+1)(y+4)}=-\frac{1}{3(y+1)}+\frac{4}{3(y+4)}$$

Replacing y by x^2, we obtain

$$\frac{x^2}{(x^2+1)(x^2+4)}=-\frac{1}{3(x^2+1)}+\frac{4}{3(x^2+4)}$$

$$\therefore\int\frac{x^2}{(x^2+1)(x^2+4)}dx=-\frac{1}{3}\int\frac{1}{(x^2+1)}dx+\frac{4}{3}\int\frac{1}{(x^2+4)}dx$$

$$=-\frac{1}{3}\tan^{-1}x+\frac{4}{3}\times\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+C$$

$$=-\frac{1}{3}\tan^{-1}x+\frac{2}{3}\tan^{-1}\left(\frac{x}{2}\right)+C$$

Ans to Q No 21 : Do yourself.

Ans to Q No 22 : Let $\sin x=t$ and $\cos x dx=dt$,we get

$$I = \int \frac{dt}{(2+t)(3+4t)}$$

Let $\frac{1}{(2+t)(3+4t)} = \frac{A}{2+t} + \frac{B}{3+4t}$ ... (i)

$\Rightarrow 1 = A(3+4t) + B(2+t)$ ... (ii)

Putting $3+4t=0$ i.e $t=-\frac{3}{4}$ in (ii), we get $1 = B\left(2-\frac{3}{4}\right) \Rightarrow B = \frac{4}{5}$

Putting $2+t=0$ i.e $t=-2$ in (ii), we get $1 = A(3-8) \Rightarrow A = -\frac{1}{5}$

Substituting the values of A and B in (i), we get

$$\frac{1}{(2+t)(3+4t)} = -\frac{1}{5}\frac{1}{(2+t)} + \frac{4}{5}\frac{1}{3+4t}$$

$$\therefore I = \int \frac{1}{(2+t)(3+4t)}dt = -\frac{1}{5}\int \frac{1}{2+t}dt + \frac{4}{5}\int \frac{1}{3+4t}dt$$

$$= -\frac{1}{5}\log|2+t| + \frac{4}{5}.\frac{1}{4}\log|3+4t| + C$$

$$= -\frac{1}{5}\log|2+\sin x| + \frac{1}{5}\log|3+4\sin x| + C$$

Ans to Q No 23 : Do yourself.

Ans to Q No 24 : Do yourself.

Ans to Q No 25 : We put $x^2 = t$ We have

$$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} = \frac{(t+1)(t+2)}{(t+3)(t+4)} = 1 + \frac{(t+1)(t+2)}{(t+3)(t+4)} - 1$$

$$= 1 + \frac{(t+1)(t+2)-(t+3)(t+4)}{(t+3)(t+4)}$$

Let $\frac{(t+1)(t+2)-(t+3)(t+4)}{(t+3)(t+4)} = \frac{A}{t+3} + \frac{B}{t+4}$

$\Rightarrow (t+1)(t+2)-(t+3)(t+4) = A(t+4) + B(t+3)$

Putting $t=-3,-4$ respectively, we get $A=2, B=-6$

$$\therefore \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} = 1 + \frac{2}{t+3} - \frac{6}{t+4}$$

$$= 1 + \frac{2}{x^2+3} - \frac{6}{x^2+4}$$

$$\Rightarrow \int \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}dx = \int dx + 2\int \frac{1}{x^2+3}dx - 6\int \frac{1}{x^2+4}dx$$

$$= x + \frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} - 3\tan^{-1}\frac{x}{2} + C.$$



1.9 MODEL QUESTIONS

A. Evaluate

1. $\int \frac{x^4}{x^2+1}dx$

2. $\int \left(1 + \frac{1}{1+x^2} - \frac{2}{\sqrt{1-x^2}} + \frac{5}{x\sqrt{x^2-1}} + 3^x\right)dx$

3. $\int \tan^{-1}\left(\frac{\sin 2x}{1+\cos 2x}\right)dx$

4. $\int \frac{4^x + 3^x}{5^x}dx$

B. Integrate the following–

1. $\int \frac{\cos 2x + x + 1}{x^2 + 2x + \sin 2x}dx$

2. $\int \frac{1 - \sin 2x}{x + \cos^2 x}dx$

3. $\int \frac{x+1}{x(x + \log^x)}dx$

4. $\int (4x+2)\sqrt{x^2+x+1}\,dx$

5. $\int \cos^5 x\,dx$

6. $\int \sin^3 x \cos^5 x\,dx$

7. $\int \frac{1}{\sqrt{16x^2+25}}dx$

8. $\int \frac{5}{x^2-8x+25}dx$

9. $\int \frac{1}{3-2x-x^2}dx$

10. $\int \frac{1}{x(x^n+1)}dx$

11. $\int \frac{1}{\sqrt{x^2+4x+2}}dx$

12. $\int \frac{1}{\sqrt{6+x-x}}dx$

13. $\int \frac{x+2}{2x^2+6x+5}dx$

14. $\int \frac{x^2+2}{x^2-5x+6}dx$

15. $\int \frac{2x+1}{\sqrt{x^2+2x-1}}dx$

16. $\int \frac{3x+1}{\sqrt{5-2x-x^2}}dx$

17. $\int \frac{1}{5+4\cos x}dx$

18. $\int \frac{1}{5-4\sin x}dx$

C. Evaluate

1. $\int \tan^{-1} x\,dx$ 2. $\int x \sin^{-1} x\,dx$ 3. $\int x^2 e^{-x}\,dx$ 4. $\int x^2 \sin^2 x\,dx$

5. $\int \log_{10}^{x} dx$ 6. $\int \log_{e}^{x} dx$ 7. $\int \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)dx$ 8. $\int e^{-x}dx \cos x\,dx$

9. $\int e^{ax}dx \cos x\,dx$ 10. $\int \sqrt{1+x-2x^2}\,dx$ 11. $\int (3x-2)\sqrt{x^2-x+1}\,dx$

12. $\int 2^x \sin x\,dx$ 13. $\int \sqrt{5x^2+8x+4}\,dx$

D. Evaluate

1. $\int \frac{3x+2}{x^3-6x^2+11x-6}dx$ 2. $\int \frac{x^2+x-1}{x^2+x-6}dx$ 3. $\int \frac{3x}{(x^2-4)(x+1)}dx$

4. $\int \frac{x}{(1+x)(1+x^2)}dx$ 5. $\int \frac{2x}{(x^2+1)(x^2+3)}dx$ 6. $\int \frac{2x+1}{(x+2)(x-3)^2}dx$

7. $\int \frac{dx}{x(x^4+2)}$ 8. $\int \frac{dx}{1-x^3}$

*** ***** ***

UNIT 2 : DEFINITE INTEGRALS

UNIT STRUCTURE

2.1 Learning Objectives

2.2 Introduction

2.3 Definite integral– Integration as the Limit of a Sum

2.4 Fundamental Theorem of Integral Calculus

2.5 Summation of Series with the Help of Integration

2.6 Evaluation of Definite Integrals by Using Properties

2.8 Let Us Sum Up

2.9 Further Readings

2.10 Answer to Check Your Progress

2.11 Model Questions

2.1 LEARNING OBJECTIVES

After going through this unit, you will be able to understand

- the meaning of definite integral as the limit of a sum,
- the fundamental theorem of integral calculus,
- the process of evaluating definite integration,
- the method of expressing the limits of sums of certain series as definite integrals and thus evaluate them,
- the properties of definite integrals

2.2 INTRODUCTION

In the preceding unit, we discussed the development of integration as the inverse of differentiation. In this unit we will introduce integration as the limit of a sum.

2.3 DEFINITE INTEGRALS – INTEGRATION AS THE LIMIT OF A SUM

Let f be a bounded function defined in $[a, b]$. We divide the interval $[a, b]$ into a finite subintervals, not necessarily of all equal length, by

means of arbitrary set of points $a = x_0, x_1, x_2,.....,x_{r-1}, x_r,..., x_{n-1}, x_n = b$ where $x_0<x_1<x_2<... ... ...<x_{r-1}<x_r<... ... ...<x_{n-1}<x_n$.

This subdivision is called a "partition" of the interval.

Let $\Delta x_r = x_r - x_{r-1}$

Then $\Delta x_1 = x_1 - x_0$, $\Delta x_2 = x_2 - x_1, ...$, $\Delta x_n = x_n - x_{n-1}$ are the respective sub-intervals in which the interval $[a,b]$ is divided. Let us take $\xi_1, \xi_2, ..., \xi_r, ..., \xi_n$ to be arbitrary points in the respective sub-intervals. Each point so selected may be an end point or an interior point of the respective sub-intervals. So we have

$x_0 \le \xi_1 \le x_1$, $x_1 \le \xi_2 \le x_2, ...$ and so on.

Now let us form the sum

$$S_n = \sum_{r=1}^{n} f(\xi_r)\Delta x_r$$

$$= f(\xi_1)\Delta x_1 + f(\xi_2)\Delta x_2 + ... + f(\xi_n)\Delta x_n$$

Here S_n will have different values corresponding to different partitions and different ways of choosing a point in a subinterval.

Now let us increase n indefinitely in such a manner that the length of the greatest of the sub-interval tends to zero.

If S_n has a limit as $n \to \infty$ and each $\Delta x_r \to 0$(for r = 1, 2, 3,... ... ..., n), and the limit is the same for all possible partitions, then the limit is called the definite integral of $f(x)$ from a to b. The definite integral is denoted by $\int_a^b f(x)dx$

Hence $\int_a^b f(x)dx = \lim_{n\to\infty}(S_n)$

$$= \lim_{\Delta x_r \to 0} \sum_{r=1}^{n} f(\xi_r)\Delta x_r \quad ..(I)$$

$$= \lim_{n\to\infty}\left[f(\xi_1)\Delta x_1 + f(\xi_2)\Delta x_2 + ... + f(\xi_n)\Delta x_n\right]$$

If the limit of S_n exists, $f(x)$ is said to be integrable over the interval $[a,b]$. The numbers a and b are called the "lower limit" and the "upper limit" respectively of the integral and $f(x)$ is called the integrand.

Note1 : Let us consider that the length of each subinterval to be equal and the length be h.

That is $\Delta x_1 = \Delta x_2 = ... = \Delta x_n (= h)$.

Further, we take the points $\xi_1, \xi_2, \ldots\ldots\ldots \xi_n$ at the right hand end points of the respective sub-intervals.

i.e. $\xi_1 = x_1 = x_0 + h = a + h$

$\xi_2 = x_2 = (a+h) + h = a + 2h$

and so on. Finally $\xi_n = x_n = b = a + nh$

Also from $b = a + nh$ we get $n = \dfrac{b-a}{h}$

For here we see that for $n \to \infty$, we have $h \to 0$.

Therefore, from (1) above we can write

$$\int_a^b f(x)dx = \lim_{n\to\infty} S_n$$

$$= \lim_{h\to 0}\left[hf(a+h) + hf(a+2h) + \ldots\ldots\ldots + hf(a+nh)\right]$$

$$= \lim_{h\to 0} h\sum_{r=1}^{n} f(a+rh)$$

Note 2: If we take the points $\xi_1, \xi_2, \ldots\ldots\ldots \xi_n$ at the left hand end points, then we get

$$\int_a^b f(x)dx = \lim_{h\to 0} h\sum_{r=0}^{n-1} f(a+rh)$$

ILLUSTRATIVE EXAMPLES

Example 1: Find the value of $\int_0^1 x^2 dx$ by the method of integration as the limit of a sum.

Solution : Here $a = 0$, $b = 1$ and $f(x) = x^2$

$\therefore b = a + nh \Rightarrow 1 = nh$

Also $f(a+rh) = (a+rh)^2 = (rh)^2$ where $r = 1,2,3,\ldots\ldots\ldots n$

Now $\int_a^b f(x)dx = \lim_{h\to 0} h\sum_{r=1}^{n} f(a+rh)$

$$\Rightarrow \int_0^1 x^2 dx = \lim_{h\to 0} h\sum_{r=1}^{n} (rh)^2$$

$$= \lim_{h\to 0} h\left\{(h)^2 + (2h)^2 + (3h)^2 + \ldots\ldots\ldots + (nh)^2\right\}$$

$$= \lim_{h\to 0} h^3\left\{1^2 + 2^2 + 3^2 + \ldots\ldots\ldots + n^2\right\}$$

$$= \lim_{h\to 0} h^3 \frac{n(n+1)(2n+1)}{6} = \lim_{h\to 0} \frac{nh(nh+h)(2nh+h)}{6}$$

$$= \lim_{h\to 0} \frac{(1+h)(2+h)}{6}$$

$$= \frac{2}{6} = 1/3$$

Example 2 : From the definition of a definite integral as the limit of a sum, evaluate $\int_a^b e^x\,dx$.

Solution: Here $f(x) = e^x$ and $b = a + nh$

$\therefore nh = b - a$ and so $n \to \infty$ as $h \to 0$.

$$\therefore \int_a^b e^x dx = \lim_{h\to 0} h\sum_{r=0}^{n-1} f(a+rh)$$

$$= \lim_{h\to 0} h\left\{e^a + e^{a+h} + e^{a+2h} + \ldots\ldots\ldots + e^{a+(n-1)h}\right\}$$

$$= \lim_{h\to 0} he^a\left(1 + e^h + e^{2h} + \ldots\ldots\ldots + e^{(n-1)h}\right)$$

$$= \lim_{h\to 0} he^a \frac{\left(e^h\right)^n - 1}{e^h - 1}$$

$$= \lim_{h\to 0} he^a \frac{e^{nh} - 1}{e^h - 1}$$

$$= \lim_{h\to 0} he^a \frac{e^{b-a} - 1}{e^h - 1}$$

$$= e^a\left(e^{b-a} - 1\right).\lim_{h\to 0} \frac{h}{e^h - 1} \qquad \left(\text{Since } \lim_{h\to 0} \frac{e^h - 1}{h} = 1\right)$$

$$= \left(e^b - e^a\right) \times \frac{1}{\lim\limits_{h\to 0} \frac{e^h - 1}{h}}$$

$$= e^b - e^a$$

2.4 FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS

If (i) $\int_a^b f(x)dx$ exists and

(ii) there exists a function $\phi(x)$ such that $\phi'(x) = f(x)$ for $x \in [a,b]$, then

$$\int_a^b f(x)dx = \phi(b) - \phi(a)$$

Proof : As discussed earlier, we can write

$$\int_a^b f(x)dx = \lim_{\Delta x_r \to 0} \sum_{r=1}^{n} f(\xi_r)\Delta x_r \quad \ldots (1)$$

Now $\phi(x)$ satisfies all the conditions of mean value theorem of differential calculus, and so there exists a point satisfying $x_{r-1} < \alpha_r < x_r$ such that

$$\begin{aligned}\phi(x_r) - \phi(x_{r-1}) &= (x_r - x_{r-1})\phi'(\alpha_r) \\ &= \Delta x_r . f(\alpha_r) \quad \ldots(2)\end{aligned}$$

Since (1) holds for every choice of ξ_r, so taking $\xi_r = \alpha_r$, we have from (1) and (2)

$$\sum_{r=1}^{n} f(\xi_r)\Delta x_r = \sum_{r=1}^{n} \{\phi(x_r) - \phi(x_{r-1})\} = \phi(b) - \phi(a)$$

and from here

$$\lim_{\Delta x_r \to 0} \sum_{r=1}^{n} f(\xi_r)\Delta x_r = \int_a^b f(x)dx = \phi(b) - \phi(a)$$

i.e., $\int_a^b f(x)dx = \phi(b) - \phi(a)$ proved.

Note: If F(*x*) is an integral of *f*(*x*), then $\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$

Here we donot write the constant of integration C. Because, if instead of F(*x*) we take $F(x) + C$ as an integral of *f*(*x*) for some value of C, then

$$\begin{aligned}\int_a^b f(x)dx &= [F(x) + C]_a^b = \{F(b) + C\} - \{F(a) + c\} \\ &= F(b) - F(a)\end{aligned}$$

ILLUSTRATIVE EXAMPLES

Example 1: Evaluate $\int_0^1 (3x^2 + x - 9)dx$

Solution: Here we first evaluate $\int (3x^2 + x - 9)dx$

then put the upper and the lower limits

So $$\int_0^1 (3x^2 + x - 9)dx = \left[3.\frac{x^3}{3} + \frac{x^2}{2} - 9x\right]_0^1$$

$$= (1)^3 + \frac{(1)^2}{2} - 9(1)$$

$$= -15/2$$

Example 2 : Evaluate $\int_0^{\pi/2} \sin^2 x\,dx$

Solution : $\int_0^{\pi/2} \sin^2 x\,dx = \int_0^{\pi/2} \frac{1}{2}(1-\cos 2x)dx$

$$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left(\frac{\pi}{2} - \frac{\sin\pi}{2}\right) - \frac{1}{2}\left(0 - \frac{\sin 0}{2}\right)$$

$$= \frac{\pi}{4}$$

Example 3: Evaluate $\int_0^{\pi/2} \frac{\cos x\,dx}{(1+\sin x)(2+\sin x)}$

Solution: Here we take $\sin x = t$

Then $\cos x\,dx = dt$

Also for $x = 0,\ t = 0$

and $x = \pi/2,\ t = 1$

$$\therefore \int_0^{\pi/2} \frac{\cos x\,dx}{(1+\sin x)(2+\sin x)} = \int_0^1 \frac{dt}{(1+t)(2+t)}$$

$$= \int_0^1 \left(\frac{1}{1+t} - \frac{1}{2+t}\right)dt$$ (Resolving into partial fraction)

$$= \left[\log(1+t) - \log(2+t)\right]_0^1$$

$$= (\log 2 - \log 3) - (\log 1 - \log 2)$$

$$= 2\log 2 - \log 3 = \log 4/3$$

Example 4: Evaluate $\int_0^{\pi/2} \frac{dx}{4+5\sin x}$

Solution : $\int_0^{\pi/2} \frac{dx}{4+5\sin x}$

$$= \int_0^{\pi/2} \frac{dx}{4\left(\cos^2\frac{x}{2} + \sin^2\frac{x}{2}\right) + 5\left(2\sin\frac{x}{2}\cos\frac{x}{2}\right)}$$

$$= \int_0^{\pi/2} \frac{\sec^2 \frac{x}{2}\, dx}{4\left(1+\tan^2 \frac{x}{2}\right) + 10 \tan x/2}$$

$$= \int_0^1 \frac{2dt}{4(1+t^2)+10t} \qquad \text{(We take here } \tan\frac{x}{2} = t\text{)}$$

$$= \int_0^1 \frac{dt}{2t^2+5t+2} = \int_0^1 \frac{dt}{2\left(t^2+\frac{5}{2}t+1\right)} = \frac{1}{2}\int_0^1 \frac{dt}{\left(t+\frac{5}{4}\right)^2 - \left(\frac{3}{4}\right)^2}$$

$$= \frac{1}{2}\left[\frac{1}{2.\frac{3}{4}} \log \frac{\left(t+\frac{5}{4}\right)-\frac{3}{4}}{\left(t+\frac{5}{4}\right)+\frac{3}{4}}\right]_0^1 = \frac{1}{3}\left[\log\left(\frac{2t+1}{2t+4}\right)\right]_0^1$$

$$= \frac{1}{3}\left(\log\frac{3}{6} - \log\frac{1}{4}\right) = \frac{1}{3}\log 2$$

2.5 SUMMATION OF SERIES WITH THE HELP OF INTEGRATION

We have already found that

$$\int_a^b f(x)dx = \lim_{h\to 0} h\sum_{r=1}^{n} f(a+rh) \text{ where } b = a+nh$$

Now, in particular, if $a = 0$, $b = 1$ then we can write

$$\int_0^1 f(x)dx = \lim_{h\to 0} h\sum_{r=1}^{n} f(rh) \qquad \text{... (i)}$$

Since $b = a + nh$ becomes $1 = nh$ i.e. $h = \frac{1}{n}$. So we get $h \to 0$ is equivalent to $n \to \infty$. Therefore from (i) we get

$$\lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r}{n}\right) = \int_0^1 f(x)dx$$

ILLUSTRATIVE EXAMPLES

Example 1: Find $\lim_{n\to\infty}\left[\frac{n}{n^2} + \frac{n}{1^2+n^2} + \frac{n}{2^2+n^2} + \ldots\ldots\ldots + \frac{n}{(n-1)^2+n^2}\right]$

Solution : Here $\lim_{n\to\infty}\left[\frac{n}{n^2}+\frac{n}{1^2+n^2}+\frac{n}{2^2+n^2}+\ldots\ldots+\frac{n}{(n-1)^2+n^2}\right]$

$$=\lim_{n\to\infty}\frac{1}{n}\left[\frac{n^2}{0^2+n^2}+\frac{n^2}{1^2+n^2}+\frac{n^2}{2^2+n^2}+\ldots\ldots+\frac{n^2}{(n-1)^2+n^2}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[\frac{1}{\left(\frac{0}{n}\right)^2+1}+\frac{1}{\left(\frac{1}{n}\right)^2+1}+\frac{1}{\left(\frac{2}{n}\right)^2+1}+\ldots\ldots+\frac{1}{\left(\frac{n-1}{n}\right)^2+1}\right]$$

(Dividing both N^r and D^r by n^2)

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n-1}\frac{1}{1+\left(\frac{r}{n}\right)^2}$$

$$=\int_0^1\frac{1}{1+x^2}dx$$

$$=\left[\tan^{-1}x\right]_0^1=\tan^{-1}(1)-\tan^{-1}(0)=\frac{\pi}{4}-0=\pi/4$$

Example 2 : Find $\lim_{n\to\infty}\left\{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}\right)\ldots\ldots\ldots\left(1+\frac{4}{n}\right)\right\}^{1/n}$

Solution : Let $P=\lim_{n\to\infty}\left\{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}\right)\ldots\ldots\ldots\left(1+\frac{4}{n}\right)\right\}^{1/n}$

$$\therefore \log P=\lim_{n\to\infty}\frac{1}{n}\left[\log\left(1+\frac{1}{n}\right)+\log\left(1+\frac{2}{n}\right)+\ldots\ldots\ldots+\log\left(1+\frac{n}{n}\right)\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\log\left(1+\frac{r}{n}\right)$$

$$=\int_0^1\log(1+x)dx$$

$$=\left[x\log(1+x)\right]_0^1-\int_0^1\frac{1}{1+x}.x\,dx$$

$$=\log 2-\int_0^1\left(1-\frac{1}{1+x}\right)dx$$

$$=\log 2-\left[x-\log(1+x)\right]_0^1=\log 2-(1-\log 2)$$

$$= 2\log 2 - 1 = \log 4 - \log e = \log\left(4/e\right)$$

$$\therefore P = 4/e$$



CHECK YOUR PROGRESS - 1

Q 1 : Find the value of the following by the method of integration as the limit of a sum

(i) $\int_a^b x\,dx$ (ii) $\int_0^1 x\,dx$ (iii) $\int_a^b x^2\,dx$

(iv) $\int_0^1 x^3\,dx$ (v) $\int_0^1 (ax+b)\,dx$

Q 2 : Evaluate the integrals

(i) $\int_0^1 xe^x\,dx$ (ii) $\int_0^1 x^2e^x\,dx$ (iii) $\int_0^{\pi/2} \sin 2x\,dx$

(iv) $\int_0^1 xe^{x^2}\,dx$ (v) $\int_0^1 \sin^{-1} x\,dx$ (vi) $\int_0^{\pi/2} x\sin x\,dx$

(vii) $\int_0^{\sqrt{e}} x\log x\,dx$ (viii) $\int_0^{\pi/2} x^3\sin x\,dx$

Q 3 : Evaluate

(i) $\int_0^{\pi/2} \cos 3\theta\cos 2\theta\,d\theta$ (ii) $\int_a^b \frac{1}{x}\log x\,dx$

(iii) $\int_0^a \frac{dx}{\sqrt{(a^2+x^2)^3}}$ (iv) $\int_0^2 \sqrt{2x-2^2}\,dx$

(v) $\int_0^1 \frac{dx}{x^2+4x+5}$ (vi) $\int_1^2 \frac{dx}{\sqrt{x^2+x-2}}$ (vii) $\int_1^e \frac{dx}{x(1+\log x)}$

(viii) $\int_1^{\log 2} \frac{e^x}{1+e^x}dx$ (ix) $\int_2^3 \frac{dx}{(x-1)\sqrt{x^2-2x}}$ (x) $\int_1^2 \frac{dx}{(x)(1+2x)^2}$

Q 4 : Show that

(i) $\int_0^{\pi/2} \sin^2 x\,dx = \int_0^{\pi/2} \cos^2 x\,dx = \frac{\pi}{4}$

(ii) $\int_0^a \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx = 2a\tan^{-1} a - \log\sqrt{1+a^2}$

(iii) $\int_1^2 (\log x)^2\,dx = 2 - 4\log 2 + 2(\log 2)^2$

(iv) $\int_0^{\pi/2} \sin x \sin 2x \, dx = 2/3$

(v) $\int_0^{\pi/2} \cos^4 x \sin^3 x dx = \frac{2}{35}$

Q 5 : Evaluate the limit–

(i) $\lim_{n\to\infty}\left[\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\ldots\ldots\ldots+\frac{1}{2n}\right]$

(ii) $\lim_{n\to\infty}\left[\frac{n}{1^2+n^2}+\frac{4}{2^2+n^2}+\ldots\ldots\ldots+\frac{1}{2n}\right]$

(iii) $\lim_{n\to\infty}\left[\frac{1}{\sqrt{(n^2-1^2)}}+\frac{1}{\sqrt{(n^2-2^2)}}+\ldots\ldots\ldots+\frac{1}{\sqrt{\left[n^2-(n-1)^2\right]}}\right]$

(iv) $\lim_{n\to\infty}\sum_{r=1}^{n}\frac{r^3}{r^4+n^4}$

2.6 EVALUATION OF DEFINITE INTEGRALS BY USING PROPERTIES

In the previous section, we have discussed some methods of evaluating definite integrals. Now we give some important properties of definite indegrals which will be useful for evaluating the definite integrals more easily. The properties are given below,

P.1 $\int_a^b f(x)dx = \int_a^b f(z)dz$

P.2 $\int_a^b f(x)dx = -\int_b^a f(x)dx$

P.3 $\int_a^b f(x)dx = \int_b^c f(x)dx + \int_c^b f(x)dx, \quad (a < c < b)$

P.4 $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

P.5 $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$

P.6 $\int_0^{na} f(x)dx = n\int_0^a f(x)dx$ if $f(a+x) = f(x)$

P.7 $\int_a^a f(x)dx = \int_0^a [f(x)+f(-x)]dx$

Proof (P.1) Let $\int f(x)dx = \phi(x)$ then $\int f(z)dz = \phi(z)$

Now $\int_a^b f(x)dx = [\phi(x)]_a^b = \phi(b) - \phi(a)$... ... ...(i)

and $\int_a^b f(z)dt = [\phi(z)]_a^b = \phi(b) - \phi(a)$... ... ...(ii)

$\therefore$ From (i) and (ii), $\int_a^b f(x)dx = \int_a^b f(z)dz$

Proof (P. 2) $\int_a^b f(x)dx = [\phi(x)]_a^b = \phi(b) - \phi(a)$ and

$$-\int_b^a f(x)dx = -[\phi(x)]_b^a = -[\phi(a) - \phi(b)] = \phi(b) - \phi(a)$$

$$\therefore \int_a^b f(x)dx = -\int_b^a f(x)dx$$

Proof (P. 3) Let $\int f(x)dx = \phi(x)$

$$\therefore \text{LHS} = \int_a^b f(x)dx = [\phi(x)]_a^b = \phi(b) - \phi(a)$$

$$\begin{aligned}\text{RHS} &= \int_a^c f(x)dx + \int_c^b f(x)dx \\ &= [\phi(x)]_a^c + [\phi(x)]_c^b \\ &= [\phi(c) - \phi(a)] + [\phi(b) - \phi(c)] \\ &= \phi(b) - \phi(a)\end{aligned}$$

$$\therefore \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

where $a < c < b$.

Corollary of (P.3) The above property can be generalised as below–

$$\int_a^b f(x)dx = \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + \ldots\ldots\ldots + \int_{c_{n-1}}^{c_n} f(x)dx + \int_{c_n}^b f(x)dx$$

where $a < c_1 < c_2 < \ldots\ldots\ldots < c_{n-1} < c_n < b$.

Proof (P.4) Here we put $a - x = z$ on the R.H.S.

$\therefore$ R.H.S. $= \int_0^a f(a - x)dx$

For $a - x = z$

$dx = -dz$

and $x = 0 \Rightarrow z = a$

$x = a \Rightarrow z = 0$

$$= \int_a^0 f(z)(-dz)$$

$$= -\int_a^0 f(z)dz = \int_0^a f(z)dz, \quad \text{by properly (P.2)}$$

$$= \int_0^a f(x)dx, \quad \text{by property (P.1)}$$

$$= \text{LHS}.$$

$$\therefore \int_0^a f(x)dx = \int_0^a f(a - x)dx$$

Proof (P.5) We can write,

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_a^{2a} f(x)dx, \text{ by property P.3 ... ...(i)}$$

In the 2nd integral of RHS of (i), we put $x = 2a - z$, then $dx = -dz$.

Also for $x = a$, $z = a$ and for $x = 2a$, $z = 0$

$\therefore$ 2nd integral of RHS of (i)

$$= \int_a^{2a} f(x)dx = \int_a^0 f(2a - z)(-dz) = -\int_a^0 f(2a - z)dx$$

$$= \int_0^a f(2a - z)dz,$$

$$= \int_0^a f(2a - x)dz,$$

$\therefore$ (i) becomes

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a - x)dx$$

Corollary 5 (i) If $f(2a - x) = f(x)$ then

$$\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx$$

Corollary 5 (ii) If $f(2a - x) = -f(x)$, then

$$\int_0^{2a} f(x)dx = 0$$

Proof of (P.6) We can write

$$\int_0^{na} f(x)dx = \int_0^a f(x)dx + \int_a^{2a} f(x)dx + \ldots\ldots\ldots + \int_{(n-1)a}^{na} f(x) \text{ ... ...(i)}$$

Now let us take $x = z + a$

Then $\int_a^{2a} f(x)dx = \int_0^a f(z + a)dz = \int_0^a f(x + a)dx = \int_0^a f(x)dx$

Similarly we can show that $\int_{2a}^{3a} f(x)dx = \int_0^a f(x)dx$ and so on.

Thus, each of the integrals on the right hand side of (i) can be shown to be equal to $\int_0^a f(x)dx$

$\therefore \int_0^{na} f(x)dx = n\int_0^a f(x)dx$

Proof (P.7) We can write

$$\int_{-a}^a f(x)dx = \int_{-a}^0 f(x)dx + \int_0^a f(x)dx \text{ ...(i)}$$

Now for the 1st integral of RHS of (i), we take $x = -z$. Then the 1st integral becomes

$$\int_{-a}^0 f(x)dx = \int_a^0 f(-z)(-dz)$$

$$= -\int_a^0 f(-z)dz$$

$$= \int_0^a f(-z)dz,$$

$$= \int_0^a f(-x)dx$$

$\therefore$ (i) becomes

$$\int_{-a}^{a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(x)dx$$

$$= \int_0^a [f(x) + f(-x)]dx$$

Corollary 7 (i) If $f(-x) = f(x)$,i.e. if f is even, then

$$\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$$

Corollary 7 (ii) If $f(-x) = -f(x)$ i.e. if f is odd, then

$$\int_{-a}^{a} f(x)dx = 0$$

ILLUSTRATIVE EXAMPLES

Example 1. Show that $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx = \frac{\pi^2}{4}$

Proof : Let $I = \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$

$$= \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1 + \{\cos(\pi - x)\}^2} dx$$

$$= \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x} dx$$

$$= \pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx - \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$$

$$= \pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx - I$$

$$\therefore 2I = \pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx$$

$$= \pi \int_1^{-1} \frac{-dz}{1+z^2}$$ (Let us put $z = \cos x$)

$$= -\pi \int_1^{-1} \frac{dz}{1+z^2}$$ Then $dz = -\sin x\, dx$

$$= \pi \int_{-1}^{1} \frac{dz}{1+z^2}$$ Also for $x = 0, z = 1$ and $x = \pi,\ z = -1$

$$= 2\pi\int_0^1 \frac{1}{1+z^2}dz$$

$$= 2\pi\left[\tan^{-1}(z)\right]_0^1$$

$$= 2\pi\left[\tan^{-1}(1) - \tan^{-1}(0)\right]$$

$$\Rightarrow 2I = 2\pi\left(\frac{\pi}{4} - 0\right)$$

$$\therefore I = \frac{\pi^2}{4} \qquad \text{i.e. } \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx = \frac{\pi^2}{4}$$

Example 2: Show that $\int_0^a \frac{dx}{x+\sqrt{(a^2-x^2)}} = \pi/4$

Solution: Let us take $x = a\sin\theta$, then $dx = a\cos\theta d\theta$

Also for $x = 0, \theta = 0$

and for $x = a, \theta = \pi/2$

Then the given integral (say I) becomes

$$I = \int_0^{\pi/2} \frac{a\cos\theta d\theta}{a\sin\theta + \sqrt{a^2 - a^2\sin^2\theta}}$$

$$= \int_0^{\pi/2} \frac{\cos\theta d\theta}{\sin\theta + \cos\theta} = \int_0^{\pi/2} \frac{\cos\left(\pi/2 - \theta\right)}{\sin\left(\frac{\pi}{2} - \theta\right) + \cos\left(\frac{\pi}{2} - \theta\right)}d\theta$$

$$= \int_0^{\pi/2} \frac{\sin\theta}{\cos\theta + \sin\theta}d\theta$$

Now I + I $= \int_0^{\pi/2} \frac{\cos\theta d\theta}{\sin\theta + \cos\theta} + \int_0^{\pi/2} \frac{\sin\theta}{\cos\theta + \sin\theta}d\theta$

$$\Rightarrow 2I = \int_0^{\pi/2} \frac{\cos\theta + \sin\theta}{\cos\theta + \sin\theta}d\theta = \int_0^{\pi/2} d\theta$$

$$= \left[\theta\right]_0^{\pi/2} = \pi/2$$

$$\therefore I = \frac{\pi}{4} \qquad \text{i.e. } \int_0^a \frac{dx}{x+\sqrt{a^2-x^2}} = \pi/4$$

Example 3: Show that $\int_0^1 \frac{\log(1+x)}{1+x^2}dx = \frac{\pi}{8}\log 2$

Solution: Let us take $x = \tan\theta$, then $dx = \sec^2\theta d\theta$

Also if $x = 0$ then $\theta = 0$

and if $x = 1$ then $\theta = \pi/4$

Then the given integral (say I) becomes

$$I = \int_0^{\pi/4} \frac{\log(1+\tan\theta)}{\sec^2\theta}\sec^2\theta\, d\theta = \int_0^{\pi/4} \log(1+\tan\theta)\, d\theta$$

$$= \int_0^{\pi/4} \log\left\{1+\tan\left(\frac{\pi}{4}-\theta\right)\right\} d\theta$$

$$= \int_0^{\pi/4} \log\left(1+\frac{1-\tan\theta}{1+\tan\theta}\right) d\theta$$

$$= \int_0^{\pi/4} \log\frac{2}{1+\tan\theta}\, d\theta$$

$$= \int_0^{\pi/4} \log 2\, d\theta - \int_0^{\pi/4} \log(1+\tan\theta)\, d\theta$$

$$= \log 2 \int_0^{\pi/4} d\theta - I$$

$$\Rightarrow 2I = (\log 2)[\theta]_0^{\frac{\pi}{4}} = (\log 2)\frac{\pi}{4} = \frac{\pi}{4}\log 2$$

$$\therefore\ I = \frac{\pi}{8}\log 2$$



CHECK YOUR PROGRESS - 2

Q. Prove the followings :

1. $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}\, dx = \frac{\pi}{4}$

2. $\int_0^{\pi/2} \log\sin x\, dx = \int_0^{\pi/2} \log\cos x\, dx = \frac{\pi}{2}\log\left(\frac{1}{2}\right)$

3. $\int_0^{\pi} \frac{x\, dx}{a^2\cos^2 x + b^2\sin^2 x} = \frac{\pi^2}{2ab}$

4. $\int_0^{\pi/4} \log(1+\tan\theta)\, d\theta = \frac{\pi}{8}\log 2$

5. $\int_{\frac{-\pi}{2}}^{\pi/2} \sin^9 x\, dx = 0$

6. $\int_0^{\pi} \sin^8 x\cos^7 x\, dx = 0$

7. $\int_{-a}^{a} \phi(y^2)\, dy = 2\int_0^{a} \phi(y^2)\, dy$

8. $\int_0^{\pi/2} \frac{\cos x - \sin x}{\cos x \sin x + 1}\, dx = 0$

9. $\int_0^{\pi/2} (2\log\sin x - \log\sin 2x)\, dx = \frac{\pi}{2}\log\frac{1}{2}$

10. $\int_0^{\pi/2} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x} dx = \frac{\pi^2}{16}$

11. $\int_0^p \frac{\sqrt{x}}{\sqrt{x} + \sqrt{p - x}} dx = \frac{p}{2}$

12. $\int_0^{2a} \frac{f(x)dx}{f(x) + f(2a - x)} = a$

13. $\int_0^{\pi/2} (a \cos^2 x + b \sin^2 x) dx = \frac{\pi}{4}(a + b)$

14. $\int_0^{\pi/2} \sin 2x \log(\tan x) dx = 0$

15. $\int_0^{\pi} \frac{x \tan x dx}{\sec x + \cos x} = \frac{\pi^2}{4}$

16. $\int_0^{\pi/2} \frac{\cos x}{1 + \cos x + \sin x} dx = \frac{\pi}{4} - \frac{1}{2} \log 2$

17. $\int_0^{\pi/2} \log\left(\frac{3 + 5 \sin x}{3 + 5 \cos x}\right) dx = 0$

18. $\int_0^{\pi} \frac{xdx}{1 + \sin x} = \pi$

19. $\int_0^1 \frac{\log x}{\sqrt{1 - x^2}} dx = -\frac{\pi}{2} \log 2$



2.8 LET US SUM UP

- Here definite integral is introduced as the limit of a sum.
- $\int_a^b f(x)dx = \lim_{h \to 0} \sum_{r=1}^{n} f(a + rh)$
- If $\int_a^b f(x)dx$ exist and there exists $\phi(x)$ such that $\phi'(x) = f(x)$ for $x \in [a, b]$, then $\int_a^b f(x)dx = \phi(b) - \phi(a)$

  It is known as fundamental theorem of integral calculus.
- Properties of Definite integrals (from propertry P-1 to P-7) are given in 2.6.



2.9 FURTHER READINGS

1) *Integral Calculus,* Kishan, Hari, Atlantic Publishers & Distributors.New Delhi -110027.

2) *Integral Calculus*, Dhami, H.S. New age International Publishers, New Delhi - 110002.

3) *Text book of Integral Calculus*, Sharma, A.K. Discovery Publishing House, New Delhi -110002.

2.10 ANSWERS TO CHECK YOUR PROGRESS

CHECK YOUR PROGRESS – 1

Ans to Q No 1 : (i) We have

$$\int_a^b f(x)dx = \lim_{h\to 0} h[f(a)+f(a+h)+f(a+2h)+\cdots+f(a+(n-1)h)]$$

Here $f(x)=x$, $a=a$, $b=b$ and $h=\dfrac{b-a}{n} \Rightarrow nh=b-a$

$$\therefore \int_a^b f(x)dx = \lim_{h\to 0} h[a+(a+h)+(a+2h)+\cdots+(a+(n-1)h)]$$

$$= \lim_{h\to 0} h[(a+a+\cdots+a)+(h+2h+3h+\cdots+(n-1)h)]$$

$$= \lim_{h\to 0} h[na+h(1+2+3+\cdots+(n-1))]$$

$$= \lim_{h\to 0} h\left[na+h\frac{n(n-1)}{2}\right]$$

$$= \lim_{h\to 0} \left[(nh)a+\frac{nh(nh-h)}{2}\right]$$

$$= \lim_{h\to 0} \left[(b-a)a+\frac{(b-a)(b-a-h)}{2}\right]$$

$$= \left[(b-a)a+\frac{(b-a)(b-a)}{2}\right]$$

$$= (b-a)\left[a+\frac{b-a}{2}\right]$$

$$= (b-a)\left[\frac{2a+b-a}{2}\right]$$

$$= \frac{(b-a)(b+a)}{2}$$

$$= \frac{b^2 - a^2}{2}$$

(ii) Do Yourself.

(iii) We have

$$\int_a^b f(x)dx = \lim_{h\to 0} h\left[f(a) + f(a+h) + f(a+2h) + \cdots + f(a+(n-1)h)\right]$$

Here $f(x) = x^2$, $\quad h = \dfrac{b-a}{n} \Rightarrow nh = b - a$

$$\therefore \int_a^b f(x)dx = \lim_{h\to 0} h\left[a^2 + (a+h)^2 + (a+2h)^2 + \cdots + (a+(n-1)h)^2\right]$$

$$= \lim_{h\to 0} h\left[na^2 + 2ah(1+2+3+\cdots+(n-1)) + h^2\left(1^2+2^2+3^2+\cdots+(n-1)^2\right)\right]$$

$$= \lim_{h\to 0} h\left[na^2 + 2ah\frac{n(n-1)}{2} + h^2\frac{n(n-1)(2n-1)}{6}\right]$$

$$= \lim_{h\to 0}\left[(nh)a^2 + a(nh)(nh-h) + \frac{1}{6}(nh)(nh-h)(2nh-h)\right]$$

$$= \lim_{h\to 0}\left[(b-a)a^2 + a(b-a)(b-a-h) + \frac{1}{6}(b-a)(b-a-h)(2(b-a)-h)\right]$$

$$= \left[(b-a)a^2 + a(b-a)^2 + \frac{2}{6}(b-a)^3\right]$$

$$= \frac{b-a}{3}\left[3a^2 + 3a(b-a) + (b-a)^2\right]$$

$$= \frac{b-a}{3}\left(3a^2 + 3ab - 3a^2 + b^2 - 2ab + a^2\right)$$

$$= \frac{1}{3}(b-a)\left(a^2 + ab + b^2\right)$$

$$= \frac{1}{3}\left(b^3 - a^3\right).$$

(iv) Do Yourself.

(v) Here $f(x) = ax + b$, $nh = 1 - 0 = 1$

$$f(rh) = arh + b$$

$$\int_0^1 (ax+b)dx = \lim_{h\to 0}\sum_{r=1}^{n}(arh+b) \quad \text{where } nh = 1$$

$$= \lim_{h\to 0} h\left[ah\sum_{r=1}^{n} r + b\sum_{r=1}^{n} 1\right]$$

$$= \lim_{h\to 0} h\left[\frac{ahn(n+1)}{2} + nb\right] \text{ where } \sum_{r=1}^{n} r = \frac{n(n+1)}{2} \text{ and } \sum_{r=1}^{n} 1 = n.$$

$$= \lim_{h\to 0}\left[\frac{a.nh.(nh+h)}{2} + nhb\right]$$

$$= \lim_{h\to 0}\left[\frac{a.1.(1+h)}{2} + 1.b\right]$$

$$= \frac{a(1+0)}{2} + b$$

$$= \frac{a}{2} + b$$

$$= \frac{1}{2}(a+2b)$$

Ans to Q No 2 : (i) We have $\int_0^1 xe^x dx = \left[xe^x\right]_0^1 - \int_0^1 1.e^x dx$

$$= \left(1.e^1 - 0.e^0\right) - \left[e^x\right]_0^1$$

$$= \left(e^1 - 0\right) - \left(e - e^0\right)$$

$$= 1 \cdot$$

(ii) Do Yourself.

(iii) Do Yourself.

(iv) Let $I = \int_0^1 xe^{x^2} dx$

We put $x^2 = t \Rightarrow 2xdx = dt$

As $x = 0, t = 0$ and $x = 1, t = 1$

$$\therefore I = \frac{1}{2}\int_0^1 e^t dt = \frac{1}{2}\left[e^t\right]_0^1$$

$$= \frac{1}{2}[e-1]$$

(v) Let $I = \int_0^1 \sin^{-1} x dx = \int_0^1 \sin^{-1} x.1 dx$

Integrating by parts, we obtain

$$I = \left[\sin^{-1} x.x\right]_0^1 - \int_0^1 \frac{1}{\sqrt{1-x^2}} x dx$$

$$=\left[\sin^{-1}x.x\right]_0^1+\frac{1}{2}\int_0^1\frac{(-2x)}{\sqrt{1-x^2}}dx$$

Let $1-x^2=t\Rightarrow -2xdx=dt$

when $x=0,t=1$ and $x=1,t=0$

$$I=\left[\sin^{-1}x.x\right]_0^1+\frac{1}{2}\int_1^0\frac{dt}{\sqrt{t}}$$

$$=\left[\sin^{-1}x.x\right]_0^1+\frac{1}{2}\left[2\sqrt{t}\right]_1^0$$

$$=\sin^{-1}(1)+\left[-\sqrt{1}\right]$$

$$\therefore I=\frac{\pi}{2}-1$$

(vi) Do Yourself.

(vii) Do Yourself.

(viii) Do Yourself.

Ans to Q No 3 : (iii) Let $x=a\tan\theta\Rightarrow dx=a\sec^2\theta d\theta$

Also, $x=0\Rightarrow\theta=0$ and $x=a\Rightarrow\theta=\frac{\pi}{4}$

$$\therefore\int_0^a\frac{dx}{\left(a^2+x^2\right)^{\frac{3}{2}}}=\int_0^{\frac{\pi}{4}}\frac{a\sec^2\theta d\theta}{\left(a^2+a^2\tan^2\theta\right)^{\frac{3}{2}}}=\frac{1}{a^2}\int_0^{\frac{\pi}{4}}\cos\theta d\theta$$

$$=\frac{1}{a^2}\left[\sin\theta\right]_0^{\frac{\pi}{4}}$$

$$=\frac{1}{a^2}\left(\frac{1}{\sqrt{2}}-0\right)$$

$$=\frac{1}{\sqrt{2}a^2}$$

(ix) Let $x-1=\frac{1}{z}\ \therefore dx=-\frac{1}{z^2}dz$

$$\therefore\int\frac{dx}{(x-1)\sqrt{x^2-2x}}=\int\frac{dx}{(x-1)\sqrt{(x-1)^2-1}}$$

$$=\int\frac{-\frac{1}{z^2}dz}{\frac{1}{z}\sqrt{\frac{1}{z^2}-1}}$$

$$= -\int \frac{dz}{\sqrt{1-z^2}}$$

$$= -\sin^{-1} z$$

$$= -\sin^{-1}\left(\frac{1}{x-1}\right)$$

$$\therefore \int_2^3 \frac{dx}{(x-1)\sqrt{x^2-2x}} = \left[-\sin^{-1}\left(\frac{1}{x-1}\right)\right]_2^3$$

$$= -\sin^{-1}\frac{1}{2} + \sin^{-1} 1$$

$$= -\frac{\pi}{6} + \frac{\pi}{2} = \frac{\pi}{3}.$$

(x) Let $\frac{1}{x(1+2x)^2} = \frac{A}{x} + \frac{B}{1+2x} + \frac{C}{(1+2x)^2}$

$$\therefore 1 = A(1+4x+4x^2) + B(x+2x^2) + Cx \quad \ldots (1)$$

Putting $x = 0$ in (1) we get $A = 1$ and putting $x = -\frac{1}{2}$ we get $C = -2$.

Equating the coefficients of x^2 on both sides we get $4A + 2B = 0$

$$\therefore B = -2$$

$$\therefore \int_1^2 \frac{dx}{x(1+2x)^2} = \int_1^2 \left[\frac{1}{x} - 2\frac{1}{1+2x} - 2\frac{1}{(1+2x)^2}\right] dx$$

$$= \left[\log x - \log(1+2x) + \frac{1}{1+2x}\right]_1^2$$

$$= \log 2 - \log 5 + \frac{1}{5} + \log 3 - \frac{1}{3}$$

$$= \log\frac{6}{5} - \frac{2}{15}$$

Answer to Q No 4 : (i) Let $I = \int_0^{\frac{\pi}{2}} \cos^2 x dx$

$$I = \int_0^{\frac{\pi}{2}} \cos^2 x dx = \int_0^{\frac{\pi}{2}} \left(\frac{1+\cos 2x}{2}\right) dx$$

$$= \left[\frac{x}{2} + \frac{\sin 2x}{4}\right]_0^{\frac{\pi}{2}}$$

$$= \left[\left(\frac{\pi}{4} + \frac{\sin \pi}{4}\right) - (0+0)\right]$$

$$= \frac{\pi}{4}$$

$$I = \int_0^{\frac{\pi}{2}} \cos^2 x dx = \int_0^{\frac{\pi}{2}} \cos^2\left(\frac{\pi}{2} - x\right) dx = \int_0^{\frac{\pi}{2}} \sin^2 x dx = \frac{\pi}{4}.$$

(ii) $\int_0^a \sin^{-1} \frac{2x}{1+x^2} dx = \int_0^a 2\tan^{-1} x dx = 2\int_0^a \tan^{-1} x.1 dx$

Now, $\int \tan^{-1} x.1 dx = \tan^{-1} x.x - \int \frac{1}{1+x^2}.x.dx$

$$= x\tan^{-1} x - \frac{1}{2}\int \frac{2x}{1+x^2} dx$$

$$= x\tan^{-1} x - \frac{1}{2}\log(1+x^2)$$

$$\therefore \int_0^a \sin^{-1} \frac{2x}{1+x^2} dx = 2\left[x\tan^{-1} x\right]_0^a - \frac{1}{2}\left[\log(1+x^2)\right]_0^a$$

$$= \left[2a\tan^{-1} a - 0.\tan^{-1} 0\right] - \frac{1}{2}\left[\log(1+a^2) - \log 1\right]$$

$$= 2a\tan^{-1} a - \frac{1}{2}\log\left(1+a^2\right).$$

$$= 2a\tan^{-1} a - \log\sqrt{1+a^2}$$

(iii) Do Yourself.

(iv) $\int_0^{\frac{\pi}{2}} \sin x \sin 2x dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} 2\sin x \sin 2x dx$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \left[\cos(2x-x) - \cos(2x+x)\right] dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos x dx - \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos 3x dx$$

$$= \frac{1}{2}[\sin x]_0^{\frac{\pi}{2}} - \frac{1}{2}\left[\frac{\sin 3x}{3}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\sin\frac{\pi}{2} - \frac{1}{6}\sin\frac{3\pi}{2}$$

$$= \frac{1}{2} + \frac{1}{6} \times 1$$

$$= \frac{2}{3}$$

(v) Do Yourself.

Ans to Q No 5: (i) Here the general term i.e., r th term $= \frac{1}{n+r}$ and r varies from 1 to n. Thus we have to find $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n+r}$.

We have $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n+r} = \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n\left\{1+\left(\frac{r}{n}\right)\right\}}$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \frac{1}{1+\left(\frac{r}{n}\right)}$$

The limits of r in this summation are 1 to n. Therefore the lower limit of integration $\lim_{n\to\infty} \frac{1}{n} = 0$, and the upper limit of integration $\lim_{n\to\infty} \frac{n}{n} = 1$.

Hence, the required limit $= \int_0^1 \frac{1}{1+x} dx = [\log(1+x)]_0^1 = \log 2$.

(ii) Here the general term i.e., r th term $= \frac{n}{n^2+r^2}$ and r varies from 1 to n.

Proceeding as in Q 4 (i), we get the required limit $= \frac{\pi}{4}$.

(iii) Here the general term $= \frac{1}{\sqrt{(n^2 - r^2)}} = \frac{1}{n} \cdot \frac{1}{\sqrt{\left\{1-\left(\frac{r}{n}\right)^2\right\}}}$, and varies from 0 to $(n-1)$.

$\therefore$ the given limit $= \lim_{n\to\infty} \sum_{r=1}^{n-1} \frac{1}{n} \frac{1}{\sqrt{\left\{1-\left(\frac{r}{n}\right)^2\right\}}}$

$= \int_0^1 \frac{1}{\sqrt{1-x^2}} dx$

$= \left[\sin^{-1} x\right]_0^1$

$= \sin^{-1} 1 - \sin^{-1} 0$

$= \frac{\pi}{2}$.

(iv) Here $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{r^3}{r^4+n^4} = \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n^4} \left\{\frac{r^3}{\left(r/n\right)^4+1}\right\}$

$= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n} \left\{\frac{\left(r/n\right)^3}{\left(r/n\right)^4+1}\right\}$

$= \int_0^1 \frac{x^3}{x^4+1} dx = \frac{1}{4}\left[\log\left(1+x^4\right)\right]_0^1 = \frac{1}{4}\log 2$.

CHECK YOUR PROGRESS – 2

Ans to Q No 1: Let $I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx$ ... (i)

Then $I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}} dx$ since $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}} dx$ ...(ii)

Adding (i) and (ii), we get

$2I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx + \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}} dx$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$$

$$= \int_0^{\frac{\pi}{2}} 1.dx$$

$$= [x]_0^{\frac{\pi}{2}} = \frac{\pi}{2} - 0$$

$$\Rightarrow I = \frac{\pi}{4}$$

Ans to Q No 2: Let $I = \int_0^{\frac{\pi}{2}} \log \sin x dx$ ... (i)

$$I = \int_0^{\frac{\pi}{2}} \log \sin\left(\frac{\pi}{2} - x\right) dx = \int_0^{\frac{\pi}{2}} \log \cos x dx \quad \text{... (ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\frac{\pi}{2}} \log \sin x dx + \int_0^{\frac{\pi}{2}} \log \cos x dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} (\log \sin x + \log \cos x) dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \log(\sin x \cos x) dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \log\left(\frac{2 \sin x \cos x}{2}\right) dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \log\left(\frac{\sin 2x}{2}\right) dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \log \sin 2x dx - \int_0^{\frac{\pi}{2}} \log 2 dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \log \sin 2x dx - \frac{\pi}{2}\log 2 \qquad \text{... (iii)}$$

Let $I_1 = \int_0^{\frac{\pi}{2}} \log \sin 2x dx$

Putting $2x = t$, we get

$$I_1 = \int_0^{\pi} \log \sin t \frac{dt}{2}$$

$$\Rightarrow I_1 = \frac{1}{2}\int_0^{\pi} \log \sin t dt$$

$$\Rightarrow I_1 = \frac{1}{2}.2\int_0^{\frac{\pi}{2}} \log \sin t dt$$

$$\Rightarrow I_1 = \int_0^{\frac{\pi}{2}} \log \sin t dt = I$$

So, from (iii), we get

$$2I = I - \frac{\pi}{2}\log 2$$

$$\Rightarrow I = -\frac{\pi}{2}\log 2$$

Hence, $\int_0^{\frac{\pi}{2}} \log \sin x dx = \int_0^{\frac{\pi}{2}} \log \cos x dx = -\frac{\pi}{2}\log 2 = \frac{\pi}{2}\log 2^{-1} = \frac{\pi}{2}\log \frac{1}{2}$

Ans to Q No 3: We have $I = \int_0^{\pi} \frac{x}{a^2 \cos^2 x + b^2 \sin^2 x} dx$ ... (1)

$$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x)}{a^2 \cos^2(\pi - x) + b^2 \sin^2(\pi - x)} dx$$

$$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x)}{a^2 \cos^2 x + b^2 \sin^2 x} dx \qquad \text{... (2)}$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi} \frac{x + \pi - x}{a^2 \cos^2 x + b^2 \sin^2 x} dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \frac{1}{a^2 \cos^2 x + b^2 \sin^2 x} dx$$

$$\Rightarrow 2I = 2\pi \int_0^{\frac{\pi}{2}} \frac{1}{a^2 \cos^2 x + b^2 \sin^2 x} dx$$

$$\Rightarrow 2I = 2\pi \int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{a^2 + b^2 \tan^2 x} dx \text{ (Dividing num. and deno.by } \cos^2 x)$$

$$\Rightarrow I = \pi \int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{a^2 + b^2 \tan^2 x} dx$$

Let $\tan x = t$, then $\sec^2 x dx = dt$

Also, $x = 0 \Rightarrow t = \tan 0 = 0$ and $x = \frac{\pi}{2} \Rightarrow t \to \infty$

$$\therefore I = \pi \int_0^{\infty} \frac{dt}{a^2 + b^2 t^2}$$

$$= \frac{\pi}{b^2} \int_0^{\infty} \frac{dt}{\left(\frac{a}{b}\right)^2 + t^2}$$

$$= \frac{\pi}{b^2} \times \frac{1}{\left(\frac{a}{b}\right)} \left[\tan^{-1} \frac{t}{\left(\frac{a}{b}\right)} \right]_0^{\infty}$$

$$\Rightarrow I = \frac{\pi}{ab} \left[\tan^{-1} \left(\frac{bt}{a} \right) \right]_0^{\infty}$$

$$= \frac{\pi}{ab} \left[\tan^{-1} \infty - \tan^{-1} 0 \right]$$

$$= \frac{\pi}{ab} \left[\frac{\pi}{2} - 0 \right]$$

$$\Rightarrow I = \frac{\pi^2}{2ab}.$$

Ans to Q No 4: Let $I = \int_0^{\frac{\pi}{4}} \log(1 + \tan\theta) d\theta$ ... (1)

$$\therefore I = \int_0^{\frac{\pi}{4}} \log\left[1+\tan\theta\left(\frac{\pi}{4}-\theta\right)\right]d\theta \quad \text{since } \int_0^a f(x)dx = \int_0^a f(a-x)dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \log\left[1+\frac{\tan\frac{\pi}{4}-\tan\theta}{1+\tan\frac{\pi}{4}\tan\theta}\right]d\theta$$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \log\left[1+\frac{1-\tan\theta}{1+\tan\theta}\right]d\theta$$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \log\left[\frac{2}{1+\tan\theta}\right]d\theta$$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \log 2\,d\theta - \int_0^{\frac{\pi}{4}} \log(1+\tan\theta)d\theta$$

$$\Rightarrow I = \log 2\int_0^{\frac{\pi}{4}} d\theta - I$$

$$\Rightarrow 2I = \log 2[\theta]_0^{\frac{\pi}{4}}$$

$$\Rightarrow 2I = \log 2\frac{\pi}{4}$$

$$\Rightarrow I = \frac{\pi}{8}\log 2.$$

Ans to Q No 5 : Let $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^9 x dx$ ... (1)

$$\sin^9(-x) = (\sin(-x))^9 = (-\sin x)^9 = -\sin^9 x$$

We know that if f is an odd function, then $\int_{-a}^{a} f(x)dx = 0$.

$$\therefore I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^9 x dx = 0$$

Ans to Q No 6 : Do yourself.

Ans to Q No 7 : Do yourself.

Ans to Q No 8 : Let $I = \int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{\cos x \sin x + 1} dx$ ... (1)

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\cos\left(\frac{\pi}{2} - x\right) - \sin\left(\frac{\pi}{2} - x\right)}{\cos\left(\frac{\pi}{2} - x\right)\sin\left(\frac{\pi}{2} - x\right) + 1} dx \quad \text{since} \quad \int_0^a f(x)dx = \int_0^a f(a-x)dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{\sin x \cos x + 1} dx \quad \text{... (2)}$$

Adding (1) and (2), we obtain

$$2I = \int_0^{\frac{\pi}{2}} \frac{0}{\sin x \cos x + 1} dx$$

$$\Rightarrow I = 0 .$$

Ans to Q No 9: Let $I = \int_0^{\frac{\pi}{2}} (2\log\sin x - \log\sin 2x)dx$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \{2\log\sin x - \log(2\sin x\cos x)\}dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \{2\log\sin x - \log\sin x - \log\cos x - \log 2\}dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} (\log\sin x - \log\cos x - \log 2)dx \quad \text{... (1)}$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \left\{\log\sin\left(\frac{\pi}{2} - x\right) - \log\cos\left(\frac{\pi}{2} - x\right) - \log 2\right\}dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} (\log\cos x - \log\sin x - \log 2)dx \quad \text{... (2)}$$

Adding (1) and (2), we obtain

$$2I = \int_0^{\frac{\pi}{2}} (-\log 2 - \log 2)dx$$

$$\Rightarrow 2I = -2\log 2\int_0^{\frac{\pi}{2}} dx$$

$$\Rightarrow 2I = -2\log 2.\frac{\pi}{2}$$

$$\Rightarrow I = \frac{\pi}{2}(-\log 2)$$

$$\Rightarrow I = \frac{\pi}{2}\log\frac{1}{2}.$$

Ans to Q No 10: Let $I = \int_0^{\frac{\pi}{2}} \frac{x\sin x\cos x}{\sin^4 x + \cos^4 x}dx$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2}-x\right)\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)}{\sin^4\left(\frac{\pi}{2}-x\right)+\cos^4\left(\frac{\pi}{2}-x\right)}dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2}-x\right)\cos x\sin x}{\cos^4 x + \sin^4 x}dx$$

$$\Rightarrow I = \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{\sin x\cos x}{\sin^4 x+\cos^4 x}dx - \int_0^{\frac{\pi}{2}} \frac{x\sin x\cos x}{\sin^4 x+\cos^4 x}dx$$

$$\Rightarrow I = \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{\sin x\cos x}{\sin^4 x+\cos^4 x}dx - I$$

$$\Rightarrow 2I = \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{\sin x\cos x}{\sin^4 x+\cos^4 x}dx$$

$$\Rightarrow 2I = \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{\tan x\sec^2 x}{1+\tan^4 x}dx \quad [\text{ Dividing Nr and Dr by } \cos^4 x]$$

$$\Rightarrow 2I = \frac{\pi}{4}\int_0^{\frac{\pi}{2}} \frac{2\tan x\sec^2 x}{1+(\tan^2 x)^2}dx$$

$$\therefore 2I = \frac{\pi}{4}\int_0^{\infty} \frac{1}{1+t^2}dt \quad \text{where } t = \tan^2 x$$

$$\Rightarrow 2I = \frac{\pi}{4}\left[\tan^{-1} t\right]_0^{\infty}$$

$$\Rightarrow 2I = \frac{\pi}{4}\left[\tan^{-1}\infty - \tan^{-1} 0\right]$$

$$\Rightarrow 2I = \frac{\pi}{4}\left[\frac{\pi}{2} - 0\right]$$

$$\Rightarrow I = \frac{\pi^2}{16}.$$

Ans to Q No 11: Let $I = \int_0^p \frac{\sqrt{x}}{\sqrt{x}+\sqrt{p-x}}dx$ ... (1)

$$\Rightarrow I = \int_0^p \frac{\sqrt{p-x}}{\sqrt{p-x}+\sqrt{x}}dx \quad \text{... (2)}$$

Adding (1) and (2),we get,

$$2I = \int_0^p \frac{\sqrt{p-x}+\sqrt{x}}{\sqrt{p-x}+\sqrt{x}}dx$$

$$\Rightarrow 2I = \int_0^p 1.dx$$

$$\Rightarrow 2I = \left[x\right]_0^p = p$$

$$\Rightarrow I = \frac{p}{2}.$$

Ans to Q No 12 : Do yourself.

Ans to Q No 13 : Let $I = \int_0^{\frac{\pi}{2}} \left(a\cos^2 x + b\sin^2 x\right)dx$ ... (1)

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \left\{a\cos^2\left(\frac{\pi}{2} - x\right) + b\sin^2\left(\frac{\pi}{2} - x\right)\right\}dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \left(a\sin^2 x + b\cos^2 x\right)dx \quad \text{... (2)}$$

Adding (1) and (2), we get

$$2I = \int_0^{\frac{\pi}{2}} \left\{a\left(\cos^2 x + \sin^2 x\right) + b\left(\sin^2 x + \cos^2 x\right)\right\}dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} (a+b)dx$$

$$\Rightarrow 2I = (a+b)\frac{\pi}{2}$$

$$\Rightarrow I = (a+b)\frac{\pi}{4}.$$

Ans to Q No 14: Let $I = \int_0^{\frac{\pi}{2}} \sin 2x \log \tan x dx$ ... (1)

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \sin 2\left(\frac{\pi}{2} - x\right) \log \tan\left(\frac{\pi}{2} - x\right) dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \sin 2x \log \cot x dx \quad \text{... (2)}$$

Adding (1 and (2), we get

$$2I = \int_0^{\frac{\pi}{2}} \sin 2x (\log \tan x + \log \cot x) dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \sin 2x \log(\tan x . \cot x) dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \sin 2x \log 1 dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \sin 2x . 0 . dx$$

$$\Rightarrow I = 0 .$$

Ans to Q No 15 : Let $I = \int_0^{\pi} \frac{x \tan x}{\sec x + \cos x} dx$ ... (1)

$$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x)\tan(\pi - x)}{\sec(\pi - x) + \cos(\pi - x)} dx$$

$$\Rightarrow I = \int_0^{\pi} \frac{-(\pi - x)\tan x}{-\sec x - \cos x} dx$$

$$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x)\tan x}{\sec x + \cos x} dx \quad \text{... (2)}$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi} \frac{\pi \tan x}{\sec x + \cos x} dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \frac{\tan x}{\sec x + \cos x} dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \frac{\sin x}{1 + \cos^2 x} dx$$

We put $\cos x = t \Rightarrow -\sin x dx = dt$

when $x = 0, t = 1$ and when $x = \pi, t = -1$

$$\therefore 2I = \pi \int_1^{-1} \frac{-dt}{1 + t^2}$$

$$\Rightarrow 2I = -\pi \left[\tan^{-1} t\right]_1^{-1}$$

$$\Rightarrow 2I = -\pi \left[-\frac{\pi}{4} - \frac{\pi}{4}\right]$$

$$\Rightarrow 2I = \frac{\pi^2}{2}$$

$$\Rightarrow I = \frac{\pi^2}{4}.$$

Ans to Q No 16 : Let $I = \int_0^{\frac{\pi}{2}} \frac{\cos x}{1 + \cos x + \sin x} dx$ ... (i)

$$\therefore I = \int_0^{\frac{\pi}{2}} \frac{\cos\left(\frac{\pi}{2} - x\right)}{1 + \cos\left(\frac{\pi}{2} - x\right) + \sin\left(\frac{\pi}{2} - x\right)} dx = \int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + \sin x + \cos x} dx \quad \text{... (ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\frac{\pi}{2}} \frac{\cos x + \sin x}{1 + \sin x + \cos x} dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \frac{1 + \cos x + \sin x - 1}{1 + \sin x + \cos x} dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \left\{1 - \frac{1}{1 + \sin x + \cos x}\right\} dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 1dx - \int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x+\cos x}dx$$

$$\Rightarrow 2I = [x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \frac{1}{1+\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}+\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}}dx$$

$$\Rightarrow 2I = \frac{\pi}{2} - \int_0^{\frac{\pi}{2}} \frac{1+\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}+2\tan\frac{x}{2}+1-\tan^2\frac{x}{2}}dx$$

$$\Rightarrow 2I = \frac{\pi}{2} - \int_0^1 \frac{2dt}{2+2t} \text{ where } t = \tan\frac{x}{2}$$

$$\Rightarrow 2I = \frac{\pi}{2} - [\log(1+t)]_0^1$$

$$\Rightarrow 2I = \frac{\pi}{2} - \log 2$$

$$\Rightarrow I = \frac{\pi}{4} - \frac{1}{2}\log 2.$$

Ans to Q No 17: Let $I = \int_0^{\frac{\pi}{2}} \log\left(\frac{3+5\sin x}{3+5\cos x}\right)dx$ ... (1)

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \log\left[\frac{3+5\sin\left(\frac{\pi}{2}-x\right)}{3+5\cos\left(\frac{\pi}{2}-x\right)}\right]dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \log\left(\frac{3+5\cos x}{3+5\sin x}\right)dx \quad \text{... (2)}$$

Adding (1) and (2), we get

$$2I = \int_0^{\frac{\pi}{2}} \left\{\log\left(\frac{3+5\sin x}{3+5\cos x}\right)+\log\left(\frac{3+5\cos x}{3+5\sin x}\right)\right\}dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \log\left(\frac{3+5\sin x}{3+5\cos x}\cdot\frac{3+5\cos x}{3+5\sin x}\right)dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \log 1\,dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 0\,dx$$

$$\Rightarrow I = 0 .$$

Ans to Q No 18 : Let $I = \int_0^{\pi} \frac{x\,dx}{1+\sin x}$ ... (1)

$$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x)dx}{1+\sin(\pi - x)}$$

$$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x)dx}{1+\sin x} \quad \text{... (2)}$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi} \frac{\pi dx}{1+\sin x}$$

$$\Rightarrow 2I = \pi\int_0^{\pi} \frac{(1-\sin x)dx}{(1+\sin x)(1-\sin x)}$$

$$\Rightarrow 2I = \pi\int_0^{\pi} \frac{1-\sin x\,dx}{1-\sin^2 x}$$

$$\Rightarrow 2I = \pi\int_0^{\pi} \frac{1-\sin x}{\cos^2 x}dx$$

$$\Rightarrow 2I = \pi\int_0^{\pi} (\sec^2 x - \tan x\sec x)dx$$

$$\Rightarrow 2I = \pi[\tan x - \sec x]_0^{\pi}$$

$$\Rightarrow 2I = \pi.2$$

$$\Rightarrow I = \pi .$$

Ans to Q No 19 : We have $I = \int_0^1 \frac{\log x}{\sqrt{1-x^2}}dx$

We put $x = \sin\theta$, $dx = \cos\theta d\theta$

Also, when $x = 0$, $\theta = 0$ and $x = 1$, $\theta = \frac{\pi}{2}$.

$$\therefore I = \int_0^{\frac{\pi}{2}} \frac{\log(\sin\theta)\cos\theta d\theta}{\cos\theta}$$

$$= \int_0^{\frac{\pi}{2}} \log(\sin\theta) d\theta$$

$$= -\frac{\pi}{2}\log 2 .$$



2.11 MODEL QUESTIONS

Q 1 : Find the values of the following from the first principle (i.e. by the method of integration as the limit of a sum)

(a) $\int_0^1 5dx$ (b) $\int_0^1 x^2 dx$ (c) $\int_0^3 (2x+1)dx$

Q 2 : Evaluate the integrals

(i) $\int_0^3 \frac{dx}{(1-2x^2)\sqrt{1-x^2}}$ (ii) $\int_0^{\pi/2} \cos^3 x \sqrt[5]{\sin x}\, dx$

(iii) $\int_0^{\pi/2} \frac{\cos x dx}{(1+\sin x)(2+\sin x)}$ (iv) $\int_0^{\pi} \sin mx \cos nx dx$

(v) $\int_0^{\pi/2} x^2 \sin x dx$ (vi) $\int_0^1 \frac{dx}{e^x + e^{-x}}$

(vii) $\int_0^{\pi/2} \frac{dx}{a^2\cos^2 x + b^2 \sin^2 x}$ (viii) $\int_0^a \sqrt{a^2 - x^2}\, dx$

Q 3 : Evaluate

(i) $\lim_{n\to\infty}\left\{\frac{1}{n} + \frac{\sqrt{n^2-1^2}}{n^2} + \ldots\ldots\ldots + \frac{\sqrt{n^2-(n-1)^2}}{n^2}\right\}$

(ii) $\lim_{n\to\infty}\left(\frac{1}{n+m} + \frac{2}{n+2m} + \ldots\ldots\ldots + \frac{1}{n+nm}\right)$

(iii) $\lim_{n\to\infty}\left(\frac{1}{3n+1} + \frac{2}{3n+2} + \ldots\ldots\ldots + \frac{1}{3n+3} + \ldots\ldots\ldots + \frac{1}{3n+n}\right)$

(iv) $\lim_{n\to\infty}\left[\frac{1}{n}+\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+\ldots\ldots\ldots+\frac{1}{8n}\right]$

(v) $\lim_{n\to\infty}\sum_{r=0}^{n}\frac{n^2}{\left(n^2+r^2\right)^{3/2}}$

Q 4 : Evaluate (by using properties of definite integrals)

(i) $\int_0^{\pi/2}\frac{x\,dx}{\sin x+\cos x}$ (ii) $\int_0^{\pi}\log(1+\cos x)\,dx$ (iii) $\int_0^{\pi}\cos^6 x\,dx$

(iv) $\int_{-1}^{1}\frac{x^2\sin^{-1}x}{\sqrt{1-x^2}}\,dx$ (v) $\int_{-1}^{1}\frac{x^2\sin^{-1}x}{\sqrt{1-x^2}}\,dx$ (vi) $\int_0^{\pi/2}\sin 2x\log\tan x\,dx$

(vii) $\int_0^{\pi}x\sin^6 x\cos^4 x\,dx$ (viii) $\int_0^{\pi/2}\frac{dx}{1+\tan x}$

Q 5 : Show that

(i) $\int_0^{\pi/2}\phi(\sin 2x)\sin x\,dx=\int_0^{\pi/2}\phi(\sin 2x)\cos x\,dx$

(ii) $\int_0^{\pi}\sin^n x\cos^{2n+1}x\,dx=0$ (iii) $\int_0^{\pi}\frac{x\sin x}{1+\sin x}\,dx=\frac{\pi}{2}(\pi-2)$

(iv) $\int_0^{\pi/2}\frac{dx}{5+4\cos x}=\frac{2}{3}\tan^{-1}\left(1/3\right)$ (v) $\int_{-\pi/2}^{\pi/2}\frac{x\cos x\,dx}{2\sin^2 x+3\cos^2 x}=0$

(vi) $\int_{-1}^{1}|x|\,dx=1$ (vii) $\int_0^{\pi/2}\frac{\cos^3 x\,dx}{\sin^3 x+\cos^3 x}=\frac{\pi}{4}$

(viii) $\int_0^{\pi}x\cos^4 x\,dx=\frac{3\pi^2}{16}$

Q 6 : If $f(x)=|x-2|+|x-3|$ then show that $\int_2^4 f(x)\,dx=3$.

*** ***** ***

UNIT 3 : REDUCTION FORMULAE

UNIT STRUCTURE

3.1 Learning Objectives

3.2 Introduction

3.3 Reduction Formulae

3.3.1 Reduction formulae for (i) $\int \sin^n x dx$, and (ii) $\int \cos^n x dx$, n being a positive integer

3.3.2 Reduction Formulae for (i) $\int \tan^n x dx$ and(ii) $\int \cot^n x dx$, n being a positive Integer

3.3.3 Reduction Formula for $\int x^n e^{ax} dx$, where n be a posititve Integer

3.3.4 Reduction Formula for $\int \cos^m x \sin x dx$

3.4 Let Us Sum Up

3.5 Further Readings

3.6 Answer to Check Your Progress

3.7 Model Questions

3.1 LEARNING OBJECTIVES

After going through this unit, you will be able to find

- reduction formulae for $\int \sin^n x dx$ and $\int \cos^n x dx$, n being a positive integer
- reduction formula for $\int \tan^n x dx$ and $\int \cot^n x dx$, n being a positive Integer
- reduction formula for $\int x^n e^{ax} dx$, and $\int \cos^m x \sin x dx$.

3.2 INTRODUCTION

In the previous unit, we discussed the definite integral and its properties. In this unit we will discuss reduction formulae.

3.3 REDUCTION FORMULA

In some cases of integration, the integrand, which is otherwise complicated, is reduced to a simpler form by the applicationof the rule of

integration by parts. In general, this process yields a new integral of the same form as the original but with an exponent which is smaller than the former one. The formula by which an integral of a complicated function involving one parameter (or more than one parameters) is connected to some integral of a simple function of the same type but of lower degree is called a "Reduction Formula".

Here we will discuss the method of obtaining some important reduction formulae.

3.3.1 Reduction formulae for (i) $\int \sin^n x\,dx$ and (ii) $\int \cos^n x\,dx$, *n* being a positive integer

(i) $\int \sin^n x\,dx = \int \sin^{n-1} x . \sin x\,dx$

$$= \sin^{n-1} x \int \sin x\,dx - \int \left\{ \frac{d}{dx}\left(\sin^{n-1} x\right) \int \sin x\,dx \right\} dx$$

(applying integration by parts)

$$= \sin^{n-1} x(-\cos x) - \int (n-1)\sin^{n-2} x . \cos x(-\cos x)\,dx$$

$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \cos^2 x\,dx$$

$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x(1-\sin^2 x)\,dx$$

$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x - (n-1)\int \sin^n x\,dx$$

Transposing the last integral from right side to the left side we get

$$n\int \sin^n x\,dx = -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\,dx$$

$$\therefore \int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx.$$

which is the required reduction formula for $\int \sin^n x\,dx$.

(ii) Here we take $\int \cos^n x\,dx = \int \cos^{n-1} x \cos x\,dx$

Then taking $\cos x$ as the 2nd function, we apply the rule of integration by parts and finally we can show that

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$$

It is the reduction formula for $\int \cos^n x\,dx$

Note : Learners are advised to complete the formula of integration by parts in the above case i.e. for $\int \cos^n x dx$, to obtain the final form of the reduction formula.

TWO IMPORTANT DEDUCTIONS

If n be a positive integer greater than 1,

$$\int_0^{\pi/2} \sin^n x dx = \int_0^{\pi/2} \cos^n x dx$$

$$= \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}.....\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2} \quad (n \text{ is even})$$

$$= \frac{n-1}{n}\frac{n-3}{n-2}\frac{n-5}{n-4}.....\frac{4}{5}.\frac{2}{3}.1 \quad (n \text{ is odd})$$

Proof : We have already shown that

$$\int \sin^n x dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x dx$$

$$\therefore \int_0^{\pi/2} \sin^n x dx = \left[-\frac{\sin^{n-1} x \cos x}{n}\right]_o^{\frac{\pi}{2}} + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x dx$$

$$= 0 + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x dx$$

$$= \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x dx \quad ... ...(i)$$

If we denote that $I_n = \int_0^{\pi/2} \sin^n x dx$ then from (i) we can write

$$I_n = \frac{n-1}{n} I_{n-2} \quad ... ...(ii)$$

Now if we change n by $n–2$, $n–4$ etc. successively then we get

$$I_{n-2} = \frac{n-3}{n-2} I_{n-4}, \qquad I_{n-4} = \frac{n-5}{n-4} I_{n-6} \text{ etc.}$$

$\therefore$ From (ii) we can write

$$I_n = \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}.....\frac{3}{4}.\frac{1}{2}.I_o$$

or $$I_n = \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}.....\frac{4}{5}.\frac{2}{3}.I_1$$

according as n is even or odd.

Now $I_0 = \int_0^{\pi/2} (\sin x)^0 dx = \int_0^{\pi/2} dx = [x]_0^{\pi/2} = \pi/2$

and $I_1 = \int_0^{\pi/2} \sin x dx = -[\cos x]_0^{\frac{\pi}{2}} = 1$

$\therefore$ We get

$$I_n = \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}\ldots\ldots\ldots\frac{3}{4}.\frac{1}{2}.\pi/2, \text{ when } n \text{ is even}$$

and $$I_n = \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}\ldots\ldots\ldots\frac{4}{5}.\frac{2}{3}.1, \text{ when } n \text{ is odd.}$$

Further we can show that

$$\int_0^{\pi/2}\sin^n x\,dx = \int_0^{\pi/2}(\sin x)^n dx = \int_0^{\pi/2}\left[\sin\left(\frac{\pi}{2}-x\right)\right]^n dx$$

$$= \int_0^{\pi/2}(\cos x)^n dx$$

$$= \int_0^{\pi/2}\cos^n x\,dx$$

Hence we have, $\int_0^{\pi/2}\cos^n x\,dx = \int_0^{\pi/2}\sin^n x\,dx$

$$= \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}\ldots\ldots\ldots\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}$$

$$= \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}\ldots\ldots\ldots\frac{4}{5}.\frac{2}{3}.1$$

according as n is even or odd.

3.3.2 Reduction Formulae For (a) $\int \tan^n x\,dx$ and (b) $\int \cot^n x\,dx$, n being a positive integer

(a) $$\int \tan^n x\,dx = \int \tan^{n-2} x.\tan^2 x\,dx$$

$$= \int \tan^{n-2} x(\sec^2 x - 1)dx$$

$$= \int \tan^{n-2} x\sec^2 x\,dx - \int \tan^{n-2} x\,dx$$

Here for the 1st integral of RHS we take

$$\tan x = t, \text{ or } \sec^2 x\,dx = dt.$$

$$\therefore \int \tan^{n-2} x\sec^2 x\,dx = \int t^{n-2}dt = \frac{t^{n-2}}{n-1} = \frac{1}{n-1}\tan^{n-1} x$$

$\therefore$ The given integral

$$\int \tan^n x\,dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\,dx$$

which is the reduction formula for $\int \tan^n x\,dx$

(b) $$\int \cot^n x\,dx = \int \cot^{n-2} x.\cot^2 x\,dx$$

$$= \int \cot^{n-2} x\left(\cos ec^2 x - 1\right)dx$$

$$= \int \cot^{n-2} x \cos ec^2 x dx - \int \cot^{n-2} x dx$$

$$= -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x dx$$

which is the reduction formula for $\int \cot^n x dx$

Reduction Formulae For (a) $\int \sec^n x dx$ and (b) $\int \cos ec^n x dx$, n being a positive integer

(a) Let us take $I_n = \int \sec^n x dx$

Then $I_n = \int \sec^n x dx = \int \sec^{n-2} x . \sec^2 x dx$

$$= \sec^{n-2} x . \tan x - \int (n-2)\sec^{n-3} x . \sec x \tan x . \tan x dx$$

(Integrating by parts where $\sec^2 x$ is taken as 2nd function)

$$I_n = \sec^{n-2} x . \tan x - (n-2)\int \sec^{n-2} x\left(\sec^2 x - 1\right)dx$$

$$= \sec^{n-2} x . \tan x - (n-2)\left[\int \sec^n x dx - \int \sec^{n-2} x dx\right]$$

$$= \sec^{n-2} x . \tan x - (n-2)(I_n - I_{n-2})$$

Now transposing we get

$$I_n = \frac{1}{n-1}\sec^{n-2} x . \tan x + \frac{n-2}{n-1} I_{n-2}$$

(b) Here $I_n = \int \operatorname{cosec}^n x dx$

$$= \int \operatorname{cosec}^{n-2} x . \operatorname{cosec}^2 x$$

Then applying the rule of integration by parts, the learners can show that

$$I_n = \frac{1}{n-1}\operatorname{cosec}^{n-2} x . \cot x + \frac{n-2}{n-1} I_{n-2}$$

3.3.3 Reduction Formulae For $\int x^n e^{ax} dx$, where n is a posititve integer

Here $I_n = \int x^n e^{ax} dx$.

Then $I_n = \int x^n e^{ax} dx$

$$= x^n . \frac{e^{ax}}{a} - \int n x^{n-1} \frac{e^{ax}}{a} dx$$

$$= \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}dx$$

$$= \frac{1}{a}x^n e^{ax} - \frac{n}{a}I_{n-1}$$

which is the reduction formula for $\int x^n e^{ax}dx$.

Reduction Formula For $\int \sin^m x \cos^n x dx$, *m, n* being Posititve Integers

Let $I_{m,n} = \int \sin^m x \cos^n x dx = \int \cos^{n-1} x\left(\sin^m x \cos x\right)dx$

Now applying integration by parts

$$I_{m,n} = \cos^{n-1} x \int \sin^m x \cos x dx - \int\left\{\frac{d}{dx}\left(\cos^{n-1} x\right)\int \sin^m x \cos x dx\right\}dx$$

$$I_{m,n} = \cos^{n-1} x.\frac{\sin^{m+1} x}{m+1} - \int(n-1)\cos^{n-2} x(-\sin x).\frac{\sin^{m+1} x}{m+1}dx$$

$$= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1}\int \cos^{n-2} x.\sin^m x.\sin^2 x dx$$

$$= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1}\int \cos^{n-2} x \sin^m x\left(1-\cos^2 x\right)dx$$

$$= \frac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \frac{n-1}{m+1}\int\left(\cos^{n-2} x \sin^m x - \sin^m x \cos^n x\right)dx$$

$$= \frac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \frac{n-1}{m+1}\int \cos^{n-2} x \sin^m x dx - \frac{n-1}{m+1}I_{m,n}$$

Then by transposing and simplifying

$$I_{m,n} = \frac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+n}\int \cos^{n-2} x \sin^m x dx$$

i.e. $I_{m,n} = \frac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+n}I_{m,n-2}$ ... ...(A), which is the reduction formula for $\int \sin^m x \cos^n x dx$

Note : If we arrange $I_{m,n}$ as below then

$$I_{m,n} = \int \sin^m x \cos^n x dx$$

$$= \int \sin^{m-1} x\left(\cos^n x \sin x\right)dx$$. Then integrating by parts we can show that

$$I_{m,n} = -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n}I_{m-2,n}$$ ... ...(B)

Important Deductions

(i) Reduction formula for $\int_0^{\pi/2} \sin^m x \cos^n x\,dx$, m and n being positive integers greater than 1. Let us assume that

$J_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x\,dx$

Now from (A) above we can write

$$\int_0^{\pi/2} \sin^m x \cos^n x\,dx = \left[\frac{\sin^{m+1} x \cos^{n-1} x}{m+n}\right]_0^{\pi/2} + \frac{n-1}{m+n}\int_0^{\pi/2} \sin^m x \cos^{n-2} x\,dx$$

$$= 0 + \frac{n-1}{m+n}\int_0^{\pi/2} \sin^m x \cos^{n-2} x\,dx$$

$$\therefore J_{m,n} = \frac{n-1}{m+n} J_{m,n-2} \quad \ldots\ldots\ldots\text{(C)}$$

Similarly from (B) we get

$$J_{m,n} = \frac{m-1}{m+n} J_{m-2,n} \quad \ldots\ldots\ldots\text{(D)}$$

(ii) Reduction formula for $\int \sin^m x \cos^n x\,dx$, where either m or n or both m and n are negative integers

We have from (A) above that

$$I_{m,n} = \frac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+n} I_{m,n-2}$$

Now changing n to $n + 2$ we get

$$I_{m,n+2} = \frac{\sin^{m+1} x \cos^{n+1} x}{m+n+2} + \frac{n+1}{m+n+2} I_{m,n}$$

Then transposing we get

$$I_{m,n} = -\frac{\sin^{m+1} x \cos^{n+1} x}{n+1} + \frac{m+n+2}{n+1} I_{m,n+2}, \quad (n+1 \neq 0) \ldots\ldots\ldots\text{(E)}$$

Similarly from (B) we get

$$I_{m,n} = \frac{\sin^{m+1} \cos^{n+1}}{m+1} + \frac{m+n+2}{m+1} I_{m+2,n}, \quad (m+1 \neq 0) \qquad \ldots\ldots\ldots\text{(F)}$$

ILLUSTRATIVE EXAMPLES

Example 1 : Evaluate $\int \sin^6 x\,dx$

Solution : We have,

$$\int \sin^n x\,dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$

So putting $n = 6$ we get

$$\int \sin^6 x dx = -\frac{1}{6}\sin^5 x \cos x + \frac{5}{6}\int \sin^4 x dx$$

$$= -\frac{1}{6}\sin^5 x \cos x + \frac{5}{6}\left[-\frac{1}{4}\sin^3 x \cos x + \frac{3}{4}\int \sin^2 x dx\right]$$

$$= -\frac{1}{6}\sin^5 x \cos - \frac{5}{24}\sin^3 x \cos x + \frac{15}{24}\int \sin^2 x dx$$

$$= -\frac{1}{6}\sin^5 x \cos x - \frac{5}{24}\sin^3 x \cos x + \frac{5}{8}\int \frac{1}{2}(1-\cos 2x)dx$$

$$= -\frac{1}{6}\sin^5 x \cos x - \frac{5}{24}\sin^3 x \cos x + \frac{5}{16}\left(x - \frac{\sin 2x}{2}\right) + C$$

Example 2 : Evaluate $\int_0^{\pi/2} \cos^{10} x dx$

Solution : Here $n = 10$ (which is even)

$$\therefore \int_0^{\pi/2} \cos^{10} x dx = \frac{9}{10}.\frac{7}{8}.\frac{5}{6}.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2} = \frac{63\pi}{512}$$

Example 3 : Find a formula for $\int_0^{\pi/4} \tan^n x dx$ and hence evaluate $\int_0^{\pi/4} \tan^6 x dx$.

Solution : We have

$$\int \tan^n x dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x dx$$

$$\therefore \int_0^{\pi/4} \tan^n x dx = \left[\frac{1}{n-1}\tan^{n-1} x\right]_0^{\pi/4} - \int_0^{\pi/4} \tan^{n-2} x dx$$

$$= \left[\frac{1}{n-1}\left(\tan \pi/4\right)^{n-1} - \frac{1}{n-1}(\tan 0)^{n-1}\right] - \int_0^{\pi/2} \tan^{n-2} x dx$$

$$= \frac{1}{n-1} - \int_0^{\pi/4} \tan^{n-1} x dx \quad \ldots\ldots\ldots (i)$$

If we take $n = 6$ in (i) then,

$$\int_0^{\pi/4} \tan^6 x dx = \frac{1}{5} - \int_0^{\pi/4} \tan^4 x dx$$

$$= \frac{1}{5} - \left[\frac{1}{3} - \int_0^{\pi/4} \tan^2 x dx\right] \quad \text{[using (i)]}$$

$$= \frac{1}{5} - \frac{1}{3} + \int_0^{\pi/4} \tan^2 x\,dx$$

$$= \frac{-2}{15} + 1 - \int_0^{\pi/4} \tan^0 x\,dx = \frac{13}{15} - \int_0^{\pi/4} dx$$

$$= \frac{13}{15} - [(x)]_0^{\pi/4} = \frac{13}{15} - \frac{\pi}{4}$$

Example 3 : If $I_n = \int_0^{\pi/4} \tan^n x\,dx$ then show that $n(I_{n-1} + I_{n+1}) = 1$.

Solution : We can write

$$\int \tan^{n+1} x\,dx = \int \tan^{n-1} x . \tan^2 x\,dx$$

$$= \int \tan^{n-1} x . (\sec^2 x - 1)dx$$

$$= \int \tan^{n-1} x \sec^2 x\,dx - \int \tan^{n-1} x\,dx$$

$$= \frac{\tan^n x}{n} - \int \tan^{n-1} x\,dx$$

$$\therefore \int_0^{\pi/4} \tan^{n+1} x\,dx = \left[\frac{\tan^n x}{n}\right]_0^{\pi/4} - \int_0^{\pi/4} \tan^{n-1} x\,dx$$

$$= \frac{1}{n} - \int_0^{\pi/4} \tan^{n-1} x\,dx$$

$$\therefore \quad I_{n+1} = \frac{1}{n} - I_{n-1}$$

$$\Rightarrow I_{n-1} + I_{n+1} = \frac{1}{n}$$

$$\Rightarrow n(I_{n-1} + I_{n+1}) = 1$$

Example 4 : Evaluate $\int_0^{\pi/2} \sin^3 x \cos^4 x \cos 2x\,dx$

Solution : $\int_0^{\pi/2} \sin^3 x \cos^4 x \cos 2x\,dx$

$$= \int_0^{\pi/2} \sin^3 x \cos^4 x (2\cos^2 x - 1)dx$$

$$= 2\int_0^{\pi/2} \sin^3 x \cos^6 x\,dx - \int_0^{\pi/2} \sin^3 x \cos^4 x\,dx$$

$$= 2\frac{6-1}{3+6}\int_0^{\pi/2} \sin^3 x \cos^4 x\,dx - \int_0^{\pi/2} \sin^3 x \cos^4 x\,dx$$

$$= \frac{10}{9}\int_0^{\pi/2} \sin^3 x \cos^4 x\,dx - \int_0^{\pi/2} \sin^3 x \cos^4 x\,dx$$

$$= \frac{1}{9}\int_0^{\pi/2} \sin^3 x \cos^4 x dx$$

$$= \frac{1}{9}\left[\frac{4-1}{3+4}\int_0^{\pi/2} \sin^3 x \cos^2 x dx\right]$$

$$= \frac{1}{21}\int_0^{\pi/2} \sin^3 x \cos^2 x dx$$

$$= \frac{1}{21}\left[\frac{2-1}{3+2}\right]\int_0^{\pi/2} \sin^3 x dx$$

$$= \frac{1}{21}\left[\frac{2-1}{3+2}\int_0^{\pi/2} \sin^3 x dx\right]$$

$$= \frac{1}{105}\int_0^{\pi/2} \sin^3 x dx$$

$$= \frac{1}{105}.\frac{2}{3}.1$$

$$= \frac{2}{315}$$

3.3.4 Reduction formula for $\int \cos^m x \sin nx dx$

Let us take $I_{m,n} = \int \cos^m x \sin x dx$

Now integrating by parts where $\cos^m x$ is taken as first function,

$$I_{m,n} = \cos^m x.\left(-\frac{\cos nx}{n}\right) - \int m\cos^{m-1} x(-\sin x).\left(-\frac{\cos nx}{n}\right)dx$$

$$= -\frac{\cos^m x \cos nx}{n} - \frac{m}{n}\int \cos^{m-1} x \sin x \cos nx dx \quad \ldots\ldots\ldots(i)$$

But we know that

$$\sin(nx - x) = \sin nx \cos x - \cos nx \sin x$$

$$\therefore \cos nx \sin x = \sin nx \cos x - \sin(n-1)x$$

Putting it in (i) we get

$$I_{m,n} = -\frac{\cos^m x \cos nx}{n} - \frac{m}{n}\int \cos^{m-1} x\{\sin nx \cos x - \sin(n-1)x\}dx$$

$$= -\frac{\cos^m x \cos nx}{n} - \frac{m}{n}\int \cos^m x \sin nx dx + \frac{m}{n} \int \cos^{m-1} x \sin(n-1)x dx$$

$$= -\frac{\cos^m x \cos nx}{n} - \frac{m}{n} I_{m,n} + \frac{m}{n} I_{m-1,\ n-1}$$

Then transpoising we get

$$(m+n)\, I_{m,n} = -\cos^m x \cos nx + m I_{m-1,n-1}$$

$$\Rightarrow I_{m,n} = -\frac{\cos^m x \cos nx}{m+n} + \frac{m}{m+n} I_{m-1,n-1}$$

Note : In the similar manner learners can obtain reduction formula for (i) $\int \cos^m x \cos nx dx$ (ii) $\int \sin^m x \cos nx dx$ (iii) $\int \sin^m x \sin nx dx$

Example : Evaluate $\int_0^{\pi/2} \cos^5 x \sin 3x dx$

Solution : We have already shown that if $I_{m,n} = \int \cos^m x \sin nx dx$ then

$$I_{m,n} = -\frac{\cos^m x \cos nx}{m+n} + \frac{m}{m+n} I_{m-1,n-1}$$

Now $\int_0^{\pi/2} \cos^5 x \sin 3x dx = \left[-\frac{\cos^5 x \cos 3x}{5+3} \right]_0^{\pi/2} + \frac{5}{5+3} \int_0^{\pi/2} \cos^4 x \sin 2x . dx$

$$= -\frac{1}{8}(-1) + \frac{5}{8} \int_0^{\pi/2} \cos^4 x . 2 \sin x \cos x dx$$

$$= \frac{1}{8} + \frac{5}{8} . 2 \int_0^{\pi/2} \cos^5 x \sin x dx$$

$$= \frac{1}{8} + \frac{5}{4} \left[\left(-\frac{\cos^6 x}{6} \right) \right]_0^{\pi/2}$$

$$= \frac{1}{8} - \frac{5}{24}(0-1) \frac{1}{8} - \frac{5}{24} \left[\cos^6 x \right]_0^{\pi/2} \Bigg|$$

$$= \frac{1}{8} + \frac{5}{24}$$

$$= \frac{1}{3}$$



CHECK YOUR PROGRESS

Q 1 : Show that $\int_0^{\pi/2} \sin^6 x\,dx = \frac{5\pi}{32}$

Q 2 : Show that $\int_0^a x^2\left(a^2 - x^2\right)^{3/2} dx = \frac{\pi a^6}{32}$

Q 3 : Show that $\int_0^a \frac{x^4}{\sqrt{a^2 - x^2}} dx = \frac{3\pi a^4}{16}$

Q 4 : If $I_n = \int \sin^n(2\theta)\,d\theta$ then show that a reduction formula for I_n is

$$I_n = -\frac{1}{2n}\sin^{n-1} 2\theta.\cos 2\theta + \frac{n-1}{n} I_{n-2}$$

Q 5 : If $I_n = \int_0^{\pi/4} \tan^n x\,dx$, then show that

$I_n + I_{n-2} = \frac{1}{n-1}$ and hence show that the value of I_5 is

$\frac{1}{2}\left(\log 2 - \frac{1}{2}\right)$.

Q 6 : Evaluate

(a) $\int \cot^5 x\,dx$ (b) $\int_0^{\pi/4} \tan^7 \theta\,d\theta$

(c) $\int \operatorname{cosec}^3 x\,dx$

Q 7 : Show that

$$\int \sec^5 x\,dx = \frac{1}{4}\sec^3 x \tan x + \frac{3}{8}\sec x \tan x + \frac{3}{8}\log\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right|$$

Q 8 : Show that $\int_0^{\pi/2} \sin^7 x\,dx = \int_0^{\pi/2} \cos^7 x\,dx = \frac{16}{35}$

Q 9: Prove that $\int_0^{\pi/2} \sin^{2m} x\,dx = \frac{(2m)!}{\{2^m.(m)!\}^2}.\frac{\pi}{2}$

Q 10 : Show that

$$\int \sin^8 x \cos^2 x\,dx = \frac{1}{512}\left\{\frac{1}{10}\sin 10x - \frac{3}{4}\sin 8x + \frac{13}{6}\sin 6x - 2\sin 4x - 7\sin 2x + 14x\right\}$$

Q 11 : Evaluate $\int_0^{\pi/2} \cos^3 x \sin 5x\,dx$.

Q 12 : Evaluate (a) $\int_0^{\pi/4} \sin^2 x \cos^4 x dx$ (b) $\int_0^1 x^{3/2}(1-x)^{3/2} dx$

Q 13 : If $I_n = \int_0^{\pi/2} x^n \sin x dx$ where $n > 1$,

then show that $I_n + n(n-1)I_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}$

Q 14 : Show that $\int_0^{\pi/2} \sin^m x \cos^n x dx = \int_0^{\pi/2} \cos^m x \sin^n x dx$

$$= \frac{1.3.5.\ldots\ldots(m-1).1.3.5.\ldots\ldots(n-1)}{2.4.6.\ldots\ldots(m+n)} \cdot \frac{\pi}{2}$$

when both m and n are even integers; and

$$= \frac{2.4.6.\ldots\ldots(m-1)}{(n+1)(n+3)\ldots\ldots(n+m)}$$

when one of the two indices (say m) is an odd integer.

Q 15 : Show that $\int x^3 e^{ax} dx = \frac{e^{ax}}{a^4}\left(a^3x^3 - 3a^2x^2 + 6ax - 6\right)$

Q 16 : Show that

$$\int \sin^m x \cos nx dx = \frac{\sin^m x \cos nx}{m+n} - \frac{m}{m+n}\int \sin^{m-1} x \cos(n-1)x dx$$

Q 17 : Show that, if $I_n = \int x^n e^{-ax} dx, \ (n \neq -1)$, then

$$aI_n = -x^n e^{-ax} + nI_{n-1}$$



3.4 LET US SUM UP

- We have derived reduction formula for $\int \sin^n x dx$ and $\int \cos^n x dx$, in 3.3.1.
- We have derived reduction formula for $\int \tan^n x dx$ and $\int \cot^n x dx$, in 3.3.2.
- We have derived reduction formula for $\int x^n e^{ax} dx$, and $\int \cos^m x \sin x dx$



3.5 FURTHER READINGS

1) *Integral Calculus,* Kishan, Hari, Atlantic Publishers & Distributors. New Delhi -110027.

2) *Integral Calculus*, Dhami, H.S. New age International Publishers, New Delhi - 110002.

3) *Text book of Integral Calculus*, Sharma, A.K. Discovery Publishing House, New Delhi -110002.



3.6 ANSWERS TO CHECK YOUR PROGRESS

CHECK YOUR PROGRESS

Ans to Q No 1 : $\int_0^{\frac{\pi}{2}} \sin^6 x dx = \frac{5\times 3\times 1}{6\times 4\times 2}\times\frac{\pi}{2} = \frac{5\pi}{32}$

Ans to Q No 2 : We put $x = a\sin\theta$, so that $dx = a\cos\theta d\theta$

Also, when $x = 0, \sin\theta = 0$ i.e., $\theta = 0$

and when $x = a, \sin\theta = 1$ i.e., $\theta = \frac{\pi}{2}$.

Then $\int_0^a x^2\left(a^2 - x^2\right)^{3/2} dx = \int_0^{\frac{\pi}{2}} a^2\sin^2\theta\left(a^2 - a^2\sin^2\theta\right)^{\frac{3}{2}} a\cos\theta d\theta$

$$= \int_0^{\frac{\pi}{2}} a^2\sin^2\theta\left(a^2\cos^2\theta\right)^{\frac{3}{2}} a\cos\theta d\theta$$

$$= \int_0^{\frac{\pi}{2}} a^2\sin^2\theta a^3\cos^3\theta a\cos\theta d\theta$$

$$= a^6\int_0^{\frac{\pi}{2}} \sin^2\theta\cos^4\theta d\theta$$

$$= a^6 \frac{3.1.1}{6.4.2}.\frac{\pi}{2}$$

$$= \frac{\pi a^6}{32}.$$

Ans to Q No 3 : We put $x = a\sin\theta$, so that $dx = a\cos\theta d\theta$

Also, when $x = 0, \sin\theta = 0$ i.e., $\theta = 0$ and when $x = a, \sin\theta = 1$ i.e., $\theta = \frac{\pi}{2}$.

Then, $\int_0^a \frac{x^4 dx}{\sqrt{(a^2 - x^2)}} = \int_0^{\frac{\pi}{2}} \frac{a^4 \sin^4\theta a\cos\theta d\theta}{\sqrt{(a^2 - a^2\sin^2\theta)}}$

$$= a^4 \int_0^{\frac{\pi}{2}} \sin^4\theta d\theta$$

$$= a^4 \frac{3.1}{4.2}.\frac{\pi}{2}$$

$$= \frac{3\pi a^4}{16}.$$

Ans to Q No 4 : Given $I_n = \int \sin^n(2\theta) d\theta = \int \sin^{n-1}(2\theta)\sin 2\theta d\theta$

Integrating by parts regarding $\sin 2\theta$ as the 2nd function, we have

$$I_n = \sin^{n-1} 2\theta\left[-\frac{1}{2}\cos 2\theta\right] - \int\left\{(n-1)\sin^{n-2} 2\theta.\cos 2\theta.2\right\}\left(-\frac{1}{2}\cos 2\theta\right)d\theta$$

$$= -\frac{1}{2}\sin^{n-1} 2\theta\cos 2\theta + (n-1)\int \sin^{n-2} 2\theta.\cos^2 2\theta d\theta$$

$$= -\frac{1}{2}\sin^{n-1} 2\theta\cos 2\theta + (n-1)\int \sin^{n-2} 2\theta.(1 - \sin^2 2\theta)d\theta$$

$$= -\frac{1}{2}\sin^{n-1} 2\theta\cos 2\theta + (n-1)\int \sin^{n-2} 2\theta d\theta - (n-1)\int \sin^n 2\theta d\theta$$

$$= -\frac{1}{2}\sin^{n-1} 2\theta\cos 2\theta + (n-1)I_{n-2} - (n-1)I_n$$

$$\Rightarrow I_n + (n-1)I_n = -\frac{1}{2}\sin^{n-1} 2\theta\cos 2\theta + (n-1)I_{n-2}$$

$$\Rightarrow nI_n = -\frac{1}{2}\sin^{n-1} 2\theta\cos 2\theta + (n-1)I_{n-2}$$

$\Rightarrow I_n = -\frac{\sin^{n-1} 2\theta \cos 2\theta}{2n} + \frac{n-1}{n} I_{n-2}$, which is the required reduction formula.

Ans to Q No 5 : Given $I_n = \int_0^{\pi/4} \tan^n x dx,$

We know that $\int \tan^n x dx = \frac{1}{n-1} \tan^{n-1} x - \int \tan^{n-2} x dx$

$$\therefore I_n = \int_0^{\frac{\pi}{4}} \tan^n x dx = \left[\frac{\tan^{n-1} x}{n-1} \right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan^{n-2} x dx$$

$$= \frac{1}{n-1} - I_{n-2} \qquad \left[I_{n-2} = \int_0^{\frac{\pi}{4}} \tan^{n-2} x dx \right]$$

or $I_n + I_{n-2} = \frac{1}{n-1}$ ... (1)

Putting $n = 5$ in the reduction formula $I_n = \frac{1}{n-1} - I_{n-2}$

we get $I_5 = \frac{1}{4} - I_3 = \frac{1}{4} - \left[\frac{1}{2} - I_1 \right] = \frac{1}{4} - \frac{1}{2} + I_1$

$$\therefore I_5 = -\frac{1}{4} + \int_0^{\frac{\pi}{4}} \tan x dx$$

$$= -\frac{1}{4} + \left[\log \sec x \right]_0^{\frac{\pi}{4}}$$

$$= -\frac{1}{4} + \left[\log \sec \frac{\pi}{4} - \log \sec 0 \right]$$

$$= -\frac{1}{4} + \left[\log \sqrt{2} - \log 1 \right]$$

$$= -\frac{1}{4} + \frac{1}{2} \log 2$$

$$= \frac{1}{2} \log 2 - \frac{1}{4}.$$

Ans to Q No 6 : (a) We know that

$$\int \cot^n x\,dx = -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x\,dx$$

Putting $n = 5$ in the above reduction formula and applying repeatedly, we have

$$\int \cot^5 x\,dx = -\frac{1}{4}\cot^4 x\,dx - \int \cot^3 x\,dx$$

$$= -\frac{1}{4}\cot^4 x\,dx - \left[-\frac{1}{2}\cot^2 x - \int \cot x\,dx\right]$$

$$= -\frac{1}{4}\cot^4 x + \frac{1}{2}\cot^2 x + \log \sin x$$

Ans to Q No 7 : Do yourself.

Ans to Q No 8 :

$$\int_0^{\frac{\pi}{2}} \sin^7 x\,dx = \int_0^{\frac{\pi}{2}} \cos^7 x\,dx = \frac{6.4.2}{7.5.3.1} = \frac{16}{35}.$$

Ans to Q No 9 : Here $2m$ is even.

$$\int_0^{\frac{\pi}{2}} \sin^{2m} x\,dx = \frac{(2m-1)(2m-3)\cdots 3.1}{(2m)(2m-2)\cdots 4.2}.\frac{\pi}{2}$$

$$= \frac{2m(2m-1)(2m-2)(2m-3)\cdots 3.2.1}{\{(2m)(2m-2)\cdots 4.2\}^2}.\frac{\pi}{2}$$

[Multiplying Nr and Dr by $2m(2m-2)(2m-4)\cdots 4.2$]

$$= \frac{(2m)!}{\{2^m m(m-1)(m-2)\cdots 2.1\}^2}.\frac{\pi}{2}$$

$$= \frac{(2m)!}{\{2^m m!\}^2}.\frac{\pi}{2}$$

Ans to Q No 10 : Do yourself.

Ans to Q No 11 : Do yourself.

Ans to Q No 12 : Do yourself.

Ans to Q No 13 : We have $I_n = \int_0^{\pi/2} x^n \sin x dx$

$$= \left[x^n.(-\cos x)\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} n.x^{n-1}.(-\cos x)dx$$

[Integrating by parts]

$$= 0 + n\int_0^{\frac{\pi}{2}} .x^{n-1} \cos x.dx$$

$$= n\left[\left[x^{n-1}.\sin x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (n-1)..x^{n-2}.\sin x.dx\right]$$,again integrating by parts

$$= n\left(\frac{1}{2}\pi\right)^{n-1} - n(n-1)\int_0^{\frac{\pi}{2}} .x^{n-2}.\sin x.dx$$

$$\therefore I_n = n\left(\frac{1}{2}\pi\right)^{n-1} - n(n-1)I_{n-2}$$

$$\therefore I_n + n(n-1)I_{n-2} = n\left(\frac{1}{2}\pi\right)^{n-1}$$

Ans to Q No 14 : Do yourself.

Ans to Q No 15 : Do yourself.

Ans to Q No 16 : Do yourself.

Ans to Q No 17 : Given, $I_n = \int x^n e^{-ax} dx$

$$= x^n \int e^{-ax} dx - \int \left\{\frac{d}{dx} x^n \int e^{-ax} dx\right\} dx$$

$$= x^n \frac{e^{-ax}}{-a} - \int nx^{n-1}\left(\frac{e^{-ax}}{-a}\right) dx$$

$$= -\frac{x^n e^{-ax}}{a} + \frac{n}{a}\int x^{n-1} e^{-ax} dx$$

$$= -\frac{x^n e^{-ax}}{a} + \frac{n}{a} I_{n-1}$$

$$\Rightarrow I_n = -\frac{x^n e^{-ax}}{a} + \frac{n}{a} I_{n-1}$$

$$\therefore aI_n = -x^n e^{-ax} + nI_{n-1}$$



3.6 MODEL QUESTIONS

Q 1 : (i) Show that $\int x^3 e^{ax} dx = \frac{e^{ax}}{a}\left(x^3 - \frac{3x^2}{a} + \frac{6x}{a^2} - \frac{6}{a^3}\right)$

(ii) If $I_n = \int e^{-x} x^n dx$ then show that $I_n = e^{-x} x^n + nI_{n-1}$

Q 2 : (i) Show that $\int_0^{\pi/2} \cos^6 x dx = \int_0^{\pi/2} \sin^6 x dx = \frac{5\pi}{32}$

(ii) Prove that $\int_0^{\pi/2} \sin^4 x \cos^5 x dx = \frac{8}{315}$

(iii) Show that $\int_0^1 x^2 \left(1 - x^2\right)^{3/2} dx = \frac{\pi}{32}$

Q 3 : Show that $\int \frac{dx}{\sin^4 \frac{x}{2}} = \frac{4}{3} \cot \frac{x}{2}$

Q 4 : If $I_n = \int x^n \cos bx dx$ and $J_n = \int x^n \sin bx dx$ then show that $bI_n = x^n \sin bx - nJ_{n-1}$.

*** ***** ***

UNIT 4 : QUADRATURE OF PLANE AREAS

UNIT STRUCTURE

4.1 Learning Objectives

4.2 Introduction

4.3 Quadrature of Plane Curves

4.3.1 Area Between Two Given Curves

4.4 Let Us Sum Up

4.5 Further Readings

4.6 Answers to Check Your Progress

4.7 Model Questions

4.1 LEARNING OBJECTIVES

After going through this unit, you will be able to find

- the area of a plane curve
- the areas between two given curves.

4.2 INTRODUCTION

Integration has a large number of applications in science, engineering and other branches of study. One of the important problems of integration is to evaluate the area of a region. In fact, integral calculus was developed in the attempt to calculate the area bounded by curves by dividing it into infinitesimal parts and then finding the sum of these parts. The process of finding the area bounded by curves is called "Quadrature".

4.3 QUADRATURE OF PLANE AREAS : CARTESIAN COORDINATES

Let $y = f(x)$ be a continuons function in the interval $[a, b]$. Let $y=f(x)$ be represented graphically by the curve PQ. Let AC and BD be two ordinates corresponding to the points $x = a$ and $x = b$. From the Fig. 4.1 we have $OA = a$ and $OB = b$.

$\therefore$ AB $= b - a$

Now let us divide AB into n equal parts (sub-intervals) where each part is equal to h.

$\therefore$ $b = a + nh$

Let the points of division of AB beA, A_1, A_2, A_3... ,B.

Let the ordinates at the points of division of AB meet the curve PQ at C, C_1, C_2,... ... ..,D.

Then the abscissa and ordinate of each of C, C_1, C_2... ... ...D are

For Points	Abscissa	Ordinate
C	a	$f(a)$
C_1	$a+h$	$f(a+h)$
C_2	$a+2h$	$f(a+2h)$
...	...	...
Finally		
D	$a+nh$	$f(a+nh)$

Now let us construct the inner rectangles with the consecutive points below the curve PQ as shown in the Fig. 4.1. The rectangles so formed are ACP_1A_1, $A_1C_1P_2A_2$... etc.

Let S denote the actual area enclosed between the curve $y = f(x)$, the two ordinates $x = a$, $x = b$ and the x-axis.

If S_1 denotes the sum of the areas of all the inner rectangles, then

$$S_1 = hf(a) + hf(a+h) + \ldots\ldots\ldots + hf\left(a + \overline{n-1}h\right)$$
$$= h\left[f(a) + f(a+h) + \ldots\ldots\ldots + f\left(a + \overline{n-1}h\right)\right]$$
$$= h\sum_{r=0}^{n-1} f(a+rh)$$

We can also construct the outer rectangles above the curves. If S_2 denotes the sum of the areas of all the outer rectangles then we can write

$$S_2 = hf(a+h) + hf(a+2h) + \ldots\ldots\ldots + hf(a+nh)$$
$$= hf(a) + hf(a+h) + \ldots\ldots\ldots + hf\left(a + \overline{n-1}h\right) + hf(a+nh) - hf(a)$$
$$= h\sum_{r=0}^{n-1} f(a+rh) + hf(a+nh) - hf(a)$$
$$= h\sum_{r=0}^{n-1} f(a+rh) + hf(b) - hf(a)$$

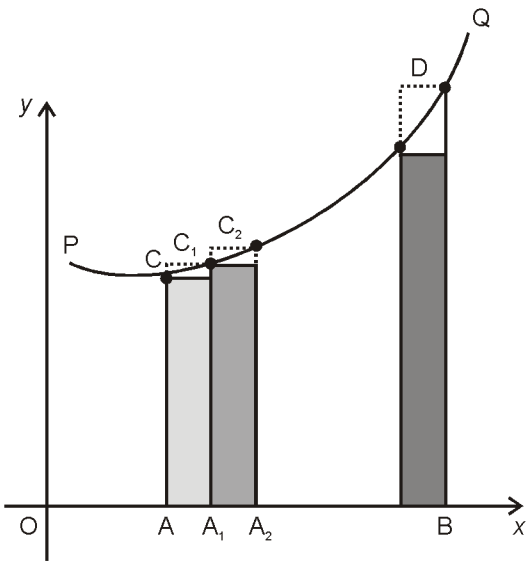


Fig. 4.1

Now it is evident that

$$S_1 < S < S_2$$

or $\sum_{r=0}^{n-1} f(a+rh) < S < \sum_{r=0}^{n-1} f(a+rh) + hf(b) - hf(a)$

If we take $n \to \infty$, then $h \to 0$ and so

$$\lim_{h\to 0} \sum_{r=0}^{n-1} f(a+rh) < S < \lim_{h\to 0} h\sum_{r=0}^{n-1} f(a+rh) + \lim_{h\to 0} h[f(b) - f(a)]$$

i.e. $\int_a^b f(x)dx < S < \int_a^b f(x)dx$

$\therefore\ S = \int_a^b f(x)dx$

Thus the definite integral $\int_a^b f(x)dx$ geometrically represents the area enclosed by the curve $y = f(x)$, the ordinates $x = a$ and $x = b$ and the x-axis.

$\therefore$ Area of ABCD $= \int_a^b f(x)dx = \int_a^b ydx$

Note : It can be shown that the area bounded by the curve $x = f(y)$, y-axis and the abscissae $y = a$ and $y = b$ is given by $\int_a^b xdy$ or $\int_a^b f(y)dy$.

ILLUSTRATIVE EXAMPLES

Example 1: Find, by integration, the area bounded by the x-axis, the y-axis and the line $y - x = 2$.

Solution : The line $y - x = 2$ cuts the y-axis at B(0, 2) and the x-axis at A(–2, 0). Therefore we are to find the area of the shaded region AOB shown in the Fig. 4.2.

$\therefore$ Required area

$$= \int_{-2}^{0} ydx = \int_{-2}^{0} (x+2)dx = \left[\frac{x^2}{2} + 2x\right]_{-2}^{0}$$

$$= [0] - [2-4] = 2\,\text{sq.unit.}$$

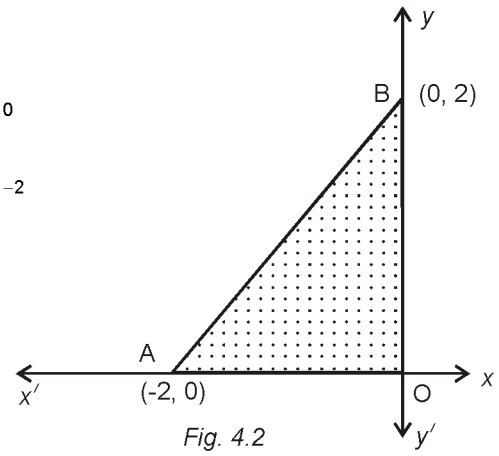


Fig. 4.2

Example 2: Find, by the method of integration, the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution : The equation of the curve is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

i.e. $y^2=\frac{b^2}{a^2}\left(a^2-x^2\right)$

The curve is symmetrical about both the axes. Hence the whole area of the ellepse is 4 times the area in the first quadrant. The area in the first quadrant is area of the region OAB.

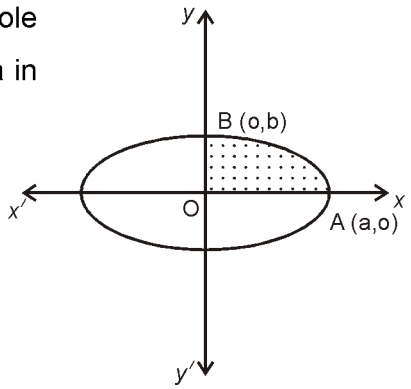


Fig. 4·3

Hence the required area

$$=4\int_0^a y\,dx$$

$$=4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx=\frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\,dx$$

$$=\frac{4b}{a}\left[\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$=\frac{4b}{a}\left[\frac{a^2}{2}\sin^{-1}(1)-\frac{a^2}{2}\sin^{-1}(0)\right]=\frac{4ab}{2}.\frac{\pi}{2}=\pi ab$$

Example 3 : Find the area bounded by the curve $y=\sin x$,the x-axis between $x=0$ to $x=2\pi$.

Solution : The graph of $y=\sin x$ between $x=0$ and $x=2\pi$ is shown in the Fig. 4.4.

From the figure it is seen that between $x=0$ and $x=\pi$, the curve lies above the *x*-axis and so ordinates are all positive. But between $x=\pi$ and $x=2\pi$ the curve lies below the *x*-axis and so the ordinates are all negative. We take absolute values to find the total area.

∴ The required area

$$=\left|\text{area OAB}\right|+\left|\text{area BCD}\right|$$

$$=\left|\int_0^{\pi}\sin x\,dx\right|+\left|\int_{\pi}^{2\pi}\sin x\,dx\right|$$

$$=\left|\left[-\cos x\right]_0^{\pi}\right|+\left|\left[-\cos x\right]_{\pi}^{2\pi}\right|$$

$$=|2|+|-2| \quad =2+2=4$$

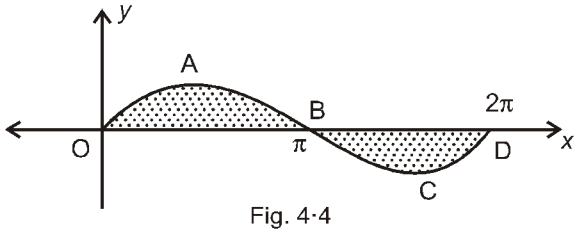


Fig. 4·4

Example 4 : Show that the area cut off a parabola by any double ordinate is two-third of the corresponding rectangles contained by that double ordinate and its distance from the vertex.

Solution : Let the parabola be $y^2 = 4ax$ and the double ordinate be $x = h$. Here ACB be the double ordinate.

$\therefore$ The required area cut off from the parabola by the double ordinate = area of AOBCA

$= 2 \times$ area of AOC (Since the parabola is symmetrical about x-axis)

$$= 2\int_{x=0}^{x=h} y\,dx$$

$$= 2\int_0^h \sqrt{4ax}\,dx$$

$$= \sqrt{a}\left[\frac{2}{3}x^{3/2}\right]_0^h = \frac{8}{3}\sqrt{a}.h^{3/2} \quad \ldots\ \ldots\ \ldots(i)$$

Here, the distance of the double ordinate ACB from the vertex $= OC = h$.

Let $AC = y_1$

$\therefore$ Coordinate of A are (h, y_1).

Since A is on the parabola, so its coordinate will satisfy the equation of the equation of the parabola $y^2 = 4ax$.

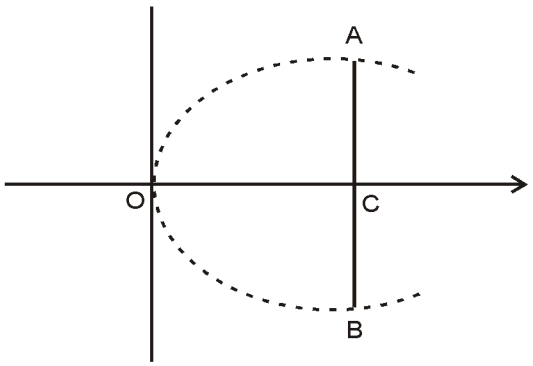


Fig. 4·5

$\therefore {y_1}^2 = 4ah \Rightarrow y_1 = 2\sqrt{ah}$

$\therefore$ Length of the double ordinate $ACB = 2y_1 = 4\sqrt{ah}$

$\therefore \frac{2}{3}\times$ area of the rectangle contained by the double ordinate ACB and its distance from the vertex

$$= \frac{2}{3} \times AB \times OC$$

$$= \frac{2}{3} \times 4\sqrt{ah} \times h = \frac{8}{3}\sqrt{a}h^{3/2} \quad \ldots\ \ldots\ \ldots(ii)$$

Hence (i) and (ii) are same.

3.3.1 Area between two given curves and two given ordinates

Let AB and CD be two curves represented by the equations $y = f_1(x)$ and $y = f_2(x)$ respectively.

Let the two ordinates be $x = a$ and $x = b$ indicated by P_1M_1 and P_2M_2 respectively.

$\therefore OM_1 = a$ and $OM_2 = b$.

Now the area included between the two curves $y = f_1(x)$ and $y = f_2(x)$, and the two ordinates $x = a$ and $x = b$ is the shaded part in the Fig. 4.6.

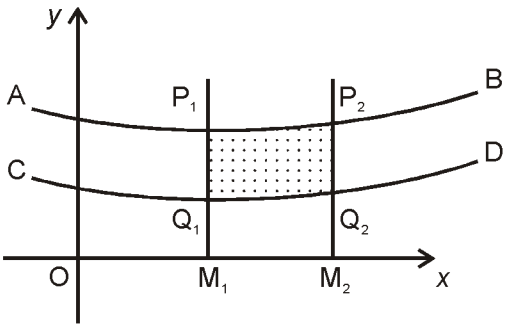


Fig. 4·6

$\therefore$ Required area = Area of $P_1M_1M_2P_2$ – area of $Q_1M_1M_2Q_2$

$$= \int_a^b f_1(x)dx - \int_a^b f_2(x)dx$$

$$= \int_a^b [f_1(x) - f_2(x)]dx$$

Example 5 : Find the area included between the curves $y^2 = 4ax$ and $y = mx$.

Solution : Here $y^2 = 4ax$ is a parabola whose vertex is at (0,0). Also $y = mx$ is a straight line passing through the origin and is inclined to x-axis at an angle θ where $\tan\theta = m$.

Now solving $y^2 = 4ax$ and $y = mx$ we get $(mx)^2 = 4ax$

$\Rightarrow x(xm^2 - 4a) = 0$

$\therefore x = 0$ or $x = \dfrac{4a}{m^2}$

$\therefore$ The two curves cut at the points

where $x = 0$ and $x = \dfrac{4a}{m^2}$.

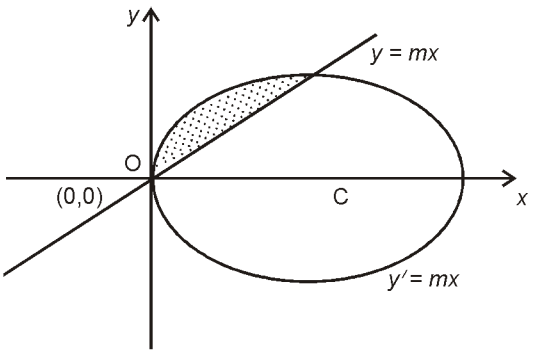


Fig. 4·7

$\therefore$ The required area is the shaded region which is included between the given curves.

$\therefore$ The required area $= \int_0^{4a/m^2} \left(\sqrt{4ax} - mx\right)dx$

$$= \left[2\sqrt{a}\frac{2}{3}x^{3/2}\right]^{4a/m^2} - \left[m\frac{1}{2}x^2\right]_0^{4a/m^2}$$

$$= \frac{4\sqrt{a}}{3}\left(\frac{4a}{m^2}\right)^{3/2} - \frac{m}{2}\left(\frac{4a}{m^2}\right)^2$$

$$= \frac{32a^2}{3m^3} - \frac{8a^2}{m^3} = \frac{8a^2}{3m^3}$$

Example 6 : Find the area of the region enclosed between the two circles $x^2 + y^2 = 1$ and $(x-1)^2 + y^2 = 1$.

Solution : Here $x^2 + y^2 = 1$ is a circle whose centre is O (0,0) and radius is 1.

For the second circle, centre is B(1,0) and radius (BO) = 1.

Now solving the given equations we find that the two curves intersect at $A\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $E\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

Now the required area is the region OEBAO, i.e. the shaded part (light shaded and dark shaded together). Here the dark shaded part (i.e. the region in the first quadrant) is symmetrical with the light shaded part (i.e. the region in the 4th quadrarnt).

$\therefore$ The required area

$= 2(\text{Area OCBAO})$

$= 2(\text{area OCAO} + \text{area CBAC})$ ... ...(i)

Area of OCAO

= area bounded by $(x-1)^2 + y^2 = 1$, x-axis and the ordinates $x = 0$ and $x = ½$.

$$= \int_0^{½} \sqrt{1-(x-1)^2}\, dx$$

Also the area of CBAC

= area bounded by $x^2 + y^2 = 1$, x-axis and the ordinates $x = ½$ and $x = 1$

$$= \int_{½}^{1} \sqrt{1-x^2}\, dx$$

$\therefore$ From (i), the total area of the region

$$= 2\left[\int_0^{1/2}\sqrt{1-(x-1)^2}\,dx + \int_{1/2}^{1}\sqrt{1-x^2}\,dx\right]$$

$$= 2\left[\frac{(x-1)\sqrt{1-(x-1)^2}}{2} + \frac{1}{2}\sin^{-1}\left(\frac{x-1}{1}\right)\right]_0^{1/2}$$

$$+ 2\left[\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1}(x)\right]_{1/2}^{1}$$

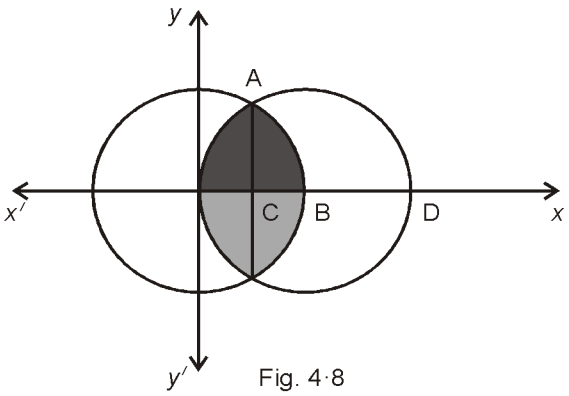


Fig. 4·8

$$= \left\{-\frac{\sqrt{3}}{4} + \sin^{-1}\left(-\frac{1}{2}\right) - \sin^{-1}(-1)\right\}$$

$$+ \left\{\sin^{-1}(1) - \frac{\sqrt{3}}{4} - \sin^{-1}\left(\frac{1}{2}\right)\right\}$$

$$= \left\{-\frac{\sqrt{3}}{4} + \sin^{-1}\left(-\frac{1}{2}\right) - \sin^{-1}(-1)\right\}$$

$$+ \left\{\sin^{-1}(1) - \frac{\sqrt{3}}{4} - \sin^{-1}\left(\frac{1}{2}\right)\right\}$$

$$= -\frac{\sqrt{3}}{4} - \frac{\pi}{6} + \frac{\pi}{2} + \frac{\pi}{2} - \frac{\sqrt{3}}{4} - \frac{\pi}{6}$$

$$= \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$$



CHECK YOUR PROGRESS - 1

Q 1 : Find the area of the region bounded by the parabola $y^2 = 4ax$ and its latus rectum.

Q 2 : Find the area of the region bounded by the upper half of the circle $x^2 + y^2 = 25$, the x-axis and the ordinates $x = -3$ and $x = 4$.

Q 3 : Find the area of the region bounded by the curve $x = y^3$, the y-axis and lines $y = 3, y = 6$.

Q 4 : Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Q 5 : Find the area of the circle $x^2 + y^2 = a^2$.

Q 6 : Find the area of the segment cut off from $y^2 = 4x$ by the line $y = x$.

Q 7 : Find the area of the region included between the parabolas $y^2 = 4ax$ and $x^2 = 4by$.

Q 8 : Find the area of the region cut off from the parabola by $4y = 3x^2$ by the line $3x - 2y + 12 = 0$.

Q 9 : Find the area of the region which is common to the circle $x^2 + y^2 = 16a^2$ and the parabola $y^2 = 6ax$.

Q 10 : Show that the area of the region bounded by the curves $y^2 = 16x$ and $x = 4$ is $\frac{128}{3}$ sq. unit.

Q 11 : Find the area bounded by the curve $x^2 = 4y$ and the straight line $x - 4y + 2 = 0$.

Q 12 : Show that the area bounded by the parabola $y^2 = 4ax$ and the lines $y = 2a$ and $x = 0$ is $\frac{2}{3}a^2$ sq. unit.

Q 13 : Find the area bounded by the curves $y = x^3$ and $y = x$.

Q 14 : Find the area of the region bounded by $x^2 = 16y$, $y = 1$, $y = 4$ and the y-axis lying in the first quadrant.

Q 15 : Find the area common to the circle $x^2 + y^2 = 4$ and the ellipse $x^2 + 4y^2 = 9$.

Q 16 : Show that the larger of the two areas into which the circle $x^2 + y^2 = (8a)^2$ is divided by the parabola $y^2 = 12ax$ is

$$\frac{16a^2}{3}\left(8\pi - \sqrt{3}\right).$$

Q 17 : Show that the area common to the two curves $x^2 - 4ax + y^2 = 0$ and $y^2 = ax$ is $\left(3\sqrt{3} + \frac{4}{3}\pi\right)a^2$.

Q 18 : Find the area, lying above x-axis and included between the circle $x^2 - 8x + y^2 = 0$ and the parabola $4x = y^2$.



4.4 LET US SUM UP

- The area enclosed by the curve $y = f(x)$, the ordinates $x = a$ and $x = b$ and the x-axis is given by $\int_a^b f(x)dx$ or $\int_a^b ydx$.
- The area bounded by the curve $x = f(y)$, y-axis and the abscissae $y = a$ and $y = b$ is given by $\int_a^b xdy$ of $\int_a^b f(y)dy$.
- The area between two curves $y = f_1(x)$ and $y = f_2(x)$ and two ordinates $x = a$ and $x = b$ is given by $\int_a^b [f_1(x) - f_2(x)]dx$.



4.5 FURTHER READINGS

1) *Integral Calculus,* Kishan, Hari, Atlantic Publishers & Distributors. New Delhi -110027.
2) *Integral Calculus*, Dhami, H.S. New age International Publishers, New Delhi - 110002.
3) *Text book of Integral Calculus*, Sharma, A.K. Discovery Publishing House, New Delhi -110002.



4.6 ANSWERS TO CHECK YOUR PROGRESS

CHECK YOUR PROGRESS – 1

Ans to Q No 1 : The latus rectum is the line perpendicular to the axis of the parabola and passing through the focus S(a,0). The parabola is symmetric about the x-axis. Limits for the upper half of the curve are $x = 0$ to $x = a$. The required area AOBSA = 2 area AOSA

$$= 2\int_0^a ydx$$

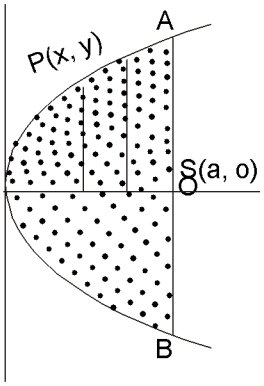


Fig. 4.9

$$= 2\int_0^a 2\sqrt{ax}dx$$

$$= 4\sqrt{a}\frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^a$$

$$= \frac{8}{3}\sqrt{a}a^{\frac{3}{2}}$$

$$= \frac{8}{3}a^2.$$

Ans to Q No 2 : Do yourself.

Ans to Q No 3 : Do yourself.

Ans to Q No 4: The area BPAO $= \int_0^a ydx$

But $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ or $y=\frac{b}{a}\sqrt{a^2-x^2}$

$$= \int_0^a \frac{b}{a}\sqrt{a^2-x^2}dx$$

$$= \frac{b}{a}\int_0^a \sqrt{a^2-x^2}dx$$

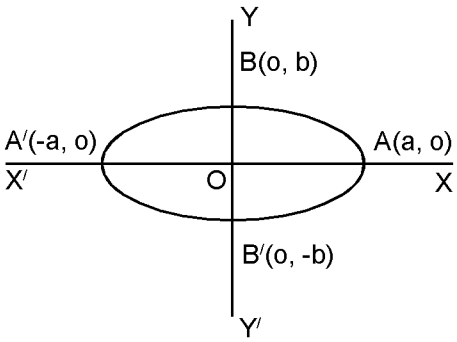


Fig. 4.10

$$= \frac{b}{a}\left[\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \frac{b}{a}.\frac{a^2}{2}\sin^{-1}1$$

$$= \frac{1}{2}ab.\frac{\pi}{2}$$

$$= \frac{\pi}{4}ab$$

$\therefore$ The whole area of the ellipse $= 4\times\frac{\pi}{4}ab = \pi ab$.

Ans to Q No 5 : The area OAB $= \int_0^a ydx$

$$= \int_0^a \sqrt{a^2-x^2}dx \qquad \left(y^2 = a^2 - x^2\right)$$

[$y = +\sqrt{a^2 - x^2}$, y being positive in the 1st and 2nd quadrants]

$$= \left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \frac{a^2}{2}.\frac{\pi}{2}$$

$$= \frac{\pi}{4}a^2$$

$\therefore$ area of the complete circle $= 4 \times \frac{\pi}{4}a^2 = \pi a^2$.

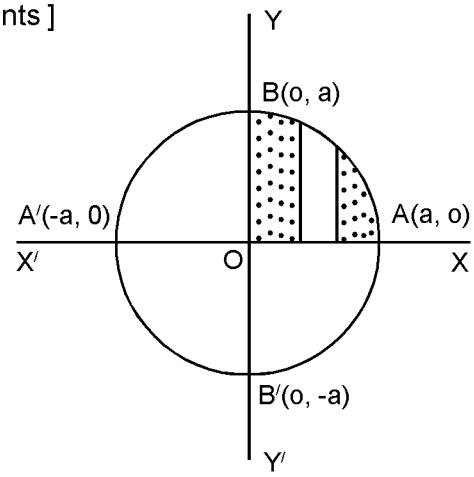


Fig. 4.11

Ans to Q No 6 : Do yourself.

Ans to Q No 7 : Solving the equations of the two given curves, we have

$$y^4 = 16a^2(4by) = 64a^2by$$

$$\therefore y(y^3 - 64a^2b) = 0$$

$$\Rightarrow y = 0, 4a^{\frac{2}{3}}b^{\frac{1}{3}}$$

when $y = 0, x = 0$

and when $y = 4a^{\frac{2}{3}}b^{\frac{1}{3}}, x = 4a^{\frac{1}{3}}b^{\frac{2}{3}}$.

A $\left(4a^{\frac{1}{3}}b^{\frac{2}{3}}, 4a^{\frac{2}{3}}b^{\frac{1}{3}}\right)$.

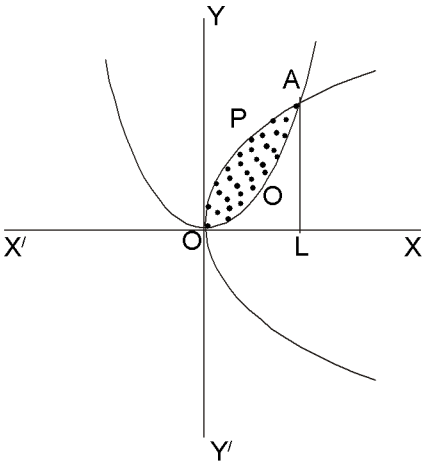


Fig. 4.12

$\therefore$ The required area = area OPAL- area OQAL.

$$= \int_0^{4a^{\frac{1}{3}}b^{\frac{2}{3}}} y\,dx \text{ , (from the curve } y^2 = 4ax) - \int_0^{4a^{\frac{1}{3}}b^{\frac{2}{3}}} y\,dx \text{ (from the curve } x^2 = 4by).$$

$$= \int_0^{4a^{\frac{1}{3}}b^{\frac{2}{3}}} \sqrt{4ax}\,dx - \int_0^{4a^{\frac{1}{3}}b^{\frac{2}{3}}} \left(\frac{x^2}{4b}\right)dx$$

$$= 2\sqrt{a}\left[\frac{2x^{\frac{3}{2}}}{3}\right]_0^{4a^{\frac{1}{3}}b^{\frac{2}{3}}} - \frac{1}{4b}\left[\frac{x^3}{3}\right]_0^{4a^{\frac{1}{3}}b^{\frac{2}{3}}}$$

$$= 2\sqrt{a}\left[8\sqrt{a}.b\right] - \frac{1}{12b}\left(64ab^2\right)$$

$$= \frac{32}{3}ab - \frac{16}{3}ab$$

$$= \frac{16}{3}ab.$$

Ans to Q No 8 : Do yourself.

Ans to Q No 9 : Solving the two equations,we get $x = 2a$, $y = \pm 2\sqrt{3}a$

Required area = 2 [area OPR + area RAP]

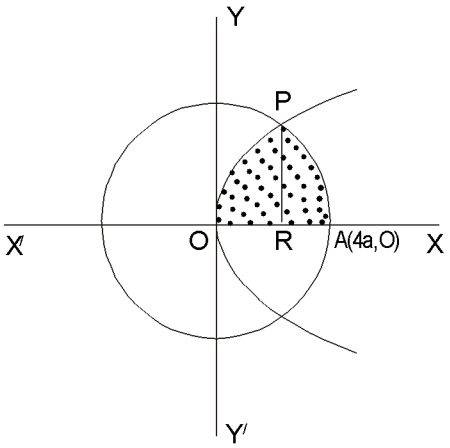


Fig. 4.13

$$= 2\int_0^{2a}\sqrt{6ax}dx + 2\int_{2a}^{4a}\sqrt{(4a)^2 - x^2}\,dx$$

$$= 2\sqrt{6a}\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^{2a} + 2\left[\frac{x}{2}\sqrt{16a^2 - x^2} + \frac{16a^2}{2}\sin^{-1}\frac{x}{4a}\right]_{2a}^{4a}$$

$$= 2\sqrt{6a}.(2a)^{\frac{3}{2}} + 2\left[8a^2.\frac{\pi}{2} - a.2\sqrt{3}a - 8a^2\frac{\pi}{6}\right]$$

$$= \frac{4a^2}{3}\left(4\pi + \sqrt{3}\right) \text{ sq.units}$$

Ans to Q No 10 : Do yourself.

Ans to Q No 11 : The equations of the given curves are

$$x^2 = 4y \quad \ldots (1)$$

and $$x = 4y - 2 \quad \ldots(2)$$

Equation (1) represents a parabola with vertex at the origin and axis along positive direction of y-axis. Equation (2) represents a straight line which meets the coordinate axes at (-2,0) and $\left(0, \frac{1}{2}\right)$ respectively.

We solve (1) and (2) simultaneously.We obtain that the points of intersection of the given parabola and the line are B(2,1) and $A\left(-1, \frac{1}{4}\right)$.

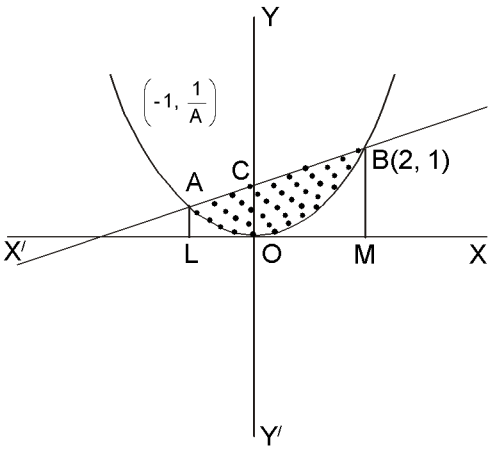


Fig. 4.14

We draw AL and BM perpendicular to x-axis.

$\therefore$ Area OBAO = Area OBCO + Area OACO ... (1)

then Area OBAO = Area OBCO - Area OMBO

$$= \int_0^2 \left(\frac{x+2}{4} - \frac{x^2}{4} \right) dx$$

$$= \left[\frac{x^2}{8} + \frac{1}{2}x - \frac{x^3}{12} \right]_0^2$$

$$= \frac{1}{4}(2 + 4) - \frac{1}{4}\left(\frac{8}{3} \right)$$

$$= \frac{5}{6}$$

Similarly, Area OACO = Area OLAC - Area OLAO

$$= \int_{-1}^0 \left(\frac{x+2}{4} - \frac{x^2}{4} \right) dx$$

$$= \left[\frac{x^2}{8} + \frac{1}{2}x - \frac{x^3}{12} \right]_{-1}^0$$

$$= \frac{1}{4}\left(\frac{1}{2} - 2 \right) - \frac{1}{12}$$

$$= \frac{1}{2} - \frac{1}{8} - \frac{1}{12}$$

$$= \frac{7}{24}$$

$\therefore$ Required area $= \frac{5}{6} + \frac{7}{24} = \frac{9}{8}$ units.

Ans to Q No 12 : The equation $y^2 = 4ax$ represents a parabola with vertex (0, 0) and axis as x-axis. The equation $y = 2a$ represents a straight line parallel to x-axis at a distance 2a from it.

$\therefore$ Required area $= \int_0^{2a} x\,dy$

$$= \int_0^{2a} \frac{y^2}{4a} dy$$

$$= \frac{1}{4a}\left[\frac{y^3}{3} \right]_0^{2a}$$

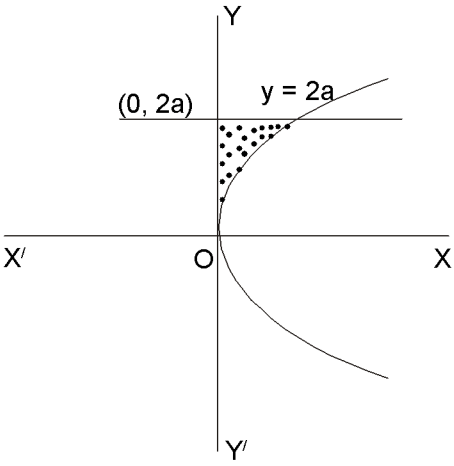


Fig. 4.15

$$= \frac{1}{4a}\left[\frac{8a^3}{3} - 0\right]$$

$$= \frac{2a^2}{3} \text{ sq. units}$$

Ans to Q No 13 : The equations of the given curves are $y = x$ ... (1)

and $y = x^3$ ... (2)

Solving (1) and (2), we find that two curves intersect at O(0,0), A(1,1) and B(-1,-1).

$\therefore$ Required area = Area BCOB + Area ODAO

Area BCOB $= \left|\int_{-1}^{0} (y_4 - y_3)dx\right|$

$= \left|\int_{-1}^{0} (x - x^3)dx\right|$ [since, $R(x, y_3)$ and $S(x, y_4)$ lie on (2) and (1)]

Area ODAO $= \int_{0}^{1} (y_2 - y_1)dx$

$= \int_{0}^{1} (x - x^3)dx$ [since, $P(x, y_1)$ and $Q(x, y_2)$ lie on (2) and (1)]

Hence, required area $= \left|\int_{-1}^{0} (x - x^3)dx\right| + \int_{0}^{1} (x - x^3)dx$

$$= \left|\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_{-1}^{0}\right| + \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_{0}^{1}$$

$$= \left|-\left(\frac{1}{2} - \frac{1}{4}\right)\right| + \left(\frac{1}{2} - \frac{1}{4}\right) = \frac{1}{2} \text{ sq. units}$$

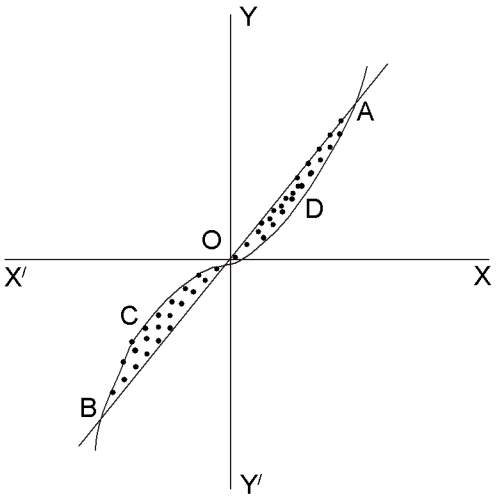


Fig. 4.16

Ans to Q No 14 : Do yourself.

Ans to Q No 15 : The equation of the circle is $x^2 + y^2 = 4$ ... (1)

and the equation of the ellipse is $x^2 + 4y^2 = 9$ ... (2)

Both the curves (1) and (2) are symmmetrical about both the axes.

Now, we solve (1) and (2) for x

$x^2 + 4(4 - x^2) = 9$ or $3x^2 = 7$ or $x^2 = \frac{7}{3}$.

$\therefore$ the x-coordinate of the point of intersection P lying in the first quadrant is $\sqrt{\frac{7}{3}}$. Also, putting $y = 0$ in $x^2 + y^2 = 4$, weget $x = 2$ at C.

$\therefore$ Required area = 4(common area lying in the first quadrant)

= 4 area OCPB

= 4 [area OBPM + area CPM]

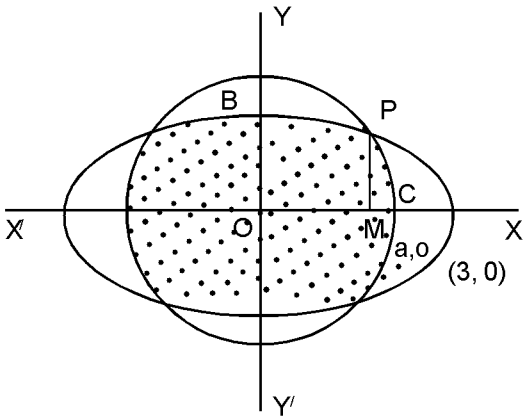


Fig. 4.17

$$= 4\left[\int_0^{\frac{7}{\sqrt{3}}} ydx, \text{for the ellipse} + \int_{\frac{7}{\sqrt{3}}}^{2} ydx, \text{for the circle}\right]$$

$$= 4\left[\int_0^{\frac{7}{\sqrt{3}}} \frac{1}{2}\sqrt{9-x^2}\,dx + \int_{\frac{7}{\sqrt{3}}}^{2} \sqrt{4-x^2}\,dx\right]$$

$$= 2\left[\frac{x\sqrt{9-x^2}}{2} + \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)\right]_0^{\frac{7}{\sqrt{3}}} + 4\left[\frac{x\sqrt{4-x^2}}{2} + 2\sin^{-1}\left(\frac{x}{2}\right)\right]_{\frac{7}{\sqrt{3}}}^{2}$$

$$= 2\left[\frac{1}{2}\sqrt{\frac{7}{3}}\sqrt{\frac{20}{3}} + \frac{9}{2}\sin^{-1}\left\{\frac{1}{2}\sqrt{\frac{7}{3}}\right\}\right] + 4\left[2\sin^{-1}(1) - \frac{1}{2}\sqrt{\frac{7}{3}}\sqrt{\frac{5}{3}} - 2\sin^{-1}\left(\frac{1}{2}\sqrt{\frac{7}{3}}\right)\right]$$

$$= \frac{2\sqrt{35}}{3} + 9\sin^{-1}\frac{\sqrt{7}}{3\sqrt{3}} + 4\pi - \frac{2}{3}\sqrt{35} - 8\sin^{-1}\frac{\sqrt{7}}{2\sqrt{3}}$$

$$= 4\pi + 9\sin^{-1}\frac{\sqrt{7}}{3\sqrt{3}} - 8\sin^{-1}\frac{\sqrt{7}}{2\sqrt{3}}$$

Ans to Q No 16 : $x^2 + y^2 = 64a^2$ is a circle with centre(0,0) and radius 8a and $y^2 = 12ax$ is a parabola whose vertex is at (0,0) and latus rectum 12a.Both the curves are symmetrical about x-axis.Solving the two equations the common point P is $\left(4a, 4a\sqrt{3}\right)$.We draw PM perpendicular from P to the y-axis.

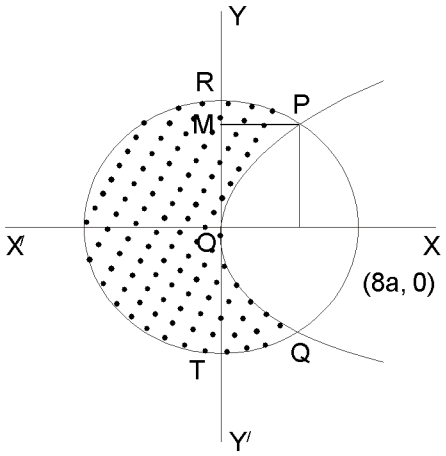


Fig. 4.18

Required area of the larger portion of the circle = Area of PRSTQOP = Area of the semi-circle RST + 2AreaOPR

$$= \frac{1}{2}.\pi.(8a)^2 + 2[\text{area OPM + area MPR }]$$

$$= \frac{1}{2}.\pi.(8a)^2 + 2\int_0^{\frac{4a}{\sqrt{3}}} x dy (\text{for } y^2 = 12ax) + 2\int_{\frac{4a}{\sqrt{3}}}^{8a} x dy (\text{for } x^2 + y^2 = 64a^2)$$

$$= \frac{1}{2}.\pi.(8a)^2 + 2\int_0^{\frac{4a}{\sqrt{3}}} \frac{y^2}{12a} dy + 2\int_{\frac{4a}{\sqrt{3}}}^{8a} \sqrt{64a^2 - y^2}\, dy$$

$$= 32\pi a^2 + \frac{1}{6a}\left[\frac{y^3}{3}\right]_0^{\frac{4a}{\sqrt{3}}} + 2\left[\frac{1}{2} y\sqrt{64a^2 - y^2} + \frac{64a^2}{2}\sin^{-1}\frac{y}{8a}\right]_{\frac{4a}{\sqrt{3}}}^{8a}$$

$$= 32\pi a^2 + \frac{1}{6a}\left[\frac{64 \times 3\sqrt{3}a^3}{3}\right] + 2\left[\left\{0 - 8a^2\sqrt{2}\right\} + 32a^2\left\{\sin^{-1}1 - \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right\}\right]$$

$$= 32\pi a^2 + \frac{32\sqrt{3}a^3}{3} - 16a^2\sqrt{3} + \frac{32}{3}a^2\pi$$

$$= \frac{128}{3}\pi a^2 - \frac{16}{3}a^2\sqrt{3}$$

$$= \frac{16}{3}a^2\left(8\pi - \sqrt{3}\right).$$

Ans to Q No 17: $y^2 = ax$ is a parabola with vertex at the origin and axis along x-axis and latus rectum a, and $x^2 - 4ax + y^2 = 0$ is a circle with centre (2a, 0) and radius 2a.

Both these curves are symmetrical about x-axis.

Solving the equations of the two curves for x, we have

$x^2 + ax = 4ax$ or $x^2 - 3ax = 0$ or $x(x-3a) = 0$

$\therefore x = 0, 3a$

Thus, the two curves intersect at the points where $x = 0$ and $x = 3a$.

Also A is the point $(4a, 0)$.

The area common to the parabola and the circle (i.e.,the shaded area)

= 2[Area OPS + Area PSA]

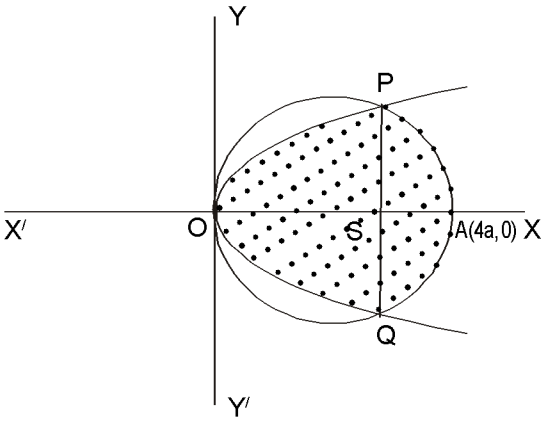


Fig. 4.19

$$= 2\left[\int_0^{3a} y\,dx, \text{from the parabola } y^2 = ax + \int_{3a}^{4a} y\,dx, \text{from the circle } x^2 + y^2 = 4ax\right]$$

$$= 2\left[\int_0^{3a} \sqrt{ax}\,dx + \int_{3a}^{4a} \sqrt{4ax - x^2}\,dx\right]$$

$$= 2\sqrt{a}\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^{3a} + 2\left[\frac{1}{2}(x-2a)\sqrt{\left\{4a^2 - (x-2a)^2\right\}} + \frac{4a^2}{2}\sin^{-1}\frac{x-2a}{2a}\right]_{3a}^{4a}$$

$$= 4a^2\sqrt{3} + 2\left[\left\{0 - \frac{1}{2}a\sqrt{3}.a\right\} + 2a^2\left\{\left(\frac{\pi}{2}\right) - \left(\frac{\pi}{6}\right)\right\}\right]$$

$$= 4\sqrt{3}a^2 - a^2\sqrt{3} + \frac{4}{3}\pi a^2$$

$$= 3\sqrt{3}a^2 + \frac{4}{3}\pi a^2$$

$$= a^2\left(3\sqrt{3} + \frac{4}{3}\pi\right).$$



4.7 MODEL QUESTIONS

Q 1 : Find the area of the quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ between the semi major and semi minor axes.

Q 2 : Find the area above the x-axis, included between the parabola $y^2=ax$ and the circle $x^2+y^2=2ax$.

Q 3 : Find the area common to the two curves $y^2=ax$ and $x^2+y^2=4ax$.

Q 4 : Find the area common between the curves $y^2=4ax$ and $x^2=4ay$.

Q 5 : Find the area between the parabola $y^2=8x$ and the line $x=2$.

*** ***** ***

UNIT 5 : RECTIFICATION OF PLANE CURVES

UNIT STRUCTURE

5.1 Learning Objectives

5.2 Introduction

5.3 Rectification of Plane Curves

5.4 Let Us Sum Up

5.5 Further Readings

5.6 Answers to Check Your Progress

5.7 Model Questions

5.1 LEARNING OBJECTIVES

After going through this unit, you will be able to

- define the term 'Rectification'
- find the length of a plane curve

5.2 INTRODUCTION

In previous unit, we have discussed quadratures of plane areas. In this unit we will discuss rectification of plane curves.

5.3 RECTIFICATION OF PLANE CURVES

The process of finding the length of an arc of a curve between two given points is called "rectification".

Let $P(x, y)$ and $Q(x + \delta x, y + \delta y)$ be two neighbouring points on the curve $y = f(x)$. Let s be the length of the arc from A to P. Also let δs be the length of the arc from P to Q.

PM and QN are perpendiculars from P and Q on x-axis. PS is perpendicular on QN. Then the chord PQ is joined. Now from the adjoining Fig. 5.1, we find

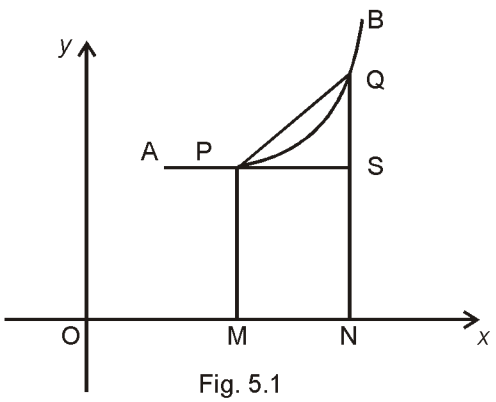


Fig. 5.1

$$(\text{Chord PQ})^2 = (\text{PS})^2 + (\text{QS})^2$$

$$\Rightarrow (\text{chord PQ})^2 = (\delta x)^2 + (\delta y)^2$$

Dividing both sides by $(\delta x)^2$, we get

$$\left(\frac{\text{Chord PQ}}{\delta x}\right)^2 = 1+\left(\frac{\delta y}{\delta x}\right)^2$$

$$\Rightarrow \left(\frac{\text{chord PQ}}{\text{arc PQ}}\cdot\frac{\text{arc PQ}}{\delta x}\right)^2 = 1+\left(\frac{\delta y}{\delta x}\right)^2$$

$$\Rightarrow \left(\frac{\text{chord PQ}}{\text{arc PQ}}\right)^2\cdot\left(\frac{\delta s}{\delta x}\right)^2 = 1+\left(\frac{\delta y}{\delta x}\right)^2$$

Now if $Q \to P$ along the curve, then $\delta x \to 0$ and so we get

$$\left(\frac{ds}{dx}\right)^2 = 1+\left(\frac{dy}{dx}\right)^2 \qquad \left[\lim_{Q\to P}\left(\frac{\text{chord PQ}}{\text{arc PQ}}\right) = 1\right]$$

Hence if s increases when x increases, then we get $\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$.

Integrating we get

$s = \int_a^x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ where the lower limit a is the abscissa of the fixed point from where s is measured (here a is the abscissa of the point A).

Hence the arc length s of the curve $y = f(x)$ included between two points for which $x = a$ and $x = b$ is equal to

$$s = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \qquad \text{where } b > a.$$

Note: If the curve is taken as $x = f(y)$ then the arc length between $y = c$ and $y = d$ is given by $\int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy, \quad d > c$.

ILLUSTRATIVE EXAMPLES

Example 1: Find the length of the circle $x^2 + y^2 = a^2$.

Solution: Here $x^2 + y^2 = a^2$

Differentiating w.r.t. x we get,

$$2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

$\therefore$ Total length of the curve (i.e. circumference)

= 4 × length of the arc in the first quadrant.

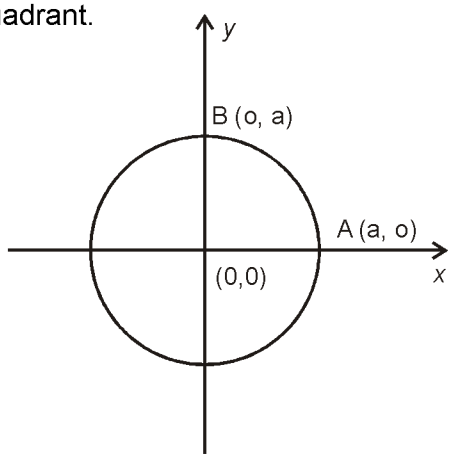


Fig. 5.2

$$= 4\int_{x=0}^{x=a} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 4\int_0^a \sqrt{\frac{a^2}{a^2 - x^2}}\, dx$$

$$= 4a\int_0^a \frac{1}{\sqrt{a^2 - x^2}}\, dx$$

$$= 4a\left[\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= 4a\left[\sin^{-1}(1) - \sin^{-1}(0)\right] = 4a\left(\frac{\pi}{2} - 0\right) = 2\pi a$$

Example 2 : Find the length of the arc of the parabola $y^2 = 4ax$ extending from the vertex to an extremity of the latus rectum.

Solution: For the parabola $y^2 = 4ax$ the coordinates of the vertex O are (0,0) and the coordinates of one extremity L of the latus rectum LSL′ are (a, 2a). Now we are to find the length of the arc from O to L.

Now, $y^2 = 4ax$

Differentiating w.r.t. *x, we get*

$$2y\frac{dy}{dx} = 4a,$$

$$\therefore \frac{dy}{dx} = \frac{2a}{y},$$

or $$\frac{dx}{dy} = \frac{y}{2a} \quad ...(i)$$

If *s* denotes the length of the parabola measured of the vertex O to any point $P(x, y)$ towards the point L, then S increases as *y* increases.

Therefore $\frac{ds}{dy}$ is positive. So we can write

Now, $$\left(\frac{ds}{dy}\right)^2 = 1 + \left(\frac{dx}{dy}\right)^2$$

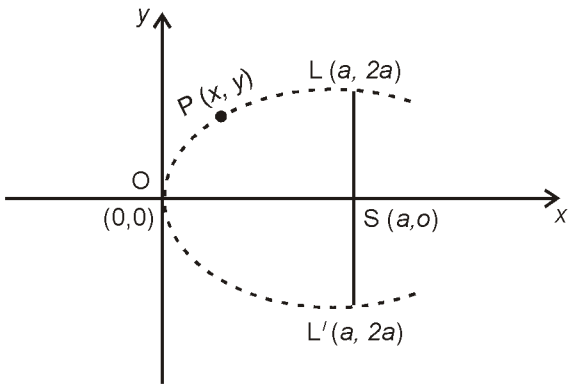


Fig. 5.3

$$= 1 + \left(\frac{y}{2a}\right)^2$$

$$= \frac{4a^2 + y^2}{4a^2}$$

$$\therefore \frac{ds}{dx} = \sqrt{\frac{4a^2 + y^2}{4a^2}} \Rightarrow ds = \frac{1}{2a}\sqrt{4a^2 + y^2}\,dy$$

$$s = \int_{y=0}^{y=2a} \frac{1}{2a}\sqrt{4a^2 + y^2}\,dy$$

$$= \frac{1}{2a}\left[\frac{y}{2}\sqrt{4a^2 + y^2} + \frac{4a^2}{2}\log\left\{y + \sqrt{4a^2 + y^2}\right\}\right]_0^{2a}$$

$$= \frac{1}{2a}\left[\frac{2a}{2}\sqrt{4a^2 + 4a^2} + 2a^2\log\left(2a + \sqrt{4a^2 + 4a^2}\right) - 2a^2\log 2a\right]$$

$$= \frac{1}{2a}\left[a.2a\sqrt{2} + 2a^2\log\left(2a + 2a\sqrt{2}\right) - 2a^2\log 2a\right]$$

$$= a\left[\sqrt{2} + \log\left(1 + \sqrt{2}\right)\right]$$

Example 3 : Find the whole length of the curve $x^{2/3} + y^{2/3} = a^{2/3}$.

Solution : The shape of the curve is as shown in the fig. 5.4. The curve is symmetrical about both the axes. The curve crosses *x*-axis at the points A(*a*,*o*) and C(–*a*,*o*). Also it crosses *y*-axis at the points B(*o*,*a*) and D(*o*,–*a*). From the figure it is seen that the arc of the curve in the first quadrant lies from A(*a*,*o*) to B(*o*,*a*).

Now, $x^{2/3} + y^{2/3} = a^{2/3}$

Differentiating w.r.t x,

$$\frac{dy}{dx} = -\left(\frac{x}{y}\right)^{-1/3} = -\left(\frac{y}{x}\right)^{1/3}$$

$$\therefore 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{y^{2/3}}{x^{2/3}} = \frac{x^{2/3} + y^{2/3}}{x^{2/3}} = \frac{a^{2/3}}{x^{2/3}}$$

$\therefore$ The whole length of the curve

$$= 4 \times \int_0^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = 4\int_0^a \sqrt{\frac{a^{2/3}}{x^{2/3}}}\,dx$$

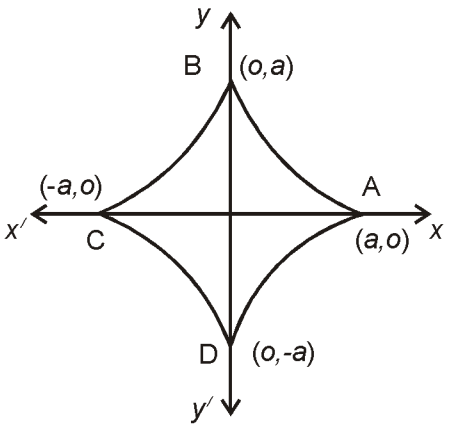


Fig. 5.4

$$= 4a^{1/3}\int_0^a x^{-1/3}dx = 4a^{1/3}\left[\frac{3}{2}x^{2/3}\right]_0^a = 6a^{1/3}.a^{2/3}$$

$$= 6a$$



CHECK YOUR PROGRESS

Q 1 : Find the perimeter of the circle $x^2 + y^2 = a^2$.

Q 2 : Find the length of the arc of the parabola $x^2 = 4ay$ cut off by its latus rectum.

Q 3 : Find the length of the arc of the parabola $y^2 = 4ax$ between the points where the line $y = 3x$ intersects the parabola.

Q 4 : Find the length of the arc of the parabola $x^2 = 4y$ from the vertex to the point where $x = 2$.

Q 5 : Find the lengths of the following curves between the points indicated with equations of the curves.

(a) $y = \log\sec x$, $x = 0$ and $x = \pi/3$

(b) $4y + \log x - 2x^2 = 0$, $x = 1$ and $x = 2$

(c) $ay^2 = x^3$, from the point (0,0) to the point (a,a)

(d) $6xy - x^4 = 3$, from $x = 1$ to $x = 2$

(e) $y = ae^x$, between the points (o,a) and (x,y).

Q 6 : Show that the length of the loop of the curve $3ay^2 = x(x-a)^2$ in $4a/\sqrt{3}$.

Q 7 : Find the total length of the curve $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$.

Q 8 : Find the length of the curve $x^{2/3} + y^{2/3} = 4$ from $x = 0$ to $x = 8$ in the first quadrant.



5.4 LET US SUM UP

- The process of finding the length of an arc of a curve between two given points is called "rectification".
- The arc length S of the curve $y = f(x)$ included between two points $x = a$ and $x = b$ is given by $S = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx,\ b > a$.
- If the curve is taken as $x = f(y)$ then the arc length between $y = c$ and $y = d$ is given by $S = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy,\ \ d > c$.



5.5 FURTHER READINGS

1) *Integral Calculus,* Kishan, Hari, Atlantic Publishers & Distributors. New Delhi -110027.

2) *Integral Calculus*, Dhami, H.S. New age International Publishers, New Delhi - 110002.

3) *Text book of Integral Calculus*, Sharma, A.K. Discovery Publishing House, New Delhi -110002.



5.6 ANSWERS TO CHECK YOUR PROGRESS

CHECK YOUR PROGRESS

Ans to Q No 1 : The equation of the curve is $x^2 + y^2 = a^2$ ... (1)

Here the curve is a circle with centre (0,0) and radius a. Also, it is symmetrical about both the axes. So, the required perimeter will be

four times the arc length lying in the first quadrant i.e.,between $x = 0$ to $x = a$.

Differentiating (1) w.r.t x, we get $2x + 2y\dfrac{dy}{dx} = 0$

or $\dfrac{dy}{dx} = -\left(\dfrac{x}{y}\right) = -\dfrac{x}{\sqrt{a^2 - x^2}}$ from (1)

$\therefore$ the required perimeter = 4 $\times$ {length of the arc in the first quadrant from (0, a) to (a, 0)}

$$= 4\int_{x=0}^{a}\sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}dx$$

$$= 4\int_{x=0}^{a}\sqrt{\left[1+\frac{x^2}{a^2-x^2}\right]}dx$$

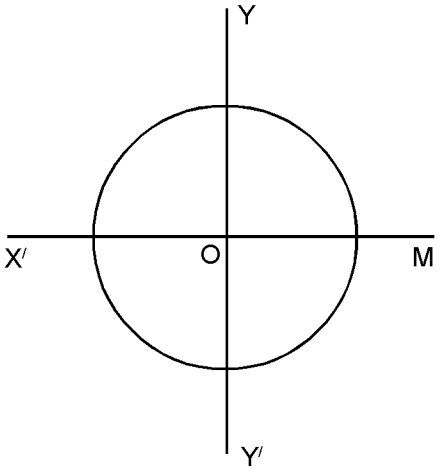
$$= 4a\int_{0}^{a}\frac{dx}{\sqrt{a^2-x^2}}dx$$

$$= 4a\left[\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= 4a\left[\sin^{-1}1 - \sin^{-1}0\right]$$

$$= 4a\left[\frac{\pi}{2} - 0\right]$$

$$= 2\pi a.$$



Ans to Q No 2 : Do yourself.

Ans to Q No 3 : We have $y^2 = 4ax$ ... (1)

and $y = 3x$ ... (2)

Substituting for x in (1) from (2), we get

$$y^2 = 4a\left(\frac{y}{3}\right)$$

or $3y^2 - 4ay = 0$

or $y(3y - 4a) = 0$

or $y = 0, \dfrac{4a}{3}$

Hence the points of intersection of the parabola and the given line are (0, 0) and $\left(\frac{4a}{9}, \frac{4a}{3}\right)$.

Also, differentiating (1) w.r.t. x, we get $2y\frac{dy}{dx} = 4a$.

$$\therefore \frac{dy}{dx} = \frac{2a}{y}$$

$$\text{or } \frac{dx}{dy} = \frac{y}{2a}$$

If S denotes the arc length of the parabola measured from the point (0, 0) to the point $\left(\frac{4a}{9}, \frac{4a}{3}\right)$, then

$$S = \int_0^{\frac{4a}{3}} \sqrt{\left[1+\left(\frac{dx}{dt}\right)^2\right]}dy$$

$$= \int_0^{\frac{4a}{3}} \sqrt{\left[1+\frac{y^2}{4a^2}\right]}dy$$

$$= \frac{1}{2a}\int_0^{\frac{4a}{3}} \sqrt{\left(y^2+4a^2\right)}dy$$

$$= \frac{1}{2a}\left[\frac{y}{2}\sqrt{\left(y^2+4a^2\right)} + \frac{4a^2}{2}\log\left\{y+\sqrt{\left(y^2+4a^2\right)}\right\}\right]_0^{\frac{4a}{3}}$$

$$= \frac{1}{2a}\left[\frac{1}{2}\frac{4}{3}a\sqrt{\left(\frac{16}{9}a^2+4a^2\right)} + 2a^2\log\left\{\frac{4}{3}a^2+\frac{2}{3}a\sqrt{13}\right\} - 2a^2\log(2a)\right]$$

$$= a\left[\frac{2}{9}\sqrt{13} + \log\left\{\frac{2}{3}+\frac{1}{3}\sqrt{13}\right\}\right]$$

Ans to Q No 4 : Do yourself.

Ans to Q No 5 : (a) The given curve is $y = \log\sec x$ ... (1)

Differentiating (1) w.r.t, x, we get

$$\frac{dy}{dx} = \frac{1}{\sec x}\sec x\tan x = \tan x$$

Now, $\left(\frac{ds}{dx}\right)^2 = 1+\left(\frac{dy}{dx}\right)^2 = 1+\tan^2 x = \sec^2 x$ ... (2)

If the arc length S of the given curve is measured from $x = 0$ in the direction of x increasing, we have

$\frac{dS}{dx} = \sec x$ or $dS == \sec x dx$

Therefore if S_1 denotes the arc length from $x = 0$ to $x = \frac{\pi}{3}$, then

$$\int_0^{S_1} dS = \int_0^{\frac{\pi}{3}} \sec x dx = \left[\log(\sec x + \tan x)\right]_0^{\frac{\pi}{3}}$$

or $S_1 = \left[\log\left(\sec\frac{\pi}{3} + \tan\frac{\pi}{3}\right) - \log 1\right] = \log\left(2+\sqrt{3}\right).$

(b) The given curve is $4y+\log x-2x^2 = 0$

or $y = \frac{1}{2}x^2 - \frac{1}{4}\log x$ ... (1)

Differentiating (1) w.r,t. x, we get

$$\frac{dy}{dx} = x - \frac{1}{4x} = \frac{4x^2-1}{4x}$$

$\therefore$ the required length of the curve $= \int_1^2 \sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}} dx$

$$= \int_1^2 \sqrt{\left\{1+\frac{\left(4x^2-1\right)}{16x^2}\right\}} dx$$

$$= \int_1^2 \frac{4x^2+1}{4x} dx$$

$$= \int_1^2 \left(x+\frac{1}{4x}\right) dx$$

$$= \left[\frac{x^2}{2} + \frac{\log x}{4}\right]_1^2$$

$$= \frac{3}{2} + \frac{1}{4}\log 2.$$

(c) The given curve is $ay^2 = x^3$ ... (1)

It is symmetrical about the x-axis.We have to find the length of the arc from

$x = 0$ to $x = a$ in the first quadrant.

Differentiating (1) w.r.t. x, we get $2ay\left(\frac{dy}{dx}\right) = 3x^2$

$$\therefore \frac{dy}{dx} = \frac{3x^2}{2ay} = \frac{3x^2}{2a\left(\frac{x^3}{a}\right)^{\frac{1}{2}}}$$, substituting for y from (1)

$$= \frac{3}{2}\left(\frac{x^{\frac{1}{2}}}{a^{\frac{1}{2}}}\right)$$

$$\therefore \text{required length} = \int_0^a \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}dx$$

$$= \int_0^a \sqrt{\left\{1+\frac{9x}{4a}\right\}}dx$$

$$= \frac{1}{2\sqrt{a}}\int_0^a (9x+4a)^{\frac{1}{2}}dx$$

$$= \frac{1}{2\sqrt{a}}\left[\frac{2}{3}..(9x+4a)^{\frac{3}{2}}.\frac{1}{9}\right]_0^a$$

$$= \frac{1}{2\sqrt{a}}\left[(13a)^{\frac{3}{2}} - (4a)^{\frac{3}{2}}\right]$$

$$= \frac{a}{27}\left[13\sqrt{13} - 8\right].$$

(d) Do yourself.

(e) Do yourself.

Ans to Q No 6 : The given curve is symmetrical about the x-axis.

At $y = 0$, we have $x = 0$ and $x = a$.

so, for the loop, x varies from 0 to a.

The equation of the given curve is $3ay^2 = x(x-a)^2$.

Taking logarthm, we have $\log 3a + 2\log y = \log x + 2\log(x-a)$

Now differentiating w.r.t x, we get $\frac{2}{y}\frac{dy}{dx} = \frac{1}{x} + \frac{2}{x-a} = \frac{3x-a}{x(x-a)}$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{(3x-a)^2}{x^2(x-a)^2}\cdot\frac{y^2}{4}$$

$$= \frac{(3x-a)^2\, x(x-a)^2}{x^2(x-a)^2.12a}$$

$$= \frac{(3x-a)^2}{12ax}$$

$$\therefore \frac{ds}{dx} = \sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}} = \sqrt{1+\frac{(3x-a)^2}{12ax}} = \frac{3x+a}{\sqrt{12ax}}$$

$\therefore$ the required length of the whole loop $= 2\int_0^a \sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}dx$

$$= 2\int_0^a \frac{3x+a}{\sqrt{12ax}}dx$$

$$= \frac{1}{\sqrt{3a}}\int_0^a \left(3x^{\frac{1}{2}} + ax^{-\frac{1}{2}}\right)dx$$

$$= \frac{1}{\sqrt{3a}}\left[3.\frac{2}{3}x^{\frac{3}{2}} + 2ax^{\frac{1}{2}}\right]_0^a$$

$$= \frac{1}{\sqrt{3a}}\left[4a^{\frac{3}{2}}\right]$$

$$= \frac{4a}{\sqrt{3}}.$$

Ans to Q No 7 : Do yourself.



5.7 MODEL QUESTIONS

Q 1 : Show that the length of the curve $y = \log \sec x$ between the points and $x = \pi/3$.

Q 2 : Find the perimeter of the circle $x^2 + y^2 = a^2$.

Q 3 : Find the perimeter of the hypo-cycloid $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$.

Q 4 : Prove that the length of the loop of the curve $3ay^2 = x(x-a)^2$ is $4a\sqrt{3}$.

Q 5 : Find the length of the arc of the parabola $x^2 = 4y$ from the vertex to the point where $x = 2$.

*** ***** ***

UNIT 6 : VOLUME OF REVOLUTION AND SURFACE OF REVOLUTION

UNIT STRUCTURE

6.1 Learning Objectives

6.2 Introduction

6.3 Volume of Solids of Revolution

6.4 Surface of Solids of Revolution

6.5 Let Us Sum Up

6.6 Further Readings

6.7 Answers to Check Your Progress

6.8 Model Questions

6.1 LEARNING OBJECTIVES

After going through this unit, you will be able to understand

- the volume of a solid of revolution
- the surface area of a solid of revolution

6.2 INTRODUCTION

In previous units,we have discussed quadratures and rectification of plane curves.In this unit we will focus on volume of solids of revolution as well as surface of solids of revolution.

6.3 VOLUMES OF SOLIDS OF REVOLUTION

Definition : Solid of revolution : If a plane curve or a plane area is revolved about a fixed line in its own plane then the body so generated by the revolution is called a solid of revolution.

Surface of revolution : If a plane curve is revolved about a fixed line in its own plane, then the surface so generated by the revolution of the curve is called a surface of revolution.

Axis of revolution : The fixed straight line about which the area or the curve revolves to form a solid or a surface is called the axis of revolution or axis of rotation.

For example, we may take a right angled triangle ABC where $\angle B = 90^0$. Now if ΔABC revolves about AB then it will generate a right circular cone. The area of the triangle will generate the volume and the hypotenuse AC will generate the curved surface of the cone.

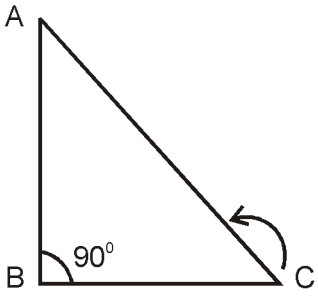


Fig. 6.1

The volume of the solid generated by the revolution of the area bounded by the curve $y = f(x)$, *x*-axis and the ordinates about the axis.

Let AB be the curve given by the equation $y = f(x)$. Let AL and BM be the two given ordinates $x = a$ and $x = b$. Let $P(x, y)$ and $Q(x + \delta x, y + \delta y)$ be any two neighbouring points on the curve. PC and QD be the perpendiculars drawn from P and Q respectively on *x*-axis. Also PS is perpendicular on QD and QR is perpendicular on CP produced.

Let the volumes of the solids generated by the revolution of the areas ALCPA and ALDQPA about *x*-axis be v and $v + \delta v$ respectively. Then δv is the volume of the solid generated by the revolution of the area PCDQP about *x*-axis.

Now $PC = y$, $QD = y + \delta y$ and $CD = \delta x$

$\therefore$ Volume generated by revolving the area $PCDS = \pi y^2 \delta x$ and the volume generated by revolving the area $RCDQ = \pi(y + \delta y)^2 \delta x$.

Also the volume generated by revolving the area $PCDQP = \delta v$, where δv lies between $\pi y^2 \delta x$ and $\pi(y + \delta y)^2 \delta x$.

i.e. $\pi y^2 \delta x \le \delta v \le \pi(y + \delta y)^2 \delta x$

$$\Rightarrow \pi y^2 \le \frac{\delta v}{\delta x} \le \pi(y + \delta y)^2$$

But if $Q \to P$ along the curve, then $\delta x \to o$ and also $\delta y \to o$ and then we get

$$\lim_{\delta x \to 0} \frac{\delta v}{\delta x} = \pi y^2$$

i.e. $$\frac{dv}{dx} = \pi y^2,$$

or $$dv = \pi y^2 dx$$

Integrating we get

$$\int_{x=a}^{b} dv = \int_{x=a}^{b} \pi y^2 dx$$

$$\Rightarrow [v]_a^b = \int_a^b \pi y^2 dx$$

$\Rightarrow$ (volume v at $x = b$) – (volume v at $x = a$)= $\int_a^b \pi y^2 dx$

$\Rightarrow$ (volume v at $x = b$) – 0 $= \int_a^b \pi y^2 dx$

$\Rightarrow$ Volume generated by the area ALMBA $= \int_a^b \pi y^2 dx$

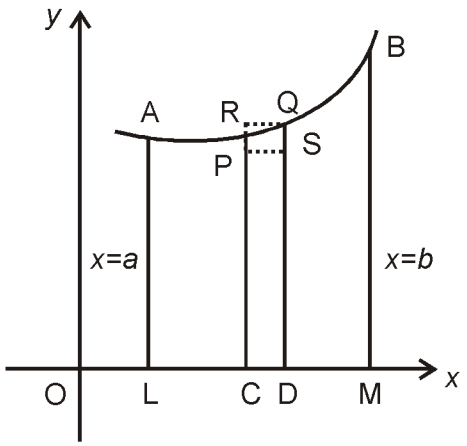


Fig. 6.2

$\therefore$ The volume of the solid generated by the revolution of the area bounded by the curve $y = f(x)$, x-axis and the ordinates $x = a$, $x = b$ about x axis is given by $\int_a^b \pi y^2 dx$.

ILLUSTRATIVE EXAMPLES

Example 1: Find the volume of a sphere of radius a.

Solution : Let $x^2 + y^2 = a^2$ be a circle whose centre is O(0,0) and radius is $OA = a$. Here OAB is the first quadrant of the circle. If this portion (or quadrant) of the circle is rotated about x-axis then it will form a hemisphere.

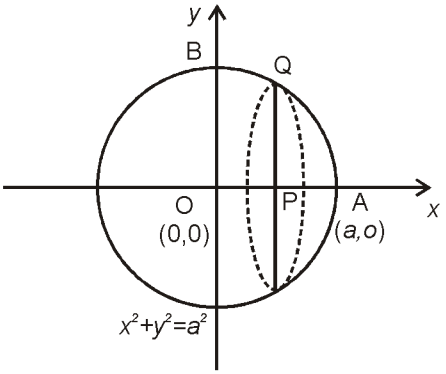


Fig. 6.3

$\therefore$ The volume of the hemisphere $= \int_0^a \pi y^2 dx$

$\therefore$ Required volume of the sphere (V)

$$= 2\int_0^a \pi y^2 dx = 2\pi \int_0^a (a^2 - x^2) dx = 2\pi \left[a^2 x - \frac{x^3}{3} \right]_0^a$$

$$= 2\pi \left(a^3 - \frac{a^3}{3} \right) = 2\pi . \frac{2a^3}{3} = \frac{4}{3}\pi a^3$$

Example 2 : Find the volume of the right circular cone obtained by the revolution of a right angled triangle about a side which contains the right angle.

Solution : Here OAB is the right angled triangle where $\angle A = 90^0$. If it is rotated about the side OA (i.e. about the x-axis) then the triangle will generate a right circular cone.

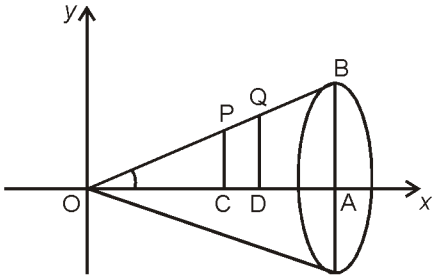


Fig. 6.4

Let $P(x, y)$ be a point on OB and let $\angle BOA = \alpha$.

The from ΔPOC, we get

$$\tan\alpha = \frac{PC}{OC} = \frac{y}{x}$$

$$\therefore y = x\tan\alpha \quad ...(i)$$

which is the equation of the line OB (OB is the hypotenuse of ΔOAB)

Now the area of the triangle is bounded by the line OB, x-axis and AB. If this area is rotated about x-axis then it will generate the right circular cone. Let OA=h be the height of the cone and AB=a be the radius of the circular base.

$$\therefore \tan\alpha = \frac{a}{h} \quad ...(ii)$$

Putting the value of $\tan\alpha$ from (i) in (ii) we get $y = x\left(\frac{a}{h}\right)$ which is the equation of the line OB expressed in terms of a and h.$\therefore$ Volume of the right circular cone (V)

$$= \int_0^h \pi y^2 dx = \int_0^h \pi \frac{x^2 a^2}{h^2} dx = \frac{\pi a^2}{h^2}\left[\frac{x^3}{3}\right]_0^h$$

$$= \frac{\pi a^2}{h^2}\frac{h^3}{3} = \frac{1}{3}\pi a^2 h$$

Example 3 : $y^2 = 4ax$ is a parabola. The latus rectum cuts off a part of the parabola and the remaining part revolves about the y axis at the vertex. Find the total volume thus generated.

Solution : The parabola $y^2 = 4ax$ is cut off by the latus rectum LSL'. The part left is LOL' which revolves about the y-axis where O is the vertex. The revolution will generate two volumes– one above x-axis and the other below x-axis where both the parts are identical. The area LMO that revolves is enclosed between the arc LO, y-axis and LM.

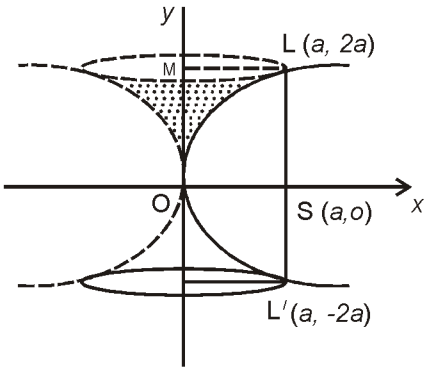


Fig. 6.5

$\therefore$ Total required volume = 2 x volume generatedby revolving the area OLM abouty y-axis.

$$= 2\int_0^{2a} \pi x^2 dy = 2\pi \int_0^{2a} \left(\frac{y^2}{4a}\right)^2 dy$$

$$= 2\pi . \frac{1}{16a^2}\left[\frac{y^5}{5}\right]_0^{2a} = 2\pi . \frac{1}{16a^2} . \frac{32a^5}{5}$$

$$= \frac{4}{5}\pi a^3 .$$



CHECK YOUR PROGRESS - 1

Q 1 : Find the volume of the solid formed by the rotation of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about the major axis.

Q 2 : Find the volume of the solid formed by the rotation of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about its minor axis.

Q 3 : Find the volume of the solid generated by the revolution of the curve $y = \sin x$ from the points where $x = 0$ to $x = 2\pi$ about the $x-$axis.

Q 4 : Find the volume of a hemisphere.

Q 5 : Find the volume of the paraboloid which is generated by the revolution about the *x*-axis of the parabola $y^2 = 4ax$ from $x = 0$ to $x = h$.

Q 6 : The astroid $x^{2/3} + y^{2/3} = a^{2/3}$ is revolved about the axis of x. Find the volume of the solid thus generated.

Q 7 : Find the volume of the solid generated by revolving about x-axis, the area bounded by the curve $y = 5x - x^2$, the line $x = 5$ and the y-axis.

Q 8 : Find the volume of the solid generated due to the revolution of $y^2 = 5x$ about the x axis between the ordinates $x = 0$ and $x = 9$.

6.4 SURFACE OF SOLIDS OF REVOLUTION

Suppose $f(x)$ be a continuous function of x where $f(x)$ does not change sign in the interval $a \le x \le b$. Now we are to find the area of the surface of the solid generated by revolving about the x-axis the area bounded by the curve $y = f(x)$ and the ordinates $x = a$ and $x = b$.

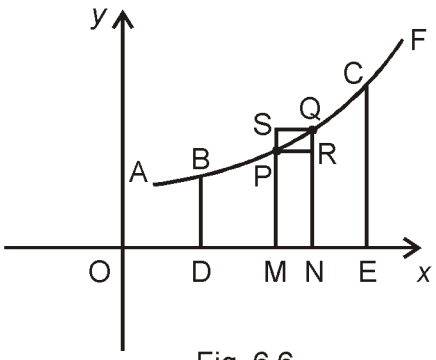


Fig. 6.6

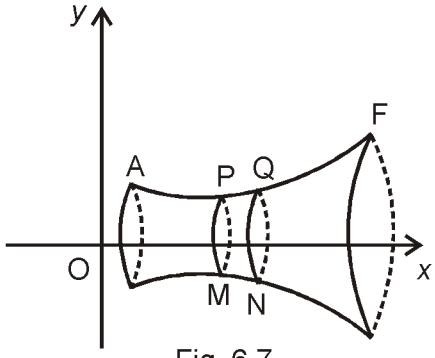


Fig. 6.7

Let the portion of the arc of the curve $y = f(x)$ which is included between the ordinates $x = a$ and $x = b$ be BC. Here the straight lines given by $x = a$ and $x = b$ are BD and CE respectively.

Let $P(x, y)$ and $Q(x + \delta x, y + \delta y)$ be two neighbouring points on $\overset{\frown}{BC}$. PM and QN are drawn perpendiculars on the x-axis PR is perpendicular on QN. QS is drawn perpendicular on extended MP.

Let length of the arc $AP = S$ and the length of the arc $PQ = \delta S$. Let A and $A + \delta A$ denote the areas of the surfaces of the solids generated by the revolution of the arc BP and arc BQ respectively about the x-axis. Then the area of the surface generated by arc PQ revolving about x-axis is δA.

Also, the lines PR and QS will generate two cylinders when they revolve about *x*-axis and the areas of the curved surfaces of these cylinders are $2\pi y\delta S$ and $2\pi(y+\delta y).\delta S$, (since Q is a point lying very very close to P, therefore PR, PQ and SQ are almost equal)

Therefore the area of the surface generated by revolving PQ about *x*-axis lies between $2\pi y\delta S$ and $2\pi(y+\delta y)\delta S$.

$$\therefore 2\pi y\delta s \le \delta A \le 2\pi(y+\delta y)\delta S$$

$$\Rightarrow 2\pi y \le \frac{\delta A}{\delta S} \le 2\pi(y+\delta y)$$

Now when $Q \to P$ along the curve then $\delta x \to 0$ and $\delta y \to 0$. Therefore

$$2\pi y \le \lim_{\delta S\to 0}\left(\frac{\delta A}{\delta S}\right) \le \lim_{\delta y\to 0} 2\pi(y+\delta y)$$

$$\therefore \frac{dA}{dS} = 2\pi y$$

$$\Rightarrow dA = 2\pi y dS$$

Integrating, $\int_a^b dA = \int_a^b 2\pi y dS$

$$\Rightarrow [A]_a^b = 2\pi\int_a^b y ds$$

$\Rightarrow$ (Value of A when $x=b$) – (value of A when $x=a$) $= 2\pi\int_a^b y ds$

$\Rightarrow$ Area of the surface generated by revolving the arc BC about *x*-axis $= 2\pi\int_{x=a}^{x=b} y ds$.

But we know that $\left(\frac{dS}{dx}\right)^2 = 1+\left(\frac{dy}{dx}\right)^2$

$$\therefore dS = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

Hence the area of the surface generated by revolving the arc BC about *x*-axis $= 2\pi\int_a^b y dS$

$$= 2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx.$$

ILLUSTRATIVE EXAMPLES

Example 1 : Find the area of the curved surface of a hemisphere of radius *a.*

Solution : Let $x^2 + y^2 = a^2$ be a circle whose centre is (0,0) and radius is a. Now if the arc AB of the circle (i.e. the quadrant AB) revolves about the radius OA (i.e. x-axis) then it will generate a hemisphere as shown in the Fig. 6.8.

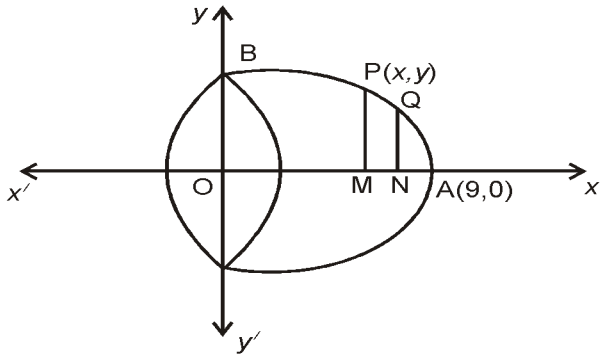


Fig. 6.8

Differentiating $x^2 + y^2 = a^2$

$$2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

$$\therefore \frac{dS}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{x^2}{y^2}} = \sqrt{\frac{x^2+y^2}{y^2}} = \sqrt{a^2/y^2} = a/y$$

$$\therefore ydS = adx$$

$$\therefore \text{ Required surface area } = 2\pi \int_{x=0}^{a} ydS = 2\pi \int_{x=0}^{a} adx$$

$$= 2\pi[ax]_0^a = 2\pi a^2$$

Example 2: Find the area of the surface formed by revolution about $x-$axis of the portion of arc of the parabola $y^2 = 4ax$ between the vertex and one extremity of the latus rectum.

Solution: Here the parabola is $y^2 = 4ax$

$$\therefore 2y\frac{dy}{dx} = 4a$$

$$\Rightarrow \frac{dy}{dx} = \frac{2a}{y}$$

$$\therefore \frac{dS}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\left(\frac{2a}{y}\right)^2} = \sqrt{\frac{y^2+4a^2}{y^2}}$$

$$= \sqrt{\frac{4ax + 4a^2}{y^2}} = \frac{2}{y}\sqrt{a(x+a)}$$

Now the coordinates of the vertex and one extremity of the latus rectum are (0,0) and (a, $2a$) respectively.

$\therefore$ Required surface area

$$= \int_{x=0}^{x=a} 2\pi y \frac{dS}{dx} dx = \int_0^a 2\pi . y . \frac{2}{y}\sqrt{a(x+a)}dx$$

$$= 4\pi\sqrt{a}\int_0^a (x+a)^{1/2}\, dx = 4\pi\sqrt{a}\left[\frac{2}{3}(x+a)^{3/2}\right]_0^a$$

$$= 4\pi\sqrt{a}.\frac{2}{3}\left[(2a)^{3/2} - (a)^{3/2}\right] = \frac{8\pi\sqrt{a}}{3}a^{3/2}\left(2\sqrt{2}-1\right)$$

$$= \frac{8}{3}\pi a^2\left(2\sqrt{2}-1\right)$$

Example 3 : Find the surface of the solid which is generated by the revolution of the ellipse $\frac{x^2}{36} + \frac{y^2}{16} = 1$ about its major axis.

Solution : The ellipse is $\frac{x^2}{36} + \frac{y^2}{16} = 1 \ldots \ldots \ldots$(i)

$\therefore a = 6$ and $b = 4$

Differentiating (i) w.r.t. x, $\frac{2x}{36} + \frac{2y}{16}\frac{dy}{dx} = 0$

$$\therefore \frac{dy}{dx} = -\frac{4x}{9y}$$

$$\therefore \frac{ds}{dx} = \sqrt{1 + \frac{16x^2}{81y^2}} = \frac{\sqrt{81y^2 + 16x^2}}{9y} = \frac{1}{9y}\sqrt{81.\frac{4}{9}(36 - x^2) + 16x^2}$$

$$= \frac{1}{9y}\sqrt{36(36 - x^2) + 16x^2} = \frac{1}{9y}\sqrt{(36)^2 - 20x^2} \ \ldots \ldots \ldots \text{(ii)}$$

Now the given ellipse is symmetrical about both the axes. For the arc of the ellipse which is lying in the first quadrant x varies from 0 to 6.

$\therefore$ The required area of the surface

$$= 2 \times \int_{x=0}^{6} 2\pi y \frac{ds}{dx} dx = 2\int_0^6 2\pi y . \frac{1}{9y}\sqrt{(36)^2 - 20x^2}\, dx$$

$$= \frac{4\pi}{9}\int_0^6 \sqrt{(36)^2 - 20x^2}\,dx = \frac{4\pi}{9}\sqrt{20}\int_0^6 \sqrt{\frac{1296}{20} - x^2}\,dx$$

$$= \frac{8\sqrt{5}\pi}{9}\int_0^6 \sqrt{\left(\frac{18}{\sqrt{5}}\right)^2 - x^2}\,dx$$

$$= \frac{8\sqrt{5}\pi}{9}\left[\frac{x}{2}\sqrt{\left(\frac{18}{\sqrt{5}}\right)^2 - x^2} + \frac{1}{2}.\frac{324}{5}\sin^{-1}\left(\frac{x\sqrt{5}}{18}\right)\right]_0^6$$

$$= \frac{8\sqrt{5}\pi}{9}\left[\left(3\sqrt{\frac{324}{5}} - 36 + \frac{162}{5}\sin^{-1}\frac{6\sqrt{5}}{18}\right) - 0\right]$$

$$= \frac{8\sqrt{5}\pi}{9}\left(3\sqrt{\frac{144}{5}} + \frac{162}{5}\sin^{-1}\frac{\sqrt{5}}{3}\right)$$

$$= \frac{8\sqrt{5}\pi}{9}\left[\frac{36}{\sqrt{5}} + \frac{162}{5}\sin^{-1}\left(\frac{\sqrt{5}}{3}\right)\right]$$

$$= 8\sqrt{5}\pi\left[\frac{4}{\sqrt{5}} + \frac{18}{5}\sin^{-1}\left(\frac{\sqrt{5}}{3}\right)\right]$$

$$= 16\pi\left[2 + \frac{9}{\sqrt{5}}\sin^{-1}\left(\frac{\sqrt{5}}{3}\right)\right]$$



CHECK YOUR PROGRESS - 2

Q 1 : Find the surface of a sphere of radius *a*.

Q 2 : Find the curved surface of the solid which is generated by the revolution about the *x*-axis of the area bounded by the *x*-axis of the parabola $y^2 = 4ax$ and the ordinate $x = h$.

Q 3 : Find the surface area of the solid generated by revolution of $y^2 = 4ax$ about *x*-axis between the extremities of the latus rectum.

Q 4 : Show that the surface area of the spherical portion of a sphere contained between two planes parallel to each other at a distance of *h* apart is $2\pi ah$.

Q 5 : Find the surface of the solid generated by the revolution of the ellipse $\frac{x^2}{16}+\frac{y^2}{4}=1$ about its major axis.

Q 6 : Find the area of the surface generated when the arc of $y=x^2$ is revolved about *y*-axis between the point $x=0$ and $x=\sqrt{2}$.

Q 7 : Find the surface area of a sphere generated by the circle $x^2+y^2=a^2$ revolving about the *x*-axis.

Q 8 : Find the surface area which is generated due to the revolution of the arc of the parabola $y^2=12x$ between $x=0$ and $x=3$ about its axis.

Q 9 : Find the area of the surface of revolution generated by revolving about the *x*-axis the arc of $y=\frac{1}{3}x^3$ from $x=0$ to $x=3$.



6.5 LET US SUM UP

- The volume of the solid generated by the revolution about *x*-axis of the area bounded by the curve $y=f(x)$, *x*-axis and the ordinates $x=a$ and $x=b$ is given by $V=\int_a^b \pi y^2 dx$.
- The area of the surface of the solid which is generated by revolving the arc of $y=f(x)$ between $x=a$ and $x=b$ about *x*-axis is given by $2\pi\int_a^b y ds$ or $2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx$.



6.6 FURTHER READINGS

1) *Integral Calculus,* Kishan, Hari, Atlantic Publishers & Distributors. New Delhi -110027.

2) *Integral Calculus*, Dhami, H.S. New age International Publishers, New Delhi - 110002.

3) *Text book of Integral Calculus*, Sharma, A.K. Discovery Publishing House, New Delhi -110002.

6.7 ANSWERS TO CHECK YOUR PROGRESS

CHECK YOUR PROGRESS – 1

Ans to Q No 1 : The equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

The ellipse is symmetrical about the x-axis and the y-axis and for the portion of the curve lying in the first quadrant x varies from 0 to a.

Therefore the required volume of the solid formed $= 2\int_0^a y^2 dx$

$$= 2\pi\int_0^a \frac{b^2}{a^2}\left(a^2-x^2\right)dx$$

from(1)

$$= 2\pi\frac{b^2}{a^2}\left[a^2x-\frac{1}{3}x^3\right]_0^a$$

$$= 2\pi\frac{b^2}{a^2}\left[a^3-\frac{1}{3}a^3\right]$$

$$= \frac{4}{3}\pi ab^3$$

Ans to Q No 2 : Do yourself.

Ans to Q No 3 : Do yourself.

Ans to Q No 4 : The hemisphere is generated by the revolution of a quadrant of the circle $x^2+y^2=a^2$ about x-axis.The limits for the volume will be from 0 to a.

$\therefore$ the required volume of the hemisphere $= \int_0^a \pi y^2 dx$

$$= \int_0^a \pi\left(a^2-x^2\right)dx$$

$$= \pi\left[a^2x - \frac{1}{3}x^3\right]_0^a$$

$$= \pi\left[a^3 - \frac{1}{3}a^3\right]$$

$$= \frac{2}{3}\pi a^3.$$

Ans to Q No 5 : The given parabola is $y^2 = 4ax$

The paraboloid is formed by the revolution of the area ODC about x-axis.

Also for the area ODC, x varies from $x = 0$ to $x = h$.

$\therefore$ the required volume $= \int_0^h \pi y^2 dx$

$$= \int_0^h \pi(4ax)dx$$

$$= 4\pi a\left[\frac{x^2}{2}\right]_0^h$$

$$= 2\pi a h^2$$

Ans to Q No 6 : The given curve is $x^{2/3} + y^{2/3} = a^{2/3}$ ... (1)

The curve is symmetrical about both the axes. The coordinates of B are (0, a) and those of A are (a, 0).

$\therefore$ the required volume $= 2\int_0^a \pi y^2 dx$, by symmetry

$$= 2\pi\int_0^a \left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)^3 dx$$

[from (1), $y^{\frac{2}{3}} = \left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)$ so that $y^2 = \left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)^3$]

$$= 2\pi\int_0^{\frac{\pi}{2}} a^2\cos^6\theta . 3a\sin^2\theta\cos\theta d\theta$$

[putting $x = a\sin^3\theta$ so that $dx = 3a\sin^2\theta\cos\theta d\theta$]

$$= 6\pi a^2 \int_0^{\frac{\pi}{2}} \sin^2\theta \cos^7\theta d\theta$$

$$= 6\pi a^2 \frac{1.6.4.2}{9.7.5.3.1} = \frac{32\pi a^3}{105}.$$

Ans to Q No 7 : Do yourself.

Ans to Q No 8 : Do yourself.

CHECK YOUR PROGRESS – 2

Ans to Q No 1 : Suppose the sphere is generated by the revolution of a semi- circle of radius a about its bounding diameter (x-axis).

Let the equation of the circle be $x^2 + y^2 = a^2$, ... (1)

the centre being the origin.

Also, for the semi-circle, x varies from -a to a.

$\therefore$ the required surface $= 2\pi \int_{-a}^{a} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$$= 2\pi \int_{-a}^{a} y . \frac{a}{y} dx$$

$$= 2\pi \int_{-a}^{a} a dx$$

$$= 2\pi a \left[x\right]_{-a}^{a}$$

$$= 2\pi a (a + a)$$

$$= 4\pi a^2 .$$

Ans to Q No 2: The given parabola is $y^2 = 4ax$ ... (1)

Differentiating (1) w.r.t. x, we get $2y\left(\frac{dy}{dx}\right) = 4a$

or $\frac{dy}{dx} = \frac{2a}{y}$

$$\therefore \frac{ds}{dx} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}} = \sqrt{\frac{y^2 + 4a^2}{y^2}} = \frac{\sqrt{y^2 + 4a^2}}{y}$$

$$= \frac{\sqrt{4ax + 4a^2}}{y}, \text{ from (1)}$$

$$= \frac{2\sqrt{a}\sqrt{x+a}}{y}$$

$\therefore$ the required curved surface $= \int_0^h 2\pi y \frac{ds}{dx} dx = 2\pi \int_0^h y . \frac{2\sqrt{a}\sqrt{x+a}}{y} dx$

$$= 4\pi\sqrt{a} \int_0^h (x+a)^{\frac{1}{2}} dx$$

$$= 4\pi\sqrt{a} \left[\frac{(x+a)^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^h$$

$$= \frac{8\pi}{3} \sqrt{a} \left[(h+a)^{\frac{3}{2}} - a^{\frac{3}{2}} \right].$$

Ans to Q No 3 : Do yourself.

Ans to Q No 4 : Let the sphere be generated by the revolution about the x-axis of the circle $x^2 + y^2 = a^2$ ... (1)

Let the two parallel planes bounding the spherical zone be formed by the revolution of the lines $x = b$ and $x = b + h$.

Then the required surface is generated by the revolution of the arc AC about x-axis.

$\therefore$ the required surface $= 2\pi \int_b^{b+h} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$

$$= 2\pi \int_b^{b+h} y . \frac{a}{y} dx$$

$$= 2\pi \int_b^{b+h} a dx$$

$$= 2\pi a [x]_b^{b+h}$$

$$= 2\pi a (b + h - h)$$

$$= 2\pi a b$$

Ans to Q No 5 : The given ellipse is $\frac{x^2}{16} + \frac{y^2}{4} = 1$ ... (1)

Comparing it with the standard form of the equation of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$,we get a=4, b=2 and the major axis is along the x-axis.So we have to revolve the curve (1) about x-axis.
Differentiating (1) w.r.t. x , we get

$$2x+8y\frac{dy}{dx}=0$$

or $$\frac{dy}{dx}=-\frac{x}{4y}$$

$$\therefore \frac{ds}{dx}=\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}=\sqrt{\left\{1+\frac{x^2}{16y^2}\right\}}=\sqrt{\frac{16y^2+x^2}{16y^2}}$$

$$=\frac{\sqrt{\{(64-4x^2)+x^2\}}}{4y}$$

$$=\frac{\sqrt{64-3x^2}}{4y} \quad \text{... (2)}$$

Now the ellipse (1) is symmetrical about both axes and for the arc of the ellipse lying in the first quadrant x varies from 0 to 4.

$\therefore$ the required surface = 2 $\times$ surface generated by the revolution of the arc in the first quadrant.

$$=2\int_0^4 2\pi y\frac{ds}{dx}dx$$

$$=4\pi\int_0^4 y.\frac{\sqrt{64-3x^2}}{4y}dx \text{ ,from (2)}$$

$$=\pi\int_0^4\sqrt{64-3x^2}\,dx$$

$$=\pi\sqrt{3}\int_0^4\sqrt{\left\{\left(\frac{8}{\sqrt{3}}\right)^2-x^2\right\}}dx$$

$$=\pi\sqrt{3}\left[\frac{x}{2}\sqrt{\left\{\left(\frac{8}{\sqrt{3}}\right)^2-x^2\right\}}-x^2+\frac{1}{2}.\frac{64}{3}\sin^{-1}\left(\frac{x\sqrt{3}}{2}\right)\right]_0^1$$

$$= \pi\sqrt{3}\left[2\sqrt{\left(\frac{64}{3}-16\right)}+\frac{32}{3}\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right]$$

$$= \pi\sqrt{3}\left[\frac{8}{\sqrt{3}}+\frac{32}{3}.\frac{\pi}{3}\right]$$

$$= 8\pi\left[1+\frac{4\pi}{3\sqrt{3}}\right]$$

Ans to Q No 6 : Do yourself.

Ans to Q No 7 : Do yourself.

Ans to Q No 8 : Do yourself.

Ans to Q No 9 : Do yourself.



6.8 MODEL QUESTIONS

Q 1 : Find the volume of the solid generated by revolving about the *y*-axis the plane area bounded by $y = 4x^2$, $x = 0$ and $y = 16$.

Q 2 : Find the volume generated by revolving the area bounded by the parabola $y^2 = 8x$ and its latus rectum about *x*-axis.

Q 3 : Find the volume generated by revolving the area cut off from the parabola $y = 4x - x^2$ by the *x*-axis about the line $y = 6$.

Q 4 : Find the volume of the solid formed by revolving about the *x*-axis, the area bounded by $\sqrt{x}+\sqrt{y}=\sqrt{a}$, *x*-axis and *y*-axis.

Q 5 : Find the volume of the solid generated by the revolution of the curve $y^2 = x^2(2-x)$ about the *x*-axis.

Q 6 : Find the area of the surface of revolution generated by revolving about the *y*-axis the arc of $x = y^3$ from $y = 0$ to $y = 1$.

Q 7 : Find the area of the surface of revolution generated by revolving a loop of the curve $x^2\left(a^2 - x^2\right) = 8a^2y^2$ about the *x*-axis.

Q 8 : Find the area of the surface generated by revolving the line $y = mx$ from $x = 0$ to $x = 2$ about x-axis.

Q 9 : The part of the parabola $y^2 = 4ax$ cut off by the latus rectum revolves about the tangent at the vertex. Find the area of the curved surface of the reel so formed.

Q 10 : Find the surface area of the solid formed by the revolution of the curve $y^2(a + x) = x^2(a - x)$ about x-axis.

*** ***** ***

UNIT 7 : DIFFERENTIAL EQUATION : FUNDAMENTAL CONCEPTS

UNIT STUCTURE

7.1 Learning objectives

7.2 Introduction

7.3 Definition

7.4 Order and Degree of Ordinary Differential Equation

7.5 Linear and Non-Linear Differential Equation

7.6 Formation of a Differential Equation

7.7 Solution of a Differential Equation

7.8 Let Us Sum Up

7.9 Further Readings

7.10 Answers To Check Your Progress

7.11 Model Questions

7.1 LEARNING OBJECTIVES

After going through this unit, you will able to understand

- definition of differential equation
- order and degree of a differential equation
- linear and non-linear differential equation
- formation of differential equation
- solution of a differential equation.

7.2 INTRODUCTION

Differential equations plays an important role in science, engineering economics, commerce and social sciences.

Differential equation is like a bridge linking Mathematics and Science with its applications.Many physical problems, when formulated in mathematical forms, lead to **differential equations.** Differential equations arise in a variety of applications, may it be in Physics, Chemistry, Biology, Anthropology, Geology, Economics etc.

In this unit, we will study some basic concepts related to differential equation, general and particular solutions of a differential equation and formation of differentialequations.

7.3 DEFINITION

By a differential equation, we mean an equation involving independent variable, dependent variable and differential coefficients of the dependent variable w.r.t independent variables.

Differential equations are of two types :

(a) Ordinary differential equation

(b) Partial differential equation

(a) Ordinary differential equation : A differential equation is called an ordinary differential equation (ode) if only ordinary differential coefficients w.r.t a single independent variable involve.

(b) Partial differential equation : A differential equation is called a partial differential equation (pde) if in it there are two or more independent variables and partial derivative w.r.t. any of these independent variables involve.

Examples : (i) $\frac{d^2y}{dx^2} + x = xe^{2x}$

(ii) $\frac{d^4x}{dt^4} + \frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^5 = e^t$

(iii) $\frac{d^3y}{dx^3} + 3\left(\frac{d^2y}{dx^2}\right)^2 + \frac{dy}{dx} = e^x$

(iv) $\frac{\partial^2 v}{\partial t^2} = k\left(\frac{\partial^3 v}{\partial t^3}\right)^2$

(v) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

(vi) $\frac{\partial^2 z}{\partial x^2} - a^2\frac{\partial^2 z}{\partial y^2} = 0$

Equations (i), (ii) and (iii) are all ordinary differential equations. Equations (iv), (v) and (vi) are all partial differential equations.

7.4 ORDER AND DEGREE OF AN ORDINARY DIFFERENTIAL EQUATION

The order of an ordinary differential equation is the order of the highest derivative present in the equation.

The degree of an ordinary differential equation is the degree of the highest order derivative, when the differential coefficients become free from fraction and radical sign.

Note : (1) The degree of a differential equation is the power of the highest order derivative occurring in a differential equation when it is written as a polynomial in differential coefficients.

(2) Order and degree (if defined) of a differential equation are always positive integers.

Examples :

(i) $\frac{d^3y}{dx^3} - y = \sin x \cos x$ is an ordinary differential equation of order 3 and degree 1.

(ii) $\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = r$ can be written as $\left(1+\left(\frac{dy}{dx}\right)^2\right)^3 = r^2\left(\frac{d^2y}{dx^2}\right)^2$ after making free from radical sign and fraction.Its order is 2 and degree 2.

(iii) Consider the differential equation $\left(\frac{d^2y}{dx^2}\right)^2 + \sin\left(\frac{dy}{dx}\right) = 0$

The highest order derivative present in the differential equation is $\frac{d^2y}{dx^2}$. So, its order is 2. Since the differential equation cannot be expressed as a polynomial in differential coefficients, so its degree is not defined.

Example 1 : In each of the following differential equations indicate its degree, wherever possible. Also, give the order of each of them.

(i) $\frac{dy}{dx}+\sin\left(\frac{dy}{dx}\right)=0$ (ii) $\frac{d^5y}{dx^5}+e^{\frac{dy}{dx}}+y^2=0$

Solution : (i) The highest order derivative present in the differential equation is $\frac{dy}{dx}$. So, it is of order 1. Clearly, the differential equation cannot be expressed as a polynomial in $\frac{dy}{dx}$. So, its degree is not defined.

(ii) The highest order derivative present in the differential equation is $\frac{d^5y}{dx^5}$. So, it is of order 5. But the differential equation is not expressible as a polynomial in $\frac{dy}{dx}$ .So,its degree is not defined.

Example 2 : Find the order and degree of the following differential equations:

(i) $\frac{d^3y}{dx^3}+\left(\frac{d^2y}{dx^2}\right)^3+\left(\frac{dy}{dx}\right)^5+y=7$ (ii) $\frac{d^2y}{dx^2}=\left[4+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{4}}$

Solution : (i) The order of the highest derivative in this equation is 3. The degree of the highest order is 1.

$\therefore$ order = 3, degree=1

(ii) Given, $\frac{d^2y}{dx^2}=\left[4+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{4}}$

To eliminate the radical in the above equation, raising to the power 4 on both sides, we get $\left(\frac{d^2y}{dx^2}\right)^4=\left[4+\left(\frac{dy}{dx}\right)^2\right]^3$.

$\therefore$ order = 2, degree=4

Example 3 : Write down the order and degree of the following differential equations.

(i) $\left(\frac{dy}{dx}\right)^3-4\left(\frac{dy}{dx}\right)+y=3e^x$ (ii) $\left(\frac{d^2y}{dx^2}\right)^3+7\left(\frac{dy}{dx}\right)^4=3\sin x$

(iii) $\frac{d^2x}{dy^2}+a^2x=0$ (iv) $\left(\frac{dy}{dx}\right)^2-3\frac{d^3y}{dx^3}+7\frac{d^2y}{dx^2}+4\left(\frac{dy}{dx}\right)-\log x=0$

(v) $\frac{d^2y}{dx^2} - \sqrt{\frac{dy}{dx}} = 0$ (vi) $\sqrt{1+x^2} = \frac{dy}{dx}$.

Solution :

The order and the degree respectively are

(i) 1 ; 3 (ii) 2 ; 3 (iii) 2 ; 1 (iv) 3 ; 1

(v) 2 ; 2 (vi) 1 ; 1

Remarks :

1. Ode $y\frac{dy}{dx} - x = 0$ of first order and first degree can be written as $ydy = xdx$.

2. Similarly, $\cos\theta\frac{dr}{d\theta} + r\sin\theta = 0$ is equivalent to $\cos\theta dr + r\sin\theta d\theta = 0$.



CHECK YOUR PROGRESS - 1

Q 1 : Find the order and degree of the following ode.

(i) $\frac{d^2y}{dx^2} + 4xy\frac{dy}{dx} = 5x$ (ii) $\left(\frac{dy}{dx}\right)^2 = \frac{1-y^2}{1+x^2}$

(iii) $\left(\frac{d^2y}{dx^2}\right)^3 - x^2\left(\frac{dy}{dx}\right)^4 = 0$ (iv) $\frac{d^3y}{dx^3} - 3\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^3 + 4y^5 = 0$

(v) $x + \frac{d^2y}{dx^2} = \sqrt{1+\frac{dy}{dx}}$ (vi) $\left(\frac{d^3y}{dx^3}\right)^{1/3} = \left(\frac{dy}{dx}\right)^{1/2}$

(vii) $y = \sqrt{x}\left(\frac{dy}{dx}\right) + k\Big/\left(\frac{dy}{dx}\right)$ (viii) $y = x\left(\frac{dy}{dx}\right) + a\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}$

(ix) $dy = (y + \sin x)dx$ (x) $\left(\frac{d^2y}{dx^2}\right)^3 + x\left(\frac{dy}{dx}\right)^5 + y = x^2$

(xi) $\frac{d^4y}{dx^4} + \sin\left(\frac{d^3y}{dx^3}\right) = 0$ (xii) $\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^2 = x\sin\left(\frac{d^2y}{dx^2}\right)$

7.5 LINEAR AND NON LINEAR DIFFERENTIAL EQUATION

A differential equation is said to be linear if the unknown function (i.e. the dependent variable) and its derivatives be of first power and be free from product of them. An ode which is not linear is called a nonlinear ode.

Examples :

(i) $\frac{d^3y}{dx^3}+2\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=e^x$ is linear ode.

(ii) $x^2\frac{d^2y}{dx^2}+2x\left(\frac{dy}{dx}\right)^2+y^2=0$ is non linear.



CHECK YOUR PROGRESS - 2

Q 1 : Classify the following equations as linear and nonlinear equations and write down their orders.

(a) $\frac{d^3y}{dx^3}+\frac{d^2y}{dx^2}\cdot\frac{dy}{dx}+y=x$ (b) $x\frac{d^4y}{dx^4}+\frac{d^2y}{dx^2}=e^x$ (c) $\frac{dy}{dx}+y^2=x^2$

7.6 FORMATION OF ORDINARY DIFFERENTIAL EQUATION

$y=\phi(x)$ is said to be solution of an ode if on substituting $y=\phi(x)$, the equation reduces to an identity. $y=e^x$ is a solution of $\frac{d^2y}{dx^2}-y=0$. It is easy to check that Ae^x, e^{-x}, Be^{-x} and Ae^x+Be^{-x} are also solutions of this equation. The last solution is called the general solution of the equation, since former solutions are obtainable from it with values (1, 0), (A, 0),(0,1), (0, B) of (A, B). Observe that the number of arbitrary constants in general solution of an ode is equal to the order of the equation. Solutions reducible from the general solution are called special solution.sometimes,a solution is obtained which is neither contains constant nor it is a special solution.Such a solution is called a singular solution.Thus,

(i) A solution of an ode is called a general solution if it contains as many constants as the order of the equation.

(ii) A solution of an ode is called a special solution if it is obtainable from the general equation.

(iii) A solution other than special solution and free from constant is called a singular solution.

Conversely, from the general solution $y = Ae^x + Be^{-x}$, we can obtain the differential equation. Since there are two arbitrary constants, we differentiate two times and obtain two equations

$$\frac{dy}{dx} = Ae^x - Be^{-x} \text{ and } \frac{d^2y}{dx^2} = Ae^x + Be^{-x}$$

Eliminating the arbitrary constants A and B,we obtain

$\frac{d^2y}{dx^2} = y$ or $\frac{d^2y}{dx^2} - y = 0$, which is the differential equation with general solution $y = Ae^x + Be^{-x}$.

This is general technique of formation of an ode from its general solution.

Example 1 : Form differential equation from

$x = a\cos nt + b\sin nt$ where a and b are arbitrary constants.

Solution : Given $x = a\cos nt + b\sin nt$ ... (1)

Differentiating (1) twice w.r.t t, we have

$$\frac{dx}{dt} = -an\sin nt + bn\cos nt$$

$$\frac{d^2x}{dt^2} = -an^2\cos nt - bn^2\sin nt$$

$$= -n^2(a\cos nt + b\sin nt)$$

$$= -n^2x \quad \text{using (1)}$$

Hence $\frac{d^2x}{dt^2} + n = 0$ is the required equation.

Example 2 : Eliminate the constants from the equation $y = e^x(A\cos x + B\sin x)$ and obtain the differential equation.

Solution : Given, $y = e^x(A\cos x + B\sin x)$ .... (1)

Differentiating (1) w.r.t. x, we have

$$\frac{dy}{dx} = e^x(A\cos x + B\sin x) + e^x(-A\sin x + B\cos x)$$

$= y + e^x(-A\sin x + B\cos x)$ ... (2)

$\therefore \frac{d^2y}{dx^2} = \frac{dy}{dx} + e^x(-A\sin x + B\cos x) + e^{-x}(-A\cos x - B\sin x)$

or $\frac{d^2y}{dx^2} = \frac{dy}{dx} + \frac{dy}{dx} - y - y$, from (1) and (2)

or $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$, which is the required differential equation.

Example 3 : Form a differential equation for $x^2 + (y-a)^2 = a^2$.

Solution : Given $x^2 + (y-a)^2 = a^2$

or $x^2 + y^2 - 2ya + a^2 = a^2$

or $x^2 + y^2 - 2ya = 0$ ... (1)

Differentiating both sides with respect to x, we get

$$2x + 2y\frac{dy}{dx} - 2a\frac{dy}{dx} = 0$$

or $x + y\frac{dy}{dx} - a\frac{dy}{dx} = 0$

or $x + y\frac{dy}{dx} - \frac{x^2 + y^2}{2y}\frac{dy}{dx} = 0$, from (1)

or $2yx + 2y^2\frac{dy}{dx} - (x^2 + y^2)\frac{dy}{dx} = 0$

or $2yx + (2y^2 - x^2 - y^2)\frac{dy}{dx} = 0$

or $2xy = (x^2 - y^2)\frac{dy}{dx}$

$\therefore \frac{dy}{dx} = \frac{2xy}{x^2 - y^2}$, which is the required differential equation.

Example 4 : Form differential equation from

(i) $x = a\sin(y + b)$ (ii) $y = ax\cos\left(\frac{n}{x} + b\right)$

(iii) $y = a\left(\sin x + \frac{1}{x}\cos x\right) + b\left(\cos x - \frac{1}{x}\sin x\right)$

(iv) $y = ae^x + be^{3x} + ce^{5x}$

Solution : (i) Differentiating $x = a\sin(y+b)$ w.r.t. y, we have

$$\frac{dx}{dy} = a\cos(y+b)$$

so, $\frac{dy}{dx} = \frac{1}{a}\sec(y+b)$

Differentiating again w.r.t.x, we have

$$\frac{d^2y}{dx^2} = \frac{1}{a}\sec(y+b)\tan(y+b).\frac{dy}{dx}$$

or, $\frac{d^2y}{dx^2} = \tan(y+b)\frac{1}{a}\sec(y+b).\frac{dy}{dx}$

or, $\frac{d^2y}{dx^2} = \tan(y+b)\left(\frac{dy}{dx}\right)^2$

or, $\frac{d^2y}{dx^2} = \frac{a\sin(y+b)}{a\cos(y+b)}\left(\frac{dy}{dx}\right)^2$

$$\therefore \frac{d^2y}{dx^2} = \frac{x}{\frac{dx}{dy}}\left(\frac{dy}{dx}\right)^2 = x\left(\frac{dy}{dx}\right)^3$$

i.e., $\frac{d^2y}{dx^2} = x\left(\frac{dy}{dx}\right)^3$ is the required differential equation.

(ii) $y = ax\cos\left(\frac{n}{x}+b\right)$ is a given relation with two arbitrary constants a and

b. Differentiating w.r.t.x , we have

$$y_1 = a\cos\left(\frac{n}{x}+b\right) + ax.\left(-\sin(\frac{n}{x}+b)\right)(-\frac{n}{x^2})$$

$$= a\cos\left(\frac{n}{x}+b\right) + \frac{an}{x}\sin(\frac{n}{x}+b)$$

or $xy_1 = y + an\sin(\frac{n}{x}+b)$

Differentiating once again, we have

$$y_1 + xy_2 = y_1 + an\cos(\frac{n}{x}+b)\left(-\frac{n}{x^2}\right)$$

or $xy_2 = -\frac{n^2}{x^3}y$

Hence, $x^4y_2 + n^2y = 0$ is the required differential equation.

(iii) $y = a\left(\sin x + \frac{1}{x}\cos x\right) + b\left(\cos x - \frac{1}{x}\sin x\right)$

$= a\sin x + b\cos x + \frac{1}{x}(a\cos x - b\sin x)$ ... (1)

is a given relation with arbitrary constants a and b.

Differentiating (1) w.r.t. x , we have

$$y_1 = a\cos x - b\sin x - \frac{1}{x^2}(a\cos x - b\sin x) + \frac{1}{x}(-a\sin x - b\cos x)$$

or $xy_1 = x(a\cos x - b\sin x) - \frac{1}{x}(a\cos x - b\sin x) - (a\sin x + b\cos x)$

$= x(a\cos x - b\sin x) - y$

or $xy_1 + y = x(a\cos x - b\sin x)$ .... (2)

Again, differentiating (2) w.r.t. x , we have

Then we have

$y_1 + xy_2 + y_1 = a\cos x - b\sin x + x(-a\sin x - b\cos x)$

or $x^2y_2 + 2xy_1 = x(a\cos x - b\sin x) - x^2(a\sin x - b\cos x)$

$= xy_1 + y - x^2\left[y - \frac{1}{x}(a\cos x - b\sin x)\right]$, using (1) and (2)

$= xy_1 + y - \left[x^2y - (xy_1 + y)\right]$

or $x^2y_2 - 2y + x^2y = 0$, which is the required differential equation.

(iv) $y = ae^x + be^{3x} + ce^{5x}$ with arbitrary constants a, b and c.

Differentiating successively, we have

$$y_1 = ae^x + 3be^{3x} + 5ce^{5x}$$

$$y_2 = ae^x + 9be^{3x} + 25ce^{5x}$$

$$y_3 = ae^x + 27be^{3x} + 125ce^{5x}$$

so, $y_1 - y = 2be^{3x} + 4ce^{5x}$

$$y_2 - y_1 = 6be^{3x} + 20ce^{5x}$$

$$y_3 - y_2 = 18be^{3x} + 100ce^{5x}$$

Eliminating b and c, we have

$$\begin{vmatrix} y_1 - y & 1 & 1 \\ y_2 - y_1 & 3 & 5 \\ y_3 - y_2 & 9 & 25 \end{vmatrix} = 0$$

or $y_3 - 9y_2 + 23y_1 - 15y = 0$ (on expanding)

This is the required differential equation.

Example 5 : (i) Find the differential equation of the family of coplanar straight lines passing through the origin.

(ii) Find the differential equation of the family of straight lines at unit distance from the origin.

Solution : (i) $y = mx$ is the equation of the family of straight lines passing through the origin. Here m is the only arbitrary constant.

Differentiating w.r.t.x, we have

$$\frac{dy}{dx} = m = \frac{y}{x}$$

or $x\frac{dy}{dx} - y = 0$ is the required equation.

(ii) $x\cos\alpha + y\sin\alpha = 1$ is the equation of family of straight lines at a unit distance from the origin.

Differentiating w.r.t.x, we have

$$\cos\alpha + \frac{dy}{dx}\sin\alpha = 0$$

From the above two equations, eliminating $\cos\alpha$, we have

$$\sin\alpha = \frac{1}{\left(y - x\frac{dy}{dx}\right)}$$

so, $\cos\alpha = -\frac{\frac{dy}{dx}}{\left(y - x\frac{dy}{dx}\right)}$

Squaring and adding, we have $1 = \frac{1}{\left(y - x\frac{dy}{dx}\right)^2}\left(1 + \left(\frac{dy}{dx}\right)^2\right)$

or $\left(y - x\frac{dy}{dx}\right)^2 = \left(\frac{dy}{dx}\right)^2 + 1$ is the required equation.

Example 6 : Find the differential equation of family of

(i) circles of radius r.

(ii) circles with centre on x-axis and touching y-axis.

Solution : (i) Equation of family of circles centred at (a, b) and radius are r are $(x-a)^2 + (y-b)^2 = r^2$ ... (1), where a and b are arbitrary constants and r is a fixed constant.Here we shall eliminate a and b to obtain the differential equation but not r.

Differentiating (1) w.r.t x, we have

$$(x-a) + (y-b)\frac{dy}{dx} = 0 \quad \text{... (2)}$$

Again, Differentiating (2) w.r.t x, we have

$$1 + \left(\frac{dy}{dx}\right)^2 + (y-b)\frac{d^2y}{dx^2} = 0 \quad \text{... (3)}$$

From (2), $(y-b) = -\dfrac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}}$ ... (4)

and from (3), $x - a = -(y-b)\dfrac{dy}{dx} = \dfrac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]\frac{dy}{dx}}{\left(\frac{d^2y}{dx^2}\right)}$ ... (5)

squaring and adding (4) and (5), we have

$$r^2 = (x-a)^2 + (y-b)^2 = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^2}{\left(\frac{d^2y}{dx^2}\right)^2}\left(1 + \left(\frac{dy}{dx}\right)^2\right)$$

or $r^2\left(\frac{d^2y}{dx^2}\right)^2 = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^3$, which is the required equation.

(ii) A circle with centre on x-axis and radius a must touch the y-axis at the origin. So, its centre is (a, 0) and radius a. The equation of family of such circles is $(x-a)^2 + y^2 = a^2$, where a is arbitrary constant.

or $x^2 - 2ax + y^2 = 0$ ... (1)

Differentiating (1) w.r.t. x, we have

$$2x - 2a + 2y\frac{dy}{dx} = 0$$

or $x - a + y\frac{dy}{dx} = 0$

or $a = x + y\frac{dy}{dx}$

Putting the value of a in (1), we get

$$x^2 - 2\left(x + y\frac{dy}{dx}\right)x + y^2 = 0$$

or $x^2 - 2x^2 - 2xy\frac{dy}{dx} + y^2 = 0$

or $(y^2 - x^2)dx - 2xydy = 0$, which is the required equation.



CHECK YOUR PROGRESS - 3

Q 1 : Form differential equation from

(i) $y^2 = Ax^2 + Bx + C$

(ii) $c(y+c)^2 = x^2$

(iii) $y = ae^{bx}$

(iv) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(v) $y^2 = a(b-x)(b+x)$

Q 2 : Form the differential equation of

(i) all parabolas each of which has a latus rectum 4a and whose axes are parallel to the x -axis.

(ii) all circles which pass through the origin and whose centres on the y-axis.

(iii) all parabolas whose axes are parallel to the axis of y.

7.7 SOLUTION OF A DIFFERENTIAL EQUATION

A function is said to be a solution of a differential equation if it satisfies the differential equation.

Thus if $y = f(x)$ be the solution, then by replacing y and its derivatives with respect to x, the given differential equation will reduce to an identity.

For example, (i) $y = c_1 \cos x + c_2 \sin x$ is the solution of the differential equation

$$\frac{d^2y}{dx^2} + y = 0.$$

Since $\frac{dy}{dx} = -c_1 \sin x + c_2 \cos x$

$$\frac{d^2y}{dx^2} = -c_1 \cos x - c_2 \sin x = -y$$

(ii) $y = e^x$ is a solution of the differential equation $\frac{dy}{dx} = y$.

since $\frac{dy}{dx} = e^x = y$.

The general (or complete) solution of a differential equation is that in which the number of independent arbitrary constants is equal to the order of the differential equation.

Thus, $y = c_1 \cos x + c_2 \sin x$ (involving two arbitrary constants c_1, c_2) is the general solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$ of second order.

A particular solution of a differential equation is that which is obtained from its general solution by giving particular values to the arbitrary constants.

For example, $y = c_1 e^x + c_2 e^{-x}$ is the general solution of the differential equation

$\frac{d^2y}{dx^2} - y = 0$, whereas $y = e^x - e^{-x}$ or $y = e^x$ are its particular solutions.

Example 1 : Verify that $y = \cos \alpha t$ satifies the differential equation $\frac{d^2y}{dt^2} = -\alpha^2 y$.

Solution : Given that $y = \cos\alpha t$ ... (1)

We differentiate (1) twice with respect to t

$$\frac{dy}{dt} = -\alpha \sin \alpha t$$

$$\frac{d^2y}{dt^2} = -\alpha^2 \cos\alpha t$$

However, we have $y = \cos\alpha t$, and so, substituting this into the equation, we have that $\frac{d^2y}{dx^2} = -\alpha^2 y$ as required.

Therefore, $y = \cos\alpha t$ is a solution of $\frac{d^2y}{dt^2} = -\alpha^2 y$.

Example 2 : Verify that the function $y = -e^{-x}$ is a solution of the differential equation $\frac{dy}{dx} = y^2 e^x$.

Solution : Given that $y = -e^{-x}$ ... (1)

we differentiate (1) with respect to x

$$\frac{dy}{dx} = e^{-x}$$

Now, $y = -e^{-x} \Rightarrow y^2 = e^{-2x}$ and so, $y^2 = e^{-x} \times e^{-x} \Rightarrow e^{-x} = y^2 e^x$.

we can now substitue this expression for e^{-x} into our differential equation.i.e., into $\frac{dy}{dx} = e^{-x}$.

This then gives, $\frac{dy}{dx} = y^2 e^x$, as required.

That is, $y = -e^{-x}$ satisfies the differential equation $\frac{dy}{dx} = y^2 e^x$.

Example3 : Show that $y = cx + \frac{a}{c}$ (where c is a constant) is a solution of the differential equation $y = x\frac{dy}{dx} + a\frac{dx}{dy}$.

Solution : Consider $y = cx + \frac{a}{c}$ ... (1)

We differentiate (1) with respect to t

$$\frac{dy}{dx} = c$$

or $\frac{dx}{dy} = \frac{1}{c}$

On substituting the values of $\frac{dy}{dx}$ and $\frac{dx}{dy}$ in R.H.S of the differential equation,

we have $x(c) + a\left(\frac{1}{c}\right) = cx + \frac{a}{c} = y$

$\Rightarrow$ R.H.S = L.H.S.

Example 4 : If $y = 3x^2 + C$ is a general solution of the differential equation $\frac{dy}{dx} - 6x = 0$,then find the particular solution when $y = 3, x = 2$.

Solution : The general solution of the differential equation is $y = 3x^2 + C$.

Now on substituting $y = 3, x = 2$,we get $3 = 12 + C$

or $C = -9$.

By substituting the value of C in the general solution, we will have

$y = 3x^2 - 9$ which is the required particular solution.



CHECK YOUR PROGRESS - 4

Q 1: Verify that the function $y = a\cos x + b\sin x$, where, $a, b \in R$ is solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$.

Q 2 : Verify that the function $y - \cos y = x$ is a solution of the differential equation $(y\sin y + \cos y + x)y' = y$.

Q 3 : Verify that the function $x + y = \tan^{-1} y$ is a solution of the differential equation $y^2 y' + y^2 + 1 = 0$.

Q 4 : Show that $y = x^{-\frac{3}{2}}$ is a solution to $4x^2 y'' + 12xy' + 3y = 0$ for $x > 0$.



7.8 LET US SUM UP

- An equation involving derivatives or differentials of one or more dependent variables with respect to one or more independent variables is called a differential equation.

Differential equation are of two types :

(i) Ordinary differential equation

A differential equation involving derivatives with respect to a single independent variable is called an ordinary differential equation.

(ii) partial differential equation

A differential equation involving partial derivatives with respect to more than one independent variables is called a partial differential equation.

- The order of the highest order derivative involved in a differential equation is called the order of the differential equation.
- The degree of a differential equation is the degree of the highest derivatives which occur in it,after the differential equation has been made free from radicals and fractions.
- A differential equation is called linear if (i) every dependent variable and every derivative involved occurs in the first degree only, and (ii) no products of dependent variables and/or derivatives occur.
- Given an equation in the variables x and y containing n arbitrary constants,we differentiate it n times to get n additional equations invloving the n arbitrary constants.we eliminate the n arbitrary constants from the above (n+1) equations to obtain a differential equation of nth order.
- A general solution of a differential equation is that solution which contains the number of arbitrary constants equal to the order of the differential equation.
- A general solution will be a particular solution when partticular values of the arbitrary constants are determined satisfying the given conditions.



7.9 FURTHER READINGS

1) Tenenbaum, Morris and Pollard, Harry. Ordinary Differential Equations, Dover Publications, Inc, N. Y 11501.

2) Differential Equations : An Introduction with Mathematica. Clay C. Ross, second Edition, Springer.

3) Differential Equations :Linear, Nonlinear, ordinary, partial. A. C. King, J. Billinggham and S.R. Otto, Cambridge-2003.

4) Differential equations with Application and Programs. S. Balachandra Rao and H. R. Anuradha, University Press (India) Limited 1996.



7.10 ANSWERS TO CHECK YOUR PROGRESS

CHECK YOUR PROGRESS – 1

Ans to Q No 1 : (i) 2,1 (ii) 1, 2 (iii) 2, 3 (iv) 3, 1 (v) 2, 2 (vi) 3, 2 (vii) 1, 2 (viii)1, 2 (ix) 1, 1 (x) 2, 3.

(xi) 4, not defined (xii) 2, not defined .

CHECK YOUR PROGRESS – 2

Ans to Q No 1 : (a) Non-linear,3 (b) Non-Linear,4 (c) Linear,1.

CHECK YOUR PROGRESS – 3

Ans to Q No 1: (i) Given $y^2 = Ax^2 + Bx + C$

On differentiating $2y\frac{dy}{dx} = 2Ax + B$

Again differentiating $2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = 2A$

On differentiating again $y\frac{d^3y}{dx^3} + \frac{dy}{dx}.\frac{d^2y}{dx^2} + 2\frac{dy}{dx}\frac{d^2y}{dx^2} = 0$

or $y\frac{d^3y}{dx^3} + 3\frac{dy}{dx}.\frac{d^2y}{dx^2} = 0$.

(ii) Given, $c(y+c)^2 = x^2$ ... (1)

Differentiating (1) w.r.t. 'x' , we get $2c(y+c).\frac{dy}{dx} = 2x$

or $c(y+c).\frac{dy}{dx} = x$ ... (2)

or $\frac{c(y+c)^2}{y+c}\cdot\frac{dy}{dx}=x$

or $\frac{x^2}{y+c}\cdot\frac{dy}{dx}=x$ From (1)

or $x\frac{dy}{dx}=y+c$

or $c=x\frac{dy}{dx}-y$.

Substiturting this value of c in (1), we get

$$\left(x\frac{dy}{dx}-y\right)\left(y+x\frac{dy}{dx}-y\right)^2=x^2$$

or $$\left(x\frac{dy}{dx}-y\right)x^2\left(\frac{dy}{dx}\right)^2=x^2$$

or $$\left(x\frac{dy}{dx}-y\right)\left(\frac{dy}{dx}\right)^2=1.$$

which is the required differential equation.

(iii) Given, $y=ae^{bx}$ ... (1)

$$\therefore\frac{dy}{dx}=abe^{bx}=by \quad \ldots \text{ (2)}$$

$$\therefore\frac{d^2y}{dx^2}=b\frac{dy}{dx}=\frac{1}{y}\cdot\frac{dy}{dx}\cdot\frac{dy}{dx}=\frac{1}{y}\left(\frac{dy}{dx}\right)^2 \quad [\text{ by (2) }]$$

or $y\frac{d^2y}{dx^2}=\left(\frac{dy}{dx}\right)^2$, which is the required differential equation.

(iv) Given, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ... (1)

Differentiating (1) w.r.t. 'x', we get $\frac{2x}{a^2}+\frac{2y}{b^2}\cdot\frac{dy}{dx}=0$

or $\frac{y}{x}\cdot\frac{dy}{dx}=\frac{-b^2}{a^2}$ ... (2)

Differentiating both sides with respect to x, we get

$$\left(\frac{y}{x}\right)\frac{d^2y}{dx^2}+\frac{\left(x\frac{dy}{dx}-y\right)}{x^2}\frac{dy}{dx}=0$$

or $xy\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2-y\frac{dy}{dx}=0$.

which is the required differential equation.

(v) Given, $y^2=a(b-x)(b+x)$ ... (1)

Differentiating (1) w.r.t. x ,we get

$$2y\frac{dy}{dx}=-2ax$$

or $y\frac{dy}{dx}=-ax$ ... (2)

Differentiating (2) w.r.t. x , we get

$$y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=-a \quad \text{... (3)}$$

From (2) and (3), we get

$$x\left\{y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right\}=y\frac{dy}{dx}$$

which is the required differential equation.

Ans to Q No 2: (i) The equation of the family of parabolas is

$(y-k)^2=4a(x-h)$,where h and k are arbitrary constants.

Differentiating this relation twice and eliminating h and k , we get the required differential equation.

(ii) If a be the radius of a circle, then its centre is the point $(0,a)$ and its equation is $x^2+(y-a)^2=a^2$

or $x^2+y^2-2ay=0$ ... (1)

Differentiating (1) w.r.t. x , we get

$$2x+2y\frac{dy}{dx}-2a\frac{dy}{dx}=0$$

or $x+y\frac{dy}{dx}=a\frac{dy}{dx}$ ... (2)

Eliminating a from (1) and (2), we get

$$x+y\frac{dy}{dx}=\frac{x^2+y^2}{2y}\frac{dy}{dx}$$

or $2xy+2y^2\frac{dy}{dx}=(x^2+y^2)\frac{dy}{dx}$

or $2xy+\left(y^2-x^2\right)\frac{dy}{dx}=0$

which is the required differential equation.

(iii) Let the equation of a parabola whose axis is parallel to y-axis be

$(x-h)^2=4a(y-k)$ ... (1) where h, k and a are arbitrary constants.

Differentiating w.r.t. x, we get

$$2(x-h)=4a\frac{dy}{dx}$$

Differentiating again w.r.t. x, we get

$$2=4a\frac{d^2y}{dx^2}$$

and again differentiating w.r.t. x,we get

$$0=4a\frac{d^3y}{dx^3}$$

$$\Rightarrow \frac{d^3y}{dx^3}=0$$

Hence $\frac{d^3y}{dx^3}=0$ is the required differential equation.

CHECK YOUR PROGRESS- 4

Ans to Q No 1 : Given, $y=a\cos x+b\sin x$ ... (1)

Differentiating both sides of (1) with respect to *x* successively, we get

$$\frac{dy}{dx}=-a\sin x+b\cos x$$

$$\frac{d^2y}{dx^2}=-a\cos x-b\sin x$$

Substituting the values of $\frac{d^2y}{dx^2}$ and y in the given differential equation,

we get

L.H.S $=(-a\cos x-b\sin x)+(a\cos x+b\sin x)=0$ = R.H.S..

Therefore, the given function is a solution of the given differential equation.

Ans to Q No 2 : Given, $y-\cos y=x$ ... (1)

Differentiating both sides with respect to x, we get

$$\frac{dy}{dx}-\frac{d}{dx}(\cos y)=\frac{d}{dx}(x)$$

$$\Rightarrow y^{/}+\sin y.y^{/}=1$$

$$\Rightarrow y^{/}(1+\sin y)=1$$

$$\Rightarrow y^{/}=\frac{1}{1+\sin y}$$

Substituting the values of x and $y^{/}$ in the given differential equation, we get

L.H.S $=(y\sin y+\cos y+x)y^{/}$

$$=(y\sin y+\cos y+y-\cos y)\times\frac{1}{1+\sin y}$$

$$=y(1+\sin y)\times\frac{1}{1+\sin y}$$

$=y$ =R.H.S..

Hence the given function is the solution of the corresponding differential equation.

Ans to Q No 3 : Given, $x+y=\tan^{-1}y$ ... (1)

Differentiating both sides with respect to x, we get

$$\frac{d}{dx}(x+y)=\frac{d}{dx}(\tan^{-1}y)$$

$$\Rightarrow 1+y^{/}=\left[\frac{1}{1+y^2}\right]y^{/}$$

$$\Rightarrow y^{/}\left[\frac{1}{1+y^2}-1\right]=1$$

$$\Rightarrow y^{/}\left[\frac{1-(1+y^2)}{1+y^2}\right]=1$$

$$\Rightarrow y^{/}\left[\frac{-y^2}{1+y^2}\right]=1$$

$$\Rightarrow y^{/}=\frac{-(1+y^2)}{y^2}$$

Substituting the value of y' the given differential equation, we get :

$$\text{L.H.S} = y^2 y' + y^2 + 1 = y^2\left[\frac{-(1+y^2)}{y^2}\right] + y^2 + 1$$

$$= -1 - y^2 + y^2 + 1$$

$$= 0 \text{ =R.H.S}$$

Hence the given function is a solution of the given differential equation.

Ans to Q No 4 : Given $y = x^{-\frac{3}{2}}$

$$\therefore y' = -\frac{3}{2}x^{-\frac{5}{2}} \text{ and } y'' = \frac{15}{4}x^{-\frac{7}{2}}.$$

Substituting the values of y, y' and y'' in the given differential equation, we get :

$$\text{L.H.S} = 4x^2\left(\frac{15}{4}x^{-\frac{7}{2}}\right) + 12x\left(-\frac{3}{2}x^{-\frac{5}{2}}\right) + 3\left(x^{-\frac{3}{2}}\right)$$

$$= 15x^{-\frac{3}{2}} - 18x^{-\frac{3}{2}} + 3x^{-\frac{3}{2}}$$

$$= 0$$

So, $y = x^{-\frac{3}{2}}$ does satisfy the differential equation and hence is a solution.



7.11 MODEL QUESTIONS

Q 1 : For the differential equations for the following

(a) $y = k\sin^{-1}x$, k parameter.

(b) $y = \alpha x + \beta x^2$, α, β parameter.

Q 2 : Find the differential equation of all the family of curves

$y = Ae^{3x} + Be^{5x}$ for different values of A and B .

Q 3 : Find the order and degree of the following ordinary differential equations.

(a) $y' = 3x^2 - 7$ (b) $\sqrt{y' + y} = \sin x$ (c) $y''' + xy'' + 2y(y')^2 + xy = 0$

(d) $\left(y'''\right)^3+\left(y''\right)^4+y=0$ (e) $y''=\left[y+\left(y'\right)^2\right]^{\frac{1}{4}}$ (f) $\left[1+\left(y'\right)^2\right]^3=ky''$

Q 4 : Classify the following equations as linear and non-linear equations and write down their orders.

(a) $\frac{d^3y}{dx^3}+\frac{d^2y}{dx^2}\cdot\frac{dy}{dx}+y=x$

(b) $x\frac{d^4y}{dx^4}+\frac{d^2y}{dx^2}=e^x$

(c) $\frac{dy}{dx}+y^2=x^2$

Q 5 : Find the differential equation of all

(a) tangent lines to the parabola $y=x^2$.

(b) ellipses centered at the origin.

(c) circles through the origin.

(d) circles tangent to y-axis.

(e) parabolas with axis parallel to the axis of y.

(f) parabolas with foci at the origin and axis along x-axis.s

Q 6 : Verify that

(a) $y=x^5+9$ satisfies the differential equation $\frac{dy}{dx}=5x^4$.

(b) $y=\cos 2x$ satisfies the differential equation $\frac{d^2y}{dx^2}+4y=0$.

(c) $y=\sqrt{x^2+1}$ satisfies the differential equation $\frac{dy}{dx}=\frac{x}{y}$.

(d) $y=x\log x$ satisfies the differential equation $x^2\frac{d^2y}{dx^2}+y=x\frac{dy}{dx}$.

(e) $y=xe^x$ satisfies the differential equation $\frac{dy}{dx}-y=e^x$.

*** ***** ***

UNIT 8 : DIFFERENTIAL EQUATION OF FIRST ORDER AND FIRST DEGREE

UNIT STUCTURE

8.1 Learning objectives

8.2 Introduction

8.3 Differential Equation of First Order and First Degree

8.4 Variable Separable Method

8.5 Homogeneous Differential Equation

8.5.1 Equations Reducible to Homogeneous form

8.6 Linear Differential Equation

8.7 Bernouli's Equation

8.8 Let Us Sum Up

8.9 Further Readings

8.10 Answers To Check Your Progress

8.11 Model Questions

8.1 LEARNING OBJECTIVES

After going through this unit, you will able to learn

- differential equation of first order and first degree
- variable separable method
- homogeneous differential equation
- linear differential equation
- Bernouli's equation.

8.2 INTRODUCTION

In this unit,we shall discuss the different methods of solving differential equations which are of the first order and first degree.These equations contain only the first order derivatives of the depedent variable w.r.t to the independent variable.The general solution will contain only one arbitrary constant.There are different methods of solving differential equations of this type depending on the form of the equation.

8.3 DIFFERENTIAL EQUATION OF FIRST ORDER AND FIRST DEGREE

Such equations are classified into four types :

1. equations where variables can be separated directly or on special substitution;
2. homogeneous differential equations and equations reducible to such form;
3. linear differential equations and equations reducible to this form;
4. exact differential equations and equations reducible to this form.

8.4 VARIABLE SEPARABLE METHOD

In a differential equation variables are said to be separated if functions of x and dx be on one side and the functions of y and dy be on the other side. Such type of equation can be reduced to the form $f(x)dx = f(y)dy$. Then the solution of the equation can be obtained on integrating both sides. The required solution is

$$\int f(x)dx = \int f(y)y + C$$

Note :

(i) When y is absent, the equation reduces to $\frac{dy}{dx} = f(x)$ and therefore the solution is

$$y = \int f(x)dx + C\,.$$

(ii) When x is absent , it reduces to $\frac{dy}{dx} = g(y)$ and in this case, the solution is $\int \frac{dy}{g(y)} = \int dx + C$.

Example 1 : Solve $\frac{dy}{dx} = e^{3x-2y} + x^2 e^{-2y}$

Solution : The given equation can be written as

$$\frac{dy}{dx} = e^{-2y}\left(e^{3x} + x^2\right)$$

or $\quad e^{2y}dy = \left(e^{3x} + x^2\right)dx$

Integrating both sides, we get $\int e^{2y}dy = \int \left(e^{3x} + x^2\right)dx$

or, $\frac{1}{2}e^{2y} = \frac{1}{3}e^{3x} + \frac{1}{3}x^3 + C$

or $3e^{2y} = 2\left(e^{3x} + x^3\right) + K$, which is the requird solution.

Example 2 : Solve $\left(e^y + 1\right)\cos x dx + e^y \sin x dy = 0$

Solution : Dividing both sides by $\left(e^y + 1\right)\sin x$, we separate the variables in the form

$$\frac{\cos x dx}{\sin x} + \frac{e^y dy}{e^y + 1} = 0$$

Integrating both sides, we get $\log \sin x + \log\left(e^y + 1\right) = \log C$

or $\log\left(\sin x\left(e^y + 1\right)\right) = \log C$

or $\sin x\left(e^y + 1\right) = C$,the required solution.

Note : If sometimes variables could not be separated then by combing some terms as total differentials it can be done.

Example 3 : Solve $y \sin 2x dx - \left(1 + y + \cos^2 x\right)dy = 0$

Solution : Here variables cannot be separated easily. We rearrange as

$$\left(y \sin 2x dx - \cos^2 x dy\right) - \left(1 + y\right)dy = 0$$

Integrating, we have

$$- y\cos^2 x - \left(y + \frac{y^2}{2}\right) = -C$$

or $y\cos^2 x + \left(y + \frac{y^2}{2}\right) = -C$, which is the required solution.

Example 4 : Solve : $x\frac{dy}{dx} = y + \left(\log x\right)\frac{dy}{dx}$

Solution : This is an equation where variables are separable but integration is not possible.

The given equation can be written as $\left(x - \log x\right)\frac{dy}{dx} = y$

$$\text{or } \frac{dy}{y} = \frac{dx}{x - \log x}$$

Integrating, $\int \frac{dy}{y} = \int \frac{dx}{x - \log x}$

or $\log y + \log C = \int \frac{dx}{x - \log x}$

or $\log(yC) = \int \frac{dx}{x - \log x}$

or $yC = e^{\int \frac{dx}{x-\log x}}$ which is the required solution.

Initial value problem : A differential equation together with an initial condition is called a initial value problem.Using the initial condition, the particular value of the arbitrary constant can be determined.

Example 5 : Find the particular solution of

$\cos y dx + (1 + 2e^{-x})\sin y dy = 0$ when $x = 0$, $y = \frac{\pi}{4}$.

Solution : Separating variables the given equation becomes

$$\frac{dx}{1 + 2e^{-x}} + \frac{\sin y dy}{\cos y} = 0$$

or $\frac{e^x dx}{e^x + 2} + \frac{\sin y dy}{\cos y} = 0$

Integrating, we have

$$\log(e^x + 2) - \log(\cos y) = \log c$$

or $e^x + 2 = c\cos y$ ... (1)

Using initial condition, $e^0 + 2 = c\cos\frac{\pi}{4}$

or $3 = c\frac{1}{\sqrt{2}}$

or $c = 3\sqrt{2}$

putting the value of c in (1), we get $e^x + 2 = 3\sqrt{2}\cos y$

or $\sec y(e^x + 2) = 3\sqrt{2}$ which is the required solution.

Note : Equations of the form $\frac{dy}{dx} = f(ax + by + c)$ can be transfomed by substitution $z = ax + by + c$.

Example 6 : Solve $\cos(x + y)dy = dx$

Let $z = x + y$ so that $\frac{dz}{dx} = 1 + \frac{dy}{dx}$.

The given ode becomes $\cos z\left(\frac{dz}{dx} - 1\right) = 1$

or $\frac{dz}{dx}.\cos z = 1+\cos z$

or $\frac{dz}{dx} = \frac{1+\cos z}{\cos z}$

or $dx = \frac{\cos z}{1+\cos z}dz$

$$= \left(1 - \frac{1}{1+\cos z}\right)dz$$

$$= \left(1 - \frac{1}{2}\sec^2\frac{z}{2}\right)dz$$

Integrating, we have

$$x = z - \tan\frac{z}{2} + C$$

or $0 = y - \tan\frac{(x+y)}{2} + C$

or $y = -C + \tan\frac{(x+y)}{2}$ is the required solution.

Example 7 : Solve $\log\left(\frac{dy}{dx}\right) = ax + by$

Solution : Substituting $z = ax + by$ and hence $\frac{dz}{dx} = a + b\frac{dy}{dx}$

we have $\frac{1}{b}\left(\frac{dz}{dx} - a\right) = e^z$

or $\frac{dz}{a+be^z} = dx$

or $\frac{e^{-z}dz}{ae^{-z}+b} = dx$

or $\frac{du}{-au} = dx$, where $u = ae^{-z} + b$ and $du = -ae^{-z}dz$

or $\log u = -ax + \log C$

or $\log\left(ae^{-z} + b\right) = -ax + \log C$

or $ae^{-(ax+by)} + b = Ce^{-ax}$

or $ae^{-ax}e^{-by} - Ce^{-ax} + b = 0$

or $ae^{-by} - C + be^{ax} = 0$

or $ae^{-by} + be^{ax} = C$, which is the required solution.

Another substitution : If $x = r\cos\theta, y = r\sin\theta$, then from $x^2 + y^2 = r^2$. we have $xdx + ydy = rdr$.

From $\tan\theta = \frac{y}{x}$, we find $\frac{xdy - ydx}{x^2} = \sec^2\theta d\theta$

or $xdy - ydx = r^2 d\theta$

This shows that an ode containing terms $x^2 + y^2$, $xdx + ydy$ and $xdy - ydx$ can be reduced to a simple form. Then method of variable separation can be applied to solve the equation.

Example 8 : Solve $\frac{xdx + ydy}{xdy - ydx} = \sqrt{\frac{a^2 - x^2 - y^2}{x^2 + y^2}}$

Solution : Substituting $x = r\cos\theta, y = r\sin\theta$, we have $xdx + ydy = rdr$, $xdy - ydx = r^2 d\theta$ and $x^2 + y^2 = r^2$.

So, the given equation reduces to

$$\frac{rdr}{r^2 d\theta} = \sqrt{\frac{a^2 - r^2}{r^2}}$$

or $$\frac{dr}{\sqrt{a^2 - r^2}} = d\theta$$

Integrating $$\int \frac{dr}{\sqrt{a^2 - r^2}} = \int d\theta$$

or $$\sin^{-1}\frac{r}{a} = \theta + C$$

or $$\frac{r}{a} = \sin(\theta + C)$$

or $$r = a(\sin\theta\cos C + \cos\theta\sin C)$$

or $$r^2 = a(r\sin\theta\cos C + r\cos\theta\sin C)$$

or $x^2 + y^2 = a(y\cos C + x\sin C)$, which is the required solution.

Equation of the form $yf_1(xy) + xf_2(xy)dy = 0$:

Substituting $z = xy$, $dz = ydx + xdy$ and $dy = \frac{xdz - zdx}{x^2}$

or $xdz - zdx = x^2 dy$

or $dz - \frac{z}{x}dx = xdy$

The variables z and x can be separated. We illustrate with the help of the following example.

Example 8 : Solve $y(1+2xy)dx + x(1-xy)dy = 0$

Solution : Substituting $z = xy$, $xdy + ydx = z$ and $xdy = dz - \frac{z}{x}dx$,

we have, $\frac{z}{x}(1+2z)dx + (1-z)\left(dz - \frac{z}{x}dx\right) = 0$

or $\left\{\frac{z}{x}(1+2z) - (1-z)\frac{z}{x}\right\}dx + (1-z)dz = 0$

or $\left\{z + 2z^2 - z + z^2\right\}\frac{dx}{x} + (1-z)dz = 0$

or $3z^2\frac{dx}{x} + (1-z)dz = 0$

or $\frac{dx}{x} + \frac{1-z}{3z^2}dz = 0$

or $\frac{dx}{x} + \frac{1}{3}\left(\frac{1}{z^2} - \frac{1}{z}\right)dz = 0$

Integrating, we have

$$\log x + \frac{1}{3}\left(-\frac{1}{z} - \log z\right) = \frac{C}{3}$$

or $$\log x - \frac{1}{3}\left(\frac{1}{xy} + \log xy\right) = \frac{C}{3}$$

or $\log\frac{x^3}{xy} - \frac{1}{xy} = C = \log k$,(say)

or $\log\frac{x^2}{ky} = \frac{1}{xy}$

or $x^2 = kye^{\frac{1}{xy}}$, which is required solution.



CHECK YOUR PROGRESS - 1

Q 1 : Solve the following differential equations :

(i) $3e^x \tan y dx + \left(1 - e^x\right)\sec^2 y dy = 0$

(ii) $\frac{dy}{dx} = e^{x+y} + x^2 e^{x^3+y}$

(iii) $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$

Q 2 : Solve the following differential equations :

(i) $\frac{dy}{dx} = (4x + y + 1)^2$

(ii) $\frac{dy}{dx} = \sin(x+y) + \cos(x+y)$

8.5 HOMOGENEOUS DIFFERENTIAL EQUATION

A function $f(x, y)$ is called a homogeneous function equation of degree n if $f(x, y)$ can be expressed as $x^n \phi\left(\frac{y}{x}\right)$ or $y^n \phi\left(\frac{x}{y}\right)$.

For example $f(x, y) = x^2 + xy + y^2 = x^2\left(1 + \frac{y}{x} + \frac{y^2}{x^2}\right)$

or $y^2\left(1 + \frac{x}{y} + \frac{x^2}{y^2}\right)$ is a homogeneous function of degree 2.

A differential equation of the form $M(x, y)dx + N(x, y)dy = 0$ is called a linear homogeneous equation if $M(x, y)$ and $N(x, y)$ are homogeneous function of the same order.This equation can be reduced to another form

$$\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)} = -\frac{\phi\left(\frac{y}{x}\right)}{\varphi\left(\frac{y}{x}\right)} = f\left(\frac{y}{x}\right).$$

Method of Solving a linear homogeneous equation $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$:

Let $z = \frac{y}{x}$ or $y = zx$ so that $\frac{dy}{dx} = \frac{dz}{dx}x + z$.

so, $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$ reduces to the form $x\frac{dz}{dx} + z = f(z)$

$$\text{or } \frac{dz}{f(z) - z} = \frac{dx}{x}$$

The solution of the equation is obtained on integration.

Example 1 : Solve the following differential equations :

(i) $\left(x^2 - y^2\right)dx - xydy = 0$

(ii) $\frac{dy}{dx} = \frac{y + \sqrt{x^2 + y^2}}{x}$

(iii) $\left(1 + e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy = 0$

(iv) $y^2 + x^2\frac{dy}{dx} = xy\frac{dy}{dx}$

Solution : (i) The given equation is $\frac{dy}{dx} = \frac{x^2 - y^2}{xy} = \frac{1 - \left(\frac{y}{x}\right)^2}{\frac{y}{x}} = f\left(\frac{y}{x}\right)$

which is a linear homogeneous equation.

Let $z = \frac{y}{x}$ or $y = zx$ so that $\frac{dy}{dx} = \frac{dz}{dx}x + z = \frac{1 - z^2}{z}$

$$\text{or} \quad x\frac{dz}{dx} = \frac{1 - 2z^2}{z}$$

$$\text{or} \quad \frac{zdz}{1 - 2z^2} = \frac{dx}{x}$$

Integrating both sides, we have $-\frac{1}{4}\log\left(1 - 2z^2\right) = \log x + \log C$

$$\text{or} \quad -\frac{1}{4}\log\left(1 - 2z^2\right) = \log Cx$$

or $\left(1-2z^2\right)^{-\frac{1}{4}} = Cx$

or $1-2\frac{y^2}{x^2} = C^{-4}x^{-4}$

or $C^4\left(x^4-2x^2y^2\right)=1$, which is the required solution.

(ii) $\frac{dy}{dx} = \frac{y+\sqrt{x^2+y^2}}{x} = \frac{y}{x}+\sqrt{1+\frac{y^2}{x^2}} = f\left(\frac{y}{x}\right)$ is a homogeneous equation.

Let $z = \frac{y}{x}$ or $y = zx$ so that $\frac{dy}{dx} = \frac{dz}{dx}x + z$

So, the given equation reduces to $x\frac{dz}{dx} + z = z + \sqrt{1+z^2}$

or $\frac{dz}{\sqrt{1+z^2}} = \frac{dx}{x}$

Integrating both sides, we have

$$\log\left(z+\sqrt{1+z^2}\right) = \log x + \log C = \log(Cx)$$

or $\frac{y}{x}+\sqrt{1+\left(\frac{y}{x}\right)^2} = Cx$

or $y+\sqrt{x^2+y^2} = Cx^2$, which is the required solution.

(iii) $\left(1+e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy = 0$ can be written as

$$\frac{dx}{dy} = -\frac{e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)}{1+e^{\frac{x}{y}}}$$

Let $z = \frac{x}{y}$, $x = yz$ and $\frac{dx}{dy} = z + y\frac{dz}{dy}$.

So the equation becomes

$$z + y\frac{dz}{dy} = -\frac{e^z(1-z)}{1+e^z}$$

or $y\frac{dz}{dy} = -\frac{e^z(1-z)}{1+e^z} - z$

or $y\frac{dz}{dy}=\frac{-e^z+ze^z-z-ze^z}{1+e^z}$

or $\frac{(1+e^z)dz}{z+e^z}=-\frac{dy}{y}$

Integrating both sides, $\log(z+e^z)=-\log y+\log C$

or $\log(z+e^z)=\log\left(\frac{C}{y}\right)$

or $z+e^z=\frac{C}{y}$

or $\frac{x}{y}+e^{\frac{x}{y}}=\frac{C}{y}$, which is the required solution.

(iv) The given equation $y^2+x^2\frac{dy}{dx}=xy\frac{dy}{dx}$ can be written in standard form

as $\frac{dy}{dx}=\frac{y^2}{xy-x^2}=\frac{\left(\frac{y}{x}\right)^2}{\frac{y}{x}-1}$. ... (1)

Let $z=\frac{y}{x}$ or $y=zx$ so that $\frac{dy}{dx}=\frac{dz}{dx}x+z$

The equation (1) becomes $x\frac{dz}{dx}+z=\frac{z^2}{z-1}$

or $x\frac{dz}{dx}=\frac{z^2}{z-1}-z$

or $x\frac{dz}{dx}=\frac{z}{z-1}$

or $\frac{(z-1)dz}{z}=\frac{dx}{x}$

or $\left(1-\frac{1}{z}\right)dz=\frac{dx}{x}$

Integrating both sides, $\int\left(1-\frac{1}{z}\right)dz=\int\frac{dx}{x}$

or $z-\log z=\log x+\log C$

or $z = \log(Cxz)$

or $\frac{y}{x} = \log\left(Cx\frac{y}{x}\right)$

or $\frac{y}{x} = \log(Cy)$

or $Cy = e^{\frac{y}{x}}$,which is the required solution.

8.5.1 Equations reducible to Homogeneous form

The non homogeneous equation of the form $\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'}$ $\left(c \neq 0, c' \neq 0\right)$ can be reduced to a homogeneous equations as follows

Case I : Suppose $\frac{a}{b} \neq \frac{a'}{b'}$ .Let $x = X + h$ and $y = Y + k$ where h and k are constants.Then $dx = dX$ and $dy = dY$ .Then the nonhomogeneous equation becomes

$$\frac{dY}{dX} = \frac{aX + bY + (ah + bk + c)}{a'X + b'Y + (a'h + b'k + c')} \quad \text{... (1)}$$

This becomes homogeneous equation choosing h and k such that $ah + bk + c = 0$ and $a'h + b'k + c' = 0$.

Here h and k are given by $\frac{h}{bc' - b'c} = \frac{k}{a'c - ac'} = \frac{1}{ab' - a'b}$

The equation (1) reduces to $\frac{dY}{dX} = \frac{aX + bY}{a'X + b'Y}$ which is homogeneous and can be solved by putting $Z = \frac{Y}{X}$.

Case II : If $\frac{a}{b} = \frac{a'}{b'} = m$,then the equation reduces to the form

$\frac{dy}{dx} = \frac{m(a'x + b'y) + c}{a'x + b'y + c'}$ .Now substituting $z = a'x + b'y$,it can be further reduced to form where variable separation is possible.In this case

$\frac{dz}{dx} = a' + b' \frac{dy}{dx}$ and the equation becomes $\frac{1}{b'}\left(\frac{dz}{dx} - a'\right) = \frac{mz + c}{z + c'}$

$$\text{or } \frac{dz}{dx} = a' + \frac{mb'z + b'c}{z + c'}$$

$$= \frac{(a' + mb')z + (a'c' + b'c)}{z + c'}$$

Separating variables and integrating both sides,we get the solution

$$\int \frac{(z + c')dz}{(a' + mb')z + (a'c' + b'c)} = x + K .$$

Example 1 : Solve $(2x + y)dx - (4x + 2y - 1)dy = 0$

Solution : The given equation can be written as

$$\frac{dy}{dx} = \frac{2x + y}{4x + 2y - 1} \text{ and in this case } \frac{2}{4} = \frac{1}{2}.$$

We put $z = 2x + y$, the equation becomes

$$\frac{dz}{dx} - 2 = \frac{z}{2z - 1}$$

$$\text{or } \frac{dz}{dx} = \frac{5z - 2}{2z - 1}$$

$$\text{or } \frac{2z - 1}{5z - 2}dz = dx$$

Integrating both sides, we have $\int \frac{2z - 1}{5z - 2}dz = \int dx + C$

$$\text{or } \int \frac{\frac{2}{5}(5z - 2) - \frac{1}{5}}{5z - 2}dz = \int dx + C$$

$$\text{or } \frac{2z}{5} - \frac{1}{25}\log(5z - 2) = x + C$$

$$\text{or } \frac{2}{5}(2x + y) - \frac{1}{25}\log(10x + 5y - 2) = x + C$$

$$\text{or } -\frac{1}{5}x + \frac{2}{5}y - \frac{1}{25}\log(10x + 5y - 2) = C$$

or $10y - 10x - \log(10x + 5y - 2) = 25C$, which is the required solution.

Example 2 : Solve $\frac{dy}{dx} = \frac{y+x-2}{y-x-4}$

Solution : In this case $\frac{a}{a'} = 1$ and $\frac{b}{b'} = -1$, so $\frac{a}{a'} \neq \frac{b}{b'}$.

Let $x = X - h$ and $y = Y - k$. where h and k are constants. Then $dx = dX$ and $dy = dY$. Then the given equation becomes

$$\frac{dY}{dX} = \frac{Y-k+X-h-2}{Y-k-X+h-4} = \frac{X+Y-(h+k+2)}{Y-X+(h-k-4)}$$

We choose h and k such that

$h+k+2=0$

$h-k-4=0$

Solving, we get $h=1$ and $k=-3$, the equation reduces to

$$\frac{dY}{dX} = \frac{X+Y}{Y-X} = \frac{1+\frac{Y}{X}}{\frac{Y}{X}-1}$$

Let $Y = ZX$ so that $\frac{dY}{dX} = Z + X\frac{dZ}{dX}$.

The equation becomes

$$Z + X\frac{dZ}{dX} = \frac{1+Z}{Z-1}$$

$$\text{or } X\frac{dZ}{dX} = \frac{1+Z}{Z-1} - Z$$

$$\text{or } X\frac{dZ}{dX} = \frac{1+Z-Z^2+Z}{Z-1}$$

$$\text{or } \frac{Z-1}{1+Z-Z^2+Z}dZ = \frac{dX}{X}$$

$$\text{or } \frac{-\frac{1}{2}(2-2Z)}{1+Z-Z^2+Z}dZ = \frac{dX}{X}$$

Integrating both sides, we have

$$\int \frac{-\frac{1}{2}(2-2Z)}{1+Z-Z^2+Z}dZ = \int \frac{dX}{X}$$

$$\text{or } -\frac{1}{2}\log(1+2Z-Z^2) = \log X + \log C$$

or $\log\left(1+2Z-Z^2\right)^{-\frac{1}{2}} = \log XC$

or $\frac{1}{\sqrt{1+2Z-Z^2}} = CX$

or $1 = CX\sqrt{1+2\frac{Y}{X}-\frac{Y^2}{X^2}} = C\sqrt{X^2+2XY-Y^2}$

or $\frac{1}{C^2} = X^2+2XY-Y^2$

or $\frac{1}{C^2} = (x+1)^2+2(x+1)(y-3)-(y-3)^2$, which is the required solution.

Example 3 : Solve $(2x+3y-6)\frac{dy}{dx} = 6x-2y-7$

Solution : The given equation can be written as

$$\frac{dy}{dx} = \frac{6x-2y-7}{2x+3y-6} \quad \ldots (1)$$

In this case $\frac{a}{a'} = \frac{6}{2} = \frac{3}{1}$ and $\frac{b}{b'} = \frac{-2}{3}$, so $\frac{a}{a'} \neq \frac{b}{b'}$.

Let $x = X+h$ and $y = Y+k$ .where h and k are constants.Then $dx = dX$ and $dy = dY$. Then (1) becomes

$$\frac{dY}{dX} = \frac{6(X+h)-2(Y+k)-7}{2(X+h)+3(Y+k)-6} = \frac{6X-2Y+6h-2k-7}{2X+3Y+2h+3k-6}$$

We choose h and k such that

$6h-2k-7=0$

$2h+3k-6=0$

solving $h = \frac{3}{2}$ and $k=1$ .The equation reduces to

$$\frac{dY}{dX} = \frac{6X-2Y}{2X+3Y}$$

We put $Y = ZX$,so that $\frac{dY}{dX} = Z+X\frac{dZ}{dX}$

$$Z+X\frac{dZ}{dX} = \frac{6-2Z}{2+3Z}$$

or $X\frac{dZ}{dX} = \frac{6-2Z}{2+3Z} - Z$

$$\text{or } X\frac{dZ}{dX} = \frac{6-4Z-3Z^2}{2+3Z}$$

$$\text{or } \int \frac{2+3Z}{6-4Z-3Z^2}dZ = \int \frac{dX}{X}$$

$$\text{or } -\frac{1}{2}\int \frac{-6Z-4}{6-4Z-3Z^2}dZ = \int \frac{dX}{X}$$

$$\text{or } -\frac{1}{2}\log\left(3Z^2+4Z-6\right) = \log X + \log C = \log CX$$

$$\text{or } \log\left(3Z^2+4Z-6\right) = \log\left(CX\right)^{-2}$$

$$\text{or } 3\left(\frac{Y}{X}\right)^2 + 4\frac{Y}{X} - 6 = C^{-2}X^{-2}$$

$$\text{or } 3\left(\frac{y-1}{x-\frac{3}{2}}\right)^2 + 4\frac{y-1}{x-\frac{3}{2}} - 6 = C^{-2}\left(x-\frac{3}{2}\right)^{-2}$$, which is the required solution.



CHECK YOUR PROGRESS - 2

Q 1 : Solve the following differential equations :

(a) $\left(x^3+3xy^2\right)dx + \left(y^3+3x^2y\right)dy = 0$

(b) $\frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x}$

(c) $\left(x^2-4xy-2y^2\right)dx + \left(y^2-4xy-2x^2\right)dy = 0$

(d) $\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3}$

8.6 LINEAR DIFFERENTIAL EQUATION

A differential equation of the form $\frac{dy}{dx} + P(x)y = Q(x)$ where $P(x)$ and $Q(x)$ are functions of x or constants is called a linear differential equation of first order.Here the dependent variable y and its derivative $\frac{dy}{dx}$ occur in the first degree.

Method of Solving a linear differential equation :

Let $R(x)$ be an integrating factor of the standard linear equation

$\frac{dy}{dx} + P(x)y = Q(x)$.Multiplying both sides by $R(x)$,we obtain

$R(x)\frac{dy}{dx} + P(x)R(x)y = Q(x)R(x)$.

It can be written as

$$\frac{d}{dx}(yR(x)) + P(x)R(x)y - y\frac{dR(x)}{dx} = Q(x)R(x)$$

Left side can be made total differential, choosing $R(x)$ such that

$$P(x)R(x)y = y\frac{dR(x)}{dx}$$

or $P(x)dx = \frac{dR(x)}{R(x)}$

On integration, we have $\log R = \int P dx$ and $R = e^{\int P dx}$.

Thus $e^{\int P dx}$ is an integrating factor of the equation.

Then the linear equation becomes

$\frac{d}{dx}\left(y.e^{\int p dx}\right) = Qe^{\int p dx}$ and its solution is

$y.e^{\int p dx} = \int \left(Qe^{\int p dx}\right)dx + C$

or $y = e^{-\int p dx}\left[\int \left(Qe^{\int p dx}\right)dx + C\right]$

Remark : Interchanging x and y ,the solution of

$\frac{dx}{dy} + P(y)x = Q(y)$ is found to be

$x = e^{-\int p dy}\left[\int \left(Qe^{\int p dy}\right)dy + C\right]$

Note : We frequently use the following properties of logarithmic and exponential functions :

(i) $e^{\log x} = x$ (ii) $e^{m\log x} = x^m$ (iii) $e^{-m\log x} = \frac{1}{x^m}$

Example 1 : Solve $\frac{dy}{dx} + 2y\cot x = 3x^2 \operatorname{cosec}^2 x$

Solution : Comparing with standard equation $\frac{dy}{dx} + Py = Q$,we have

$P = 2\cot x$, $Q = 3x^2 \operatorname{cosec}^2 x$

Integrating factor (I.F) $= e^{\int p dx} = e^{\int 2\cot x dx} = e^{2\log \sin x} = \sin^2 x$

The solution is given by

$$y \sin^2 x = \int 3x^2 \operatorname{cosec}^2 x \sin^2 x dx$$

$$= 3\int x^2 dx$$

$$= x^3 + C$$

or $y = \operatorname{cosec}^2 x\left(x^3 + C\right)$.

Example 2 : Solve $\frac{dy}{dx} - y\tan x = e^x$

Solution : Comparing with standard equation $\frac{dy}{dx} + Py = Q$, we have

$P = -\tan x$, $Q = e^x$.

Integrating factor (I.F) $= e^{-\int \tan x dx} = e^{\log \cos x} = \cos x$

The solution of the given differential equation is given by

$$y.\cos x = \int \left(e^x.\cos x\right)dx + C$$

$$= \frac{1}{2}e^x\left(\sin x + \cos x\right) + C$$

or $y = \frac{1}{2}e^x \sec x\left(\sin x + \cos x + 2C\right)$

Example 3 : Solve $\cos^2 x\frac{dy}{dx} + y = \tan x$

Solution : The given equation can be written in standard form of linear equation as $\frac{dy}{dx} + \sec^2 xy = \tan x \sec^2 x$

Comparing with standard equation $\frac{dy}{dx} + Py = Q$, we have

$P = \sec^2 x$, $Q = \tan x \sec^2 x$.

Here I.F $= e^{\int \sec^2 x dx} = e^{\tan x}$

The solution of the given differential equation is given by

$$y.e^{\tan x} = \int e^{\tan x}.\tan x \sec^2 x dx + C$$

$$= e^{\tan x}\left(\tan x - 1\right) + C$$

or $y = \left(\tan x - 1\right) + Ce^{-\tan x}$.

Example 4 : Solve $\left(1 - x^2\right)\frac{dy}{dx} + 2xy = x\sqrt{1 - x^2}$

Solution : The equation can be written in the standard form as

$$\frac{dy}{dx}+\frac{2x}{\left(1-x^2\right)}y=\frac{x\sqrt{1-x^2}}{\left(1-x^2\right)}=\frac{x}{\sqrt{1-x^2}}$$

Comparing with standard equation $\frac{dy}{dx}+Py=Q$,we have

$$P=\frac{2x}{\left(1-x^2\right)},\ Q=\frac{x}{\sqrt{1-x^2}}.$$

Here I.F $=e^{\int\frac{2x}{1-x^2}dx}=e^{-\log\left(1-x^2\right)}=\frac{1}{1-x^2}$

The solution is given by

$$y.\frac{1}{1-x^2}=\int\frac{1}{1-x^2}\frac{x}{\sqrt{1-x^2}}dx+C$$

$$=\int\frac{x}{\left(1-x^2\right)^{\frac{3}{2}}}dx+C$$

$$=\frac{1}{\sqrt{1-x^2}}+C$$

or $y=-\sqrt{1-x^2}+C\left(1-x^2\right)$.

Example 6 : Solve $\left(1+y^2\right)dx=\left(\tan^{-1}y-x\right)dy$

Solution : The given equation can be written as

$$\frac{dx}{dy}+\frac{x}{1+y^2}=\frac{\tan^{-1}y}{1+y^2}$$

Here I.F $=e^{\int\frac{dy}{1+y^2}}=e^{\tan^{-1}y}$

Thus the solution is $x.e^{\tan^{-1}y}=\int\frac{\tan^{-1}y}{1+y^2}.e^{\tan^{-1}y}dy$

$$=\int u.e^u du \text{ if } u=\tan^{-1}x.$$

$$=ue^u-e^u+C$$

$$=\left(\tan^{-1}y-1\right)e^{\tan^{-1}y}+C$$

or $x=\tan^{-1}y-1+Ce^{-\tan^{-1}y}$.

Example 7 : Solve $\left(x+\tan y\right)dy=\sin 2ydx$

Solution : The given equiation can be written as

$$\frac{dx}{dy} - \cos ec2yx = \frac{\tan y}{\sin 2y} = \frac{1}{2}\sec^2 y$$

Here I.F $= e^{\int pdy} = e^{-\int \cos ecydy} = e^{-\frac{1}{2}\log \tan y} = e^{\log \sqrt{\cot y}} = \sqrt{\cot y}$

So, the required solution is

$$x\sqrt{\cot y} = \frac{1}{2}\int \sec^2 y\sqrt{\cot y}dy + C$$

$$= \frac{1}{2}\int (\tan y)^{-\frac{1}{2}} d(\tan y) + C$$

$$= \frac{1}{2}\frac{(\tan y)^{\frac{1}{2}}}{\frac{1}{2}} + C = \sqrt{\tan y} + C$$

or $x = \tan y + C\sqrt{\tan y}$

Example 8 : Solve $(x + y + a)\frac{dy}{dx} = y^2 + b$

Solution : The given equiation can be written as

$$\frac{dx}{dy} = \frac{x + y + a}{y^2 + b}$$

or $$\frac{dx}{dy} = \frac{x}{y^2 + b} + \frac{y + a}{y^2 + b}$$

or $$\frac{dx}{dy} - \frac{x}{y^2 + b} = \frac{y + a}{y^2 + b}$$

Here I.F $= e^{-\int \frac{1}{y^2+b}dy} = e^{-\frac{1}{\sqrt{b}}\tan^{-1}\frac{y}{\sqrt{b}}}$

The required solution is

$$x.e^{-\frac{1}{\sqrt{b}}\tan^{-1}\frac{y}{\sqrt{b}}} = \int \frac{y + a}{y^2 + b}.e^{-\frac{1}{\sqrt{b}}\tan^{-1}\frac{y}{\sqrt{b}}}dy + C.$$

$$= -\int (y + a).e^{-\frac{1}{\sqrt{b}}\tan^{-1}\frac{y}{\sqrt{b}}} d\left(\frac{1}{\sqrt{b}}\tan^{-1}\frac{y}{\sqrt{b}}\right)y + C.$$

$$= -(y + a).e^{-\frac{1}{\sqrt{b}}\tan^{-1}\frac{y}{\sqrt{b}}} + \int e^{-\frac{1}{\sqrt{b}}\tan^{-1}\frac{y}{\sqrt{b}}} d\left(\frac{1}{\sqrt{b}}\tan^{-1}\frac{y}{\sqrt{b}}\right)y + C.$$

$$= -(y + a).e^{-\frac{1}{\sqrt{b}}\tan^{-1}\frac{y}{\sqrt{b}}} + e^{-\frac{1}{\sqrt{b}}\tan^{-1}\frac{y}{\sqrt{b}}} + C$$

or $x = -(y+a)+1+Ce^{\frac{1}{\sqrt{b}}\tan^{-1}\frac{y}{\sqrt{b}}}$

or $x+y+a-1 = Ce^{\frac{1}{\sqrt{b}}\tan^{-1}\frac{y}{\sqrt{b}}}$



CHECK YOUR PROGRESS - 3

Q 1 : Solve the following differential equations :

(a) $x\cos x\frac{dy}{dx}+y(x\sin x+\cos x)=1$

(b) $x(1-x^2)dy+(2x^2y-y-ax^3)dx=0$

(c) $(1+x+xy^2)dy+(y+y^3)dx=0$

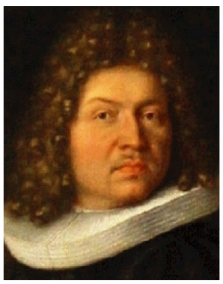
(d) $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}}-\frac{y}{\sqrt{x}}\right)\frac{dx}{dy}=1$

(e) $(y\log y)dx+(x-\log y)dy=0$

8.7 BERNOULI'S EQUATION

The equation $\frac{dy}{dx}+Py=Qy^n$, where P, Q are functions of x is called Bernoulli's equation.

This equation is reducible to the linear differential equation $\frac{dz}{dx}+P(1-n)z=Q(1-n)$, by substitution $z=y^{1-n}$ which can be solved easily.



James Bernoulli (1655–1705) was a Swiss Mathematician

Differentiating $z=y^{1-n}$ w.r.t x, we have $\frac{dz}{dx}=(1-n)y^{-n}\frac{dy}{dx}$.

The Bernoull's equation then reduces to

$$\frac{1}{1-n}y^n\frac{dz}{dx}+P.y=Qy^n$$

or $\frac{dz}{dx}+Pz(1-n)=Q(1-n)$, which is linear.

I.F $=e^{\int(1-n)Pdx}$

The required solution is given by

$$z.e^{\int(1-n)Pdx}=\int e^{\int(1-n)Pdx}.Q(1-n)dx+C$$

Note : The equation is named after James Bernoulli (1655–1705).

Example 1 : Solve $\cos x \frac{dy}{dx} = y \sin x + y^3 \cos^2 x$

Solution : The given equation can be written as

$$\frac{dy}{dx} - y \frac{\sin x}{\cos x} = y^3 \cos x$$

Dividing by y^3, $y^{-3} \frac{dy}{dx} - y^{-2} \frac{\sin x}{\cos x} = \cos x$

We put $z = y^{-2}$ so that $\frac{dz}{dx} = -2y^{-3} \frac{dy}{dx}$

The equation becomes

$$-\frac{1}{2}\frac{dz}{dx} - z \frac{\sin x}{\cos x} = \cos x$$

or $\frac{dz}{dx} + 2z \frac{\sin x}{\cos x} = -2 \cos x$, which is linear.

$$\therefore \text{I. F } = e^{\int \frac{2 \sin x}{\cos x} dx} = \sec^2 x$$

The required solution is

$$z.\sec^2 x = \int \sec^2 x(-2 \cos x)dx = -2\int \sec x dx$$

or $z.\sec^2 x = -2 \log(\sec x + \tan x) + C$

or $y^{-2} \sec^2 x = -2 \log(\sec x + \tan x) + C$

or $\sec^2 x + 2y^2 \log(\sec x + \tan x) = Cy^2$

Example 2 : Solve $\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$

Solution : Dividing the given equation by $\cos^2 y$, we have

$$\sec^2 y \frac{dy}{dx} + 2x \sin y \sec y = x^3$$

or $\sec^2 y \frac{dy}{dx} + 2x \tan y = x^3$

we put $z = \tan y$ so that $\frac{dz}{dx} = \sec^2 y \frac{dy}{dx}$

The equation reduces to

$$\frac{dz}{dx} + 2xz = x^3, \text{ which is linear.}$$

$$\text{I.F } = e^{\int x dx} = e^{x^2}$$

The required solution is

$$z.e^{x^2} = \int e^{x^2}.x^3 dx$$

$$= \int e^u .u \frac{du}{2} \qquad (u = x^2)$$

$$= \frac{1}{2} ue^u - \frac{1}{2}\int e^u du$$

$$= \frac{1}{2}\left(ue^u - e^u\right) + C$$

or $\tan y.e^{x^2} = \frac{1}{2}x^2 e^{x^2} - \frac{1}{2}e^{x^2} + C$

or $\tan y = \frac{1}{2}x^2 - \frac{1}{2} + Ce^{-x^2} = \frac{1}{2}\left(x^2 - 1\right) + Ce^{-x^2}$.

Example 3 : Solve $\frac{dz}{dx} + \frac{z}{x}\log z = \frac{z}{x}(\log z)^2$

Solution : Dividing the given equation by z, we have

$$\frac{1}{z}\frac{dz}{dx} + \frac{1}{x}\log z = \frac{1}{x}(\log z)^2$$

we put $v = \log z$, $\frac{dv}{dx} = \frac{1}{z}\frac{dz}{dx}$

The equation becomes

$$\frac{dv}{dx} + \frac{1}{x}v = \frac{1}{x}v^2$$

or $\frac{1}{v^2}\frac{dv}{dx} + \frac{1}{x}\frac{1}{v} = \frac{1}{x}$

Again, we put $t = \frac{1}{v}$ so that $\frac{dt}{dx} = -\frac{1}{v^2}\frac{dv}{dx}$

Again the equation becomes

$$\frac{dt}{dx} - \frac{1}{x}t = -\frac{1}{x}$$

I.F $= e^{-\int \frac{1}{x}dx} = e^{-\log x} = \frac{1}{x}$

The required solution is

$$\frac{1}{x}.t = \int \frac{1}{x}.\left(-\frac{1}{x}\right)dx = \frac{1}{x} + C$$

or $t = 1 + Cx$

or $\frac{1}{\log z} = 1 + Cx$

or $\log z = \frac{1}{1+Cx}$

or $z = e^{\frac{1}{1+Cx}}$

Example 4 : Solve $x\frac{dy}{dx} + y = xy^3$

Solution : Dividing the given equation by xy^3, we have

$$\frac{1}{y^3}\frac{dy}{dx} + \frac{1}{xy^2} = 1$$

we put $z = \frac{1}{y^2}$, so that $\frac{dz}{dx} = -2y^3\frac{dy}{dx}$.

The equation becomes

$$-\frac{1}{2}\frac{dz}{dx} + \frac{1}{x}z = 1$$

or $\frac{dz}{dx} + \left(\frac{-2}{x}\right)z = -2$

$\text{I.F} = e^{-\int\frac{2}{x}dx} = e^{-2\log x} = \frac{1}{x^2}$

The required solution is

$$z.\frac{1}{x^2} = \int \frac{1}{x}.(-2)dx$$

$$= \frac{2}{x} + C$$

or $\frac{1}{x^2y^2} = \frac{2}{x} + C$

Example 5 : Solve $(x+a)\frac{dy}{dx} - 3y = (x+a)^5$

Solution : We put $z = x + a$ so that $\frac{dz}{dx} = 1$ and $\frac{dy}{dx} = \frac{dy}{dz}.\frac{dz}{dx} = \frac{dy}{dz}$

The given equation becomes

$$z\frac{dy}{dz} - 3y = z^5$$

$$\text{or } \frac{dy}{dz}+\left(\frac{-3}{z}\right)y=z^4$$

$$\text{I.F}=e^{\int\left(\frac{-3}{z}\right)dz}=e^{-3\log z}=\frac{1}{z^3}$$

The required solution is

$$y.\frac{1}{z^3}=\int\frac{1}{z^3}.z^4dz$$

$$=\int zdz$$

$$=\frac{z^2}{2}+C$$

$$\text{or } \frac{y}{(x+a)^3}=\frac{1}{2}(x+a)^2+C$$

$$\text{or } y=\frac{1}{2}(x+a)^5+C(x+a)^3.$$



CHECK YOUR PROGRESS - 4

Q 1 : Solve the following differential equations :

(a) $x\frac{dy}{dx}+y\log y=xye^x$

(b) $\left(x^2y^3+xy\right)\frac{dy}{dx}=1$

(c) $\frac{dy}{dx}+\frac{x}{1-x^2}y=x\sqrt{y}$

(d) $xy\left(1+xy^2\right)\frac{dy}{dx}=1$



8.8 LET US SUM UP

- The solution of the differential equation of the type $\frac{dy}{dx}=f(x)g(y)$ is obtained after separating the variables and integrating both sides.
- A differential equation of the form $M(x,y)dx+N(x,y)dy=0$ is called a linear homogeneous equation if $M(x,y)$ and $N(x,y)$ are

homogeneous functions of the same order. This equation can be reduced to another form $\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)} = -\frac{\phi\left(\frac{y}{x}\right)}{\varphi\left(\frac{y}{x}\right)} = f\left(\frac{y}{x}\right)$.

The solution of a homogeneous differential equation is obtained by substituting $y = vx$ or $x = vy$ and then separating the variables.

- The solution of the first order linear equation $\frac{dy}{dx} + Py = Q$ is

$$y.e^{\int p dx} = \int \left(Qe^{\int p dx}\right)dx + C$$

$e^{\int P dx}$ is called an integrating factor of the equation. Similarly, the solution of the first order linear equation

$\frac{dx}{dy} + P(y)x = Q(y)$ is $xe^{\int P dy} = \int \left(Qe^{\int P dy}\right)dy + C$

$e^{\int P dy}$ is called an integrating factor of the equation.

- The equation $\frac{dy}{dx} + Py = Qy^n$, where P, Q are functions of x is called the Bernoull's equation.

This equation is reducible to the linear differential equation $\frac{dz}{dx} + P(1-n)z = Q(1-n)$, by substitution $z = y^{1-n}$ which can be solved easily.



8.9 FURTHER READINGS

1) Tenenbaum, Morris and Pollard, Harry. Ordinary Differential Equations, Dover Publications, Inc, N.Y 11501.
2) Differential Equations : An Introduction with Mathematica. Clay C. Ross, second Edition, Springer.
3) Differential Equations : Linear, Nonlinear, ordinary, partial. A.C. King, J. Billinggham and S.R. Otto, Cambridge-2003.
4) Differential equations with Application and Programs. S. Balachandra Rao and H.R. Anuradha, University Press(India) Limited 1996.

8.10 ANSWERS TO CHECK YOUR PROGRESS

Check your progress -1

Ans to Q No 1: (i) Separating the variables, we get $\frac{3e^x}{1-e^x}dx+\frac{\sec^2 y}{\tan y}dy=0$

Integrating both sides,we get $\int\frac{3e^x}{1-e^x}dx+\int\frac{\sec^2 y}{\tan y}dy=\log C$

or $-3\log(1-e^x)+\log(\tan y)=\log C$

or $\log(\tan y)=\log(1-e^x)^3+\log C$

or $\tan y=C(1-e^x)^3$.

(ii) Given, $\frac{dy}{dx}=e^{x+y}+x^2e^{x^3+y}$

or $\frac{dy}{dx}=e^y\left(e^x+x^2e^{x^3}\right)$

Separating the variables, we get $e^{-y}dy=\left(e^x+x^2e^{x^3}\right)dx$ ... (1)

Integrating (1), $\int e^{-y}dy=\int e^x dx+\int x^2e^{x^3}dx$

or $-e^{-y}=e^x+\frac{1}{3}\int e^t dt+C$,putting $x^3=t$

or $-e^{-y}=e^x+\frac{1}{3}e^t+C$

or $-e^{-y}=e^x+\frac{1}{3}e^{x^3}+C$.

(iii) Separating the variables,the given differential equation can be written

as $\frac{dy}{1+y^2}=\frac{dx}{1+x^2}$ ... (1)

Integrating both sides of (1),we get

$$\int\frac{dy}{1+y^2}=\int\frac{dx}{1+x^2}$$

or $\tan^{-1}y=\tan^{-1}x+C$

which is the general solution of the equation (1).

Ans to Q No 2 : (i) Let $4x + y + 1 = v$ ... (1)

Differentiating (1) with respect to x, we get $4 + \frac{dy}{dx} = \frac{dv}{dx}$

or $\frac{dy}{dx} = \frac{dv}{dx} - 4$ ... (2)

using (1) and (2), the given equation becomes $\frac{dv}{dx} - 4 = v^2$

or $\frac{dv}{dx} = 4 + v^2$

Now, separating the variables x and v, $dx = \frac{dv}{4 + v^2}$

Integrating both sides, we get $x + C' = \frac{1}{2} \times \tan^{-1}\left(\frac{v}{2}\right)$

or $2x + 2C' = \tan^{-1}\left(\frac{v}{2}\right)$

or $2x + C = \tan^{-1}\left(\frac{v}{2}\right)$, where $C = 2C'$

or $v = 2\tan(2x + C)$

or $4x + y + 1 = 2\tan(2x + C)$.

(ii) Let $x + y = v$ ... (1)

Differentiating (1) w.r.t x, $1 + \frac{dy}{dx} = \frac{dv}{dx}$

or $\frac{dy}{dx} = \frac{dv}{dx} - 1$ ... (2)

From (1) and (2), the given equation becomes

$\frac{dv}{dx} - 1 = \sin v + \cos v$

or $\frac{dv}{dx} = 1 + \sin v + \cos v$

or $\frac{dv}{dx} = 1 + 2\sin\frac{v}{2}\cos\frac{v}{2} + 2\cos^2\frac{v}{2} - 1$

or $\frac{dv}{dx} = 2\cos^2\frac{v}{2}\left[1 + \tan\frac{v}{2}\right]$

Separating the variables, we get $dx = \dfrac{dv}{2\cos^2\dfrac{v}{2}\left[1+\tan\dfrac{v}{2}\right]}$

$$\text{or } dx = \frac{\frac{1}{2}\sec^2\frac{v}{2}dv}{1+\tan\frac{v}{2}}$$

Integrating both sides,we get $x + C = \log\left[1+\tan\dfrac{v}{2}\right]$

$$\text{or } x + C = \log\left[1+\tan\left(\frac{x+y}{2}\right)\right]$$

Check your progress - 2

Ans to Q No 1: (a) Given, $\dfrac{dy}{dx} = -\dfrac{x^3+3xy^2}{y^3+3x^2y} = -\dfrac{1+3\left(\frac{y}{x}\right)^2}{\left(\frac{y}{x}\right)^3+3\left(\frac{y}{x}\right)}$ ... (1)

Let $\dfrac{y}{x} = v$ i.e., $y = vx$ so that $\dfrac{dy}{dx} = v + x\left(\dfrac{dv}{dx}\right)$ ... (2)

From (1) and (2), $v + x\dfrac{dv}{dx} = -\dfrac{1+3v^2}{v^3+3v}$

or $x\dfrac{dv}{dx} = -\dfrac{1+3v^2}{v^3+3v} - v$

or $x\dfrac{dv}{dx} = -\dfrac{v^4+6v^2+1}{v^3+3v}$

or $4\dfrac{dx}{x} = -\dfrac{4v^3+12v}{v^4+6v^2+1}dv$

Integrating both sides, we get

$$4\log x = -\log\left(v^4+6v^2+1\right)+\log C$$

or $\log x^4 = \log\left[\dfrac{C}{v^4+6v^2+1}\right]$

or $x^4\left(v^4+6v^2+1\right) = C$

or $y^4+6x^2y^2+x^4 = C$, which is the required solution.

(b) Let $\frac{y}{x} = v$, i.e., $y = vx$ so that $\frac{dy}{dx} = v + x\left(\frac{dv}{dx}\right)$ ... (1)

Using (1), the given equation becomes

$$v + x\frac{dv}{dx} = v + \tan v$$

or $$\frac{dx}{x} = \frac{\cos v}{\sin v}dv$$

Integrating both sides, we get

$$\log x + \log C = \log \sin v$$

or $$Cx = \sin v$$

or $$Cx = \sin\left(\frac{y}{x}\right).$$

(c) The given equation can be written as

$$\frac{dy}{dx} = \frac{x^2 - 4xy - 2y^2}{2x^2 + 4xy - y^2}$$

Let $\frac{y}{x} = v$, i.e., $y = vx$ so that $\frac{dy}{dx} = v + x\left(\frac{dv}{dx}\right)$ ... (1)

Using (1), the given equation becomes

$$v + x\frac{dv}{dx} = \frac{1 - 4v - 2v^2}{2 + 4v - v^2}$$

or $$x\frac{dv}{dx} = \frac{1 - 4v - 2v^2}{2 + 4v - v^2} - v$$

or $$x\frac{dv}{dx} = \frac{1 - 6v - 6v^2 + v^3}{2 + 4v - v^2}$$

or $$3\frac{dx}{x} + \frac{3\left(v^2 - 4v - 2\right)}{v^3 - 6v^2 - 6v + 1}dv = 0$$

Integrating, we get

$$3\log x + \log\left(v^3 - 6v^2 - 6v + 1\right) = \log C$$

or $$x^3\left(v^3 - 6v^2 - 6v + 1\right) = C$$

or $$x^3\left\{\left(\frac{y}{x}\right)^3 - 6\left(\frac{y}{x}\right)^2 - 6\left(\frac{y}{x}\right) + 1\right\} = C$$

or $$y^3 - 6xy^2 - 6x^2y + x^3 = C.$$

(d) In this case $\frac{a}{a'} = \frac{1}{2}$ and $\frac{b}{b'} = \frac{2}{1}$, so $\frac{a}{a'} \neq \frac{b}{b'}$.

Let $x = X + h$ and $y = Y + k$, where h and k are constants. Then $dx = dX$ and $dy = dY$. Then the given equation becomes

$$\frac{dY}{dX} = \frac{X + 2Y + (h + 2k - 3)}{2X + Y + (2h + k - 3)} \quad \ldots (1)$$

We choose h, k so that $h + 2k - 3 = 0$ and $2h + k - 3 = 0$ ... (2)

Solving (2), we get $h = 1, k = 1$

From (1), $\frac{dY}{dX} = \frac{X + 2Y}{2X + Y} = \frac{1 + \left(\frac{2Y}{X}\right)}{2 + \left(\frac{Y}{X}\right)}$ ... (3)

We take $\frac{Y}{X} = v$, i.e., $Y = vX$

Therefore, $\frac{dY}{dX} = v + X\frac{dv}{dX}$ ... (4)

From (3) and (4), we have

$$v + X\frac{dv}{dX} = \frac{1 + 2v}{2 + v}$$

or $X\frac{dv}{dX} = \frac{1 + 2v}{2 + v} - v = \frac{1 - v^2}{2 + v}$

or $\frac{dX}{X} = \frac{(2 + v)dv}{(1 - v)(1 + v)} = \left[\frac{1}{2}\left(\frac{1}{1 + v}\right) + \frac{3}{2}\left(\frac{1}{1 - v}\right)\right]dv$, resolving into partial fractions.

Integrating, we have $\log X + \log C = \frac{1}{2}[\log(1 + v) - 3\log(1 - v)]$

or $2\log CX = \log\frac{1 + v}{(1 - v)^3}$

or $X^2C^2 = \frac{1 + v}{(1 - v)^3}$

or $X^2C^2\left(1 - \frac{Y}{X}\right)^3 = 1 + \frac{Y}{X}$

or $C^2(X - Y)^3 = X + Y$

or $C^2\{x - 1 - (y - 1)\}^3 = x - 1 + y - 1$

or $C'\{x-y\}^3 = x+y-2$, taking $C' = C^2$.

Check your progress - 3

Ans to Q No 1: (a) Re-writing the given equation, we have

$$\frac{dy}{dx} + \left(\tan x + \frac{1}{x}\right)y = \frac{\sec x}{x} \quad \ldots (1)$$

I.F $= e^{\int\left(\tan x+\frac{1}{x}\right)dx} = e^{\log\sec x+\log x} = e^{\log(x\sec x)} = x\sec x$

Hence the required solution is

$$yx\sec x = \int \sec^2 x dx + C$$

or $yx\sec x = \tan x + C$.

(b) Re-writing the given equation, we have

$$x(1-x^2)\frac{dy}{dx} + y(2x^2-1) = ax^3$$

or $$\frac{dy}{dx} + \frac{(2x^2-1)}{x(1-x^2)}y = \frac{ax^2}{1-x^2} \quad \ldots (1)$$

Here $P = \frac{(2x^2-1)}{x(1-x^2)} = -\frac{1}{x} - \frac{1}{2(x+1)} - \frac{1}{2(x-1)}$ and $Q = \frac{ax^2}{1-x^2}$ ... (2)

$$\int P dx = -\int\left[\frac{1}{x} + \frac{1}{2(x+1)} + \frac{1}{2(x-1)}\right]dx$$

$$= -\left[\log x + \frac{1}{2}\times\log(x+1) + \frac{1}{2}\times\log(x-1)\right]$$

$$= -\left[\log x + \frac{1}{2}\times\log(x^2-1)\right]$$

$$= -\log\left[x(x^2-1)^{\frac{1}{2}}\right]$$

$$= \log\left[x(x^2-1)^{\frac{1}{2}}\right]^{-1}$$

I.F $= e^{\int P dx} = e^{\log\left[x(x^2-1)^{\frac{1}{2}}\right]^{-1}} = \left\{x(x^2-1)^{\frac{1}{2}}\right\}^{-1} = \frac{1}{\left\{x(x^2-1)^{\frac{1}{2}}\right\}}$

Hence required solution is

$$\frac{y}{x\left(x^2-1\right)^{\frac{1}{2}}}=\int\frac{ax^2}{1-x^2}\times\frac{1}{x\left(x^2-1\right)^{\frac{1}{2}}}dx+C$$

or $$\frac{y}{x\left(x^2-1\right)^{\frac{1}{2}}}=C-a\int\frac{x}{\left(x^2-1\right)^{\frac{3}{2}}}dx$$

or $$\frac{y}{x\left(x^2-1\right)^{\frac{1}{2}}}=C-\frac{a}{2}\int\frac{dt}{t^{\frac{3}{2}}}$$, Putting $x^2-1=t$ and $2xdx=dt$

or $$\frac{y}{x\left(x^2-1\right)^{\frac{1}{2}}}=C-\frac{a}{2}\left[\frac{t^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)}\right]$$

or $$\frac{y}{x\left(x^2-1\right)^{\frac{1}{2}}}=C+\frac{a}{\sqrt{t}}$$

or $$\frac{y}{x\left(x^2-1\right)^{\frac{1}{2}}}=C+\frac{a}{\left(x^2-1\right)^{\frac{1}{2}}}$$

or $y=ax+Cx\left(x^2-1\right)^{\frac{1}{2}}$

(c) Re-writing the given equation, we have

$$\frac{dx}{dy}+\frac{1+x\left(1+y^2\right)}{y\left(1+y^2\right)}=0$$

or $$\frac{dx}{dy}+\frac{x}{y}=-\frac{1}{y\left(1+y^2\right)}$$

Here $P=\frac{1}{y}$ and $Q=-\frac{1}{y\left(1+y^2\right)}$

I.F $=e^{\int\frac{1}{y}dy}=e^{\log y}=y$

Hence required solution is

$$xy=-\int\frac{1}{y\left(1+y^2\right)}.ydy+C$$

or $xy=-\tan^{-1}y+C$.

(d) Re-writing the given equation, we have

$$\frac{dy}{dx}+\frac{1}{\sqrt{x}}y=\frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$

Comparing with $\frac{dy}{dx}+Py=Q$, we get

$$P=\frac{1}{\sqrt{x}}, Q=\frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$

Then I.F $=e^{\int Pdx}=e^{\int\frac{1}{\sqrt{x}}dx}=e^{2\sqrt{x}}$
Hence required solution is

$$y.e^{2\sqrt{x}}=\int\frac{e^{-2\sqrt{x}}}{\sqrt{x}}.e^{2\sqrt{x}}dx+C$$

or $y.e^{2\sqrt{x}}=\int\frac{1}{\sqrt{x}}dx+C$

or $y.e^{2\sqrt{x}}=2\sqrt{x}+C$.

(e) Re-writing the given equation, we have

$$\frac{dx}{dy}+\frac{x}{y\log y}=\frac{1}{y}$$

Comparing with $\frac{dx}{dy}+Px=Q$, we get

$$P=\frac{1}{y\log y}, \quad Q=\frac{1}{y}$$

Then I.F $=e^{\int Pdy}=e^{\int\frac{1}{y\log y}dy}=e^{\log(\log y)}=\log y$
Hence, the solution is

$$x\log y=\int\frac{1}{y}.\log y dy+C$$

or $x\log y=\frac{1}{2}(\log y)^2+C$

Check your progress - 4

Ans to Q No 1: (a) Dividing by xy, the given becomes $\frac{1}{y}\frac{dy}{dx}+\frac{1}{x}\log y=e^x$ ... (1)

Let $\log y = v$ so that $\frac{1}{y}\frac{dy}{dx} = \frac{dv}{dx}$ ... (2)

Using (2), (1) reduces to $\frac{dv}{dx} + \frac{1}{x}v = e^x$, which is linear differential equation.

Here $P = \frac{1}{x}$ and $Q = e^x$

I.F $= e^{\int \frac{1}{x}dx} = e^{\log x} = x$

Hence the required solution is

$v.x = \int xe^x dx + C$

or $v.x = xe^x - \int e^x dx + C$

or $v.x = xe^x - e^x + C$

or $x\log y = e^x(x-1) + C$

(b) The given equation can be written as

$$\frac{dx}{dy} = x^2y^3 + xy$$

or $$\frac{dx}{dy} - yx = y^3x^2$$

or $$x^{-2}\left(\frac{dx}{dy}\right) - x^{-1}y = y^3 \quad \text{... (1)}$$

We put $x^{-1} = v$ and $-x^{-2}\left(\frac{dx}{dy}\right) = \frac{dv}{dy}$

(1) becomes $-\frac{dv}{dy} - yv = y^3$

or $\frac{dv}{dy} + yv = -y^3$, which is linear differential equation.

I.F $= e^{\int ydy} = e^{\frac{y^2}{2}}$

Hence the required solution is

$v.e^{\frac{y^2}{2}} = -\int y^3 e^{\frac{y^2}{2}} dy + C = -2\int te^t dt + C$, putting $\frac{y^2}{2} = t$ and $ydy = dt$

or $v.e^{\frac{y^2}{2}} = -2\left[t.e^t - \int 1.e^t dt\right] + C$

or $v.e^{\frac{y^2}{2}} = -2\left(te^t - e^t\right) + C$

or $x^{-1}.e^{\frac{y^2}{2}} = -2e^{\frac{y^2}{2}}\left(\frac{y^2}{2} - 1\right) + C$ as $x^{-1} = v$

or $\frac{1}{x} = -2\left(\frac{y^2}{2} - 1\right) + Ce^{-\frac{y^2}{2}}$

(c) Dividing by $\sqrt{y}$, the given equation becomes

$$y^{-\frac{1}{2}}\frac{dy}{dx} + \frac{x}{1-x^2}y^{\frac{1}{2}} = x \quad \text{... (1)}$$

Putting $y^{\frac{1}{2}} = z$ so that $\frac{1}{2}y^{-\frac{1}{2}}\frac{dy}{dx} = \frac{dz}{dx}$ or $y^{-\frac{1}{2}}\frac{dy}{dx} = 2\frac{dz}{dx}$

Equation (1) becomes

$$2\frac{dz}{dx} + \frac{x}{1-x^2}z = x$$

or $\frac{dz}{dx} + \frac{x}{2\left(1-x^2\right)}z = \frac{x}{2}$ which is linear in z.

I.F $= e^{\int \frac{x}{2\left(1-x^2\right)}dx} = e^{-\frac{1}{4}\int \frac{-2x}{1-x^2}dx} = e^{-\frac{1}{4}\log\left(1-x^2\right)} = \left(1-x^2\right)^{-\frac{1}{4}}$

Hence the required solution is

$$z.\left(1-x^2\right)^{-\frac{1}{4}} = \int \frac{x}{2}\left(1-x^2\right)^{-\frac{1}{4}}dx + C$$

or $z.\left(1-x^2\right)^{-\frac{1}{4}} = -\frac{1}{4}\int \left(1-x^2\right)^{-\frac{1}{4}}\left(-2x\right)dx + C$

or $z.\left(1-x^2\right)^{-\frac{1}{4}} = -\frac{1}{4}\frac{\left(1-x^2\right)^{\frac{3}{4}}}{\frac{3}{4}} + C$

or $z = -\frac{1}{3}\left(1-x^2\right) + C\left(1-x^2\right)^{\frac{1}{4}}$

or $\sqrt{y} = -\frac{1}{3}\left(1-x^2\right)+C\left(1-x^2\right)^{\frac{1}{4}}$.

(d) The given equation can be written as $\frac{dx}{dy} - yx = y^3x^2$

Dividing by x^2, we have $x^{-2}\frac{dx}{dy} - yx^{-1} = y^3$ ... (1)

Putting $x^{-1} = z$ so that $-x^{-2}\frac{dx}{dy} = \frac{dz}{dx}$ or $x^{-2}\frac{dx}{dy} = -\frac{dz}{dx}$

Equation (1) becomes

$$-\frac{dz}{dx} - yz = y^3$$

or $\frac{dz}{dx} + yz = -y^3$ which is linear in z.

I.F $= e^{\int ydx} = e^{\frac{1}{2}y^2}$

Hence the required solution is

$$z.e^{\frac{1}{2}y^2} = \int -y^3e^{\frac{1}{2}y^2}dy + C$$

or $z.e^{\frac{1}{2}y^2} = \int -y^2e^{\frac{1}{2}y^2}.ydy + C$

or $z.e^{\frac{1}{2}y^2} = -\int 2te^t dt + C$ where $t = \frac{y^2}{2}$

or $z.e^{\frac{1}{2}y^2} = -2\left[te^t - \int e^t dt\right] + C$

or $z.e^{\frac{1}{2}y^2} = -2\left(te^t - e^t\right) + C$

or $z.e^{\frac{1}{2}y^2} = -2e^{\frac{1}{2}y^2}\left(\frac{1}{2}y^2 - 1\right) + C$

or $z = -2\left(\frac{1}{2}y^2 - 1\right) + Ce^{-\frac{1}{2}y^2}$

or $z = 2 - y^2 + Ce^{-\frac{1}{2}y^2}$.



8.11 MODEL QUESTIONS

Q 1 : Solve the following differential equations :

(a) $\frac{y}{x}.\frac{dy}{dx} = \sqrt{1+x^2+y^2+x^2y^2}$

(b) $e^y\left(1+x^2\right)\frac{dy}{dx} - 2x\left(1+e^y\right) = 0$

(c) $\frac{dy}{dx} - x\tan\left(y-x\right) = 1$

(d) $\sin\left(x+y\right)dy = dx$

Q 2 : Solve the following differential equations :

(a) $x\left(x-y\right)\frac{dy}{dx} = y\left(x+y\right)$

(b) $x\frac{dy}{dx} = y + x\cos^2\frac{y}{x}$

Q 3 : Solve the following differential equations :

(a) $\left(2x-2y+5\right)\frac{dy}{dx} = x-y+3$

(b) $\left(6x+2y-10\right)\frac{dy}{dx} - 2x - 9y + 29 = 0$

(c) $\left(3y-7x+7\right)dx + \left(7y-3x+3\right)dy = 0$

(d) $\left(2x^2+3y^2-7\right)xdx - \left(3x^2+2y^2-8\right)ydy = 0$

Q 4 : Solve the following differential equations :

(a) $\left(1+x^3\right)\frac{dy}{dx} + 6x^2y = 1+x^2$

(b) $\left(x+2y^3\right)\frac{dy}{dx} = y$

(c) $\frac{dy}{dx} = \frac{y}{2y\log y + y - x}$

(d) $\sqrt{1-y^2}\,dx = \left(\sin^{-1}y - x\right)dy$

Q 5 : Solve the following differential equations :

(a) $2\frac{dy}{dx} = \frac{y}{x} + \frac{y^2}{x^2}$

(b) $(x+1)\frac{dy}{dx} + 1 = 2e^{-y}$

(c) $\frac{dy}{dx} + \frac{y \log y}{x} = \frac{y(\log y)^2}{x^2}$

(d) $y(2xy + e^x)dx = e^x dy$

Q 6 : Solve the following differential equations :

(a) $\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$

(b) $xdy + ydx + \frac{xdy - ydx}{x^2 + y^2} = 0$

(c) $(a^2 - 2xy - y^2)dx - (x + y)^2 dy = 0$

(d) $(2ax + by)ydx + (ax + 2by)xdy = 0$

*** ***** ***

UNIT 9 : EXACT DIFFERENTIAL EQUATION OF FIRST ORDER AND FIRST DEGREE

UNIT STUCTURE

9.1 Learning objectives

9.2 Introduction

9.3 Exact differential Equation

9.4 Equations reducible to Exact Equations

9.5 Let Us Sum Up

9.6 Further Readings

9.7 Answers To Check Your Progress

9.8 Model Questions

9.1 LEARNING OBJECTIVES

After going through this unit, you will able to learn

- Exact differential equation
- Equations reducible to Exact differential equation.

9.2 INTRODUCTION

In previous unit,we have discussed ordinary differential equation of first order and first degree. We have also discussed the different methods of solving differential equations which are of the first order and first degree. This unit will focus on exact differential equations of first order and first degree as well as equations reducible to exact differential equation of first order and first degree.

9.3 EXACT DIFFERENTIAL EQUATION

A differential equation $M(x,y)dx+N(x,y)dy=0$ is said be an exact equation if left hand expression of this equation is total differential of a function $u(x,y)$, say, that is if $M(x,y)dx+N(x,y)dy=0$ be exact, then $M(x,y)dx+N(x,y)dy=du(x,y)$ and solution of this equation is $u(x,y)=$ constant.

So, an exact differential equation can also be defined as an equation obtainable by differentiating an equation in x and y ,without any other powers.

The necessary and sufficient condition for $M(x,y)dx+N(x,y)dy=0$ to be an exact differential equation :

Theorem : The necessary and sufficient condition for $M(x,y)dx+N(x,y)dy=0$ to be an exact is $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$.

Proof : Suppose $M(x,y)dx+N(x,y)dy=0$ is exact. Then there is a function $u(x,y)$ such that

$$du=Mdx+Ndy$$

$$\text{or } \frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy=Mdx+Ndy$$

Equating the coefficients of dx and dy , we have

$M=\frac{\partial u}{\partial x}$ and $N=\frac{\partial u}{\partial y}$

So, $\frac{\partial M}{\partial y}=\frac{\partial^2 u}{\partial y\partial x}$ and $\frac{\partial N}{\partial x}=\frac{\partial^2 u}{\partial x\partial y}$

Assuming the condition for equality of $\frac{\partial^2 u}{\partial y\partial x}=\frac{\partial^2 u}{\partial x\partial y}$,

we have $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$.

Conversely suppose $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$

Let $\int Mdx=V$, then $\frac{\partial V}{\partial x}=M$ and $\frac{\partial M}{\partial y}=\frac{\partial^2 V}{\partial y\partial x}=\frac{\partial^2 V}{\partial x\partial y}=\frac{\partial N}{\partial x}$.

Then $\frac{\partial}{\partial x}\left(N-\frac{\partial V}{\partial y}\right)=0\Rightarrow N-\frac{\partial V}{\partial y}$ is independent of x.

So, $\frac{\partial N}{\partial x}=\frac{\partial}{\partial x}\left(\frac{\partial V}{\partial y}\right)$. Hence $N=\frac{\partial V}{\partial y}+\phi'(y)$.

Therefore $Mdx+Ndy=\left(\frac{\partial V}{\partial x}dx+\frac{\partial V}{\partial y}dy\right)+\phi'(y)dy$

$$= dV + d(\phi(y))$$

$$= dU \text{ where } U = V + \phi(y)$$

So, $M(x,y)dx + N(x,y)dy = 0$ is exact and its solution is $V + \phi(y) =$ constant.

or $\int\limits_{y=\text{constant}} Mdx + \int\left(N - \frac{\partial V}{\partial y}\right)dy = \text{constant}$.

Note : If $M(x,y)dx + N(x,y)dy = 0$ be exact, then its solution is

$$\int\limits_{y=\text{constant}} Mdx + \int(N - \text{term containing } x)dy = C$$

Example 1 : Solve $(x^2 - 4xy - 2y^2)dx + (y^2 - 4xy - 2x^2)dy = 0$

Solution : Here $M = x^2 - 4xy - 2y^2$ and $N = y^2 - 4xy - 2x^2$

So, $\frac{\partial M}{\partial y} = -4x - 4y$ and $\frac{\partial N}{\partial x} = -4y - 4x$

i.e., $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

So, the given equation is exact.

The solution of the equation is

$\int\limits_{y=\text{constant}} (x^2 - 4xy - 2y^2)dx + \int y^2 dy = 0$ (From N delete the terms containing x)

or $\frac{x^3}{3} - 2x^2y - 2y^2x + \frac{y^3}{3} = C$.

Example 2 : Solve $\frac{dy}{dx} + \frac{ax + hy + g}{hx + by + f} = 0$

Solution : The given equation can be written as

$$(ax + hy + g)dx + (hx + by + f)dy = 0$$

Here $M = ax + hy + g$ and $N = hx + by + f$

So, $\frac{\partial M}{\partial y} = h = \frac{\partial N}{\partial x}$

Hece the equation is exact.

The solution is

$$\int\limits_{y=\text{constant}} Mdx + \int(\text{terms free from } x \text{ in N})dy = C$$

or $\int(ax + hy + g)dx + \int(by + f)dy = C$

or $\frac{ax^2}{2}+hyx+gx+\frac{by^2}{2}+fy=C$

or $ax^2+2hxy+by^2+2gx+2fy=2C$, which is the required solution.

Example 3 : Solve $y\sin 2xdx-\left(1+y+\cos^2 x\right)dy=0$

Solution : Here $M=y\sin 2x$ and $N=-1-y-\cos^2 x$

So, $\frac{\partial M}{\partial y}=\sin 2x=\frac{\partial N}{\partial x}$

Hence the given equation is exact.

The solution is

$$\int_{y=\text{constant}} y\sin 2xdx+\int_{(\text{terms free from } x \text{ in N})}(-1-y)dy=C$$

or $-\frac{1}{2}y\cos 2x-y-\frac{y^2}{2}=C$

or $y\cos 2x+2y+y^2+2C=0$, which is the required solution.



CHECK YOUR PROGRESS - 1

Q 1 : Solve the following differential equations :

(a) $\left(y^2e^{xy^2}+4x^3\right)dx+\left(2xye^{xy^2}-3y^2\right)dy=0$

(b) $\left\{y\left(1+\frac{1}{x}\right)+\cos y\right\}dx+\left(x+\log x-x\sin y\right)dy=0$

(c) $\left[\cos x\tan y+\cos(x+y)\right]dx+\left[\sin x\sec^2 y+\cos(x+y)\right]dy=0$

(d) $\left(5x^4+3x^2y^2-2xy^3\right)dx+\left(2x^3y-3x^2y^2-5y^4\right)dy=0$

9.4 EQUATIONS REDUCIBLE TO EXACT EQUATIONS

If an equation $M(x,y)dx+N(x,y)dy=0$ becomes an exact when multiplied by a function $\mu(x,y)$ then $\mu(x,y)$ is called an integrating factor (I.F).

For example, $xdy-ydx=0$ is not an exact equation,

since $\frac{\partial M}{\partial y}=\frac{\partial}{\partial y}(-y)=-1$ but $\frac{\partial N}{\partial x}=\frac{\partial x}{\partial x}=1$.

Multiplying by a function $\mu(x,y)$ it can be made an exact equation and it can be done in many ways.

(i) $\mu_1(x,y)=\frac{1}{x^2}$ is an I.F since $\frac{1}{x^2}(xdy-ydx)=d\left(\frac{y}{x}\right)$

(ii) $\mu_2(x,y)=\frac{1}{y^2}$ is an I.F since $\frac{1}{y^2}(xdy-ydx)=-d\left(\frac{x}{y}\right)$

(iii) $\mu_3(x,y)=\frac{1}{xy}$ is an I.F since $\frac{1}{xy}(xdy-ydx)=d\left[\log\frac{y}{x}\right]$

(iv) $\mu_4(x,y)=\frac{1}{x^2+y^2}$ is an I.F since $\frac{1}{x^2+y^2}(xdy-ydx)$

$$=\frac{\frac{xdy-ydx}{x^2}}{1+\left(\frac{y}{x}\right)^2}$$

$$=\frac{d\left(\frac{y}{x}\right)}{1+\left(\frac{y}{x}\right)^2}=d\left(\tan^{-1}\frac{y}{x}\right)$$

Proposition : The number of integrating factors is infinite.

Proof : Suppose $\mu(x,y)$ is an integrating factor of $Mdx+Ndy=0$.

So, there is a $u(x,y)$ such that $\mu Mdx+\mu Ndy=du$.

Multiplying both sides by any function $f(u)$,we have

$(\mu f(u))Mdx+(\mu f(u))Ndy=f(u)du=\phi'(u)du$,say

Then RHS $=d\phi(x,y)$, a total differential.

So, $(\mu f(u))[Mdx+Ndy]$ is a total differential.

Hence, $\mu f(u)$ is an I.F of $Mdx+Ndy=0$ .Since f is an arbitrary function,the I.F, $\mu f(u)$ can be chosen in an infinite ways.

To solve a differential equation $Mdx+Ndy=0$,difficulty lies in finding an I.F to make it an exact equation.Here we discuss some methods of finding an I.F.

I. Method of Inspection to find an I.F.

To find I.F by inspection, terms of the given equation are to be arranged so as to obtain some total differentials. For example

(i) $d(xy) = ydx + xdy$ (ii) $d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2}$

(iii) $d\left(\frac{y}{x}\right) = \frac{xdy - ydx}{x^2}$ (iv) $d\left(\frac{1}{xy}\right) = \frac{-(xdy + ydx)}{x^2y^2}$

(v) $\frac{1}{2}d(x^2 + y^2) = xdx + ydy$ (vi) $\frac{1}{2}d(\log(x^2 + y^2)) = \frac{xdx + ydy}{x^2 + y^2}$

(vii) $d\left(\log\frac{x}{y}\right) = \frac{ydx - xdy}{xy}$ (viii) $d\left(\log\frac{y}{x}\right) = \frac{xdy - ydx}{xy}$

(ix) $d\left(\tan^{-1}\frac{x}{y}\right) = \frac{ydx - xdy}{x^2 + y^2}$ (x) $d\left(\tan^{-1}\frac{y}{x}\right) = \frac{xdy - ydx}{x^2 + y^2}$

Example 1 : Solve (i) $y(2xy + e^x)dx = e^x dy$

(ii) $xdx + ydy = a(x^2 + y^2)dy$

(iii) $(2xy^2 - y)dx + xdy = 0$

(iv) $xdy - ydx = \sqrt{x^2 + y^2}\,dx$

(v) $xdx = y(x^2 + y^2 - 1)dy$

(vi) $ye^{\frac{x}{y}}dx = \left(xe^{\frac{x}{y}} + y^2\right)dy$

(vii) $ydx - xdy + \log x dx = 0$

Solution : (i) $ye^x dx - e^x dy + 2xy^2 dx = 0$

Dividing by y^2, we have

$$\frac{ye^x dx - e^x dy}{y^2} + 2xdx = 0$$

or $d\left(\frac{e^x}{y}\right) + d(x^2) = 0$

Integrating, the requird solution is

$$\frac{e^x}{y} + x^2 = C$$

(ii) $xdx + ydy = a\left(x^2 + y^2\right)dy$

or $$\frac{2xdx+2ydy}{x^2+y^2} = 2ady$$

or $$\frac{d\left(x^2 + y^2\right)}{x^2 + y^2} = 2ady$$

Integrating, the requird solution is

$$\log\left(x^2 + y^2\right) = 2ay + C$$

(iii) $2xy^2dx + \left(xdy - ydx\right) = 0$

or $2xdx + \frac{xdy - ydx}{y^2} = 0$

or $d\left(x^2\right) - d\left(\frac{x}{y}\right) = 0$

Integrating, the requird solution is

$$x^2 - \frac{x}{y} = C$$

or $x^2y - x = Cy.$

(iv) $xdy - ydx = \sqrt{x^2 + y^2}\,dx$

or $$\frac{xdy - ydx}{\sqrt{x^2 + y^2}} = dx$$

or $$\frac{\frac{xdy - ydx}{x^2}}{\sqrt{1+\left(\frac{y}{x}\right)^2}} = \frac{1}{x}dx$$

Integrating, the requird solution is

$$\int \frac{d\left(\frac{y}{x}\right)}{\sqrt{1+\left(\frac{y}{x}\right)^2}} = \int \frac{1}{x}dx$$

or $$\log\left(\frac{y}{x} + \sqrt{1+\left(\frac{y}{x}\right)^2}\right) = \log x + \log C$$

or $\log\left(\frac{y}{x}+\sqrt{1+\left(\frac{y}{x}\right)^2}\right)=\log(Cx)$

or $\frac{y}{x}+\sqrt{1+\left(\frac{y}{x}\right)^2}=Cx$

or $y+\sqrt{x^2+y^2}=Cx^2$.

(v) $xdx=y\left(x^2+y^2-1\right)dy$

or $xdx+ydy=y\left(x^2+y^2\right)dy$

or $\frac{d\left(x^2+y^2\right)}{x^2+y^2}=2ydy$

Integrating, the requird solution is

$\log\left(x^2+y^2\right)=y^2+C$

(vi) $ye^{\frac{x}{y}}dx=\left(xe^{\frac{x}{y}}+y^2\right)dy$

or $e^{\frac{x}{y}}\left(ydx-xdy\right)=y^2dy$

or $e^{\frac{x}{y}}\frac{ydx-xdy}{y^2}=dy$

or $e^{\frac{x}{y}}d\left(\frac{x}{y}\right)=dy$

Integrating, the requird solution is

$e^{\frac{x}{y}}=y+C$

(vii) $ydx-xdy+\log xdx=0$

or $-\frac{xdy-ydx}{x^2}+\frac{\log x}{x^2}dx=0$

or $-d\left(\frac{y}{x}\right)-\log x.\frac{1}{x^2}dx=0$

Integrating, the requird solution is

$-\frac{y}{x}-\left[-\log x.\frac{1}{x}-\int\frac{1}{x}.\frac{-1}{x}dx\right]=C$

or $y+\log x+1=Cx$

or $y+\log x+1-Cx=0$.

II. Solution of homogeneous equation $Mdx+Ndy=0$.

Proposition : If $Mdx+Ndy=0$ be a homogeneous equation of degree r, then $\frac{1}{Mx+Ny}$ is an integrating factor provided $Mx+Ny\neq 0$.

Proof : Let $M=x^r\phi\left(\frac{y}{x}\right)$ and $N=x^r\psi\left(\frac{y}{x}\right)$. Multiplying by $\frac{1}{Mx+Ny}$ and substituting for M and N, we have

$$\frac{x^r\phi\left(\frac{y}{x}\right)}{x^{r+1}\phi\left(\frac{y}{x}\right)+yx^r\psi\left(\frac{y}{x}\right)}dx+\frac{x^r\psi\left(\frac{y}{x}\right)}{x^{r+1}\phi\left(\frac{y}{x}\right)+x^r y\psi\left(\frac{y}{x}\right)}dy=0$$

It is enough to check the exactness of this equation

$$\frac{\partial}{\partial y}\left(\frac{x^r\phi\left(\frac{y}{x}\right)}{x^{r+1}\phi\left(\frac{y}{x}\right)+yx^r\psi\left(\frac{y}{x}\right)}\right)$$

$$=\frac{\phi'\left(\frac{y}{x}\right).\frac{1}{x}\left\{x\phi\left(\frac{y}{x}\right)+y\psi\left(\frac{y}{x}\right)\right\}-\phi\left(\frac{y}{x}\right)\left\{x\phi'\left(\frac{y}{x}\right).\frac{1}{x}+\psi\left(\frac{y}{x}\right)+\frac{y}{x}\psi'\left(\frac{y}{x}\right)\right\}}{\left(x\phi\left(\frac{y}{x}\right)+y\psi\left(\frac{y}{x}\right)\right)^2}$$

$$=\frac{\phi'\phi+\frac{y}{x}\phi'\psi-\phi'\phi-\phi\psi-\frac{y}{x}\phi\psi'}{(x\phi+y\psi)^2}$$

$$=\frac{\frac{y}{x}(\phi'\psi-\phi\psi')-\phi\psi}{(x\phi+y\psi)^2} \quad \ldots (1)$$

Similarly, $\frac{\partial}{\partial x}\left(\frac{\psi\left(\frac{y}{x}\right)}{x\phi\left(\frac{y}{x}\right)+y\psi\left(\frac{y}{x}\right)}\right)$

$$= \frac{\psi'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right)\left\{x\phi\left(\frac{y}{x}\right)+y\psi\left(\frac{y}{x}\right)\right\}-\psi\left(\frac{y}{x}\right)\left\{\phi\left(\frac{y}{x}\right)+x\phi'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right)+y\phi'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right)\right\}}{\left(x\phi\left(\frac{y}{x}\right)+y\psi\left(\frac{y}{x}\right)\right)^2}$$

$$= \frac{-\frac{y}{x}\phi\psi' - \frac{y^2}{x^2}\psi\psi' - \phi\psi + \psi\phi'.\frac{y}{x} + \frac{y^2}{x^2}\psi\psi'}{\left(x\phi\left(\frac{y}{x}\right)+y\psi\left(\frac{y}{x}\right)\right)^2}$$

$$= \frac{\frac{y}{x}\left(\phi'\psi - \psi'\phi\right) - \phi\psi}{\left(x\phi + y\psi\right)^2} \quad \text{... (2)}$$

From (1) and (2), we see that the condition of exactness is satisfied. Hence

$\frac{1}{Mx + Ny}$ is an integrating factor.

Example 2 : Solve $\left(x^2y - 2xy^2\right)dx - \left(x^3 - 3x^2y\right)dy = 0$

Solution : This equation is homogeneous in x and y.

$$\text{I.F} = \frac{1}{Mx+Ny} = \frac{1}{\left(x^2y - 2xy^2\right)x + \left(-x^3 + 3x^2y\right)y}$$

$$= \frac{1}{x^3y - 2x^2y^2 - x^3y + 3x^2y^2}$$

$$= \frac{1}{x^2y^2}$$

Multiplying the given equation by $\frac{1}{x^2y^2}$, we have

$$\left(\frac{1}{y} - \frac{2}{x}\right)dx + \left(-\frac{x}{y^2} + \frac{3}{y}\right)dy = 0$$

which is an exact equation since $\frac{\partial}{\partial y}\left(\frac{1}{y} - \frac{2}{x}\right) = -\frac{1}{y^2}$ and

$\frac{\partial}{\partial x}\left(-\frac{x}{y^2} + \frac{3}{y}\right) = -\frac{1}{y^2}$.

The solution is

$$\int_{y=\text{constant}} \left(\frac{1}{y} - \frac{2}{x}\right)dx + \int_{(\text{terms independent of x in coefficient of dy})} \frac{3}{y}dy = C$$

or $\frac{x}{y} - 2\log x + 3\log y = C$.

Example 3 : Solve $y^2 dx + \left(x^2 - xy - y^2\right)dy = 0$

Solution : It is a homogeneous equation.

$$\text{I.F} = \frac{1}{Mx + Ny} = \frac{1}{xy^2 + x^2 y - xy^2 - y^3}$$

$$= \frac{1}{y\left(x^2 - y^2\right)}$$

Multiplying the equation by $\frac{1}{y\left(x^2 - y^2\right)}$, we have

$$\frac{y}{x^2 - y^2}dx + \left(\frac{1}{y} - \frac{x}{x^2 - y^2}\right)dy = 0$$

Now,

$$\frac{\partial}{\partial y}\left(\frac{y}{x^2 - y^2}\right) = -\frac{1}{2}\log\left(x^2 - y^2\right), \frac{\partial}{\partial x}\left(\frac{1}{y} - \frac{x}{x^2 - y^2}\right) = -\frac{1}{2}\log\left(x^2 - y^2\right)$$

$\therefore$ the equation is exact.

The solution is

$$\int_{y=\text{constant}} \frac{y}{x^2 - y^2}dx + \int_{\text{terms free from } x \text{ in coeff.of } dy} \frac{1}{y}dy = C$$

or $\frac{y}{2y}\log\frac{x - y}{x + y} + \log y = \log C$

or $\log y\sqrt{\frac{x - y}{x + y}} = \log C$

or $y\sqrt{\frac{x - y}{x + y}} = C$

or $y^2\left(x - y\right) = C^2\left(x + y\right)$.

III (A): Integrating factor for an equation of type

$f_1(xy)ydx + f_2(xy)xdy = 0$ is $\frac{1}{xyf_1(xy) - xyf_2(xy)}$ if the denominator is not zero.

Proof : Multiplying the given equation by $\frac{1}{xyf_1(xy) - xyf_2(xy)}$, we have

$$\frac{yf_1(xy)dx}{xy\{f_1(xy)-f_2(xy)\}}+\frac{xf_2(xy)dx}{xy\{f_1(xy)-f_2(xy)\}}=0$$

We check that this is an exact equation.

$$\frac{\partial M}{\partial y}=\frac{\partial}{\partial y}\left(\frac{f_2}{x(f_1-f_2)}\right)=\frac{x^2f_1'(f_1-f_2)-f_1\{x^2(f_1'-f_2')\}}{x^2(f_1-f_2)^2}$$

$$=\frac{f_1f_2'-f_1'f_2}{(f_1-f_2)^2}$$

$$\frac{\partial N}{\partial x}=\frac{\partial}{\partial x}\left(\frac{f_1}{y(f_1-f_2)}\right)=\frac{y^2f_2'(f_1-f_2)-f_2\{y^2(f_1'-f_2')\}}{y^2(f_1-f_2)^2}$$

$$=\frac{f_1f_2'-f_1'f_2}{(f_1-f_2)^2}$$

So, $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$. Hence the given equation is an exact equation and it can be solved by the method already discussed above.

III (B) : With the help of the equation

$$\int\frac{f(xy)+F(xy)}{f(xy)-F(xy)}\frac{d(xy)}{xy}+\log\frac{x}{y}=C$$

Show that $\frac{1}{xy\{f(xy)-F(xy)\}}$ is an integrating factor of $f(xy)ydx+F(xy)xdy=0$. Hence solve

$(x^2y^2+xy+1)ydx-x(x^2y^2-xy+1)dy=0$

It is enough to show that

$\frac{yf(xy)dx+xF(xy)dy}{xy\{f(xy)-F(xy)\}}$ is a total differential.

$$\frac{yf(xy)dx+xF(xy)dy}{xy\{f(xy)-F(xy)\}}$$

$$=\frac{\frac{1}{2}\left\{(xyf(xy)+xyF(xy))\left(\frac{dx}{x}+\frac{dy}{y}\right)+(xyf(xy)-xyF(xy))\left(\frac{dx}{x}-\frac{dy}{y}\right)\right\}}{xyf(xy)-xyF(xy)}$$

$$=\frac{1}{2}\frac{xyf(xy)+xyF(xy)}{xyf(xy)-xyF(xy)}\frac{d(xy)}{xy}+\frac{1}{2}\frac{ydx-xdy}{xy}$$

$$= \frac{1}{2}\left\{\frac{f(xy)+F(xy)}{f(xy)-F(xy)}\frac{d(xy)}{xy}+\frac{d\left(\frac{x}{y}\right)}{\left(\frac{x}{y}\right)}\right\}$$

So, $\int \frac{f(xy)ydx+F(xy)xdy}{xyf(xy)-xyF(xy)} = \frac{1}{2}\left[\int \frac{f(xy)+F(xy)}{f(xy)-F(xy)}\frac{d(xy)}{xy}+\log\frac{x}{y}\right]$

$$= \frac{1}{2}C$$

So, $\frac{f(xy)ydx+F(xy)xdy}{xyf(xy)-xyF(xy)} = d\left(\frac{1}{2}C\right)$, a total differential.

Hence $\frac{yf(xy)dx+xF(xy)dy}{xy\{f(xy)-F(xy)\}}=0$ is an exact equation and

$\frac{1}{xy\{f(xy)-F(xy)\}}$ is an integrating factor of $f(xy)ydx+F(xy)xdy=0$.

Second part : Applying the preceding result, the integrating factor of $(x^2y^2+xy+1)ydx-x(x^2y^2-xy+1)dy=0$ is

$$\frac{1}{xy\{x^2y^2+xy+1+x^2y^2-xy+1\}} = \frac{1}{xy(2x^2y^2+2)}$$

$$\therefore \quad \frac{x^2y^2+xy+1}{2x(x^2y^2+1)}dx+\frac{xy-x^2y^2-1}{2y(x^2y^2+1)}dy=0$$

or $\left(\frac{y}{2(1+x^2y^2)}+\frac{1}{2x}\right)dx+\left(\frac{x}{2(1+x^2y^2)}-\frac{1}{2y}\right)dy=0$

So, its solution is

$$\frac{1}{2y}\int\frac{ydx}{1+x^2y^2}+\frac{1}{2}\int\frac{dx}{x}-\frac{1}{2}\int\frac{dy}{y}=\log C$$

or $\frac{1}{2}\tan^{-1}(xy)+\frac{1}{2}\log\left(\frac{x}{y}\right)=\log C$

or $C^2y=xe^{\tan^{-1}(xy)}$, the required solution.

Example 4 : Solve $(1+xy)ydx+(1-xy)xdy=0$

Solution : It is of the form $f_1(xy)ydx+f_2(xy)xdy=0$.

$$\text{I.F} = \frac{1}{Mx - Ny} = \frac{1}{xy(1+xy) - xy(1-xy)} = \frac{1}{2x^2y^2}$$

Multiplying the given equation by $\frac{1}{2x^2y^2}$, we have

$$\frac{1+xy}{2x^2y}dx + \frac{1-xy}{2xy^2}dy = 0$$

This is an exact equation and its solution is

$$\int_{y=\text{constant}} \left(\frac{1}{2x^2y} + \frac{1}{2x}\right)dx + \int -\frac{1}{2y}dy = \log C$$

or $-\frac{1}{2xy} + \frac{1}{2}\log x - \frac{1}{2}\log y = \log C$

or $-\frac{1}{xy} = \log\left(\frac{C^2y}{x}\right)$

or $\frac{C^2y}{x} = e^{-\frac{1}{xy}}$

or $C^2y = xe^{-\frac{1}{xy}}$.

Example 5 : Solve $y(xy + 2x^2y^2)dx + x(xy - x^2y^2)dy = 0$

Solution : It is of the form $f_1(xy)ydx + f_2(xy)xdy = 0$.

So, I.F = $\frac{1}{xy\{f(xy) - F(xy)\}} = \frac{1}{xy(xy + 2x^2y^2) - xy(xy - x^2y^2)}$

$$= \frac{1}{3x^3y^3}$$

Multipling the given equation throughout by the I.F,we get

$$\frac{y(1+2xy)}{x^2y^2}dx + \frac{x(1-xy)}{x^2y^2}dy = 0$$

or $\left(\frac{1}{x^2y} + \frac{2}{x}\right)dx + \left(\frac{1}{xy^2} - \frac{1}{y}\right)dy = 0$

The equation is exact and its solution is

$$\int_{y=\text{constant}} \left(\frac{1}{x^2y} + \frac{2}{x}\right)dx + \int -\frac{1}{y}dy = \log C$$

or $-\frac{1}{xy}+2\log x-\log y=\log C$

or $-\frac{1}{xy}=\log\frac{Cy}{x^2}$

or $\frac{Cy}{x^2}=e^{-\frac{1}{xy}}$

or $Cy=x^2e^{-\frac{1}{xy}}$

IV (A) If $\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=f(x)$ be a function of x or constant, then show that $e^{\int f(x)dx}$ is an integrating factor of $Mdx+Ndy=0$.

Proof : It is enough to show that $\left(Me^{\int f(x)dx}\right)dx+\left(Ne^{\int f(x)dx}\right)dy=0$ is an exact equation, equivalently left side in a total differential.

$$\frac{\partial}{\partial y}\left(Me^{\int f(x)dx}\right)=e^{\int f(x)dx}\frac{\partial M}{\partial y}$$

and $\frac{\partial}{\partial x}\left(Ne^{\int f(x)dx}\right)=e^{\int f(x)dx}\frac{\partial N}{\partial x}+Ne^{\int f(x)dx}.f(x)$

$$=e^{\int f(x)dx}\left(\frac{\partial N}{\partial x}+\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)$$, replacing $Nf(x)$ by $\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}$

$$=e^{\int f(x)dx}\frac{\partial M}{\partial y}$$

So, $\frac{\partial}{\partial y}\left(Me^{\int f(x)dx}\right)=\frac{\partial}{\partial x}\left(Ne^{\int f(x)dx}\right)$. Hence $e^{\int f(x)dx}$ is an integrating factor of $Mdx+Ndy=0$.

IV (B). If $\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}=f(y)$ is a function of y then $e^{\int f(y)dy}$ is an integrating factor of $Mdx+Ndy=0$.

Proof : $\frac{\partial}{\partial y}\left(e^{\int f(y)dy}M\right)=e^{\int f(y)dy}.f(y)M+e^{\int f(y)dy}\frac{\partial M}{\partial y}$

$$=e^{\int f(y)dy}\left\{\frac{\partial M}{\partial y}+f(y)M\right\}$$

$$=e^{\int f(y)dy}\left\{\frac{\partial M}{\partial y}+\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right\}$$

$$=e^{\int f(y)dy}.\frac{\partial N}{\partial x}$$

$$=\frac{\partial\left\{e^{\int f(y)dy}.N\right\}}{\partial x}$$

So, $Mdx+Ndy=0$ is an exact equation and $e^{\int f(y)dy}$ is an integrating factor of $Mdx+Ndy=0$.

Example 6 : Solve : (i) $\left(xy^2-e^{\frac{1}{x^3}}\right)dx-x^2ydy=0$

(ii) $\left(xy^3+y\right)dx+2\left(x^2y^2+x+y^4\right)dy=0$

Solution : (i) $M=xy^2-e^{\frac{1}{x^3}}$ and $N=-x^2y$

$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{2xy+2xy}{-x^2y}=-\frac{4}{x}$, function of x.

I.F $=e^{-\int\frac{4}{x}dx}=e^{-4\log x}=x^{-4}$

Multiplying both sides by x^{-4}, we have

$\left(x^{-3}y^2-x^{-4}e^{\frac{1}{x^3}}\right)dx-x^{-2}ydy=0$, an exact equation.

Required solution is

$$y^2\int x^{-3}dx-\int e^{\frac{1}{x^3}}.x^{-4}dx+0=C$$

or $\frac{y^2x^{-2}}{-2}+\frac{1}{3}\int e^u du=C$ [Putting $u=x^{-3}$, $du=-3x^{-4}dx$]

or $-\frac{1}{2}\frac{y^2}{x^2}+\frac{1}{3}e^{x^{-3}}=C$.

(ii) Here, $M=xy^3+y$ and $N=2\left(x^2y^2+x+y^4\right)$

So, $\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)=\frac{1}{y\left(xy^2+1\right)}\left(4xy^2+2-3xy^2-1\right)=\frac{1}{y}=f(y)$ is a function of y.

So, I.F $=e^{\int f(y)dy}=e^{\log y}=y$

Multiplying the equation by y, we have

$\left(xy^4+y^2\right)dx+2\left(x^2y^3+xy+y^5\right)dy=0$, which is an exact equation.

The solution of the equation is

$$y^4\int xdx+y^2\int dx+2\int y^5dy=C$$

or $\frac{1}{2}x^2y^4+xy^2+\frac{1}{3}y^6=C$.

IV(c) Show that the equation $\left(Pdx+Qdy\right)e^{\int f(x)dx}=0$ satisfies the conditions of exactness if $\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}+Qf(x)$.

From this show that I.F can be determined if $\frac{1}{Q}\left[\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right]$ is a function of x. Hence solve $\left(x^3+xy^4\right)dx+2y^3dy=0$.

Let $M=Pe^{\int f(x)dx}$ and $N=Qe^{\int f(x)dx}$

$\frac{\partial M}{\partial y}=\frac{\partial P}{\partial y}e^{\int f(x)dx}$ and $\frac{\partial N}{\partial x}=Qe^{\int f(x)dx}.f(x)+\frac{\partial Q}{\partial x}e^{\int f(x)dx}$

So, condition of exactness is satisfied if

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

or $\frac{\partial P}{\partial y}e^{\int f(x)dx}=Qe^{\int f(x)dx}.f(x)+\frac{\partial Q}{\partial x}e^{\int f(x)dx}$

or $\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}+Qf(x)$

For second part, we see that $\left(Pdx+Qdy\right)e^{\int f(x)dx}$ is exact, equivalentley $e^{\int f(x)dx}$ is an I.F of $Pdx+Qdy=0$ if

$$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}+Qf(x)$$

i.e., $\frac{1}{Q}\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=f(x)$, a function of x.

To solve : $\left(x^3+xy^4\right)dx+2y^3dy=0$

Here $P=x^3+xy^4$ and $Q=2y^3$.

So, $\frac{1}{Q}\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=\frac{1}{2y^3}\left(4xy^3-0\right)=2x=f(x)$

I.F $=e^{\int f(x)dx}=e^{\int 2xdx}=e^{x^2}$

Multiplying both sides of the equation by e^{x^2}, we have

$e^{x^2}\left(x^3+xy^4\right)dx+e^{x^2}.2y^3dy=0$

The equation is exact.

The solution of the equation is

$\int e^{x^2}.x^2.xdx+y^4\int e^{x^2}xdx=C$

or $\frac{1}{2}\int e^u udu+\frac{1}{2}y^4\int e^u du=C \qquad \left(u=x^2\right)$

or $\frac{1}{2}\left[ue^u-\int e^u du+y^4e^u\right]=C$

or $ue^u-e^u+y^4e^u=2C$

or $e^{x^2}\left(x^2-1+y^4\right)=k$.

(V) For the equation of the type

$x^ay^b\left(mydx+nxdy\right)+x^cy^d\left(pydx+qxdy\right)=0$,

the integrating factor is x^hy^k.

Multiplying by x^hy^k throughout we have

$x^{a+h}y^{b+k}\left(mydx+nxdy\right)+x^{c+h}y^{d+k}\left(pydx+qxdy\right)=0$ which is of the form

$Mdx+Ndy=0$ with

$$M=mx^{a+h}y^{b+k+1}+px^{c+h}y^{d+k+1}$$

and $N=nx^{a+h+1}y^{b+k}+qx^{c+h+1}y^{d+k}$.

From the condition of exactness, we have

$$mx^{a+h}.(b+k+1)y^{b+k}+px^{c+h}.(d+k+1)y^{d+k}$$
$$=n(a+h+1)x^{a+h}y^{b+k}+q(c+h+1)x^{c+h}y^{d+k}$$

Comparing coefficients, we have

$m(b+k+1)=n(a+h+1)$ and $p(d+k+1)=q(c+h+1)$

or $\frac{b+k+1}{n}=\frac{a+h+1}{m}$ and $\frac{d+k+1}{q}=\frac{c+h+1}{p}$.

Solving these two equations, values of h and k can be obtained so as to obtain the integrating factor.

Example 7 : Solve $y(xy+2x^2y^2)dx+x(xy-x^2y^2)dy=0$

Solution : Writing the equation in the standard form

$x^a y^b(mydx+nxdy)+x^c y^d(pydx+qxdy)=0$, we have

$$xy(ydx+xdy)+x^2y^2(2ydx-xdy)=0.$$

Let $x^h y^k$ be an integrating factor, then

$\frac{1+h+1}{1}=\frac{1+k+1}{1}$ and $\frac{2+h+1}{2}=\frac{2+k+1}{-1}$

or $h=k$ and $-3-h=6+2k$.

Solving, we have $h=k=-3$. So, integrating factor is $\frac{1}{x^3y^3}$.

$\therefore$ the equation becomes

$\left(\frac{1}{x^2y}+\frac{2}{x}\right)dx+\left(\frac{1}{xy^2}-\frac{1}{y}\right)dy=0$, which is an exact equation.

Its solution is

$$\frac{1}{y}\left(-\frac{1}{x}\right)+2\log x-\log y=C$$

or $2\log x-\log y-\frac{1}{xy}=C$.

Example 8 : Solve $x(4ydx+2xdy)+y^3(3ydx+5xdy)=0$

Solution : Let $x^h y^k$ be an integrating factor of the given equation. So,

$$\frac{1+h+1}{4}=\frac{k+1}{2} \text{ and } \frac{h+1}{3}=\frac{3+k+1}{5}$$

or $2k+2=2+h$ and $12+3k=5h+5$

or $h=2k$ and $5h-3k=7$

Solving, $h=2$ and $k=1$.So, x^2y is an integrating factor of the equation.

Multiplying the equation by x^2y throughout, we have,

$(4x^3y^2+3x^2y^5)dx+(2x^4y+5x^3y^4)dy=0$, which is an exact equation.

The solution of this equation is

$$4y^2\int x^3dx+3y^5\int x^2dx+0=C$$

or $y^2x^4+x^3y^5=C$.



CHECK YOUR PROGRESS - 2

Q 1 : Solve the following differential equations :

(i) $ydx - xdy + (1+x^2)dx + x^2 \sin ydy = 0$

(ii) $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$

(iii) $(2x^2y^2 + y)dx - (x^3y - 3x)dy = 0$

(iv) $(xy\sin xy + \cos xy)ydx + (xy\sin xy - \cos xy)xdy = 0$

(v) $(x^2 + y^2 + x)dx + xydy = 0$



9.5 LET US SUM UP

- The necessary and sufficient condition for $M(x,y)dx + N(x,y)dy = 0$ to be an exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.
- If $M(x,y)dx + N(x,y)dy = 0$ be exact, then its solution is

$$\int_{y=\text{constant}} Mdx + \int (N - \text{term containing } x)dy = C$$

- A function $\mu(x,y) \neq 0$ which is used to convert a differential equation of the form $M(x,y)dx + N(x,y)dy = 0$ (which is not exact) into an exact equation is called Integrating factor (I.F) of the equation $M(x,y)dx + N(x,y)dy = 0$.



9.6 FURTHER READINGS

1) Tenenbaum, Morris and Pollard, Harry. Ordinary Differential Equations, Dover Publications, Inc, N.Y 11501.

2) Differential Equations : An Introduction with Mathematica. Clay C. Ross, second Edition, Springer.

3) Differential Equations : Linear, Nonlinear, ordinary, partial. A.C. King, J. Billinggham and S.R. Otto, Cambridge-2003.

4) Differential equations with Application and Programs. S. Balachandra Rao and H.R. Anuradha, University Press(India) Limited 1996.

9.7 ANSWERS TO CHECK YOUR PROGRESS

Check your progress -1

Ans to Q No 1: (a) Comparing the given equation with $Mdx + Ndy = 0$, we get

$M = y^2 e^{xy^2} + 4x^3$ and $N = 2xye^{xy^2} - 3y^2$

Also, $\frac{\partial M}{\partial y} = 2ye^{xy^2} + y^2.2xye^{xy^2}$

and $\frac{\partial N}{\partial x} = 2ye^{xy^2} + y^2.2xye^{xy^2}$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Hence, the given equation is exact and so its solution is

$$\int_{y=\text{constant}} Mdx + \int (\text{terms free from } x \text{ in N})dy = C$$

or $\int \left(y^2 e^{xy^2} + 4x^3\right)dx + \int \left(-3y^2\right)dy = C$

or $y^2 \times \frac{1}{y^2} \times e^{xy^2} + 4 \times \frac{1}{4} \times x^4 - 3 \times \frac{y^3}{3} = C$

or $e^{xy^2} + x^4 - y^3 = C$.

(b) Do yourself.

(c) Here $M = \cos x \tan y + \cos(x + y)$

$N = \sin x \sec^2 y + \cos(x + y)$

$\therefore \frac{\partial M}{\partial y} = \cos x \sec^2 y - \sin(x + y) = \frac{\partial N}{\partial x}$

Thus the given equation is exact and its solution is

$$\int_{y=\text{constant}} Mdx + \int (\text{terms free from } x \text{ in N})dy = C$$

or $\int [\cos x \tan y + \cos(x+y)]dx = C$

or $\sin x \tan y + \sin(x+y) = C$.

(d) Do Yourself

Required solution is

$x^5 + x^3y^2 - x^2y^3 - y^5 = C$.

Check your progress - 2

Ans to Q No 1: (i) Dividing the given equation by x^2, we get

$$\frac{ydx - xdy}{x^2} + \left(\frac{1+x^2}{x^2}\right)dx + \sin y dy = 0$$

or $-\frac{xdy - ydx}{x^2} + \left(\frac{1}{x^2} + 1\right)dx + \sin y dy = 0$

or $-d\left(\frac{y}{x}\right) + \left(1 + \frac{1}{x^2}\right)dx + \sin y dy = 0$

or $-\frac{y}{x} + x - \left(\frac{1}{x}\right) - \cos y = c$

or $-y + x^2 - 1 - x\cos y = cx$, which is the required solution.

(ii) The given equation is homogeneous in x and y with

$M = x^2y - 2xy^2$ and $N = -x^3 + 3x^2y$

Now, $Mx + Ny = x^3y - 2x^2y^2 - x^3y + 3x^2y^2 = x^2y^2 \neq 0$

I.F $= \frac{1}{Mx + Ny} = \frac{1}{x^2y^2}$

Multiplying throughout by $\frac{1}{x^2y^2}$, the given equation becomes

$\left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0$ which is exact.

The required solution is

$$\int_{y=\text{constant}} \left(\frac{1}{y} - \frac{2}{x}\right)dx + \int \frac{3}{y}dy = C$$

or $\frac{x}{y} - 2\log x + 3\log y = C$.

(iii) Given $\left(2x^2y^2 + y\right)dx + \left(3x - x^3y\right)dy = 0$ ... (1)

Multiplying both sides of (1) by I.F $x^h y^k$, we get

$\left(2x^{h+2}y^{k+2} + x^h y^{k+1}\right)dx + \left(3x^{h+1}y^k - x^{h+3}y^{k+1}\right)dy = 0$ ... (2)

which must be exact. Comparing (2) with $Mdx + Ndy = 0$, we have

$M = 2x^{h+2}y^{k+2} + x^h y^{k+1}$ and $N = 3x^{h+1}y^k - x^{h+3}y^{k+1}$

For (2) to be exact, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\Rightarrow 2(k+2)x^{h+2}y^{k+1} + (k+1)x^h y^k = 3(h+1)x^h y^k - (h+3)x^{h+2}y^{k+1}$

$\Rightarrow 2(k+2) = -(h+3)$ and $(k+1) = 3(h+1)$

$\Rightarrow h + 2k = -7$ and $3h - k = -2$

$\Rightarrow h = -\frac{11}{7}$ and $k = -\frac{19}{7}$

Hence an I.F of (1) is $x^{-\frac{11}{7}}y^{-\frac{19}{7}}$. Multiplying (1) by $x^{-\frac{11}{7}}y^{-\frac{19}{7}}$, we have

$$\left(2x^{\frac{3}{7}}y^{-\frac{5}{7}} + x^{-\frac{11}{7}}y^{-\frac{12}{7}}\right)dx + \left(3x^{-\frac{4}{7}}y^{-\frac{19}{7}} - x^{\frac{10}{7}}y^{-\frac{12}{7}}\right)dy = 0$$

which must be exact. Hence, as usual the required solution is

$$\frac{2x^{\frac{10}{7}}y^{-\frac{5}{7}}}{\frac{10}{7}} + \frac{x^{-\frac{4}{7}}y^{-\frac{12}{7}}}{-\frac{4}{7}} = \frac{7C}{20}$$

or $4x^{\frac{10}{7}}y^{-\frac{5}{7}} - 5x^{-\frac{4}{7}}y^{-\frac{12}{7}} = C$

(iv) Do yourself.

Required solution is

$\frac{x}{y}\sec xy = C$

(v) Do yourself.

Required solution is

$3x^4 + 6x^2y^2 + 4x^3 = C$.



9.8 MODEL QUESTIONS

Q 1 : Solve the following differential equations :

(a) $\frac{dy}{dx}+\frac{y\cos x+\sin y+y}{\sin x+x\cos y+x}=0$

(b) $xdy+ydx+\frac{xdy-ydx}{x^2+y^2}=0$

(c) $\left(a^2-2xy-y^2\right)dx-(x+y)^2dy=0$

(d) $(2ax+by)ydx+(ax+2by)xdy=0$

Q 2 : Solve the following differential equations :

(a) $\left(x^2-3xy+2y^2\right)dx+x(3x-2y)dy=0$

(b) $\left(x^2-2xy+5y^2\right)dx+\left(2y^2-7xy+x^2\right)dy=0$

(c) $(2x-y)e^{\frac{y}{x}}dx+\left(y+xe^{\frac{y}{x}}\right)dy=0$

(d) $y-x\frac{dy}{dx}=x+y\frac{dy}{dx}$

(e) $\left(2x^3-y^3\right)dy-y^2xdx=0$

(f) $\left(x^4+y^4\right)dx-xy^3dy=0$

(g) $y^2dx+\left(x^2-xy-y^2\right)dy=0$

(h) $\left(y^4-2x^3y\right)dx+\left(x^4-2xy^3\right)dy=0$

*** ***** ***

UNIT 10 : FIRST ORDER AND HIGHER DEGREE DIFFERENTIAL EQUATIONS

UNIT STUCTURE

10.1 Learning objectives

10.2 Introduction

10.3 Differential Equation of First Order and Higher Degree

10.3.1 Equation Solvable for p

10.3.2 Equation Solvable for x

10.3.3 Equation Solvable for y

10.4 Clairaut's form

10.4.1 Equations Reducible to Clairaut's form

10.5 Let Us Sum Up

10.6 Further Readings

10.7 Answers To Check Your Progress

10.8 Model Questions

10.1 LEARNING OBJECTIVES

After going through this unit, you will able to know

- differential equation of first order and higher degree
- equation solvable for p
- equation solvable for x
- equation solvable for y
- clairaut's form.

10.2 INTRODUCTION

In unit 9, we have discussed the methods of solving linear equations in which dependent variable and its derivatives are of first order and first degree. In this unit, we shall consider non linear equations in which dependent variable and its derivatives are of first order and of higher degree. There is no specific method to solve all such equations. Here,some typical types of equations are considered to describe the methods of solution of such equations.

10.3 EQUATION OF FIRST ORDER AND HIGHER DEGREE

Differential equations of the first order but of a higher degree can be solved by one or more of the following four methods :

(a) Equations solvable for p, $\left(p = \frac{dy}{dx}\right)$

(b) Equations solvable for y : $y = f(x, p)$

(c) Equations solvable for x : $x = f(y, p)$

(d) Clairaut's equation : $y = px + f(p)$

10.3.1 Equation solvable for p

Consider a differential equation of first order but of the nth degree :

$$p^n + f_1(x,y)p^{n-1} + f_2(x,y)p^{n-2} + ... + f_{n-1}(x,y)p + f_n(x,y) = 0 \quad .....(1)$$

which is solvable for p.

Splitting up the left side into n linear factors, we have

$$[p - \phi_1(x,y)][p - \phi_2(x,y)]..[p - \phi_n(x,y)] = 0$$

Equating each factor to zero,we have

$$p = \phi_1(x,y),\ p = \phi_2(x,y),...,\ p = \phi_n(x,y).$$

Each of these n equations is of first order and first degree, which can be solved by a method discussed in unit 5.

Suppose $F_1(x,y,c) = 0, F_2(x,y,c) = 0,...,F_n(x,y,c) = 0$ are solutions of these n equations. These n solutions constitute the general solution of (1). The general solution can also be written as $F_1(x,y,c).F_2(x,y,c)...F_n(x,y,c) = 0$.

Example 1 : Solve $p^2 + p(e^x + e^{-x}) + 1 = 0$

Solution : From the given equation

$$p^2 + p(e^x + e^{-x}) + 1 = 0$$

$$\text{or } p(p + e^x) + e^{-x}(p + e^x) = 0$$

$$\text{or } (p + e^x)(p + e^{-x}) = 0$$

we have $p + e^x = 0$, or, $p + e^{-x} = 0$

solving, we have $y + e^x + c_1 = 0$,or , $y - e^{-x} + c_2 = 0$

The given equation is of first order and hence its general solution contains only one arbitrary constant. So, we replace c_1 and c_2 by single constant c. Since each of these two solutions is a solution of the given equation, the required solution is $(y + e^x + c)(y - e^{-x} + c) = 0$.

Example 2 : Solve, $p^2 + 2py\cot x = y^2$

Solution: Solving for p, we get

$$p = \frac{1}{2}\left(-2y\cot x \pm \sqrt{4y^2\cot^2 x + 4y^2}\right)$$

$$= -y\cot x \pm y\,\mathrm{cosec}\,x \quad \ldots (1)$$

From (1), $\frac{dy}{dx} = -y\cot x + y\,\mathrm{cosec}\,x$

or $\frac{dy}{dx} = y(\mathrm{cosec}\,x - \cot x)$

or $\frac{dy}{y} = (\mathrm{cosec}\,x - \cot x)dx$

Integrating, we have

$$\log y = \log\tan\frac{x}{2} - \log\sin x + \log c$$

or $$y = c\left(\frac{\tan\frac{x}{2}}{\sin x}\right) = \frac{c}{\left(2\cos^2\frac{x}{2}\right)}$$

or $y(1 + \cos x) = c \quad \ldots (2)$

Again from (1), $\frac{dy}{dx} = -y(\cot x + \mathrm{cosec}\,x)$

or $\frac{dy}{y} = -(\cot x + \mathrm{cosec}\,x)dx$

Integrating, we have $\log y = -\log\sin x - \log\tan\frac{x}{2} + \log c$

or $$y = \frac{c}{2\sin^2\frac{x}{2}}$$

or $y(1 - \cos x) = c \quad \ldots (3)$

combining (2) and (3), we get

$y(1 \pm \cos x) = c$ is the required solution.

Example 3 : Solve $\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$

Solution : The given equation is $p - \frac{1}{p} = \frac{x}{y} - \frac{y}{x}$

or $p^2 - p\left(\frac{x}{y} - \frac{y}{x}\right) - 1 = 0$

or $p\left(p - \frac{x}{y}\right) + \frac{y}{x}\left(p - \frac{x}{y}\right) = 0$

or $\left(p - \frac{x}{y}\right)\left(p + \frac{y}{x}\right) = 0$

or $p = \frac{x}{y}, p = -\frac{y}{x}$

Solving $p = \frac{x}{y} \Rightarrow \frac{dy}{dx} = \frac{x}{y}$

Integrating, we get, $x^2 - y^2 = c$

Again solving $p = -\frac{y}{x} \Rightarrow \frac{dy}{dx} + \frac{y}{x} = 0$

or $\frac{dy}{y} + \frac{dx}{x} = 0$

Integrating, we get $xy = c$

So, $x^2 - y^2 = c$ and $xy = c$ constitute the required solution.

Example 4 : Solve $p^3 + 3xp^2 - y^3p^2 - 3xy^3p = 0$

Solution : The given equation is

$p^3 + 3xp^2 - y^3p^2 - 3xy^3p = 0$

or $p^2(p + 3x) - py^3(p + 3x) = 0$

or $(p^2 - py^3)(p + 3x) = 0$

or $p(p - y^3)(p + 3x) = 0$

(i) If $p = 0$, then $\frac{dy}{dx} = 0$ or $dy = 0$, i.e., $y = c$.

(ii) If $p - y^3 = 0$, then $\frac{dy}{dx} = y^3$ or $\frac{dy}{y^3} = dx$

Integrating we get, $-\frac{1}{2y^2} = x + c$

or $2xy^2 + 2cy^2 + 1 = 0$

(iii) If $p + 3x = 0$ then $\frac{dy}{dx} + 3x = 0$ or $dy + 3xdx = 0$

Integrating we get, $y + \frac{3}{2}x^2 = c$

or $2y + 3x^2 - 2c = 0$

Hence the solution of the given differential equation is

$$(y - c)(2xy^2 + 2cy^2 + 1)(2y + 3x^2 - 2c) = 0 .$$



CHECK YOUR PROGRESS - 1

Q 1 : Solve the following differential equations :

(i) $p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0$

(ii) $p^2y^2\cos^2\alpha - 2pxy\sin^2\alpha + y^2 - x^2\sin^2\alpha = 0$

(iii) $p^2 - px + 1 = 0$

(iv) $x^2p^2 + xyp - 6y^2 = 0$

10.3.2 Equation solvable for x

Suppose an equation of first order and higher degree can be transformed into the form : $x = f(y, p)$ ... (1)

Differentiating (1) w.r.t ' y ', it can be transformed again to the form

$$\frac{1}{p} = \frac{dx}{dy} = \phi\left(y, p, \frac{dp}{dx}\right)$$

Suppose it is possible to solve the new differential equation in y and p.

Let its solution be $F(y, p, c) = 0$.

Eliminating p from this equation with the help of the given equation, we obtain the required solution.

Example 1 : Solve $x = 4(p + p^2)$

Solution : Given, $x = 4(p + p^2)$ ... (1)

Differentiating (1) w.r.t y, we have

$$\frac{1}{p} = 4(1+2p)\frac{dp}{dy}$$

or $dy = 4p(1+2p)dp$

Integrating, we get

$$y = 4\int (p+2p^2)dp = \frac{4}{2}p^2 + \frac{8}{3}p^3 + c$$

$y = 2p^2 + \frac{8}{3}p^3 + c$ which is the required solution.

Example 2 : Solve $4yp^2 - 2px + y = 0$

Solution : The given equation can be written as

$$x = 2py + \frac{y}{2p} \quad \ldots (1)$$

Differentiating (1) w.r.t y yields

$$\frac{1}{p} = 2\frac{dp}{dy}.y + 2p + \frac{2p - y2\frac{dp}{dy}}{4p^2}$$

$$= 2\frac{dp}{dy}.y + 2p + \frac{1}{2p} - \frac{y}{2p^2}\frac{dp}{dy}$$

or $0 = 2p + 2y\frac{dp}{dy} - \frac{1}{2}\frac{1}{p} - \frac{y}{2p^2}\frac{dp}{dy}$

$$= 4p^2\left(\frac{1}{2p} + \frac{y}{2p^2}\frac{dp}{dy}\right) - \left(\frac{1}{2p} + \frac{y}{2p^2}\frac{dp}{dy}\right)$$

$$= \left(\frac{1}{2p} + \frac{y}{2p^2}\frac{dp}{dy}\right)(4p^2 - 1)$$

Solving, we get $1 + \frac{y}{p}\frac{dp}{dy} = 0$

or $\frac{dy}{y} + \frac{dp}{p} = 0$

Integrating, we have $yp = c$.

Subtituting $p = \frac{c}{y}$ in the given equation, we have

$$4y.\frac{c^2}{y^2} - 2.\frac{c}{y}.x + y = 0$$

or $4c^2 - 2cx + y^2 = 0$ which is the required general solution.

Example 3 : Solve $y = 2px + y^2p^3$

Solution : The given equation can be written as

$$x = \frac{y}{2p} - \frac{y^2p^2}{2} \quad \text{... (1)}$$

Differentiating (1) w.r.t y, we have

$$\frac{1}{p} = \frac{2p - 2y\frac{dp}{dy}}{4p^2} - \frac{2yp^2 + y^2 2p\frac{dp}{dy}}{2}$$

$$= \frac{1}{2}\frac{p - y\frac{dp}{dy}}{p^2} - yp^2 - y^2p\frac{dp}{dy}$$

$$\text{or } 0 = \frac{1}{2p} - \frac{y\frac{dp}{dy}}{2p^2} - yp^2 - y^2p\frac{dp}{dy} - \frac{1}{p}$$

$$\text{or } 0 = \frac{1}{2p^2}y\frac{dp}{dy} + yp^2 + y^2p\frac{dp}{dy} + \frac{1}{2p}$$

$$\text{or } 0 = yp\left(y\frac{dp}{dy} + p\right) + \frac{1}{2p^2}\left(y\frac{dp}{dy} + p\right)$$

$$\text{or } 0 = \left(y\frac{dp}{dy} + p\right)\left(yp + \frac{1}{2p^2}\right)$$

Solving, $y\frac{dp}{dy} + p = 0$, separating variables, we have

$$\log py = \log c$$

or, $py = c$

$$\text{or } p = \frac{c}{y}$$

Eliminating p from the given equation, we have

$$y = \frac{2cx}{y} + y^2.\frac{c^3}{y^3}$$

or $y^2 = 2cx + c^3$ which is required solution.



CHECK YOUR PROGRESS - 2

Q 1 : Solve the following differential equations :

(i) $y^2 \log y = xyp + p^2$

(ii) $p = \tan\left(x - \frac{p}{1+p^2}\right)$ (iii) $p^3 - 4xyp + 8y^2 = 0$

10.3.3 Equation solvable for y

Suppose a differential equation of first order and higher degree takes the form $y = f(x, p)$ ... (1)

Differentiating (1) w.r.t x, we have

$$p = \frac{dy}{dx} = \phi\left(x, p, \frac{dp}{dx}\right)$$

If it is possible to solve this equation and $F(x, p, c) = 0$ ... (2) be a solution, then on elimination of p between (1) and (2), we get the required solution. If the elimination of p is not possible then solving (1) and (2) for x and y ,we obtain $x = F_1(p, c)$ and $y = F_2(p, c)$ as the solution.

Example 1 : Solve $y = px + ap(1-p)$

Solution : Given, $y = px + ap(1-p)$ ... (1)

Differentiating (1) w.r.t. x, we get

$$p = \frac{dp}{dx}x + p + a\frac{dp}{dx} - 2p\frac{dp}{dx}a$$

$$\text{or } \frac{dp}{dx}(x + a - 2ap) = 0$$

So, $\frac{dp}{dx} = 0 \Rightarrow p = \text{constant} = c$

and $p = \frac{1}{2a}(x + a)$

Eliminating p, we have $y = cx + ac(1-c)$ which is the general solution.

and $y = \frac{1}{2a}(x + a)x + \frac{a}{2a}(x + a)\left(1 - \frac{x + a}{2a}\right)$

$$= \frac{1}{2a}(x+a)\left\{x+a-\frac{ax+a^2}{2a}\right\}$$

$$= \frac{1}{2a}(x+a)^2 - \frac{1}{4a}(x+a)^2$$

$$= \frac{1}{4a}(x+a)^2$$

$$4ay = (x+a)^2$$

Example 2 : Solve $y - 2px = \tan^{-1}(xp^2)$

Solution : Given $y - 2px = \tan^{-1}(xp^2)$ ... (1)

Differentiating (1) w.r.t x, we get

$$p = 2\frac{dp}{dx}x + 2p + \frac{1}{1+x^2p^4}\left(p^2 + x2p\frac{dp}{dx}\right)$$

or $0 = \left(p + 2x\frac{dp}{dx}\right) + \frac{p}{1+x^2p^4}\left(p + 2x\frac{dp}{dx}\right)$

or $0 = \left(p + 2x\frac{dp}{dx}\right)\left(1 + \frac{p}{1+x^2p^4}\right)$

This gives $p + 2x\frac{dp}{dx} = 0$ and p is given by $p^4x^2 + p + 1 = 0$

$$\Rightarrow \frac{dp}{p} + \frac{1}{2}\frac{dx}{x} = 0$$

Integrating, $\log p + \frac{1}{2}\log x = \log c$

or $px^{\frac{1}{2}} = c$

or $p = \frac{c}{\sqrt{x}}$

Eliminating p from the given equation, the required solution is

$$y - 2c\sqrt{x} = \tan^{-1} c^2 .$$

Example 3 : Solve $y = 2px + p^n$

Solution : Given, $y = 2px + p^n$ ... (1)

Differentiating (1) w.r.t x, we get

$$p = 2\frac{dp}{dx}x + 2p + np^{n-1}\frac{dp}{dx}$$

or $0 = 2\frac{dp}{dx}x + p + np^{n-1}\frac{dp}{dx}$

or $0 = 2x + p\frac{dx}{dp} + np^{n-1}$

or $\frac{2x}{p} + \frac{dx}{dp} = -np^{n-2}$

Here I.F $= e^{2\log p} = p^2$ and solution is $x.p^2 = -\int np^n dp$

$$\text{or } x.p^2 = -n\frac{p^{n+1}}{n+1} + c$$

$$\text{or } x = -\frac{n}{n+1}p^{n-1} + cp^{-2}$$

Also substituting for x in the given equation, we have

$$y = -\frac{2n}{n+1}p^n + 2cp^{-1} + p^n$$

$$= \frac{2c}{p} + \frac{1-n}{1+n}p^n$$

The required solution is given by

$x = \frac{c}{p^2} - \frac{n}{n+1}p^{n-1}$ and $y = \frac{2c}{p} + \frac{1-n}{1+n}p^n$.



CHECK YOUR PROGRESS - 3

Q 1 : Solve :

(i) $y + px = x^4p^2$

(ii) $y = x + a\tan^{-1}p$

(iii) $y = p\tan p + \log(\cos p)$

10.4 CLAIRAUT'S FORM

An equation of the form $y = px + f(p)$ ... (1) is known as Clairaut's equation.

Differentiating (1) w.r.t x, we have

$$p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

or $0 = \left(x + f'(p)\right)\frac{dp}{dx}$

$\therefore \quad \frac{dp}{dx} = 0$ or $x + f'(p) = 0 \ldots$ (ii)

From $\frac{dp}{dx} = 0$, we have $p = c$ ... (iii)

Eliminating p, from (i) and (iii), we get $y = cx + f(c)$ ... (iv)

which is the general solution of (i).



Alexis Claude Clairaut (13 May 1713 – 17 May 1765) was a prominent French mathematician.

Thus we see that the general solution of the Clairaut's equation is obtained by replacing p by c.

Note : Clairaut's equation is named after the French mathematician is Alexis Claude Clairaut (1713-1765).

Example 1: Solve the following differential equations :

(a) $y = px + \frac{a}{p}$

Solution : The given equation is in Clairaut's form. So replacing p by c its solution is $y = cx + \frac{a}{c}$.

(b) $p = \log(px - y)$

Solution : Given, $p = \log(px - y)$

or $e^p = px - y$

or $y = px - e^p$, which is Clairaut's form. So replacing p by c, its solution is $y = pc - e^c$.

(c) $p = \tan(px - y)$

Solution : Given, $p = \tan(px - y)$

or $\tan^{-1} p = px - y$

or $y = px - \tan^{-1} p$ which is Clairaut's form. So replacing p by c, its solution is $y = Cx - \tan^{-1} C$.

(d) $y^2 + x^2\left(\frac{dy}{dx}\right)^2 - 2xy\left(\frac{dy}{dx}\right) = 4\left(\frac{dx}{dy}\right)^2$

Solution : The given equation can be written as

$$y^2 + x^2 p^2 - 2xy\,\frac{1}{p} = \frac{4}{p^2}$$

or $(y - px)^2 = \dfrac{4}{p^2}$

or $y = px \pm \left(\dfrac{2}{p}\right)$ which is Clairaut's form. So replacing p by c, its solution is $y = Cx \pm \left(\dfrac{2}{C}\right)$

or $(y - Cx)^2 = \left(\dfrac{2}{C}\right)^2$

or $y^2 + x^2C^2 - 2xyC = \dfrac{4}{C^2}$.



CHECK YOUR PROGRESS - 4

Q 1 : Solve the following differential equations :

(i) $y = px + p - p^3$

(ii) $y = xp + \sqrt{1 + p^2}$

(iii) $y = xp + ap(1 - p)$

(iv) $y = xp + \sin^{-1} p$

(v) $y = xp + e^p$

(vi) $y = xp + p^2$

(vii) $y = xp + \dfrac{p}{p-1}$

(viii) $y = px + \sqrt{1 - p^2}$

10.4.1 Equations reducible to Clairaut's form

Many differential equations of the first order but of higher degree can be reduced to Clairaut's form with the help of suitable substitutions. However, there is no specific method of selecting such substitutions. It depends upon practice and experience.

Example 1 : Transform the equation $x^2(y-px)=p^2y$ into Clairaut's form and then solve it.

Solution : Substituting $x^2=u$ i.e., $2xdx=du$

and $y^2=v$ i.e., $2ydy=dv$

So, $\frac{dv}{du}=\frac{\frac{dv}{dy}}{\frac{du}{dx}}.\frac{dy}{dx}=\frac{y}{x}\frac{dy}{dx}$

Hence, $p=\frac{dy}{dx}=\frac{x}{y}\frac{dv}{du}=\sqrt{\frac{u}{v}}\frac{dv}{du}$

The given equation transforms to

$$u\left(\sqrt{v}-\sqrt{\frac{u}{v}}\sqrt{u}\frac{du}{dv}\right)=\frac{u}{v}\left(\frac{dv}{du}\right)^2.\sqrt{v}$$

or $u\left(\sqrt{v}-\frac{u}{\sqrt{v}}\frac{du}{dv}\right)=\frac{u}{\sqrt{v}}\left(\frac{dv}{du}\right)^2$

or $u\sqrt{v}.\frac{\sqrt{v}}{u}-\frac{u^2}{\sqrt{v}}\frac{\sqrt{v}}{u}\frac{dv}{du}=\left(\frac{dv}{du}\right)^2$

or $v-u\frac{dv}{du}=\left(\frac{dv}{du}\right)^2$

or $v=u\frac{dv}{du}+\left(\frac{dv}{du}\right)^2$, which is Clairaut's form.

The general solution is given by replacing $\frac{dv}{du}$ by a constant C.

Hence its general solution is

$$v=uC+C^2$$

i.e., $\sqrt{y}=\sqrt{x}C+C^2$.

Example 2 : Solve $y^2(y-px)=x^4p^2$, transforming to Clairaut's form with the help of substitutions $u=\frac{1}{x}$ and $v=\frac{1}{y}$.

Solution : From $u=\frac{1}{x}$, we get $\frac{du}{dx}=-\frac{1}{x^2}$ and

from $v=\frac{1}{y}$, we get $\frac{dv}{dy}=-\frac{1}{y^2}$.

or $\frac{dy}{dx}=\frac{y^2}{x^2}\frac{dv}{du}$

The given equation transforms to

$$\frac{1}{v^2}\left(\frac{1}{v}-\frac{1}{u}\frac{u^2}{v^2}\frac{dv}{du}\right)=\frac{1}{u^4}\frac{u^4}{v^4}\left(\frac{dv}{du}\right)^2$$

or $$v^2\left(\frac{1}{v}-\frac{u}{v^2}\frac{dv}{du}\right)=\left(\frac{dv}{du}\right)^2$$

or $$v-u\frac{dv}{du}=\left(\frac{dv}{du}\right)^2$$

or $v=u\frac{dv}{du}+\left(\frac{dv}{du}\right)^2$, which is Clairaut's form.

The general solution is given by replacing $\frac{dv}{du}$ by a constant C.

Hence its general solution is

$$v=uC+C^2$$

$$\text{or}\frac{1}{y}=\frac{1}{x}C+C^2$$

or $x=Cy+C^2xy$.

Example 3 : Solve $(px-y)(py+x)=2p$

Solution : Substituting $x^2=u$ i.e., $2xdx=du$

and $y^2=v$ i.e., $2ydy=dv$

We get

$$\frac{dv}{du}=\frac{\frac{dv}{dy}}{\frac{du}{dx}}.\frac{dy}{dx}=\frac{y}{x}\frac{dy}{dx}$$

Hence, $p=\frac{dy}{dx}=\frac{x}{y}\frac{dv}{du}=\sqrt{\frac{u}{v}}\frac{dv}{du}$

The given equation becomes

$$\left(\sqrt{\frac{u}{v}}.\frac{dv}{du}.\sqrt{u}-\sqrt{v}\right)\left(\sqrt{\frac{u}{v}}\frac{dv}{du}\sqrt{v}+\sqrt{u}\right)=2\sqrt{\frac{u}{v}}\frac{dv}{du}$$

or $$\left(\frac{u}{\sqrt{v}}\frac{dv}{du}-\sqrt{v}\right)\left(\sqrt{u}\frac{dv}{du}+\sqrt{u}\right)=2\sqrt{\frac{u}{v}}\frac{dv}{du}$$

or $$u\frac{dv}{du}-v=\frac{2\frac{dv}{du}}{1+\frac{dv}{du}}$$

or $$v=u\frac{dv}{du}-\frac{2\frac{dv}{du}}{1+\frac{dv}{du}}$$, which is of Clairaut's form.

So, its general solution is

$$v=uC-\frac{2C}{1+C}$$

or $$y^2=x^2C-\frac{2C}{1+C}.$$



CHECK YOUR PROGRESS - 5

Q 1: Reduce to Clairaut's form with the help of given substitution and solve.

(i) $xp^2-2yp+x+2y=0\ \left(u=x^2,v=y-x\right)$

(ii) $(px-y)(py+x)=a^2p\ \left(u=x^2,v=y^2\right)$

(iii) $x^2(y-px)=p^2y\ \ \left(u=x^2,v=y^2\right)$

(iv) $x^2p^2+yp(2x+y)+y^2=0\ \left(v=xy,u=y\right)$

(v) $xp^2-2yp+x+2y=0\ \ \left(u=x^2,v=y-x\right)$



10.5 LET US SUM UP

- Differential equations of the first order but of a higher degree can be solved by one or more of the following four methods :

(a) Equations solvable for p , $p = \frac{dy}{dx}$

(b) Equations solvable for y : $y = f(x, p)$

(c) Equations solvable for x : $x = f(y, p)$

(d) Clairaut's equation : $y = px + f(p)$

- Consider a differential equation of first order but of the nth degree :

$p^n + f_1(x, y)p^{n-1} + f_2(x, y)p^{n-2} + ... + f_{n-1}(x, y)p + f_n(x, y) = 0$

which is solvable for p .

Splitting up the left side into n linear factors,we have

$[p - \phi_1(x, y)][p - \phi_2(x, y)]...[p - \phi_n(x, y)] = 0$

Equating each factors to zero,we have

$p = \phi_1(x, y)$, $p = \phi_2(x, y)$,..., $p = \phi_n(x, y)$.

Suppose $F_1(x, y, c) = 0, F_2(x, y, c) = 0, ..., F_n(x, y, c) = 0$ are solutions of these n equations.These n solutions constitute the general solution .The general solution can also be written as

$F_1(x, y, c).F_2(x, y, c)...F_n(x, y, c) = 0$.

- Suppose an equation of first order and higher degree can be transformed into the form : $x = f(y, p)$ ... (1)

Differentiating (1) w.r.t 'y', it can be transformed again to the form

$$\frac{1}{p} = \frac{dx}{dy} = \phi\left(y, p, \frac{dp}{dx}\right)$$

Suppose it is possible to solve the new differential equation in y and p .

Let its solution be $F(y, p, c) = 0$.

Eliminating p from this equation with the help of the given equation, we obtain the required solution.

- Suppose a differential equation of first order and higher degree takes the form $y = f(x, p)$ ... (1)

Differentiating (1) w.r.t x , we have

$$p = \frac{dy}{dx} = \phi\left(x, p, \frac{dp}{dx}\right)$$

If it is possible to solve this equation and $F(x, p, c) = 0$ ... (2) be a solution then on elimination of p between (1) and (2), we get the

required solution.If the elimination of p is not possible then solving (1) and (2) for x and y, we obtain $x = F_1(p,c)$ and $y = F_2(p,c)$ as the solution.

- The differential equation of the form $y = px + f(p)$ is known as Clairaut's equation. The general solution of Clairaut's equation is obtained by replacing p by c. Some complex differential equations can be reduced to Clairaut's form with the help of proper substitutions.



10.6 FURTHER READINGS

1) Tenenbaum, Morris and Pollard, Harry. Ordinary Differential Equations, Dover Publications, Inc, N.Y 11501.
2) Differential Equations : An Introduction with Mathematica. Clay C.Ross, second Edition, Springer.
3) Differential Equations : Linear, Nonlinear, ordinary, partial. A.C. King, J. Billinggham and S. R. Otto, Cambridge-2003.
4) Differential equations with Application and Programs. S. Balachandra Rao and H. R. Anuradha, University Press (India) Limited 1996.



10.7 ANSWERS TO CHECK YOUR PROGRESS

Check your progress -1

Ans to Q No 1: (i) The given equation can be re-written as

$$p(p^2 + 2xp - y^2 p - 2xy^2) = 0$$

or $p\{p(p+2x) - y^2(p+2x)\} = 0$

or $p(p+2x)(p-y^2) = 0$

or $p = 0, p + 2x = 0, p - y^2 = 0$

Integrating, $y = C$, $y + x^2 = C$ and $x + \frac{1}{y} = C$

Hence, the combined solution is

$$\left(y - C\right)\left(y + x^2 - C\right)\left(x + \frac{1}{y} - C\right) = 0.$$

(ii) Given, $(py)^2 - 2(py)x\tan^2\alpha + \left(y^2\sec^2\alpha - x^2\tan^2\alpha\right) = 0$

$$\therefore py = \frac{2x\tan^2\alpha \pm \sqrt{\left\{4x^2\tan^4\alpha - 4\left(y^2\sec^2\alpha - x^2\tan^2\alpha\right)\right\}}}{2}$$

or $py = x\tan^2\alpha \pm \sqrt{\left\{x^2\tan^2\alpha\left(\tan^2\alpha + 1\right) - y^2\sec^2\alpha\right\}}$

or $y\frac{dy}{dx} = x\tan^2\alpha \pm \sec\alpha\sqrt{x^2\tan^2\alpha - y^2}$

or $ydy - x\tan^2\alpha dx = \pm\sec\alpha\sqrt{\left(x^2\tan^2\alpha - y^2\right)}dx$

or $\pm\frac{x\tan^2\alpha dx - ydy}{\sqrt{x^2\tan^2\alpha - y^2}} = -\sec\alpha dx$

Integrating, $\pm\sqrt{\left(x^2\tan^2\alpha - y^2\right)} = C - x\sec\alpha$

Squaring, we get

$x^2\tan^2\alpha - y^2 = C^2 - 2Cx\sec\alpha + x^2\sec^2\alpha$

or $x^2 + y^2 - 2Cx\sec\alpha + C^2 = 0$.

(iii) Do yourself

(iv) Do yourself.

Check your progress -2

Ans to Q No 2: (i) Solving for x, $x = \frac{y\log y}{p} - \frac{p}{y}$ ... (1)

Differentiating (1) w.r.t. y, we get

$$\frac{1}{p} = \left(\log y + y.\frac{1}{y}\right)\frac{1}{p} - y\log y.\frac{1}{p^2}\frac{dp}{dy} - \left[-\frac{1}{y^2}.p + \frac{1}{y}\frac{dp}{dy}\right]$$

or $\frac{1}{p} = \frac{\log y}{p} + \frac{1}{p} + \frac{p}{y^2} - \frac{dp}{dy}\left(\frac{y\log y}{p^2} + \frac{1}{y}\right)$

or $0 = \frac{p}{y}\left(\frac{y\log y}{p^2} + \frac{1}{y}\right) - \frac{dp}{dy}\left(\frac{y\log y}{p^2} + \frac{1}{y}\right)$

or $0 = \left(\frac{y\log y}{p^2} + \frac{1}{y}\right)\left(\frac{p}{y} - \frac{dp}{dy}\right)$

Omitting the first factor, we have $\frac{p}{y} - \frac{dp}{dy} = 0$

or $\frac{dp}{p} = \frac{dy}{y}$

Integrating, $\log p = \log y + \log C$

or $p = Cy$ ... (2)

Putting the value of p given by (2) in (1), we get

$x = \frac{\log y}{C} - C$

or $\log y = Cx + C^2$.

(ii) Solving for x, we have $x = \tan^{-1} p + \frac{p}{1+p^2}$ ... (1)

Differentiating both sides w.r.t. y,

$$\frac{dx}{dy} = \frac{1}{p} = \frac{1}{1+p^2}\cdot\frac{dp}{dy} + \frac{\left(1+p^2\right)-2p^2}{\left(1+p^2\right)^2}\cdot\frac{dp}{dy}$$

or $\frac{1}{p} = \frac{2\left(1+p^2\right)-2p^2}{\left(1+p^2\right)^2}\cdot\frac{dp}{dy}$

or $dy = \frac{2p}{\left(1+p^2\right)^2}dp$

Integrating, $y = C - \frac{1}{1+p^2}$ ... (2)

Equations (1) and (2) together constitute the general solution.

(iii) Given , $p^3 - 4xyp + 8y^2 = 0$ where $p = \frac{dy}{dx}$ ... (1)

Solving (1) for x, $x = \left(\frac{1}{4y}\right)p^2 + \left(\frac{1}{p}\right)(2y)$ ... (2)

Differentiating (2) w.r.t. y, we get

$$\frac{1}{p} = -\frac{p^2}{4y^2} + \frac{2p}{4y}\frac{dp}{dy} + \frac{2}{p} - \frac{2y}{p^2}\frac{dp}{dy}$$

or $\frac{p^2}{4y^2} - \frac{1}{p} - \frac{dp}{dy}\left(\frac{p}{2y} - \frac{2y}{p^2}\right) = 0$

or $\left(\frac{p^2}{4y^2}-\frac{1}{p}\right)-\frac{2y}{p}\frac{dp}{dy}\left(\frac{p^2}{4y^2}-\frac{1}{p}\right)=0$

or $\left(\frac{p^2}{4y^2}-\frac{1}{p}\right)\left(1-\frac{2y}{p}\frac{dp}{dy}\right)=0$ ... (3)

Neglecting the first factor which doesnot involve $\frac{dp}{dy}$, (3) reduces to

$$1-\frac{2y}{p}\frac{dp}{dy}=0$$

or $\left(\frac{2}{p}\right)dp=\left(\frac{1}{y}\right)dy$

Integrating, $2\log p=\log y+\log C'$

or $p^2=C'y$ ... (4)

From (1), $p\left(p^2-4xy\right)=-8y^2$

Squaring, $p^2\left(p^2-4xy\right)^2=64y^4$

or $C'y\left(C'y-4xy\right)^2=64y^4$, using (4)

or $4Cy\left(4Cy-4xy\right)^2=64y^4$ where $C'=4C$

or $C\left(C-x\right)^2=y$.

Check your progress -3

Ans to Q No 1: (i) Given, $y=-px+x^4p^2$ ... (1)

Differentiating both sides w.r.t. x,

$$\frac{dy}{dx}=p=-p-x\frac{dp}{dx}+4x^3p^2+2x^4p\frac{dp}{dx}$$

or $2p+x\frac{dp}{dx}-2px^3\left(2p+x\frac{dp}{dx}\right)=0$

or $\left(2p+x\frac{dp}{dx}\right)\left(1-2px^3\right)=0$

or $2p+x\frac{dp}{dx}=0$, discarding the factor $\left(1-2px^3\right)$

or $\frac{dp}{p}+2\frac{dx}{x}=0$

Integrating, $\log p + 2\log x = \log C$

or $\log px^2 = \log C$

or $px^2 = C$

or $p = \dfrac{C}{x^2}$

Putting this value of p in(1), we have $y = -\dfrac{C}{x} + C^2$ which is the required solution.

(ii) Given, $y = x + a\tan^{-1} p$ ... (1)

Differentiating both sides w.r.t. x, we get

$$p = 1 + \frac{a}{1+p^2}\frac{dp}{dx}$$

or $p - 1 = \dfrac{a}{1+p^2}\dfrac{dp}{dx}$

or $dx = \dfrac{a}{(p-1)(1+p^2)}dp$

or $dx = \dfrac{a}{2}\left[\dfrac{1}{p-1} - \dfrac{1+p}{1+p^2}\right]dp$, by partial fraction

or $dx = \dfrac{a}{2}\left[\dfrac{1}{p-1} - \dfrac{1}{1+p^2} - \dfrac{p}{1+p^2}\right]dp$

Integrating, $x = \dfrac{a}{2}\left[\log(p-1) - \tan^{-1} p - \dfrac{1}{2}\log(1+p^2) + \log C\right]$

or $x = \dfrac{a}{2}\left[\log\left\{C(p-1)(1+p^2)^{-\frac{1}{2}}\right\} - \tan^{-1} p\right]$ ... (2)

Putting the value of x given by (2) in (1), we have

$$y = \frac{a}{2}\left[\log\left\{C(p-1)(1+p^2)^{-\frac{1}{2}}\right\} + \tan^{-1} p\right] \quad \text{... (3)}$$

(2) and (3) together form the required solution in parametric form, p being a parameter.

(iii) Given, $y = p\tan p + \log(\cos p)$ ... (1)

Differentiating both sides w.r.t. x, we get

$$p = \left[\tan p + p\sec^2 p + \left\{\frac{1}{\cos p}\right\}(-\sin p)\right]\left(\frac{dp}{dx}\right)$$

or $p = [\tan p + p\sec^2 p - \tan p]\left(\frac{dp}{dx}\right)$

or $p = p\sec^2 p\left(\frac{dp}{dx}\right)$

or $dx = \sec^2 p dp$

Integrating, $x = \tan p + C$ ... (2)

(1) and (2) form the solution in parametric form, p being a parameter.

Check your progress - 4

(i) Given, $y = px + p - p^3$

This is of the Clairaut's form.

Hence putting C for p, the required solution is $y = Cx + C - C^3$

(ii) Given, $y = xp + \sqrt{1+p^2}$

This is of the Clairaut's form.

Hence putting C for p, the required solution is $y = Cx + \sqrt{1+C^2}$.

(iii) Given, $y = xp + ap(1-p)$

This is of the Clairaut's form.

Hence putting C for p, the required solution is $y = Cx + aC(1-C)$.

(iv) Given, $y = xp + \sin^{-1} p$

This is of the Clairaut's form.

Hence putting C for p, the required solution is $y = Cx + \sin^{-1} C$.

(v) Given, $y = xp + e^p$

This is of the Clairaut's form.

Hence putting C for p, the required solution is $y = Cx + e^C$.

(vi) Given, $y = xp + p^2$

This is of the Clairaut's form.

Hence putting C for p, the required solution is $y = Cx + C^2$.

(vii) Given, $y = px + \frac{p}{p-1}$.

This is of the Clairaut's form.

Hence putting C for p ,the required solution is $y = Cx + \frac{C}{C-1}$.

(viii) Given, $y = px + \sqrt{1-p^2}$

This is of the Clairaut's form.

Hence putting C for p ,the required solution is $y = Cx + \sqrt{1 - C^2}$.

Check your progress - 5

Ans to Q No 1: (i) Given, $xp^2 - 2yp + x + 2y = 0$ ... (1)

Also, given $y - x = v$ and $x^2 = u$ ... (2)

Differentiating w.r.t. x

$\frac{dy}{dx} - 1 = \frac{dv}{dx}$ and $2x = \frac{du}{dx}$

or $p - 1 = \frac{dv}{du}$ and $2x = \frac{du}{dx}$

$\therefore \frac{dv}{du} = \frac{\frac{dv}{dx}}{\frac{du}{dx}} = \frac{p-1}{2x}$

or $P = \frac{p-1}{2x}$

or $p = 2xP + 1$

Put this in the given equation. Then, we get

$x(2xP+1)^2 - 2y(2xP+1) + x + 2y = 0$

or $2x^2P^2 - 2(y - x)P + 1 = 0$

or $2uP^2 - 2vP + 1 = 0$, using (2)

or $v = uP + \frac{1}{2P}$ which is of Clairaut's form.

So, its general solution is $v = uC + \frac{1}{2C}$

or $y - x = x^2C + \frac{1}{2C}$

or $2C^2x^2 - 2C(y - x) + 1 = 0$.

(ii) Given, $(px - y)(py + x) = a^2p$

or $p^2xy + p(x^2 - y^2 - a^2) - xy = 0$ ... (1)

Substituting $x^2 = u$ i.e., $2xdx = du$ and $y^2 = v$ i.e., $2ydy = dv$

So, $\frac{dv}{du} = \frac{\frac{dv}{dx}}{\frac{du}{dx}} \cdot \frac{dy}{dx} = \frac{y}{x}\frac{dy}{dx}$

$$\frac{y}{x}\frac{dy}{dx} = \frac{dv}{du} \Rightarrow p = \frac{x}{y}\frac{dv}{du} \quad \ldots (2)$$

Putting the value of p in (2) from (1), we have

$$xy\frac{x^2}{y^2}\left(\frac{dv}{du}\right)^2 + \frac{x}{y}\left(\frac{dv}{du}\right)\left(x^2 - y^2 - a^2\right) - xy = 0$$

or $x^2\left(\frac{dv}{du}\right)^2 + \frac{dv}{du}\left(x^2 - y^2 - a^2\right) - y^2 = 0$

If we put $\frac{dv}{du} = P$, and $x^2 = u \quad y^2 = v$

$uP^2 + \left(u - v - a^2\right)P - v = 0$

or $uP(P+1) - v(P+1) - Pa^2 = 0$

or $v = uP - \frac{P}{P+1}a^2$, which is of Clairaut's form.

So, its general solution is $v = uC - \frac{C}{C+1}a^2$

or $y^2 = x^2C - \frac{C}{C+1}a^2$.

(iii) Given, $x^2\left(y - px\right) = p^2y$

or $y - px = \frac{yp^2}{x^2}$

or $y^2 = pxy + \left(\frac{py}{x}\right)^2$

or $y^2 = \left(\frac{py}{x}\right)x^2 + \left(\frac{py}{x}\right)^2 \quad \ldots (1)$

Substituting $x^2 = u$ i.e., $2xdx = du$ and $y^2 = v$ i.e., $2ydy = dv \ldots (2)$

So, $\frac{dv}{du} = \frac{\frac{dv}{dx}}{\frac{du}{dx}} \cdot \frac{dy}{dx} = \frac{y}{x}\frac{dy}{dx}$

or $P = \frac{y}{x}p \ldots (3)$

Using (2) and (3), (1) becomes $v = Pu + P^2$ which is of Clairaut's form.

So, its general solution is $v = Cu + C^2$

(iv) Given equation is $x^2p^2 + yp(2x + y) + y^2 = 0$ ... (1)

Given $y = u$ and $xy = v$ ... (2)

Differentiation (2), $dy = du$ and $xdy + ydx = dv$

$$\therefore \frac{xdy + ydx}{dy} = \frac{dv}{du}$$

or $$x + y\frac{dx}{dy} = \frac{dv}{du}$$

or $$x + \frac{y}{p} = P$$

or $$\frac{y}{p} = P - x$$

or $$p = \frac{y}{P - x}$$ where $p = \frac{dy}{dx}$, $P = \frac{dv}{du}$.

Putting $p = \frac{y}{P - x}$ in (1), we have

$$\frac{x^2y^2}{(P - x)^2} + \frac{y^2}{P - x}(2x + y) + y^2 = 0$$

or $x^2 + (P - x)(2x + y) + (P - x)^2 = 0$

or $Py - xy + P^2 = 0$

or $v = uP + P^2$, using (2), ... (3), which is of Clairaut's form.

So, its general solution is $v = uC + C^2$

or $xy = yC + C^2$.

(v) Given equation is $xp^2 - 2yp + x + 2y = 0$ ... (1)

Also, given $y - x = v$ and $x^2 = u$ ... (2)

Differentiating (2), $dy - dx = dv$ and $2xdx = du$

$$\therefore \frac{dy - dx}{2xdx} = \frac{dv}{du}$$

or $$\frac{dy}{dx} - 1 = 2x\frac{dv}{du}$$

or $p - 1 = 2xP$, where $p = \frac{dy}{dx}$, $P = \frac{dv}{du}$

or $p = 1 + 2xP$

Putting $p = 1 + 2xP$ in (1), we get

$x(1+2xP)^2 - 2(1+2xP) + x + 2y = 0$

or $2x^2P^2 - 2(y-x)P + 1 = 0$

or $2uP^2 - 2vP + 1 = 0$, using (2)

or $v = uP + \frac{1}{2P}$, which is of Clairaut's form.

So, its general solution is $v = uC + \frac{1}{2C}$

or $y - x = Cx^2 + \frac{1}{2C}$.



10.8 MODEL QUESTIONS

Q 1 : Solve the following differential equations :

(a) $xyp^2 + (x^2 + xy + y^2)p + x^2 + xy = 0$

(b) $p^3(x+2y) + 3p^2(x+y) + p(y+2x) = 0$

(c) $4y^2p^2 + 2pxy(3x+1) + 3x^3 = 0$

(d) $y = x\left\{p + (1+p^2)^{\frac{1}{2}}\right\}$

(e) $p^2 + 2xp - 3x^2 = 0$

(f) $p^2 - p(e^x + e^{-x}) + 1 = 0$

(g) $yp^2 - (x-y)p - x = 0$

(h) $xyp^2 - (x^2 - y^2)p - xy = 0$

(i) $p(p+y) = x(x+y)$

(j) $y = x\left(p + \sqrt{1+p^2}\right)$

(k) $xyp^2 - (x^2 + y^2)p + xy = 0$

Q 2 : Solve the following differential equations :

(a) $4(xp^2 + yp) = y^4$

(b) $xp^3 = a + bp$

(c) $y = 3px + 4p^3$

(d) $\sqrt{1+p^2}\,x + p = \sqrt{1+p^2}\,a$

(e) $p^3 y + 2px - y = 0$

(f) $x - yp - ap^2 = 0$

Q 3 : Solve the following differential equations :

(a) $y - px + x - \left(\frac{y}{b}\right) = a$

(b) $e^{p-y} = p^2 - 1$

(c) $p - y = \log\left(p^2 - 1\right)$

(d) $4p^3 + 3xp = yp$

(e) $y = p\sin p + \cos p$

(f) $y = xp^2 + p$

(g) $x - yp = ap^2$

(h) $xp^2 - 2yp + ax = 0$

Q 4 : Solve the following differential equations :

(a) $px - y + p^3 = \frac{m^3}{p^3}$

(b) $p^2\left(x^2 - a^2\right) - 2pxy + y^2 + a^4 = 0$

(c) $y^2 - 2pxy + p^2\left(x^2 - 1\right) = m^2$ s

(d) $\left(x^2 - 1\right)p^2 - 2xyp + y^2 = 1$

(e) $xp^2 - yp + a = 0$

(f) $4yp^2 + 2xp - y = 0$

(g) $y = xp + p^3$

Q 5 : Solve the following differential equations :

(a) $(px - y)(x - py) = 2p \ \left(u = x^2, v = y^2\right)$

(b) $y = 2px + 6y^2p^2 \ (y^3 = v\ )$

*** ***** ***

UNIT 11 : LINEAR DIFFERENTIAL EQUATION OF HIGHER ORDER

UNIT STUCTURE

11.1 Learning objectives

11.2 Introduction

11.3 Linear Differential Equations with Constant Coefficients

11.3.1 Rules of Finding Complementary Function

11.3.2 Methods of Finding Particular Integral

11.4 Let Us Sum Up

11.5 Further Readings

11.6 Answers To Check Your Progress

11.7 Model Questions

11.1 LEARNING OBJECTIVES

After going through this unit, you will able to know

- linear differential equation of higher order with constant coefficients,

11.2 INTRODUCTION

In unit 10, we discussed different types of first order and higher degree nonlinear differential equations and their solutions. In this unit, we shall discuss about solution of linear differential equation of higher order with constant coefficients.

11.3 LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

An ordinary linear differential equation of nth order with constant coefficients is of the form

$$\frac{d^n y}{dx^n}+P_1\frac{d^{n-1}y}{dx^{n-1}}+P_2\frac{d^{n-2}y}{dx^{n-2}}+\cdots+P_{n-1}\frac{dy}{dx}+P_n y=X$$

where $P_1, P_2, \cdots, P_n$ are constants and X is a function of x or a constant.

If the coefficient of $\frac{d^n y}{dx^n}$ is not unity,we divide each term by that coefficient.

We consider two forms of the equation,viz,(i) $X=0$, (ii) X is a function of x only or a constant.

Theorem 11.3.1 : If $y=f(x)$ is the general solution of the equation

$$\frac{d^n y}{dx^n}+P_1\frac{d^{n-1}y}{dx^{n-1}}+P_2\frac{d^{n-2}y}{dx^{n-2}}+\cdots+P_{n-1}\frac{dy}{dx}+P_n y=0 \quad \text{... (1)}$$

and $y=\phi(x)$ be a solution of the equation

$$\frac{d^n y}{dx^n}+P_1\frac{d^{n-1}y}{dx^{n-1}}+P_2\frac{d^{n-2}y}{dx^{n-2}}+\cdots+P_{n-1}\frac{dy}{dx}+P_n y=X \quad \text{... (2)}$$

then $y=f(x)+\phi(x)$ is the general solution of the equation (2).

Proof : Since $y=f(x)$ is the general solution of (1),

$$\therefore \frac{d^n f}{dx^n}+P_1\frac{d^{n-1}f}{dx^{n-1}}+P_2\frac{d^{n-2}f}{dx^{n-2}}+\cdots+P_{n-1}\frac{df}{dx}+P_n f=0$$

Again, since $y=\phi(x)$ be a solution of (2),

$$\therefore \frac{d^n \phi}{dx^n}+P_1\frac{d^{n-1}\phi}{dx^{n-1}}+P_2\frac{d^{n-2}\phi}{dx^{n-2}}+\cdots+P_{n-1}\frac{d\phi}{dx}+P_n\phi=X$$

Now, $$\frac{d^n(f+\phi)}{dx^n}+P_1\frac{d^{n-1}(f+\phi)}{dx^{n-1}}+P_2\frac{d^{n-2}(f+\phi)}{dx^{n-2}}+\cdots+P_{n-1}\frac{d(f+\phi)}{dx}+P_n(f+\phi)=0+X=X$$

$\therefore y=f(x)+\phi(x)$ is a solution of the equation (2).

Theorem 11.3.2 : If $y=y_1, y=y_2, \ldots, y=y_n$ are solutions of the equation

$$\frac{d^n y}{dx^n}+P_1\frac{d^{n-1}y}{dx^{n-1}}+P_2\frac{d^{n-2}y}{dx^{n-2}}+\cdots+P_{n-1}\frac{dy}{dx}+P_n y=0 \quad \text{... (1)}$$

then $y=C_1y_1+C_2y_2+\cdots+C_ny_n$ is a solution of the equation (1).

Here $C_1, C_2, \cdots, C_n$ are arbitrary constants.

Proof : We have

$$C_1\left(\frac{d^n y_1}{dx^n}+P_1\frac{d^{n-1}y_1}{dx^{n-1}}+\cdots+P_{n-1}\frac{dy_1}{dx}+p_n y_1\right)+$$

$$C_2\left(\frac{d^n y_2}{dx^n}+P_1\frac{d^{n-1}y_2}{dx^{n-1}}+\cdots+P_{n-1}\frac{dy_2}{dx}+p_n y_2\right) +,\cdots,+$$

$$\left(\frac{d^n y_n}{dx^n}+P_1\frac{d^{n-1}y_n}{dx^{n-1}}+\cdots+P_{n-1}\frac{dy_n}{dx}+p_n y_n\right)$$

$= C_1\times 0 + C_2\times 0 +,\cdots,+C_n\times 0$

$= 0$

$\therefore y = C_1y_1 + C_2y_2 +\cdots+C_ny_n$ is a solution of (1).

This solution contains n arbitrary constants.So,it is the general or complete solution of the equation (1).

Note (a) From the above two results, it is clear that the complete solution of (2) consists of two parts.The first part is the general solution of (1), called the complementary function (C.F). It contains n arbitrary constants. The second part is a solution of (2), called particular integral (P.I). It does not contain an arbitrary constant.

Thus,the complete or general solution of (2) is

$$y = \text{C.F} + \text{P.I}$$

(b) If $y_1(x)$ is a solution of a linear differential equation,then $Cy_1(x)$ is also a solution of the equation.

If $y_1(x)$ and $y_2(x)$ are two solutions of a linear differential equation, then $C_1y_1(x)+C_2y_2(x)$ is also a solution of the equation.

Operator D and its properties :

Let $D=\frac{d}{dx}$ and $\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}(Dy)=D(Dy)=D^2y$ and in general $\frac{d^ny}{dx^n}=D^ny$. So the equation (1) can be written as

$$\left(D^n+P_1D^{n-1}+\cdots+P_n\right)y=X.$$

If m and n are positive integer,then

$$D^mD^n=D^nD^m=D^{m+n}.$$

Result : If α and β are scalars then

$$(D-\alpha)(D-\beta)y=(D-\beta)(D-\alpha)y=\left(D^2-(\alpha+\beta)D+\alpha\beta\right)y$$

Proof : $(D-\alpha)(D-\beta)y=(D-\alpha)(Dy-\beta y)$

$$= D(Dy - \beta y) - \alpha(Dy - \beta y)$$
$$= D^2 y - \beta Dy - \alpha Dy + \alpha\beta y$$
$$= D^2 y - (\alpha + \beta)Dy + \alpha\beta y$$

Similarly, $(D - \beta)(D - \alpha)y = D^2 y - (\alpha + \beta)Dy + \alpha\beta y$
$$= (D^2 - (\alpha + \beta)D + \alpha\beta)y$$

11.3.1 Rules of Finding Complementary Function

Consider the linear equation

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + k_n y = 0$$

where $k_1, k_2, \cdots, k_n$ are constants and right hand expression $X = 0$ and it is called the auxiliary equiation of

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + k_n y = X$$

In terms of operator D, it can be written as

$$(D^n + k_1 D^{n-1} + \cdots + k_n)y = X$$

Let $y = e^{mx}$ be a solution of $(D^n + k_1 D^{n-1} + \cdots + k_n)y = 0$ ... (3)

Then $Dy = me^{mx}, D^2 y = m^2 e^{mx}, \ldots, D^n y = m^n e^{mx}$ and equation (3) becomes $(m^n + k_1 m^{n-1} + k_2 m^{n-2} + \cdots + k_n)e^{mx} = 0$

$$\Rightarrow m^n + k_1 m^{n-1} + k_2 m^{n-2} + \cdots + k_n = 0 \quad (\because e^{mx} \neq 0) \quad \ldots (4)$$

case I : If $m_1, m_2, \cdots, m_n$ be real and distinct roots of (4), then (3) can be written as

$$(D - m_1)(D - m_2)\cdots(D - m_n)y = 0 \quad \ldots (5)$$

(3) will be satisfied by $(D - m_n)y = 0$. Solving it

$$\frac{dy}{dx} = m_n y \text{ ,we get}$$

or $$\frac{dy}{y} = m_n dx$$

or $$\log y = m_n x + \log c$$

or $$\log \frac{y}{c} = m_n x$$

or $$y = e^{m_n x}$$

Since the order of the factors in (5) are independent of their position (3) will be satisfied by the solutions of $(D-m_i)y=0$, $1\le i\le n$.

i.e., $y=c_ie^{m_ix}\ (1\le i\le n)$

So, the general solution of (3) i.e., the complementary function is

$$y=c_1e^{m_1x}+c_2e^{m_2x}+\cdots+c_ne^{m_nx}$$

Csae II : If two roots are equal,say, $m_1=m_2$.then complete solution of (3) is $y=(c_1+c_2x)e^{m_1x}+c_3e^{m_3x}+\cdots+c_ne^{m_nx}$

Similarly, If $m_1=m_2=m_3,m_4,m_5,\cdots,m_n$ are roots, then complete solution is $y=(c_1+c_2x+c_3x^2)e^{m_1x}+c_4e^{m_4x}+c_5e^{m_5x}+\cdots+c_ne^{m_nx}$

Case III : If one pair of roots be imaginary i.e $m_1=\alpha+i\beta$ and $m_2=\alpha-i\beta$ then the complete solution is

$$y=c_1e^{(\alpha+i\beta)x}+c_2e^{(\alpha-i\beta)x}+c_3e^{m_3x}+\cdots+c_ne^{m_nx}$$

$$y=e^{\alpha x}\left[c_1(\cos\beta x+i\sin\beta x)+c_2(\cos\beta x-i\sin\beta x)\right]+c_3e^{m_3x}+\cdots+c_ne^{m_nx}$$

$$=e^{\alpha x}\left[c_1\cos\beta x+c_2\sin\beta x\right]+c_3e^{m_3x}+\cdots+c_ne^{m_nx}$$

where $C_1=c_1+c_2$ and $C_2=i(c_1-c_2)$

Case IV : If two pairs of imaginary roots are equal, i.e., $m_1=m_2=\alpha+i\beta$, $m_3=m_4=\alpha-i\beta$, then as in caseII, the complete solution is

$$y=e^{\alpha x}\left[(c_1+c_2x)\cos\beta x+(c_3+c_4x)\sin\beta x\right]+c_5e^{m_5x}+\cdots+c_ne^{m_nx}$$

Example 1: Solve (i) $2D^2y+9Dy-18y=0$

(ii) $\dfrac{d^3y}{dx^3}-\dfrac{dy}{dx}=0$

(iii) $(D^4-7D^2+6D)y=0$

Solution : (i) Auxiliary equation is $2m^2+9m-18=0$

or $2m^2+12m-3m-18=0$

or $2m(m+6)-3(m+6)=0$

or $(2m-3)(m+6)=0$

or $m=\dfrac{3}{2},-6$

So, $\dfrac{3}{2}$ and - 6 are distinct realroots. Hence, the solution is

$y = Ae^{\frac{3}{2}x} + Be^{-6x}$.

(ii) Auxiliary equation is $m^3 - m = 0$

or $m(m+1)(m-1) = 0$

or $m = 0, -1, 1$

Hence the complete solution is $y = A + Be^{-x} + Ce^{x}$.

(iii) Auxiliary equation is $m^4 - 7m^2 + 6m = 0$

By inspection $m = 0$ and 1 are two roots.

So, the equation can be written as

$$m(m^3 - 7m + 6) = 0$$

or $m(m-1)(m^2 + m - 6) = 0$

or $m(m-1)(m+3)(m-2) = 0$

So, m = 0, 1, 2, –3 are roots of the equation.

Hence its solution is $y = A + Be^{x} + Ce^{2x} + De^{-3x}$.

Example 2 : Solve (i) $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 9x = 0$

(ii) $(D^3 - 3mD^2 + 3m^2D - m^3)y = 0$

(iii) $\frac{d^4y}{dx^4} - 4\frac{d^3y}{dx^3} + 4\frac{d^2y}{dx^2} = 0$

(iv) $(D^5 - D^4 - 2D^3 + 2D^2 + D - 1)y = 0$

Solution : (i) Auxiliary equation is $m^2 + 6m + 9 = 0$

or $(m+3)^2 = 0$

$\therefore \quad m = 3, -3$ are repeated roots.

So, the complete solution is $y = (Ax + B)e^{-3x}$.

(ii) Auxiliary equation is $u^3 - 3mu^2 + 3m^2u - u^3 = 0$

or $(u - m)^3 = 0$

So, $u = m, m, m$ are repeated roots.

The general solution is $y = (Ax^2 + Bx + C)e^{mx}$.

(iii) Auxiliary equation is $m^4 - 4m^3 + 4m^2 = 0$

or $m^2(m^2 - 4m + 4) = 0$

or $m^2(m-2)^2 = 0$

So, $m = 0, 0$ and 2, 2 are roots of the equation.

Hence the general solution is

$$y = (Ax + B)e^{0.x} + (Cx + D)e^{2x}$$

$$= Ax + B + (Cx + D)e^{2x}$$

(iv) Auxiliary equation is $m^5 - m^4 - 2m^3 + 2m^2 + m - 1 = 0$

By inspective $m = \pm 1$ are two roots so that $(m^2 - 1)$ is a factor.

$$m^5 - m^4 - 2m^3 + 2m^2 + m - 1 = (m^2 - 1)(m^3 - m^2 - m + 1)$$

$$= (m^2 - 1)\{m^2(m-1) - (m-1)\}$$

$$= (m^2 - 1)(m-1)(m^2 - 1)$$

$$= (m+1)(m-1)(m-1)(m+1)(m-1)$$

$$= (m+1)^3 (m-1)^2$$

$\therefore$ m = 1,1,1,-1,-1 are five roots of the auxiliary equation.

So, the complete solution of the equation is

$$y = (Ax^2 + Bx + C)e^x + (Dx + E)e^{-x}.$$

Example 3 : Solve (i) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 10y = 0$

(ii) $(D^2 + n^2)y = 0$

(iii) $D^3y + D^2y + Dy + y = 0$

(iv) $(D^4 + 8D^2 + 16)y = 0$

(v) $(D^3 - 8)y = 0$

(vi) $D^2 - 2mD + (m^2 + n^2)y = 0$

(ii) $\frac{d^5y}{dx^5} + 4\frac{d^2y}{dx^2} = 0$

Solution : (i) Auxiliary equation is $m^2 + 2m + 10 = 0$

So, $m = \frac{-2 \pm i6}{2} = -1 \pm 3i$

are two complex roots. So the solution of the equation is

$$y = (A\cos 3x + B\sin 3x)e^{-x}$$

(ii) Auxiliary equation is $m^2 + n^2 = 0$

or $m = 0 \pm in$ are complex conjugate roots.

Hence the complete solution is $y = A\cos nx + B\sin nx$.

(iii) Auxiliary equation is $m^3 + m^2 + m + 1 = 0$

or $m^2(m+1) + (m+1) = 0$

or $(m^2+1)(m+1)=0$

or $m=-1, m=0\pm i$ are roots.

So, the solution is $y=Ae^{-x}+B\cos x+C\sin x$.

(iv) Auxiliary equation is $m^4+8m^2+16=0$

or $(m^2+4)^2=0$

or $m=\pm 2i,\pm 2i$ are repeated complex roots.

so, the solution is $y=(Ax+B)\cos 2x+(Cx+D)\sin 2x$.

(v) Auxiliary equation is $m^3-8=0$

or $(m-2)(m^2+2m+4)=0$

or $m=2,\dfrac{-2\pm i2\sqrt{3}}{2}$

or $m=2,-1\pm i\sqrt{3}$ are roots of the equation.

Hence the solution is $y=Ae^{2x}+(B\cos\sqrt{3}x+C\sin\sqrt{3}x)e^{-x}$.

(vi) Auxiliary equation is $t^2-2mt+(m^2+n^2)=0$

so, $t=\dfrac{2m\pm\sqrt{4m^2-4m^2-4n^2}}{2}=m\pm in$ are

complex conjugate roots.

So, the complete solution is $y=e^{mx}(A\cos nx+B\sin nx)$.

(vii) Auxiliary equation is $m^5+4m^2=0$

or $m^2\left(m^3+\left(4^{\frac{1}{3}}\right)^3\right)=0$

Roots of the equation are $m=0,0,-4^{\frac{1}{3}},\dfrac{1}{2}\left(4^{\frac{1}{3}}\pm\sqrt{4^{\frac{2}{3}}-4.4^{\frac{2}{3}}}\right)$

i.e., $m=0,0,-4^{\frac{1}{3}},2^{-\frac{1}{3}}\left(1\pm i\sqrt{3}\right)$

So, the complete solution is

$$y=(Ax+b)e^{o.x}+Ce^{-4^{\frac{1}{3}}x}+e^{2^{-\frac{1}{3}}x}\left(D\cos(2^{-\frac{1}{3}}\sqrt{3})x+E\sin(2^{-\frac{1}{3}}\sqrt{3})x\right)$$

$$=Ax+B+Ce^{-\sqrt[3]{4}x}+e^{-\sqrt[3]{2}x}\left(D\cos\frac{\sqrt{3}}{\sqrt[3]{2}}x+E\sin\frac{\sqrt{3}}{\sqrt[3]{2}}x\right).$$



CHECK YOUR PROGRESS - 1

Q 1 : Solve the following differential equations :

(i) $\frac{d^3y}{dx^3} - 7\frac{dy}{dx} - 6y = 0$

(ii) $\frac{d^4y}{dx^4} + 13\frac{d^2y}{dx^2} + 36y = 0$

(iii) $\left(D^2+1\right)^2(D-1)y = 0$

(iv) $\left(D^4 - 2D^3 + 2D^2 - 2D + 1\right)y = 0$

(v) $\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} + 6y = 0$

(vi) $\left(D^3 + 3D^2 + 3D + 1\right)y = 0$

(vii) $\left(D^4 - 5D^2 + 4\right)y = 0$

(viii) $\left(D^4 - 81\right)y = 0$

(ix) $\left(D^6 - 1\right)y = 0$

(x) $\left(D^4 + 2D^3 + 3D^2 + 2D + 1\right)y = 0$

11.3.2 Methods of finding particular integral

We have already discussed that the complete integral or general solution of $\left(D^n + k_1D^{n-1} + k_2D^{n-2} + \cdots + k_n\right)y = X$ ... (1) has two parts i.e., complementary function (C.F) and particular integral (P.I).

General solution (G.S) = C.F +P.I.

(I) can be written as $f(D)y = X$ ... (2)

Particular integral of (2) $= \frac{1}{f(D)}X$

Result I $\frac{1}{D}X = \int X dx$

suppose $\frac{1}{D}X = v$ so, $D\left(\frac{1}{D}\right)X = Dv$

or $X = Dv = \frac{dv}{dx}$

Hence, $\int X dx = v = \frac{1}{D}X$

Note : $\frac{1}{D^2}X = \frac{1}{D}\left(\frac{1}{D}X\right) = \frac{1}{D}\int X dx = \int\left(\int X dx\right)dx$

For example, $\frac{1}{D^2}(\sin x) = \frac{1}{D}\left(\int \sin x dx\right) = \frac{1}{D}(-\cos x) = -\sin x$.

Result II $\frac{1}{D-a}(X) = e^{ax}\int Xe^{-ax}dx$

Let $\frac{1}{D-a}(X) = y$

So, $(D-a)\frac{1}{D-a}(X) = (D-a)y$

or $(D-a)y = X$

or $\frac{dy}{dx} - ay = X$, which is a linear equation with I.F $= e^{-ax}$.

So, its solution is $y = e^{ax}\int Xe^{-ax}dx$.

Hence, $\frac{1}{D-a}(X) = e^{ax}\int Xe^{-ax}dx$.

Similarly, $\frac{1}{D+a}(X) = e^{-ax}\int Xe^{ax}dx$

We discuss the methods of finding P.I for $X(x) = e^{ax}, \sin(ax+b), \cos(ax+b), x^m, e^{ax}v(x)$. Finally we discuss the general method to find P.I for any function of x.

Casel (a) when $X(x) = e^{ax}$, $\frac{1}{f(D)}e^{ax} = \frac{e^{ax}}{f(a)}$ if $f(a) \neq 0$.

$$f(D) = \left(D^n + k_1 D^{n-1} + k_2 D^{n-2} + \cdots + k_{n-1} + k_n\right)e^{ax}$$

$$= a^n e^{ax} + k_1 a^{n-1} e^{ax} + k_2 a^{n-2} e^{ax} + \cdots + k_{n-1} a e^{ax} + k_n e^{ax}$$

$$= f(a)e^{ax}$$

So, $\frac{1}{f(D)}\left[f(D)e^{ax}\right] = \frac{1}{f(D)}\left[f(a)e^{ax}\right] = f(a)\frac{1}{f(D)}e^{ax}$

or $e^{ax} = f(a)\frac{1}{f(D)}e^{ax}$

so, $\frac{1}{f(D)}e^{ax} = \frac{e^{ax}}{f(a)}$ if $f(a) \neq 0$.

(b) If $f(a)=0$, $\frac{1}{f(D)}e^{ax}=x\frac{1}{f'(a)}e^{ax}$

$f(a)=0$ implies $D-a$ is a factor of $f(D)=0$.

so, $f(D)=(D-a)\phi(D)$ and $\phi(a)\neq 0$.

$$\frac{1}{f(D)}e^{ax}=\frac{1}{D-a}.\frac{1}{\phi(D)}e^{ax}=\frac{1}{D-a}.\frac{1}{\phi(a)}e^{ax}$$

$$=\frac{1}{\phi(a)}\frac{1}{D-a}e^{ax}$$

$$=\frac{1}{\phi(a)}e^{ax}\int e^{-ax}.e^{ax}dx \quad \text{(Result II)}$$

$$=x\frac{1}{\phi(a)}e^{ax}$$

$f(D)=(D-a)\phi(D)$

Differentiating, $f'(D)=\phi(D)+\phi'(D)(D-a)$

or $f'(a)=\phi(a)+\phi'(a)(a-a)$

so, $f'(a)=\phi(a)+\phi'(a)\times 0$

$f'(a)=\phi(a)$

Hence $\frac{1}{f(D)}e^{ax}=x\frac{1}{f'(a)}e^{ax}$ provided $f'(a)\neq 0$.

Note : If $f'(a)=0$, it can similarly be shown that $\frac{1}{f(D)}e^{ax}=x^2\frac{1}{f''(a)}e^{ax}$ provided $f''(a)\neq 0$.

Example 1 : Find $\frac{1}{(D-2)^3}e^{2x}$

Solution : We know that $\frac{1}{D-a}(X)=e^{ax}\int Xe^{-ax}dx$

$$\frac{1}{(D-2)}e^{2x}=e^{2x}\int e^{-2x}e^{2x}dx$$

$$=e^{2x}x$$

So, $\frac{1}{(D-2)^2}e^{2x}=\frac{1}{D-2}xe^{2x}$

$$= \frac{1}{2} e^{2x} \int x e^{2x} e^{-2x} dx$$

$$= \frac{e^{2x} x^2}{2}$$

and $\frac{1}{(D-2)^3} e^{2x} = \frac{1}{2} \frac{1}{D-2} \left(x^2 e^{2x} \right)$

$$= \frac{1}{2} e^{2x} \int \left(x^2 e^{2x} \right) e^{-2x} dx$$

$$= \frac{x^3 e^{2x}}{6}$$

Example 2 : Solve $\left(D^2 + 5D + 6 \right) y = e^x$

Solution : The auxiliary equation is $m^2 + 5m + 6 = 0$

or $(m+2)(m+3) = 0$

or $m = -2, -3$

The complementary function is $Ae^{-2x} + Be^{-3x}$

$$\text{P. I} = \frac{1}{D^2 + 5D + 6} e^x$$

$$= \frac{1}{1+5+6} e^x = \frac{e^x}{12}$$

G.S is $y = \text{C.F} + \text{P.I} = Ae^{-2x} + Be^{-3x} + \frac{e^x}{12}$.

Example 3 : Solve $(D+2)(D-1)^2 y = e^{-2x} + 2\sinh x$

Solution : The auxiliary equation is $(m+2)(m-1)^2 = 0$ and its roots are -2,1,1.

C.F $= Ae^{-2x} + (Bx + C)e^x$

$$\text{P.I} = \frac{1}{(D+2)(D-1)^2} \left(e^{-2x} + 2\sinh x \right)$$

$$= \frac{1}{(D+2)(D-1)^2} \left(e^{-2x} + e^x - e^{-x} \right)$$

$$= \frac{1}{(D+2)(D-1)^2} e^{-2x} + \frac{1}{(D+2)(D-1)^2} e^x - \frac{1}{(D+2)(D-1)^2} e^{-x}$$

$$= \frac{1}{9} \frac{1}{(D+2)} e^{-2x} + \frac{1}{3} \frac{1}{(D-1)^2} e^x - \frac{1}{4} e^{-x} \left[\frac{1}{f(D)} e^{ax} = \frac{e^{ax}}{f(a)} \text{ if } f(a) \neq 0 \right]$$

$$= \frac{1}{9}xe^{-2x} + \frac{1}{3}\frac{x^2}{2!}e^{x} - \frac{1}{4}e^{-x} \qquad \left[\frac{1}{f(D)}e^{ax} = x\frac{1}{f'(a)}e^{ax}\right]$$

$$= \frac{x}{9}e^{-2x} + \frac{x^2}{6}e^{x} - \frac{1}{4}e^{-x}$$

G.S is $y = Ae^{-2x} + (Bx + C)e^{x} + \frac{x}{9}e^{-2x} + \frac{x^2}{6}e^{x} - \frac{1}{4}e^{-x}$.

Example 4 : Solve $(D^3 - 2D^2 + D)y = e^{-x}$

The auxiliary equation is $m^3 - 2m^2 + m = 0$

or $m(m^2 - 2m + 1) = 0$

or $m(m-1)^2 = 0$

The roots of the equation are 0,1,1.

C.F $= Ae^{o.x} + (Bx + C)e^{x} = A + (Bx + C)e^{x}$

P.I $= \frac{1}{D(D-1)^2}e^{-x} = \frac{e^{-x}}{-1.4} = \frac{-1}{4}e^{-x}$

The general solution is $y = A + (Bx + C)e^{x} - \frac{1}{4}e^{-x}$.

Case II When $X = \sin(ax + b)$ or $\cos(ax + b)$

$$\frac{1}{f(D^2)}\sin(ax + b) = \frac{1}{f(-a^2)}\sin(ax + b) \text{ if } f(-a^2) \neq 0$$

Proof :

Let $f(D^2) = (D^2)^n + k_1(D^2)^{n-1} + k_2(D^2)^{n-2} + \cdots + k_{n-1}D^2 + k_n$

$D(\sin(ax + b)) = a\cos(ax + b)$ and

$D^2(\sin(ax + b)) = (-a^2)\sin(ax + b)$

$D^3(\sin(ax + b)) = (-a^3)\cos(ax + b)$ and

$D^4(\sin(ax + b)) = a^4\cos(ax + b) = (-a^2)^2\sin(ax + b)$

and in general $(D^2)^r(\sin(ax + b)) = (-a^2)^r\sin(ax + b)$

So, $f(D^2)\sin(ax + b) = f(-a^2)(\sin(ax + b))$

as $\sin(ax + b) = f(-a^2)\frac{1}{f(D^2)}\sin(ax + b)$ so that

$$\frac{1}{f(D^2)}\sin(ax + b) = \frac{\sin(ax + b)}{f(-a^2)}, \text{if } f(-a^2) \neq 0$$

Similarly, $\frac{1}{f(D^2)}\cos(ax+b)=\frac{\cos(ax+b)}{f(-a^2)}$, if $f(-a^2)\neq 0$

If $f(-a^2)=0$, $f(D^2)=(D^2+a^2)\phi(D^2)$ and

$$\frac{1}{f(D^2)}\sin(ax+b)=\frac{1}{D^2+a^2}\frac{1}{\phi(D^2)}\sin(ax+b)$$

$$=\frac{1}{\phi(-a^2)}\frac{1}{D^2+a^2}\sin(ax+b)$$

$$=\frac{1}{\phi(-a^2)}x\sin(ax+b)$$

$$=\frac{1}{f'(-a^2)}x\sin(ax+b)$$

and $\frac{1}{f(D^2)}\cos(ax+b)=\frac{1}{f'(-a^2)}x\cos(ax+b)$ if $f'(-a^2)\neq 0$.

Example 5 : Solve $(D^2+3D+2)y=4\cos^2 x$

Solution : The auxiliary equation is $m^2+3m+2=0$ and its roots are $m=-1,-2$.

$$\text{C.F} = Ae^{-x}+Be^{x}$$

$$\text{P.I} = \frac{1}{D^2+3D+2}4\cos^2 x$$

$$=\frac{1}{D^2+3D+2}2(1+\cos 2x)$$

$$=\frac{1}{D^2+3D+2}2+\frac{1}{D^2+3D+2}2\cos 2x$$

$$=\left(\frac{1}{D+1}-\frac{1}{D+2}\right)2+2.\frac{D^2+2-3D}{(D^2+2)^2-9D^2}\cos 2x$$

$$=e^{-x}\int e^{x}2dx-e^{-2x}\int e^{2x}.2dx+\frac{1}{20}(-4\cos 2x+6\sin 2x+2\cos 2x)$$

$$\left[\ \frac{1}{D-a}(X)=e^{ax}\int Xe^{-ax}dx\ .\ \right]$$

$$=2-1+\frac{1}{10}(3\sin 2x-\cos 2x)$$

$$= 1 + \frac{1}{10}(3\sin 2x - \cos 2x)$$

G.S is $y = Ae^{-x} + Be^{-2x} + 1 + \frac{1}{10}(3\sin 2x - \cos 2x)$

Example 6 : Solve $(D^3 + 2D^2 + D)y = e^{2x} + \sin 2x$

Solution : The auxiliary equation is $m^3 + 2m^2 + m = 0$

or $m(m^2 + 2m + 1) = 0$

or $m = 0, -1, -1$

C.F $= A + (Bx + C)e^{-x}$

$$\text{P.I} = \frac{1}{D^3 + 2D^2 + D}(e^{2x} + \sin 2x)$$

$$= \frac{1}{D^3 + 2D^2 + D}e^{2x} + \frac{1}{D^3 + 2D^2 + D}\sin 2x$$

$$= \frac{1}{18}e^{2x} + \frac{1}{-2^2.D + 2(-2^2) + D}\sin 2x$$

$$= \frac{1}{18}e^{2x} + \frac{1}{-8 - 3D}\sin 2x$$

$$= \frac{1}{18}e^{2x} - \frac{3D - 8}{9D^2 - 64}\sin 2x$$

$$= \frac{1}{18}e^{2x} + \frac{1}{100}(6\cos 2x - 8\sin 2x)$$

G.S is $y =$ C.F +P.I

$$= A + (Bx + C)e^{-x} + \frac{1}{18}e^{2x} + \frac{1}{100}(6\cos 2x - 8\sin 2x).$$

Example 7 : Solve $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 3x = \sin t$

Solution : The auxiliary equation is $m^2 + 2m + 3 = 0$

or $m = \frac{1}{2}(-2 \pm \sqrt{-8})$

or $m = -1 \pm i\sqrt{2}$

C.F $= e^{-t}(A\cos\sqrt{2}t + B\sin\sqrt{2}t)$

$$\text{P.I} = \frac{1}{D^2 + 2D + 3}\sin t$$

$$= \frac{1}{-1^2 + 2D + 3} \sin t$$

$$= \frac{2D - 2}{4D^2 - 4} \sin t$$

$$= -\frac{1}{8}(2\cos t - 2\sin t)$$

G. S is $x = e^{-t}\left(A\cos\sqrt{2}t + B\sin\sqrt{2}t\right) - \frac{1}{4}(\cos t - \sin t)$.

Case III : When $X = x^m$, then P.I $= \frac{1}{f(D)} x^m = [f(D)]^{-1} x^m$

$$= [1 + \phi(D)]^{-1} x^m$$

Expand $[1 + \phi(D)]^{-1}$ upto the term D^m and neglect the powers of D higher than D^m, since $D^{m+i}(x^m) = 0$.

Example 8 : Solve (i) $(D^3 + 3D^2 + 2D)y = x^2$

(ii) $\frac{d^2y}{dx^2} + 4y = x^2 + \cos 2x$

Solution : (i) The auxiliary equation is $m^3 + 3m^2 + 2m = 0$

or $m(m^2 + 3m + 2) = 0$

or $m(m+1)(m+2) = 0$

or $m = 0, -1, -2$

C.F $= A + Be^{-x} + Ce^{-2x}$

P.I $= \frac{1}{D^3 + 3D^2 + 2D} x^2$

$$= \frac{1}{2D} \cdot \frac{1}{\left(\frac{1}{2}D^2 + \frac{3}{2}D + 1\right)} x^2$$

$$= \frac{1}{2D}\left[1 + \left(\frac{3}{2}D + \frac{1}{2}D^2\right)\right]^{-1} x^2$$

$$= \frac{1}{2D}\left[1 - \left(\frac{3}{2}D + \frac{1}{2}D^2\right) + \left(\frac{3}{2}D + \frac{1}{2}D^2\right)^2 - \cdots\right] x^2$$

$$= \frac{1}{2D}\left[1 - \frac{3}{2}D + \left(\frac{9}{4} - \frac{1}{2}\right)D^2\right]x^2$$

$$= \frac{1}{2D}\left[x^2 - 3x + \frac{7}{4} \times 2\right]$$

$$= \frac{1}{2}\int\left(x^2 - 3x + \frac{7}{2}\right)dx$$

$$= \frac{x^3}{6} - \frac{3x^2}{4} + \frac{7}{4}x$$

General solution is $y = A + Be^{-x} + Ce^{-2x} + \frac{x^3}{6} - \frac{3x^2}{4} + \frac{7}{4}x$.

(ii) The auxiliary equation is $m^2 + 4 = 0$

or $m = \pm 2i$

C.F $= A\cos 2x + B\sin 2x$

P.I $= \frac{1}{D^2 + 4}\left(x^2 + \cos 2x\right)$

$$= \frac{1}{D^2 + 4}x^2 + \frac{1}{D^2 + 4}\cos 2x \quad \text{... (i)}$$

Now, $\frac{1}{D^2 + 4}x^2 = \frac{1}{4}\left(1 + \frac{D^2}{4}\right)^{-1}x^2 = \frac{1}{4}\left(1 - \frac{D^2}{4}\right)x^2$

$$= \frac{1}{4}x^2 - \frac{1}{16} \times 2$$

$$= \frac{1}{4}x^2 - \frac{1}{8}$$

Let $Y = \frac{1}{D^2 + 4}\cos 2x$ and $Z = \frac{1}{D^2 + 4}\sin 2x$ so that

$$Y + iZ = \frac{1}{D^2 + 4}\left(\cos 2x + i\sin 2x\right)$$

$$= \frac{1}{D^2 + 4}e^{2xi}$$

$$= \frac{1}{4i}\left[\frac{1}{D - 2i} - \frac{1}{D + 2i}\right]e^{2xi}$$

$$= \frac{1}{4i}\left[\frac{1}{D-2i}e^{2xi} - \frac{1}{D+2i}e^{2xi}\right]$$

$$= \frac{1}{4i}\left[\frac{1}{D-2i}e^{2xi} - \frac{e^{2xi}}{4i}\right] \quad \left(\frac{1}{f(D)}e^{ax} = \frac{e^{ax}}{f(a)}\right)$$

$$= \frac{1}{4i}\left[e^{2xi}\int e^{-2xi}.e^{2xi}dx - \frac{e^{2xi}}{4i}\right] \quad \left(\frac{1}{D-a}(X) = e^{ax}\int Xe^{-ax}dx\right)$$

$$= \frac{1}{4i}\left[x(\cos 2x + i\sin 2x) - \frac{1}{4i}(\cos 2x + i\sin 2x)\right]$$

$$= -\frac{x}{4}i\cos 2x + \frac{1}{4}x\sin 2x + \frac{1}{16}\cos 2x - \frac{i}{16}\sin 2x$$

$$= \left(\frac{1}{4}x\sin 2x + \frac{1}{16}\cos 2x\right) + i\left(-\frac{1}{16}\sin 2x - \frac{x}{4}\cos 2x\right)$$

Equating real and imaginary parts, we get

$$Y = \frac{1}{D^2+4}\cos 2x = \frac{1}{4}x\sin 2x + \frac{1}{16}\cos 2x$$

The complete solution is

$$y = A\cos 2x + B\sin 2x + \frac{1}{4}x^2 - \frac{1}{8} + \frac{1}{4}x\sin 2x + \frac{1}{16}\cos 2x.$$

Case IV : when $X = e^{ax}V(x)$.If u be a function of x, then

$$D(e^{ax}u) = e^{ax}Du + ae^{ax}u = e^{ax}(D+a)u$$

$$D^2(e^{ax}u) = (ae^{ax}Du + e^{ax}D^2u) + (ae^{ax}Du + a^2e^{ax}u)$$

$$= e^{ax}(D^2 + 2aDu + a^2u) = e^{ax}(D+a)^2u$$

so, $f(D)(e^{ax}u) = \{D^n + k_1D^{n-1} + k_2D^{n-2} + \cdots + k_0\}(e^{ax}u)$

$$= e^{ax}(D+a)^n u + k_1e^{ax}(D+a)^{n-2}u + \cdots + e^{ax}k_0u$$

$$= e^{ax}f(D+a)u$$

$$= e^{ax}V \quad \text{where } V = f(D+a)u$$

and $\frac{1}{f(D)}e^{ax}V = e^{ax}u = e^{ax}\frac{1}{f(D+a)}V$

Thus $\frac{1}{f(D)}(e^{ax}V) = e^{ax}\frac{1}{f(D+a)}V$.

Example 9 : Solve $(D^2 - 2D + 4)y = e^x\cos x$

Solution : The auxiliary equation is $m^2 - 2m + 4 = 0$

$$\text{or } m = \frac{2 \pm \sqrt{-12}}{2} = 1 \pm i\sqrt{3}$$

$$\text{C.F } = e^x\left(A\cos\sqrt{3}x + B\sin\sqrt{3}x\right)$$

$$\text{P. I } = \frac{1}{D^2 - 2D + 4}\left(e^x\cos x\right)$$

$$= e^x \frac{1}{(D+1)^2 - 2(D+1) + 4}\cos x \quad \left(\frac{1}{f(D)}\left(e^{ax}V\right) = e^{ax}\frac{1}{f(D+a)}V\right)$$

$$= e^x \frac{1}{D^2 + 3}\cos x$$

$$= e^x \frac{1}{-1^2 + 3}\cos x \quad \left(\frac{1}{f(D^2)}\cos(ax+b) = \frac{\cos(ax+b)}{f(-a^2)}\right)$$

$$= \frac{e^x}{2}\cos x$$

General solution is $y = e^x\left(A\cos\sqrt{3}x + B\sin\sqrt{3}x\right) + \frac{e^x}{2}\cos x$.

Example 10 : Solve $\left(D^2 - 1\right)y = x\sin 3x + \cos x$

Solution : The auxiliary equation is $m^2 - 1 = 0$

$$\text{or } m = \pm 1$$

$$\text{C.F } = Ae^x + Be^{-x}$$

$$\text{P.I } = \frac{1}{D^2 - 1}\left(x\sin 3x + \cos x\right)$$

$$= \frac{1}{D^2 - 1}x\sin 3x + \frac{1}{D^2 - 1}\cos x$$

Let $Z = \frac{1}{D^2 - 1}x\sin 3x$ and $Y = \frac{1}{D^2 - 1}x\cos 3x$

so that $Y + iZ = \frac{1}{D^2 - 1}\left(xe^{i3x}\right)$

$$= e^{i3x}\frac{1}{(D + i3)^2 - 1}x$$

$$= e^{i3x}\frac{1}{D^2 + 6iD - 10}x$$

$$= \frac{e^{i3x}}{-10}\left(1 - \frac{D^2 + 6iD}{10}\right)^{-1} x$$

$$= \frac{e^{i3x}}{-10}\left[1 - \frac{1}{10}\left(D^2 - 6iD\right)\right]x$$

$$= \frac{e^{i3x}}{-10}\left(x + \frac{3}{5}i\right)$$

$$= \frac{1}{-10}\left(\cos 3x + i\sin 3x\right)\left(x + \frac{3}{5}i\right)$$

$$= \frac{1}{-10}\left(x\cos 3x + ix\sin 3x\right) + \frac{1}{-10}\left(\frac{3}{5}i\cos 3x + i^2\frac{3}{5}\sin 3x\right)$$

$$= \frac{1}{-10}\left(x\cos 3x - \frac{3}{5}\sin 3x\right) - i\frac{1}{10}\left(x\sin 3x + \frac{3}{5}\cos 3x\right)$$

so, equating real and imaginary parts, we get

$$Z = -\frac{1}{10}\left(x\sin 3x + \frac{3}{5}\cos 3x\right)$$

and $\dfrac{1}{D^2 - 1}\cos x = -\dfrac{1}{2}\cos x$

General solution is

$$y = Ae^x + Be^{-x} - \frac{1}{10}\left(x\sin 3x + \frac{3}{5}\cos 3x\right) - \frac{1}{2}\cos x\,.$$

Example 11 : Solve $\left(D^2 - 2D + 1\right)y = xe^x \sin x$

Solution : The auxiliary equation is $m^2 - 2m + 1 = 0$

or $(m-1)^2 = 0$

or $m = 1,1$

$$\text{C.F} = \left(Ax + B\right)e^x$$

$$\text{P.I} = \frac{1}{D^2 - 2D + 1}xe^x \sin x$$

$$= \frac{1}{(D-1)^2}xe^x \sin x$$

$$= e^x \frac{1}{(D+1-1)^2}x\sin x$$

$$= e^x \frac{1}{D^2}x\sin x$$

$$= e^x \frac{1}{D}\int x\sin x dx$$

$$= e^x \frac{1}{D}\left[-x\cos x + \int \cos x dx\right]$$

$$= e^x \frac{1}{D}\left[-x\cos x + \sin x\right]$$

$$= e^x \int\left[-x\cos x + \sin x\right]dx$$

$$= -e^x \int x\cos x dx - \int \sin x dx$$

$$= -e^x\left[x\sin x - 2\int \sin x dx\right]$$

$$= -e^x\left(x\sin x + 2\cos x\right)$$

General solution is $y = (Ax+B)e^x - e^x(x\sin x + 2\cos x)$.

Example 12 : Solve $\left(D^4 + 2D^2 + 1\right)y = x^2\cos x$

Solution : The auxiliary equation is $m^4 + 2m^2 + 1 = 0$

or $\left(m^2+1\right)^2 = 0$

or $m = \pm i, \pm i$

C.F $= (A+Bx)\cos x + (C+Dx)\sin x$

P.I $= \dfrac{1}{\left(D^2+1\right)^2}x^2\cos x$

Let $Y = \dfrac{1}{\left(D^2+1\right)^2}x^2\cos x$ and $Z = \dfrac{1}{\left(D^2+1\right)^2}x^2\sin x$

Now, $Y + iZ = \dfrac{1}{\left(D^2+1\right)^2}e^{ix}x^2$

$$= e^{ix}\frac{1}{\left\{(D+i)^2+1\right\}^2}x^2$$

$$= e^{ix}\frac{1}{\left(D^2+2iD\right)^2}x^2$$

$$= \frac{e^{ix}}{-4D^2}\left(1+\frac{D}{2i}\right)^{-2}x^2$$

$$= e^{ix}\frac{1}{-4D^2}\left(1+\frac{D}{i}+\frac{3D^2}{-4}\right)x^2$$

$$= \frac{e^{ix}}{-4}\frac{1}{D^2}\left(x^2 + \frac{2x}{i} - \frac{3}{2}\right)$$

$$= \frac{1}{-4}(\cos x + i\sin x)\left(\frac{x^4}{12} - i\frac{x^3}{3} - \frac{3}{4}x^2\right)$$

$$= \frac{1}{-4}\left(\frac{x^4}{12}\cos x - i\frac{x^3}{3}\cos x - \frac{3}{4}x^2\cos x + i\frac{x^4\sin x}{12} + \frac{x^3}{3}\sin x - \frac{3}{4}ix^2\sin x\right)$$

So, Y = real part $= -\frac{x^4}{48}\cos x + \frac{3x^2}{16}\cos x - \frac{x^3}{12}\sin x$

$$= -\frac{1}{48}(x^4\cos x - 9x^2\cos x + 4x^3\sin x)$$

General solution is

$$y = (A + Bx)\cos x + (C + Dx)\sin x - \frac{1}{48}(x^4\cos x - 9x^2\cos x + 4x^3\sin x)$$

Case V : when X is any function of x.

$$\text{P.I} = \frac{1}{f(D)}X$$

Let $f(D) = (D - m_1)(D - m_2)(D - m_3)\cdots(D - m_n)$

and $\frac{1}{f(D)} = \frac{A_1}{(D - m_1)} + \frac{A_2}{(D - m_2)} + \frac{A_3}{(D - m_3)} + \cdots + \frac{A_n}{(D - m_n)}$ (say)

So,

$$\frac{1}{f(D)}X = A_1\frac{1}{(D - m_1)}X + A_2\frac{1}{(D - m_2)}X + \cdots + A_n\frac{1}{(D - m_n)}X$$

$$= A_1e^{m_1x}\int e^{-m_1x}Xdx + A_2e^{m_2x}\int e^{-m_2x}Xdx + \cdots + A_ne^{m_nx}\int e^{-m_nx}Xdx$$

Example 13 : Solve $(D^2 + a^2)y = \tan ax$

Solution : The auxiliary equation is $m^2 + a^2 = 0$

or $m = \pm ia$

$$\text{C.F} = A\cos ax + B\sin ax$$

$$\text{P.I} = \frac{1}{D^2 + a^2}\tan ax$$

$$= \frac{1}{2ia}\left[\frac{1}{D - ia} - \frac{1}{D + ia}\right]\tan ax$$

$$= \frac{1}{2ia}\left[e^{iax}\int e^{-iax}\tan axdx - e^{-iax}\int e^{iax}\tan axdx\right]$$

$$= \frac{1}{2ai}\left[e^{aix}\int(\cos ax - i\sin ax)\frac{\sin ax}{\cos ax}dx - e^{-aix}\int(\cos ax + i\sin ax)\frac{\sin ax}{\cos ax}dx \right]$$

$$= \frac{1}{2ia}\left[e^{iax}\left(\int \sin axdx - i\int \frac{\sin^2 ax}{\cos ax} \right) - e^{-iax}\left(\int \sin axdx + i\int \frac{\sin^2 ax}{\cos ax} \right) \right]$$

$$= \frac{1}{2ai}[e^{aix}\left(\frac{-\cos ax}{a} \right) - e^{aix}i$$

$$\int(\sec ax - \cos ax)dx - e^{-aix}\left(\frac{-\cos ax}{a} \right) - e^{-aix}i\int(\sec ax - \cos ax)dx]$$

$$= \frac{1}{2a^2 i}(-\cos ax)\times 2i\sin ax - \frac{2\cos ax}{2a^2}[\log(\sec ax + \sin ax) + \sin ax]$$

$$= -\frac{1}{a^2}\sin 2ax - \frac{\cos ax}{a^2}\log(\sec ax + \tan ax) - \frac{\sin 2ax}{2a^2}$$

$$= -\frac{3}{2a^2}\sin 2ax - \frac{\cos ax}{a^2}\log(\sec ax + \tan ax)$$

General solution is

$$y = A\cos ax + B\sin ax - \frac{3}{2a^2}\sin 2ax - \frac{\cos ax}{a^2}\log(\sec ax + \tan ax).$$

Case VI : When $X = x\phi(x)$

$$\frac{1}{f(D)}x\phi(x) = \left\{ x - \frac{1}{f(D)}f'(D) \right\}\frac{1}{f(D)}\varphi(x)$$

Proof : We have $D(x\phi) = xD\phi + \phi$

$$D^2(x\phi) = D\phi + xD^2\phi + D\phi = xD^2\phi + 2D\phi$$

Similarly $D^n(x\phi) = xD^n\phi + nD^{n-1}\phi = xD^n\phi + \left(\frac{dD^n}{dD} \right)\phi$

Hence $f(D)(x\phi) = xf(D)\phi + f'(D)\phi$

Substituting $f(D)\phi = \phi_1$ and hence $\phi = \frac{1}{f(D)}\phi_1$

we have from preceding step

$$f(D)\left\{ x\frac{1}{f(D)}\phi_1 \right\} = x\phi_1 + f'(D)\frac{1}{f(D)}\phi_1$$

and $x\frac{1}{f(D)}\phi_1 = \frac{1}{f(D)}(x\phi_1) + \frac{1}{f(D)}f'(D)\frac{1}{f(D)}\phi_1$

So, $\frac{1}{f(D)}(x\phi_1) = x\frac{1}{f(D)}\phi_1 - \frac{1}{f(D)}.f'(D)\frac{1}{f(D)}\phi_1$

Replacing ϕ_1 by ϕ ,we have

$$\frac{1}{f(D)}x\phi(x) = \left\{x - \frac{1}{f(D)}f'(D)\right\}\frac{1}{f(D)}\varphi(x).$$

Example 14 : Solve $(D^2-1)y = x\sin x + (1+x^2)e^x$

Solution : The auxiliary equation is $m^2 - 1 = 0$

or $m = \pm 1$

$$\text{C.F} = Ae^x + Be^{-x}$$

$$\text{P.I} = \frac{1}{D^2-1}\left[x\sin x + (1+x^2)e^x\right]$$

$$= \frac{1}{D^2-1}x\sin x + \frac{1}{D^2-1}(1+x^2)e^x$$

$$= \left[\left\{x - \frac{1}{D^2-1}.\frac{d}{dD}(D^2-1)\right\}\frac{1}{D^2-1}\sin x\right] + e^x\frac{1}{(D+1)^2-1}(1+x^2)$$

$$= \left\{x - \frac{1}{D^2-1}.2D\right\}\frac{1}{-1^2-1}\sin x + e^x\frac{1}{D^2+2D}(1+x^2)$$

$$= -\frac{x\sin x}{2} + \frac{1}{D^2-1}\cos x + e^x\frac{1}{2D}\left(1+\frac{D}{2}\right)^{-1}(1+x^2)$$

$$= -\frac{x\sin x}{2} + \frac{1}{D^2-1}\cos x + e^x\frac{1}{2D}\left(1-\frac{D}{2}+\frac{D^2}{4}\right)(1+x^2)$$

$$= -\frac{x\sin x}{2} - \frac{1}{2}\cos x + e^x\frac{1}{2D}\left(1+x^2-\frac{1}{2}.2x+\frac{1}{4}.2\right)(1+x^2)$$

$$= -\frac{1}{2}(x\sin x + \cos x) + \frac{e^x}{2}\frac{1}{D}\left(x^2 - x + \frac{3}{2}\right)$$

$$= -\frac{1}{2}(x\sin x + \cos x) + \frac{e^x}{2}\left(\frac{x^3}{3} - \frac{x^2}{2} + \frac{3}{2}x\right)$$

$$= -\frac{1}{2}(x\sin x + \cos x) + \frac{xe^x}{12}(2x^2 - 3x + 9)$$

The general solution is

$$y = Ae^x + Be^{-x} - \frac{1}{2}(x\sin x + \cos x) + \frac{xe^x}{12}(2x^2 - 3x + 9).$$



CHECK YOUR PROGRESS - 2

Q 1 : Solve the following differential equations :

(a) $\left(4D^2+12D+9\right)y=144e^{-x}$

(b) $\frac{d^2y}{dx^2}-6\frac{dy}{dx}+9y=6e^{3x}+7e^{-2x}-\log 2$

Q 2 : Solve the following differential equations :

(a) $\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=\cos 2x$

(b) $\left(D^2+4\right)y=\sin^2 x$

Q 3 : Solve the following differential equations :

(a) $\left(D^3+8\right)y=x^4+2x+1$

(b) $(D-2)^2y=8\left(e^{2x}+\sin 2x+x^2\right)$

Q 4 : Solve the following differential equations :

(a) $\left(D^2-2D+1\right)y=x^2e^{3x}$

(b) $\left(D^2-3D+2\right)y=xe^x+\sin 2x$

Q 5 : Solve the following differential equations :

(a) $\frac{d^2y}{dx^2}+4y=\tan 2x$

(b) $\frac{d^2y}{dx^2}+y=\sec^2 x$



11.4 LET US SUM UP

- The complete solution of the equation

$$\frac{d^ny}{dx^n}+P_1\frac{d^{n-1}y}{dx^{n-1}}+P_2\frac{d^{n-2}y}{dx^{n-2}}+\cdots+P_n\frac{dy}{dx}=X$$

or $\left(D^n+P_1D^{n-1}+P_2D^{n-2}+\cdots+P_n\right)y=X$ has two parts. The first part is complementary function (C.F) obtained in general observing the nature of roots of the equation :

$$m^n+P_1m^{n-1}+P_2m^{n-2}+\cdots+P_n=0$$

This equation is known as auxiliary equation (A.E).

- Methods of finding C.F :

(i) If the roots are real and distinct i.e., $m_1, m_2, \cdots, m_n$

then C.F $= C_1 e^{m_1 x} + C_2 e^{m_2 x} + \cdots + C_n e^{m_n x}$

(ii) If out of n roots r roots are real and repeated, say, m and the remaining are non-repeated ,say, $m_{r+1}, m_{r+2}, \cdots, m_n$ then C.F

$= \left(C_1 x^r + C_2 x^{r-1} + \cdots + C_{r-1} x + C_r\right) e^{mx} + C_{r+1} e^{m_{r+1} x}$

$+ C_{r+2} e^{m_{r+2} x} + \cdots + C_n e^{(m_r + n - r)x}$.

For example if the roots are $m, m, m, m_4, m_5, \cdots, m_{n-3}$,then

C.F $= e^{mx}\left(C_1 + C_2 x + C_3 x^2\right) + C_4 e^{m_4 x} + \cdots + C_{n-3} e^{m_{n-3} \cdot x}$

(iii) If one pair of roots are imaginary and remaining are real and distinct,say, $\alpha + i\beta, \alpha - i\beta, m_3, m_4, \cdots, m_n$,then

C.F $= e^{\alpha x}\left(C_1 \cos \beta x + C_2 \sin \beta x\right) + C_3 e^{m_3 x} + \cdots + C_n e^{m_n x}$

(iv) If two pairs are complex and repeated and the remaining are different,say, $\alpha \pm i\beta, \alpha \pm i\beta, m_5, m_6, \cdots, m_n$,then C.F

$= e^{\alpha x}\left\{\left(C_1 + C_2 x\right)\cos \beta x + \left(C_3 + C_4 x\right)\sin \beta x\right\} + C_5 e^{m_5 x} + \cdots + C_n e^{m_n x}$

(v) One pair real and repeated, one pair complex and the remaining are real and distinct,say, $m_1, m_1, m_2 \pm i m_3, m_5, m_6, \cdots, m_n$ .then C.F

$= \left(C_1 + C_2 x\right) e^{m_1 x} + e^{m_2 x}\left(C_3 \cos m_3 x + C_4 \sin m_3 x\right) + C_5 e^{m_5 x} + \cdots + C_n e^{m_n x}$.

- The second part of the general solution of

$\left(D^n + P_1 D^{n-1} + P_2 D^{n-2} + \cdots + P_n\right) y = X$ ... $(*)$ is a paticular solution which is called particular integral (P.I) of $(*)$.

The general solution of $(*)$ is given by $y =$ C.F +P.I.

If y is a P.I of $(*)$, then $f(D) y = X$ .It is written as $y = \frac{1}{f(D)} X$.

- Methods of finding P.I :

(i) If $f(D) = \frac{1}{D - \alpha}$,P.I $= \frac{1}{D - \alpha} X = e^{\alpha x} \int e^{-\alpha x} X dx$.

and $\frac{1}{D} X = \int X dx$

(ii) If $f(D) = (D - \alpha_1)(D - \alpha_2) \cdots (D - \alpha_n)$ and $\alpha_1, \alpha_2, \cdots, \alpha_n$ are distinct.

then $\frac{1}{f(D)}X = \frac{1}{(D-\alpha_1)(D-\alpha_2)\cdots(D-\alpha_n)}(X)$

$= \left(\frac{A_1}{D-\alpha_1} + \frac{A_2}{D-\alpha_2} + \cdots + \frac{A_n}{D-\alpha_n}\right)(X)$ (Resolving into partial fraction)

$= A_1 e^{\alpha_1 x}\int e^{-\alpha_1 x}X dx + \cdots + A_n e^{\alpha_n x}\int e^{-\alpha_n x}X dx$

(iii) If $X = e^{ax}$ then $\frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}$ if $f(a) \neq 0$

If $f(a) = 0$, $\frac{1}{f(D)}e^{ax} = x\frac{1}{f'(a)}e^{ax}$

If $f'(a) = 0$, it can similarly be shown that $\frac{1}{f(D)}e^{ax} = x^2\frac{1}{f''(a)}e^{ax}$

provided $f''(a) \neq 0$ and so on.

(iv) When $X = \sin(ax+b)$ or $\cos(ax+b)$

$\frac{1}{f(D^2)}\sin(ax+b) = \frac{1}{f(-a^2)}\sin(ax+b)$ if $f(-a^2) \neq 0$

Similarly, $\frac{1}{f(D^2)}\cos(ax+b) = \frac{\cos(ax+b)}{f(-a^2)}$, if $f(-a^2) \neq 0$

If $f(-a^2) = 0$, $\frac{1}{f(D^2)}\sin(ax+b) = \frac{1}{f'(-a^2)}x\sin(ax+b)$

and $\frac{1}{f(D^2)}\cos(ax+b) = \frac{1}{f'(-a^2)}x\cos(ax+b)$ if $f'(-a^2) \neq 0$.

(v) If $X = e^{ax}\phi(x)$, then

$\frac{1}{f(D)}e^{ax}\phi(x) = e^{ax}\frac{1}{f(D+a)}\phi(x)$

(vi) If $X = x\phi(x)$, then

$\frac{1}{f(D)}X = \left(x - \frac{1}{f(D)}.f'(D)\right)\frac{1}{f(D)}\phi(x)$

(vii) If $X = x^m$, then

$\frac{1}{f(D)}X = \frac{1}{D^n(1+P_1D+P_2D^2+\cdots)}X = \frac{1}{D^n}(1+P_1D+P_2D^2+\cdots)^{-1}X.$

where $u = \int \frac{-y_2 X}{W} dx$ and $v = \int \frac{y_1 X}{W} dx$, $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$.

General solution = C.F + P.I.



11.5 FURTHER READINGS

1) Tenenbaum, Morris and Pollard, Harry. Ordinary Differential Equations, Dover Publications, Inc, N.Y 11501.
2) Differential Equations: An Introduction with Mathematica. Clay C.Ross, second Edition, Springer.
3) Differential Equations: Linear, Nonlinear, ordinary, partial. A.C. King, J.Billinggham and S. R. Otto, Cambridge-2003.
4) Differential equations with Application and Programs. S. Balachandra Rao and H.R. Anuradha, University Press (India) Limited 1996.



11.6 ANSWERS TO CHECK YOUR PROGRESS

CHECK YOUR PROGRESS – 1

Ans to Q No 1: (i) The given equation is $(D^3 - 7D - 6)y = 0$

Auxiliary equation is $m^3 - 7m - 6 = 0$

or $(m+1)(m+2)(m-3) = 0$

or $m = -1, -2, 3$

Hence complete solution is $y = C_1 e^{-x} + C_2 e^{3x} + C_3 e^{-2x}$.

(ii) The given equation is $(D^4 + 13D^2 + 36)y = 0$

The auxiliary equation is $m^4 + 13m^2 + 36 = 0$

or $m^4 + 4m^2 + 9m^2 + 36 = 0$

or $(m^2 + 4)(m^2 + 9) = 0$

or $m^2+4=0, m^2+9=0$

or $m=\pm 2i, m=\pm 3i$

The complete solution is

$y=C_1\cos 2x+C_2\sin 2x+C_3\cos 3x+C_4\sin 3x$.

where C_1, C_2, C_3, C_4 and C_5 are arbitrary constants.

(iii) The auxiliary equation is $(m^2+1)^2(m-1)=0$

or $m=\pm i, \pm i, 1$

The complete solution is $y=$ C.F +P.I

$y=(C_1+xC_2)\cos x+(C_3+xC_4)\sin x+C_5e^x$

(iv) Here auxiliary equiation is $m^4-2m^3+2m^2-2m+1=0$

or $(m-1)^2(m^2+1)=0$

or $m=1,1,\pm i$

Hence complete solution is $y=(C_1+xC_2)+(C_3\cos x+C_4\sin x)$.

(v) The given equation can be re-written as

$(D^3+6D^2+11D+6)y=0,$ where $D=\dfrac{d}{dx}$.

The auxiliary equation is $m^3+6m^2+11m+6=0$

or $(m+1)(m+2)(m+3)=0$

or $m=-1,-2,-3$

Hence complete solution is $y=C_1e^{-x}+C_2e^{-2x}+C_3e^{-3x}$.

(vi) The auxiliary equation is $m^3+3m^2+3m+1=0$

or $(m+1)^3=0$

or $m=-1,-1,-1$

Hence complete solution is $y=(C_1+C_2x+C_3x^2)e^{-x}$.

(vii) The auxiliary equation is $m^4-5m^2+4=0$

or $(m^2-4)(m^2-1)=0$

or $m^2=4$ and $m^2=1$

or $m=2,-2,1-1$

Hence complete solution is $y=C_1e^{2x}+C_2e^{-2x}+C_3e^x+C_4e^{-x}$.

(viii) Here auxiliary equation is $m^4-81=0$

or $(m^2+9)(m^2-9)=0$

or $(m+3i)(m-3i)(m+3)(m-3)=0$

or $m=\pm 3i, 3, -3$

Hence complete solution is $y = C_1 e^{3x} + C_2 e^{-3x} + C_3 \cos 3x + C_4 \sin 3x$.

(ix) Here auxiliary equation is $m^6 - 1 = 0$

or $\left(m^2\right)^3 - (1)^3 = 0$

or $\left(m^2 - 1\right)\left(m^4 + m^2 + 1\right) = 0$

or $\left(m^2 - 1\right)\left\{\left(m^4 + 2m^2 + 1\right) - m^2\right\} = 0$

or $(m-1)(m+1)\left\{\left(m^2 + 1\right)^2 - m^2\right\} = 0$

or $(m-1)(m+1)\left(m^2 + 1 + m\right)\left(m^2 + 1 - m\right) = 0$

or $m = 1, -1, \dfrac{-1 \pm (1-4)^{\frac{1}{2}}}{2}, \dfrac{1 \pm (1-4)^{\frac{1}{2}}}{2}$

$$m = 1, -1, -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}, \frac{1}{2} \pm \frac{i\sqrt{3}}{2}$$

Hence complete solution is

$$y = C_1 e^{x} + C_2 e^{-x} + e^{-\frac{x}{2}}\left[C_3 \cos\frac{x\sqrt{3}}{2} + C_4 \sin\frac{x\sqrt{3}}{2}\right] + e\frac{x}{2}\left[C_5 \cos\frac{x\sqrt{3}}{2} + C_6 \sin\frac{x\sqrt{3}}{2}\right].$$

(x) Here auxiliary equation is $m^4 + 2m^3 + 3m^2 + 2m + 1 = 0$

or $\left(m^4 + 2m^3 + m^2\right) + \left(2m^2 + 2m + 1\right) = 0$

or $\left(m^2 + m\right)^2 + 2\left(m^2 + m\right) + 1 = 0$

or $\left(m^2 + m + 1\right)^2 = 0$

or $m^2 + m + 1 = 0$ (twice)

or $\therefore m = \dfrac{\left\{-1 \pm (1-4)^{\frac{1}{2}}\right\}}{2} = -\left(\dfrac{1}{2}\right) \pm i\left(\dfrac{\sqrt{3}}{2}\right)$ (twice)

Hence complete solution is

$$y = e^{-\frac{x}{2}}\left[(C_1 + C_2 x)\cos\left(\frac{\sqrt{3}x}{2}\right) + (C_3 + C_4 x)\sin\left(\frac{\sqrt{3}x}{2}\right)\right]$$

CHECK YOUR PROGRESS – 2

Ans to Q No 1: (a) Here auxiliary equation is $(2m+3)^2 = 0$

or $m = -\dfrac{3}{2}, -\dfrac{3}{2}$

C.F $= (C_1 + C_2 x)e^{-\frac{3x}{2}}$

and P.I $= \dfrac{1}{4D^2+12D+9}144e^{-3x}$

$= 144\dfrac{1}{(2D+3)^2}e^{-3x}$

$= \dfrac{144}{(-6+3)^2}e^{-3x}$

$= 16e^{-3x}$

Hence the required solution is $y = (C_1 + C_2 x)e^{-\frac{3x}{2}} + 16e^{-3x}$.

(b) The given equation can be written as

$(D^2 - 6D + 9)y = 6e^{3x} + 7e^{-2x} - \log 2$

A. E is $m^2 - 6m + 9 = 0$

or $(m-3)^2 = 0$

or $m = 3,3$

C.F $= (C_1 + C_2 x)e^{3x}$

P.I $= \dfrac{1}{D^2-6D+9}\left(6e^{3x} + 7e^{-2x} - \log 2\right)$

$= \dfrac{1}{D^2-6D+9}6e^{3x} + \dfrac{1}{D^2-6D+9}7e^{-2x} - \dfrac{1}{D^2-6D+9}\log 2$

$= x\dfrac{1}{2D-6}6e^{3x} + \dfrac{1}{4+12+9}7e^{-2x} - \log 2\dfrac{1}{D^2-6D+9}e^{ox}$

$= x\dfrac{1}{2}6e^{3x} + \dfrac{7}{25}e^{-2x} - \log 2\left(\dfrac{1}{9}\right)$

$= 3x^2e^{3x} + \dfrac{7}{25}e^{-2x} - \dfrac{1}{9}\log 2$

Complete solution is

$y = (C_1 + C_2 x)e^{3x} + 3x^2e^{3x} + \dfrac{7}{25}e^{-2x} - \dfrac{1}{9}\log 2\left(\dfrac{1}{9}\right)$

Ans to Q No 2 : (a) Given, $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y = \cos 2x$

The given equation can be written as $(D^2 + D + 1)y = \cos 2x$ ...(1)

Auxiliary equation is $m^2 + m + 1 = 0$

or $m = \dfrac{-1 \pm i\sqrt{3}}{2}$

$$\text{C.F} = e^{-\frac{x}{2}}\left[C_1 \cos\frac{\sqrt{3}}{2}x + C_2 \sin\frac{\sqrt{3}}{2}x\right]$$

$$\text{P.I} = \frac{1}{D^2+D+1}\cos 2x$$

$$= \frac{1}{-2^2+D+1}\cos 2x$$

$$= \frac{1}{D-3}\cos 2x$$

$$= \frac{D+3}{D^2-9}\cos 2x$$

$$= \frac{D+3}{-2^2-9}\cos 2x$$

$$= \frac{1}{-13}(D+3)\cos 2x$$

$$= \frac{1}{-13}[D(\cos 2x)+3\cos 2x]$$

$$= \frac{1}{-13}[-2\sin 2x+3\cos 2x]$$

Hence the required solution is

$$y = e^{-\frac{x}{2}}\left[C_1 \cos\frac{\sqrt{3}}{2}x + C_2 \sin\frac{\sqrt{3}}{2}x\right] + \frac{1}{13}[2\sin 2x - 3\cos 2x]$$

(b) Given, $(D^2+4)y = \sin^2 x$ ... (1)

Here auxiliary equation is $m^2+4=0$

or $m = \pm 2i$

Hence C.F $= C_1 \cos 2x + C_2 \sin 2x$

$$\text{P.I} = \frac{1}{D^2+4}\sin^2 x$$

$$= \frac{1}{2}\frac{1}{D^2+4}(1-\cos 2x)$$

$$= \frac{1}{2}\left[\frac{1}{D^2+4}e^{0.x} - \frac{1}{D^2+4}\cos 2x\right]$$

$$= \frac{1}{2}\left[\frac{1}{0+4}e^{0.x} - \left(\frac{x}{2\times 2}\right)\sin 2x\right]$$

$$= \frac{1}{8} - \frac{1}{8} x \sin 2x$$

Hence the required solution is $y = C_1 \cos 2x + C_2 \sin 2x + \frac{1}{8} - \frac{1}{8} x \sin 2x$

Ans to Q No 3 : (a) Given, $\left(D^3 + 8\right) y = x^4 + 2x + 1$

The auxiliary equation is $m^3 + 8 = 0$

or $(m+2)\left(m^2 - 2m + 4\right) = 0$

or $m = -2, 1 \pm i\sqrt{3}$

C.F $= C_1 e^{-2x} + e^x \left(C_2 \cos x\sqrt{3} + C_3 \sin x\sqrt{3}\right)$

P.I $= \frac{1}{D^3 + 8}\left(x^4 + 2x + 1\right)$

$$= \frac{1}{8\left(1 + \frac{1}{8} D^3\right)}\left(x^4 + 2x + 1\right)$$

$$= \frac{1}{8}\left(1 + \frac{1}{8} D^3\right)^{-1}\left(x^4 + 2x + 1\right)$$

$$= \frac{1}{8}\left(1 - \frac{1}{8} D^3 + \right)\left(x^4 + 2x + 1\right)$$

$$= \frac{1}{8}\left[\left(x^4 + 2x + 1\right) - \frac{1}{8} D^3\left(x^4 + 2x + 1\right)\right]$$

$$= \frac{1}{8}\left[\left(x^4 + 2x + 1\right) - 3x\right]$$

(b) Given, $(D-2)^2 y = 8\left(e^{2x} + \sin 2x + x^2\right)$ ... (1)

Auxiliary equation is $(m-2)^2 = 0$

or $m = 2, 2$

C.F $= (C_1 + C_2 x) e^{2x}$

P.I $= \frac{1}{(D-2)^2}\left(e^{2x} + \sin 2x + x^2\right)$

$$= \frac{1}{(D-2)^2} e^{2x} + \frac{1}{(D-2)^2} \sin 2x + \frac{1}{(D-2)^2} x^2 \quad \text{... (2)}$$

Now, $\frac{1}{(D-2)^2} e^{2x} = x . \frac{1}{2(D-2)} e^{2x}$

$$= x^2 . \frac{1}{2} e^{2x}$$

$$= \frac{x^2}{2} e^{2x}$$

$$\frac{1}{(D-2)^2} \sin 2x = \frac{1}{D^2 - 2D + 4} \sin 2x$$

$$= \frac{1}{-2^2 - 2D + 4} \sin 2x$$

$$= -\frac{1}{2D} \sin 2x$$

$$= -\frac{1}{4} \int \sin 2x dx$$

$$= -\frac{1}{4} \left(-\frac{\cos 2x}{2} \right)$$

$$= \frac{1}{8} \cos 2x$$

$$\frac{1}{(D-2)^2} x^2 = \frac{1}{(2-D)^2} x^2 = \frac{1}{4\left(1 - \frac{D}{2}\right)^2} x^2 = \frac{1}{4} \left(1 - \frac{D}{2}\right)^{-2} x^2$$

$$= \frac{1}{4} \left[1 - 2\left(-\frac{D}{2} \right) + \frac{(-2)(-3)}{2} \left(\frac{D}{2} \right)^2 - \cdots \right] x^2$$

$$= \frac{1}{4} \left[1 + D + \frac{3}{4} D^2 + \cdots \right] x^2$$

$$= \frac{1}{4} \left(x^2 + 2x + \frac{3}{2} \right)$$

From (2), P.I $= \frac{x^2}{2} e^{2x} + \frac{1}{8} \cos 2x + \frac{1}{4} \left(x^2 + 2x + \frac{3}{2} \right)$

Hence the required solution is

$$y = (C_1 + C_2 x) e^{2x} + \frac{x^2}{2} e^{2x} + \frac{1}{8} \cos 2x + \frac{1}{4} \left(x^2 + 2x + \frac{3}{2} \right).$$

Ans to Q No 4 : (a). Given, $(D^2 - 2D + 1) y = x^2 e^{3x}$

Auxiliary equation is $m^2 - 2m + 1 = 0$

or $m = 1, 1$

C.F $= (C_1 + C_2 x)e^x$

P.I $= \dfrac{1}{D^2 - 2D + 1} x^2 e^{3x}$

$= \dfrac{1}{(D-1)^2} x^2 e^{3x}$

$= e^{3x} \dfrac{1}{(D+3-1)^2} x^2$

$= \dfrac{1}{4} e^{3x} \dfrac{1}{\left(1 + \dfrac{D}{2}\right)^2} x^2$

$= \dfrac{1}{4} e^{3x} \left(1 + \dfrac{D}{2}\right)^{-2} x^2$

$= \dfrac{1}{4} e^{3x} \left[1 - 2\left(\dfrac{D}{2}\right) + 3\left(\dfrac{D}{2}\right)^2 + \cdots\right] x^2$

$= \dfrac{1}{4} e^{3x} \left[x^2 - Dx^2 + \dfrac{3}{4} D^2 x^2 + \cdots\right]$

$= \dfrac{1}{4} e^{3x} \left[x^2 - 2x + \dfrac{3}{2}\right]$

Hence the required solution is

$$y = (C_1 + C_2 x)e^x + \frac{1}{4} e^{3x} \left[x^2 - 2x + \frac{3}{2}\right]$$

(b) Given, $(D^2 - 3D + 2)y = xe^x + \sin 2x$ ... (1)

Auxiliary equation is $m^2 - 3m + 2 = 0$

or $m = 1, 2$

C.F $= C_1 e^x + C_2 e^{2x}$

P. I $= \dfrac{1}{D^2 - 3D + 2} (xe^x + \sin 2x)$

$= \dfrac{1}{D^2 - 3D + 2} xe^x + \dfrac{1}{D^2 - 3D + 2} \sin 2x$

$= e^x \dfrac{1}{(D+1)^2 - 3(D+1) + 2} x + \dfrac{1}{-2^2 - 3D + 2} \sin 2x$

$$= e^x \frac{1}{D^2 - D} x - \frac{1}{3D+2} \sin 2x$$

$$= e^x \frac{1}{(-D)(1-D)} x - \frac{(3D-2)}{9D^2 - 4} \sin 2x$$

$$= -e^x \frac{1}{D} (1-D)^{-1} x - \frac{(3D-2)}{9(-2^2) - 4} \sin 2x$$

$$= -e^x \frac{1}{D} [1 + D + \cdots] x + \frac{1}{40} (3D \sin 2x - 2 \sin 2x)$$

$$= -e^x \frac{1}{D} [x + 1] + \frac{1}{40} (6 \cos 2x - 2 \sin 2x)$$

$$= -e^x \int (x+1) dx + \frac{1}{20} (3 \cos 2x - \sin 2x)$$

$$= -e^x \left(\frac{x^2}{2} + x \right) + \frac{1}{20} (3 \cos 2x - \sin 2x)$$

Hence the complete solution is

$$y = C_1 e^x + C_2 e^{2x} - e^x \left(\frac{x^2}{2} + x \right) + \frac{1}{20} (3 \cos 2x - \sin 2x).$$

Ans to Q No 5: (a).Given, $\frac{d^2 y}{dx^2} + 4y = \tan 2x$

The auxiliary equation is $m^2 + 4 = 0$

or $m = \pm 2i$

C.F $= A \cos 2x + B \sin 2x$

P.I $= \frac{1}{D^2 + 4} \tan 2x$

$$= \frac{1}{4i} \left[\frac{1}{D - 2i} - \frac{1}{D + 2i} \right] \tan 2x$$

$$= \frac{1}{4i} \left[e^{2xi} \int e^{-2xi} \tan 2x dx - e^{-2xi} \int e^{2xi} \tan 2x dx \right]$$

$$= \frac{1}{4i} [(\cos 2x + i \sin 2x) \int (\cos 2x - i \sin 2x) \tan 2x dx -$$

$$(\cos 2x - i \sin 2x) \int (\cos 2x + i \sin 2x) \tan 2x dx]$$

$$= \frac{1}{4i} [(\cos 2x + i \sin 2x) \left\{ -\frac{1}{2} \cos 2x - \frac{1}{2} \log(\sec 2x + \tan 2x) + \frac{1}{2} \sin 2x \right\} -$$

$$(\cos 2x - i\sin 2x)\left\{-\frac{1}{2}\cos 2x + \frac{1}{2}\log\left(\sec 2x + \tan 2x - \frac{1}{2}\sin 2x\right)\right\}]$$

$$= \frac{1}{4i}\left[\frac{1}{2} - \frac{i}{2}e^{2ix}\log(\sec 2x + \tan 2x) - \frac{1}{2} - \frac{i}{2}e^{-2ix}\log(\sec 2x + \tan 2x)\right]$$

$$= \frac{1}{4i}\left[-\frac{i}{2}2\cos 2x\log(\sec 2x + \tan 2x)\right]$$

$$= -\frac{1}{4}\cos 2x\log(\sec 2x + \tan 2x)$$

The complete solution is

$$y = A\cos 2x + B\sin 2x - \frac{1}{4}\cos 2x\log(\sec 2x + \tan 2x).$$

(b) The given equation can be written as $(D^2 + 1)y = \sec^2 x$

Auxiliary equation is $m^2 + 1 = 0$

or $m = \pm i$

C.F $= C_1\cos x + C_2\sin x$

P.I $= \dfrac{1}{D^2 + 1}\sec^2 x$

$$= \frac{1}{(D - i)(D + i)}\sec^2 x$$

$$= \frac{1}{2i}\left[\frac{1}{D - i} - \frac{1}{D + i}\right]\sec^2 x \quad ...(1)$$

Now, $\dfrac{1}{D - i}\sec^2 x = e^{ix}\int e^{-ix}\sec^2 x dx$

$$= e^{ix}\int \frac{\cos x - i\sin x}{\cos^2 x}dx$$

$$= e^{ix}\int (\sec x - i\sec x\tan x)dx$$

$$= e^{ix}[\log(\sec x + \tan x) - i\sec x] \quad ... (2)$$

Replacing i by $-i$, we have

$$\frac{1}{D + i}\sec^2 x = e^{-ix}[\log(\sec x + \tan x) + i\sec x] \quad ... (3)$$

From (1), (2) and (3), we have

$$\text{P. I} = \frac{1}{2i}\left[e^{ix}\log(\sec x + \tan x) - ie^{ix}\sec x - e^{-ix}\log(\sec x + \tan x) - ie^{-ix}\sec x\right]$$

$$= \frac{e^{ix} - e^{-ix}}{2i}\log(\sec x + \tan x) - \frac{e^{ix} + e^{-ix}}{2}\sec x$$

$$= \sin x \log(\sec x + \tan x) - \cos x \sec x$$

$$= \sin x \log(\sec x + \tan x) - 1$$

The complete solution is

$$y = C_1 \cos x + C_2 \sin x + \sin x \log(\sec x + \tan x) - 1$$



11.7 MODEL QUESTION

Q 1 : Solve the following differential equations :

(a) $(D^4 - 2D^3 + 5D^2 - 8D + 4)y = 0$

(b) $(D^5 - 13D^3 + 26D^2 + 82D + 104)y = 0$

Q 2 : Solve the following differential equations :

(a) $(D^2 + 9)y = \sec 3x$

(b) $(D^2 - 9D + 18)y = e^{e^{-3x}}$

(c) $(D^3 - D^2 - 4D + 4)y = e^{3x}$

(d) $(D^2 - 2kD + k^2)y = e^{kx}$

(e) $(D^2 - a^2)y = \cos mx$

(f) $(D^2 + 1)y = \cos x \sin 3x$

(g) $(D^2 + 2D + 1)y = 2x + x^2$

(h) $(D^4 - 2D^3 + 5D^2 - 8D + 4)y = x^2$

(i) $(D^3 - 7D - 6)y = (x + 1)e^{2x}$

(j) $(D^2 - 1)y = e^x(1 + x^2)$

(k) $(D^2 - 5D + 6)y = x \sin 3x$

(l) $(D^2 + 5D + 6)y = e^{-2x} + 5 \sin 4x$

*** ***** ***

UNIT 12 : LINEAR HOMOGENEOUS DIFFERENTIAL EQUATION OF HIGHER ORDER

UNIT STUCTURE

12.1 Learning Objectives

12.2 Introduction

12.3 Linear Homogeneous Equation

12.3.1 Equation Reducible to Homogeneous form

12.4 Method of Variation of Parameters

12.5 Let Us Sum Up

12.6 Further Readings

12.7 Answers To Check Your Progress

12.8 Model Questions

12.1 LEARNING OBJECTIVES

After going through this unit, you will able to know

- linear homogeneous equation
- equation reducible to homogeneous form
- method of variation of parameters.

12.2 INTRODUCTION

In unit 11, we have discussed about solution of linear differential equation of higher order with constant coefficients. In this unit, we will discuss solution of linear homogeneous differential equations. We shall also discuss the method of variation of parametres for solving linear differential equation with constant coefficients.

12.3 LINEAR HOMOGENEOUS EQUATION

A differential equation of the form

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_2 x^2 \frac{d^2 y}{dx^2} + a_1 x \frac{dy}{dx} + a_0 y = X \quad \text{... (1)}$$

where $a_0, a_1, a_2, \cdots, a_{n-1}, a_n$ are constants and X is a function of x is called a homogeneous linear equation.

The equation is reducible to a linear equation with constants coefficients with the help of substitution $x = e^z$ which can be solved the methods described in the section 7.3.

Let $x = e^z$ i.e., $z = \log x$, then

$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x} \frac{dy}{dz}$ so that $x \frac{dy}{dx} = \frac{dy}{dz} = Dy$ where $D = \frac{d}{dz}$

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d^2 y}{dz^2}\frac{dz}{dx}$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2 y}{dz^2}$$

or $x^2 \frac{d^2 y}{dx^2} = -\frac{dy}{dz} + \frac{d^2 y}{dz^2} = \left(D^2 - D\right)y = D(D-1)y$

Similarly, $x^3 \frac{d^3 y}{dx^3} = \left(D^2 - D\right)y = D(D-1)(D-2)y$

and in general $x^n \frac{d^n y}{dx^n} = D(D-1)(D-2)\cdots(D-n+1)y$

Subtituting for $x\frac{dy}{dx}, x^2\frac{d^2 y}{dx^2}$,... the equation (1) reduces to a linear equation with constants coefficients which is solvable by the methods already discussed.

Example 1 : Solve $x^2 \frac{d^2 y}{dx^2} - x\frac{dy}{dx} + y = \log x$

Solution : Let $x = e^z$ i.e., $z = \log x$

$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x} \frac{dy}{dz}$ so that $x \frac{dy}{dx} = \frac{dy}{dz} = Dy$ where $D = \frac{d}{dz}$

so, $x^2\frac{d^2y}{dx^2} = -\frac{dy}{dz} + \frac{d^2y}{dz^2} = \left(D^2 - D\right)y = D(D-1)y$

The given equation reduces

$$\left(D(D-1) - D + 1\right)y = z$$

or $(D-1)^2 = y$

C.F $= (Az + B)e^z = (A\log x + B)x$

P.I $= \frac{1}{(D-1)^2}z$

$= (1-D)^{-2}z$

$= (1+2D)z$

$= z + 2$

$= \log x + 2$

The general solution is $y = (A\log x + B)x + \log x + 2$.

Example 2 : Solve $x^3\frac{d^3y}{dx^3} + 3x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = x^3\log x$

Solution : Let $x = e^z$ i.e., $z = \log x$

$$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz}$$ so that $x\frac{dy}{dx} = \frac{dy}{dz} = Dy$ where $D = \frac{d}{dz}$

so, $x^2\frac{d^2y}{dx^2} = -\frac{dy}{dz} + \frac{d^2y}{dz^2} = \left(D^2 - D\right)y = D(D-1)y$

The given equation reduces

$[D(D-1)(D-2) + 3D(D-1) + D)]y = e^{3z}z$

or $D^3y = e^{3z}z$

C.F $= \left(A + Bz + Cz^2\right)e^{0.z} = A + B\log x + C(\log x)$

P.I $= \frac{1}{D^3}\left(e^{3z}z\right)$

$= e^{3z}\frac{1}{(D+3)^3}z$

$= e^{3z}\left(1 + \frac{D}{3}\right)^{-3}z$

$= \frac{e^{3z}}{27}(1-D)z$

$$= \frac{e^{3z}}{27}(z-1)$$

$$= \frac{x^3}{27}(\log x - 1)$$

The general solution is $y = A + B\log x + C(\log x)^2 + \frac{x^3}{27}(\log x - 1)$.

12.3.1 Equations Reducible to Homogeneous form

An equation of the form

$$(a+bx)^n \frac{d^n y}{dx^n} + P_1(a+bx)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + P_{n-1}(a+bx)\frac{dy}{dx} + P_n y = X$$

where $P_1, P_2, \cdots, P_n$ are all constants,can be reduced to the homogeneous linear form by putting $a+bx = e^z$ and can be solved by the method discussed in 7.4.

Example 3 : Solve $(1+x)^2\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 2\sin\log(1+x)$

Solution : Let $1+x = e^z$ i.e., $z = \log(1+x)$

$$\frac{dy}{dx} = \frac{dy}{dz}.\frac{1}{1+x} \text{ or } (1+x)\frac{dy}{dx} = \frac{dy}{dz} = Dy$$

similarly, $(1+x)^2\frac{d^2y}{dx^2} = D(D-1)y$

The given equation becomes

$$D(D-1)y + Dy + y = 2\sin z$$

or $(D^2+1)y = 2\sin z$

Auxiliary equation is $m^2 + 1 = 0$

or $m = \pm i$

C.F $= A\cos z + B\sin z = A\cos\log(1+x) + B\sin\log(1+x)$

$$\text{P.I} = \frac{1}{D^2+1}(2\sin z)$$

$$= 2\text{Im}\left(\frac{1}{D^2+1}e^{iz}\right)$$

$$= 2\text{Im}\left(e^{iz}\frac{1}{(D+i)^2+1}1\right)$$

$$= 2\mathrm{Im}\left(e^{iz}\frac{1`}{D^2+2iD}1\right)$$

$$= 2\mathrm{Im}\left(e^{iz}\frac{1}{2iD}\left(1+\frac{D}{2i}\right)^{-1}1\right)$$

$$= 2\mathrm{Im}\left(e^{iz}\frac{1}{2iD}1\right)$$

$$= 2\mathrm{Im}\left(e^{iz}\frac{z}{2i}\right)$$

$$= \mathrm{Im}\left(\frac{z\cos z+2i\sin z}{i}\right)$$

$$= -z\cos z$$

$$= -\log(1+x)\cos\log(1+x)$$

The general solution is

$$y = A\cos\log(1+x)+B\sin\log(1+x)-\log(1+x)\cos\log(1+x).$$



CHECK YOUR PROGRESS - 1

Q 1 : Solve the following differential equations :

(a) $x^3\frac{d^3y}{dx^3}+2x^2\frac{d^2y}{dx^2}+2y=10\left(x+\frac{1}{x}\right)$

(b) $\left(x^2D^2-3xD+4\right)y=2x^2$

(c) $\left(x^2D^2-xD+4\right)y=\cos(\log x)+x\sin(\log x)$

(d) $(x+a)^2\frac{d^2y}{dx^2}-4(x+a)\frac{dy}{dx}+6y=x$

12.4 METHOD OF VARIATION OF PARAMETERS

Theorem : For the equation $y''+Py'+Qy=X$ where P,Q and X being functions of x, the P.I is given by $-y_1\int\frac{y_2X}{W}dx+y_2\int\frac{y_1X}{W}dx$. Here y_1 and y_2 are solutions of $y''+Py'+Qy=0$ and $W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$, the Wronskian of y_1, y_2 named after the Polish mathematician and philospher Hoene Wronsky.



Hoene Wronsky (1776-1853) was a Polish mathematician and philospher

Proof : Let the complementary function of $y'' + Py' + Qy = X$ be $y = c_1 y_1 + c_2 y_2$. Here c_1 and c_2 can be regarded as parametric functions $u(x)$, $v(x)$.

Let the P.I be $y = u(x)y_1 + v(x)y_2$

Differentiating w.r.t x, we get

$$y' = uy_1' + u'y_1 + vy_2' + v'y_2$$

Choose u & v such that $u'y_1 + v'y_2 = 0$ so that

$$y' = uy_1' + vy_2'$$

and $y'' = u'y_1' + uy_1'' + v'y_2' + vy_2''$

Since y_1, y_2 are roots of $y'' + Py' + Qy = 0$, we have

$$y_1'' + Py_1' + Qy_1 = 0$$

and $$y_2'' + Py_2' + Qy_2 = 0$$

The given equation becomes

$$u'y_1' + uy_1'' + v'y_2' + vy_2'' + P\left(uy_1' + vy_2'\right) + Qy = X$$

or $$u(y_1'' + Py_1' + Qy_1) + v(y_2'' + Py_2' + Qy_2)$$
$$-Q\left(uy_1 + vy_2\right) + u'y_1' + v'y_2' + Qy = X$$

or $$u'y_1' + v'y_2' - Q\left(uy_1 + vy_2\right) + Qy = X$$

or $$u'y_1' + v'y_2' - Qy + Qy = X \quad (\ y = uy_1 + vy_2)$$

or $$u'y_1' + v'y_2' - X = 0 \quad \ldots (1)$$

Also, $u'y_1 + v'y_2 + 0.X = 0$ ... (2)

Solving (1) and (2), we get

$$\frac{u'}{y_2} = \frac{v'}{-y_1} = \frac{X}{y_1'y_2 - y_1y_2'} = \frac{X}{-W}$$

Hence $u' = \frac{X}{-W}y_2$ and $v' = \frac{X}{W}y_1$

i.e., $u = -\int \frac{y_2 X}{W}dx$ and $v = \int \frac{y_1 X}{W}dx$

and P. I $= -y_1\int \frac{y_2 X}{W}dx + y_2\int \frac{y_1 X}{W}dx$

Hence proved.

Example 4 : Solve $\frac{d^2y}{dx^2} + 4y = \tan 2x$

Solution : Auxiliary equation is $m^2+4=0$

or $m=\pm 2i$

C.F $=A\cos 2x+B\sin 2x$

$=Ay_1+By_2$ where $y_1=\cos 2x$ and $y_2=\sin 2x$ and $X=\tan 2x$.

Wronskian, $W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=\begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}=2$.

$$\text{P. I}=-y_1\int\frac{y_2X}{W}dx+y_2\int\frac{y_1X}{W}dx$$

$$=-\cos 2x\int\frac{\sin 2x\tan 2x}{2}dx+\sin 2x\int\frac{\cos 2x\tan 2x}{2}dx$$

$$=-\frac{\cos 2x}{2}\int\frac{1-\cos^2 2x}{\cos 2x}dx+\sin 2x\int\frac{\sin 2x}{2}dx$$

$$=-\frac{1}{2}\cos 2x\int(\sec 2x-\cos 2x)dx-\frac{1}{4}\sin 2x\cos 2x$$

$$=-\frac{1}{2}\cos 2x\left\{\frac{1}{2}\log(\sec 2x+\tan 2x)-\frac{1}{2}\sin 2x\right\}-\frac{1}{4}\sin 2x\cos 2x$$

$$=-\frac{1}{4}\cos 2x\left\{\frac{1}{2}\log(\sec 2x+\tan 2x)-\sin 2x\right\}-\frac{1}{4}\sin 2x\cos 2x$$

$$=-\frac{1}{4}\cos 2x\log(\sec 2x+\tan 2x)$$

The general solution is

$$y=A\cos 2x+B\sin 2x-\frac{1}{4}\cos 2x\log(\sec 2x+\tan 2x).$$

Example 5 : Solve $\frac{d^2y}{dx^2}+a^2y=\sec ax$

Solution: Auxiliary equation is $m^2+a^2=0$

or $m=\pm ia$

C.F $=A\cos ax+B\sin ax$

Let $y_1=\cos ax$ and $y_2=\sin ax$

Wronskian, $W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=\begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix}=a$ and $X=\sec ax$.

$$\text{P. I}=-y_1\int\frac{y_2\sec ax}{a}dx+y_2\int\frac{y_1\sec ax}{a}dx$$

$$= -\frac{1}{2}\cos ax\int \frac{\sin ax}{\cos ax}dx + \frac{\sin ax}{a}\int dx$$

$$= \frac{1}{a^2}\cos ax \log\cos ax + \frac{x}{a}\sin ax$$

The general solution is

$$y = A\cos ax + B\sin ax + \frac{1}{a^2}\cos ax\log\cos ax + \frac{x}{a}\sin ax.$$

Example 6 : Solve by the method of variation of parameters :

$$y'' - 2y' + y = e^x \log x$$

Solution : The given equation can be written as

$$\left(D^2 - 2D + 1\right)y = e^x \log x$$

The auxiliary equation is $m^2 - 2m + 1 = 0$

or $(m-1)^2 = 0$

or $m = 1,1$

C. F $= (Ax + B)e^x$

Let $y_1 = e^x, y_2 = xe^x, X = e^x$ and $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix}$

$$= e^{2x}\begin{vmatrix} 1 & x \\ 1 & 1+x \end{vmatrix} = e^{2x}$$

P. I $= -y_1\int \frac{y_2 X}{W}dx + y_2\int \frac{y_1 X}{W}dx$

$$= -e^x\int \frac{xe^{2x}\log x}{e^{2x}}dx + xe^x\int \frac{e^x . e^x \log x}{e^{2x}}dx$$

$$= -e^x\int x\log x dx + xe^x\int \log x dx$$

$$= -e^x\left\{\log x.\frac{x^2}{2} - \int \frac{1}{x}.\frac{x^2}{2}dx\right\} + xe^x\left\{x\log x - \int dx\right\}$$

$$= -e^x\left\{\frac{x^2}{2}\log x - \frac{x^2}{4}\right\} + xe^x(\log x - 1)$$

$$= \frac{1}{4}x^2e^x\{2\log x + 1 + 4(\log x - 1)\}$$

$$= \frac{1}{4}x^2e^x(2\log x - 3)$$

The general solution is

$$y = (Ax+B)e^x + \frac{1}{4}x^2e^x(2\log x - 3).$$

Example 7 : Solve by method of varation of parameters :

$$\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$$

Solution : Given, $\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$

A. E is $m^2 - 1 = 0$

or $m^2 = 1$

or $m = \pm 1$

C.F $= C_1e^x + C_2e^{-x}$

and $y^1.y_2^{/} - y_1^{/}.y_2 = -e^x.e^{-x} - e^x.e^{-x} = -2$

$$u = \int \frac{-y^2X}{y_1.y_2^{/} - y_1^{/}.y_2}dx = -\int \frac{e^{-x}}{-2} \times \frac{2}{1+e^x}dx$$

$$= \int \frac{e^{-x}}{1+e^x}dx$$

$$= \int \frac{dx}{e^x(1+e^x)}$$

$$= \int \left(\frac{1}{e^x} - \frac{1}{1+e^x}\right)dx$$

$$= \int e^{-x}dx - \int \frac{e^{-x}}{e^{-x}+1}dx$$

$$= -e^{-x} + \log(e^{-x}+1)$$

$$v = \int \frac{y_1X}{y_1.y_2^{/} - y_1^{/}.y_2}dx = \int \frac{e^{-x}}{-2} \times \frac{2}{1+e^x}dx$$

$$= \int \frac{e^x}{e^x+1}dx$$

$$= -\log(e^x+1)$$

P.I $= u.y_1 + v.y_2$

$$= \left[-e^{-x} + \log(e^{-x}+1)\right]e^x - e^{-x}\log(1+e^x)$$

$$= -1 + e^x\log(e^{-x}+1) - e^{-x}\log(1+e^x)$$

The general solution is

$$y = C_1 e^x + C_2 e^{-x} - 1 + e^x \log\left(e^{-x}+1\right) - e^{-x}\log\left(1+e^x\right).$$



CHECK YOUR PROGRESS - 2

Q 1 : Apply the method of variation of parameters to solve.

(i) $y_2 + a^2 y = \cot ax$

(ii) $y_2 - 2y_1 + y = xe^x \log x, x > 0$

(iii) $\left(D^2 - 2D + 2\right)y = e^x \tan x$

(iv) $y^{//} + 3y^{/} + 2y = x + \cos x$



12.5 LET US SUM UP

- A differential equation of the form

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_2 x^2 \frac{d^2 y}{dx^2} + a_1 x \frac{dy}{dx} + a_0 y = X$$

...(1)

where $a_0, a_1, a_2, \cdots, a_{n-1}, a_n$ are constants and X is a function of x is called a homogeneous linear equation.

The equation is reducible to a linear equation with constants coefficients with the help of substitution $x = e^z$ which can be solved the methods described in the section 12.3.

Let $x = e^z$ i.e., $z = \log x$

$$x\frac{dy}{dx} = \frac{dy}{dz} = Dy$$

$$x^2 \frac{d^2 y}{dx^2} = -\frac{dy}{dz} + \frac{d^2 y}{dz^2} = \left(D^2 - D\right)y = D(D-1)y$$

Similarly, $x^3 \dfrac{d^3 y}{dx^3} = \left(D^2 - D\right)y = D(D-1)(D-2)y$

and in general $x^n \dfrac{d^n y}{dx^n} = D(D-1)(D-2)\cdots(D-n+1)y$

The substitution of these values in the given equation reduces the given homogeneous equation to a differential equation with constants coefficients.

- Method of variation of parameter to find particular integral of

$P\frac{d^2y}{dx^2}+Q\frac{dy}{dx}+Ry=X$ is as follows :

Let complementary function(C.F) $=Ay_1+By_2$

then particular integral(P.I) $=uy_1+vy_2$

where $u=\int\frac{-y_2X}{W}dx$ and $v=\int\frac{y_1X}{W}dx$, $W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$.

General solution =C.F +P.I.



12.6 FURTHER READINGS

1) Tenenbaum, Morris and Pollard, Harry. Ordinary Differential Equations, Dover Publications, Inc, N.Y 11501.

2) Differential Equations: An Introduction with Mathematica. Clay C.Ross, second Edition, Springer.

3) Differential Equations: Linear, Nonlinear, ordinary, partial. A.C. King, J.Billinggham and S. R. Otto, Cambridge-2003.

4) Differential equations with Application and Programs. S. Balachandra Rao and H.R. Anuradha, University Press (India) Limited 1996.



12.7 ANSWERS TO CHECK YOUR PROGRESS

CHECK YOUR PROGRESS – 1

Ans to Q No 1 : (b) Given that $\left(x^2D^2-3xD+4\right)y=2x^2$ ... (1)

Let $x=e^z$ or $z=\log x$ and $D_1=\frac{d}{dz}$.

Then (1) becomes

$\{D_1(D_1-1)-3D_1+4\}y = 2e^{2z}$

or $(D_1-2)^2 y = 2e^{2z}$

Auxiliary equation is $(m-2)^2 = 0$

or $m = 2,2$

C. F $= (C_1 + C_2 z)e^{2z} = (C_1 + C_2 z)(e^z)^2 = (C_1 + C_2 \log x)x^2$

P. I $= \dfrac{1}{(D_1-2)^2} 2e^{2z}$

$= 2\dfrac{z^2}{2!}e^{2z} = z^2(e^z)^2$

$= (\log x)^2 x^2$

Complete solution is

$y = (C_1 + C_2 \log x)x^2 + (\log x)^2 x^2$.

(c) Putting $x = e^z$ and denoting $\dfrac{d}{dz}$ by D_1, the given equation becomes

$[D_1(D_1-1)-D_1+4]y = \cos z + e^z \sin z$

or $\left[D_1^{\,2}-2D_1+4\right]y = \cos z + e^z \sin z$

Auxiliary equation is $m^2 - 2m + 4 = 0$

or $m = 1 \pm \sqrt{3}i$

C.F $= e^z\left[C_1 \cos \sqrt{3}z + C_2 \sin \sqrt{3}z\right]$

$= x\left[C_1 \cos \sqrt{3}(\log x) + C_2 \sin \sqrt{3}(\log x)\right]$

P.I $= \dfrac{1}{D_1^{\,2}-2D_1+4}\cos z + \dfrac{1}{D_1^{\,2}-2D_1+4}e^z \sin z$

$= \dfrac{1}{-1^2-2D_1+4}\cos z + e^z \dfrac{1}{(D_1+1)^2-2(D_1+1)+4}\sin z$

$= \dfrac{1}{3-2D_1}\cos z + e^z \dfrac{1}{D_1^{\,2}+4}\sin z$

$= \dfrac{(3+2D_1)}{9-4D_1^{\,2}}\cos z + e^z \dfrac{1}{-1^2+3}\sin z$

$= \dfrac{3\cos z - 2\sin z}{9+4} + \dfrac{e^z \sin z}{2}$

$$= \frac{1}{13}\left[3\cos(\log x) - 2\sin(\log x)\right] + \frac{x}{2}\sin(\log x)$$

Hence the complete solution is

$$y = x\left[C_1\cos\sqrt{3}(\log x) + C_2\sin\sqrt{3}(\log x)\right] + \frac{1}{13}\left[3\cos(\log x) - 2\sin(\log x)\right] + \frac{x}{2}\sin(\log x)$$

(d) Putting $x + a = e^z$ and denoting $\frac{d}{dx}$ by D ,the equation becomes

$$\left[D(D-1) - 4D + 6\right]y = e^z - a$$

or $\left[D^2 - 5D + 6\right]y = e^z - a$

Auxiliary equation is $m^2 - 5m + 6 = 0$

or $m = 3, 2$

C.F $= C_1e^{3z} + C_2e^{2z} = C_1(x+a)^3 + C_2(x+a)^2$

P.I $= \frac{1}{D^2 - 5D + 6}.e^z - \frac{1}{D^2 - 5D + 6}.a$

$$= \frac{1}{(D-3)(D-2)}.e^z - \frac{1}{(D-3)(D-2)}.a$$

$$= \frac{e^z}{(1-3)(1-2)} - a\frac{1}{(D-3)(D-2)}e^{0z}$$

$$= \frac{1}{3}e^z - a\frac{1}{(0-3)(0-2)}e^{0z}$$

$$= \frac{1}{3}e^z - a\frac{1}{6}e^{0z}$$

$$= \frac{1}{3}e^z - \frac{1}{6}a$$

$$= \frac{1}{3}(x+a) - \frac{1}{6}a$$

Hence, the complete solution is

$$y = C_1(x+a)^3 + C_2(x+a)^2 + \frac{1}{3}(x+a) - \frac{1}{6}a$$

CHECK YOUR PROGRESS – 2

Ans to Q No 1 : (i) Given, $y_2 + a^2y = \cot ax$ ... (1)

Auxiliary equation is $m^2 + a^2 = 0$

or $m = \pm ia$

C.F $= C_1\cos ax + C_2\sin ax$

Here $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix} = a \neq 0$

Then P. I $= u.y_1 + v.y_2$

where $u = -\int \frac{y_2.X}{W}dx = -\int \frac{\sin ax \cot ax}{a}dx = -\frac{1}{a}\int \cos axdx = -\frac{\sin ax}{a^2}$

and $v = \int \frac{y_1.X}{W}dx = \int \frac{\cos ax \cot ax}{a}dx = \frac{1}{a}\int \frac{1-\sin^2 ax}{\sin ax}dx$

$$= \frac{1}{a}\int (\operatorname{cosec} ax - \sin ax)dx$$

$$= \frac{1}{a}\int (\operatorname{cosec} ax - \sin ax)dx = \frac{1}{a^2}\left(\log \tan \frac{ax}{2} + \cos ax\right)$$

P. I $= \cos ax \times \left(-\frac{1}{a^2}\right) \times \frac{1}{a^2} \times \sin ax + \sin ax \times \frac{1}{a^2} \times \left(\log \tan \frac{ax}{2} + \cos ax\right)$

$$= \frac{1}{a^2} \times \log \tan\left(\frac{ax}{2}\right)$$

Hence, the general solution of (1) is y = C.F+P.I.

i.e., $y = C_1 \cos ax + C_2 \sin ax + \frac{1}{a^2} \times \log \tan\left(\frac{ax}{2}\right)$.

(ii) Given, $(D^2 - 2D + 1)y = xe^x \log x, x > 0$, $D = \frac{d}{dx}$

Auxiliary equation is $(m-1)^2 = 0$

or $m = 1,1$

C.F $= (C_1 + C_2 x)e^x = C_1 e^x + C_2 xe^x$

Let $y_1 = e^x$, $y_2 = xe^x$. Also, $X = xe^x \log x$

Here, $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^{2x} \neq 0$

Then, P. I $= u.y_1 + v.y_2$

where $u = -\int \frac{y_2.X}{W}dx = -\int \frac{xe^x.xe^x \log x}{e^{2x}}dx = -\int x^2 \log xdx$

$$= -\left[\log x.\frac{x^3}{3} - \int \frac{1}{x}.\frac{x^3}{3}dx\right]$$

$$= -\left[\log x.\frac{x^3}{3} - \frac{1}{9}x^3\right]$$

and $v = \int \frac{y_1.X}{W}dx = \int \frac{e^x.xe^x \log x}{e^{2x}}dx = \int x \log x dx$

$$= \frac{x^2}{2}\log x - \int \frac{x^2}{2}.\frac{1}{x}dx$$

$$= \frac{x^2}{2}\log x - \frac{x^2}{4}$$

P.I $= -e^x\left\{\frac{x^3}{3}\log x - \frac{x^3}{9}\right\} + xe^x\left\{\frac{x^2}{2}\log x - \frac{x^2}{4}\right\}$

$$= x^3e^x \log x\left(\frac{1}{2} - \frac{1}{3}\right) - x^3e^x\left(\frac{1}{4} - \frac{1}{9}\right)$$

$$= \left(\frac{1}{6}\right) \times x^3e^x \log x - \left(\frac{5}{36}\right) \times x^3e^x$$

Hence, the general solution is y = C.F+P.I.

$$\therefore\ y = C_1e^x + C_2xe^x + \left(\frac{1}{6}\right) \times x^3e^x \log x - \left(\frac{5}{36}\right) \times x^3e^x.$$

(iii) Do Yourself.

The required solution is

$$y = e^x(C_1 \cos x + C_2 \sin x) - e^x \cos x \log(\sin x + \tan x)$$

(iv) Do youself.

The required solution is

$$y = C_1e^{-x} + C_2e^{-2x} + \frac{x}{2} - \left(\frac{3}{4}\right) + \left(\frac{1}{10}\right) \times (3\sin x - \cos x).$$



12.8 MODEL QUESTIONS

Q 1 : Solve the following differential equations

(a) $(3x+2)^2\frac{d^2y}{dx^2} + 5(3x+2)\frac{dy}{dx} - 3y = x^2 + x + 1$

(b) $x^3\frac{d^3y}{dx^3}+3x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=\log x+x$

(c) $\left(x^3D^3-3x^2D^2+6xD-6\right)y=(\log x)^2$

(d) $(x+1)^2\frac{d^2y}{dx^2}-3(x+1)+4y=x$

(e) $\left\{(1+2x)^2D^2-6(1+2x)D+16\right\}y=8(1+2x)^2$

Q 2 : Solve the following differential equations by the method of variation of parameters :

(a) $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=2x$

(b) $\frac{d^2y}{dx^2}+y=\operatorname{cosec}x\cot x$

(c) $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+3y=x+\sin x$

(d) $\left(D^2-9\right)y=e^{2x}+x$

(e) $\frac{d^2y}{dx^2}+y=\sin^2 x$

*** ***** ***

UNIT 13 : PARTIAL DIFFERENTIAL EQUATION

UNIT STUCTURE

13.1 Learning Objectives

13.2 Introduction

13.3 Formation of Partial Differential Equation

13.3.1 Formation of Partial Differential Equation by Elimination of Arbitrary Constants

13.3.2 Formation of Partial Differential Equation by Elimination of Arbitrary Functions

13.4 Let Us Sum Up

13.5 Further Readings

13.6 Answers To Check Your Progress

13.7 Model Questions

13.1 LEARNING OBJECTIVES

After going through this unit, you will able to learn

- how to form a partial differential equation
- solution of first order and first degree partial differential equation.

13.2 INTRODUCTION

Learners are already introduced to partial derivative. Here, we shall discuss the formation of partial differential equations by eliminating either arbitrary constants or arbitary functions.

A differential equation which involves partial derivatives is called a partial differential equation. So, in partial differential equations there are more than one independent variables.

Examples :

$$\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=1 \qquad \ldots (1)$$

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0 \qquad \ldots (2)$$

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z \qquad \ldots (3)$$

$$\frac{\partial^2 z}{\partial x\partial y}=xy^2 \qquad \ldots (4)$$

Order and degree of partial differential equation :

The **order** of a partial differential equation is the order of the highest ordered derivative present in the equation.

The **degree** of a partial differential equation is the degree of the highest order partial derivative occurring in the equation.

Example : (1) $\frac{\partial z}{\partial x}+\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial x\partial y}=xy^2$ is of order two and degree one.

(2) $\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=y^2$ is of order one and degree one.

Usually, x and y are treated as independent variables and z as dependent variable so that $z=f(x,y)$.

$\frac{\partial z}{\partial x}$ means derivative of z w.r.t x treating y as constant.

For example, if $z=ax^2+2hxy+by^2$, then $\frac{\partial z}{\partial x}=2ax+2hy$.

$\frac{\partial z}{\partial y}$ means derivative of z w.r.t y treating x as constant.

Note : Throughout we shall follow the following symbols

$$p=\frac{\partial z}{\partial x},\ q=\frac{\partial z}{\partial y},\ r=\frac{\partial^2 z}{\partial x^2},\ s=\frac{\partial^2 z}{\partial x\partial y},\ t=\frac{\partial^2 z}{\partial y^2}.$$

Linear and Nonlinear Partial Differential Equations :

A partial differential equation is called a linear partial differential equation if z (the dependent variable) and all its partial derivatives appear in the first degree and product of z and its derivatives donot appear in the equation.

A partial differential equation which is not linear is called a nonlinear partial differential equation.

8.3 FORMATION OF PARTIAL DIFFERENTIAL EQUATION

An ordinary differential equation can be formed by eliminating arbitrary constants.

A partial differential equation can be formed by the elimination of arbitrary constants or arbitrary functions.

13.3.1 Formation of partial differential equation by elimination of arbitrary constants

Let $f(x, y, z, a, b) = 0$ ... (1) be an equation involving two arbitrary constants a and b.

Differentiating (1) partially with respect to x and y ,we get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = 0$$

or $$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}p = 0 \quad \text{... (2)}$$

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} = 0$$

or $$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}q = 0 \quad \text{... (3)}$$

Eliminating a and b from (1), (2), (3), we get an equation of the form

$$F(x, y, z, p, q) = 0$$

which is a partial differential equation of first order.

Note :

(1) If the number of arbitrary constants to be eliminated is less than the number of independent variables, then the elimination of arbitrary constants give rise to more than one partial differential equation of order one.

(2) If the number of arbitrary constants to be eliminated is equal to the number of independent variables, the partial differential that forms is of first order.

(3) If the number of arbitrary constants to be eliminated is more than the number of independent variables, the partial differential equation so obtained is of second or higher order.

Example 1 : Form partial differential equations by eliminating arbitrary constants from the following equations :

(a) $z = a(x+y)+b$ (b) $z = ax + a^2y^2 + b$

Solution : (a) Given $z = a(x+y)+b$ ... (1)

Differentiating (1) partially with respect to x and y, we get

$$\frac{\partial z}{\partial x} = a \quad \text{... (2)} \quad \text{and} \quad \frac{\partial z}{\partial y} = a \quad \text{... (3)}$$

Eliminating a between these, we get

$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}$, which is the required partial differential equation.

(b) Given, $z = ax + a^2y^2 + b$ ... (1)

Differentiating (1) partially with respect to x and y, we get

$$\frac{\partial z}{\partial x} = a \quad \text{... (2)} \quad \text{and} \quad \frac{\partial z}{\partial y} = 2a^2y \quad \text{... (3)}$$

putting the value of a from (2) in (3), we get

$\frac{\partial z}{\partial y} = 2\left(\frac{\partial z}{\partial x}\right)^2 y$, which is the required differential equation.

Example 2 : Form the partial differential equation from

$(x-a)^2 + (y-b)^2 + z^2 = r^2$ where a and b are arbitrary constants

Solution : Differentiating the given equation partially w.r.t x and y respectively

we have $2(x-a) + 2zp = 0$ ... (1)

and $2(y-b) + 2zq = 0$ ... (2)

From (1), $x - a = -zp$

From (2), $y - b = -zq$

Putting the values of $(x-a)$ and $(y-b)$ in the given equation,we have

$$(-zp)^2 + (-zq)^2 + z^2 = r^2$$

or $z^2(p^2 + q^2 + 1) = r^2$

which is the required equation.

Example 3 : Eliminate a and b from $z = axe^y + \frac{1}{2}a^2e^{2y} + b$ and form a partial differential equation.

Solution : Given, $z = axe^y + \frac{1}{2}a^2e^{2y} + b$ ... (1)

Differentiating (1) partially with respect to x and y, we get

$$\frac{\partial z}{\partial x} = ae^y \quad \text{... (2)}$$

and $\frac{\partial z}{\partial y} = axe^y + a^2e^{2y} = x\left(ae^y\right) + \left(ae^y\right)^2$ ... (3)

substituting the value of ae^y from (2) in (3), we get

$\frac{\partial z}{\partial y} = x\left(\frac{\partial z}{\partial x}\right) + \left(\frac{\partial z}{\partial x}\right)^2$,which is the required equation.

Example 4 : Form a partial differential equation by eliminating the arbitrary constants a and b from $\log(az-1) = x + ay + b$.

Solution : Given, $\log(az-1) = x + ay + b$ ... (1)

Differentiating (1) partially w.r.t. 'x', we get

$$\frac{a}{az-1}\frac{\partial z}{\partial x} = 1 \quad \text{... (2)}$$

Differentiating (1) partially w.r.t. 'y', we get

$$\frac{a}{az-1}\frac{\partial z}{\partial y} = a \quad \text{... (3)}$$

From (3), $az - 1 = \frac{\partial z}{\partial y}$ so that $a = \frac{1 + \frac{\partial z}{\partial y}}{z}$ ... (4)

putting the above values of $az-1$ and a in (2), we have

$$\frac{1 + \left(\frac{\partial z}{\partial y}\right)}{z\left(\frac{\partial z}{\partial y}\right)} \cdot \frac{\partial z}{\partial x} = 1$$

or $1 + \left(\frac{\partial z}{\partial y}\right)\frac{\partial z}{\partial x} = z\left(\frac{\partial z}{\partial y}\right)$ which is the required equation.



CHECK YOUR PROGRESS - 1

Q 1 : Form partial differential equations from the following by eliminating arbitrary constants :

(i) $z = ax + by + a^2 + b^2$

(ii) $z = a(x+y) + b(x-y) + abt + c$

(iii) $z = (x+a)(y+b)$

(iv) $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

Q 2 : Find the partial differential equations of all spheres of fixed radius having their centres in the xy -plane.

Q 3 : Find the partial differential equations of all spheres whose centres lie on the z-axis.

13.3.2 Formation of partial differential equation by elimination by arbitrary functions

(a) **One arbitrary function**

Let $z = f(u)$ ... (1) where $f(u)$ is an arbitrary function of u and u is a given function of x, y, z i.e., $u = u(x, y, z)$.

Differentiating (1) partially w.r.t. x and y by chain rule

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial z}\frac{\partial z}{\partial x} \quad \text{... (2)}$$

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial z}\frac{\partial z}{\partial y} \quad \text{... (3)}$$

By eliminating the arbitrary function $f(u)$ from (1),(2),(3), we get a partial differential equation of first order.

(b) **Two arbitrary functions :**

Differentiating twice or more, the elimination process results in a partial differential equation of 2nd or higher order.

Note :

(1) If the partial differential equation is obtained by elimination of arbitrary functions, then the order of the partial differential equation, in general, is equal to the number of arbitrary functions eliminated.

(2) When n is the number of arbitrary functions, one may get several partial differential equations.

Example 5 : Find the partial differential equations by eliminating the arbitrary functions from the following :

(i) $z = f\left(x^2 - y^2\right)$

Solution : Differentiating z partially w.r.t. to x and y,

$$p = \frac{\partial z}{\partial x} = f'\left(x^2 - y^2\right)2x \quad \text{... (1)}$$

and $$q = \frac{\partial z}{\partial y} = f'\left(x^2 - y^2\right)(-2y) \quad \text{... (2)}$$

Dividing (1) by (2),

$$\frac{p}{q} = -\frac{x}{y}$$

or $py + qx = 0$ which is the required equation.

(ii) $z = yf(x) + xg(y)$

Differentiating z partially w.r.t. to x and y respectively, we get

$$p = \frac{\partial z}{\partial x} = yf'(x) + g(y)$$

$$q = \frac{\partial z}{\partial y} = f(x) + xg'(y)$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = f'(x) + g'(y)$$

$$\therefore \quad xys = xy\left(f'(x) + g'(y)\right) \quad \text{... (1)}$$

Also $px + qy - z = \left(yf'(x) + g(y)\right)x + \left(f(x) + xg'(y)\right)y - z$

$$= xyf'(x) + xg(y) + yf(x) + xyg'(y) - yf(x) - xg(y)$$

$$= xy\left(f'(x) + g'(y)\right) \quad \text{... (2)}$$

From (1) and (2)

which is the required differential equation.

Example 6 : Find partial differential equation from

$$z = f(x + at) + g(x - at) \quad \text{... (i)}$$

Solution : Differentiating (i) partially w.r.t. x and t,

$$\frac{\partial z}{\partial x} = f'(x + at) + g'(x - at) \quad \text{.... (ii)}$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x+at) + g''(x-at) \quad \text{... (iii)}$$

$$\frac{\partial z}{\partial t} = af'(x+at) - ag'(x-at)$$

$$\frac{\partial^2 z}{\partial t^2} = a^2\{f''(x+at) + g''(x-at)\} \quad \text{... (iv)}$$

$$= a^2 \frac{\partial^2 z}{\partial x^2}$$

The required differential equation is

$$\frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2}.$$

Example 7: Form the PDE by eliminating the arbitrary function from

$$z = f\left(x^2 + y^2 + z^2\right)$$

Solution : Differentiating partially w.r.t. x and y, we have

$$\frac{\partial z}{\partial x} = f'\left(x^2 + y^2 + z^2\right) \times \left(2x + 2z\frac{\partial z}{\partial x}\right) \quad \text{... (i)}$$

$$\frac{\partial z}{\partial y} = f'\left(x^2 + y^2 + z^2\right) \times \left(2y + 2z\frac{\partial z}{\partial y}\right) \quad \text{... (ii)}$$

Dividing (i) by (ii),

$$\frac{\frac{\partial z}{\partial x}}{\frac{\partial z}{\partial y}} = \frac{\left(2x + 2z\frac{\partial z}{\partial x}\right)}{\left(2y + 2z\frac{\partial z}{\partial y}\right)}$$

or $\frac{\partial z}{\partial x} 2\left(y + z\frac{\partial z}{\partial y}\right) = 2\left(x + z\frac{\partial z}{\partial x}\right)\frac{\partial z}{\partial y}$

or $\frac{\partial z}{\partial x}\left(y + z\frac{\partial z}{\partial y}\right) = \frac{\partial z}{\partial y}\left(x + z\frac{\partial z}{\partial x}\right)$

or $y\frac{\partial z}{\partial x} + z\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} = x\frac{\partial z}{\partial y} + z\frac{\partial z}{\partial y}\frac{\partial z}{\partial x}$

or $y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 0$, which is the required equation.

Elimination of Arbitrary Function F

Let $F(u,v) = 0$ ... (1)

where $u = u(x, y, z)$ and $v = v(x, y, z)$ are given functions of x, y, z.

Differentiating (1) partially w.r.t. x by chain rule, we get

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right)+\frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right)=0 \quad \text{... (2)}$$

Similarly, differentiating (1) partially w.r.t. y by chain rule, we get

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial y}\right)+\frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\frac{\partial z}{\partial y}\right)=0 \quad \text{... (3)}$$

Eliminating $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v}$ from (2) and (3), we get

$$\begin{vmatrix} \frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial x} & \frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\frac{\partial z}{\partial x} \\ \frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial y} & \frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\frac{\partial z}{\partial y} \end{vmatrix}=0 \quad \text{.... (4)}$$

Now, expanding the determinant (4) results in a P.D.E. which is free of the arbitrary function F as

$$\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right)\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\frac{\partial z}{\partial y}\right)-\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial y}\right)\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right)=0$$

or $\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}\right)p+\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}\right)q=\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}\right)$

which is a first order linear P.D.F.

Example 8: Obtain the partial differential equation by eliminating arbitrary function from

$$f\left(x^2+y^2, z-xy\right)=0$$

Solution : Let $u = x^2 + y^2$ and $v = z - xy$ so that $f(u, v) = 0$

Differentiating partially w.r.t. x and y, we have

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}p\right)+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p\right)=0$$

or $\frac{\partial f}{\partial u}(2x)+\frac{\partial f}{\partial v}(-y+p)=0$ ... (1)

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q\right)+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q\right)=0$$

or $\frac{\partial f}{\partial u}(2y)+\frac{\partial f}{\partial v}(-x+q)=0$ ....(2)

Eliminating $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ from (1) and (2), we get

$$\begin{vmatrix} 2x & -y+p \\ 2y & -x+q \end{vmatrix}=0$$

or $2x(-x+q)-2y(-y+p)=0$

or $xq-yp=x^2-y^2$, which is the required PDE.

Example 9 : Obtain the partial differential equation by eliminating arbitrary function from

$$F(xy+z^2, x+y+z)=0$$

Solution : Let $u=xy+z^2$ and $v=x+y+z$ so that $F(u,v)=0$.

Differentiating partially w.r.t. x and y, we have

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}p\right)+\frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p\right)=0$$

or $\frac{\partial F}{\partial u}(y+2zp)+\frac{\partial F}{\partial v}(1+p)=0$ ... (1)

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q\right)+\frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q\right)=0$$

or $\frac{\partial F}{\partial u}(x+2zq)+\frac{\partial F}{\partial v}(1+q)=0$ ....(2)

Eliminating $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v}$ from (1) and (2), we get

$$\begin{vmatrix} y+2zp & 1+p \\ x+2zq & 1+q \end{vmatrix}=0$$

or $(1+q)(y+2zp)-(1+p)(x+2zq)=0$

or $(2z-x)p+(y-2z)q=x-y$



CHECK YOUR PROGRESS - 2

Q 1 : Form partial differential equations by eliminating arbitrary functions from the following :

(i) $z = f\left(\frac{y}{x}\right)$ (ii) $z = f(xy)$

(iii) $e^x f(y-x) = z$ (iv) $z = f(x^2 - y) + g(x^2 + y)$

(v) $z = f_1(2x+y) + f_2(x-2y)$ (vi) $f(xyz, x+y+z) = 0$



13.4 LET US SUM UP

- An equation containing one or more partial derivatives of an unknown function of two or more independent variables is known as a partial differential equation.
- The order of a partial differential equation is defined as the order of the highest partial derivative occuring in the partial differential equation.
- The degree of a partial differential equation is the degree of the highest order derivative which occurs in it after the equation has been rationalised, i.e., made free from radicals and fractions so far as derivatives are concerned.
- A partial differential equation is said to be linear if the dependent variable z and its derivatives are of degree one and product of z and its derivatives donot appear in the equation.
- The partial differential equations may be formed
  (i) by the elimination of arbitrary constants from a given equation,
  (ii) by the elimination of arbitrary functions of the variables.
- If the number of arbitary constants to be eliminated is equal to the number of the independent variables, the partial differentials that arise are of the first order.
- If the number of arbitrary constants to be eliminated is more than the number of independent variables, the partial differential equations obtained are of second or higher order.

- If the partial differential equation is obtained by elimination of arbitrary functions, then the order of the partial differential equation is equal to the number of arbitrary functions eliminated.



13.5 FURTHER READINGS

1) Bhamra. K. S. Partial Differential Equations : An Introductory Treatment with Applications. PHL Learning Private Limited, New Delhi - 110001.
2) Amaranasth, T. Partial Differential Equations, 2nd edition, Jones and Bartlett Publishers.
3) C. Evans, Lawrence. Partial Differential Equations, 2nd edition, American Mathematical Society.
4) Ioannis P Stavroulakis and Stepan A Tersia. Partial differential Equations : An Introduction with Mathematica and Maple, 2nd edition, World scientific Publishing Co. Pt. Ltd, 2004.
5) Donald Greenspan. Introduction to Partial Differential Equation, Mc Graw Hill, Inc.



13.6 ANSWERS TO CHECK YOUR PROGRESS

Check your progress -1

Ans to Q No 1 : (i) Given, $z = ax + by + a^2 + b^2$ ... (1)

Differentiating (1) partially w.r.t. x , y we get

$\frac{\partial z}{\partial x} = a$ ... (2) and $\frac{\partial z}{\partial y} = b$ ... (3)

Putting the values of a and b from (2) , (3) in (1), we get

$z = px + qy + p^2 + q^2$ where $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$.

which is required partial differential equation.

(ii) Given, $z = a(x+y)+b(x-y)+abt+c$ ... (1)

Differentiating (1) partially w.r.t. x, y and t, we get

$$\frac{\partial z}{\partial x} = a+b, \frac{\partial z}{\partial y} = a-b, \frac{\partial z}{\partial t} = ab \quad \text{... (2)}$$

Since $(a+b)^2-(a-b)^2 = 4ab$ s

We get, $\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = 4\frac{\partial z}{\partial t}$ (using (2))

which is required differential equation.

(iii) we have $z = (x+a)(y+b)$ ... (1)

$$\frac{\partial z}{\partial x} = y+b \quad \text{and} \quad \frac{\partial z}{\partial y} = x+a$$

substituting in (1), we have $z = \frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}$ which is required equation.

(iv) we have $2z = \frac{x^2}{a^2}+\frac{y^2}{b^2}$ ... (1)

Differentiating (1) partially w.r.t. x and y, we get

$$2\frac{\partial z}{\partial x} = \frac{2x}{a^2}$$

or $\frac{1}{x}\frac{\partial z}{\partial x} = \frac{1}{a^2}$

or $\frac{1}{a^2} = \frac{1}{x}\frac{\partial z}{\partial x} = \frac{p}{x}$ ... (2)

and $2\frac{\partial z}{\partial y} = \frac{2y}{b^2}$

or $\frac{1}{y}\frac{\partial z}{\partial y} = \frac{1}{b^2}$

or $\frac{1}{b^2} = \frac{1}{y}\frac{\partial z}{\partial y} = \frac{q}{y}$ ... (3)

Substituting the values of $\frac{1}{a^2}$ and $\frac{1}{b^2}$ from (2) and (3) in (1), we get

$2z = px+qy$ which is required partial differential equation.

Ans to Q No 2 : The equation of any sphere of fixed radius 'λ' having centre $(h,k,0)$ in the xy-plane is given by

$(x-h)^2+(y-k)^2+(z-0)^2=\lambda^2$

or $(x-h)^2+(y-k)^2+z^2=\lambda^2$ ... (1) where h and k are arbitrary constants.

Differentiating (1) partially w.r.t. x and y, we get

$$2(x-h)+2z\frac{\partial z}{\partial x}=0$$

or $(x-h)=-z\frac{\partial z}{\partial x}$ ... (2)

and $2(y-k)+2z\frac{\partial z}{\partial y}=0$

or $(y-k)=-z\frac{\partial z}{\partial y}$ ... (3)

Substituting the values of $(x-h)$ and $(y-k)$ from (2) and (3) in (1),

we get, $z^2\left(\frac{\partial z}{\partial x}\right)^2+z^2\left(\frac{\partial z}{\partial y}\right)^2+z^2=\lambda^2$

or $z^2\left[\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1\right]=\lambda^2$, which is required partial differential equation.

Ans to Q No 3 : The equation of any sphere of fixed radius 'λ' having centre $(0,0,k)$ in the z-axis is given by

$(x-0)^2+(y-0)^2+(z-k)^2=\lambda^2$

or $x^2+y^2+(z-k)^2=\lambda^2$ ... (1) where k is an arbitrary constant.

Differentiating (1) partially w.r.t. x and y, we get

$$2x+2(z-k)\frac{\partial z}{\partial x}=0$$

or $x+(z-k)\frac{\partial z}{\partial x}=0$

or $x+(z-k)p=0$

or $z-k=-\frac{x}{p}$ ... (2)

and $2y + 2(z-k)\frac{\partial z}{\partial y} = 0$

or $y + (z-k)\frac{\partial z}{\partial y} = 0$

or $y + (z-k)q = 0$

or $z - k = -\frac{y}{q}$ ... (3)\

From (2) and (3), we get $-\frac{x}{p} = -\frac{y}{q}$

or $xq - yp = 0$, which is required partial differential equation.

Check your progress - 2

Ans to Q No 1 : (i) Given, $z = f\left(\frac{y}{x}\right)$ ... (1)

Differentiating (1) partially w.r.t. x and y, we get

$$\frac{\partial z}{\partial x} = f'\left(\frac{y}{x}\right) \times \left(-\frac{y}{x^2}\right)$$

or $f'\left(\frac{y}{x}\right) = -\left(\frac{x^2}{y}\right) \times \frac{\partial z}{\partial x}$ ... (2)

and $\frac{\partial z}{\partial y} = f'\left(\frac{y}{x}\right) \times \left(\frac{1}{x}\right)$

or $f'\left(\frac{y}{x}\right) = x\left(\frac{\partial z}{\partial y}\right)$ ... (3)

Eliminating $f'\left(\frac{y}{x}\right)$ between (2) and (3), we have

$$-\frac{x^2}{y}\frac{\partial z}{\partial x} = x\frac{\partial z}{\partial y}$$

or $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$, which is required partial differential equation.

(ii) Given, $z = f(xy)$ ... (1)

Differentiating (1) partially w.r.t. x and y, we get

$$\frac{\partial z}{\partial x} = f'(xy)y \quad \text{... (2)}$$

$$\frac{\partial z}{\partial y} = f'(xy)x \quad \text{... (3)}$$

Dividing (2) by (3), we get

$$\frac{\frac{\partial z}{\partial x}}{\frac{\partial z}{\partial y}} = \frac{y}{x}$$

or $\frac{p}{q} = \frac{y}{x}$

or $xp - yq = 0$.

(iii) Given, $e^x f(y-x) = z$ ... (1)

Differentiating (1) partially w.r.t. x, we get

$$e^x f(y-x) - e^x f'(y-x) = \frac{\partial z}{\partial x}$$

or $z - e^x f'(y-x) = p$, using (1)

or $z - p = e^x f'(y-x)$ ... (2)

Again, differentiating (1) partially w.r.t. y, we get

$$e^x f(y-x) + e^x f'(y-x) = \frac{\partial z}{\partial y}$$

or $z + z - p = q$, using (1) and (2)

or $p + q = 2z$.

(iv) Given, $z = f(x^2 - y) + g(x^2 + y)$ ... (1)

Differentiating (1) partially w.r.t. x and y, we have

$$\frac{\partial z}{\partial x} = 2xf'(x^2 - y) + 2xg'(x^2 + y)$$

or, $\frac{\partial z}{\partial x} = 2x\{f' 2(x^2 - y) + g'(x^2 + y)\}$ ... (2)

and $\frac{\partial z}{\partial y} = -f'(x^2 - y) + g'(x^2 + y)$ ... (3)

Differentiating (2) and (3) partially w.r.t. x and y respectively, we have

$$\frac{\partial^2 z}{\partial x^2} = 2\{f'(x^2 - y) + g'(x^2 + y)\} + 4x^2\{f''(x^2 - y) + g''(x^2 + y)\} \quad \text{...(4)}$$

and $\frac{\partial^2 z}{\partial y^2} = f''(x^2 - y) + g''(x^2 + y)$ ... (5)

Again (2) $\Rightarrow f'(x^2 - y) + g'(x^2 + y) = \frac{1}{2x} \times \frac{\partial z}{\partial x}$ ... (6)

Using (5) and (6) in (4), we get

$$\frac{\partial^2 z}{\partial x^2} = 2 \times \frac{1}{2x}\frac{\partial z}{\partial x} + 4x^2 \frac{\partial^2 z}{\partial y^2}$$

or $x\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial x} + 4x^3 \frac{\partial^2 z}{\partial y^2}$

which is the required partial differential equation.

(v) Do Yourself.

(vi) Let $u = x+y+z$ and $v = xyz$ so that $f(u, v) = 0$.

Differentiating partially w.r.t. x and y, we have

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}p\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}p\right) = 0$$

or $\frac{\partial f}{\partial u}(1+p) + \frac{\partial f}{\partial v}(yz + xyp) = 0$ ... (1)

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}q\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}q\right) = 0$$

$$\frac{\partial f}{\partial u}(1+q) + \frac{\partial f}{\partial v}(xz + xyq) = 0 \qquad \text{....(2)}$$

Eliminating $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ from (1) and (2), we get

$$\begin{vmatrix} 1+p & yz+xyp \\ 1+q & xz+xyq \end{vmatrix} = 0$$

or $(xz + xyq)(1+p) - (1+q)(yz + xyp) = 0$

or $x(z-y)p + (x-z)yq + z(x-y) = 0$.



13.7 MODEL QUESTIONS

Q 1 : Eliminating the arbitrary constants from the following equations form partial differential equations.

(a) $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ (b) $(x-a)^2+(y-b)^2=z^2-c$

Q 2 : Form partial differential equations by eliminating the arbitrary functions from the following equations.

(a) $z=F(x^2+y^2)$ (b) $z=e^{my}\phi(x-y)$

(c) $lx+my+nz=\phi(x^2+y^2+z^2)$

Q 3 : Eliminate the arbitrary functions from

$y=f(x-at)+xF(x-at)+x^2\phi(x-at)$

Q 4 : Obtain the partial differential equation of all planes which are at a constant distance p from the origin.

Q 5 : Obtain a partial differential equation by eliminating arbitrary functions f from $z=x+y+f(xy)$.

*** ***** ***

UNIT 14 : LINEAR PARTIAL DIFFERENTIAL EQUATION

UNIT STUCTURE

14.1 Learning Objectives

14.2 Introduction

14.3 Solution of first order and first degree partial differential equation

14.4 Let Us Sum Up

14.5 Further Readings

14.6 Answers To Check Your Progress

14.7 Model Questions

14.1 LEARNING OBJECTIVES

After going through this unit, you will able to find

- solution of first order and first degree partial differential equation.

14.2 INTRODUCTION

We have already discussed the formation of partial differential equations by eliminating either arbitrary constants or arbitary functions in previous unit. In this unit, we shall proceed to solution of first order and first degree partial differential equation.

14.3 SOLUTION OF FIRST ORDER AND FIRST DEGREE PARTIAL DIFFERENTIAL EQUATION

A differential equation involving $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$ only and no higher order is called of degree one.If the degree of p and q are unity, then it is called a linear partial differential equation of order one.

Consider $Pp + Qq = R$ where P, Q and R are functions of x, y and dependent variable z.

Working Rule for solving $Pp + Qq = R$ **:**

Step 1 : We put the given linear partial differential equation of the first order in the standard form

$$Pp + Qq = R \qquad \text{... (1)}$$

Step 2 : We write down Lagrange's auxiliary equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \qquad \text{... (2)}$$

Step 3 : We solve (2) by using some well known method.Let $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ be two independent solutions of (2).

Step 4 : The general solution of (1) is then written in one of the following three equivalent forms :

$\phi(u, v) = 0, u = \phi(v)$ and $v = \phi(u)$, ϕ being an arbitrary function.

Example 10 : Solve $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$

Solution : Here auxiliary equations are

$$\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy} \qquad \text{... (1)}$$

Each of these equations =

$$\frac{dx - dy}{x^2 - y^2 - z(y - x)} = \frac{dy - dz}{y^2 - z^2 - x(z - y)} = \frac{dz - dx}{z^2 - xy - x^2 + yz}$$

so, $$\frac{d(x - y)}{(x - y)(x + y + z)} = \frac{d(y - z)}{(y - z)(y + z + x)} = \frac{d(z - x)}{(z - x)(x + y + z)}$$

or $$\frac{d(x - y)}{(x - y)} = \frac{d(y - z)}{(y - z)} = \frac{d(z - x)}{(z - x)}$$

Now $$\frac{d(x - y)}{(x - y)} = \frac{d(y - z)}{(y - z)}$$

Integrating, $\log(x - y) = \log(y - z) + \log C_1$

$$\text{or } \frac{x - y}{y - z} = C_1 \qquad \text{... (2)}$$

Again from $$\frac{d(y - z)}{y - z} = \frac{d(z - x)}{z - x}$$

Integrating we get $\log(y - z) - \log(z - x) = \log C_2$

$$\text{or } \frac{y - z}{z - x} = C_2 \qquad \text{... (3)}$$

$\therefore$ the solution is $f\left(\frac{x-y}{y-z}, \frac{y-z}{z-x}\right)=0$.

Example 11 : Solve $y^2 p - xyq = x(z-2y)$

Solution : The auxiliary equations are

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

Taking the first two ratios, we get

$$\frac{dx}{y} = \frac{dy}{-x}$$

or $xdx + ydy = 0$

Integrating, we get $\frac{x^2}{2} + \frac{y^2}{2} = \frac{C_1}{2}$

or $x^2 + y^2 = C_1$

Again taking $\frac{dy}{-y} = \frac{dz}{z-2y}$

or $-2ydy + zdy + ydz = 0$

or $-y^2 + yz = C_2$

Then, $yz - y^2 = f\left(x^2 + y^2\right)$ is the required solution.

Example 12 : Solve $xzp + yzq = xy$

Solution : Auxiliary equations are $\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{xy}$

From first two $\frac{dx}{xz} = \frac{dy}{yz} \Rightarrow \frac{dx}{x} = \frac{dy}{y}$

Integrating $\log x = \log y + \log C_1$

or $\frac{x}{y} = C_1$

Similarly from the last two $\frac{y}{z} = C_2$

Hence the general solution is $\phi\left(\frac{x}{y}, \frac{y}{z}\right)=0$.

Example 13 : Solve $\frac{y^2 z}{x} p + xzq = y^2$

Solution : The given equation can be written as

$$y^2zp + x^2zq = xy^2$$

Auxiliary equations are

$$\frac{dx}{y^2z} = \frac{dy}{x^2z} = \frac{dz}{xy^2}$$

From first two, $x^2dx = y^2dy$,i.e. $x^3 - y^3 = C_1$

From first and third , $xdx = zdz$,i.e. $x^2 - z^2 = C_2$

Hence $\phi\left(x^3 - y^3, x^2 - z^2\right) = 0$ is the required solution.

Example 14 : Solve $yzp + xzq = xy$

Solution : The auxiliary equations are

$$\frac{dx}{yz} = \frac{dy}{xz} = \frac{dz}{xy} \quad \text{... (1)}$$

From (1), $\frac{dx}{y} = \frac{dy}{x}$

$$\Rightarrow xdx = ydy$$

$$\Rightarrow x^2 - y^2 = C_1$$

Similarly, From (1), $\frac{dy}{z} = \frac{dz}{y}$

$$\Rightarrow ydy = zdz$$

$$\Rightarrow y^2 - z^2 = C_2$$

Hence, $\phi\left(x^2 - y^2, y^2 - z^2\right) = 0$ is the required solution.

Method of Multipliers :

Let the auxiliary equations of the given equation be

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Let l, m, n be constants or functions of x, y, z then we have

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{ldx + mdy + ndz}{lP + mQ + nR}$$

l, m, n are chosen in such a way that $lP + mQ + nR = 0$

Thus we get $ldx + mdy + ndz = 0$

Solving this differential equation, we get $u = C_1$

Similarly, we choose another set of multipliers (l_1, m_1, n_1), we get $v = C_2$.

Hence, the required solution is $f(u, v) = 0$.

Example 15 : Solve : $\frac{y-z}{yz}p + \frac{z-x}{zx}q = \frac{x-y}{xy}$

Solution : The given equation can be written as

$(xy - zx)p + (yz - xy)q = xz - yz$

The auxiliary equations are

$$\frac{dx}{xy-zx} = \frac{dy}{xy-zx} = \frac{dz}{yz-xy} = \frac{dx+dy+dz}{0}$$

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

Now, $dx + dy + dz = 0 \Rightarrow x + y + z = C_1$

and $\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0 \Rightarrow \log x + \log y + \log z = \log C_2$

$$\Rightarrow xyz = C_2$$

Hence the general solution is $\phi(x + y + z, xyz) = 0$

Example 16 : Solve $(mz - ny)\frac{\partial z}{\partial x} + (nx - lz)\frac{\partial z}{\partial y} = (ly - mx)$

Solution : Given, $(mz - ny)\frac{\partial z}{\partial x} + (nx - lz)\frac{\partial z}{\partial y} = (ly - mx)$

Here, the auxiliary equations are

$$\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx}$$

using multipliers x, y, z we get

each fraction $= \frac{xdx + ydy + zdz}{x(mz-ny) + y(nx-lz) + z(ly-mx)} = \frac{xdx + ydy + zdz}{0}$

$$\therefore xdx + ydy + zdz = 0 \quad \ldots (1)$$

Integrating (1), we get $x^2 + y^2 + z^2 = C_1 \quad \ldots (2)$

Again using multipliers l, m, n we get each fraction

$$= \frac{ldx + mdy + ndz}{l(mz - ny) + m(nx - lz) + n(ly - mx)} == \frac{ldx + mdy + ndz}{0}$$

$\therefore ldx + mdy + ndz = 0$ ... (3)

Integrating (2), we get $lx + my + nz = C_2$ ... (4)

Hence from (2) and (4), the required solution is

$x^2 + y^2 + z^2 = f(lx + my + nz)$.



CHECK YOUR PROGRESS

Q 1: Solve the following partial differential equations :

(a) $yq - xp = z$

(b) $p + q = \frac{z}{a}$

(c) $x^2(y - z)p + (z - x)y^2q = z^2(x - y)$

(d) $z(xp - yq) = y^2 - x^2$

(e) $x(y^2 + z)p - y(x^2 + z)q = z(x^2 - y^2)$



14.4 LET US SUM UP

- The partial differential equation of the form $Pp + Qq = R$, where P, Q and R are functions of x, y, z is called the standard form of the linear partial differential equation of the first order and is called Lagrange's linear partial differential equation of first order.
- To solve the equation $Pp + Qq = R$, we apply following steps

  (i) We form the auxiliary equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

  (ii) We solve the auxiliary equations by the method of grouping or the method of multipliers or both to get two independent solutions $u = a$ and $v = b$, where a, b are arbitrary constants.

(iii) Then $\phi(u,v)=0$ or $u=f(v)$ is the general solution of the equation $Pp+Qq=R$.



14.5 FURTHER READINGS

1) Bhamra. K. S. Partial Differential Equations : An Introductory Treatment with Applications. PHL Learning Private Limited, New Delhi - 110001.
2) Amaranasth, T. Partial Differential Equations, 2nd edition, Jones and Bartlett Publishers.
3) C. Evans, Lawrence. Partial Differential Equations, 2nd edition, American Mathematical Society.
4) Ioannis P Stavroulakis and Stepan A Tersia. Partial differential Equations : An Introduction with Mathematica and Maple, 2nd edition, World scientific Publishing Co. Pt. Ltd, 2004.
5) Donald Greenspan. Introduction to Partial Differential Equation, Mc Graw Hill, Inc.



14.6 ANSWERS TO CHECK YOUR PROGRESS

Check your progress

Ans to Q No 1 : (a) Here the auxiliary equations are

$$\frac{dx}{-x}=\frac{dy}{y}=\frac{dz}{z} \quad \ldots \text{ (1)}$$

From the first two of (1), $\frac{dx}{-x}=\frac{dy}{y}$

or $-\log x=\log y-\log C_1$

or $\log x+\log y=\log C_1$

or $\log xy = \log C_1$

or $xy = C_1$ ... (2)

From the last two of (1), $\frac{dy}{y} = \frac{dz}{z}$

or $\log y = \log z + \log C_2$

or $\log y - \log z = \log C_2$

or $\log\left(\frac{y}{z}\right) = \log C_2$

or $\frac{y}{z} = C_2$ ... (3)

From (2) and (3), we get $f\left(xy, \frac{y}{z}\right) = 0$ which is required solution.

(b) The auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{z/a}$$

Taking the first two terms we have $dx = dy$

Integrating both sides, we get

$$x - y = c_1$$

Again taking the last two terms we have $\frac{dy}{a} = \frac{dz}{z}$

Integrating both sides, we get

$$\frac{y}{a} = \log z + \log c_2$$

or $\frac{y}{a} = \log(zc_2)$

or $e^{\frac{y}{a}} = zc_2$

or $z = ce^{\frac{y}{a}}$

$\therefore$ The general solution is $z = e^{\frac{y}{a}} f(y - x)$

(c) The auxiliary equations are

$$\frac{dx}{x^2(y-z)} = \frac{dy}{(z-x)y^2} = \frac{dz}{z^2(x-y)}$$

Using $\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{z^2}$ as multipliers

$$\text{Each fraction} = \frac{\frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2}}{0}$$

$$\therefore \frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2} = 0$$

Integrating , $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = C_1$ ... (1)

Again, using $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers

$$\text{Each fraction} = \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{0}$$

$$\therefore \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

Integrating, $\log x + \log y + \log z = \log C_2$

or $\log xyz = \log C_2$

or $xyz = C_2$ ... (2)

From (1) and (2), we get

$f\left(xyz, \frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = 0$ which is required solution.

(d) The given equation can be written as

$$zxp - zyq = y^2 - x^2$$

The auxiliary equations are $\frac{dx}{zx} = \frac{dy}{-zy} = \frac{dz}{y^2 - x^2}$ ... (1)

Using x, y, z as multipliers, we have

$$\text{Each fraction} = \frac{xdx + ydy + zdz}{0}$$

$$\therefore xdx + ydy + zdz = 0$$

Integrating, $x^2 + y^2 + z^2 = C_1$ ... (2)

From the first two members, we have

$ydx + xdy = 0$

or $d(xy) = 0$

or $xy = C_2$ ... (3)

From (2) and (3), we get

$f\left(xy, x^2 + y^2 + z^2\right) = 0$ which is required solution.

(e) The auxiliary equations are

$$\frac{dx}{x\left(y^2 + z\right)} = \frac{dy}{-y\left(x^2 + z\right)} = \frac{dz}{z\left(x^2 - y^2\right)}$$

using $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers,

$$\text{Each fraction} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$$\therefore \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

Integrating, $\log x + \log y + \log z = \log c_1$

or $\log xyz = \log c_1$

or $xyz = c_1$

Again using $x, y, -1$ as multipliers,

$$\text{Each fraction} = \frac{xdx + ydy - dz}{0}$$

$$\therefore 2xdx + 2ydy - 2dz = 0$$

Integrating, we get $x^2 + y^2 - 2z = c_2$

Hence the general solution is

$$f\left(x^2 + y^2 - 2z, xyz\right) = 0 .$$



14.7 MODEL QUESTIONS

Q 1 : Solve the following differential equations :

(a) $p\tan x + q\tan y = \tan z$

(b) $px^2 + qy^2 = z^2$

(c) $(x+2z)p + (4zx - y)q = 2x^2 + y$

(d) $p + q = 1$

(e) $(y^2 + z^2)p - xyq = -3x$

(f) $y^2 p - xyq = x(z - 2y)$

(g) $p + 3q = 5z + \tan(y - 3x)$

(h) $x^2 p + y^2 q = nxy$

(i) $(x^2 - y^2 - z^2)p + 2xyq = 2xz$

(j) $(z^2 - 2yz - y^2)p + (xy + zx)q = xy - zx$.

*** ***** ***



REFERENCES

- *Integral Calculus,* B.C. Das and B.N. Mukherjee, U. N. Dhur & Sons Pvt. Ltd., Kolkata.
- *Integral Calculus*, K. C. Maity & Ram Krishna Ghosh, New Central Book Agency (P) Ltd., Kolkata.
- *Integral Calculus*, P. N. Chatterji, Rajhans Prakashan Mandir, Meerut.
- Ordinary and partial differential equation, Raisinghania, M. D. S.Chand & company Ltd, New Delhi -110055
- Advanced differential Equations, Sharma, A. K. Discovery Publishing House, New Delhi - 110002
- Ramana, B. V. Higher Engineering Mathematics,11th edition, Tata McGraw Hill Education private Ltd, New Delhi.