FOUR YEAR BACHELOR'S DEGREE COURSE

(CBCS in Accordance with NEP 2020)

SEMESTER - VI

MECHANICS AND ELEMENTS OF STATISTICS

MATHEMATICS

Hiranya Chandra Bhuyan School of Science and Technology

KRISHNA KANTA HANDIQUI STATE OPEN UNIVERSITY

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COURSE INTRODUCTION

'Mechanics and Elements of Statistics' is the sixth discipline specific course in Mathematics of Four Year UG Programme.

The course consists of fourteen units.

In unit 1, We discuss force and its types.

In unit 2,We deals with Lami's theorem.

In unit 3,,We discuss moments and couples.

In unit 4,,We deals with friction and properties of friction.

In unit 5,We discuss Kinematics of a particle ,rectilinear motion under constant acceleration and Newton's Laws of Motion

In unit 6,we deals with simple harmonic motion, geometrical representation of simple harmonic motion and motion of a simple pendulum

In unit 7, We discuss measure of central tendency and various measures of central tendency e.g mean,median and mode.

In unit 8, We discuss measure of dispersion and various measures of dispersion e.g range,quartile deviation, mean deviation and standard deviation.

In unit 9, We discuss about moments ,kurtosis and skewness.

In unit 10,we deals with the theory of probability.We also discuss some Interesting facts about probability.

In unit 11,we discuss the concept conditional probability,Multiplication theorem of probability,Law of probability and Baye's theorem.

In unit 12,we discuss random variable and its probability distribution.

In unit 13,we discuss binomial distribution and poisson distribution.

In unit 14,we discuss normal distribution and properties of normal distribution.

While going through a unit, you will notice some along-side boxes, which have been included to help you know some of the difficult, unseen terms. At the end of each section, you will get "CHECK YOUR PROGRESS" questions. These have been designed to self-check your progress of study. It will be better if you solve the problems put in these boxes immediately after you go through the sections of the units and then match your answers with "ANSWERS TO CHECK YOUR PPROGRESS" given at the end of each unit.

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UNIT 1: FORCES

UNIT STRUCTURE

1.1 Learning Objectives

1.2 Introduction

1.3 Forces

1.4 Types of Forces

1.5 Magnitude and direction of the resultant of two forces acting on a particle

1.6 Let Us Sum Up

1.7 Further Readings

1.8 Answers To Check Your Progress

1.9 Model Questions

1.1 LEARNING OBJECTIVES

After going through this unit, you will able to know

- forces
- types of forces
- magnitude and direction of the resultant of the forces acting on a particle

1.2 INTRODUCTION

Mechanics, the oldest of all physical sciences, deals with the conditions of rest or motion of the objects around us. Mechanics is divided into two branches, viz., statics and Dynamics. Statics may be called the science of rest or equilibrium of bodies, whereas Dynamics may be called the science of motion of bodies. The most fundamental concept in statics as well as in Dynamics is force. In this unit, we will study force and its different types. Also, we will deduce the magnitude and direction of the resultant of the forces acting on a particle.

1.3 FORCES

A force is that which changes or tends to change the state of rest of a body or of its uniform motion in a straight line

A force has three characteristics, viz., (i) a magnitude, (ii) a direction, (iii) a point of application.

A force has both magnitude and direction. So it is a vector quantity.

Let $\overrightarrow{AB}$ denote a force $\vec{F}$. It is acting on a certain body. The length

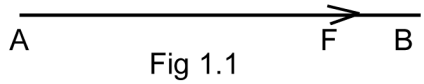


Fig 1.1

AB is the magnitude of the force. The arrowhead denotes the direction of the force. The initial point A of the line segment AB represents the point of application of the force. The line along the direction of the force is called its line of action.

Units of measurements

There are different systems of units for measuring physical quantities, viz., (i) CGS (centimentre-gram-second) system, (ii) MKS (metre-kilogram-second) system, (iii) FPS (foot-pound-second) system.

MKS system is nothing but an extension of CGS system. It is used throughout the world nowadays.

The different systems are shown below

	Mass	Length	Time	Force
CGS	gram (g)	Centimetre (cm)	second (s)	Dyne
MKS	kilogram (kg)	metre (m)	Second (s)	Newton (N)
FPS	pound (lb)	foot (ft)	Second (s)	Poundal (lb. wt.)

1.4 TYPES OF FORCES

(i) Attraction and Repulsion

Pull or push without actual contact of the body or without any visible medium is called attraction or repulsion respectively.

Examples : attraction of a body by earth, attraction or repulsion of a magnet by another.

The force with which the earth attracts a body is called the weight of the body.

(ii) Tension and Thrust

If a body is pulled by means of a string, the force exerted throughout the string is called tension.

If a body is pushed by means of a rod, the force exerted throughout the rod is called thrust.

(iii) Action and Reaction

If a body exerts a force upon another body, the second body also exerts a force on the first body. These two forces are equal in magnitude, but opposite in direction. Moreover

they have a common line of action. Thus, action and reaction are equal and opposite.

(iv) Friction

If a body slides or rolls on another, one or both being rough, there is always a force which tends to resist motion. Friction is the force which resists the sliding of one body over another.

Parallelogram of forces

If two or more forces act on a particle, their total effect is always equivalent to a single force. This single force is called the resultant of the forces. If two forces are acting at a point, their resultant can be obtained by the draw of parallelogram of forces, which is the foundamental theorem of statics.

Law of parallelogram of forces

If two forces acting at a point are represented in magnitude and direction by the sides of a parallelogram drawn from that point, their resultant is represented in magnitude and direction by the diagonal of the parallelogram drawn from that point.

Thus, if the forces $\vec{P}$ and $\vec{Q}$ acting at a point O of a body be represented in magnitude and direction by $\vec{OA}$ and $\vec{OB}$ respectively, their resultant $\vec{R}$ will be represented by the diagonal $\vec{OC}$ drawn through O. Thus, we have

$$\vec{OC} = \vec{OA} + \vec{AC} = \vec{OA} + \vec{OB} \qquad \because \vec{AC} = \vec{OB}$$

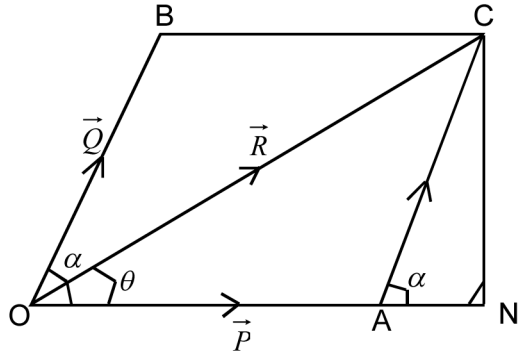


Fig 1.2

1.5 MAGNITUDE AND DIRECTION OF THE RESULTANT OF TWO FORCES ACTING AT A POINT

Let the forces P and Q act at a point O of a body. Let α be the angle between two forces.

Let $\vec{OA}$ and $\vec{OB}$ represent the forces P and Q. We complete the parallelogram OACB.

By the parallelogram law of forces, the resultant R is represented by the diagenal $\vec{OC}$ in magnitude and direction. Let θ be the angle made by the line of action of the resultant with that of the force P.

Case I α is an acute angle

From C, we draw CN perpendicular on OA produced.

From the figure, we have

$\angle CAN = \angle BOA = \alpha$

From the right angled triangle CNA,

$\sin \angle CAN = \frac{CN}{AC}$

$\cos \angle CAN = \frac{AN}{AC}$

$\therefore\ CN = AC \sin \angle CAN = OB \sin \angle CAN$

$\therefore\ CN = Q \sin \alpha$

$AN = AC \cos \angle CAN = OB \cos \angle CAN$

$\therefore\ AN = Q \cos \alpha$

$\therefore$ ON = OA + AN = P + Q cos α

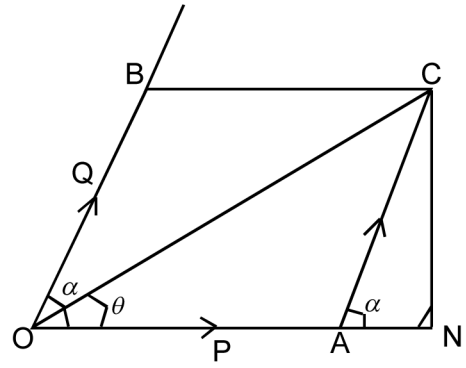


Fig 1.3

Now, from the right angled triangle ONC

$OC^2 = ON^2 + CN^2$

or, $R^2 = (P + Q \cos \alpha)^2 + Q^2 \sin^2 \alpha$

or, $R^2 = P^2 + 2PQ \cos \alpha + Q^2 \sin^2 \alpha$

$\therefore\ R^2 = P^2 + 2PQ \cos \alpha + Q^2$ . . . (1) $(\cos^2 \alpha + \sin^2 \alpha = 1)$

From (1), we get the magnitude of the resultant R.

Again, from the right angled triangle ONC,

$\cos \theta = \frac{ON}{OC} = \frac{P + Q \cos \alpha}{R}$ i.e. $R \cos \theta = P + Q \cos \alpha$

$\sin \theta = \frac{CN}{OC} = \frac{Q \sin \alpha}{R}$ i.e. $\therefore R \sin \theta = Q \sin \alpha$ . . . (2)

From (2), we get the direction of the resultant R.

Also from (2), $\tan \theta = \frac{\theta \sin \alpha}{P + Q \cos \alpha}$

Case 2 α is an obtuse angle

From C, we draw CN perpendicular on OA.

From the figure, we have $\angle CAN = 180^0 - \alpha$

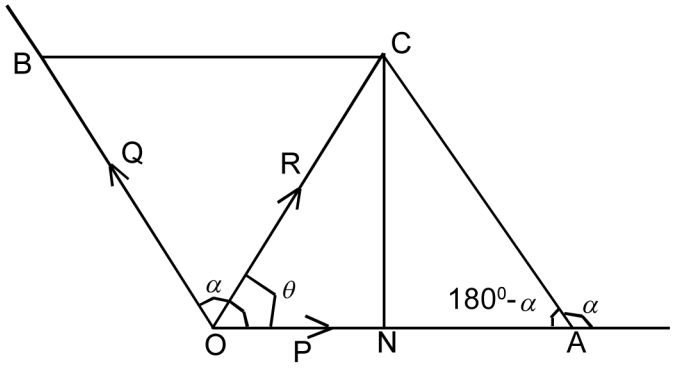


Fig 1.4

From the right angled triangle CNA, $\sin \angle CAN = \frac{CN}{AC}$

$\cos \angle CAN = \frac{AN}{AC}$

$\therefore\ CN = AC \sin \angle CAN = OB \sin(180^0 - \alpha) = Q \sin \alpha$

$AN = AC \cos \angle CAN = OB \cos(180^0 - \alpha) = -Q \cos \alpha$

$\therefore\ ON = OA - NA = P - (-Q \cos \alpha) = P + Q \cos \alpha$

Now from the right angled triangle ONC.

$OC^2 = ON^2 + CN^2$

or $R^2 = (P + Q \cos \alpha)^2 + Q^2 \sin 2\alpha$

$\therefore\ R^2 = P^2 + 2PQ \cos \alpha + Q^2$ . . . (1)

From (1), we get the magnitude of the resultant R.

Again, from the right angled triangle ONC,

$\cos \theta = \frac{ON}{OC} = \frac{P + Q \cos \alpha}{R}$ i.e. $R \cos \theta = P + Q \cos \alpha$ . . . (2)

$\sin \theta = \frac{CN}{OC} = \frac{Q \sin \alpha}{R}$ i.e. $R \sin \theta = Q \sin \alpha$

From (2), we get the direction of the resultant R.

Also from (2), $\tan \theta = \frac{\theta \sin \alpha}{P + Q \cos \alpha}$

Corollaries

1. If $\alpha = 0$ i.e, if the forces act in the same straight line in the same direction, then from (1) we have

$$R^2 = P^2 + 2PQ\cos 0 + Q^2 = P^2 + 2PQ + Q^2$$

$$= (P+Q)^2$$

$$\therefore\ R = P + Q$$

i.e. the magnitude of the resultant = the sum of the magnitude of the forces.

In this case, the magnitude of the resultant is greatest.

2. If $\alpha = 180^0$, i.e., if the forces act in the same straight line in the opposite direction, them from (1), we have

$$R^2 = P^2 + 2PQ\cos 180^0 + Q^2$$

$$= P^2 - 2PQ + Q^2 = (P \sim Q)^2$$

$$\therefore\ R = P \sim Q$$

i.e. the magnitude of the resultant = the difference of the magnitude of the forces.

In this case, the magnitude of the resultant is least.

3. If P = Q, i.e., if the magnitudes of the forces are equal, then from (1), we have

$$R^2 = P^2 + 2P^2\cos\alpha + P^2$$

$$= 2P^2(1+\cos\alpha) = 4P^2\cos^2\frac{\alpha}{2}$$

$$\therefore\ R = 2P\cos\frac{\alpha}{2}$$

Again, from (3), we have

$$\tan\theta = \frac{Q\sin\alpha}{Q + Q\cos\alpha} = \frac{\sin\alpha}{1+\cos\alpha}$$

$$= \frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{2\cos^2\frac{\alpha}{2}}$$

$$= \tan\frac{\alpha}{2}$$

i.e. $\theta = \frac{\alpha}{2}$

Thus, if the magnitudes of the two forces are equal, the resultant bisects the angle between the two forces.

4. If $\alpha = 90^0$, i.e., if the forces are orthogonal (perpendicular) to each other, then from (1), we have

$$R^2 = P^2 + 2PQ\cos 90^0 + Q^2 = P^2 + Q^2$$

i.e. $R = (P^2 + Q^2)^{\frac{1}{2}}$

Again, from (3), we have

$$\tan\theta = \frac{Q}{P}$$

Note : P and Q are called the component forces (or components) of the force R.

Components and resolved parts

Let the force R be given. It acts at O along OD. Let OX and OY be two given direction on different sides of OD.

Let OD make angle α with OX and β with OY.

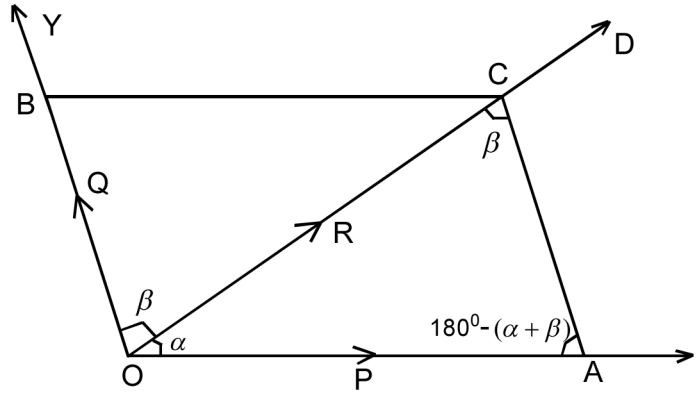


Fig 1.5

Let the force R be represented by OC completely. Through C, we draw BC parallel to OX, and AC parallel to OY respectively. We complete the parallelogram OACB. Let the force P be represented by OA completely and the force Q be represented by OB completely.

Then, P and Q are the components of R and R is the resultant of P and Q.

From the triangle OAC, we have

$$\frac{OA}{\sin\angle OCA} = \frac{AC}{\sin\angle AOC} = \frac{OC}{\sin\angle OAC}$$

or $\frac{P}{\sin\beta} = \frac{Q}{\sin\alpha} = \frac{R}{\sin(180^0 - (\alpha+\beta))}$

or, $\frac{P}{\sin\beta} = \frac{Q}{\sin\alpha} = \frac{R}{\sin(\alpha+\beta)}$

$$\therefore P = \frac{R\sin\beta}{\sin(\alpha+\beta)}$$

$$Q = \frac{R\sin\alpha}{\sin(\alpha+\beta)}$$

Thus, if R, α, β are given, we can determine P, Q uniquely

Deductions : If $\alpha+\beta = 90^0$ then $\beta = 90^0 - \alpha$.

$$\therefore\ P = \frac{R\sin(90^0-\alpha)}{\sin 90^0} = R\cos\alpha$$

$Q = R \sin \alpha$

$R\cos\alpha$ and $R\sin\alpha$ are called the resolved parts or resolutes of the force R.

Corollary : If $\alpha = 0$,

R cosα = R cos 0 = R

and R sinα = R sin0 = 0

Thus, we conclude that the resolved part of a force along its own direction is the force itself and the resolved part along its perpendicular direction is zero.

Example Resolve a force of 18N along a direction making an angle of 30^0 with the force and a second direction perpendicular to the force.

Solution

Resolved part along the 1st direction = 18 cos30^0N

$$= 18\times\frac{\sqrt{3}}{2}N = 9\sqrt{3}N$$

Resolved part along the 2nd direction = 18 sin30^0N

$$= 18\times\frac{1}{2}N = 9N$$

Theorem on resolved parts

The algebraic sum of the resolved parts of two forces acting at a point in any direction in their plane is equal to the resolved part of their resultant in the same direction.

Proof Let the forces P and Q act at a point O of a body. Let $\vec{OA}$ and $\vec{OB}$ represent the forces P and Q respectively. We complete the parallelogram OACB. The resultant R is represented by the diagonal $\overrightarrow{OC}$ in magnitude and direction.

Let OX and OY be two given directions perpendicular to each other in the plane of the forces.

Let α, β, θ be the angles made by P, Q, R with the direction OX respectively.

We draw AL, BM, CN perpendiculars on OX, and AE perpendicular on CN.

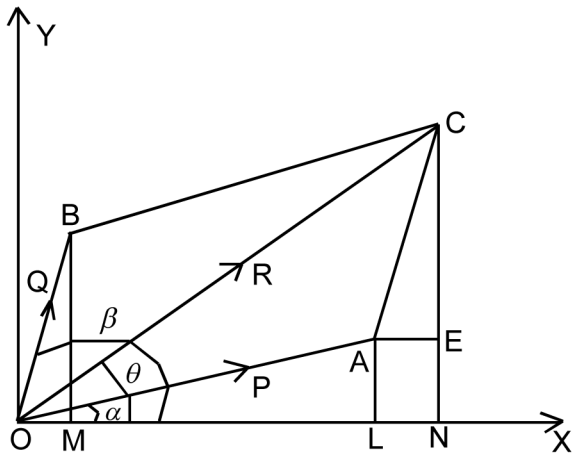


Fig 1.6

Clearly the triangles OBM and ACE are congruent

$\therefore$ OM = AE

From the right angled triangle OLA

$$\cos\alpha = \frac{OL}{OA} \qquad \therefore\ OL = OA\cos\alpha$$

From the right angled triangle ONC, we have

$$\cos\theta = \frac{ON}{OC} \qquad \therefore\ ON = OC\cos\theta$$

From the right angled triangle OMB, we have

$$\cos\beta = \frac{OM}{OB} \qquad \therefore\ OM = OB\cos\beta$$

Now, ON = OL + LN

or, ON = OL + AE

or, ON = OL + OM

or, OC cosθ = OAcosα + OBcosβ

i.e. Rcosθ = Pcosα + Qcosβ

i.e. resolved part of R along OX

= resolved part of P along OX

+ resolved part of Q along OX.

Extension: The above result can be extended to any finite number of forces. Thus, the algebraic sum of the resolved parts of any number of forces acting at a point in any direction in their plane is equal to the resolved part of their resultant in the same direction.

Resultant of a finite number of forces acting at a point

Let P_1, P_2, P_3, ..., P_n be n number of forces acting at a point O.

We choose any two suitable perpendicular lines OX and OY in the plane of the forces.

Let the line of action of the forces P_1, P_2, P_3, ..., P_n make angles $\alpha_1, \alpha_2, \alpha_3, ... \alpha_n$ with OX respectively.

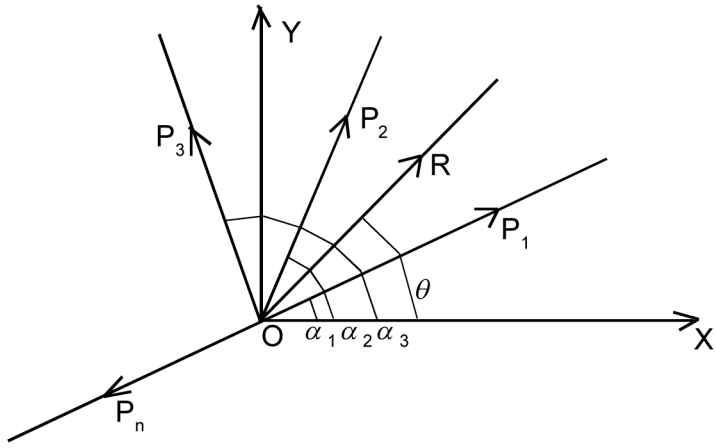


Fig 1.7

Let R be the resultant of the forces, and its line of action make angle θ with OX sums of the resolved parts of the forces P_1, P_2, P_3, ..., $P_n$along OX and OY respectively.

Therefore,

$$R\cos\theta = X = P_1\cos\alpha_1 + P_2\cos\alpha_2 + ... + P_n\cos\alpha_n$$

$$R\sin\theta = Y = P_1\sin\alpha_1 + P_2\sin\alpha_2 + ... + P_n\sin\alpha_n$$

$$\therefore R^2 = X_2 + Y_2 \quad (1)$$

$$\tan\theta = \frac{Y}{X} \quad (2)$$

From (1), we get the magnitude of the resultant.

From (2), we get the direction of the resultant.

Note : Here X and Y cannot be zero simultaneously. If X = 0, Y = 0, then R = 0, and in this case the system of forces will be in equilibrium.

Example 1 : Find the angle between two forces of equal magnitude acting at a point so that their resultant is also equal to either of them in magnitude.

Solution : Let the magnitude of each of the forces be P.

$\therefore$ the magnitude of the resultant is also P.

Let α be the angle between the forces.

$\therefore$ $P^2 = P^2 + P^2 + 2PP\cos\alpha$

i.e. $1 + 2\cos\alpha = 0$

i.e. $\cos\alpha = -\frac{1}{2}$

$\therefore$ $\alpha = 120^0$

$\therefore$ the angle between the forces is 120^0.

Example 2 : The line of action of the resultant of two forces of magnitude P, Q acting at point divides the angle between them in the ratio 1 : 2. Show that the magnitude of then resultant is $\frac{P^2 - Q^2}{Q}$.

Solution : Let θ be the angle between the forces P and Q.

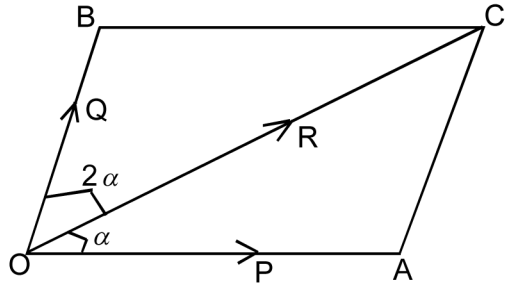


Fig 1.8

It is given that the resultant divides the angle between the forces in the ratio 1 : 2.

$\therefore$ if $\angle COA = \alpha,\ \angle COB = 2\alpha$

$\therefore \theta = \alpha + 2\alpha = 3\alpha$

Now $R^2 = P^2 + Q^2 + 2PQ\cos\theta$

or, $R^2 = P^2 + Q^2 + 2PQ\cos^3\alpha$

or, $R^2 = P^2 + Q^2 + 2PQ(4\cos^3\alpha - 3\cos\alpha)$

Again, $\tan\alpha = \frac{Q\sin 3\alpha}{P + Q\cos 3\alpha}$

or, $(P + Q\cos 3\alpha)\tan\alpha = Q\sin 3\alpha$

or, $(P + Q(4\cos^3\alpha - 3\cos\alpha)\frac{\sin\alpha}{\cos\alpha} = Q(3\sin\alpha - 4\sin^3\alpha)$

or, $P\frac{\sin\alpha}{\cos\alpha} = 2Q\sin\alpha$

or, $\cos\alpha = \frac{P}{2Q}$

$$\therefore R^2 = P^2 + Q^2 + 2PQ\left(4\left(\frac{P}{2Q}\right)^3 - \frac{3P}{2Q}\right)$$

$$= P^2 + Q^2 + \frac{P^4}{Q^2} - 3P^2$$

$$= \frac{P^2Q^2 + Q^4 + P^4 - 3P^2Q^2}{Q^2}$$

$$= \frac{P^4 + Q^4 - 2P^2Q^2}{Q^2}$$

$$= \frac{\left(P^2 - Q^2\right)^2}{Q^2}$$

$$\therefore \ R = \frac{P^2 - Q^2}{Q}$$

Example 3 : The magnitudes of the greatest and least resultants of two forces of given magnitude acting at a point are P_1 and P_2 respectively. Prove that when the forces act at an angle 2α , their resultant is of magnitude. $\left(P_1^2\cos^2\alpha + P_2^2\sin^2\alpha\right)^{\frac{1}{2}}$

Solution : Let the forces be P and Q.

Now $P_1 = P + Q$

$P_2 = P - Q$, if we assume P > Q.

$\therefore$ $2P = P_1 + P_2$ i.e., $P = \frac{P_1 + P_2}{2}$

$2Q = P_1 - P_2$ i.e., $Q = \frac{P_1 - P_2}{2}$

Let the resultant of P and Q be R.

$$\therefore \ R^2 = P^2 + Q^2 + 2PQ\cos 2\alpha$$

$$= \frac{1}{4}\left(P_1 + P_2\right)^2 + \frac{1}{4}\left(P_1 - P_2\right)^2 + 2.\frac{P_1 + P_2}{2}.\frac{P_1 - P_2}{2}.\cos 2\alpha$$

$$= \frac{1}{4}\left[2P_1^2 + 2P_2^2 + 2(P_1^2 - P_2^2)\cos 2\alpha\right]$$

$$= \frac{1}{4}\left[2P_1^2(1 + \cos 2\alpha) + 2P_2^2(1 - \cos 2\alpha\right]$$

$$= \frac{1}{4}\left[2P_1^2.2\cos^2\alpha + 2P_2^2.2\sin^2\alpha\right]$$

$$= P_1^{\,2}\cos^2\alpha + P_2^{\,2}\sin^2\alpha$$

$$\therefore R = \left(P_1^{\,2}\cos^2\alpha + P_2^{\,2}\sin^2\alpha\right)^{\frac{1}{2}}$$

Example 4 : The magnitude of the resultant of two forces P and Q acting at an angle θ is equal to (2n+1)$\sqrt{P^2+Q^2}$. When they act at an angle $(90^0-\theta)$, the magnitude of the resultant is (2n–1)$\sqrt{P^2+Q^2}$. Show that

$$\tan\theta = \frac{n-1}{n+1}$$

Solution : We have

$$(2n+1)^2\left(\sqrt{P^2+Q^2}\right)^2 = P^2+Q^2+2PQ\cos\theta$$

i.e. $(4n^2+4n+1)(P^2+Q^2)-(P^2+Q^2) = 2PQ\cos\theta$

i.e. $(4n^2+4n+1-1)(P^2+Q^2) = 2PQ\cos\theta$

i.e. $4n+(n+1)(P^2+Q^2) = 2PQ\cos\theta$ (1)

Again, $(2n-1)^2\left(\sqrt{P^2+Q^2}\right)^2 = P^2+Q^2+2PQ\cos(90^0-\theta)$

i.e. $(4n^2-4n+1)(P^2+Q^2)-(P^2+Q^2) = 2PQ\sin\theta$

i.e. $(4n^2-4n+1-1)(P^2+Q^2) = 2PQ\sin\theta$

i.e. $4n(n-1)(P^2+Q^2) = 2PQ\sin\theta$ (2)

From (1) and (2),

$$\frac{2PQ\sin\theta}{2PQ\cos\theta} = \frac{4n(n-1)(P^2+Q^2)}{4n(n+1)(P^2+Q^2)}$$

i.e. $\tan\theta = \dfrac{n-1}{n+1}$

Example 5 : Two forces of magnitude P and 2P act on a particle. If the angle between the forces is 120^0, find the angle made by the resultant with the greater force.

Solution : Let θ be the angle made by the resultant with the force 2P (the greater force).

$$\therefore \tan\theta = \frac{P\sin 120^0}{2P+P\cos 120^0}$$

$$= \frac{\frac{\sqrt{3}\,P}{2}}{2P-\frac{P}{2}} = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$$

$$\therefore \theta = 30^0$$

$\therefore$ the resultant makes an angle of 30^0 with the greater force.

CHECK YOUR PROGRESS

Q 1: Two forces of magnitude 2N and 3N act at a point at an angle 60^0 to each other. find the magnitude of the resultant and the angle made by the resultant with the direction of the force 2N.

Q 2: Two forces of equal magnitudes 4N and 4N act at a point at an angle 120^0 to each other. Find the resultant and the angle made by the resultant with either of the forces.

Q 3: Resultant of two forces acting at a point at right angle is 8N. If the resultant makes an angle of 60^0 with one of the forces, find the forces.

Q 4: The resultant of two forces of magnitudes P and 2P acting at a point is $P\sqrt{3}$. Find the angle between the forces and the angle made by the resultant with the force of P.

Q 5: Two forces of equal magnitude act on a particle. Find the angle between them when the square of their resultant is equal to three times their product.

Q 6: Two forces act at a point and are such that if the direction of one is reversed, the direction of the resultant is turned through a right angle. Prove that the two forces are equal in magnitude.

Q 7: The resultant of two forces P and Q acting at a point is equal to Q in magnitude and that of P and 2Q (acting in the same direction) is also Q. Show that $P = \sqrt{3}Q$ and the angle between the forces is 150^0.

Q 8: The resultant of two forces P and Q is $\sqrt{3}Q$ and the resultant is at an angle of 30^0 with P. Show that either P = Q or P = 2Q.

Q 9: The resultant of two forces P and Q acting at a point is R; if Q is doubled, R is doubled. If Q is reversed, R is again doubled. Show that

$P : Q : R = \sqrt{2} : \sqrt{3} : \sqrt{2}$

1.6 LET US SUM UP

- A force is that which changes or tends to change the state of rest of a body or of its uniform motion in a straight line.
- A force has three characteristics, viz, (i) a magnitude, (ii) a direction, (iii) a point of application.

- A force has both magnitude and direction. So, it is a vector quantity.
- There are different systems of units for measuring physical quantities, viz, (i) CGS system, (ii) MKS system, (iii) FPS system.
  MKS system is nothing but an extension of CGS system. It is used throughout the world nowadays.
- There are different types of forces, viz, attraction and repulsion; tension and thru; action and reaction; friction.
- If two forces acting at a point are represented in magnitude and direction by the sides of a parallelogram drawn from that point, their resultant is represented in magnitude and direction by the diagonal of the parallelogram drawn from that point. This is the fundamental theorem of statics.
- The magnitude R of two forces P and Q acting at a point at an angle α is given by
  $R^2 = P^2 + 2PQ\cos\alpha + Q^2$.
  The direction of the resultant R is given by the relations
  $R\cos\theta = P + Q\cos\alpha$
  $R\sin\theta = Q\sin\alpha$
- The algebraic sum of the resolved parts of two forces acting at a point in any direction in their plane is equal to the resolved part of their resultant in the same direction.

1.7 FURTHER READINGS

1) Statics and Dynamics with Background Mathematics, Adrian Roberts, Cambridgee University Press.
2) Enginering Machanics : Statics and Dynamics by R. G. Hibbeler and Ashok Gupta.
3) Engineering Mechanics : Statics and Dynamics by S. Rajasekarn and Sankara subramanian. Vikas Publishing House Pvt. Ltd.

1.8 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q No 1: We have

$$R^2 = 2^2 + 2\times 2\times 3\cos 60^0 + 3^2$$

$$= 4 + 2\times 2\times 3\times\frac{1}{2} + 9 = 19$$

$$\therefore\ R = \sqrt{19}\ N$$

Also $\tan\theta = \dfrac{3\sin 60^0}{2 + 3\cos 60^0}$ (Here P = 2, Q = 3)

$$= \frac{\frac{3\sqrt{3}}{2}}{2+3\times\frac{1}{2}} = \frac{3\sqrt{3}}{7}$$

$$\therefore \theta = \tan^{-1}\frac{3\sqrt{3}}{7}$$

Ans to Q No 2: We know that if the forces are of equal magnitude P, then

$$R = 2p\cos\frac{\alpha}{2}, \quad \text{and } \theta = \frac{\alpha}{2}$$

$$\therefore\ R = 2\times 4\times\cos\frac{120^0}{2}N$$

$$= 2\times 4\times\cos 60^0 N = 2\times 4\times\frac{1}{2}N = 4N$$

$$\theta = \frac{120^0}{2} = 60^0$$

Ans to Q No 3: We have $R^2 = P^2 + Q^2$

and $\tan\theta = \frac{Q}{P}$

Now, $8^2 = P^2 + Q^2$ i.e. $P^2 + Q^2 = 64$

$\tan 60^0 = \frac{Q}{P}$ i.e. $\sqrt{3} = \frac{Q}{P}$ i.e. $Q = \sqrt{3}P$

$\therefore\ P^2 + 3P^2 = 64$ or $4P^2 = 64$ or $P^2 = 16$

$\therefore\ P = 4\ N$

and $Q = 4\sqrt{3}\,N$

Ans to Q No 4: Let α be the angle between the forces.

$$\therefore\ \left(P\sqrt{3}\right)^2 = P^2 + 2\times P\times 2P\cos\alpha + (2P)^2$$

or, $3P^2 = P^2 + 4P^2\cos\alpha + 4P^2$

or, $2\cos\alpha = -1$

or, $\cos\alpha = -\frac{1}{2}$

$$\therefore \alpha = 120^0$$

Let θ be the angle made by the resultant with the force P.

$$\therefore\ P\sqrt{3}\sin\theta = 2P\sin 120^0 (R\sin\theta = Q\sin\alpha)$$

or, $\sqrt{3}\sin\theta = 2\times\frac{\sqrt{3}}{2}$

or, $\sin\theta = 1$

$\therefore\ \theta = 90^0$

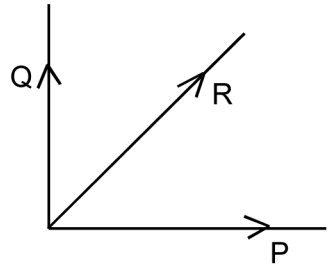
Ans to Q No 5: Let R be the resultant and α be the angle between the forces.

Let the magnitude of each force be P.

$\therefore\ R^2 = P^2 + 2PP\cos\alpha + P^2$

$= 2P^2 + 2P^2\cos\alpha = 2P^2(1+\cos\alpha)$

Also $R^2 = 3PP = 3P^2$



$\therefore\ 3P^2 = 2P^2(1+\cos\alpha)$

or $1+\cos\alpha = \frac{3}{2}$

or $\cos\alpha = \frac{1}{2}$

$\therefore\ \alpha = 60^0$

$\therefore$ the angle between the forces is 60^0.

Ans to Q No 6: Let two forces P and Q act at a point O. Let α be the angle between the forces.

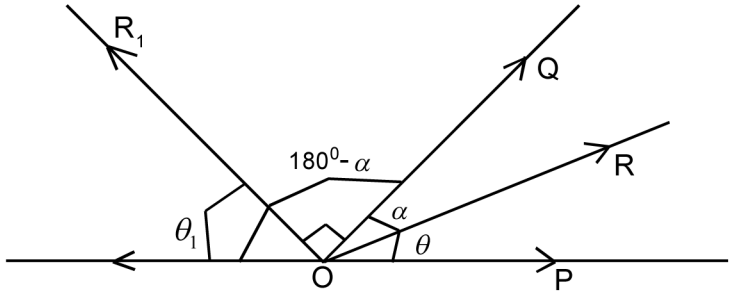


Fig 1.9

Let θ be the angle made by the resultant with the force P.

$\therefore \tan\theta = \frac{Q\sin\alpha}{P+Q\cos\alpha}$

Let the force P be reversed. Then the resultant is turned through a right angle (given).

Let the new resultant make an angle θ_1, with P (when reversed).

Now, $\theta + 90^0 + \theta_1 = 180^0$

$\therefore\ \theta_1 = 180^0 - 90^0 - \theta = 90^0 - \theta$

Now $\tan\theta_1 = \frac{Q\sin(180^0-\alpha)}{P+Q\cos(180^0-\alpha)}$

or, $\tan(90^0 - \theta) = \dfrac{Q\sin\alpha}{P - Q\cos\alpha}$

or, $\cot\theta = \dfrac{Q\sin\alpha}{P - Q\cos\alpha}$ (2)

From (1) and (2)

or, $\dfrac{Q\sin\alpha}{P + Q\cos\alpha} \cdot \dfrac{Q\sin\alpha}{P - Q\cos\alpha} = \tan\theta \,.\, \cot\theta$

or, $\dfrac{Q^2\sin^2\alpha}{P^2 - Q^2\cos^2\alpha} = 1$

or, $P^2 - Q^2\cos^2\alpha = Q^2\sin^2\alpha$

or, $P^2 = Q^2(\sin^2\alpha + \cos^2\alpha) = Q^2$

$\therefore$ P = Q

Ans to Q No 7: Let α be the angle between the forces.

$\therefore Q^2 = P^2 + 2PQ\cos\alpha + Q^2$

or, $P^2 + 2PQ\cos\alpha = 0$

or, $P(P + 2Q\cos\alpha) = 0$

or, $P + 2Q\cos\alpha = 0$

or, $2Q\cos\alpha = -P$ (1)

Again, $Q^2 = P^2 + 4PQ\cos\alpha + 4Q^2$

or, $P^2 + 4PQ\cos\alpha + 3Q^2 = 0$

or, $P^2 + 2P(2Q\cos\alpha) + 3Q^2 = 0$

or, $P^2 + 2P(-P) + 3Q^2 = 0$ from (1)

or, $P^2 = 3Q^2$

$\therefore P = \sqrt{3}\,Q$

$\therefore$ from (1) $2Q\cos\alpha = -\sqrt{3}\,Q$

$\therefore \cos\alpha = -\dfrac{\sqrt{3}}{2}$

or, $\cos\alpha = -\cos 30^0$

or, $\cos\alpha = -\cos(180^0 - 30^0)$

$\therefore \alpha = 150^0$

i.e. the angle between the forces is 150^0.

Ans to Q No 8: Let α be the angle between the force.

$\therefore \left(\sqrt{3}\,Q\right)^2 = P^2 + 2PQ\cos\alpha + Q^2$

or, $3Q^2 = P^2 + 2Q\cos\alpha + Q^2$

or, $P^2 + 2PQ\cos\alpha - 2Q^2 = 0$ (1)

Again $\sqrt{3}\,Q\cos 30^0 = P + Q\cos\alpha$ $(R\cos\theta = P + Q\cos\alpha)$

or, $\sqrt{3}\,Q \,.\, \frac{\sqrt{3}}{2} = P + Q\cos\alpha$

or, $3Q = 2P + 2Q\cos\alpha$

or, $2Q\cos\alpha = 3Q - 2P$ (2)

From (1) and (2),

$P^2 + P(3Q - 2P) - 2Q^2 = 0$

or, $P^2 + 3P\,Q - 2P^2 - 2Q^2 = 0$

or, $P^2 - 3PQ + 2Q^2 = 0$

or $(P - Q)(P - 2Q) = 0$

$\therefore P = Q$ or $P = 2Q$

Ans to Q No 9: Let α be the angle between P and Q.

$\therefore R^2 = P^2 + 2PQ\cos\alpha + Q^2$ (1)

Also, $(2R)^2 = P^2 + 2P(2Q\cos\alpha) + (2Q)^2$

or, $4R^2 = P^2 + 4PQ\cos\alpha + 4Q^2$ (2)

Again, $(2R)^2 = P^2 + 2PQ\cos(180^0 - \alpha) + Q^2$

or, $4R^2 = P^2 - 2PQ\cos\alpha + Q^2$ (3)

Multiplying (1) by 2, we get

$2R^2 = 2P^2 + 4PQ^2\cos\alpha + 2Q^2$ (4)

Subtracting (4) from (2), we get

$2R^2 = -P^2 + 2Q^2$

or $P^2 - 2Q^2 + 2R^2 = 0$ (5)

Again, adding (1) and (3), we get

$5R^2 = 2P^2 + 2Q^2$

or, $2P^2 + 2Q^2 - 5R^2 = 0$ (6)

From (5) and (6), we get

$$\frac{P^2}{10-4} = \frac{Q^2}{-(-5-4)} = \frac{R^2}{4+2}$$

or, $\frac{P^2}{6}=\frac{Q^2}{9}=\frac{R^2}{6}$

or, $\frac{P^2}{2}=\frac{Q^2}{3}=\frac{R^2}{2}$

$\therefore \frac{P}{\sqrt{2}}=\frac{Q}{\sqrt{3}}=\frac{R}{\sqrt{2}}$

?1.9 MODEL QUESTIONS

Q 1: Two forces P and Q act at a point at an angle θ and their resultant R makes an angle α with P. If P = 24, R = 36, α = 90^0, find Q.

Q 2: Two unequal forces act at a point. Show that their resultant is nearer the greater force.

Q 3: The magnitude of the resultant of two forces each of magnitude P acting at a point is R. If $R^2 = 3P^2$. Show that the forces act at an angle of 60^0.

Q 4: The resultant of two forces P and Q acting at a point at an angle θ is R. If the first force is doubled in magnitude. Show that the magnitude of the new resultant is $\sqrt{2P^2+2R^2-Q^2}$.

Q 5: The resultant of forces P and Q acting at a point at angle 90^0, θ and $90^0-\theta$ are of magnitude R, nR and (n + 2) R respectively. Show that $\frac{n+3}{n-1}$.

Q 6: The resultant of two forces P and Q acting at a point is of magnitude P. If the magnitude of the first force is doubled, show that the new resultant is of magnitude $\sqrt{4P^2-Q^2}$.

Q 7: The resultant of two forces P and Q acting at a point is R. If the direction of the second force is reversed and its magnitude changes to $(R^2 - P^2)/Q$, Show that the magnitude of the resultant is not altered.

Q 8: If the resultant of two forces acting on a particle be at right angle to one of them and its magnitude be one-third the magnitude of the other, Show that the magnitude of the larger force to that of the smaller force is $3:2\sqrt{2}$.

Q 9: R is the resultant of the forces P and Q acting at the point. If a transversal cuts the lines of action of P, Q, R at the points L, M, N respectively. Show that $\frac{P}{OL}+\frac{Q}{OM}=\frac{R}{ON}$.

Q 10: The magnitude of the resultant of two forces of equal magnitude acting at angle 2θ is twice that when they act at angle 2ϕ. Show that $\cos\theta = 2\cos\phi$.

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UNIT 2: COPLANAR FORCES

UNIT STRUCTURE

2.1 Learning Objectives

2.2 Introduction

2.3 Lami's Theorem

2.4 Equilibrium of a particle under several coplanar forces

2.5 Parallel Forces

2.6 Let Us Sum Up

2.7 Further Readings

2.8 Answers To Check Your Progress

2.9 Model Questions

2.1 LEARNING OBJECTIVES

After going through this unit, you will able to-

- Lami's Theorem
- equilibrium of a particle under several coplanar forces
- parallel forces

2.2 INTRODUCTION

If two forces act on equal masses under similar conditions and produce equal accelerations, the forces are said to be equal. If a number of forces act on a body, the body may move or may be at rest. If the body remains at rest, the body is said to be in equilibrium, and the forces are also said to be in equilibrium. In this unit, we will study Lami's theorem, equilibrium of a particle under several coplanar forces and parallel forces.

2.3 LAMI'S THEOREM

If three forces acting at a point are in equilibrium, each force is proportional to the sine of the angle between the other two forces.

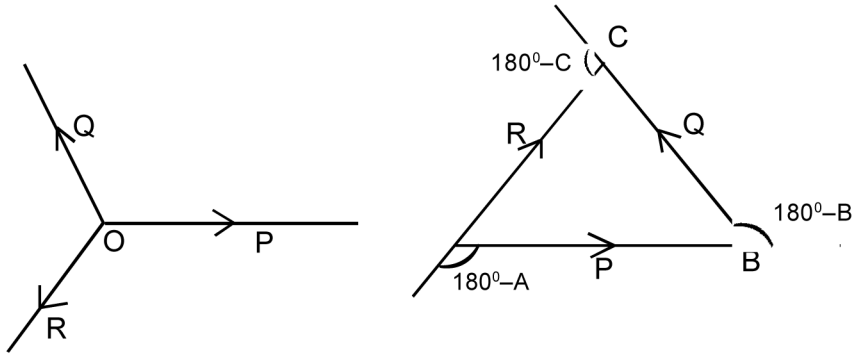


Fig 2.1

Let three forces P, Q, R acting on O be at equilibrium.

It is required to prove that $\frac{\widehat{P}}{\sin(Q,R)} = \frac{\widehat{Q}}{\sin(R,P)} = \frac{\widehat{R}}{\sin(P,Q)}$

Where (Q, R) means angle between Q and R, $\frown$

(R, P) means angle between R and P, $\frown$

(P, Q) means angle between P and Q. $\frown$

Let ABC be a triangle whose sides AB, BC, CA are parallel to the directions of P, Q, R respectively.

$\therefore$ the lengths AB, BC, CA are proportional to the magnitudes of P, Q, R respectively.

$$\therefore \frac{AB}{P} = \frac{BC}{Q} = \frac{CA}{R} \qquad (1)$$

Again from trigonometry, we know that

$$\therefore \frac{AB}{Sin\,C} = \frac{BC}{Sin\,A} = \frac{CA}{Sin\,B} \qquad (2)$$

From (1) and (2),

$$\frac{P}{Sin\,C} = \frac{Q}{Sin\,A} = \frac{R}{Sin\,B}$$

i.e. $$\frac{P}{Sin(180^0 - C)} = \frac{Q}{Sin(180^0 - A)} = \frac{R}{Sin(180^0 - B)}$$

Now, $180^0 - C$ = angle between Q and R

$180^0 - A$ = angle between R and P

$180^0 - B$ = angle between P and Q

$$\therefore \frac{\widehat{P}}{Sin(Q\,P)} = \frac{\widehat{Q}}{Sin(R,\,P)} = \frac{\widehat{R}}{Sin(P\,Q)}$$

2.4 EQUILIBRIUM OF A PARTICLE UNDER SEVERAL COPLANAR FORCES

We consider the following three cases.

Case I : Two forces are acting on a particle. The particle will be in equilibrium, i.e., the forces are in equilibrium if

(i) they have equal magnitude,

(ii) they act along the same line,

and (iii) they act in opposite directions.

Case II : Three non-colinear forces are acting on a particle. The particle will be in equilibrium, i.e., the forces are in equilibrium if

(i) the forces can be represented by the sides of a triangle taken in order

(ii) each force is proportional to the sine of the angle between the other two forces (known as Lami's Theorem)

or (iii) if the sum of their resolved along each of two perpendicular directions in their plane is zero.

Case III : A finite number of forces, no two of which are colinear, are acting on a particle. The particle will be in equilibrium, i.e., the forces are in equilibrium if

(i) the forces can be represented by the sides of a polygon taken in order.

or (ii) the forces are coplanar and the sum of their resolved parts along each of two perpendicular directions in their plane is zero.

or (iii) the point at which the given forces act is the centroid of the extremities of the line segments representing the forces.

Example 1 : Examine whether three forces whose magnitudes are in the proportion 5 : 3 : 2, acting on a particle, can be kept in equlibrium under any circumstance.

Solution : Let the magnitudes of the forces be 5k, 3k, 2k.

Equilibrium is possible when the force of magnitude 5k acts in one direction and the forces of magnitudes 3k and 2k act in the opposite direction along the same straight line.

CHECK YOUR PROGRESS

Q 1: Examine whether three forces whose magnitudes are in the proportion 3 : 2 : 1 acting on a particle, can be kept in equilibrium under any circumstance.

Q 2: Examine whether three forces whose magnitudes are in the proportion 3 : 5 : 10 acting on a particle can be kept in equilibrium.

Q 3: Three equal forces acting at a point are equally inclined to each other. Show that they are in equilibrium.

Q 4: Three forces P, Q, R act at the in centre I of the triangle ABC in the direction IA, IB, IC respectively. If the forces are in equilibrium, prove that $P^2 : Q^2 : R^2 = a(b + c – a) : b(c + a –b) : c(a + b + c)$.

Q 5: Three forces P, Q, R act at the centroid G of the triangle ABC in the directions GA, GB, GC respectively. If the forces are in equilibrium, prove that

P : Q R = GA : GB : GC

Q 6: Three forces P, Q, R act at the circumcentre O of the triangle ABC in the directions OA, OB, OC respectively. If the forces are in equilibrium, prove that P : Q : R.

$$= \frac{1}{b^2}+\frac{1}{c^2}-\frac{a^2}{b^2c^2} : \frac{1}{c^2}+\frac{1}{a^2}-\frac{b^2}{c^2a^2} : \frac{1}{a^2}+\frac{1}{b^2}-\frac{c^2}{a^2b^2}$$

Q 7: Three non-collinear forces acting at a point O are in equilibrium. A circle is drawn through O to cut the lines of action of the forces in A, B, C respectively. Prove that the forces are proportional to the sides of the triangle ABC.

Q 8: Forces acting at a point are represented in magnitude and direction by 2AB, 3BC, 2CD, DA, CA and DB where ABCD is a quadrilateral. Show that the forces are in equilibrium.

Q 9: Forces acting at a point are represented in magnitude and direction by AB, 2BC, 2CD, 2DE, 2EF, FA and FB where ABCDEF is a hexagon. Show that the forces are in equilibrium.

Q 10: ABCDE is a pentagon. Show that the forces acting at a point and represented in magnitude and direction by AB, 3BC, 2CA, CD, DE, 2EB and AE are in equilibrium.

2.5 PARALLEL FORCES

Two forces are said to be parallel if their lines of action are parallel. They are said to be like parallel if they act in the same direction, whereas they are called unlike parallel if they act in opposite directions.

Clearly the forces are not concurrent. They are assumed to act on a rigid body.

Resultant of two like parallel forces

Let P, Q be two like parallel forces acting at the points A, B respectively of a rigid body.

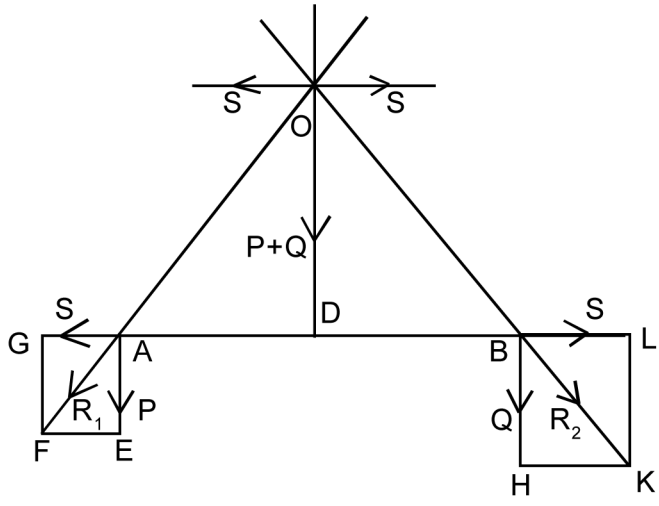


Fig 2.2

Let P, Q be represented by AE, BH respectively.

We join AB and introduce two equal and opposite force S, S at A, B represented by AG, BL respectively.

Since the forces balance each other, they will have no effect on the system.

We complete the parallelograms AEFG and BHKL.

The resultant of P and S is R_1 along AF, and that of Q and S is R_2 along BK.

We produce FA and KB to meet at O.

We draw OD parallel to AE (or BH) to meet AB at D.

The resultants R_1 and R_2 may be supposed to act at O. Thus, R_1 at O is equivalent to P at O along OD and S at O parallel to AG. Also R_2 at O is equivalent to Q at O along OD and S at O parallel to BL.

The two forces S, S at O balance each other. So, we are left with P + Q along OD, which is the resultant of the like parallel forces P, and Q.

Now, we find the position of D where the line of action of P + Q meets AB.

Clearly, the triangles AFE and OAD are similar.

$$\therefore \frac{AE}{OD} = \frac{EF}{DA}$$

or, $\frac{P}{OD} = \frac{S}{DA}$ i.e., $\frac{P}{S} = \frac{OD}{DA}$ (1)

Also the triangles BHK and ODB are similar.

$$\therefore \frac{BH}{OD} = \frac{HK}{DB}$$

or, $\frac{Q}{OD} = \frac{S}{DB}$ i.e., $\frac{Q}{S} = \frac{OD}{DB}$ (2)

From (1) and (2), $\frac{P}{S} \cdot \frac{S}{Q} = \frac{OD}{DA} \cdot \frac{DB}{OD}$

i.e. $\frac{P}{Q} = \frac{DB}{DA}$

Thus, D divides AB internally in the inverse ratio of the forces.

Note : If P = Q, the resultant is 2P and in this case, D is the middle point of AB.

Resultant of two unequal, unlike parallel forces

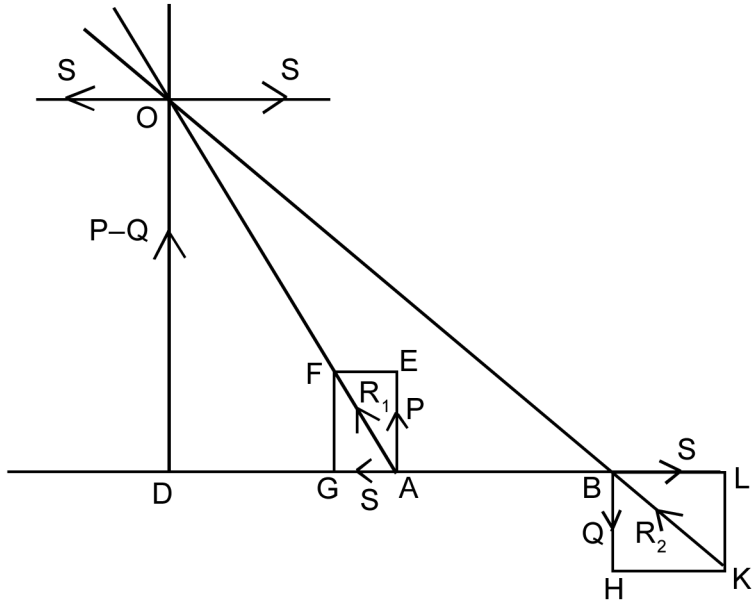


Fig 2.3

Let P, Q be two unlike parallel forces acting at the points A, B respectively of a right body, where P > Q.

Let P, Q be represented by AE, BH respectively.

We join AB and introduce two equal and opposite forces S, S at A, B represented by AG, BL respectively.

Since the forces balance each other they will have no effect on the system.

We complete the parallelograms AEFG and BHKL.

The resultant of P and S is R_1 along AF, and that of Q and S is R_2 along KB.

We produce AF and KB to meet at O.

We draw OD parallel to AE (or BH) to meet BA produced at D.

The resultant R_1 and R_2 may be supposed to act at O. Thus, R_1 at O is equivalent to P at O along DO, and S at O parallel to AG. Also R_2 at O is equivalent to Q at O along OD and parallel to BL.

The two forces S, S at O balance each other. So, we are left with P– Q along DO, which is the resultant the unlike parallel forces P and Q.

Now we find the position of D where the line of action of P–Q meets BA produced.

Clearly, the triangles AFG, and AOD are similar.

$$\therefore \frac{GF}{DO} = \frac{AG}{AD}$$

or, $\frac{GF}{AG} = \frac{DO}{AD}$ i.e., $\frac{P}{S} = \frac{DO}{AD}$ (1)

Also, the triangles BHK and ODB are similar.

$$\therefore \frac{BH}{OD} = \frac{HK}{DB}$$

or, $\frac{BH}{HK} = \frac{OD}{DB}$ i.e., $\frac{Q}{S} = \frac{OD}{DB}$ (2)

From (1) and (2), $\frac{P}{S} \cdot \frac{Q}{S} = \frac{DO}{AD} \cdot \frac{BD}{OD}$

i.e., $\frac{P}{Q} = \frac{DB}{DA}$

Thus, D divides BA externally in the inverse ratio of the forces.

Example 2 : Two like parallel forces 11 newton and 19 newton are acting at A and B of a rigid body. If AB = 10m, find the magnitude and the point of application of the resultant.

Solution : Let P = 11 newton

Q = 19 newton

$\therefore$ R = (P + Q) newton

= (11 + 19) newton

= 30 newton.

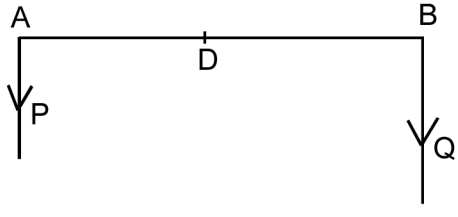


Fig 2.4

Let D be the point of application of the resultant.

$\therefore$ P. DA = Q . DB

or, 11. DA = 19. (10 – DA)

or, 11. DA = 190 – 19DA

or, 30DA = 190

$$\therefore DA = \frac{19}{3} m$$

CHECK YOUR PROGRESS

Q 1: Two like parallel forces P and Q acting at A and B have a resultant R acting at D on AB. If P = 6 newton, AD = 9m and DB = 8m, find Q and R.

Q 2: The position of the resultant of two like parallel forces remains unaltered, when the positions of the forces are interchanged. Show that the forces are equal in magnitude.

Q 3: Two unlike parallel forces P and Q acting at A and B have a resultant R acting at D on AB produced. If P = 5 newton, AD = 12m, AB = 9m, find Q and R.

Q 4: P and Q are two unlike parallel forces where P > Q. If the magnitude of each of the force is increased by the same amount, prove that the line of action of the resultant will move further off from P.

Q 5: P and Q are two like parallel forces. If P is moved parallel to itself through a distance x, prove that the resultant moves through a distance.

$$\frac{Px}{P+Q}.$$

Q 6: Three like parallel forces P, Q, R are acting at the angular points A, B, C of the triangle ABC. If the line of action of the resultant passes through

(i) the centroid, prove that P = Q = R,

(ii) the circumcentre, prove that

$$\frac{P}{\sin 2A} = \frac{Q}{\sin 2B} = \frac{R}{\sin 2C}$$

(iii) the incentre, prove that

$$\frac{P}{BC} = \frac{Q}{CA} = \frac{R}{AB}$$

(iv) the orthocentre, prove that

$$\frac{P}{\tan A} = \frac{Q}{\tan B} = \frac{R}{\tan C}$$

2.9 LET US SUM UP

- If a number of forces act on a body, the body may move or may be at rest. If the body remains at rest, the body is said to be in equilibrium, and the forces are also said to be in equilibrium.
- If three forces acting on a particle are in equilibrium, each force is proportional to the sine of the angle between the other two forces. (This is konwn as Lami's Theorem)
- Two forces are said to be parallel if their lines of action are parallel. They are said to be like parallel if they act in the same direction, whereas they are called unlike parallel if they act in opposite directions. They are assumed to act on a rigid body.

2.10 FURTHER READING

1) Statics and Dynamics with Background Mathematics, Adrian Roberts, Cambridge University Press.
2) Enginering Machanics : Statics and Dynamics by R. G. Hibbeler and Ashok Gupta.
3) Engineering Mechanics : Statics and Dynamics by S. Rajasekarn and Sankara subramanian. Vikas Publishing House Pvt. Ltd.

2.11 ANSWERS TO CHECK YOUR PROGRESS

CHECK YOUR PROGRESS 1

Ans to Q No 1: Let the magnitudes of the forces be 3k, 2k, k.

Equilibrium is possible when the force of magnitude 3k acts in one direction and the forces of magnitudes 2k and k act in the opposite direction along the same straight line.

Ans to Q No 2: Let the magnitudes of the forces be 3k, 5k, 10k.

Equilibrium is not possible as the forces cannot be represented by the sides of a triangle. It is not possible to draw a triangle with sides 3k, 5k, 10k units, as $3k + 5k < 10k$.

Ans to Q No 3: We are given that three equal forces acting at a point are equally inclined to each other. Therefore, the forces can be represented by the sides of a triangle taken in order. Therefore, the forces are in equilibrium.

Ans to Q No 4: Given that the forces are in equilibrium.

$\therefore$ by Lami's theorem,

$$\frac{P}{\sin\angle BIC}=\frac{Q}{\sin\angle CIA}=\frac{R}{\sin\angle AIB}$$

Now $\angle BIC+\frac{1}{2}\angle B+\frac{1}{2}\angle C=180^0$

$$\therefore\ \angle BIC=180^0-\frac{1}{2}(\angle B+\angle C)$$

$$=180^0-\frac{1}{2}(180^0-\angle A)\quad\left[\because\angle A+\angle B+\angle C=180^0\right]$$

$$=90^0+\frac{A}{2}$$

Similarly $\angle CIA=90^0+\frac{B}{2}$, $\angle AIB=90^0+\frac{C}{2}$

$$\therefore\ \frac{P}{\sin\left(90^0+\frac{A}{2}\right)}=\frac{Q}{\sin\left(90^0+\frac{B}{2}\right)}=\frac{R}{\sin\left(90^0+\frac{C}{2}\right)}$$

$$\Rightarrow\frac{P}{\cos\frac{A}{2}}=\frac{Q}{\cos\frac{B}{2}}=\frac{R}{\cos\frac{C}{2}}$$

$$\Rightarrow\frac{P^2}{\cos^2\frac{A}{2}}=\frac{Q^2}{\cos^2\frac{B}{2}}=\frac{R^2}{\cos^2\frac{C}{2}}$$

Now, $\cos A=2\cos^2\frac{A}{2}-1$

$$\therefore 2\cos^2\frac{A}{2}=1+\cos A$$

$$=1+\frac{b^2+c^2-a^2}{2bc}$$

$$=\frac{2bc+b^2+c^2-a^2}{2bc}$$

$$=\frac{(b+c)^2-a^2}{2bc}$$

$$=\frac{(b+c+a)\,(b+c-a)}{2bc}$$

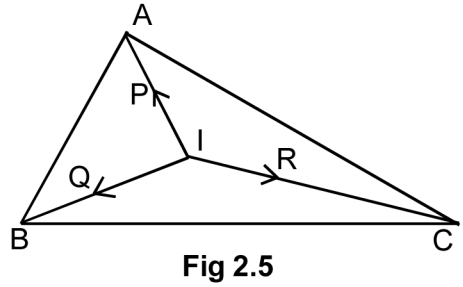


Fig 2.5

Simarly,

$$\cos^2\frac{B}{2}=\frac{(c+a+b)(c+a-b)}{2ca}$$

$$\cos^2\frac{C}{2}=\frac{(a+b+c)(a+b-c)}{2ab}$$

$$\therefore\ P^2\ :\ Q^2\ :\ R^2$$

$$=\frac{(b+c+a)(b+c-a)}{4bc}:\frac{(c+a+b)(c+a-b)}{4ca}:\frac{(a+b+c)(a+b-c)}{4ab}$$

$$=a(b+c-a):b(c+a-b):c(a+b-c)$$

Ans to Q No 5: Given that the forces are in equilibrium.

$\therefore$ by Lami's theorem

$$\frac{P}{\sin\angle BGC}=\frac{Q}{\sin\angle CGA}=\frac{R}{\sin\angle AGB}$$

We know that area of the triangle GBC

= area of the triangle GCA

= area of the triangle GAB

$$\therefore\ \frac{1}{2}GB.GC\sin\angle BGC=\frac{1}{2}GC.GA\sin\angle CGA$$

$$=\frac{1}{2}GA.GB\sin\angle AGB$$

$$\Rightarrow\frac{\sin\angle BGC}{GA}=\frac{\sin\angle CGA}{GB}=\frac{\sin\angle AGB}{GC}\quad(2)$$

From (1) and (2)

$$\frac{P}{GA}=\frac{Q}{GB}=\frac{R}{GC}$$

Ans to Q No 6: Given that the forces are in equilibrium.

$\therefore$ by Lami's theorem

$$\frac{P}{\sin\angle BOC}=\frac{Q}{\sin\angle COA}=\frac{R}{\sin\angle AOB}$$

$$\Rightarrow\frac{P}{\sin 2A}=\frac{Q}{\sin 2B}=\frac{R}{\sin 2C}$$

$$\Rightarrow\frac{P}{2\sin A\cos A}=\frac{Q}{2\sin B\cos B}=\frac{R}{2\sin C\cos C}$$

$$\Rightarrow\frac{P}{\sin A\cos A}=\frac{Q}{\sin B\cos B}=\frac{R}{\sin C\cos C}\quad(1)$$

Again, we know that

$$\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}=k,\ \text{say}$$

$\therefore \sin A = ka,\ \sin B = kb,\ \sin C = kc$

$\therefore$ from (1),

$$\frac{P}{ka\cos A} = \frac{Q}{kb\cos B} = \frac{R}{kc\cos C}$$

i.e. $$\frac{P}{a\cos A} = \frac{Q}{b\cos B} = \frac{R}{c\cos C} \qquad (2)$$

Now $$a\cos A = a\frac{b^2 + c^2 - a^2}{2bc}$$

$$= \frac{abc}{2}\ \frac{b^2 + c^2 - a^2}{b^2c^2}$$

$$= \frac{abc}{2}\left(\frac{1}{b^2} + \frac{1}{c^2} - \frac{a^2}{b^2c^2}\right)$$

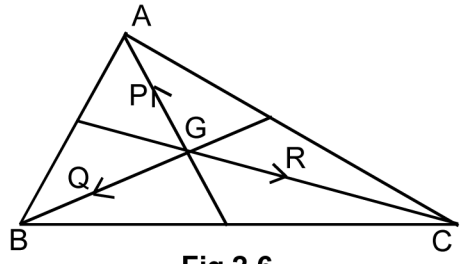


Fig 2.6

Similarly,

$$b\cos B = \frac{abc}{2}\left(\frac{1}{c^2} + \frac{1}{a^2} - \frac{b^2}{c^2a^2}\right)$$

$$c\cos C = \frac{abc}{2}\left(\frac{1}{a^2} + \frac{1}{b^2} - \frac{c^2}{a^2b^2}\right)$$

$\therefore$ from (2)

$$\text{P : Q : R} = \frac{abc}{2}\left(\frac{1}{b^2} + \frac{1}{c^2} - \frac{a^2}{b^2c^2}\right) : \frac{abc}{2}\left(\frac{1}{c^2} + \frac{1}{a^2} - \frac{b^2}{c^2a^2}\right) : \frac{abc}{2}\left(\frac{1}{a^2} + \frac{1}{b^2} - \frac{c^2}{a^2b^2}\right)$$

$$= \left(\frac{1}{b^2} + \frac{1}{c^2} - \frac{a^2}{b^2c^2}\right) : \left(\frac{1}{c^2} + \frac{1}{a^2} - \frac{b^2}{c^2a^2}\right) : \left(\frac{1}{a^2} + \frac{1}{b^2} - \frac{c^2}{a^2b^2}\right)$$

Ans to Q No 7: Given that the forces are in equilibrium.

$\therefore$ by Lami's theorem

$$\frac{P}{\sin\angle BOD} = \frac{Q}{\sin\angle DOA} = \frac{R}{\sin\angle AOB}$$

Now $\angle BOD + \angle BOC = 180^0$

$\angle DOA + \angle AOC = 180^0$

$\angle AOB + \angle BOA = 180^0$

$\therefore \sin\angle BOD = \sin\left(180^0 - \angle BOC\right)$

$= \sin\angle BOC$

$= \sin\angle BAC$

$$\sin\angle DOA = \sin\left(180^0 - \angle AOC\right)$$
$$= \sin\angle AOC$$
$$= \sin\angle ABC$$
$$\sin\angle AOB = \sin\left(180^0 - \angle BCA\right)$$
$$= \sin\angle BCA$$

$\therefore$ from (1)

$$\frac{P}{\sin\angle BAC} = \frac{Q}{\sin\angle ABC} = \frac{R}{\sin\angle BCA} \qquad (2)$$

Again, from the triangle ACB

$$\frac{\sin\angle BAC}{BC} = \frac{\sin\angle ABC}{CA} = \frac{\sin\angle BCA}{AB} \qquad (3)$$

$\therefore$ from (2) and (3)

$$\frac{P}{BC} = \frac{Q}{CA} = \frac{R}{AB}$$

8 and 9. Left as exercise

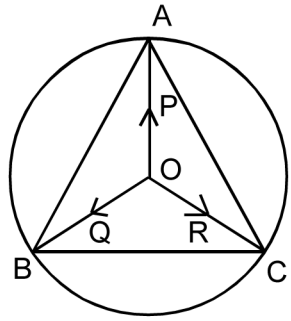


Fig 2.7

Ans to Q No 10: We have

$$\overrightarrow{AB} + 3\overrightarrow{BC} + 2\overrightarrow{CA} + \overrightarrow{CD} + \overrightarrow{DE} + 2\overrightarrow{EB} + \overrightarrow{AE}$$
$$= \left(\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE}\right) + 2\overrightarrow{BC} + 2\overrightarrow{CA} + 2\overrightarrow{EB} + \overrightarrow{AE}$$
$$= \overrightarrow{AE} + 2\overrightarrow{BC} + 2\overrightarrow{CA} + 2\overrightarrow{EB} + \overrightarrow{AE}$$
$$= 2\overrightarrow{AE} + 2\overrightarrow{BC} + 2\overrightarrow{CA} + 2\overrightarrow{EB}$$
$$= 2\left(\overrightarrow{CA} + \overrightarrow{AE} + \overrightarrow{EB} + \overrightarrow{BC}\right)$$
$$= 2\left(\overrightarrow{CE} + \overrightarrow{EB} + \overrightarrow{BC}\right)$$
$$= \vec{O}$$

$\therefore$ the forces are in equilibrium.

CHECK YOUR PROGRESS 2

Ans to Q No 1: P = 6 newton

AD = 9m

DB = 8m

$\therefore$ Now, P. DA = Q. DB

or, 6 x 9 = Q x 8

$\therefore Q = \frac{3\times 9}{4} = \frac{27}{4}$ newton

$\therefore R = \left(6 + \frac{27}{4}\right)$ newton

$= \frac{51}{4}$ newton

Ans to Q No 2: Let P, Q be two like parallel forces acting at A, B respectively.

Let the resultant act at D on AB

$\therefore$ P. AD = Q. BD (1)

Let the positions of the forces be interchanged.

Now, it is given that the resultant acts at D.

$\therefore$ Q. AD = P. BD (2)

From (1) and (2),

$\frac{P}{Q} = \frac{Q}{P}$

i.e., $P^2 = Q^2$

i.e., P = Q

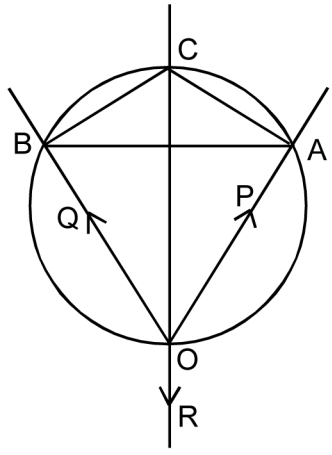


Fig 2.8

Ans to Q No 3: P = 5 newton

AD = 12 m

AB = 9m

P. DA = Q. DB

or, 5 x 12 = Q x 3

$\therefore Q = \frac{5 \times 12}{3} = 20$ newton

$\therefore$ R = Q – P

= (20 – 5) newton = 15 newton.

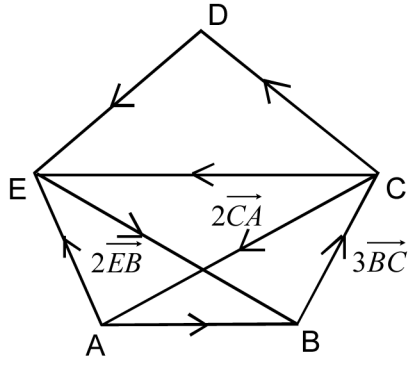


Fig 2.9

Ans to Q No 4: Let P, Q be two unlike parallel forces acting at A, B respectively, where P > Q.

Let the resultant act at D on BA produced.

$\therefore$ P. AD = Q. BD

or P. AD = Q.(AB + AD)

$\therefore$ (P – Q) AD = Q. AB

$\therefore AD = \frac{Q}{P-Q} AB$ (1)

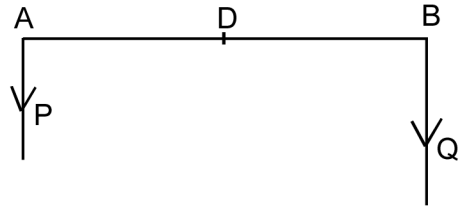


Fig 2.10

Let the magnitude of each force be increased by x.

Let the new resultant act at E on BA produced

$\therefore$ (P + x). AE = (Q + x). BE

or, (P + x) AE = (Q + x) (AB + AE)

or, (P + x – Q – x) AE = (Q + x) . AB

$\therefore$ (P – Q). AE = (Q + x). AB

$\therefore AE = \frac{Q+x}{P-Q}.AB$

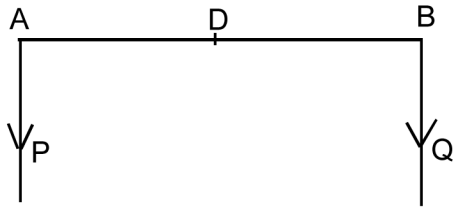


Fig 2.11

Now, $\frac{Q+x}{P-Q}.AB = \frac{Q}{P-Q}.AB + \frac{x}{P-Q}.AB > \frac{Q}{P-Q}AB$

i.e. AE > AD.

Ans to Q No 5: Left as exercise

6 (iii) Let I be the incentre of the triangle ABC, where AD is the bisector of the angle BAC.

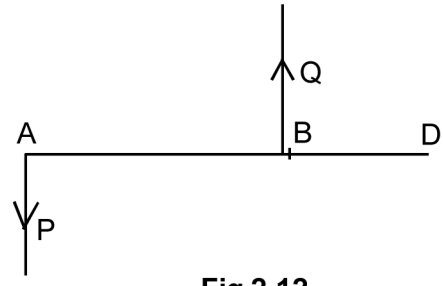


Fig 2.12

$\therefore \frac{AB}{AC} = \frac{BD}{DC}$ (1)

It is given that the resultant passes through the incentre I.

$\therefore$ the resultant of Q at B and R at C must pass through D.

$\therefore$ Q. BD = R. DC, i.e., $\frac{Q}{R} = \frac{DC}{BD}$ (2)

$\therefore$ from (1) and (2)

$\frac{Q}{R} = \frac{AC}{AB}$ i.e. $\frac{Q}{CA} = \frac{R}{AB}$

Similarly, $\frac{P}{BC} = \frac{Q}{CA}$

$\therefore \frac{P}{BC} = \frac{Q}{CA} = \frac{R}{AB}$

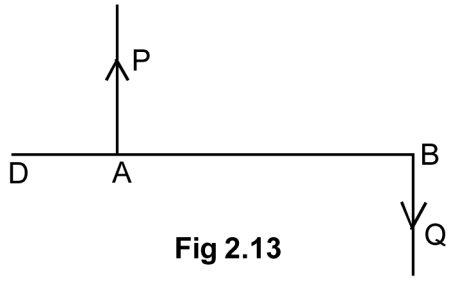


Fig 2.13

Ans to Q No 6: (i), (ii), (iv), Left as exercise.

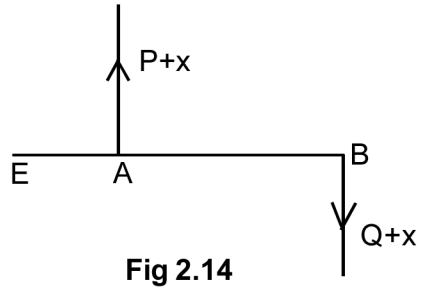


Fig 2.14

?2.12 MODEL QUESTIONS

Q 1: Three coplanar forces can act at a point in any manner. Can they produce equilibrium if their magnitudes are (i) 2, 3, 5 (ii) 3, 4, 5 (iii) 1, 3, 5 ?

Q 2: Forces P, Q, R act at the orthocentre O of Δ ABC along OA, OB, OC respectively and are in equibrium. Show that P : Q : R = a : b : c.

Q 3: Forces P, Q, R acting on a particle are in equibrium. If the angle between P and Q is double the angle between P and R. Show that $R^2 = Q(Q - P)$.

Q 4: Two like parallel forces P and Q act at A and B respectively and their resultant R acts at C on AB. Find :

(i) R and AC if P = 12 newton, Q = 8 newton, AB = 10m

(ii) Q and AB if P = 13 newton, R = 20 newton, AC = 14m

(iii) P and Q if R = 12 newton, AC = 0.5m, AB = 3m.

Q 5: Two unlike parallel forces P and Q act at A and B respectively and their resultant R acts at C on AB (produced). Find :

(i) Q and R if P = 5 newton, AB = 9m, AC = 12m

(ii) P and Q if AB = 2.5m, R = 5 newton, AC = 5m

(iii) AB and R if P = 910 newton, Q = 210 newton, AC = 3m.

Q 6: Two like parallel forces P and Q (P > Q) act at A and B respectively. If their positions are interchanged, prove that the resultant is displaced on AB through a distance $\frac{P-Q}{P+Q}$ AB.

Q 7: Two unlike parallel forces P and Q (P > Q) act at two points x meter apart. If the direction of the greater force is reversed, prove that the position of the resultent is displaced through a distance $\frac{2PQx}{P^2-Q^2} m$.

*** ***** ***

UNIT 3: MOMENT AND COUPLES

UNIT STRUCTURE

3.1 Learning Objectives

3.2 Introduction

3.3 Moment of a force about a point

3.4 Couples

3.5 Let Us Sum Up

3.6 Further Readings

3.7 Answers To Check Your Progress

3.8 Model Questions

3.1 LEARNING OBJECTIVES

After going through this unit, you will able to-

- moment of a force about a point
- couples

3.2 INTRODUCTION

In this unit, we will study moments and couples.

3.3 MOMENTS OF A FORCE ABOUT A POINT

The moment of a force about a given point is defined to be the product of the force and the perpendicular distance from the point to the line of action of the force.

Let P be a force acting on a rigid body. Let it tend to rotate the body about the fixed point O. Then the moment of P about the point O = pP.

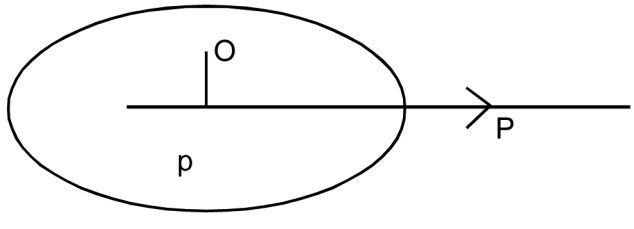


Fig 3.1

Moment is a vector quantity.

The moment of a force about a point is considered to be positive if the tendency of

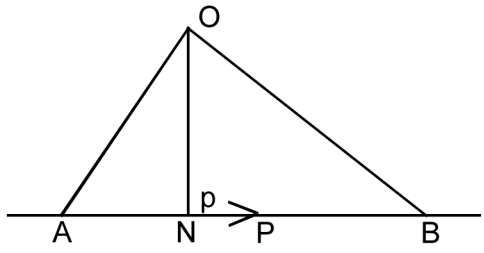


Fig 3.2

rotation is in the counter-clockwise direction and negative if the tendency of rotation is in the clockwise direction.

Units of Moment : The units of moment are expressed as dyne-cm, newton-m, poundal-ft.

Geometrical Meaning of moment of a force about a point

Let P be a force acting on a rigid body. Let it tend to rotate the body about the fixed point O.

Let AB represent the force P.

Moment of P about O

= pP.

Again, area of the triangle OAB

$$= \frac{1}{2} pP$$

$\therefore$ Moment of P about O

= 2 (area of the triangle OAB).

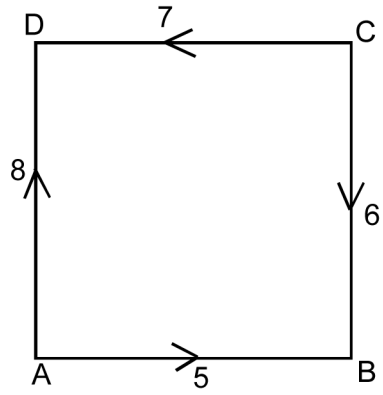


Fig 3.3

Thus, the moment of a force about a point is twice the area of the triangle whose base is the line-segment representing the force and whose vertex is the point about which the moment is taken.

Example 3 : ABCD is a square of side 4m. Force 5, 6, 7, 8 newton act along the sides AB, CB, CD, AD. Find the sum of their moments about A, B, C, D.

Solution : Sum of the moments about A

= (0 x 5 – 4 x 6 + 4 x 7 x 0 x 8) newton - m

= 4 newton - m

Sum of the moments about B

= (0 x 5 + 0 x 6 + 4 x 7 – 4 x 8) newton - m

= – 4 newton - m

Sum of the moments about C

= (0 x 6 + 0 x 7 – 4 x 8 + 4 x 5)

= – 12 newton - m

Sum of the moments about D

= (0 x 7 + o x 8 + 4 x 6) newton - m

= – 4 newton - m

Example 4 : ABC is an isosceles triangle board, right angled at C. It is placed on a smooth table. A force 2 newton acts at C along CA and a force 3 newton acts at B along CB. If AC = BC = 1m, find the sum of the moments about A, B and D, where D is the middle point of AB.

Solution : D is the middle point of AB

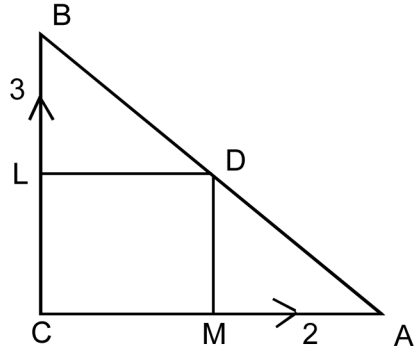


Fig 3.4

$$\therefore DL = \frac{1}{2}CA = \frac{1}{2}m$$

$$DM = \frac{1}{2}BC = \frac{1}{2}m$$

Sum of the moments about A

= (0 x 2 – 1 x 3) newton – m = –3 newton -m

Sum of the moments about B

= (1 x 2 + 0 x 3) newton – m = 2 newton -m

Sum of the moments about D

= $\left(\frac{1}{2}\times 2 - \frac{1}{2}\times 3\right)$ newton m = –0.5 newton -m

CHECK YOUR PROGRESS

Q 1: ABCD is a square of side 4m. Forces 4, 3, 6, 10 newton act along AB, BC, DC, DA respectively. Find the algebraic sum of the

moments of the forces about A, B, C, D, O, where O is the centre of the square.

Q 2: ABCD is a square of side 4m. Forces 4, 3, 2, 10 newton act along CB, BA, DA, DB respectively. Find the algebraic sum of the moments of the forces about C.

Q 3: ABCD is a square of side 4 m. Force 3, 5, 3, 5 newton act along AB, CB, DC, DA respectively. Find the algebraic sum of the moments of the forces about B, C and O, where O is the centre of the square.

3.4 COUPLES

Two equal and unlike parallel forces (having different lines of action) acting on a rigid body form a couple.

The perpendicular distance between the forces of a couple is called the arm of the couple.

The product of one of the forces of a couple and its arm is called the moment of the couple.

The unit of a couple is the same as that of a moment.

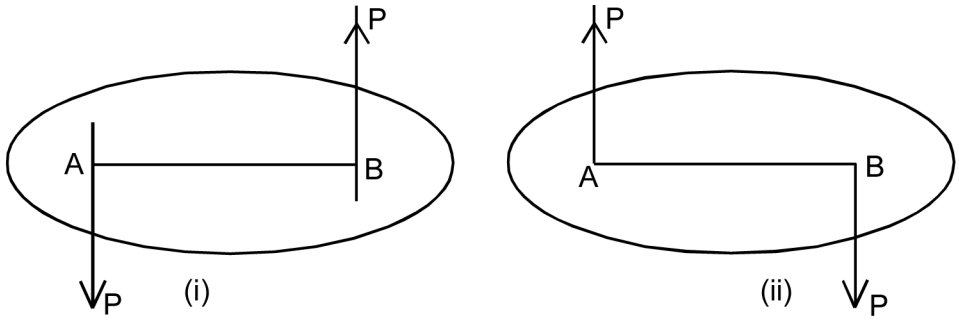


Fig 3.5

Let a couple consist of two forces of magnitude P. Let AB(=p) be the perpendicular distance between the forces. The couple is generally written as (P, P).

The moment of a couple is considered to be positive if the tendency of rotation is in the counter-clockwise direction and negative if the tendency of rotation is in the clockwise direction.

Now, we state a theorem, without proof, which is required to obtain the resultant of any number of couples in a plane.

Theorem : The resultant of two couples acting on a right body in the same plane is couple whose moment is the algebraic sum of the moments of the couples.

Example 5 : Forces equal to 3, 5, 3, 5 newton respectively act along the sides of a square

taken in order. Find their resultant.

Solution : Let a be the length of each side of the square.

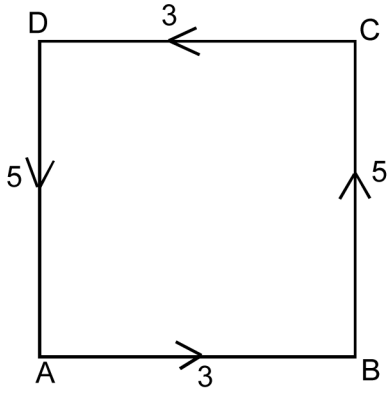


Fig 3.6

From the figure, we see that the force 3 newton along AB and 3 newton along CD form a couple. Its moment is 3a.

Similarly, the force 5 newton along BC and 5 newton along DA form a couple whose moment is 5a.

$\therefore$ the resultant of the given system of forces is (3a + 5a), i.e., 8a.

Example 6 : Forces equal to 8, 7, 8, 7 newton respectively act along the sides of a square taken in order. Find their resultant.

(Left as an exercise)

3.5 LET US SUM UP

- The moment of a force about a given point is defined to be the product of the force and the perpendicular distance from the point to the line of action of the force.
- The moment of a force is considered to be positive if the tendency of rotation is in the counter-clockwise direction and negative if the tendency of rotation is in the clockwise direction.
- Two equal and unlike parallel forces (having different lines of action) acting on a rigid body form a couple.

  The perpendicular distance between the forces of a couple is called the arm of the couple.

  The product of one of the forces of a couple and its arm is called the moment of the couple.

3.6 FURTHER READING

1) Statics and Dynamics with Background Mathematics, Adrian Roberts, Cambridge University Press.

2) Enginering Machanics : Statics and Dynamics by R. G. Hibbeler and Ashok Gupta.

3) Engineering Mechanics : Statics and Dynamics by S. Rajasekarn and Sankara subramanian. Vikas Publishing House Pvt. Ltd.

3.7 ANSWERS TO CHECK YOUR PROGRESS

CHECK YOUR PROGRESS

Ans to Q No 1: ∵ O is the centre of the square ABCD, therefore its distance from the sides of the square $= \frac{1}{2} \times 4m = 2m$

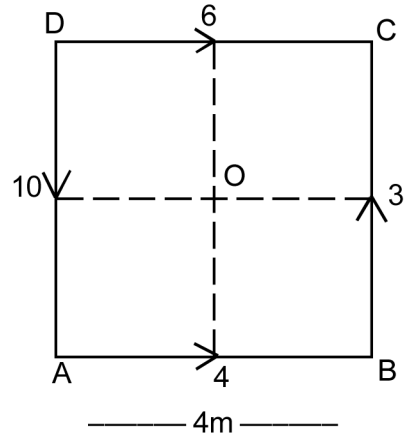


Fig 3.7

Sum of the moments about A

= 0 x 4 + 4 x 3 – 4 X 6 + 0 x 10 = –12 newton m

Sum of the moments about B

= 0 x 4 + 0 x 3 – 4 x 6 + 4 x 10 = 16 newton m.

Sum of the moments about C

= 0 x 3 + 0 x 6 + 4 x 10 + 4 x 4

= 56 newton m

Sum of the moments about D = 0 x 6 + 0 x 10 + 4 x 4 + 4 x 3

= 28 newton m

Sum of the moments about O = 2 x 4 + 2 x 3 – 2 x 6 + 2 x 10

= 22 newton m

Ans to Q No 2: Left as exercise.

Ans to Q No 3: Left as exercise.

3.8 MODEL QUESTIONS

Q 1: ABCD is a square of sides 1.5 m each. Forces of magnitude 4, 3, 2 and $5\sqrt{2}$ act along CB, BA, DA and DC respectively. Find the sum of the moments of the forces about C.

Q 2: Forces having magnitudes 4, 7, 4 and 7 units act along the sides of a square taken in order, prove that the resultant of the forces is a couple whose moment is of magnitude 11a units, a being the length of each side of the square.

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UNIT 4 : FRICTION

UNIT STRUCTURE

4.1 Learning Objectives

4.2 Introduction

4.3 Friction

- 4.3.1 Definition
- 4.3.2 Types of Friction
- 4.3.3 Limiting Friction
- 4.3.4 Co-efficient of Friction
- 4.3.5 Laws of Friction

4.4 Let Us Sum Up

4.5 Further Reading

4.6 Answers To Check Your Progress

4.7 Model Questions

4.1 LEARNING OBJECTIVES

After going through this unit, you will able to know

- definition of friction
- types of friction
- limiting friction & co-efficient of friction
- laws of friction.

4.2 INTRODUCTION

In previous units, we have already discussed forces ,types of forces and coplanar forces. in this unit we will study friction, type of friction, limiting friction and co-efficient of friction. In this unit, we will also discuss the laws of friction.

4.2 FRICTION

4.3.1 Definition

If a body slides or rolls on another, there is a force which tends to resist

motion. This resistance is called the force of friction. Friction is a force which resist the sliding of one body over another. It is a self adjusting force.

4.3.2 Types of Friction

Friction of the following two types–

1) Statical friction
2) Dynamical friction

1) Statical friction : The force of friction acting on a body at rest is called Statical friction.

2) Dynamical friction : The force of friction a acting on a body when it is actually in motion is called dynamical friction. It is also called kinetic friction. Dynamical friction is of the following two types.

(a) Sliding friction : When a body slides over another body, the friction experienced by the body is called sliding friction.

(b) Rolling friction : When a body rolls over another body the friction experiences by the body is called rolling friction.

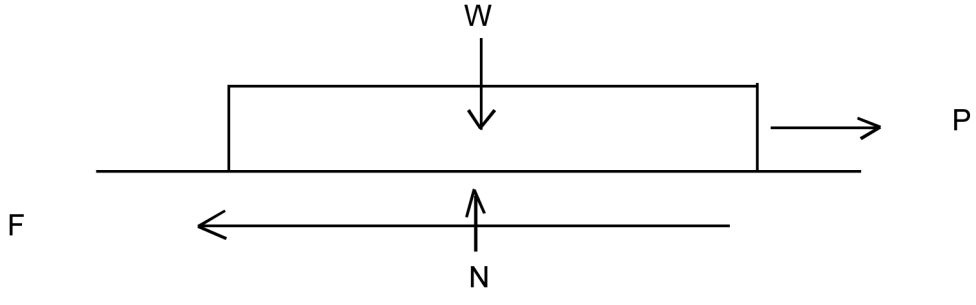
4.3.3 Limiting Friction

The force of friction acting on a body when it is on the point of motion is called limiting friction. limiting friction is equal in magnitude and opposite in direction to the force which tends to produce motion

4.3.4 Co-efficient of Friction

Limiting friction between two bodies bears a constant ratio to the normal reaction between them. This ratio is called the coefficient of friction.

It depend on the nature of the surfaces in contact.



Co-efficience of friction = $\frac{F}{N}$

Where F is limiting friction and N is normal reaction between the contact surfaces.

Co-efficient of friction is denoted by μ.

There $\mu = \frac{F}{N}$

4.3.5 Laws of Friction

1. The direction of the force of friction is always opposite to that in which the point of contact has the tendency to move.
2. The magnitudes of the force of friction is such that it is sufficient to prevent the motion of the point of contact, subject to a certain minimum limit.
3. Limiting friction is independent of the areas in contact.

CHECK YOUR PROGRESS

Q 1: Define friction?

Q 2: Define friction?

Q 3: Define co-efficient of friction?

4.4 LET US SUM UP

- If a body slides or rolls on another, there is a force which tends to resist motion. This resistance is called the force of friction.
- Friction is a force which resists the sliding of one body over another. It is a self adjusting force.
- Friction is of two types : (a) Statical friction (b) Dynamical friction
- Dynamical friction is of two types– (a) Sliding friction (b) Rolling friction.
- The force of friction acting on a body when it is on the point of motion is called limiting friction.
- Limiting friction between the body bears a constant ratio to the normal reaction between them. This ratio is called the co-efficient of friction (μ).

  i.e., $\mu = \frac{F}{N}$

  where F = limiting friction

  N = Normal reaction between the contact surfaces.

4.5 FURTHER READING

1) Statics and Dynamics with Background Mathematics, Adrian Roberts, Cambridge University Press.
2) Enginering Machanics : Statics and Dynamics by R. G. Hibbeler and Ashok Gupta.
3) Engineering Mechanics : Statics and Dynamics by S. Rajasekarn and Sankara subramanian. Vikas Publishing House Pvt. Ltd.

4.6 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q No 1: See section 4.3.1

Ans to Q No 2: See section 4.3.5

Ans to Q No 3: See Section 4.3.4

4.7 MODEL QUESTIONS

Q 1: Explain statical friction and dynamical friction.

Q 2: Define limiting friction.

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UNIT 5: KINEMATICS OF A PARTICLE

UNIT STRUCTURE

5.1 Learning Objectives

5.2 Introduction

5.3 Kinematics of a Particle and related terms

5.4 Relative velocity

5.5 Angular velocity

5.6 Rectilinear motion under constant acceleration

5.7 Newton's Laws of Motion

5.8 Let Us Sum Up

5.9 Further Readings

5.10 Answers To Check Your Progress

5.11 Model Questions

5.1 LEARNING OBJECTIVES

After going through this unit, you will able to learn

- kinematics of a particle and related terms
- rectilinear motion under constant acceleration
- Newton's laws of motion.

5.2 INTRODUCTION

Dynamics is a branch of Mechanics. It deals with motion of bodies under the action of forces.

Dynamics is generally divided into two parts, – Kinetics and Kinematics. In this unit, we will study kinematics of a particle and related terms; angular velocity, relative velocity etc. Also, we will study rectilinear motion under constant acceleration. We will also study Newton's laws of motion.

5.3 KINEMATICS OF A PARTICLE AND RELATED TERMS

Kinematics deals with motion of material bodies without taking into consideration of their mass and the forces which cause it.

Matter : Matter is anything that occupies space and is perceptible by our senses.

Mody : A body (or material body) is a limited portion of matter.

Volume : The amount of space occupied by body is called its volume.

Mass : Mass is the quantity of matter in a body.

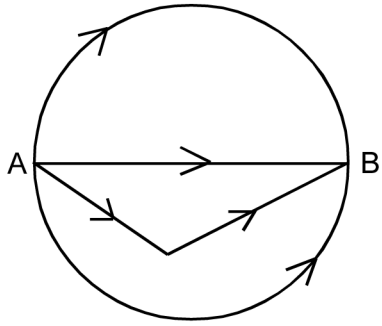
Particle : Particle is a body of negligible mass. It may be identified with a moving geometrical point.

Position : A point at which a particle may be found at a certain time is known as position.

Motion and Rest : Motion is change of position of a body with time. When it is not changing its position, it is said to be at rest.

Displacement : If a particle moves along a certain path, its position changes. The change of position between two points, whatever may be the path, is called displacement.

In the adjoining figure, A is the initial position and B is the final position of the particle. $\overrightarrow{AB}$ is the displacement.



We note that displacement has three characteristics, viz, (a) magnitude – represented by the length AB, (b) directions – along AB, (c) sense, – the displacement from A to B.

Thus, displacement is a vector quantity.

Velocity : The velocity of a moving particle is its rate of change of displacement having a finite direction and magnitude.

Velocity is a vector quantity.

Speed : Rate of change of position is speed. It has no specified direction.

Speed is a scalar quantity.

Note : In rectilinear motion, speed and velocity are identical.

Units of magnitude of velocity : If the displacement of a particle is measured in metre and time in second, the unit of speed is one metre per second. It is denoted by m/

s. If the displacement of a particle is measured in kilometre and time in hour ,the unit of speed is one kilometre per hour. It is denoted by km/h.

It may be noted that 18km/h = 5 m/s.

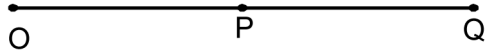
Uniform velocity

If a particle moves along the same straight line in the same sense, and passes over equal distances in equal intervals of time, its velocity is said to be uniform.

Analytical Expression for Velocity

Let a moving point P be at a distance x from a fixed point O at time t. Let the particle be at Q at time $t+\delta t$, where $OQ = x+\delta x$

$\therefore PQ = \delta x$



Velocity υ of P at time t = $\lim_{\delta t \to o} \frac{\delta x}{\delta t}$ ($\because$ velocity is the rate of change of displacement)

$= \frac{dx}{dt}$

i.e., $\upsilon = \frac{dx}{dt}$.

For uniform velocity, $\upsilon = \frac{dx}{dt}$ = constant.

For non-uniform velocity, $\upsilon = \frac{dx}{dt} \neq$ a constant.

Acceleration

The rate of change of velocity of a moving particle is called its acceleration.

If a particle moves in a straight line and its velocity increases, the acceleration is positive; and if the velocity decreases, the acceleration is negative. A negative acceleration is said to be retardation.

Units of acceleration

Unit of acceleration is one metre or one contimetre per second change of velocity per second. Symbolically it is written as 1m/s^2 or 1 cm/s^2. The first “per second” gives the rate of change of velocity, and the second “per second” gives the unit of time in which the change of velocity occurs.

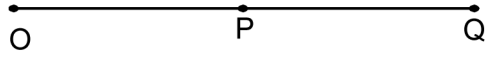
Analytical Expression for Acceleration

Let υ be the velocity of a particle at time t, and $\upsilon+\delta\upsilon$ be its velocity at time $t+\delta t$. Acceleration f at time t

$= \lim_{\delta t \to 0} \frac{(\upsilon+\delta\upsilon)-\upsilon}{\delta t}$ ($\because$ acceleration is the rate of change of velocity)

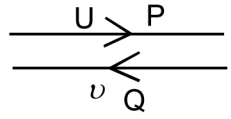
$= \lim_{\delta t \to 0} \frac{\delta \upsilon}{\delta t}$

$= \frac{d\upsilon}{dt}$

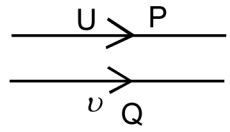


i.e., $f = \frac{d\upsilon}{dt}$

Now $f = \frac{d\upsilon}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}$



Again $f = \frac{d\upsilon}{dt} = \frac{d\upsilon}{dx}\frac{dx}{dt} = \upsilon\frac{d\upsilon}{dx}$



Thus, $f = \frac{d\upsilon}{dt} = \frac{d^2x}{dt^2} = \upsilon\frac{d\upsilon}{dx}$

5.4 RELATIVE VELOCITY

Let two points P and Q move in the same place. The rate of change of position of Q as seen from P is defined as the relative velocity of Q with respect to P.

If P move with velocity u and Q move with velocity υ in the opposite direction of P then, velocity of P relative to Q is $= u - (-\upsilon) = u + \upsilon$

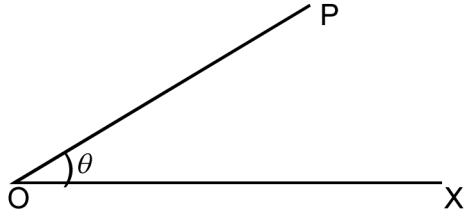
If P and Q move in the same direction with their respective velocities u and υ then, velocity of P relative to Q is $= u - \upsilon$

5.5 ANGULAR VELOCITY

The rate of change of the angle made by the line joining a fixed point to the moving point on a plane with any straight line drawn in a fixed direction in that plane is called the angular velocity of the moving point about the fixed point.

Let O be a fixed point on a fixed line OX in a plane.

Let P be the position of a particle moving in the plane at time t.



Let $\angle XOP = \theta$. It is measured positive in the anti-clockwise sense.

The rate of change of θ with respect to time t, i.e., $\frac{d\theta}{dt}$ is the angular velocity of the particle at P about the point O. Thus, the rate at which the angle θ is increasing is the angular velocity of the particle about O.

Angular velocity is measured in radians per second. Thus, if ω is the angular velocity of a particle, then

$$\omega = \frac{d\upsilon}{dt} \text{ rad/s}$$

The rate of change of ω with respect to time t, i.e., $\frac{d\omega}{dt}$, i.e., $\frac{d^2\theta}{dt^2}$ is the acceleration in the sense of θ increasing.

CHECK YOUR PROGRESS

Q 1: A particle, moving in a straight line travels a distance x in time t. If $x = 3t^2 + 2t$, find velocity and acceleration after 4 seconds (x is in metres, t in seconds).

Q 2: A particle, moving in a straight line, travels a distance x in time t (x is in metres, t in seconds). If $x = 9t^2 - t^3 - 4$, find (i) its initial acceleration, (ii) its velocity when its acceleration is zero, (iii) its distance from the starting point when it again comes to rest.

Q 3: A particle moves in a straight line in such a way that its distance from a fixed point O in the line at the end of t seconds is given by $x = t^3 + 18t + 11$ (measured in cm).

Find its distance from O, its velocity and acceleration at the end of 5 seconds.

Q 4: The velocity of a moving particle on a line at a distance x from a fixed point in its line of motion is given by $\vartheta^2 = 2(x\sin x + \cos x)$.

Find its acceleration.

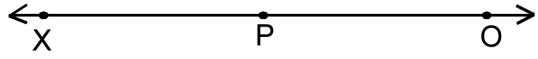
Q 5: The velocity of a moving particle at a distance x from a fixed point in its line of action is given by $v^2 = ax^2 + 2bx + c$, where a, b, c are constants.Prove that the acceleration is proportional to the distance from the fixed point.

5.6 RECTILINEAR MOTION UNDER CONSTANT ACCELERATION

The motion of a particle which takes place along a straight line is known as rectilinear motion.

Let a particle move in a straight line under constant acceleration.

Let OX be the straight line in which the particle moves.



Let the particle be at O at the start of the motion. O is called the initial position of the particle.

Let P be the position of the particle at time t. Let OP = x.

Let the particle start with velocity u and let υ be its velocity at P.

Let f be the constant acceleration.

Now, we prove the following relations.

I. $\boxed{v = u + ft}$

We have $f = \dfrac{d\upsilon}{dt}$

$\therefore\ dv - fdt$

$\therefore \int dv = f\int dt$

or $v = ft + c$, where c is a constant of integration.

When t = 0, $v = u$

$\therefore$ u = f0 + c or c = u

$\therefore\ v = ft + u$ i.e. $\boxed{v = u + ft}$

II. $\boxed{s = ut + \dfrac{1}{2} f t^2}$

We have $v = u + f\,t$

Now, $v = \dfrac{dx}{dt}$

$\therefore\ \dfrac{dx}{dt} = u + f\,t$

or $dx = udt + ftdt$

$\therefore\ \int dx = u\int dt + f\int t\,d$

$\therefore\ x = ut + \dfrac{ft^2}{2} + c$, where c is a constant

When t = 0, x = 0

$\therefore$ 0 = 0 + 0 + c i.e. c = 0

$\therefore\ x = ut + \frac{1}{2} f t^2$

or $\boxed{s = ut + \frac{1}{2} f t^2}$

where s is the distance or space (s for space) described by the particle in time t.

III. $\boxed{v^2 = u^2 + 2fs}$

We have

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

Also $\frac{dv}{dt} = f$

$\therefore\ v\frac{dv}{dx} = f$

or, $v dv = f\, dx$

$\therefore\ \int v dv = f \int dx$

$\therefore\ \frac{v^2}{2} = fx + c$, where c is a constant of integration.

When $v = u$ x = 0

$\therefore\ \frac{d^2}{2} = 0 + c$ i.e. $C = \frac{u^2}{2}$

$\therefore\ \frac{v^2}{2} = fx + = \frac{u^2}{2}$

or $v^2 = u^2 + 2fx$

or $\boxed{v^2 = u^2 + 2fs}$

IV. Expression for the distance travelled by a particle in n th second of its motion in a straight line.

Let s_n and s_{n-1} denote the distances described by the particle in n seconds and (n–1) seconds respectively.

$\therefore$ the distance travelled in the nth second

= s_n – s_{n-1}

$$=\left(un+\frac{1}{2}fn^2\right)-\left(u(n-1)+\frac{1}{2}f(n-1)^2\right)$$

$$=u+\frac{1}{2}f(n-1)$$

Theorem :The average velocity is the mean of the initial and the final velocities and is equal to the velocity at the middle of the interval.

Proof Average velocity $=\frac{total\ distance\ traversed}{total\ time\ taken}$

$$=\frac{s}{t}$$

$$=\frac{ut+\frac{1}{2}ft^2}{t}$$

$$=u+\frac{1}{2}ft$$

$$=\frac{2u+ft}{2}$$

$$=\frac{u+(u+ft)}{2}$$

$$=\frac{u+v}{2}$$

= mean of initial and final velocities.

Again, average velocity $=\frac{s}{t}$

$$=\frac{ut+\frac{1}{2}ft^2}{t}$$

$$=u+\frac{1}{2}ft$$

$$=u+f\left(\frac{t}{2}\right)$$

= Velocity at time $\frac{t}{2}$

= Velocity at the middle of the interval.

Note : If the particle moves under variable acceleration or if the particle moves in a curved path, the above relations do not hold good.

Example 1: A particle moving in a straight line with acceleration 5m/s^2 describes 200 metres in 5 seconds. What are its initial and final velocities?

Solution : Let the initial velocity be u m/s and the final velocity be v m/s.

Now $200 = u5 + \frac{1}{2} \times 5 \times 5^2$ $\left(s = ut + \frac{1}{2} ft^2 \right)$

or, $200 = 5u + \frac{125}{2}$

or, $10u = 400 - 125$

or, $10u = 275$

$\therefore u = 27.5$

Againg $v = 27.5 + 5 \times 5$ $(v = u + ft)$

= 52.5

$\therefore$ initial velocity = 27.5 m/s

final velocity = 52.5 m/s

Example 2 : A particle is moving in a straight line with uniform acceleration. In the 11th and 15th seconds from the commencement, it moves through 720 and 960 cms respectively. Find its initial velocity and the acceleration with which it moves.

Solution : Let the initial velocity and the acceleration be u cm/s and f cm/s^2 respectivaly.

$\therefore 720 = u + \frac{1}{2} f(2 \times 11 - 1)$

or $720 = u + \frac{21}{2} f$ (1)

and $960 = u + \frac{1}{2} f(2 \times 15 - 1)$

or $960 = u + \frac{29}{2} f$ (2)

Subtracting (1) from (2), we get

240 = 4f or f = 60

$\therefore$ from (1), $720 = u + \frac{21}{2} \times 60$

$\therefore$ u = 720 – 630 = 90

$\therefore$ initial velocity = 90 cm/s

acceleration = 60 cm/s^2

Example 3 : Prove that, for a particle moving with uniform acceleration f in a straight line,

$$f = 2\left(\frac{s'}{t'} - \frac{s}{t}\right) / \left(t + t'\right)$$

where s is the space deseribed in t seconds and s/ during the next t/ seconds.

Solution : Let u be the initial velocity and v be the velocity after time t.

$$\therefore\ v = u + f\,t \qquad (1)$$

$$s = ut + \frac{1}{2} f\,t^2 \qquad (2)$$

$$s' = v\,t' + \frac{1}{2} f\,t'^2$$

i.e. $s' = (u + ft)t' + \frac{1}{2} f\,t'^2$ (3)

From (2), $\frac{s}{t} = u + \frac{1}{2} f\,t$ (4)

From (3), $\frac{s'}{t'} = u + ft + \frac{1}{2} f\,t'$ (5)

Subtracting (4) from (5), we get

$$\frac{s'}{t'} - \frac{s}{t} = \frac{1}{2} ft + \frac{1}{2} f\,t'$$

or, $2\left(\frac{s'}{t'} - \frac{s}{t}\right) = f\left(t + t'\right)$

$$\therefore\ f = 2\left(\frac{s'}{t'} - \frac{s}{t}\right) / \left(t + t'\right)$$

CHECK YOUR PROGRESS

Q 1: A particle which is moving along a straight line at the rate of 60 m/s is brought to rest in 3 minutes with uniform retardation. Find the retardation and also the distance that the particle travels before coming to rest. [Retardation is negative acceleration. When velocity decreases, its rate of change is negative.]

Q 2: A bullet fired into a target loses half its velocity after penetrating 3 cm. How much further will it penetrate?

Q 3: If x_1 and x_2 be the spaces described by a particle moving on a straight line during the pth and (p+q)th seconds of its motion respectively, show that its acceleration is $\frac{x_2 - x_1}{q}$.

Q 4: A train moving on a straight path and stopping at two stations 2 km apart takes 4 minutes on the journey from one of the stations to the other. Assuming that its motion is first that of uniform acceleration a and then that of uniform retardation b, prove that $\frac{1}{a}+\frac{1}{b}=4$

Q 5: A particle starts with initial velocity u to move in a straight line and passes over the first half of the given distance with a uniform acceleration f_1 and over the second half with uniform acceleration f_2. Show that the final velocity is the same as if the whole distance was traversed with uniform acceleration $\frac{1}{2}(f_1+f_2)$.

Q 6: If a particle moving on a straight line with uniform acceleration describes successive equal distances in times t_1, t_2, t_3 respectively, show that $\frac{1}{t_1}-\frac{1}{t_1}+\frac{1}{t_3}=\frac{3}{t_1+t_1+t_3}$.

5.7 NEWTON'S LAWS OF MOTION

Sir Isaac Newton (1642-1727) formulated the following statements. These are known as Laws of Motion.

First Law : Every body continues in its state of rest or of uniform motion in a straight line unless acted on by some external force.

Second Law : The rate of change of momentum is proportional to the applied force and takes place in the direction in which the force acts.

Third Law : To every action, there is an equal and opposite reaction.

The above laws are taken as axioms. These axioms are based on experience and observations. There are no direct proofs of these laws.

Momentum

The momentum of a moving particle at any instant is the product of its mass and its velocity at that instant.

The momentum of a body is a vector quantity.

Unit of momentum : The unit of momentum is defined as "unit mass moving with unit velocity".

In CGS system, the unit of momentum is gm-cm per sec.

In FPS system, the unit of momentum is lb-ft per sec.

If a body of mass m gm moves with a velocity u cm/s, the momentum is mu gm-cm/s

The formula P = mf

Let a force P act on a body of mass m. Let the force change the velocity of the body continuously.

Let v be the velocity and f the acceleration produced at any instant t.

Then rate of change of momentum

$$= \frac{d}{dt}(mv)$$

$$= m\frac{dv}{dt} + v\frac{dm}{dt}$$

= mf [For constant mass m, $\frac{dm}{dt} = 0$]

The above acceleration is along the line of action of P.

By Newton's second Law of Motion, P is proportional to mf.

$\therefore$ P = kmf, where k is a constant

We choose a unit of force so that k = 1.

Let unit force be that force which acting on unit mass produces unit acceleration. Thus, we have

P = 1, if m = 1 and f = 1

$\therefore$ k = 1

$\therefore$ P = mf

Units of Force

Newton's Second Law of Motion gives us a method of measuring forces by applying them to a standard mass and observing their applications.

1 Newton is that force which gives a mass of 1 kg an acceleration of 1 m/s^2.

1 dyne is that force which gives a mass of 1 gm an acceleration of 1 cm/s^2.

1 poundal is that force which gives a mass of 1 lb an acceleration of 1 ft/s^2.

The above units, viz, Newton, dyne and poundal are known as absolute units of force. They do not depend upon the value of g, the acceleration due to gravrty.

The weight of a body is the force with which the earth attracts the body.

Let W be weight of a mass m. The acceleration produced by the force W in the mass m is the acceleration of due to granty.

Using the formula P = mf, we get W = mg.

1. When the mass is 1 gm, the weight is g dyne.
2. When the mass is 1 kg, the weight is g newton.
3. When the mass is 1 lb, the weight is g poundal.

We have the following relations

1 gm wt = g dyne (g = 981)

1 kg wt = g newton (g = 9.81)

1 lb wt = g poundal (g = 32.2)

The above units, viz, gm wt, kg wt, lb wt are called gravitational units of force. They depend on the value of g, the acceleration due to g.

Note : I newton = 1 kg x 1 m/s^2 = 10^3 gm x 10^2 cm/s^2 = 10^5 dyne

1 poundal = 13800 dyne = 0.138 newton.

Example 1 : What is the weight of a body if its mass is 1 gm ?

Solution : The required weight

= 981 dyne

Example 2 : What is the weight of a body if its mass is 1 kg ?

Solution : The required weight

= 9.81 N

Example 3 : What is the mass of a body if its weight is 29.4 N ?

Solution : The required mass

$\frac{29.4}{9.8}$ kg = 3 kg (g = 9.8)

Example 4 : A force of 8N acts on a body of mass 2 kg. What is the acceleration ?

Solution : Here P = 8N, m = 2kg

Using the formula P = mf, we have

$$f = \frac{p}{m} = \frac{8}{2} m/s^2 = 4m/s^2$$

CHECK YOUR PROGRESS

Q 1: A mass of 5 gm is accelerating at 981 cm/s^2. What is the force acting on it ?

Q 2: A particle of mass 675 g is accelerating at 16 cm/s^2. Find the force acting on it in newton.

Q 3: A body of moves 45 kg is accelerating at 6m/s^2. Find the force acting on it.

Q 4: Find the weight of mass of 2 ton on the earth. (1 ton = 1000 kg)

Q 5: A particle of mass 20 g is acted upon by a constant force, which produces a velocity of 15 cm/s in 5 seconds. If the particle is initially at rest, find the magnitude of the force.

Q 6: A uniform force of 520 dyne, acting for half a minute, changes the velocity of a body moving in a straight line from 2.9 m/s to 3.5 m/s. Find the mass of the body.

Q 7: A constant force of 200 dyne changes the velocity of a body moving in a straight line from 250 m/s to 300 m/s in one minute. Find the mass of the body.

Q 8: A mass m gram is acted on by a constant force of P dyne under which, in t second, it moves a distance x cm from rest, and acquires a velocity v cm per second. Show that $x = \frac{1}{2}\frac{mv^2}{p}$

5.8 LET US SUM UP

- If a particle moves along a certain path, its position changes. The change of position between two points, whatever may be the path, is called displacement. Displacement is a vector quantity.
- The velocity of a moving particle is its rate of change of displacement having a finite direction and magnitude.
  Velocity is a vector quantity.
- Rate of change of position is speed. It has no specified direction. Speed is a scalar quantity.
- In rectilinear motion, speed and velocity are identical.
- If a particle moves along the same straight line in the same sense, and passes over equal distances in equal intervals of time, its velocity is said to be uniform.
- The rate of change of velocity of a moving particle is called its acceleration.
- A negative acceleration is said to be reardation.
- Let two points P and Q move in the same plane. The rate of change of position of Q as seen from P is defined as the relative velocity of Q with respect to P.

- The rate of change of the angle made by the line joining a fixed point to the moving point on a plane with any straight line drawn in a fixed direction in that plane is called the angular velocity of the moving point about the fixed point.
  Angular velocity is measured in radian per second.
- The motion of a paiticle which takes place along a straight line is known as rectilinear motion.
- Let a particle start with a velocity u and move along a straight line with uniform acceleration f. Let s be the displacement at the end of time t, and v be the velocity at that time. Then
  1. $v = u + ft$
  2. $s = ut + \frac{1}{2} ft^2$
  3. $v^2 = u^2 + 2fs$
- The average velocity is the mean of the initial and the final velocities and is equal to the velocity at the middle of the interval.
- Newtons Laws of Motion are as follows :
  First Law : Every body continues in its state of rest or of uniform motion in a straight line unless acted on by some external force.
  Second Law : The rate of change of momentum is proportional to the applied force and takes place in the direction in which the force acts.
  Third Law : To every action, there is an opposite and equal reaction.
- The momentum of a moving particle at any instant is the product of its mass and its velocity at that instant.
- P = mf, where m is the mass of a body, P the applied force acting on it and f the acceleration produced by P.
- P = mf is satisfied when
  (1) P is in newton, m in kg, f in m/s^2

or (2) P is in dyne, m in gram, f in cm/s^2

or (3) P is in poundal, m in lb, f in ft/s^2.

5.9 FURTHER READING

1) Statics and Dynamics with Background Mathematics, Adrian Roberts, Cambridge University Press.

2) Enginering Machanics : Statics and Dynamics by R. G. Hibbeler and Ashok Gupta.

3) Engineering Mechanics : Statics and Dynamics by S. Rajasekarn and Sankara subramanian. Vikas Publishing House Pvt. Ltd.

5.10 ANSWERS TO CHECK YOUR PROGRESS

CHECK YOUR PROGRESS 1

Ans to Q No 1: Here $x = 3t^2 + 2t$

$$\therefore\ v = \frac{dx}{dt} = 6t + 2$$

$$f = \frac{d^2x}{dt^2} = 6$$

$\therefore$ velocity after 4 seconds = (6 x 4 + 2) m/s

= 26 m/s

acceleration after 4 seconds = 6 m/s^2

Ans to Q No 2: Here $x = 9t^2 - t^3 - 4$

$$\therefore\ v = \frac{dx}{dt} = 18t - 3t^2$$

$$f = \frac{d^2x}{dt^2} = 18 - 6t$$

(i) Initial acceleration

= Acceleration (at t = 0)

= (18 – 6 x 0) m/s^2

= 18m/s^2

(ii) When acceleration is zero; $\frac{d^2x}{dt^2} = 0$

i.e. 18 – 6t = 0

i.e. t = 3

$\therefore$ Velocity at t = 3

= (18 x 3 – 3 x 3^2) m/s

= 27m/s

(iii) When the particle comes to rest,

$$v = \frac{dx}{dt} = 0$$

This $\Rightarrow$ $18t - 3t^2 = 0 \Rightarrow t(6 - t) = 0 \Rightarrow t = 0$ or $t = 6$.

When $t = 0$, $x = 0$

When $t = 6$, $x = 9 \times 6^2 - 6^3 - 4 = 104$

$\therefore$ the required distance = 104m

Ans to Q No 3: Here $x = t^3 + 18t + 11$

$$\therefore\ v = \frac{dx}{dt} = 3t^2 + 18$$

$$f = \frac{d^2x}{dt^2} = 6t$$

At the end of 5 seconds,

$x = (5^3 + 18 \times 3 + 11)$ cm = 190 cm

$v = (3 \times 5^2 + 18)$ cm/s = 93 cm/s

$f = 6 \times 5$ cm/s^2 = 30 cm/s^2.

Ans to Q No 4: Here $v^2 = 2(x\sin x + \cos x)$

$$\therefore\ 2v\frac{dv}{dx} = 2(1\sin x + x\cos x - \sin x)$$

$$\Rightarrow\ v\frac{dv}{dx} = x\cos x$$

$\therefore$ acceleration = $x \cos x$

Ans to Q No 5: Here $v^2 = ax^2 + 2bx + c$

$$\therefore\ 2v\frac{dv}{dx} = 2ax + 2b$$

$$\Rightarrow v\frac{dv}{dx} = ax + b$$

$\therefore$ a and b are constants, therefore acceleration is proportional to the distance from the origin.

CHECK YOUR PROGRESS 2

Ans to Q No 1: Let f m/s^2 be the uniform acceleration and x m be the distance travelled by the particle before coming to rest.

Now $0 = 60 + 180f$ ($v = u + ft$, $t = 30 \times 6 = 180$s)

or $f = -\frac{1}{3}$

$\because$ f is negative, $\therefore$ the retardation is $\frac{1}{3}$ m/s^2.

Again $0^2 = 60^2 + 2.\left(-\frac{1}{3}\right).x$

or x = 5400

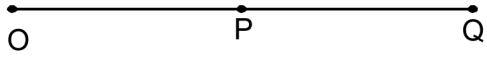
$\therefore$ the required distance is 5400 m

Ans to Q No 2: Let the bullet be fired with velocity u. Let OA = 3 cm

$\therefore$ Velocity at $A = \frac{1}{2}u$

Now, using the formula $v^2 = u^2 + 2fs$ we have $\left(\frac{u}{2}\right)^2 = u^2 + 2f3$

or, $-\frac{3u^2}{4} = 6f$



$\therefore\ f = -\frac{u^2}{8}$

Let the bullet penetrate upto B, i.e., will come to rest at B, i.e., velocity at B is zero.

$\therefore$ using the formula $v^2 = u^2 + 2fs$, we have

$0 = \left(\frac{u}{2}\right)^2 + 2\left(-\frac{u^2}{8}\right)s$, where AB = s

or, $\frac{u^2}{4} = \frac{u^2}{4}s$

$\therefore\ s = 1$

$\therefore$ the required distance is 1 cm.

Ans to Q No 3: Let the acceleration be f, and the initial velocity be u.

$\therefore\ x_1 = u + \frac{1}{2}f(2p-1)$ (1)

$x_2 = u + \frac{1}{2}f(2p+2q-1)$ (2)

Subtracting (1) from (2), we get

$x_2 - x_1 = \frac{1}{2}f.2q$

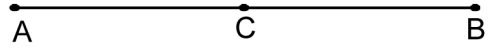
$\therefore\ f = \frac{x_2 - x_1}{q}$

Thus, the required acceleration is $\frac{x_2 - x_1}{q}$.

Ans to Q No 4: The train stops at both the stations A and B.

$\therefore$ the initial velocity as well as the final velocity is zero

We have AB = 2



Let the train travel AC (= s_1) in time t_1 with uniform acceleration a and CB (= s_2) in time t_2 with uniform retardation b.

Clearly $s_1 + s_2 = 2$,

$t_1 + t_2 = 4$

Let the velocity at C be v.

$\therefore\ v = 0 + at_1$ i.e. $t_1 = \frac{v}{a}$

$0 = v - bt_2$ i.e. $t_2 = \frac{v}{b}$

$\therefore\ t_1 + t_2 = v\left(\frac{1}{a} + \frac{1}{b}\right)$

i.e. $v\left(\frac{1}{a} + \frac{1}{b}\right) = 4$ _____(1)

Again, $v^2 = 0 + 2as_1$ i.e. $s_1 = \frac{v^2}{2a}$

$0 = v^2 - 2bs_2$ i.e. $s_2 = \frac{v^2}{2b}$

$\therefore\ s_1 + s_2 = \frac{v^2}{2}\left(\frac{1}{a} + \frac{1}{b}\right)$

i.e. $\frac{v^2}{2}\left(\frac{1}{a} + \frac{1}{b}\right) = 2$

i.e. $v^2\left(\frac{1}{a} + \frac{1}{b}\right) = 4$ _____(2)

From (1) and (2), we get

$v = 1$

$\therefore$ from (1), $\frac{1}{a} + \frac{1}{b} = 4$

Ans to Q No 5: Let the given distance be s. Let v be the final velocity and v_1 be the velocity at the end of the first half.

$\therefore\ v_1^2 = u^2 + 2f_1.\frac{s}{2} = u^2 + f_1 s$

$$v^2 = v_1^{\,2} + 2f_2 . \frac{s}{2}$$

$$= u^2 + f_1 s + f_2 s$$

$$= u^2 + (f_1 + f_2)s$$

$$= u^2 + 2 \times \frac{1}{2}(f_1 + f_2)s$$

Thus, the result is proved.

Ans to Q No 6: Let u be the velocity at the beginning of t_1, and let f be the acceleration.

$\therefore$ at the beginning of t_2, the velocity is $u + ft_1$.

Again, at the beginning of t_3, the velocity is $(u + ft_1) + ft_2$ i.e. $u + f(t_1 + t_2)$

Let the equal distance be s.

$$\therefore\ s = ut_1 + \frac{1}{2} f t_1^{\,2} \qquad (1)$$

$$s = (u + ft_1)t_2 + \frac{1}{2} f t_2^{\,2} \qquad (2)$$

$$s = u + f(t_1 + t_2)t_3 + \frac{1}{2} f t_3^{\,2} \qquad (3)$$

Also $s + s + s = u(t_1 + t_2 + t_3) + \frac{1}{2} f (t_1 + t_2 + t_3)^2$

i.e. $3s = u(t_1 + t_2 + t_3) + \frac{1}{2} f (t_1 + t_2 + t_3)^2 \qquad (4)$

From (1), $\frac{s}{t_1} = u + \frac{1}{2} f t_1$

From (2), $\frac{s}{t_2} = u + f t_1 + \frac{1}{2} f t_2$

From (3), $\frac{s}{t_3} = u + f(t_1 + t_2) + \frac{1}{2} f t_3$

From (4), $\frac{3s}{t_1 + t_2 + t_3} = u + \frac{1}{2} f (t_1 + t_2 + t_3)$

Clearly $\frac{s}{t_1} - \frac{s}{t_2} + \frac{s}{t_3} = \frac{3s}{t_1 + t_2 + t_3}$

i.e. $\frac{1}{t_1} - \frac{1}{t_2} + \frac{1}{t_3} = \frac{3}{t_1 + t_2 + t_3}$

CHECK YOUR PROGRESS 3

Ans to Q No 1: Here m = 5 gm,

f = 981 cm/s^2

$\therefore$ P = 5 x 981 dyne

= 4905 dyne

Ans to Q No 2: Here m = 675 g

f = 16 cm/s^2

$\therefore$ P = 675 x 16 dyne

= 10800 dyne

We know that

10^5 dyne = 1 newton

$\therefore$ 10800 dyne = $\frac{10800}{10^5}$ newton

= 0.108 newton

$\therefore$ P = 0.108 newton

Ans to Q No 3: Here m = 45 kg

f = 6 m/s^2

$\therefore$ P = 45 x 6 newton

= 270 newton

Ans to Q No 5: Let f cm/s^2 be the acceleration.

Now from the result $v = u + ft$ we have

16 = 0 + f5

$\therefore$ f = 3 cm/s^2

Also m = 20 g

$\therefore$ P = 20 x 3 dyne

= 60 dyne

Ans to Q No 6: Change in velocity while moving for $\frac{1}{2}$ minute = (3.5 – 2.9)

m/s = 0.6 m/s

$\therefore f = \frac{2 \times 0.6 \times 100}{60} cm/s^2$

= 2 cm/s^2

P = 520 dyne (given)

$$\therefore\ m = \frac{P}{f} = \frac{520}{2} g = 260g$$

Ans to Q No 8: We have $v^2 = 0 + 2fx$, where f is the acceleration

(using $v^2 = u^2 + 2fs$)

$$\therefore\ f = \frac{v^2}{2x}$$

Againg P = mf

or, $P = m\frac{v^2}{2x}$

$$\therefore\ x = \frac{mv^2}{2p}$$

?5.11 MODEL QUESTIONS

Q 1: A particle moves along a otraight line such that its displacement x from a fixed point on the line at time t is given by $x = t^3 - 9t^2 + 24t + b$. Determine (i) the instant when the acceleration becomes zero, (ii) The velocity of the particle at that instant.

Q 2: If the distance s is given by $s = at^2 + bt + c$, where t is the time and a, b, c are constants, prove that $4a(s-c) = v^2 - b^2$, where v is the velocity.

Q 3: A point moves with uniform acceleration and v_1, v_2, v_3 donote the average velocities in three successive. intervals t_1, t_2, t_3; prove that $\frac{v_1 - v_2}{v_2 - v_3} = \frac{t_1 - t_2}{t_2 + t_3}$.

Q 4: A particle moving in a straight line is at a distance of x meter from a given point on the line after t seconds from start where $x = t^3 - 2t - 16$. Find its velocity at the end of 3 seconds and acceleration of the end of 4 seconds.

Q 5: A particle moves in a straight line according to the law $v = x^2$. Show that, its acceleration is $2v^{3/2}$.

Q 6: A particle starts from rest and moves in a straight line first with uniform acceleration 'a' and then with uniform reterdation 'b'. If it comes to rest in time t after describing a distance s, prove that $t^2 = 2s\left(\frac{1}{a} + \frac{1}{b}\right)$.

Q 7: What is the mass of a body whose weight is 19.6 N ?

Q 8: What is the magnitude of the force that acts on a mass of 500 gm and produces acceleration $10 m/s^2$?

Q 9: A body of mass m gm at rest is acted upon by a uniform force P gmwt for t seconds. Prove that the velocity aquired is $\frac{pgt}{m} cm/s$ and the distance described is $\frac{pgt^2}{2m} cm$.

Q 10: A force of 8000 dyne acting on a body increases its velocity from 20 cm/s to 50m/s in 6 secs. Find the mass of the body.

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UNIT 6: SIMPLE HARMONIC MOTION AND RELATED PROBLEMS

UNIT STRUCTURE

6.1 Learning Objectives

6.2 Introduction

6.3 Simple Harmonic Motion

6.3.1 Geometrical Representation of Simple Harmonic Motion

6.3.2 Motion of a Simple Pendulum

6.4 Let Us Sum Up

6.5 Further Readings

6.6 Answers To Check Your Progress

6.7 Model Questions

6.1 LEARNING OBJECTIVES

After going through this unit, you will able to learn

- the equation of simple harmonic motion
- the velocity of the particle under simple harmonic motion
- position of the particle moving under simple harmonic motion, periodic time, frequency, amplitude etc.
- geometrical representation of simple harmonic motion
- the motion of a simple pendulum.

6.2 INTRODUCTION

Sometimes we find special types of motion, viz, the motion of a simple pendulum oscillating through a very small angle, the motion of a body tied to an extremity of a spring or elastic spring the other extremity of which being fixed etc. Such motions are simple harmonic motion. Here acceleration is considered to be proportional to the distance.

6.3 SIMPLE HARMONIC MOTION

If a particle moves in a straight line such that its acceleration is always directed

towards a fixed point in the line and is proportional to the distance from the point, then the particle is said to execute Simple Harmonic Motion (S.H.M).

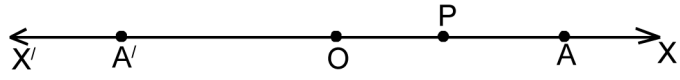


Fig 6.1

Let O be the fixed point on the line X/OX. Let a particle start from rest at A. Let P be the position of the particle at any time t, where OP = x since P is taken on the positive side, therefore x > o.

Then $\frac{dx}{dt}$ is the velocity and $\frac{d^2x}{dt^2}$ is the acceleration of the particle at time t acting in the direction in which x increases.

Since the acceleration is directed towards the fixed point O and is proportional to the distance of the particle from O, hence the equation of motion is $\frac{d^2x}{dt^2} = -\mu x$ – (1)

where μ is a positive constant.

The equation (1) is true whether P lies on the right or left of O.

Let v be the velocity of the particle at time t. Therefore,

$$v = \frac{dx}{dt}, \quad \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

$$\therefore v\,\frac{dv}{dx} = \frac{d^2x}{dt^2}$$

$\therefore$ (1) can be written as

$$v\,\frac{dv}{dx} = -\mu x$$

or, $v\,dv = -\mu x dx$

$$\therefore \int v dv = -\int \mu x dx$$

or $\frac{v^2}{2} = -\frac{\mu x^2}{2} + c$ where C is a constant of integration.

Let OA = a

Since the particle starts from rest at A, therefore $v = 0$ when x = a

$$\therefore 0 = -\frac{\mu a^2}{2} + C$$

i.e. $c = \frac{\mu a^2}{2}$

$$\frac{v^2}{2} = -\frac{\mu x^2}{2} + \frac{\mu a^2}{2}$$

i.e. $v^2 = \mu(a^2 - x^2)$

$\therefore v = -\sqrt{\mu(a^2 - x^2)}$ (2)

We note that x decreases when t increases. Therefore $v = \frac{dx}{dt}$ is taken to be negative.

From (2), we have

$$\frac{dx}{dt} = -\sqrt{\mu}\ \sqrt{a^2 - x^2}$$

or $$\sqrt{\mu}\ dt = -\frac{1}{\sqrt{a^2 - x^2}}dx$$

$$\therefore \int \sqrt{\mu}\ dt = \int\left(-\frac{1}{\sqrt{a^2 - x^2}}\right)dx$$

or $\sqrt{\mu}\ dt = \cos^{-1}\frac{x}{a} + k$, where k is a constant of integration.

Since the particle starts from rest at A, therefore t = 0 when $x = a$

$$\therefore\ 0 = \cos^{-1}\frac{a}{a} + k$$

i.e. $k = 0$

$$\therefore\ \sqrt{\mu}\ t = \cos^{-1}\frac{x}{a}$$

or $x = a\cos\sqrt{\mu}\ t$ ... (3)

Note :

1. From (2), we get the velocity of the moving particle at any distance x. It is $-\sqrt{\mu}\ \sqrt{a^2 - x^2}$.
2. From (3), we get the distance of the moving particle from the fixed point at any time t. It is $a\cos\sqrt{\mu}\ t$.
3. When x = 0, from (2), we get $v = -\sqrt{\mu}\ a$. i.e. the velocity of the moving particle at the fixed point O is $-\sqrt{\mu}\ a$ which is maximum. This implies that the particle does not stop at O. It passes through O on the other side OX/. Let A/ be a point on the line on the other side such that OA/ = OA = a.

   Clearly, from (2), $v = 0$ when x = a or x = -a, i.e. the velocity is zero at A or at A/.

From $A^{/}$, the particle retraces its path, passes through O and again it instantaneously comes to rest at A. Thus, the motion of the particle is oscillatory about the point O (from A to $A^{/}$, and then back from $A^{/}$ to A). It is repeated over and over again. Such type of motion is known as Simple Harmonic Motion. The point O is known as the centre of oscillation or the centre of mean position.

Amplitude

The maximum distance OA or $OA^{/}$(=a) to which the particle moves on either side of O is called the amplitude. Thus, the maximum displacement traversed by the particle on either side of the centre of oscillation is called the amplitude.

Periodic Time

The time for one complete oscillation, i.e. from A to $A^{/}$, and back to A, is called the periodic time or the time of a complete oscillation.

Putting x = 0 in (3), we get

$0 = a\cos\sqrt{\mu}\,t$ [We note that at the point O, x = 0]

$\therefore \cos\sqrt{\mu}\,t = 0$

or $\sqrt{\mu}\,t = \frac{\pi}{2}$

$\therefore t = \frac{\pi}{2\sqrt{\mu}}$

$\therefore$ the time taken by the particle from A to O is $\frac{\pi}{2\sqrt{\mu}}$

Now, time from A to O

= time from O to $A^{/}$

= time from $A^{/}$ to O

= time from O to A.

$\therefore$ periodic time $= 4\times\frac{\pi}{2\sqrt{\mu}} = \frac{2\pi}{\sqrt{\mu}}$

We note that the periodic time $\frac{2\pi}{\sqrt{\mu}}$ does not contain a. Thus, the periodic time is independent of the amplitude a.

Frequency

The number of complete oscillations per unit time is called the frequency of oscillation.

Let n denote the frequency any T denote the periodic time.

$\therefore\ nT = 1$

or $n = \frac{1}{T} = \frac{\sqrt{\mu}}{2\pi}$

or $\mu = 4n^2\pi^2$

Example 1 : The position of a particle moving in a straight time is given by x = a cos nt + b sin nt. Prove that the particle executes simple harmonic motion. Find the periodic time and the amplitude.

Solution : We have

x = a cos nt + b sin nt.

Differentiating with respect to t, we get

$$\frac{dx}{dt} = -an \sin nt + bn \cos nt$$

Again, differentiating we get

$$\frac{d^2x}{dt^2} = -an^2 \cos nt - bn^2 \sin nt$$

i.e. $\frac{d^2x}{dt^2} = -n^2x$ (1)

From equation (1), we see that the acceleration is proportional to the distance and it is directed towards the fixed point. Therefore, the motion executed by the particle is simple harmonic.

Periodic time $= \frac{2\pi}{\sqrt{\mu}} = \frac{2\pi}{\mu}$ (Here $\mu = n^2$)

We have

$$v = \frac{dx}{dt} = -an \sin t + bn \cos nt$$

Again, we know that

amplitude = the distance between the centre of oscillation and the position of instantaueous rest where velocity is zero.

Now, when $v = 0$

–a n sin nt + bn cos nt = 0

or an sin nt = bn cos nt

or $\frac{\sin nt}{bn} = \frac{\cos nt}{an} = k$, (say)

$\therefore$ sin nt = kbn, con nt = kan

Now $\cos^2 nt + \sin^2 nt = 1$

or $k^2n^2(a^2 + b^2) = 1$

or $k = \dfrac{1}{n\sqrt{a^2+b^2}}$

$\therefore \sin nt = \dfrac{b}{\sqrt{a^2+b^2}}, \quad \cos nt = \dfrac{a}{\sqrt{a^2+b^2}}$

$\therefore$ amplitude $= a \cdot \dfrac{a}{\sqrt{a^2+b^2}} = b \cdot \dfrac{b}{\sqrt{a^2+b^2}}$

$= \dfrac{a^2+b^2}{\sqrt{a^2+b^2}} = \sqrt{a^2+b^2}$

Example 2 : The velocity of a particle moving on a straight line at any time is given by $v^2 = 40x - 4x^2 - 64$. Show that the motion is simple harmonic.

Solution : We have $v^2 = 40x - 4x^2 - 64$

Differentiating both sides with respect to x, we get.

$$2v\frac{d^2v}{dx} = 40 - 8x$$

or $$v\frac{dv}{dx} = 20 - 4x$$

or $$\frac{d^2x}{dt^2} = -4(x-5)$$

Taking x – 5 = y, we get

$$\frac{d^2y}{dt^2} = -4y \qquad (1)$$

From equation (1), we see that the acceleration is proportional to the distance and it is directed towards the fixed point. Therefore, the motion executed by the particle is simple harmonic.

Example 3 : A particle is moving with simple harmonic motion. At the end of three successive seconds, the distance of the particle from its mean position, measured in the same direction are 1, 5 and 5. Show that the period of a complete oscillation is $\dfrac{2\pi}{\theta}$, where $\cos\theta = \dfrac{3}{5}$.

Solution : We have $x = a\cos\sqrt{\mu}\,t \qquad (1)$

Now, considering the conditions of question, we get

$1 = a\cos\sqrt{\mu}\,t$ (2)

$5 = a\cos\sqrt{\mu}\,(t+1)$ (3)

$5 = a\cos\sqrt{\mu}\,(t+2)$ (4)

Adding (2) and (4), we get

1 + 5 $= a[\cos\sqrt{\mu}\,t + \cos\sqrt{\mu}\,(t+2)]$

$$= 2a\cos\frac{\sqrt{\mu}(t+2)+\sqrt{\mu}\,t}{2}.\ \cos\frac{\sqrt{\mu}(t+2)-\sqrt{\mu}\,t}{2}$$

$$= 2a\cos\sqrt{\mu}(t+1)\cos\sqrt{\mu}$$

or $6 = 2\times 5\cos\sqrt{\mu}$, using (3)

$$\therefore\ \cos\sqrt{\mu} = \frac{6}{2\times 5} = \frac{3}{5}$$

$$\therefore\ \sqrt{\mu} = \cos^{-1}\frac{3}{5} = \theta,\quad \text{(say)}$$

$$\therefore\ \text{periodic time} = \frac{2\pi}{\sqrt{\mu}} = \frac{2\pi}{\theta} \text{ where } \cos\theta = \frac{3}{5}$$

Example 4: The speed v of a point P which moves in a straight line is given by the relation $v^2 = a - bx^2$, where x is the distance of the point P from a fixed point on the path, a and b being constants. Show that the motion of P is simple narmonic and determine its amplitude and period.

Solution : Given that $v^2 = a - bx^2$.

Differentiating with respect to x, we get

$$2v\frac{dv}{dx} = -2bx$$

or $$\frac{vdv}{dx} = -bx$$

or $$\frac{d^2v}{dt^2} = -bx \qquad \text{(i)} \qquad \left[\because \frac{vdv}{dx} = \frac{d^2x}{dt^2}\right]$$

This equation is the hundard form of S.H.M. Hence the motion of the given particle is simple hermonic and its period $= \frac{2\pi}{b}$ $[\mu = b]$.

Also from $v^2 = a - bx^2$ when $v = 0$, we have

$$a - bx^2 = 0$$

or $x = \sqrt{\frac{a}{b}}$

We know that the amplitude of the motion is the value of x when v is zero, provided accolaration is zero at $v = 0$.

$\therefore$ amplitude, of the motion $= \sqrt{\frac{a}{b}}$

CHECK YOUR PROGRESS

Q 1: A particle is moving on a straight line and its velocity at a distance x from the origin is $\lambda\sqrt{a^2 - x^2}$. Show that the motion is simple harmonic.

Q 2: The velocity of a particle moving on a straight line at any time is given by $v^2 = \lambda^2(-x^2 + 10ax - 16a^2)$, where a and λ are positive constants. Show that the motion is simple harmonic.

Q 3: A particle is moving on a straight line with simple harmonic motion. At the end of three successive seconds, the distances of the particle from its mean position, measured in the same direction are a, b and c respectively. Show that the period of a complete oscillation is $\frac{2\pi}{\cos^{-1}\frac{a+c}{2b}}$

Q 4: If V and f be the velocity and acceleration at any instant in a simple harmonic motion and T be the periodic time, show that $f^2T^2 + 4\pi^2V^2$ is a constant.

Q 5: A particle moving on a straight line with simple harmonic motion has velocities v_1, v_2 at distances x_1, x_2 respectively from the centre of the path. Show that the period of motion is $2\pi\sqrt{\frac{x_1^2 - x_2^2}{v_2^2 - v_1^2}}$ and the amplitude of motion is $\sqrt{\frac{v_2^2 x_1^2 - v_1^2 x_2^2}{v_2^2 - v_1^2}}$

Q 6: At what distance from the centre of oscillation will the velocity in a simple harmonic motion be half of the maximum?

Q 7: A particle is executing simple harmonic motion. Show that the particle requires one sixth of its period to move from the position of maximum displacement to

one in which the displacement is half the amplitude.

Q 8: A body moving on a straight line OAB has its velocity zero when at points A and B whose distance from O are a and b respectively, and has velocity v when half-way between them. Show that the complete period is $\frac{\pi(b-a)}{v}$.

Q 9: A particle moving under simple harmonic motion on a straight line has velocities v_1, v_2 and accelerations f_1, f_2 in two of its positions on the path. If d is the distance between the two positions, show that $d = \frac{v_1^2 \sim v_2^2}{f_1 + f_2}$.

Q 10: A particle moving with S.H.M. has a velocity of 8m/s when it is at a distance of 3m from the centre of oscillation, and has a velocity of 6m/s when it is at a distance of 4m from the centre. Find its maximum velocity and periodic time.

6.3.1 Geometrical Representation of Simple Harmonic Motion

If a particle describes a circle with constant angular velocity, then the foot of the perpendicular from the particle on a diameter moves ander Simple Harmonic Motion. This is known as the geometrical respresentation of S.H.M.

Let a particle P describe a circle round O as centre and constant angular velocity w.

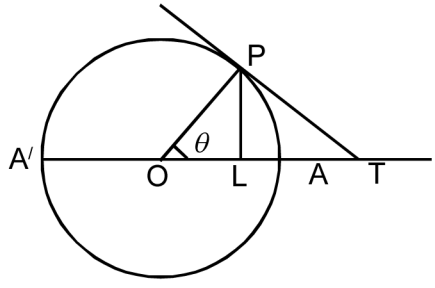


Fig 6.2

Let L be the foot of the perpendicular drawn from P on the fixed diameter A/OA.

When P is at A, L is also at A. As P moves from A towards A/ along the circle, L moves from A to A/ along the diameter AOA/. Finally, when P is at A/, L is also at A/. When P moves further from A/ towards A along the circle, L retraces its path from A/ to A along the diameter A/OA. Thus, the particle P oscillates from A to

$A^{/}$ and back.

Let a be the radius of the circle and P be the point (x, y) with O as origin and $A^{/}OA$ as the x-axis.

$$\therefore\ x = a\cos\theta$$

Now, the only acceleration of P towards O along the normal PO is w^2a.

The component of this acceleration along OA is

$w^2a\cos\theta$ = - w^2x towards O.

Thus, the equation of motion of the point L is

$$\frac{d^2x}{dt^2} = -w^2x$$

which is Simple Harmonic Motion.

Clearly, Periodic time $= \frac{2\pi}{\sqrt{w^2}} = \frac{2\pi}{w}$.

6.3.2 Motion of a Simple Pendulum

A heavy small particle suspended from a fixed point by a light inextensible string, and made to oscillate in a vertical plane, is called a simple pendulum. For small oscillation the motion is approximately rectilinear.

Let C be the fixed point. Let a heavy small particle be suspended from C by a light inextensible string. Let the length of the string be l.

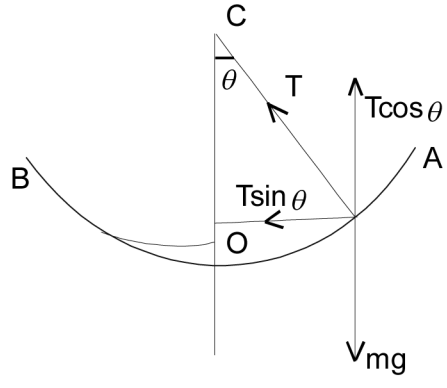


Fig 6.3

Let m be the mass of the particle.

Let the position of the particle at any time t be at P. when the particle is at P, let the string make an angle θ with the vertical, where θ is a very small angle.

Let PO be the horizontal displacement of the particle from its mean position

O (the displacement is considered to be horizontal). Let OP = x.

Let T be the tension of the string.

Resolving horizontally and vertically, we have

$$T\cos\theta = mg \qquad (1)$$

$$-T\sin\theta = m\frac{d^2x}{dt^2} \qquad (2)$$

Since θ is very small, $\therefore$ we take $\cos\theta = 1$

$\therefore$ from (1), T = mg

$\therefore$ from (2) $-mg\sin\theta = m\frac{d^2x}{dt^2}$

or, $\frac{d^2x}{dt^2} = -g\sin\theta \qquad (3)$

Now from the right angled triangle COP,

$$\sin\theta = \frac{OP}{CP} = \frac{x}{l}$$

$\therefore$ from (3), $\frac{d^2x}{dt^2} = -\frac{g}{l}x \qquad (4)$

From equation (4), we see that the acceleration is proportional to the distance and it is directed towards O. Therefore, the motion executed by the particle is simple harmonic.

Periodic time $= \frac{2\pi}{\sqrt{\mu}} = \frac{2\pi}{\sqrt{\frac{g}{l}}} = 2\pi\sqrt{\frac{l}{g}}$

6.4 LET US SUM UP

- If a particle moves in a straight line such that its acceleration is always directed towards a fixed point in the line and is proportional to the distance from the point, them the particle is said to execute Simple Harmonic Motion (S.H.M.).
- The velocity of a moving particle under S.H.M. at any distance x is $-\sqrt{\mu}\sqrt{a^2 - x^2}$.
- The distance of a moving particle under S.H.M. from the fixed point at any time t is a cos $a\cos\sqrt{\mu}\,t$.
- The maximum displacement traversed by a particle under S.H.M. on either side of

the centre of oscillation is called the amplitude.

- The time for one complete oscillation is called the periodic time or the time of a complete oscillation. The periodic time is $\frac{2\pi}{\sqrt{\mu}}$, which is independent of the amplitude a.
- The number of complete oscillations per unit time is called the frequency of oscillation.
- If a particle describes a circle with constant angular velocity, then the foot of the perpendicular from the particle on a diameter moves under Simple Harmonic Motion. This is known as the geometrical representation of S.H.M.
- The motion of a simple pendulum is simple harmonic and its periodic time is $2\pi\sqrt{\frac{l}{g}}$, where l is the length of the pendulum and g, the acceleration due to gravity.

6.5 FURTHER READINGS

1) Statics and Dynamics with Background Mathematics, Adrian Roberts, Cambridge University Press.
2) Enginering Machanics : Statics and Dynamics by R. G. Hibbeler and Ashok Gupta.
3) Engineering Mechanics : Statics and Dynamics by S. Rajasekarn and G.Sankara subramanian. Vikas Publishing House Pvt. Ltd.

6.6 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q No 1: We have $v = \lambda\sqrt{a^2 - x^2}$

or $v^2 = \lambda^2\sqrt{a^2 - x^2}$

Differentiating both sides with respect to x, we get

$$2v\frac{dv}{dx} = \lambda^2(0 - 2x)$$

or $$v\frac{dv}{dx} = -\lambda^2 x$$

or $$\frac{d^2x}{dt^2} = -\lambda^2 x \quad (1)$$

From equation (1), we see that the acceleration is proportional to the distance and it

is directed towards the fixed point. Therefore, the motion executed by the particle is simple harmonic.

Ans to Q No 2: We have $v^2 = \lambda^2(-x^2 + 10ax - 16a^2)$

Differentiating both sides with respect to x, we get

$$2v\frac{dv}{dx} = \lambda^2(-2x + 10a)$$

or, $$v\frac{dv}{dx} = -\lambda^2(x - 5a)$$

or, $$\frac{d^2x}{dt^2} = -\lambda^2(x - 5a)$$

Taking x – 5a = y, we get

$$\frac{d^2y}{dt^2} = -\lambda^2 y \qquad (1)$$

From equation (1), we see that the acceleration is proportional to the distance and it is directed towards the fixed point. Therefore, the motion executed by the particle is simple harmonic.

Ans to Q No 3: We have $x = a\cos\sqrt{\mu}\,t \qquad (1)$

Now, considering the conditions of question, we get

$$a = a\cos\sqrt{\mu}\,t \qquad (2)$$

$$b = a\cos\sqrt{\mu}\,(t+1) \qquad (3)$$

$$c = a\cos\sqrt{\mu}\,(t+2) \qquad (4)$$

Adding (2) and (4), we get

$$a + c = a[\cos\sqrt{\mu}\,t + \cos\sqrt{\mu}\,(t+2)]$$

$$= 2a\cos\frac{\sqrt{\mu}\,(t+2) + \sqrt{\mu}\,t}{2}\cos\frac{\sqrt{\mu}\,(t+2) - \sqrt{\mu}\,t}{2}$$

$$= 2a\cos\sqrt{\mu}\,(t+1)\cos\sqrt{\mu}$$

$$= 2b\cos\sqrt{\mu}, \text{ using (3)}$$

$$\therefore \cos\sqrt{\mu} = \frac{a+c}{2b}$$

or, $$\sqrt{\mu} = \cos^{-1}\frac{a+c}{2b}$$

$$\therefore \text{ periodic time } = \frac{2\pi}{\sqrt{\mu}} = \frac{2\pi}{\cos^{-1}\frac{a+c}{2b}}$$

Ans to Q No 4: We have $V = -\sqrt{\mu}\sqrt{a^2 - x^2}$

or, $V^2 = \mu\left(a^2 - x^2\right)$ (1)

$f = -\mu x$

or, $f^2 = \mu^2 x^2$

$T = \dfrac{2\pi}{\sqrt{\mu}}$

or, $T^2 = \dfrac{4\pi^2}{\mu}$

Now, $f^2T^2 + 4\pi^2 v^2$

$= \mu^2x^2.\dfrac{4\pi^2}{\mu} + 4\pi^2\mu\left(a^2 - x^2\right)$

$= 4\pi^2\mu x^2 + 4\pi^2\mu a^2 - 4\pi^2\mu x^2$

$= 4\pi^2\mu a^2$ which is a constant.

Ans to Q No 5: We have $v_1^{\,2} = \mu\left(a^2 - x_1^{\,2}\right)$ (1)

$v_2^{\,2} = \mu\left(a^2 - x_2^{\,2}\right)$ (2)

$\therefore\ v_2^{\,2} - v_1^{\,2} = \mu\left(a^2 - x_2^{\,2}\right) - \mu\left(a^2 - x_1^{\,2}\right)$

$= \mu\left(x_1^{\,2} - x_2^{\,2}\right)$

$\therefore\ \mu = \dfrac{v_2^{\,2} - v_1^{\,2}}{x_1^{\,2} - x_1^{\,2}}$

$\therefore$ periodic time $= \dfrac{2\pi}{\sqrt{\mu}} = 2\pi\sqrt{\dfrac{x_1^{\,2} - x_2^{\,2}}{v_2^{\,2} - v_1^{\,2}}}$

2nd Part Multiplying (1) by $x_2^{\,2}$ and (2) by $x_1^{\,2}$, We get

$v_1^{\,2}x_2^{\,2} = \mu a^2 x_2^{\,2} - \mu x_1^{\,2}x_2^{\,2}$ (3)

$v_2^{\,2}x_1^{\,2} = \mu a^2 x_1^{\,2} - \mu x_2^{\,2}x_1$ (4)

Subtracting (3) from (4), we get

$\mu a^2\left(x_1^{\,2} - x_2^{\,2}\right) = v_2^{\,2}x_1^{\,2} - v_1^{\,2}x_2^{\,2}$

or, $a^2 = \dfrac{v_2^{\,2}x_1^{\,2} - v_1^{\,2}x_2^{\,2}}{\mu\left(x_1^{\,2} - x_2^{\,2}\right)}$

or, $a = \dfrac{\sqrt{v_2^{\,2}x_1^{\,2} - v_1^{\,2}x_2^{\,2}}}{\sqrt{\mu}\ \sqrt{x_1^{\,2} - x_2^{\,2}}}$

$$= \frac{\sqrt{v_2^{\,2} x_1^{\,2} - v_1^{\,2} x_2^{\,2}}}{\sqrt{x_1^{\,2} - x_2^{\,2}}} \cdot \frac{\sqrt{x_1^{\,2} - x_2^{\,2}}}{\sqrt{v_2^{\,2} - v_1^{\,2}}}$$ using

$$= \frac{\sqrt{v_2^{\,2} x_1^{\,2} - v_1^{\,2} x_2^{\,2}}}{\sqrt{v_2^{\,2} - v_1^{\,2}}}$$

Ans to Q No 6: We have $v = \sqrt{\mu}\sqrt{a^2 - x^2}$

Considering the condition of question,

$v = \frac{1}{2}$ (maximum velocity)

or, $\sqrt{\mu}\sqrt{a^2 - x^2} = \frac{1}{2}\sqrt{\mu}\ a$

or, $\sqrt{a^2 - x^2} = \frac{1}{2}a$

or, $a^2 - x^2 = \frac{a^2}{4}$

or, $x^2 = a^2 - \frac{a^2}{4}$

or, $x^2 = \frac{3a^2}{4}$

$\therefore x = \frac{\sqrt{3}\,a}{2}$

Ans to Q No 7: We have $x = a\cos\sqrt{\mu}\,t$

Considering the condition of question, we get $x = \frac{1}{2}a$

or, $a\cos\sqrt{\mu}\,t = \frac{1}{2}a$

or, $\cos\sqrt{\mu}\,t = \frac{1}{2}$

or, $\sqrt{\mu}\,t = \frac{\pi}{3}$

or, $t = \frac{\pi}{3\sqrt{\mu}}$

$\therefore t = \frac{1}{6} \cdot \frac{2\pi}{\sqrt{\mu}} = \frac{1}{6}$ (periodic time)

Ans to Q No 8: We have velocity at A = 0

Velocity at B = 0

$\therefore$ A and B are the position of instantaneous rest.

Now AB = OB – OA

= b – a

$\therefore$ amplitude $= \frac{1}{2}AB = \frac{1}{2}(b-a)$

Let $\frac{1}{2}(b-a) = \alpha$

We have $V^2 = \mu\left(\alpha^2 - x^2\right)$

When x = 0, $V = v$

$\therefore\ v^2 = \mu\alpha^2 = \frac{\mu}{4}(b-a)^2$

$\therefore\ \mu = \frac{4v^2}{(b-a)^2}$, or $\sqrt{\mu} = \frac{2v}{b-a}$

$\therefore$ periodic time $= \frac{2\pi}{\sqrt{\mu}} = \frac{2\pi(b-a)}{2v}$

$= \frac{\pi(b-a)}{v}$

Ans to Q No 9: We have $\frac{d^2x}{dt^2} = -\mu x$

and $v^2 = \mu\left(\alpha^2 - x^2\right)$

Let the distances of the two positions from the centre of oscillation be x_1, x_2 respectively. We assume that $x_2 > x_1$

$\therefore\ f_1 = -\mu x_1,\ f_2 = -\mu x_2$ (1)

$v_1^{\,2} = \mu\left(a^2 - x_1^{\,2}\right),\ v_2^{\,2} = \mu\left(a^2 - x_2^{\,2}\right)$ (2)

$\therefore\ v_1^{\,2} - v_2^{\,2} = \mu\left(a^2 - x_1^{\,2}\right) - \mu\left(a^2 - x_2^{\,2}\right)$

$= \mu\left(a^2 - x_1^{\,2} - a^2 - x_2^{\,2}\right)$

$= \mu\left(x_2^{\,2} - x_1^{\,2}\right)$

$= \mu\left(x_2 - x_1\right)\left(x_2 + x_1\right)$

$= \mu d\left(x_2 + x_1\right)$ (3)

From (1) $f_1 + f_2 = -\mu\left(x_1 + x_2\right)$

or, $\mu = -\frac{f_1 + f_2}{x_1 + x_2}$ (4)

From (3) and (4),

$$v_1^2 - v_2^2 = -\frac{f_1 + f_2}{x_1 + x_2}\, d\left(x_1 + x_2\right)$$

or $d = -\dfrac{v_2^2 - v_1^2}{f_1 + f_2}$

If we assume $x_1 > x_2$, $\quad d = \dfrac{v_1^2 - v_2^2}{f_1 + f_2}$

Thus, $\quad d = \dfrac{v_1^2 \sim v_2^2}{f_1 + f_2}$

Ans to Q No 10: We have $v^2 = \mu\left(a^2 - x^2\right)$ (1)

When x = 3, v = 8

from (1) $64 = \mu\left(a^2 - 9\right)$ (2)

When x = 4, $v = 6$

$\therefore$ from (1) $36 = \mu\left(a^2 - 16\right)$ (3)

Dividing (3) by (2), we get

$$\frac{36}{64} = \frac{a^2 - 16}{a^2 - 9}$$

or, $\dfrac{9}{16} = \dfrac{a^2 - 16}{a^2 - 9}$

or, $16a^2 - 256 = 9a^2 - 81$

or, $7a^2 = 175$

or, $7a^2 = 175$

or, $a^2 = 25$

or, a = 5.

$\therefore$ from (2), $64 = \mu\left(25 - 9\right)$

or $16\mu = 64$

$\therefore \mu = 4$

$\therefore$ Maximum velocity $= \sqrt{\mu}\, a$

= 2 x 5 m/s = 10 m/s

Periodic time $= \dfrac{2\pi}{\sqrt{\mu}} = \dfrac{2\pi}{2} = \pi$

?6.7 MODEL QUESTIONS

Q 1: A particle executing simple harmonic motion has the velocities 10m/sec and 8m/sec at distances 6 m and 8m respectively from the centre of oscillation. Find the periodic time and the amplitude of the motion.

Q 2: A particle is moving under simple harmonic motion and its amplitude is a and the periodic time is T. Show that the maximum velocity of the particle is $\frac{2\pi a}{T}$.

Q 3: A particle moving under simple harmonic motion in a straight line, has velocities v_1, v_2, v_3 respectively at distances x_1, x_2, x_3, from the centre of oscillation. Prove that $x_1^2\left(v_2^2 - v_3^2\right) + x_2^2\left(v_3^2 - v_1^2\right) + x_3^2\left(v_1^2 - v_2^2\right) = 0$

Q 4: A particle is executing simple harmonic motion with initial conditions $x = x_0$ and $v = v_0$ at t = 0. Show the amplitude is $\left(x_0^2 + \frac{v_0^2}{n^2}\right)^{\frac{1}{2}}$

Q 5: A particle is executing simple harmonic motion. It the velocities at distances a, b, c from a fixed point which is not the centre of oscillation are u, v, w respectively, show that the periodic time T is given by

$$\frac{4\pi^2}{T^2}(b-c)(c-a)(a-b) = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ u^2 & v^2 & w^2 \end{vmatrix}.$$

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UNIT 7 : MEASURES OF CENTRAL TENDENCY

UNIT STRUCTURE

7.1 Learning Objectives

7.2 Introduction

7.3 Arithmetic Mean (or Mean)

7.4 Geometric Mean

7.5 Harmonic Mean

7.6 Median

7.7 Mode

7.8 Quartile

7.9 Let Us Sum Up

7.10 Further Readings

7.11 Answers To Check Your Progress

7.12 Model Questions

7.1 LEARNING OBJECTIVES

After going through this unit, you will able to learn -

- arithmetic mean
- geometric mean
- harmonic mean
- median
- mode
- quartile

7.2 INTRODUCTION

By a measure of central tendency, we mean ‘a summary measure that attempts to describe a whole set of data with a single value that represents the middle or centre of its distribution. There are five measures of central tendency, viz, arithmetic mean (or simply mean), geometric mean, harmonic mean, median, and mode. These measures are of common use. We will discuss these measures in this unit. Moreover we will discuss quartile which is required for our subsequent discussion.

7.3 ARITHMETIC MEAN

By the word mean, we mean arithmetic mean.

I. Mean for ungrouped data

Let there be n observations $x_1, x_2, x_3, \ldots\ldots, x_n$. The mean of these observations, denoted by $\bar{x}$, is defined by

$$\bar{x} = \frac{x_1 + x_2 + \ldots\ldots + x_n}{n} = \frac{\sum_{i=1}^{n} x_i}{n}$$

Note : $\sum$ is the eighteenth letter (capital) of the Greek alphabet. It is pronounced as 'sigma'.

Example 1 : The heights of 8 persons in centimetres are 155, 161, 164, 166, 167, 169, 171, 177. Find the mean height.

Solution : $\bar{x} = \frac{155+161+164+166+167+169+171+177}{8}$

$= \frac{1330}{8}$ = 166.25

$\therefore$ Mean height = 166.25 cm

II. Mean for discrete frequency distribution

Let there be n observations $x_1, x_2, \ldots., x_n$ occurring with frequencies $f_1, f_2, \ldots\ldots$, f_n respectively. The mean of these observations, denoted by $\bar{x}$, is defined by

$$\bar{x} = \frac{f_1x_1 + f_2x_2 + \ldots\ldots + f_nx_n}{f_1 + f_2 + \ldots\ldots + f_n}$$

$$= \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i} = \frac{\sum_{i=1}^{n} f_i x_i}{N}, \quad \text{Where } N = \sum_{i=1}^{n} f_i$$

Example 2 : Find the mean of the following frequency distribution.

x_i	3	5	7	9	11	13
f_i	6	8	15	25	8	4

Solution : $\bar{x} = \frac{6\times3+8\times5+15\times7+25\times9+8\times11+4\times13}{6+8+15+25+8+4}$

$= \frac{18+40+105+225+88+52}{6+8+15+25+8+4}$

$$= \frac{528}{66} = 8$$

III. Mean for grouped or continuous frequency distribution

Mean for grouped or continuous frequency distribution is given by

$$\bar{x} = \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i} = \frac{\sum_{i=1}^{n} f_i x_i}{N}$$

where x_i is taken as the mid-value of the corresponding class.

Example 3 : Find the mean for the following frequency distribution.

Marks	0–10	10–20	20–30	30–40	40–50	50–60
No. of students	15	17	19	27	19	12

Solution : [It is a continuous frequency distribution. Upper limits of each class are excluded from the respective class. For example 10 is not included in the class 0–10; it is included in the class 10–20.]

Class interval	Frequency f_i	Mid-value x_i	$f_i x_i$
0–10	15	5	75
10–20	17	15	255
20–30	19	25	475
30–40	27	35	945
40–50	19	45	855
50–60	12	55	660

$$N = \sum f_i = 109, \quad \sum f_i x_i = 3265$$

$$\therefore \bar{x} = \frac{\sum f_i x_i}{N} = \frac{3265}{109}$$

= 29.954 (approx)

Note : In the above example f_i's and x_i's are large. So, $f_i x_i$'s are larger. In order to avoid big calculations, we use a short method to obtain the mean.

Short Method.

In this method, the mean $\bar{x}$ is given by

$$\bar{x} = a + \frac{1}{N}\sum_{i=1}^{n} f_i (x_i - a),$$

Where a is an arbitrary number, called the assumed mean.

In case of grouped or continuous frequency distribution having equal class-interval a shorter method is used. In this case, we put

$d_i = \frac{x_i - a}{h}$, where h is the width of the class-interval

The mean is given by

$$\bar{x} = a + h\frac{\sum f_i d_i}{N}$$

Example 4 : Find the mean of the frequency distribution given in Example 3 by short method.

Solution : We take a = 35. h = 10

Class interval	Frequency f_i	Mid-value x_i	$d_i = \frac{x_i - 35}{10}$	$f_i d_i$
0–10	15	5	–3	–45
10–20	17	15	–2	–34
20–30	19	25	–1	–19
30–40	27	35	0	0
40–50	19	45	1	19
50–60	12	55	2	24

$N = \sum f_i = 109$

$\sum f_i d_i = -55$

$\therefore \bar{x} = 35 + 10 \times \left(\frac{-55}{109}\right)$

$= 35 - \frac{550}{109}$

= 35 – 5.0459

= 29.9541

Propertice of Arithmetic Mean

1. The algebraic sum of the deviations of a set of values from their arithmetic mean is zero.

Proof : Let there be n observations $x_1, x_2, \ldots, x_n$ occurring with frequencies $f_1, f_2, \ldots, f_n$ respectively. Then the mean $\bar{x}$ is given by

$$\bar{x} = \frac{f_1x_1 + f_2x_2 + \ldots\ldots + f_nx_n}{f_a + f_2 + \ldots\ldots + f_n}$$

$$= \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i} = \frac{\sum_{i=1}^{n} f_i x_i}{N}, \qquad \text{where } N = \sum_{i=1}^{n} f$$

It is required to prove that $\sum_{i=1}^{n} f_i(x_i - \bar{x}) = 0$

Now, $\sum_{i=1}^{n} f_i(x_i - \bar{x})$

$$= f_1(x_1 - \bar{x}) + f_2(x_2 - \bar{x}) + \ldots\ldots + f_n(x_n - \bar{x})$$

$$= (f_1x_1 + f_2x_2 + \ldots\ldots + f_nx_n) - (f_1 + f_2 + \ldots\ldots + f_n)\,\bar{x}$$

$$= \sum_{i=1}^{n} f_i x_i - \sum_{i=1}^{n} f_i\,\bar{x}$$

$$= N\bar{x} - N\bar{x}$$

= 0

2. The sum of the squares of the deviations of a set of values is minimum when taken about mean.

Proof : Let there be n observations $x_1, x_2, \ldots\ldots x_n$ occurring with frequencies $f_1, f_2, \ldots\ldots, f_n$ respectively. The mean $\bar{x}$ is given by

$$\bar{x} = \frac{f_1x_1 + f_2x_2 + \ldots\ldots + f_nx_n}{f_1 + f_2 + \ldots\ldots + f_n}$$

$$= \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i} == \frac{\sum_{i=1}^{n} f_i x_i}{N_i}, \qquad N = \sum_{i=1}^{n} f_i$$

It is required to prove that $\sum_{i=1}^{n} f_i(x_i - a)^2$ is minimum when $a = \bar{x}$, where a is an arbitrary number.

Let $z = \sum_{i=1}^{n} f_i(x_i - a)^2$

$$= f_1(x_1 - a)^2 + f_2(x_2 - a)^2 + \ldots\ldots + f_n(x_n - a)^2$$

$$= f_1(x_1 - \bar{x} + \bar{x} - a)^2 + f_2(x_2 - \bar{x} + \bar{x} - a)^2 + \ldots\ldots + f_n(x_n - \bar{x} + \bar{x} - a)^2$$

$$= f_1[(x_1 - \bar{x})^2 + (\bar{x} - a)^2 + 2(x_1 - x)(\bar{x} - a)^2]$$

$$+ f_2[(x_2 - \bar{x})^2 + (\bar{x} - a)^2 + 2(x_2 - \bar{x})(\bar{x} - a)] + \ldots\ldots\ldots\ldots$$

$$+ f_n[(x_n - \bar{x})^2 + (\bar{x} - a)^2 + 2(x_n - \bar{x})(\bar{x} - a)]$$

$$= f_1(x_1-\bar{x})^2 + f_2(x_2-\bar{x})^2 + \ldots\ldots + f_n(x_n-\bar{x})^2 + (\bar{x}-a)^2(f_1+f_2+\ldots\ldots+f_n)$$
$$+ 2(\bar{x}-a)[f_1(x_1-\bar{x}) + f_2(x_2-\bar{x}) + \ldots\ldots + f_n(x_n-\bar{x})]$$
$$= \sum_{i=1}^{n} f_i(x_i-\bar{x})^2 + (\bar{x}-a)^2 \sum_{i=1}^{n} f_i + 2(\bar{x}-a) \sum_{i=1}^{n} f_i\,(x_i-\bar{x})$$

We know that $\sum_{i=1}^{n} f_i = N$ and $\sum_{i=1}^{n} f_i(x_i-\bar{x})^2 = o$ (property 1)

$$\therefore\ z = \sum_{i=1}^{n} f_i(x_i-a)^2 = \sum_{i=1}^{n} f_i(x_i-\bar{x})^2 + (\bar{x}-a)^2 N$$

If $a = \bar{x}$, $\qquad (\bar{x}-a)^2 = 0$

$\therefore$ z is minimum if $a = \bar{x}$ and the minimum value is $\sum_{i=1}^{n} f_i(x_i-\bar{x})^2$.

7.4 GEOMETRIC MEAN

Let there be n observations $x_1, x_2, \ldots\ldots, x_n$; where x_i's are non-zero and positive. The geometric mean of these observations, denoted by G, is defined by

$$G = (x_1 x_2 \ldots\ldots x_n)^{\frac{1}{n}}$$

$$\therefore \log G = \log(x_1 x_2 \ldots\ldots x_n)^{\frac{1}{n}}$$
$$= \frac{1}{n}\log\,(x_1 x_2 \ldots\ldots x_n)$$
$$= \frac{1}{n}\log\,[\log x_1 + \log x_2 + \ldots\ldots + \log x_n] \qquad (1)$$

$\therefore$ G = antilog $\left[\frac{1}{n}(\log x_1 + \log x_2 + \ldots\ldots + \log x_n)\right]$

Again, let there be n observations $x_1, x_2, \ldots\ldots, x_n$ occurring with frequencies $f_1, f_2, \ldots\ldots, f_n$ respectively. The geometric mean of these observations, denoted by G, is defined by

$$G = \left(x_1^{f_1} x_2^{f_2} \ldots\ldots x_n^{fn}\right)^{\frac{1}{N}}, \qquad \text{where } N = \sum_{i=1}^{n} f_i$$

$$\therefore \log G = \log\left(x_1^{f_1} x_2^{f_2} \ldots\ldots x_n^{fn}\right)^{\frac{1}{N}}$$
$$= \frac{1}{N}\log\left(x_1^{f_1} x_2^{f_2} \ldots\ldots x_n^{fn}\right)$$
$$= \frac{1}{N}\left[\log\, x_1^{f_1} + \log\, x_2^{f_2} + \ldots\ldots + \log\, x_n^{fn}\right]$$

$$= \frac{1}{N}\left[f_1 \log x_1 + f_2 \log x_2 + \ldots + f_n \log x_n\right] \qquad (2)$$

$\therefore$ G = antilog $\left[\frac{1}{n}\left(f_1 \log x_1 + f_2 \log x_2 + \ldots + f_n \log x_n\right)\right]$

From (1) and (2) we see that the logarithm of the geometric mean of n observations is equal to the arithmetic mean of the logarithms of the observations.

7.5 HARMONIC MEAN

Let there be n observations $x_1, x_2, \ldots, x_n$, where x_i, s are non-zero.

The harmonic mean of these observations, denoted by H, is defined by

$$H = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}} = \frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}}$$

Again, let there be in observations $x_i, x_2, \ldots, x_n$ occurring with frequencies $f_1, f_2, \ldots, f_n$ respectively. The harmonic mean of these observations, denoted by H, is defined by

$$H = \frac{N}{\frac{f_1}{x_1} + \frac{f_2}{x_2} + \ldots + \frac{f_n}{x_n}} \qquad \text{where } N = \sum_{i=1}^{n} f_i$$

Example 5 : Calculate the arithmetic mean (AM), the geometric mean (GM), and the harmonic mean (HM) of 7, 8, 49, and show that AM > GM > HM.

Solution : $AM = \frac{7+8+49}{3} = \frac{64}{3} = 21.3$ (approx)

$$GM = \left(7 \times 8 \times 49\right)^{\frac{1}{3}} = \left(7 \times 2^3 \times 7^2\right)^{\frac{1}{3}} = \left(2^3 \times 7^3\right)^{\frac{1}{3}} = 2 \times 7 = 14$$

$$HM = \frac{3}{\frac{1}{7} + \frac{1}{8} + \ldots + \frac{1}{49}} = \frac{3}{\frac{113}{392}} = 3 \times \frac{392}{113} = 10.41 \text{ (approx)}$$

Clearly AM > GM > HM

CHECK YOUR PROGRESS

Q 1: Find the mean of the following observations 14.9, 10.8, 12.3, 23.3

Q 2: Find the mean from the following data

Marks	5	10	15	20	25	30	35	40
No. of students	5	7	9	10	8	6	3	2

Q 3: Find the mean height from the following frequency distribution.

Height (in cm)	0-10	10-20	20-30	30-40	40-50	50-60
No. of students	42	44	58	35	26	15

Q 4: Calculate the arithmetic mean (AM), the geometric mean (GM), the harmonic mean (HM) of 7, 27, 49, and hence show that AM>GM>HM.

Q 5: Find the average speed if the first km is travelled at 10 km an hour, the next km at 8 km an hour, and the third km at 4 km an hour.

Q 6: Find the mean of the squares of the first n natural numbers.

Q 7: The numbers 3.2, 5.8, 7.9, and 4.5 have frequences x, (x + 2), (x – 3), and (x + 6) respectively. If their mean is 4.876; find the value of x.

7.6 MEDIAN

By 'median' of a distribution, we mean the valus of the variable which divides the distribution into two equal parts.

I. Median for ungrouped data

Let there be n observations

First we arrange the observations in ascending (or descending) order of magnitude.

(a) Let n be odd.

In this case, the $\left(\frac{n+1}{2}\right)$th term is the median, i.e. median = $\left(\frac{n+1}{2}\right)$th term.

(b) Let n be even.

In this case, the mean of the two middle terms, i.e., the mean of the $\frac{n}{2}$th and $\left(\frac{n}{2}+1\right)$th terms is the median.

Note : If n is even, any value lying between the two middle values can be taken as median. It is a convention to take the mean of the two middle terms as the median.

Example 6 : Find the median of the following observations.

5, 19, 42, 11, 50, 30, 21, 0, 52, 36, 27.

Solution : We arrange the observations in ascending order of magnitude as follows. 0, 5,

11, 19, 21, 27, 30, 36, 42, 50, 52.

There are 11 observations (odd number of observations).

$\therefore$ Median = $\left(\frac{11+1}{2}\right)$ th i.e., 6th observation = 27

Example 7 : Find the median of the following observations

6, 10, 4, 3, 9, 11, 22, 18.

Solution : We arrange the observations in ascending order of magnitude as follows 3, 4, 6, 9, 10, 11, 18, 22.

There are 8 observations (even number of observatons).

$\therefore$ Median = Mean of the $\frac{8}{2}$ th and $\left(\frac{8}{2}+1\right)$ th observations

= Mean of the 4th and 5th observations

$= \frac{9+10}{2} = \frac{19}{2} = 9.5$

II. Median for discrete frequency distribution

Let there be n observations $x_1, x_2, \ldots, x_n$ with frequencies $f_1, f_2, \ldots, f_n$ respectively. In this case median is obtained by considering the cumulative frequencies.

First we find $N = \sum_{i=1}^{n} f$. Then we obtain $\frac{N}{2}$. We obtain the cumulative frequency (c.f.) just greater than $\frac{N}{2}$. The corresponding value of x is the median.

Example 8 : Find the median for the following frequency distribution.

x_i	20	9	25	50	40	80
f_i	6	4	16	7	8	2

Solution :

x_i(in anscending order)	frequency	$f_i$cumulative (c.f.)
9	4	4
29	6	10
25	16	26
40	8	34
50	7	41
80	2	43

$N = \Sigma f_i = 43$, $\frac{N}{2} = 43 = 21.5$

The c.f. just greater than 21.5 in 25. The corresponding value of x is 25. $\therefore$ Median = 25.

III. Median for grouped or continuous frequency distribution

Median for grouped or continuous frequency distribution is given by

Median = $L + \frac{\frac{N}{2} - c}{f} \times h$

Where L = the lower limit of the median class,

f = the frequency of the median class,

c = the c.f. of the class preceding the median class,

h = the width of the median class.

By 'median class', we mean the class corresponding to the c.f. just greater than $\frac{N}{2}$, where $N = \Sigma f_i$

Example 9 : Find the median for the following frequency distribution.

Class interval	20-30	30-40	40-50	50-60	60-70
Frequency	5	10	20	5	3

Solution :

Class intervel	Frequency	c.f
20-30	5	5
30-40	10	15
40-50	20	35
50-60	5	40
60-70	3	43

Here $N = \Sigma f_i = 43$, $\frac{N}{2} = \frac{43}{2} = 21.5$

The class corresponding to the c.f. just greater than 21.5 is 40-50.

$\therefore$ Median class : 40 – 50

L = 40, f = 20, c = 15, h = 20

Median $= L + \frac{\frac{N}{2} - c}{f} \times h$

$$= 40 + \frac{21.5 - 15}{20} \times 10$$

$$= 40 + \frac{6.5}{2}$$

= 40 + 3.25

= 43.25

CHECK YOUR PROGRESS

Q 1: Find the median of the following observations.

25, 30, 21, 55, 47, 10, 15, 17, 45, 35

Q 2: Find the median height from the following frequency distribution.

Heights (in cm)	150	155	158	160	164	168
No. of students	10	14	8	15	7	11

Q 3: Find the median for the following frequency distribution.

Class-interval	0-5	5-10	10-15	15-20	20-25
frequency	7	10	16	32	24

Q 4: The median of the observations 8, 11, 13, 15, x+1, x+3, 30, 35, 40, 43 arranged in ascending order is 22. Find x.

Q 5: Find the median of the following distribution.

Wages (in Rs.)	2200-2300	2300-2400	2400-2500	2500-2600	2600-2700
No. of workers	3	5	20	10	5

7.7 MODE

Mode is 'the most frequently occuring value in a distribution.'

Let us consider the following observations.

6, 5, 3, 4, 3, 7, 8, 5, 9, 5, 4.

Here the most frequently occuring value is 5.

$\therefore$ Mode of the distribution is 5.

Note : If there are more than one value each with the highest frequency, than the mode is not defined.

II. Mode for discrete frequency distribution

In case of discrete frequency distribution, mode is the value of the variable corresponding to the maximum frequency.

Let us consider the following example.

x_i	1	2	3	4	5	6	7
f_i	10	15	20	35	16	3	1

Here, maximum frequency = 35, and the corresponding value of x is 4. $\therefore$ mode = 4.

This method is not applicable in any one (or more) of the following cases.

1. The maximum frequency is repeated.
2. The maximum frequency occurs at the very beginning or at the end of the distribution.
3. There are irregularties in the distribution.

In such cases, mode is obtained by the method of grouping. We will not discuss this method.

III. Mode for grouped or continuous frequency distribution :

Mode for grouped or continuous frequency distribution is given by

$$\text{Mode} = L + \frac{f_m - f_1}{2f_m - f_1 - f_2} \times h,$$

Where L = the lower limit of the modal class,

h = the width of the modal class.

f_1 = the frequency of the class preceding the modal class.

f_2 = the frequency of the class succeeding the modal class.

f_m = the frequency of the modal class.

By 'modal class', we mean the class having maximum frequency or the class in which the mode lies.

Example : Find the mode for the following distribution.

Class-interval	0-5	5-10	10-15	15-20	20-25
Frequency	7	10	16	32	24

Solution : Here, the maximum frequency is 32.

$\therefore$ the modal class is 15-20.

$\therefore$ L = 15, f_m = 32, f_1 = 16, f_2 = 24, h = 5

$$\therefore \text{Mode} = L + \frac{f_m - f_1}{2f_m - f_1 - f_2} \times h$$

$$= 15 + \frac{32-16}{2\times 32-16-24}\times 5$$

$$= 15 + \frac{16}{24}\times 5$$

$$= 15 + \frac{2}{3}\times 5 \quad = 15 + \frac{10}{3} = 18.3$$

Empirical relation between mean, median and mode

A distribution in which mean = median = mode is called a symmetrical distribution. If mean, median and mode are nor equal, the distribution is said to be asymmetrical or skewed. In moderately asymmetrical distribution, there exists a relationship between mean, median and mode. Statistician karl pearson expressed the relationship as

Example: If mode=255, median=24,25 find mean.

Solotion: Mode = 3 median – 2 mean.

or, 25.5 = 3 x 24.25 – 2 mean

or, 2 mean = 72.75 – 25.5

or, 2 mean = 47.35

$\therefore$ mean $= \frac{47.25}{2}$

= 23.625

CHECK YOUR PROGRESS

Q 1: Find the mode of the following distribution

Heights (in cm)	0-10	10-20	20-30	30-40	40-50	50-60
No. of plants	42	44	58	35	26	15

Q 2: If mode = 16, mean = 20.2, find the median.

Q 3: If the values of mode and mean are 60, and 66 respectively, find the value of the median.

Q 4: If mode = 50.04, mean = 45, what is median?

Q 5: Find the mean, median and mode of the following observations

1, 2, 2, 3, 4, 7, 9.

7.8 QUARTILE

Quartiles are those value of the variable which divide the sequence of observations into four equal parts. Each part contains equal number of items. The first point is known as first quartile (Q_1), the second point is known as second quartle (Q_2, median), the third point is known as third quartile (Q_3).

Computation of quartiles

I. Quartiles for ungrouped data

Let there be n observations– $x_1, x_2, \ldots, x_n$.

We arrange the observations in ascending order of magnitude. Let the total number of observations be N. Then

Q_1 = Value of the $\left(\frac{N+1}{4}\right)$th observation.

Q_2 = Value of the $\left(\frac{N+1}{2}\right)$th observation, if n is odd

= mean of the values of $\frac{N}{2}$ th and $\left(\frac{N+1}{2}\right)$th observations, if n is even.

Q_3 = Value of the $3\left(\frac{N+1}{4}\right)$th observation.

II. Quartiles for grouped or continuous frequency distribution.

We compute the cumulative frequency (c.f.) of the given distribution

Let $N = \sum_{i=1}^{n} f_i$

We compute $\frac{iN}{4}$, where i = 1 for Q_1, i = 2 for Q_2 (or median), i = 3 for Q_3.

We find the cumulative frequency just greater than $\frac{iN}{4}$ and the corresponding class. The corresponding class is called the quartile class.

We use the following formula to compute Q_1, Q_2, and Q_3.

$$Q_i = L + \frac{\frac{iN}{4} - c}{f} \times h$$

i = 1, 2, 3

Where L = the lower limit of the quartile class,

f = the frequency of the quartile class,

c = the c.f. of the class preceding the quartile class,

h = the width of the quartile class.

7.9 LET US SUM UP

- Let there be n observations $x_1, x_2, \ldots, x_n$. The mean of these n observations, denoted by $\bar{x}$, is defined by

$$\bar{x} = \frac{x_1 + x_2 + \ldots + x_n}{n} = \frac{\sum_{i=1}^{n} x_i}{n}$$

Let there be n observations $x_1, x_2, \ldots, x_n$ occurring with frequencies $f_1, f_2, \ldots, f_n$ respectively. The mean of these observations is defined by

$$\bar{x} = \frac{f_1x_1 + f_2x_2 + \ldots + f_nx_n}{f_1 + f_2 + \ldots + f_n} = \frac{\sum_{i=1}^{n} f_ix_i}{\sum_{i=1}^{n} f_i} = \frac{\sum_{i=1}^{n} f_ix_i}{N}$$

Mean for grouped or continuous frequency distribution is given by

$$\bar{x} = \frac{\sum_{i=1}^{n} f_ix_i}{N}, \quad N = \sum_{i=1}^{n} f$$

where x_i taken as the mid-value of the corresponding class.

- By 'median' of a distribution, we mean the value of the variable which divides the distribution into two equal parts.

We arrange the observations in ascending (or descending) order of magnitude. If there are n observations and n is odd, the $\left(\frac{n+1}{2}\right)$th term is the median. If n is even, the mean of the $\frac{n}{2}$th and $\left(\frac{n}{2}+1\right)$th terms is the median.

Median for grouped or continuous frequency distribution is given by

$$\text{Median} = L + \frac{\frac{N}{2} - c}{f} \times h$$

Where L = the lower limit of the median class,

f = the frequency of the median class,

c = the c.f. of the class preceding the median class,

h = the width of the median class.

By 'median class', we mean the class corresponding to the c.f. just greater than $\frac{N}{2}$, where $N = \Sigma f_i$

- Mode is 'the most frequently occurring value in a distribution'.

  Mode for grouped or continuous frequency distribution is given by

  $$\text{Mode} = L + \frac{f_m - f_1}{2f_m - f_1 - f_2} \times h$$

Where L = the lower limit of the modal class,

h = the width of the modal class,

f_1 = the frequency of the class preceding the modal class.

f_2 = the frequency of the class succeeding the modal class,

f_m = the frequency of the modal class.

By 'modal class', we mean the class having maximum frequency or the class in which the mode lies.

7.10 FURTHER READING

1) Statistical Methods by N.G. Das, McGraw Hill Education (India) Private Ltd. 2009
2) Probability and Statistics by G.Shanker Rao, University Press (India) private Ltd. 2011
3) Introduction to probability and Statistics by Seymour Lipschutz, John J.Schiller, Tata McGraw Hill Education private Ltd. 2005
4) Probability, Random Process, Queseing theory and Statistics by Dr. S.Rajasekaran, Prof.A.Chandrasekaran & G.Kavitha, Dhanam Publications, Chennai-60042
5) Probability and Statistics by G.Balaji, G.Balaji Publishers, Chennai-33

7.11 ANSWERS TO CHECK YOUR PROGRESS

Check Your Progress 1

Ans to Q No 1: $\bar{x} = \frac{14.9 + 10.8 + 12.3 + 23.3}{4} = \frac{61.3}{4} = 15.325$

Ans to Q No 2:

$$\bar{x} = \frac{5\times5 + 7\times10 + 9\times15 + 10\times20 + 8\times25 + 6\times30 + 3\times35 + 2\times40}{5 + 7 + 9 + 10 + 8 + 6 + 3 + 2}$$

$$= \frac{25+70+135+200+200+180+105+80}{50}$$

$$= \frac{995}{50} = 19.9$$

Ans to Q No 3:

Class interval	frequency (f_i)	Mid-value (x_i)	$f_i x_i$
0-10	42	5	210
10-20	44	15	660
20-30	58	25	1450
30-40	35	35	1225
40-50	26	45	1170
50-60	15	55	825

$N = \Sigma f_i = 220$, $\Sigma f_i x_i = 5540$

$\therefore \ \bar{x} = \frac{\Sigma f_i x_i}{N} = \frac{5540}{220}$ = 25.18 (approx)

Ans to Q No 4: $AM = \frac{7+27+49}{3} = \frac{83}{3}$ = 27.67 (approx)

$$GM = (7+27+49)^{\frac{1}{3}} = \left(7 \times 3^3 \times 7^2\right)^{\frac{1}{3}} = \left(3^3 \times 7^3\right)^{\frac{1}{3}} = 3 \times 7 = 21$$

$$HM = \frac{3}{\frac{1}{7}+\frac{1}{27}+\frac{1}{49}} = \frac{3}{\frac{265}{1323}} = 3 \times \frac{1323}{265}$$

$= \frac{3969}{265}$ = 14.98 (approx)

$\therefore$ AM > GM > HM

Ans to Q No 5: 1st km is travelled in $\frac{1}{10}$ hour

2nd km is travelled in $\frac{1}{8}$ hour

3rd km is travelled in $\frac{1}{4}$ hour

$\therefore$ The distance (3 km) is travelled in $\left(\frac{1}{10}+\frac{1}{8}+\frac{1}{4}\right)$ hour

$\therefore$ average speed $= \frac{3}{\frac{1}{10}+\frac{1}{8}+\frac{1}{4}}$ km/hour

$$= \frac{3}{\frac{19}{40}} \text{ km/hour}$$

$$= 3 \times \frac{40}{19} \text{ km/hour}$$

= 6.316 km/hour (approx)

Note : Average speed is the harmonic mean of 10, 8, 4.

Ans to Q No 6: $\bar{x} = \frac{1^2 + 2^2 + 3^2 + \ldots + n^2}{n}$

$$= \frac{n(n+1)(2n+1)}{6n}$$

$$= \frac{(n+1)(2n+1)}{6}$$

Ans to Q No 7: $4.876 = \frac{(3.2)x + (5.8)(x+2) + (7.9)(x-3) + (4.5)(x+6)}{x + x + 2 + x - 3 + x + 6}$

or, $4.876 = \frac{(21.4)x + 11.6 - 23.7 + 27}{4x + 5}$

or, (4.876) (4x + 5) = (21.4)x + 14.9

or, (19.504)x + 24.38 = (21.4)x + 14.9

or, (21.4)x – (19.504)x = 24.38–14.9

or, (1.896)x = 9.48

$\therefore x = \frac{9.48}{1.896} = 5$

Check Your Progress 2

Ans to Q No 1: We arrange the observations in ascending order of magnitude as follows.

10, 15, 17,21, 25, 30, 35, 45, 47, 55

There are 10 observations (even number of observations).

$\therefore$ Median = mean of the $\frac{10}{2}$ th and $\left(\frac{10}{2}+1\right)$ th observations

= mean of the 5th and 6th observations

$$= \frac{25+30}{2} = \frac{55}{2} = 27.5$$

Ans to Q No 2: Try yourself

Ans to Q No 3: Try yourself

Ans to Q No 4: The observations in ascending order are 8, 11, 13, 15, x+1, x+3, 30, 35,

40, 43.

There are 10 observations

$\therefore$ median = mean of the 5th and 6th observations

$$= \frac{x+1+x+3}{2} = \frac{2x+4}{2}$$

= x + 2.

Now, median = 22 (given)

$\therefore$ x + 2 = 22

$\therefore$ x = 20

Ans to Q No 5: Try yourself

Check Your Progress 3

Ans to Q No 1: Try yourself

Ans to Q No 2: Try yourself

Ans to Q No 3: Try yourself

Ans to Q No 4: Mode = 50.04, mean = 45 (given)

We know that

mode = 3 median – 2 mean

or, 50.04 = 3 median – 2 x 45

or, 3 median = 50.04 + 90

or, 3 median = 140.04

$\therefore$ Median = $\frac{140.04}{3}$ = 46.68

Ans to Q No 5: The observations are

1, 2, 2, 3, 4, 7, 9

mean $= \frac{1+2+2+3+4+7+9}{7} = \frac{28}{7} = 4$

There are 7 observations.

median $= \left(\frac{7+1}{2}\right)$ th observation

= 4th observation

= 3

mode = 2

Note : mode < median < mean.

?7.12 MODEL QUESTIONS

Q 1: Let there be n observations 1, 2, 3, ..., n with frequencies 1, 2, 3, ..., n respectively. Find the arithmetic mean.

Q 2: Find the arithmetic mean (AM), the geometric mean (GM), and the harmonic mean (HM) of 2, 4, 16 and 32. Hence show that AM > GM > HM.

Q 3: The algebraic sum of the deviations of 20 observations measured from 30 is 2. Find the mean of these observations.

Q 4: The mean height of 15 students is 154 cm. It was found later on that while calculating the mean, the reading 155 cm was wrongly read as 125 cm. Find the correct mean height.

Q 5: Find the mean, the median, the mode from the following distribution.

Class-interval	0-10	10-20	20-30	30-40	40-50
frequency	3	10	17	7	6

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UNIT 8 : MEASURES OF DISPERSION

UNIT STRUCTURE

8.1 Learning Objectives

8.2 Introduction

8.3 Concept of Dispersion

- 8.3.1 Range
- 8.3.2 Quartile Deviation
- 8.3.3 Mean Deviation
- 8.3.4 Standard Deviation
- 8.3.5 Coefficient of Variation

8.4 Let Us Sum Up

8.5 Further Readings

8.6 Answers To Check Your Progress

8.7 Model Questions

8.1 LEARNING OBJECTIVES

After going through this unit, you will able to learn

- concept of dispersion
- range
- quartile deviation
- mean deviation
- standard deviation
- coefficient of variation.

8.2 INTRODUCTION

Dispersion means 'the degree of scatter of data'. Measures of dispersion provide 'certain specific information concerning the scatter or dispersion of values in a distribution'. These measures viz. range, quartile deviation, mean deviation, standard deviation will be discussed in this unit.

8.3 CONCEPT OF DISPERSION

Let us consider the following two sets of observations.

A. 25, 26, 27, 28, 29

B. 3, 7, 13, 50, 60

In each of A, and B, there are 5 observations. The mean in each case is 27. But the observations in A are more concentrated than observations in B. Though the mean is the same, the two sets are differently constituted. The greatest variation from the mean in A is 4; whereas in B it is 35. Thus we see that the measures of central tendency do not give a clear idea of a distribution. These measues must be supplemented by other measures. Dispersion is one of such measures.

8.3.1 Range

Range is the difference between the maximum and the minimum values of the observations. Thus,

Range= M–m, where M is the maximum value and m is the minimum value.

Example 1: Find the range of the following observations.

23, 16, 14, 29, 5.6, 27, 13, 59

Solution : We arrange the data in ascending order.

13, 14, 16, 23, 27, 29, 56, 59

Here maximum value = 59

Minimum value = 13

$\therefore$ Range = 59 – 13 = 46

Note : Though range can be calculated easily, it is not a reliable measure. It is not based on the entire set of data. It depends upon the two extremes. In the above example, the two extremes are 13, and 59. Range is based upon these two.

Coefficient of range : Coefficient of range is defined as follows.

Coefficient of range $= \frac{M-m}{M+m}$, where M is the maximum value, and m is the minimum value.

Example 2: Find the coefficient of range of the observations given in Example 1.

Solution : Here M = 59, m = 13

$\therefore$ Coefficient of range = $\frac{59-13}{59+13} = \frac{46}{72} = \frac{23}{36}$

8.3.2 Quartile Deviation

The difference between the third quartile and the first quartile is called interquartile range, and half the interquartile range is called quartile deviation (QD). Thus quartile deviation Q is given by

$$QD = Q = \frac{Q_3 - Q_1}{2},$$

Where Q_3 is the third quartile, and Q_1 is the first quartile.

Coefficient of quartile deviation is defined as follows.

$$\text{Coefficient of quartile deviation} = \frac{Q_3 - Q_1}{Q_3 + Q_1}$$

Example 3: Find the quartile deviation for the following data

Class-interval (Marks)	0-10	10-20	20-30	30-40	40-50	50-60
No. of students	4	6	20	10	7	3

Solution : We know that $Q_1 = L + \frac{\frac{N}{4} - c}{f} \times h$,

Where L is the lower limit of the first quartile class,

N is the total number of frequencies,

c is the cumulative frequency of the class just above the quartile class,

f is the frequency of the quartile class,

h is the class interval.

Now,

Class interval	Number of students (f)	c.f.
0-10	4	4
10-20	6	10
20-30	20	30
40-50	7	47
50-60	3	50

$\therefore$ N = 50, $\frac{N}{4} = \frac{50}{4} = 12.5$

12.5 is contained the minimum c.f. 30.

$\therefore$ 1st quartile class : 20 – 30

L = 20, c = 10, f = 20, h = 10

$$\therefore Q_1 = 20 + \frac{12.5 - 10}{20} \times 10$$

$$= 20 + \frac{2.5}{2} = 20 + 1.25 = 21.25$$

Again, $Q_3 = L + \frac{\frac{3N}{4} - c}{f} \times h$

$$\frac{3N}{4} = \frac{3 \times 50}{4} = 37.5$$

$\therefore$ 3rd quartile class : 30 – 40

L = 30, c = 30, f = 10 h = 10

$$\therefore Q_3 = 30 + \frac{37.5 - 30}{10} \times 10$$

= 30 + 7.5 = 37.5

$\therefore$ Quartile deviation = $\frac{Q_3 - Q_1}{2}$

$$= \frac{37.5 - 21.25}{2}$$

$$= \frac{16.25}{2}$$

= 8.125

Note : Quartile deviation is not affected by the extreme values.

CHECK YOUR PROGRESS 1

Q 1: Write down one advantage and disadvantage of range.

Q 2: Find the range and coefficient of range of the following observations

43, 13, 26, 11, 39, 56, 7, 98, 96, 34, 65, 42 .

Q 3: Is quartile deviation affected by the extreme value ?

Q 4: What is two range of the following observations ?

6, 6, 6, 6, 6, 6, 6, 6, 6.

Q 5: Find the quartile deviation for the following data.

Class-interval	0-10	10-20	20-30	30-40	40-50
No.of students	5	8	15	16	6

8.3.3 Mean Deviation

Mean deviation is the mean of the numerical deviations from the mean, median or mode.

Mean deviation for ungrouped data

Let $x_1, x_2, x_3, \ldots\ldots x_n$ be n observations

Let $\bar{x}$ be the mean, out M be the median of the distribution.

(A) Mean deviation about the mean, denoted by MD ($\bar{x}$), is given by

$$MD(\bar{x}) = \frac{\sum_{i=1}^{n} |x_i - \bar{x}|}{n}$$

(B) Mean deviation about the mode, denoted by MD(M), is given by

$$MD(M) = \frac{\sum_{i=1}^{n} |x_i - M|}{n}$$

Example 4: Find the mean deviation (a) about the mean, (b) about the median for the following observations

4, 5, 7, 8, 9, 13, 16, 18

Solution : (a) $\bar{x} = \frac{4+5+7+8+9+13+16+18}{8} = \frac{80}{8} = 10$

$\therefore$ the values of $(x_i - \bar{x})$ are –6, –5, –3, –2, –1, 3, 6, 8.

$\therefore$ the values of $|x_i - \bar{x}|$ are 6, 5, 3, 2, 1, 3, 6, 8.

$$\therefore MD(\bar{x}) = \frac{\sum_{i=1}^{8} |x_i - \bar{x}|}{8}$$

(b) The observations are in ascending order :

4, 5, 7, 8, 9, 13, 16, 18.

There are 8 observations

Median $M = \frac{1}{2}\left(\frac{8}{2}\text{th observation} + \left(\frac{8}{2}+1\right)\text{th observation}\right)$

$= \frac{1}{2}$(4th observation + 5th observation)

$= \frac{1}{2} \times (8+9) = \frac{17}{2} = 8.5$

$\therefore$ the values of (x_i–M) are

–4.5, –3.5, –1.5, –0.5, 0.5, 4.5, 7.5, 9.5

$\therefore$ the values of $|x_i - M|$ are

4.5, 3.5, 1.5, 0.5, 0.5, 4.5, 7.5, 9.5

$$\therefore\ MD(M) = \frac{\sum_{i=1}^{8} |x_i - M|}{8}$$

$$= \frac{4.5+3.5+1.5+0.5+0.5+4.5+7.5+9.5}{8}$$

$$= \frac{32}{2} = 4$$

Mean deviation for discrete frequency distribution :

Let $x_1, x_2, x_3, \ldots\ldots, x_n$ be n observations occurring with frequencies $f_1, f_2, f_3, \ldots\ldots, f_n$ respectively.

(A) Mean deviation about the mean, denoted by $MD(\bar{x})$, is given by

$$MD(\bar{x}) = \frac{\sum_{i=1}^{n} f_i |x_i - \bar{x}|}{\sum_{i=1}^{n} f_i}, \quad \text{Where}$$

$$\bar{x} = \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i} = \frac{\sum_{i=1}^{n} f_i x_i}{N}, \quad N = \sum_{i=1}^{n} f_i$$

(B) Mean deviation about the mode, denoted by MD (M), is given by

$$MD(M) = \frac{\sum_{i=1}^{n} f_i |x_i - M|}{N}$$

Example 5: Find the mean deviation (a) about the mean, (b) about the median for the following distribution

x_i	6	12	18	24	30	86
f_i	5	4	11	6	4	6

Solution :

(a)

x_i	f_i	$f_i x_i$	$x_i - \bar{x}$	$\lvert x_i - \bar{x} \rvert$	$f_i \lvert x_i - \bar{x} \rvert$
6	5	30	–15	15	75
12	4	48	–9	9	36
18	11	198	–3	3	33
24	6	144	3	3	18
30	4	120	-	9	36
36	6	216	15	15	90

$N = \sum_{i=1}^{6} f_i = 36, \quad \sum_{i=1}^{6} f_i x_i = 756$

$\therefore \ \bar{x} = \frac{\Sigma f_i x_i}{N} = \frac{756}{36} = 21$

$\therefore \ MD(\bar{x}) = \frac{\Sigma f_i \mid x_i - \bar{x} \mid}{N} = \frac{288}{36} = 8$

(B)

x_i	f_i	cf	x_i-M	$\lvert x_i$-M$\rvert$	$f_i \lvert x_i$-M$\rvert$
6	5	5	–12	12	60
12	4	9	–6	6	24
18	11	20	0	0	0
24	6	26	6	6	36
30	4	30	12	12	48
36	6	35	18	18	104

N = 36, an even number

$\therefore$ Median M = $\frac{1}{2}$ [$\frac{36}{2}$th observation + $\left(\frac{36}{2}+1\right)$th observation]

= $\frac{1}{2}$ [18th observation +19th observation]

$= \frac{1}{2}(18+18) = 18$

$\therefore \ MD(M) = \frac{\sum_{i=1}^{6} f_i \mid x_i - M \mid}{N} = \frac{272}{36} = 7.5$

Mean deviation for continuous frequency distribution :

(A) Mean deviation about the mean, denoted by MD $(\bar{x})$, is given by

$MD(\bar{x}) = \frac{\Sigma f_i \mid x_i - \bar{x} \mid}{N}$, where $N = \Sigma f_i$

(B) Mean deviation about the median, denoted by MD (M), is given by

$$MD(M) = \frac{\Sigma f_i \mid x_i - M \mid}{N}$$

Example 6: Find the mean deviation about the mean for the following distribution.

Class-interval (Marks)	0-10	10-20	20-30	30-40	40-50	50-60
Frequency (Number of students)	4	8	14	16	4	2

Solution :

Class-interval	Frequency (f_i)	Mid point (x_i)	f_ix_i	$x_i - \bar{x}$	$\lvert x_i - \bar{x} \rvert$	$f_i\lvert x_i - \bar{x} \rvert$
0–10	6	5	30	–22	22	132
10–20	8	15	120	–12	12	96
20–30	14	25	350	–2	2	28
30–40	16	35	560	8	8	128
40–50	4	45	180	18	18	72
50–60	2	55	110	28	28	56

$N = \Sigma f_i = 50, \quad \Sigma f_i x_i = 1350$

$\therefore \ \bar{x} = \frac{\Sigma f_i x_i}{N} = \frac{1350}{50} = 27$

$MD(\bar{x}) = \frac{\Sigma f_i \mid x_i - \bar{x} \mid}{N}$

$= \frac{512}{50} = 10.24$

Example 7: Find the mean deviation about the median for the following distribution

Class-interval	0-10	10-20	20-30	30-40	40-50
Frequency	5	10	20	5	10

Solution :

Class-interval	frequency (f_i)	Midpoint (x_i)	c.f.	$\lvert x_i - M \rvert$	$f_i\lvert x_i - M \rvert$
0–10	5	5	5	20	100
10–20	10	15	15	10	100
20–30	20	25	35	0	0
30–40	5	35	40	10	50
40–50	10	45	50	20	200

Here $N = \Sigma f_i = 50 \qquad \therefore \ \frac{N}{2} = \frac{50}{2} = 25$

The cumulative frequency just greater than 25 is 35, and the corresponding class-interval is 20–30.

$\therefore$ Median class is 20–30

Now, median $M = L + \frac{\frac{N}{2} - c}{f} \times h$

Here L = 20, $\frac{N}{2}=25$ c = 15, f = 20, h = 10

$$\therefore M = 20+\frac{25-15}{20}\times 10 \quad = 20+\frac{10}{20}\times 10 = 20+5 = 25$$

$$\therefore MD(M) = \frac{\Sigma f_i \mid x_i - M \mid}{N} = \frac{450}{50} = 9$$

CHECK YOUR PROGRESS

Q 1: Find the mean deviation about the mean for the following observations.

1, 2, 3, 4, 5, 6, 7

Q 2: Find the mean deviation about the median for the following observations

2, 3, 4, 7, 8, 9, 10, 11, 14

Q 3: Find the mean deviation (a) about the mean, (b) about the median for the following distribution.

x_i	5	7	9	10	12	15
f_i	8	6	2	2	2	6

Q 5: The mean of 2, 4, 6, k, 8 and 15 is 7. Find the mean deviation (a) about the mean, (b) about the median.

Q 6: Find the mean deviation about the mean for the following distribution.

Class-interval	0–10	10–20	20–30	30–40	40–50
frequency	5	10	20	5	10

Q 6: Find the mean deviation about the median for the following distribution.

Class-interval	0–10	10–20	20–30	30–40	40–50
frequency	5	8	15	16	6

8.3.4 Standard Deviation

(A) Standard deviation for ungrouped data :

Let $x_1, x_2, x_3, \ldots\ldots, x_n$ be n observations. Let $\bar{x}$ be the mean.

The standard deviation (SD), denoted by σ (Greek letter small 'sigma'), is defined by $\sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - x)^2}$

The square of the standard deviation is called variance. Thus,

Variance $= \sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(xi - \bar{x})^2$

Formula for σ can be simplified as follows

$$\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left(x_i^{\,2} - 2x_i\bar{x} + \bar{x}^2\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n}x_i^{\,2} - \frac{2\bar{x}}{n}\Sigma x_i + \frac{1}{n}\Sigma\bar{x}^2$$

$$= \frac{1}{n}\sum_{i=1}^{n}x_i^{\,2} - 2\bar{x}\,\bar{x} + \frac{1}{n}n\bar{x}^2$$

$$= \frac{1}{n}\sum_{i=1}^{n}x_i^{\,2} - 2\bar{x}^{\,2} + \bar{x}^2$$

$$= \frac{\sum_{i=1}^{n}x_i^{\,2}}{n} - \bar{x}^{\,2}$$

Thus, $SD = \sigma = \sqrt{\frac{\Sigma\bar{x}_i^{\,2}}{n} - \bar{x}^2}$

Variance $= \sigma^2 = \frac{\Sigma\bar{x}_i^{\,2}}{n} - \bar{x}^2$

Example 8: Find the SD of the first five natural numbers.

Solution : The first five natural numbers are 1, 2, 3, 4, 5.

$$\bar{x} = \frac{1+2+3+4+5}{5} = \frac{15}{5} = 3$$

$$\therefore\ \bar{x}^2 = 3^2 = 9$$

$$\frac{\Sigma x_i^{\,2}}{n} = \frac{1^2+2^2+3^2+4^2+5^2}{5} = \frac{55}{5} = 11$$

$$\therefore\ \frac{\Sigma x_i^{\,2}}{n} - \bar{x}^2 = 11 - 9 = 2$$

$\therefore SD = \sigma = \sqrt{2}$

Example 9: The standard deviation of the observations a, 3, 6, 11 is 3.5. Find a.

Solution : We have $\bar{x} = \frac{a+3+6+11}{4} = \frac{a+20}{4}$

$$\frac{\Sigma x_i^2}{n} = \frac{a^2+3^2+6^2+11^2}{4} = \frac{a^2+9+36+121}{4} = \frac{a^2+166}{4}$$

Now $\sigma^2 = \frac{\Sigma x_i^2}{n} - \bar{x}^2$

or, $(3.5)^2 = \frac{a^2+166}{4} - \frac{(a+20)^2}{4}$

or, $\frac{49}{4} = \frac{4(a^2+166)-(a^2+40a+400)}{16}$

or, $49 = \frac{3a^2-40a+264}{4}$

or, $3a^2 - 40a + 264 = 196$

or, $3a^2 - 40a + 68 = 0$

or, $3a^2 - 6a - 34a + 68 = 0$

or, $3a(a-2) - 34(a-2) = 0$

or, $(a-2)(3a-34) = 0$

$\therefore$ $a - 2 = 0$ or $3a - 34 = 0$

$\therefore$ $a = 2$ or $a = \frac{34}{3}$

Two properties

Property : If each of the observations $x_1, x_2, x_3, \ldots, x_n$ is increased by a non-zero constant k, the standard deviation as well as the variance remains the same.

Pf: Let the observations be $x_1, x_2, \ldots, x_n$.

$$\therefore \bar{x} = \frac{x_1 + x_2 + \ldots + x_n}{n}$$

$$\therefore SD = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

Now the new observations are

$y_1 = x_1 + k$, $y_2 = x_2 + k$, ..., $y_n = x_n + k$

$\therefore$ new mean $\bar{y} = \frac{(x_1+k)+(x_2+k)+\ldots+(x_n+k)}{n}$

$$= \frac{(x_1 + x_2 + \ldots + x_n) + (k + k + \ldots + k)}{n}$$

$$= \frac{x_1 + x_2 + \ldots + x_n}{n} + \frac{nk}{n}$$

$$= \bar{x} + k$$

Now, new $SD = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(y_i - y)^2}$

$$= \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i + k - \bar{x} - k)^2}$$

$$= \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

= old SD.

Thus, the SD remains the same.

Property : If each of the observations $x_1, x_2, \ldots, x_n$ is multiplied by a non-zero constant k, the standard deviation is also multiplied by the same constant k.

Proof: Let the observations be $x_1, x_2, \ldots, x_n$.

$\therefore \bar{x} = \frac{x_1 + x_2 + \ldots + x_n}{n}$

$\therefore SD = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}$

Now, the new observations are

$y_1 = kx_1, y_2 = kx_2, \ldots, y_n = kx_n$

$\therefore$ new mean $\bar{y} = \frac{kx_1 + kx_2 + \ldots + kx_n}{n}$

$$= \frac{k(x_1 + x_2 + \ldots + x_n)}{n}$$

$$= k\bar{x}$$

Now, new $SD = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2}$

$$= \sqrt{\frac{1}{n}\sum_{i=1}^{n}(k x_i - k\bar{x})^2}$$

$$= \sqrt{\frac{1}{n}\sum_{i=1}^{n}k^2(x_i - \bar{x})^2}$$

$$= \sqrt{k^2\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

$$= k\sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

= k (old SD)

Note : If each of the observations $x_1, x_2, \ldots, x_n$ is multiplied by a non-zero constant, the variance is multiplied by k^2.

(B) Standard deviation for discrete frequency distribution

Let $x_1, x_2, \ldots, x_n$ be n observations occurring with frequencies $f_1, f_2, \ldots \ldots, f_n$ respectively.

The SD is defined by

$$SD = \sigma = \frac{1}{n}\sum_{i=1}^{n} f_i(x_i - \bar{x})^2, \qquad \text{where } N = \Sigma f$$

Example 10: FInd the standard deviation for the following distribution

x_i	5	15	25	35	45
f_i	5	4	6	3	2

Solution :

x_i	f_i	f_ix_i	$x_i - \bar{x}$	$(x_i - \bar{x})^2$	$f_i(x_i - \bar{x})^2$
5	5	25	–16.5	272.25	136.25
15	4	60	–6.5	42.25	169
25	6	150	3.5	12.25	73.5
35	3	105	13.5	182.25	546.75
45	2	90	23.5	552.25	1104.5

Now, $N = \Sigma f_i = 20, \ \Sigma f_i x_i = 430$

$$\therefore \bar{x} = \frac{430}{20} = 21.5$$

$$\therefore \Sigma f_i(x_i - \bar{x})^2 = 3255$$

$$\therefore SD = \sqrt{\frac{3255}{20}} = \sqrt{162.75}$$

(C) Statdard deviation for continuous frequency distribution.

The standard deviation for a continuous frequency distribution is given by

$$SD = \sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n} f_i(x_i - \bar{x})^2}$$

where $N = \sum_{i=1}^{n} f_i$, x_i is the midpoint of the i th class.

Example 11: Find the standard deviation for the following distribution.

Class-interval	0–10	10–20	20–30	30–40	40–50
Frequency	5	10	20	5	10

Solution :

Class-interval	frequency f_i	Midpoint x_i	$f_i x_i$	$x_i - \bar{x}$	$(x_i - \bar{x})^2$	$f_i(x_i - \bar{x})^2$
0–10	5	5	25	–21	441	2205
10–20	10	15	150	–11	121	1210
20–30	20	25	500	–1	1	20
30–40	5	35	175	9	81	405
40–50	10	45	450	19	361	3610

Now, $N = \Sigma f_i = 50, \ \Sigma f_i x_i = 1300$

$$\therefore \bar{x} = \frac{\Sigma f_i x_i}{N} = \frac{1300}{50} = 26$$

Now, $\Sigma f_i (x_i - \bar{x})^2 = 7450$

$$\therefore SD = \sigma = \sqrt{\frac{1}{N}\Sigma f_i (x_i - \bar{x})^2} = \sqrt{\frac{7450}{50}} = \sqrt{149}$$

Note : Standard deviation explains the average amount of variation on either side of the mean. It is considered to be the best measure of dispersion. It is widely used.

There is an empirical (based on experience) relation between quartile deviation, mean deviation, and standard deviation. It is given by

6 QD = 5 MD = 4 SD

CHECK YOUR PROGRESS

Q 1: The standard deviation of 5 observations is 13.25. If each of the observations is increased by 3, what is the new standard deviation?

Q 2: The standard deviation of 7 observations is 15.75. If each of the observations is multiplied by 2, what is the new standard deviation?

Q 3: The standard deviation of 5 observations is 27.45. What is the (a) quartile deviation, (b) mean deviation?

Q 4: Find the standard deviation of the following observations

13, 16, 17, 22

Q 5: Find the SD of the first n natural numbers.

Q 6: The mean and variance of the observations 1, 2, a, 6 and b are 4.4, and 8.24 respectively. Find the values of a and b.

Q 7: Find the SD for the following distribution

x_i	6	10	14	18	20
f_i	2	4	7	12	5

Q 8: Find the SD for the following distribution

Class-interval	0–10	10–20	20–30	30–40	40–50
Frequency	2	4	7	12	5

8.3.5 Coefficient of Variation

Let the mean and the standard deviation of a distribution be $\bar{x}$ and σ respectively. The coefficient of variation (CV) is defined by

$$CV = \frac{\sigma}{\bar{x}} \times 100, \quad \bar{x} \neq 0$$

This measure was introduced by statistician karl pearson.

Note : 1. CV is a unitless measure of dispersion.

2. CV is the best measure to compare the variability of two distributions.

3. A distribution with less CV is considered more consistent (less variable).

4. A distribution with more CV is considered less consistent (more variable).

Example 12: The mean and standard deviation of heights of adults (age 30 years), and children (age 10 months) are given below -

	Adults	Children
Mean	160 cm	60 cm
SD	10 cm	5 cm

Show that there is a greater variation in the heights of children than that of adults.

Solution :

$$\text{CV (adults)} = \frac{\sigma}{\bar{x}} \times 100 = \frac{10}{160} \times 100 = 6.25$$

CV (children) $\frac{\sigma}{\bar{x}} \times 100 = \frac{5}{60} \times 100 = 8.33$

Now CV (children) > CV (adults)

$\therefore$ there is greater variation in the heights of children than that of adults.

CHECK YOUR PROGRESS

Q 1: The mean blood pressure of 25 boys is 120 mm Hg, and SD is 10 mm Hg. The mean height and SD of the same set of boys is 160 cm and 5 cm respectively. Show that blood pressure is more variable than height.

8.4 LET US SUM UP

- Measures of dispersion provide 'certain specific information concerning the scatter or dispersion of values in a distribution'.
- Range, Quartile deviation, Mean Deviation, Standard Deviation are measures of dispersion.
- Range is the difference between the maximum and the minimum values of the observations.
- The difference between the third quartile and the first quartile is called interquartile range, and half the interquartile range is called quartile deviation. Quartile deviation is not affected by the extreme values.
- Mean deviation is the mean of the numerical deviations from the mean, median, or mode.
- The standard deviation (SD) of the observations $x_1, x_2, \ldots, x_n$; denoted by σ (Greek letter small 'sigma'); is defined by

$$SD = \sigma = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2}$$

- SD of observations $x_1, x_2, \ldots, x_n$ occurring with frequencies $f_1, f_2, \ldots, f_n$ respectively is defined by

$$SD = \sigma = \sqrt{\frac{1}{n} \sum_{i=1}^{n} fi(x_i - \bar{x})^2}, \quad \text{where } N = \sum_{i=1}^{n} f_i$$

- Square of SD is called variance.
- Coefficient of variation (CV) is defined by $CV = \frac{\sigma}{\bar{x}} \times 100, \quad \bar{x} \neq 0$

  Where σ is the SD, and $\bar{x}$ the mean of the distribution.

- CV is a unitless measure of dispersion.
- A distribution with less CV is considered more consistent.
- A distribution with more CV is considered less consistent.

8.5 FURTHER READING

1) Statistical Methods by N.G. Das, McGraw Hill Education (India) Private Ltd. 2009
2) Probability and Statistics by G.Shanker Rao, University Press (India) private Ltd. 2011
3) Introduction to probability and Statistics by Seymour Lipschutz, John J.Schiller, Tata McGraw Hill Education private Ltd. 2005
4) Probability, Random Process, Queseing theory and Statistics by Dr. S.Rajasekaran, Prof.A.Chandrasekaran & G.Kavitha, Dhanam Publications, Chennai-60042
5) Probability and Statistics by G.Balaji, G.Balaji Publishers, Chennai-33

8.6 ANSWERS TO CHECK YOUR PROGRESS

Check Your Progress 1

Ans to Q No 1: Advantage : Range can be calculated easily.

Disadvantage : It is not a reliable measure. It is not based on the entire set of data. It depends upon the two extremes.

Ans to Q No 2: We arrange the data in ascending order.

7, 11, 13, 26, 34, 39, 42, 43, 56, 65, 96, 98.

Here, maximum value = 98

minimum value = 7

$\therefore$ Range = 98 – 7 = 91

Ans to Q No 3: Quartile deviation is not affected by the extreme values.

Ans to Q No 4: In this case, all the observations are identical.

Maximum value = Minimum value = 6.

$\therefore$ Range = 6 – 6 = 0

Ans to Q No 5:

Class interval	Number of students (f)	c.f.
0-10	5	5
10-20	8	13
20-30	15	28
30-40	16	44
40-50	6	50

Now, N = 50, $\frac{N}{4} = \frac{50}{4} = 12.5$

$\therefore$ 1st Quartile class : 10–20

$\therefore$ L = 10, c = 5, f = 8, h = 10

Now $Q_1 = L + \frac{\frac{N}{4} - c}{f} \times h$

$$= 10 + \frac{12.5 - 5}{8} \times 10$$

= 10 + 9.375 = 19.375

Now, $\frac{3N}{4} = \frac{3 \times 50}{4} = 37.5$

$\therefore$ 3rd Quartile class : 30–40

L = 30, c = 28, f = 16, h = 10

$$Q_3 = L + \frac{\frac{3N}{4} - c}{f} \times h$$

$$= 30 + \frac{37.5 - 28}{16} \times 10$$

= 30 + 5.9375 = 35.9375

$\therefore$ Quartile deviation $= \frac{Q_3 - Q_1}{2}$

$$= \frac{35.9375 - 19.375}{2}$$

$$= \frac{16.5625}{2} = 8.28125$$

Check Your Progress 2

Ans to Q No 1: $\bar{x} = \frac{1+2+3+4+5+6+7}{7} = \frac{28}{7} = 4$

The values of $(x_i - \bar{x})$ are

–3, –2, –1, 0, 1, 2, 3

The values of $|x_i - \bar{x}|$ are

3, 2, 1, 0, 1, 2, 3

$$\therefore MD(\bar{x}) = \frac{\sum_{i=1}^{7} |x_i - \bar{x}|}{7} = \frac{3+2+1+0+1+2+3}{7}$$

$$= \frac{12}{7}$$

Ans to Q No 2: The observations are in ascending order

2, 3, 4, 7, 8, 9, 10, 11, 14

There are 9 observations

Median $M = \left(\frac{9+1}{2}\right)$ th observation

= 5th observation = 8

The values of $(x_i - M)$ are –6, –5, –4, –1, 0, 1, 2, 3, 6

The values of $|x_i - M|$ are 6, 5, 4, 1, 0, 1, 2, 3, 6

$$\therefore MD(M) = \frac{\Sigma |x_i - M|}{9}$$

$$= \frac{6+5+4+1+0+1+2+3+6}{9}$$

$$= \frac{28}{9}$$

Ans to Q No 3: We have

(a)

x_i	f_i	$f_i x_i$	$x_i - \bar{x}$	$\lvert x_i - \bar{x} \rvert$	$f_i \lvert x_i - \bar{x} \rvert$
5	8	40	–4	4	32
7	6	42	–2	2	12
9	2	18	0	0	0
10	2	20	1	1	2
12	2	24	3	3	6
15	6	90	6	6	36

$N = \Sigma f_i = 26, \ \Sigma f_i x_i = 234$ $\therefore \bar{x} = \frac{\Sigma f_i x_i}{N} = \frac{234}{26} = 9$

$$\therefore MD(\bar{x}) = \frac{\Sigma f_i |x_i - \bar{x}|}{N} = \frac{88}{26} = 3.38$$

(b)

x_i	f_i	cf	x_i-M	$\lvert x_i$-M$\rvert$	$f_i\lvert x_i$-M$\rvert$
5	8	8	–2	2	16
7	6	14	0	0	0
9	2	16	2	2	4
10	2	18	3	3	6
12	2	20	5	5	10
15	6	26	8	8	48

N = 26, an even number

$\therefore$ Median = $\frac{1}{2}\left[\ \frac{26}{2}\text{th observation} + \left(\frac{26}{2}+1\right)\text{th observation}\ \right]$

$= \frac{1}{2}\left[\ 13\text{th observation} + 14\text{th observation}\ \right]$

$= \frac{1}{2}(7+7) = 7$

$$MD(M) = \frac{\Sigma f_i \mid x_i - M \mid}{N} = \frac{84}{20} = 3.23$$

Ans to Q No 4: We have

$$7 = \frac{2+4+6+k+8+15}{6}$$

or, $7 = \frac{35+k}{6}$ or 42 = 35 + k, or k = 7

$\therefore$ the observations are 2, 4, 6, 7, 8, 15.

(a) We have $\bar{x} = 7$

$\therefore$ the values of $(x_i - \bar{x})$ are

–5, –3, –1, 0, 1, 8

$\therefore$ the values of $\left|x_i - \bar{x}\right|$ are

5, 3, 1, 0, 1, 8

$$\therefore MD(\bar{x}) = \frac{\Sigma \mid x_i - \bar{x} \mid}{6} = \frac{5+3+1+0+1+8}{6} = \frac{18}{6} = 3$$

(b) The observations are in ascending order :

2, 4, 6, 7, 8, 15

There are 6 observations.

Median M = $\frac{1}{2}\left[\ \frac{6}{2}\text{ th observation} + \left(\frac{6}{2}+1\right)\text{th observation}\ \right]$

$= \frac{1}{2}$ (3rd observation +4th observation)

$= \frac{1}{2} \times (6+7) = \frac{13}{2} = 6.5$

The values of $(x_i - M)$ are

$-4.5, -2.5, -0.5, 0.5, 1.5, 8.5$

$\therefore$ the values of $|x_i - M|$ are

$4.5, 2.5, 0.5, 0.5, 1.5, 8.5$

$$\therefore MD(M) = \frac{\Sigma |x_i - M|}{6} = \frac{4.5+2.5+0.5+0.5+1.5+8.5}{6}$$

$$= \frac{18}{6} = 3$$

Ans to Q No 5:

Class-interval	Frequency f_i	Mid point x_i	$f_i x_i$	$x_i - \bar{x}$	$\lvert x_i - \bar{x} \rvert$	$f_i \lvert x_i - \bar{x} \rvert$
0–10	5	5	25	–21	21	105
10–20	10	15	150	–11	11	110
20–30	20	25	500	–1	1	20
30–40	5	35	175	9	9	45
40–50	10	45	450	19	19	190

$$N = \Sigma f_i = 50$$

$$\Sigma f_i x_i = 1300$$

$$\therefore \bar{x} = \frac{\Sigma f_i x_i}{N} = \frac{1300}{50}$$

$$= 26$$

$$\therefore MD(\bar{x}) = \frac{\Sigma f_i \left| x_i - \bar{x} \right|}{N}$$

$$= \frac{470}{50}$$

$$= 9.4$$

Ans to Q No 6:

Class-interval	Frequency (f_i)	Mid point (x_i)	cf	$\|x_i-M\|$	$f_i\|x_i-M\|$
0–10	6	5	5	23	115
10–20	8	15	13	13	104
20–30	15	25	28	3	45
30–40	16	35	44	7	112
40–50	6	45	50	17	102

Here $N = \Sigma f_i = 50$ $\quad \therefore \frac{N}{2} = \frac{50}{2} = 25$

The cumulative frequency just greater than 25 is 28, and the corresponding class-interval is 20–30. $\therefore$ Median class is 20–30.

Now, median $M = L + \frac{\frac{N}{2} - c}{f} \times h$

Here L = 20, $\frac{N}{2} = 25$, c = 13, f = 15, h = 10

$\therefore \; M = 20 + \frac{25-13}{15} \times 10 \;\; = 20 + \frac{12}{15} \times 10 \;\; = 20 + 8 = 28$

$\therefore MD(M) = \frac{\Sigma f_i \,|\, x_i - M \,|}{N} = \frac{478}{50} = 9.56$

Check Your Progress 3

Ans to Q No 1: We know that if each of the observations is increased by a constant, the standard deviation remains the same.

$\therefore$ the new standard deviation is also 13.25.

Ans to Q No 2: The new standard deviation = 5 x 15.75 = 78.75

Ans to Q No 3: We know that the quartile deviation, mean deviation and standard deviation are connected by 6 QD = 5 MD = 4SD

Here SD = 27.45

$\therefore$ 6 QD = 4 x 27.45

or, QD = $\frac{109.8}{6} = 18.3$

Ans to Q No 4: We know that $\sigma^2 = \frac{\Sigma x_i^{\,2}}{N} - \bar{x}^2$

Now, $\bar{x} = \frac{13+16+17+22}{4} = \frac{68}{4} = 17$

$$\therefore\ \sigma^2 = \frac{13^2+16^2+17^2+22^2}{4} - 17^2$$

$$= \frac{169+256+289+484}{4} - 289$$

$$= \frac{1198}{4} - 289 = 299.5 - 289 = 10.5$$

$$\therefore\ SD = \sigma = \sqrt{10.5}$$

Ans to Q No 5: We have $\sigma^2 = \frac{\Sigma x_i^{\,2}}{N} - \bar{x}^2$

Now $\bar{x} = \frac{1+2+3+\ldots\ldots+n}{n} = \frac{n(n+1)}{2n} = \frac{n+1}{2}$

$$\frac{\Sigma x_i^{\,2}}{n} = \frac{1^2+2^2+3^2+\ldots\ldots+n^2}{n}$$

$$= \frac{n(n+1)(2n+1)}{n} = \frac{(n+1)(2n+1)}{6}$$

$$\therefore\ \sigma^2 = \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4}$$

$$= \frac{2(n+1)(2n+1) - 3(n+1)^2}{12}$$

$$= \frac{(n+1)[2(2n+1) - 3(n+1)]}{12}$$

$$= \frac{(n+1)(4n+2-3n-3)}{12}$$

$$= \frac{(n+1)(n-1)}{12} = \frac{n^2-1}{12}$$

$$\therefore\ SD = \sigma = \sqrt{\frac{n^2-1}{12}}$$

Ans to Q No 6: We have $\frac{1+2+a+6+b}{5} - 4.4$

or, 9 + a + b = 4.5 x 5 = 22

$\therefore$ a + b = 22 – 9 = 13 (1)

Again, $\frac{1^2+2^2+a^2+6^2+b^2}{5} = (4.4)^2 = 8.24$

or, 1 + 4 + a^2 + 36 + b^2 – 19.36 x 5 = 8.24 x 5

or, 41 + a^2 + b^2 – 96.8 = 41.2

or, $a^2 + b^2 = 97$ (2)

From (1), $b = 13 - a$

$\therefore$ from (2), $a^2 + (13 - a)^2 = 97$

or, $a^2 + 169 - 26a + a^2 = 97$

or, $2a^2 - 26a + 72 = 0$

or, $a^2 - 13a + 36 = 0$

or, $a^2 - 4a - 9a + 36 = 0$

or, $a(a-4) - 9(a-4) = 0$

or, $(a-4)(a-9) = 0$

$\therefore$ $a = 4$, or $a = 9$

If $a = 4$, $b = 13 - 4 = 9$

If $a = 9$, $b = 13 - 9 = 4$

$\therefore$ the required observations are 4, 9.

Ans to Q No 7: Try yourself.

Ans to Q No 8: Try yourself.

Check Your Progress 4

Ans to Q No 1: CV (blood pressure) $= \frac{\sigma}{\bar{x}} \times 100$

$= \frac{10}{120} \times 100 = 8.3$ (approx)

CV (height) $= \frac{\sigma}{\bar{x}} \times 100 = 3.125$

$= \frac{5}{100} \times 100 = 3.125$

now CV (blood pressure) > CV (height)

$\therefore$ blood pressure is more variable than height.

8.7 MODEL QUESTIONS

Q 1: In a class test 5 students obtain the following marks out of 20 in a particular subject

A	B	C	D	E
10	12	14	12	18

Calculate the standard deviation of these marks.

Q 2: Calculate the mean deviation about (a) mean, and (b) median from the following data.

Class-interval	20–30	30–40	40–50	50–60	60–70
frequency	120	201	150	75	25

Q 3: The mean and variance of 7 observations are 8 and 16 respectively. If 5 of these observations are 6, 8, 10, 12, 14; find the other two observations.

Q 4: The scores of two cricketers A and B in five matches are given below.

A : 58 59 60 54 65

B : 82 12 44 08 64

Which cricketer may be considered as more consistent?

Q 5: Coefficients of variation of two sets of observations are 58 and 69. Their standard deviations are 21.2 and 15.6 respectively. FInd their arithmetic means.

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UNIT 9 : MOMENTS, SKEWNESS AND KURTOSIS

UNIT STRUCTURE

9.1 Learning Objectives

9.2 Introduction

9.3 Moments

9.3.1 Relation Between Raw and Central Moments

9.4 Skewness

9.4.1 Measures of Skewness

9.5 Kurtosis

9.5.1 Measures of Kurtosis

9.6 Let Us Sum Up

9.7 Further Reading

9.8 Answers To Check Your Progress

9.9 Model Questions

9.1 LEARNING OBJECTIVES

After going through this unit, you will able to learn

- moments
- relation between raw moment and central moment
- skewness
- measures of skewness
- kurtosis
- measures of kurtosis.

9.2 INTRODUCTION

The notion of moments in statistics comes from the notion of moments in mechanics. In mechanics, by moment of a force we mean a force f multiplied by a perpendicular distance d, i.e., fd. The analogouy quantities in statistics are f_i, and $x_i - a$. Moment, skewness and kurtosis are important concepts of statistics.

9.3 MOMENTS

Let a variable x take the values $x_1, x_2, \ldots, x_n$ with frequencies $f_1, f_2, \ldots, f_n$

respectively.

The r th moment of x about any point a, denoted by μ_r', is defined by

$$\mu'_r = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i - \bar{x})^r, \quad N = \sum_{i=1}^{n} f_i \qquad (1)$$

μ'_r is called the r th **raw** moment.

The r th moment of x about the mean $\bar{x}$, denoted by μ_r, is defined by

$$\mu_r = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i - \bar{x})^r \qquad (2)$$

μ_r is called the r th **central** moment.

Note : For ungrouped data, the corresponding formulas for raw moment and central moment are given by

$$\mu'_r = \frac{1}{N}\sum_{i=1}^{n}(x_i - a)^r \text{ and } \mu_r = \frac{1}{N}\sum_{i=1}^{n}(x_i - \bar{x})^r$$

respectively.

Deductions

(A) Putting r = 0 in (1), we get

$$\mu'_0 = \frac{1}{N}\sum_{i=1}^{n} f_i, \quad (x_i - a)^0 = 1$$

$$= \frac{1}{N} N = 1$$

Putting r = 0 in (2), we get

$$\mu_0 = \frac{1}{N}\sum_{i=1}^{n} f_i = \frac{1}{N} N = 1$$

Thus, $\mu'_0 = \mu_0 = 1$, i.e. the 0 th raw as well as central moment is unity

(B) Putting r = 1 in (1), we get

$$\mu_1' = \frac{1}{N}\Sigma f_i(x_i - a)$$

$$= \frac{1}{N}\Sigma(f_i x_i - f_i a)$$

$$= \frac{1}{N}\left[\Sigma f_i x_i - \Sigma f_i a\right]$$

Now $\bar{x} = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{\Sigma f_i x_i}{N} \Rightarrow \Sigma f_i x_i = N\bar{x}$

$$\Sigma f_i a = a\Sigma f_i = aN$$

$$\therefore\ \mu_1' = \frac{1}{N}(N\bar{x} - aN) = \bar{x} - a = d,\ \text{(say)}$$

Thus, the 1st raw moment is $\bar{x} - a$

Putting r = 1 in (2), we get

$$\mu_1 = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i - \bar{x})$$

$$= \frac{1}{N}\sum_{i=1}^{n}(f_i x_i - f_i\bar{x})$$

$$= \frac{1}{N}\left(\Sigma f_i x_i - \Sigma f_i\bar{x}\right)$$

$$= \frac{1}{N}\left(N\bar{x} - N\bar{x}\right) = 0$$

Thus, the 1st central moment is zero.

(C) Putting r = 2 in (1), we get

$$\mu_2' = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i - a)^2$$

Putting r = 2 in (2), we get

$$\mu_2 = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i - \bar{x})^2 = a^2$$

Thus, the 2nd central moment is the variance.

9.3.1 Relation Between Raw and Central Moments

We have

$$\mu_2 = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i - \bar{x})^r \quad \text{where } N = \sum_{i=1}^{n} f_i$$

We put $X_i = x_i - a,\ \ d = \bar{x} - a$

$$\therefore\ X_i - d = (x_i - a) - (\bar{x} - a) = x_i - \bar{x}$$

$$\therefore\ \mu_r = \frac{1}{N}\Sigma f_i(X_i - d)^r$$

$$= \frac{1}{N}\Sigma f_i\left(X_i^r - {}^rC_1 X_i^{r-1} d + {}^rC_2 X_i^{r-2} d^2 - {}^rC_3 X_i^{r-3} d^3 + \ldots\ldots + (-1)^r d^r\right)$$

$$= \frac{1}{N}\left[\Sigma f_i X_i^r - {}^rC_1 d\,\Sigma f_i X_i^{r-1} + {}^rC_2 d^2 \Sigma f_i X_i^{r-2} - {}^rC_3 d^3 \Sigma f_i X_i^{r-3} + \ldots\ldots + (-1)^r d^r \Sigma f_i\right]$$

$$= \frac{1}{N}\Sigma f_i(x_i - a)^r - {}^rC_1 d\,\frac{1}{N}\Sigma f_i(x_i - a)^{r-1} + {}^rC_2 d^2\,\frac{1}{N}\Sigma f_i(x_i - a)^{r-2}$$

$$-^{r}C_3 d^3 \frac{1}{N}\Sigma f_i (x_i - a)^{r-3} + \ldots\ldots + (-1)^r d^r \frac{1}{N}\Sigma f_i$$

$$= \mu'_r - {}^{r}C_1 d\,\mu'_{r-1} + {}^{r}C_2 d^2 \mu'_{r-2} - {}^{r}C_3 d^3 \mu'_{r-3} + \ldots\ldots + (-1)^r d^r$$

$$= \mu'_r - {}^{r}C_1 \mu'_{r-1}\mu'_1 + {}^{r}C_2 d^2 \mu'_{r-2}\mu_1'^{2} - {}^{r}C_3 d^3 \mu'_{r-3}\mu_1'^{3} + \ldots\ldots + (-1)^r \mu_1'^{r} \quad (3)$$

Deductions

Putting r = 2 in (3), we get

$$\mu_2 = \mu_2' - {}^{2}C_1 \mu'_1 \mu_1' + {}^{2}C_2 \mu'_0 \mu_1'^{2}$$

$$= \mu_2' - 2\mu_1'^{2} + \mu_1'^{2}, \quad (\mu'_0 = 1)$$

$$= \mu_2' - \mu_1'^{2} \qquad (4)$$

Putting r = 3 in (3), we get

$$\mu_3 = \mu_3' - {}^{3}C_1 \mu_2' \mu_1' + {}^{3}C_2 \mu_1' \mu_1'^{2} - {}^{3}C_3 \mu_0' \mu_1'^{3}$$

$$= \mu_3' - 3\mu_2' \mu_1' + 3\mu_1'^{3} - \mu_1'^{3}$$

$$= \mu_3' - 3\mu_2' \mu_1' + 2\mu_1'^{3} \qquad (5)$$

Putting r = 4 in (3), we get

$$\mu_4 = \mu_4' - {}^{4}C_1 \mu_3' \mu_1' + {}^{4}C_2 \mu_2' \mu_1'^{2} - {}^{4}C_3 \mu_1' \mu_1'^{3} + {}^{4}C_4 \mu_{4-4}' \mu_1'^{4}$$

$$= \mu_4' - 4\mu_3' \mu_1' + 6\mu_2' \mu_1'^{2} - 4\mu_1'^{4} + \mu_1'^{4}$$

$$= \mu_4' - 4\mu_3' \mu_1' + 6\mu_2' \mu_1'^{2} - 3\mu_1'^{4} \qquad (6)$$

Note : In the right hand side of (4), the coefficients are 1, –1; and their sum is 0. In the right hand side of (5), the coefficients are 1, –3, 2; and their sum is 0. In the right hand side of (6), the coefficients are 1, –4, 6, –3; and their sum is 0. In general, the sum of the coefficients in the right hand side of (3) is 0 as shown below.

$1 - {}^{r}C_1 + {}^{r}C_2 - {}^{r}C_3 + \ldots\ldots + (-1)^r = (1-1)^r = 0$

Example 1: The first moment of a variable 1 is 5. Find the mean.

Solution : We have

$$\mu_1' = \frac{1}{N}\Sigma f_i (x_i - 1), \quad (a = 1)$$

$$= \frac{1}{N}(\Sigma f_i x_i - \Sigma f_i)$$

$$= \frac{1}{N}\Sigma f_i x_i - \frac{1}{N}\Sigma f_i, \quad \left(\frac{\Sigma f_i x_i}{N} = \bar{x},\ \Sigma f_i = N\right)$$

$$= \bar{x} - 1$$

Now, $\bar{x} - 1 = 5 \Rightarrow \bar{x} = 6$ mean = 6

Example 2: The first two moments of a variable about 2 are 1, and 16 respectively. Find the mean and the variance.

Solution : We have $\mu_1' = \frac{1}{N}\Sigma f_i(x_i - 2)$

$$\mu_2' = \frac{1}{N}\Sigma f_i(x_i - 2)^2$$

Now, $\mu_1' = \frac{1}{N}(\Sigma f_i x_i - 2f_i)$

$$= \frac{\Sigma f_i x_i}{N} - \frac{2\Sigma f_i}{N}$$

$$= \bar{x} - 2$$

Now, $\mu_1' = 1$, (given)

$\therefore \bar{x} - 2 = 1$

$\Rightarrow \bar{x} = 3$

$\therefore$ mean = 3

Now, $\mu_2' = \frac{1}{N}\Sigma f_i(x_i - 2)^2$

$$= \frac{1}{N}\Sigma f_i(x_i - 3 + 1)^2$$

$$= \frac{1}{N}\Sigma f_i\left((x_i - 3)^2 + 2(x_i - 3) + 1\right)$$

$$= \frac{\Sigma f_i(x_i - 3)^2}{N} + \frac{2\Sigma f_i x_i}{N} - 5\frac{\Sigma f_i}{N}$$

Now, $\mu_2' = 16$, (given); and $\mu_2' = \frac{\Sigma f_i(x_i - 3)^2}{N}$, $(\bar{x} = 3)$

$\therefore 16 = \mu_2 + 2\bar{x} - 5$

or, $16 = \mu_2 + 2 \times 3 - 5$

$\therefore \mu_2 = 15$

We know that the 2nd central moment is the variance.

$\therefore$ Variance $= \mu_2 = 15$

Sheppard's Corrections to moments for grouped frequency distributions :

While computing moments for grouped frequency distributions, we assume that the frequencies are concentrated at the midpoints of the classes. This assumption introduces some errors. These are called grouping errors. Statistician William Flectword Sheppard (1863-1936) has developed a method for adjusting

the moments, known as sheppard's corrections. Sheppard's corrections for the first four orders are given given below.

μ_1' (corrected) = μ_1',

μ_2' (corrected) $= \mu_2' - \frac{h^2}{12}$,

μ_3' (corrected) = $\mu_3' - \frac{h^2}{4}\mu_1'$

μ_4' (corrected) $= \mu_4' - \frac{h^2}{2}\mu_2' + \frac{7}{240}h^4$

and, μ_2 (corrected) $= \mu_2 - \frac{h^2}{12}$

μ_3 (corrected) = μ_3

μ_4 (corrected) $= \mu_4 - \frac{h^2}{12}\mu_2 + \frac{7}{240}h^4$

where h is the width of the class-interval,

Pearson's β and γ coefficients :

There are four important coefficients or moment-ratios due to Karl Pearson (1857-1936).

These are as follows

$\beta_1 = \frac{\mu_3^2}{\mu_2^3}$, $\gamma_1 = +\sqrt{\beta_1}$

$\beta_2 = \frac{\mu_4}{\mu_2^2}$, $\gamma_2 = \beta_2 - 3$

Note : These four coefficients are related to the shape of the curve. These are pure numbers.

CHECK YOUR PROGRESS

Q 1: The numbers of bacteria in 1ml of blood from 5 persons are 2, 3, 7, 8, 10. Find the first, second, third and fourth moments about the mean.

Q 2: The first four moments of a distribution about the value 5 of the variable are 2, 20, 40 and 50. Find the corresponding moments about the mean.

Q 3: The first four moments of a distribution about the value 4 of the variable are 1, 4, 10 and 45. Show that the mean is 5 and the variance is 3 and μ_3 and μ_4 are 0 and 26 respectively.

Q 4: The first four moments of a distribution about the value 5 of the variable are –4, 22, –117 and 560. Find the corresponding moments about zero.

Q 5: The first four central moments of a distribution are 0, 2.5, 0.7 and 18.75. Find β_1, and β_2.

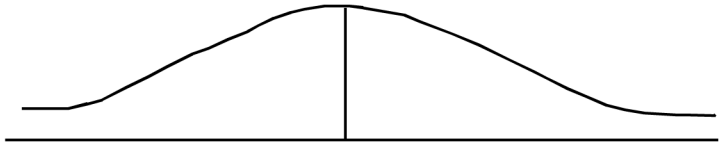
9.4 SKEWNESS

A distribution is said to be symmetrical if mean, median and mode are identical. Skewness means 'lack of symmetry'.

A symmetrical distribution when plotted on a graph gives a perfectly bell-shaped curve. If the frequency curve is not symmetrical, then the distribution is skewed.

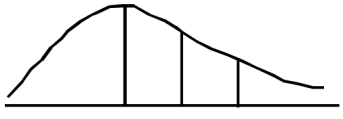
If the curve of the distribution has a longer tail towards the right, it is said to be positively skew. In this case, mean > median > mode.

If the curve of the distribution has a longer tail towards the left, it is said to be negatively skew. In this case mean < median < mode.



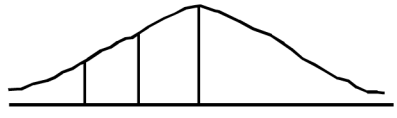
Mean = Mode = Median

Symmetrical curve



M_0 M_d M

Positively Skew



M M_d M_0

Negatively Skew

[Source : **Biostatistics** by Dr. P N Arora and Dr. P K Malhan (Himalya Publishing House, 2010)]

9.4.1 Measures of Skewness

The object of measuring skewness is to estimate (1) the extent to which, and (2) the direction in which the curve is distorted from the symmetrical curve.

The first measure of skewness = Mean – Mode

The second measure of skewness

$= (Q_3 - \text{Median}) - (\text{Median} - Q_1)$

$= Q_3 + Q_1 - 2Q_2$.

Karl Pearson's Coefficient of Skewness

We know that if a distribution is positively skew, then

mean > median > mode;

whereas if a distribution is negatively skew, then

mean < median < mode;

The difference (mean – mode) divided by the standard deviation is taken as coefficient of skewness. Thus

$$S_k = \frac{\text{mean} - \text{mode}}{\text{standard deviation}}$$

This is known as pearson's first coefficient of skewness.

We know that a set of data may not have a single mode. It may have two modes, called bimodal. It may have several modes, called multimodal. In this case it is said that the mode is ill defined. If the mode is ill defined, then using the empirical relation, mode = 3 median – 2 mean, for a moderately asymmetrical distribution, the second coefficient of skewness, as suggested by Karl Pearson, is given by

$$S_k = \frac{3(\text{mean} - \text{mode})}{\text{standard deviation}}$$

Note: The above two coefficients of skewness are pure numbers, since both the numerator and denominator have the same dimension. The value of first coefficient of skewness lies between –3 and 3; and that of the second lies between –1 and 1.

Again, based upon moments, coefficient of skewness is given by

$$S_k = \frac{\sqrt{\beta_1}(\beta_2 + 3)}{2(5\beta_2 - 6\beta_1 - 9)}$$

where β_1, and β_2 have been defined earlier.

Now, if $\beta_1 = 0$ or $\beta_2 = 0$, then $S_k = 0$. But by definition $\beta_2 = \frac{\mu_4}{\mu_2^2}$, which cannot be negative. $\therefore \beta_2$ cannot be –3. Thus, $S_k = 0$ if and only if $\beta_1 = 0$. Thus, for a symmetrical distribution, $\beta_1 = 0$.

Example 3: Karl Pearson's first coefficient of skewness of a distribution's is 0.32, its standard deviation is 6.5 and the mean is 29.6. Find the mode of the distribution.

Solution : We have

$$\frac{29.6 - \text{mode}}{6.5} = 0.32$$

or, 29.6 – mode = 2.08

$\therefore$ mode = 29.6 – 2.08

i.e, mode = 27.52

9.5 KURTOSIS

The word kurtosis has been derived from the Greek word 'kurtos' meaning 'curved, arching.' If there are two distributions, they may have two same mean, same standard deviation. Also they may be equally skew. But one of them may be more peaked than the other. This measure, called 'convexity or flatness of the curve' by Karl Pearson, is known as kurtosis of the curve. It gives an idea about the peakedness of the curve.

9.5.1 Measures of Kurtosis

Kurtosis is measured by the coefficient β_2 or its derivation γ_2 given by

$$\beta_2 = \frac{\mu_4}{\mu_2^2}, \quad \gamma_2 = \beta_2 - 3$$

Note : 1. When the value of $\beta_2 > 3$ the curve is said to be leptokurtic.

2. When the value of $\beta_2 = 3$, the curve is said to be normal or mesokurtic.

3. When the value of $\beta_2 < 3$, the curve is said to be platykurtic.

Example 4: The first four central moments of a distribution are 0, 2.5, 0.7 and 18.75. Find the kurtosis of the distribution.

Solution : We have $\mu_1 = 0, \ \mu_2 = 2.5, \ \mu_3 = 0.7, \ \mu_4 = 18.75$

Measure of kurtosis $\beta_2 = \frac{\mu_4}{\mu_2^2}$

$$= \frac{18.75}{(2.5)^2} = \frac{18.75}{6.25} = 3$$

Note : Since $\beta_2 = 3$, the curve is normal.

CHECK YOUR PROGRESS

Q 1: In a distribution, $Q_3 + Q_1 = 70$, $Q_3 - Q_1 = 30$, and the median is 40. Find the measure of skewness.

Q 2: The first four moments of a distribution about x = 4 are 1, 4, 10 and 45 respectively. Show the mean is 5. Find the corresponding moments about the mean. Calculate β_1 and β_2 and comment on their values.

Q 3: In a certain distribution, mean = 45, median = 48, coefficient of skewness = – 0.4. Find the standard deviation.

Q 4: Karl Pearson's coefficient of skewness of a distribution is 0.4. Its standard deviation is 8, and the mean is 30. Find the mode and the median of the distribution.

Q 5: The first four central moments of a distribution are 0, 2.3, 0.9 and 15.65. Test the kurtosis of the distribution.

Q 6: Show that the following distributions have the same degree of skewness.

Distribution A : Mean 100, Median 90, SD 10

Distribution B : Mean 90, Median 80, SD 10

9.6 LET US SUM UP

- Let a variable x take the values $x_1, x_2, \ldots, x_n$ with frequencies $f_1, f_2, \ldots, f_n$ respectively.

  The r th moment of x about any point a, denoted by μ_r', is defined by

  $$\mu_r' = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i - a)^r, \quad N = \sum_{i=1}^{n} f_i$$

  μ_r' is called the r th raw moment.

  The r th moment of x about the mean $\bar{x}$, denoted by μ_r, is defined by

  $$\mu_r = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i - \bar{x})^r$$

  μ_r is called the r th central moment.
- A distribution is said to be symmetrical if mean, median and mode are identical. Skewness means 'lack of symmetry'.
- A symmetrical distribution when plotted on a graph gives a perfectly bell-shaped curve. If the frequencey curve is not symmetrical, then the distribution is skew.
- If the curve of a distribution has a longer tail towards the right, it is said to be positively skew. In this case, mean > median > mode.
- If the curve of a distribution has a longer tail towards the left, it is said to be negatively

skew. In this case, mean < median < mode.

- Karl Pearson's first coefficient of skewness is given by

$$S_k = \frac{\text{mean} - \text{mode}}{\text{standard deviation}}$$

- Karl Pearson's second coefficient of skewness is given by

$$S_k = \frac{3(\text{mean} - \text{mode})}{\text{standard deviation}}$$

- If there are two distribution, they may have the same mean, same standard deviation. Also they may be equally skew. But one of them may be more peaked than the other. This measure, called 'convexity or flatness of the curve 'by Karl Pearson, is known as kurtosis of the curve.
- Kurtosis is measured by the coefficient β_2 or its derivation γ_2 given by

$$\beta_2 = \frac{\mu_4}{\mu_2^2}, \quad \gamma_2 = \beta_2 - 3$$

9.7 FURTHER READING

1) Statistical Methods by N.G. Das, McGraw Hill Education (India) Private Ltd. 2009
2) Probability and Statistics by G.Shanker Rao, University Press (India) private Ltd. 2011
3) Introduction to probability and Statistics by Seymour Lipschutz, John J.Schiller, Tata McGraw Hill Education private Ltd. 2005
4) Probability, Random Process, Queueing theory and Statistics by Dr. S.Rajasekaran, Prof.A.Chandrasekaran & G.Kavitha, Dhanam Publications, Chennai-60042
5) Probability and Statistics by G.Balaji, G.Balaji Publishers, Chennai-33

9.8 ANSWERS TO CHECK YOUR PROGRESS

Check Your Progress 1

Ans to Q No 1: Here $\bar{x} = \frac{2+3+7+8+10}{5} = 6$

We know that first moment about the mean is zero.

$\therefore \ \mu_1 = 0$

$$\mu_2 = \frac{1}{N}\Sigma(x_i - \bar{x})^2$$

$$= \frac{(2-6)^2 + (3-6)^2 + (7-6)^2 + (8-6)^2 + (10-6)^2}{5}$$

$$= \frac{16+9+1+4+16}{5} = \frac{46}{5} = 9.2$$

$$\mu_3 = \frac{(2-6)^3 + (3-6)^3 + (7-6)^3 + (8-6)^3 + (10-6)^3}{5}$$

$$= \frac{-64-27+1+8+64}{5} = \frac{-18}{5} = -3.6$$

$$\mu_4 = \frac{(2-6)^4 + (3-6)^4 + (7-6)^4 + (8-6)^4 + (10-6)^4}{5}$$

$$= \frac{256+81+1+16+256}{5} = \frac{610}{5} = 122$$

Ans to Q No 2: We have

$\mu_1' = 2,\ \mu_2' = 20,\ \mu_3' = 40,\ \mu_4' = 50$ given

Now $\mu_1 = 0$ (since the first moment about the mean is zero)

$$\mu_2 = \mu_2' - \mu_1'^2 = 20 - 4 = 16$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$$

$$= 40 - 3 \times 20 \times 2 + 2 \times 2^3$$

$$= 40 - 120 + 16 = -64$$

$$\mu_4 = \mu_4' - 3\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$$

$$= 50 - 4 \times 40 \times 2 + 6 \times 20 \times 2^2 - 3 \times 2^4$$

$$= 50 - 320 + 480 - 48 = 162$$

Ans to Q No 3: We have $\mu_1' = 1,\ \mu_2' = 4,\ \mu_3' = 10,\ \mu_4' = 45$

Now, $\mu_1' = \frac{1}{N}\Sigma f_i(x_i - 4)$

or, $\mu_1' = \bar{x} - 4$, after simplification

$\therefore\ \bar{x} = \mu_1' + 4 = 1 + 4 = 5$

We know that the second central moment is the variance.

Now, $\mu_2 = \mu_2' - \mu_1'^2 = 4 - 1 = 3$

$\therefore$ variance = 3

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$$

$$= 10 - 3 \times 4 \times 1 + 2 \times 1$$

$$= 10 - 12 + 2 = 0$$

$$\mu_4 = \mu_4' - 3\mu_3'\mu_1' + 2\mu_2'\mu_1'^2 - 3\mu_1'^4$$

$$= 45 - 4 \times 10 \times 1 + 6 \times 4 \times 1 - 3 \times 1$$

$$= 45 - 40 + 24 - 3 = 26$$

Ans to Q No 4: We have $\mu_1' = -4,\ \mu_2' = 22,\ \mu_3' = -117$

$\mu_4' = 560,$ (given)

Now, $\mu_1' = \frac{1}{N}\Sigma f_i(x_i - 5)$

or, $-4 = \frac{\Sigma f_i x_i}{N} - 5\frac{\Sigma f_i}{N}$

or, $-4 = \bar{x} - 5$

$\therefore\ \bar{x} = \frac{\Sigma f_i x_i}{N} = -4 + 5 = 1$

$\therefore\ \left(\mu_1'\right)_0 = \frac{1}{N}\Sigma f_i(x_i - 0) = \frac{\Sigma f_i x_i}{N} = 1$

Again, $\frac{1}{N}\Sigma f_i(x_i - 5)^2 = \mu_2'$

or, $\frac{1}{N}\Sigma f_i(x_1^2 - 10x_i + 25)^2 = 22$

or, $\frac{1}{N}\Sigma f_i x_i^2 - 10\frac{\Sigma f_i x_i}{N} + 25\frac{\Sigma f_i}{N} = 22$

or, $\frac{1}{N}\Sigma f_i x_i^2 - 10 \times 1 + 25 - 22$

or, $\frac{1}{N}\Sigma f_i x_i^2 = 22 + 10 - 25 = 7$

$\therefore\ \left(\mu_2'\right)_0 = \frac{1}{N}\Sigma f_i(x_i - 0)^2 = \frac{1}{N}\Sigma f_i x_i^2 = 7$

Again, $\frac{1}{N}\Sigma f_i(x_i - 5)^3 = \mu_3'$

or, $\frac{1}{N}\Sigma f_i(x_i^3 - 15x_i^2 + 75x_i - 125) = -117$

or, $\frac{1}{N}\Sigma f_i x_i^3 - 15\frac{\Sigma f_i x_i^2}{N} + 75\frac{\Sigma f_i x_i}{N} - 125\frac{\Sigma f_i}{N} = -117$

or, $\frac{1}{N}\Sigma f_i x_i^3 - 15 \times 7 + 75 \times 1 - 125 = -117$

or $\frac{1}{N}\Sigma f_i x_i^3 = 117 + 105 - 75 + 125 = 38$

$$\therefore (\mu_3')_0 = \frac{1}{N}\Sigma f_i (x_i - 0)^3$$

$$= \frac{1}{N}\Sigma f_i x_i^3 = 38$$

Again, $\frac{1}{N}\Sigma f_i (x_i - 5)^4 = \mu_4'$

or, $\frac{1}{N}\Sigma f_i (x_i^4 - 20x_i^3 + 150x_i^2 - 500x_i + 625) = 560$

or, $\frac{1}{N}\Sigma f_i x_i^4 - 20\frac{1}{N}\Sigma f_i x_i^3 + 150\frac{1}{N}\Sigma f_i x_i^2 - 500\frac{1}{N}\Sigma f_i x_i + 625\frac{1}{N}\Sigma f_i = 560$

or, $\frac{1}{N}\Sigma f_i x_i^4 - 20 \times 38 + 150 \times 7 - 500 \times 1 + 625 = 560$

or $\frac{1}{N}\Sigma f_i x_i^4 - 760 + 1050 - 500 + 625 = 560$

$$\therefore \frac{1}{N}\Sigma f_i x_i^4 = 560 + 760 - 1050 + 500 - 625 \ = 1820 - 1675 = 145$$

$$\therefore (\mu_4')_0 = \frac{1}{N}\Sigma f_i (x_i - 0)^4$$

$$= \frac{1}{N}\Sigma f_i x_i^4 = 145$$

Ans to Q No 5: We have

$$\mu_1 = 0, \ \mu_2 = 2.5, \ \mu_3 = 0.7, \ \ \mu_4 = 18.75$$

Now, $\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{0.49}{15.625} = 0.03136$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{18.75}{6.25} = 3$$

Check Your Progress 2

Ans to Q No 1: We have $Q_3 + Q_1 = 70$, $Q_3 - Q_1 = 30$

$\therefore$ $2Q_3 = 100 \Rightarrow Q_3 = 50$

$2Q_1 = 40 \Rightarrow Q_1 = 20$

Also, median = $Q_2 = 40$.

$\therefore$ measure of skewness = $Q_3 + Q_1 - 2Q_2$

= 50 + 20 – 2 x 40

= –10

Ans to Q No 2: We have

$\mu_1' = 1,\ \mu_2' = 4,\ \mu_3' = 10,\ \mu_4' = 5$ (given)

Now $\mu_1' = \frac{1}{N}\Sigma f_i(x_i - 4)$

or, $1 = \frac{\Sigma f_i x_i}{N} - 4\frac{\Sigma f_i}{N}$

or, $1 = \bar{x} - 4$

$\therefore\ \bar{x} = 1 + 4 = 6$

Now, $\mu_1 = 0$ (since the first moment about the mean is zero)

$\mu_2 = \mu_2' - \mu_1'^2 = 4 - 1 = 3$

$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$

= 10 – 3 x 4 x 1 + 2 x 1 = 10 –12 + 2 = 0

$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$

= 45 – 4 x 10 x 1 + 6 x 4 x 1 – 3 x 1

= 45 – 40 + 24 – 3

= 26

Now $\beta_1 = \frac{\mu_3^2}{\mu_2^2} = \frac{0}{27} = 0$

$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{26}{9} < 3$

Since $\beta_1 = 0$, the distribution is symmetrical.

Since $\beta_2 < 3$, the curve is platykurtic

Ans to Q No 3: mean = 45, median = 48

coefficient of skewness = – 0.4 (given)

We know that

Coefficient of skewness $= \frac{3(\text{mean} - \text{median})}{\text{standard deviation}}$

$\therefore\ -0.4 = \frac{3 \times (45 - 48)}{SD}$

$\therefore\ SD = \frac{9}{0.4} = 22.5$

Ans to Q No 4: Coefficient of skewness = 0.4

SD = 8, mesn = 30

We know that

Coefficient of skewness $= \dfrac{\text{mean} - \text{median}}{\text{SD}}$

or, $0.4 = \dfrac{30 - \text{mode}}{8}$

or, 3.2 = 30 – mode

$\therefore$ mode = 30 – 3.2 = 26.8

Again, coefficient of skewness $= \dfrac{3(\text{mean} - \text{median})}{\text{SD}}$

or, $0.4 = \dfrac{3(30 - \text{median})}{8}$

or 3.2 = 90 – 3 (median)

$\therefore$ 3(median) = 90 – 3.2 = 86.8

$\therefore$ median $= \dfrac{86.8}{3} = 28.93$

Ans to Q No 5: We have $\mu_1 = 0,\ \mu_2 = 2.3,\ \mu_3 = 0.9,\ \mu_4 = 15.65$

Measure of kurtosis $\beta_2 = \dfrac{\mu_4}{\mu_2^2}$

$$= \frac{15.65}{(2.3)^2}$$

$$= \frac{15.65}{5.29} = 2.9584$$

Note : $\because\ \beta_2 < 3$, the curve is platykurtic.

Ans to Q No 6: For distribution A :

Coefficient of skewness $= \dfrac{3(\text{mean} - \text{median})}{\text{SD}}$

$$= \frac{3 \times (100 - 90)}{10} = 3$$

For distribution B :

Coefficient of skewness $= \dfrac{3(\text{mean} - \text{median})}{\text{SD}}$

$$= \frac{3 \times (90 - 80)}{10} = 3$$

$\therefore$ distributions A and B have the same degree of skewness.

?9.9 MODEL QUESTIONS

Q 1: What is meant by skewness ? What is the object of measuring skewness?

Q 2: Write short notes on

(a) Sheppard's correction to moments

(b) Skewness and kurtosis.

Q 3: Karl Pearson's coefficient of skewness of a distribution is 0.5. If mean and mode are respectively 40 and 36, find the coefficient of variation. If the median is 42, find the standard deviation.

Q 4: Karl Pearson's coefficient of skewness of a distribution is 0.32. Its standard deviation is 6.5 and mean is 29.6. Find the median and the mode of the distribution. If the mode of the above distribution is 24.8, find the standard deviation.

Q 5: The standard deviation of a symmetrical distribution is 5. What must be the value of the fourth moment about the mean in order that the distribution be (i) leptokurtic, (ii) mesokurtic and (iii) platykurtic?

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UNIT 10 : PROBABILITY

UNIT STRUCTURE

10.1 Learning Objectives

10.2 Introduction

10.3 Basic Terminologies

10.4 Classical Definition of Probability

10.5 Axiomatic Approach to Probability

10.6 Some Theorems on Probability

10.6.1 Addition theorem on Probability

10.7 Let Us Sum Up

10.8 Further Reading

10.9 Answers To Check Your Progress

10.10 Model Questions

10.1 LEARNING OBJECTIVES

After going through this unit, you will able to learn

- the meaning of probability
- basic terminologies of probability
- classical definition of probability
- axiomatic approach to probability
- some theorems on probability.

10.2 INTRODUCTION

'Probability is the measure of the likelihood that an event will occur.' In this unit, we will state the classical as well as the axiomatic definition of probability. We will also discuss some theorems on probability.

10.3 BASIC TERMINOLOGIES

We will discuss some basic terminologies which are required for definition of probability.

1. **Random experiment :** A random experiment is an action or an experiment which occurs under identical conditions. The result of a random experiment is called an outcome.

Some examples of random experiment are given below.

(a) 'Tossing a coin' is a random experiment. If a fair coin is tossed, there are two possible outcomes, viz., head (H) and tail (T). Though we know that there are two possible outcomes, we cannot predict which one will occur.

(b) 'Throwing a die' is also a random experiment. A die is a solid cube, six faces of which are marked with 1, 2, 3, 4, 5, 6 dots. If a die is thrown, there are 6 possible outcomes, viz., 1, 2, 3, 4, 5, 6.

(c) 'Drawing a card from a well-shuffled pack of cards' is another example of random experiment. A pack of cards contains 52 cards in four suits, viz., Spades, Clubs, Hearts and Diamonds. Spades, and clubs are black-faced cards. On the other hand Hearts and Diamonds are red-faced cards. Each suit consists of 13 cards, of which nine cards are numbered from 2 to 10, an ace, a king, a queen, a jack (or knave). Thus, there are 4 aces, 4 kings, 4 queens and 4 jacks. Aces, kings, queens and jacks are called face cards.

2. **Trial :** The performance of a random experiment is called a trial.

3. **Event :** A collection of possible outcomes of an experiment is called an event.

4. **Equally likely events :** Events are said to be equally likely when there is no reason to expect any one of the events in preference over the others.

   In the random experiment of tossing a coin, the event of getting a **Head** and the event of getting a **Tail** are equally likely events.

5. **Mutually exclusive events :** Events are said to be mutually exclusive if the occurrence of one of them rules out the occurrence of all the others.

   In the random experiment of tossing a coin, the event of getting a **Head** and the event of getting a **Tail** are mutually exclusive events.

   In the random experiment of drawing a card, the event of getting a king and the event of getting a heart are not mutually exclusive events. There is a king of hearts.

6. **Exhaustive events :** Two or more events are said to be exhaustive if there is a certain chance of occurrence of at least one of them when they are all considered together. It may be noted that exhaustive events may or may not be equally likely and mutually exclusive.

7. **Favourable events :** Outcomes which entail the occurence of an event is said to be favourable to the event. In the random experiment of throwing a die, the numbers 2, 3 and 5 are favourable to the appearance of a prime number.

10.4 CLASSICAL DEFINITION OF PROBABILITY

Let there be n equally likely, mutually exclusive and exhaustive outcomes and m of them are favourable to an event E. Then the probability of the occurrence of the event E, denoted by P(E), is defined by

$$P(E) = \frac{number\ of\ outcomes\ favomable\ to occurence\ of\ E}{number\ of\ all\ possible outcomes}$$

$$= \frac{m}{n}$$

Odds of an event : Let there be m outcomes favourable to an event E, and n outcomes unfavourable to E. Then

(i) odds in favour of $E = \frac{m}{n}$

(ii) odds against $E = \frac{n}{m}$

(iii) $P(E) = \frac{m}{m+n}$

Example 1 : The odds in favour of occurrence of an event are $\frac{3}{14}$. Find the probability of the occurrence of this event.

Solution : Number of favourable outcomes = 3

Number of unfavourable outcomes = 14

$\therefore$ Total number of outcomes = 3 + 14 = 17

$\therefore$ the required probability $= \frac{3}{17}$

Example 2 : If the probability of the occurrence of an event is $\frac{6}{19}$, find the

(i) odds in favour of its occurrence,

(ii) odds against its occurrence.

Solution : Let the favourable outcomes be m and the unfavourable outcomes be n.

$\therefore$ the probability $= \frac{m}{m+n}$

But $\frac{m}{m+n} = \frac{6}{19}$, (given)

or 19m = 6m + 6n

or 13m = 6n

$\therefore \frac{m}{n} = \frac{6}{13}$

$\therefore$ (i) odds in favour $= \frac{m}{n} = \frac{6}{13}$

(ii) odds against $= \frac{n}{m} = \frac{13}{6}$

10.5 AXIOMATIC APPROACH TO PROBABILITY

The set of all possible outcomes in a random experiment is called a **sample space.** Generally it is denoted by S. Elements of a sample space are called sample points.

Two coins are flipped together. Then the outcomes are HH, HT, TH, TT. HH means head on the first coin as well as head on the second coin.HT means head on the first coin and tail on the second coin, and so on. Thus, S = {HH, HT, TH, TT}. Here the sample points are HH, HT, TH, TT, i.e. there are 4 sample points.

A subset of a sample space is called an event.

Definition : Let S be the sample space in a random experiment. Let E be an event associated with this random experiment. The probability of the occurrence of the event E, denoted by P(E), is defined by $P(E) = \frac{n(E)}{n(S)}$

$$= \frac{number\ of\ elements\ in E}{number\ of\ elements\ in\ S}$$

Note : $P(E) = \frac{n(E)}{n(S)} \geq 0, \because n(E) \geq 0$

Example 3: Two coins are tossed together. Find the probability of getting.

(i) one head,

(ii) two heads,

(iii) at least one head,

(iv) no head.

Solution : Let S be the sample space.

$\therefore$ S = {HH, HT, TH, TT}

(i) Let E_1 be the event of getting one head

$\therefore E_1$ = {HT, TH}

$\therefore P(E_1) = \frac{n(E_1)}{n(S)} = \frac{2}{4} = \frac{1}{2}$

(ii) Let E_2 be the event of getting 2 heads.

$\therefore$ E_2 = {HH}

$\therefore\ P(E_2) = \frac{n(E_2)}{n(S)} = \frac{1}{4}$

(iii) Let E_3 be the event of getting at least one head.

$\therefore$ E_3 = {HT, TH, HH}

$\therefore\ P(E_3) = \frac{n(E_3)}{n(S)} = \frac{3}{4}$

(v) Let E_4 be the event of getting no head.

$\therefore$ E_4 = {TT}

$\therefore\ P(E_4) = \frac{n(E_4)}{n(S)} = \frac{1}{4}$

Now, we discuss some terminologies again.

Impossible event : Let S be a sample space in a random experiment.

Clearly $\phi \subseteq S$

$\therefore$ ϕ is an event. It is called an **impossible event.**

Example : A die is thrown.

S = { 1, 2, 3, 4, 5, 6 }, the set of all outcomes.

Let E be the event of getting a number less than 1.

Clearly $E = \phi$.

$\therefore$ E is an impossible event.

Note : Probability of an impossible event is zero.

Let S be a sample space in a random experiment

Let E be an impossible event, i.e. $E = \phi$

$\therefore\ P(E) = \frac{n(E)}{n(S)} = \frac{n(\phi)}{n(S)} = \frac{0}{n(S)} = 0$

Sure event : Let S be a sample space in a random experiment.

Clearly $S \subseteq S$

$\therefore$ S is an event. It is called a sure event.

Example : A die is thrown.

S = { 1, 2, 3, 4, 5, 6 }

Let E be the event of getting a number less than 7.

Clearly E = 6

$\therefore$ E is a sure event.

Note : Probability of a sure event is one

Let S be a sample space in a random experiment.

Let E be a sure event, i.e., E = S

$$\therefore P(E) = \frac{n(E)}{n(S)} = \frac{n(S)}{n(S)} = 1$$

Elementary and Compound events : If a sample space contains one sample point only, then it is called an elementary (or simple) event.

An event which is not elementary is called a compound (or composite) event.

Mutually exclusive events : Two events E_1 and E_2 are said to be mutually exclusive if $E_1 \cap E_2 = \phi$

Note : (1) $E_1 \cap E_2$ is also written as E_1E_2.

(2) If $E_1, E_2, \ldots, E_n$ are events associated with a random experiment, then $E_1, E_2, \ldots, E_n$ are mutually exclusive if $E_i \cap E_j = \phi$, for $i \neq j$.

Thus, E_1, E_2, E_3 are mutualy exclusive if $E_1 \cap E_2 = \phi$, $E_2 \cap E_3 = \phi$, $E_3 \cap E_1 = \phi$

Mutually exhaustive events : Two events E_1 and E_2 are said to be mutually exhaustive if $E_1 \cup E_2 = S$, where S is the sample space.

Note : (1) $E_1 \cup E_2$ can be denoted by $E_1 + E_2$ if $E_1 \cap E_2 = \phi$

(2) If $E_1, E_2, \ldots, E_n$ are events associated with a random experiment, then $E_1, E_2, \ldots, E_n$ are mutually exhaustive if $E_1 \cup E_2 \cup \ldots \cup E_n = S$ where S is the sample space.

Complementary event : Let S be a sample space, and $E \subseteq S$, i.e., E is an event. Clearly E^c is also an event, as $E^c \subseteq S$. E^c is called the complementary event. Generally it is denoted by $\overline{E}$.

Note : If w is an outcome of a trial, and $w \in E$, we say that the event E has occurred. On the other hand, if $w \notin E$, we say that the event E has not occurred

Clearly,

(i) the event E occurs if $\overline{E}$ does not occur,

(ii) the event $\overline{E}$ occurs if E does not occur.

Properties

I. Let S be a sample space. Let $E_1 \subseteq S$, $E_2 \subseteq S$ i.e., E_1, E_2 are two events.

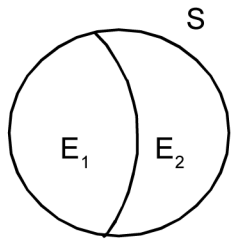
If E_1, E_2 are mutually exclusive and exhaustive events, then $P(E_1) + P(E_2) = 1$.

Proof : We have $E_1 \cap E_2 = \phi$

and $E_1 \cup E_2 = S$

Let n(S) = x, and n(E_1) = y

$\therefore$ n(E_2) = x – y, $\because E_1 \cap E_2 = \phi,\ E_2 \cup E_2 = S$



Now $P(E_1) + P(E_2) = \frac{n(E_1)}{n(S)} + \frac{n(E_2)}{n(S)}$

$= \frac{y}{x} + \frac{x-y}{x}$

$= \frac{y+x-y}{x} + \frac{x}{x} = 1$

II. Let S be a sample space, and $E \subseteq S$, i.e., E is an event. Then $P(E) + P(\overline{E}) = 1$, i.e. $P(\overline{E}) = 1 - P(E)$

Proof : We know that E and $\overline{E}$ are mutually exclusive and exhaustive events.

$\therefore$ by the previous result, $P(E) + P(\overline{E}) = 1$

i.e. $P(\overline{E}) = 1 - P(E)$

III. Let S be a sample sapce, and E_1, E_2 be two events such that $E_1 \subseteq E_2$. Then $P(E_1) \le P(E_2)$

Proof : We have $E_1 \subseteq E_2$

$\therefore\ n(E_1) \le n(E_2)$, ($E_1$, E_2 are finite)

or, $\frac{n(E_1)}{n(S)} \le \frac{n(E_2)}{n(S)}$

$\therefore\ P(E_1) \le P(E_2)$

IV. Let S be a sample space, and E be an event, i.e. $E \subseteq S$. Then $0 \le P(E) \le 1$

Proof : We know that

$\phi \subseteq E$ and $E \subseteq S$

$\therefore P(\phi) \le P(E)$ and $P(E) \le P(S)$

or, $0 \le P(E)$ and $P(E) \le 1$

$\therefore\ 0 \le P(E) \le 1$

Example 4 : Two dice are thrown. Find the probability of getting a total of 9.

Solution : Let S be the sample space

$\therefore$ S = { (1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), ...

. . . . . .

. . . . . .

. . . . . .

. . . . . .

(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)}

[Here (1, 1) means 1 in the first die, 1 in the second; (1, 2) means 1 in the first die, 2 in the second die, and so on]

$\therefore$ n (S) = 6 x 6 = 36.

Let E be the event of getting a total of 9.

$\therefore$ E = { (3, 6), 4, 5), (5, 4), (6, 3) }

$\therefore$ n (E) = 4

$$\therefore P(E) = \frac{n(E)}{n(S)} = \frac{4}{36} = \frac{1}{9}$$

Example 5 : A letter of the English alphabet is chosen at random. Find the probability that the letter so chosen.

(i) is a consonant,

(ii) precedes o and is a consonant (in alphabetical order),

(iii) follows o and is a consonant.

Solution : Let S be the sample space

$\therefore$ S = {a, b, c, ..., x, y, z}

Clearly n(S) = 26

(i) Let E_1 be the event of getting a consonant. We know that 21 consonants in the English alphabet.

$$\therefore P(E_1) = \frac{n(E_1)}{n(S)} = \frac{21}{26}$$

(ii) Let E_2 be the event of getting a consonant that precedes o.

$\therefore$ E_2 = {b, c, d, f, g, h, j, k, l, m, n}

$n(E_2) = 11$

$$\therefore P(E_2) = \frac{n(E_2)}{n(S)} = \frac{11}{26}$$

(iii) Let E_3 be the event of getting a consonant that follows o.

$\therefore$ E_3 = {p, q, r, s, t, v, w, x, y, z}

$n(E_3) = 10$

$$\therefore P(E_3) = \frac{n(E_3)}{n(S)} = \frac{10}{26} = \frac{5}{13}$$

Example 6 : 4 cards are drawn at random from a pack of well-shuffled pack of 52 cards. Find the probability that 2 are kings and 2 are queens.

Solution : 4 cards can be drawn from a pack of 52 cards in $^{52}C_4$ ways.

Let S be the sample space.

$\therefore\ n(S) = {}^{52}C_4$

Let E be the event of getting 2 kings and 2 queens.

2 kings can be drawn in $^4C_2$ ways

2 queens can be drawn in $^4C_2$ ways

$\therefore$ 2 kings and 2 queens can be drawn in $^4C_2 \times {}^4C_2$ ways.

$\therefore n(E) = {}^4C_2 \times {}^4C_2$

$$\therefore P(E) = \frac{n(E)}{n(S)}$$

$$= \frac{{}^4C_2 \times {}^4C_2}{{}^{52}C_4}$$

CHECK YOUR PROGRESS

Q 1: Find the probability of getting a club from a well-shuffled pack of cards.

Q 2: A bag contains 6 red, 8 black, 10 yellow, and 1 green ball. Find the probability of drawing (i) one red ball, (ii) one black ball, (iii) one white ball.

Q 3: A coin is tossed 3 times in succession. FInd the probability of getting (i) 2 heads, (ii) at least 2 heads.

Q 4: A number is chosen at random from the first 150 natural numbers. Find the probability that the number chosen is a multiple of 5.

Q 5: A single letter is selected at random the word PROBABILITY. Find the probability that it is a vowel.

Q 6: Two dice are tossed simultaneously, FInd the probability of getting a total (i) of 6, (ii) of 12, (iii) of 18.

Q 7: Two cards are drawn at random from a well-shuffled pack of 52 cards. Find the probability that one is a king and the other is a queen.

Q 8: Two cards are drawn at random from a well-shuffled pack of 52 cards. Find the probability of getting 2 hearts.

Q 9: A bag contains 4 white, 5 black, and 6 green balls. If 2 balls are drawn at random. Find the probability that none of them is black.

Q 10: A letter of the English alphabet is chosen at random. Find the probability that the letter so chosen precedes ‘h’ and is a vowel.

Q 11: What is the probability that a leap year selected at random will contain 53 Mondays?

10.6 SOME THEOREMS ON PROBABILITY

10.6.1 Addition Theorem of Probability

I. Let S be a sample space in a random experiment. Let E_1, E_2 be two events. Then

$P(E_1 \text{ or } E_2) = P(E_1) + P(E_2) - P(E_1 \text{ and } E_2)$

i.e., $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

Proof : $P(E_1 \cup E_2) = \dfrac{n(E_1 \cup E_2)}{n(S)}$

$$= \frac{n(E_1) + n(E_2) - (E_1 \cap E_2)}{n(S)}$$

$$= \frac{n(E_1)}{n(S)} + \frac{n(E_2)}{n(S)} - \frac{(E_1 \cap E_2)}{n(S)}$$

$$= P(E_1) + P(E_2) - P(E_1 \cap E_2$$

Deduction : If E_1, E_2 are mutually exclusive, then $E_1 \cap E_2 = \phi$ and $\therefore\ P(E_1 \cap E_2) = P(\phi) = 0$. Thus, for two mutually exclusive events E_1 and E_2,

$P(E_1 \cup E_2) = P(E_1) + (E_2)$

In this case we write

$P(E_1 + E_2) = P(E_1) + P(E_2)$

II. Let S be a sample space in a random experiment. Let E_1, E_2, E_3 be three events. Then

$P(E_1 \text{ or } E_2 \text{ or } E_3)$

$= P(E_1 \cup E_2 \cup E_3)$

$= P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2) - P(E_2 \cap E_3) - P(E_3 \cap E_1) + P(E_1 \cap E_2 \cap E_3)$

Proof : We have

$P(E_1 \cup E_2 \cup E_3)$

$$= \frac{n(E_1 \cup E_2 \cup E_3)}{n(S)}$$

$$= \frac{n(E_1)}{n(S)} + \frac{n(E_2)}{n(S)} + \frac{n(E_3)}{n(S)} - \frac{n(E_1 \cap E_2)}{n(S)} - \frac{n(E_2 \cap E_3)}{n(S)} - \frac{n(E_3 \cap E_1)}{n(S)} + \frac{n(E_1 \cap E_2 \cap E_3)}{n(S)}$$

$$= P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2) - P(E_2 \cap E_3) - P(E_3 \cap E_1) + P(E_1 \cap E_2 \cap E_3)$$

Deduction I : If E_1, E_2, E_3 are mutually exclusive events, then

$E_1 \cap E_2 = \phi,\ E_2 \cap E_3 = \phi,\ E_3 \cap E_1 = \phi,\ E_1 \cap E_2 \cap E_3 = \phi$

Thus, for three mutually exclusive events E_1, E_2 and E_3,

$P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3)$

i.e., $P(E_1+E_2+E_3) = P(E_1) + P(E_2) + P(E_3)$

Similarly, if E_1, E_2, E_3, ..., E_n are mutually exclusive events.

$P(E_1 + E_2 + \ldots\ldots + E_n) = P(E_1) + P(E_2) + \ldots\ldots P(E_n)$.

II : Let E_1, E_2 be two events associated with a random experiment S. Then

(i) $P\left(\overline{E_1 \cup E_2}\right) = P\left(\overline{E_1} \cap \overline{E_2}\right)$

(ii) $P\left(\overline{E_1 \cap E_2}\right) = P\left(\overline{E_1} \cup \overline{E_2}\right)$

Proof : We have $E_1 \subseteq S,\ E_2 \subseteq S$

By De Morgan's Laws, we know that

$\left(E_1 \cup E_2\right)^c = E_1^{\,c} \cap E_2^{\,c}$

i.e., $\overline{E_1 \cup E_2} = \overline{E}_1 \cap \overline{E}_2$

$\therefore P\left(\overline{E_1 \cup E_2}\right) = P\left(\overline{E}_1 \cap \overline{E}_2\right)$

Also $\left(E_1 \cap E_2\right)^c = E_1^{\,c} \cup E_2^{\,c}$

i.e. $\left(\overline{E_1 \cap E_2}\right) = \overline{E}_1 \cap \overline{E}_2$

$\therefore\ P\left(\overline{E_1 \cap E_2}\right) = \left(\overline{E}_1 \cap \overline{E}_2\right)$

Example 7 : Two dice are tossed together. Find the probability of getting a doublet or a total of 8.

Solution : Let S be the sample space.

Clearly n(S) = 36 [See Example 4]

Let E_1 be the event of getting a doublet. [Doublet means 'either of two dice, each of which, when thrown, has the same number of spots on the face lying uppermost'.]

$\therefore$ E_1 = { (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6) }

Let E_2 be the event of getting a total of 8.

$\therefore$ E_2 = { (2, 6), (3, 5), (4, 4), (5, 3), (6, 2) }

$E_1 \cap E_2 = \{(4, 4)\}$

Thus $n(E_1) = 6$, $n(E_2) = 5$, $n(E_1 \cap E_2) = 1$

$\therefore$ $P(E_1$ or $E_2)$

$= P(E_1 \cup E_2)$

$= P(E_1) + P(E_2) - P(E_1 \cap E_2)$

$$= \frac{n(E_1) + n(E_2) + n(E_3) - n(E_1 \cap E_2) - n(E_2 \cap E_3) - n(E_3 \cap E_1) + n(E_1 \cap E_2 \cap E_3)}{n(S)}$$

$$= \frac{n(E_1)}{n(S)} + \frac{n(E_2)}{n(S)} - \frac{n(E_1 \cap E_2)}{n(S)}$$

$$= \frac{6}{36} + \frac{5}{36} - \frac{1}{36}$$

$$= \frac{6+5-1}{36}$$

$$= \frac{10}{36}$$

$$= \frac{5}{18}$$

Example 8 : Two dice are tossed. Find the probability of getting an odd number in the first die or a total of 8.

Solution : Let S be the sample space

$\therefore$ n(S) = 36

Let E_1 be the event of getting an odd number in the first die.

$\therefore$ $n(E_1) = 3 \times 6 = 18$

[There are 3 odd numbers, viz., 1, 3, 5. Any one of them will appear in the first die. Any one of 1, 2, 3, 4, 5, 6 will appear in the second die.]

Let E_2 be the event of getting a total of 8.

$\therefore$ $E_2 = \{(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)\}$

$n(E_2) = 5$

$E_1 \cap E_2 = \{(3, 5),(5, 7)\}$

$n(E_1 \cap E_2) = 2$

$\therefore P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

$$= \frac{n(E_1)}{n(S)} + \frac{n(E_2)}{n(S)} - \frac{n(E_1 \cap E_2)}{n(S)}$$

$$= \frac{18}{36} + \frac{5}{36} - \frac{2}{36}$$

$$= \frac{18+5-2}{36} = \frac{21}{36} = \frac{7}{12}$$

Example 9 : Let E_1, E_2, E_3 be three mutually exclusive and exhaustive events associated with a random experiment. If $P(E_1) = \frac{3}{2}P(E_2)$ and $P(E_3) = \frac{1}{2}P(E_2)$, find

$P(E_1)$, $P(E_2)$, and $P(E_3)$

Solution : Let S be the sample spare

$\therefore E_1 \cup E_2 \cup E_3 = S$ ($\because$ E_1, E_2, E_3 are exhaustive events)

$\therefore P(E_1 \cup E_2 \cup E_3) = P(S)$

or, $P(E_1) + P(E_2) + P(E_3) = P(S)$ ($\because E_1$, E_2, E_3 are mutually exclusive events)

or, $\frac{3}{2}P(E_2) + P(E_2) + \frac{1}{2}P(E_2) = 1$

or, $\left(\frac{3}{2} + 1 + \frac{1}{2}\right)P(E_2) = 1$

or, $3P(E_2) = 1$

$\therefore P(E_2) = \frac{1}{3}$

$\therefore P(E_1) = \frac{3}{2}.P(E_2) = \frac{3}{2} \cdot \frac{1}{3} = \frac{1}{2}$

$P(E_3) = \frac{1}{2}.P(E_2) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$

Example 10 : Let E_1, E_2, be two events associated with a random experiment.

If $P(E_1) = \frac{1}{2}$, $P(E_2) = \frac{1}{3}$ and $P(E_1 \cap E_2) = \frac{1}{6}$, find

(i) $P\left(\overline{E_1 \cap E_2}\right)$, (ii) $P\left(\overline{E_1 \cup E_2}\right)$

Solution : (i) $P\left(\overline{E_1 \cap E_2}\right)$

$= 1 - p\left(E_1 \cap E_2\right)$ $[P(\overline{E}) = 1 - P(E)]$

$= 1 - \frac{1}{6} = \frac{5}{6}$

(ii) $P\left(\overline{E_1 \cup E_2}\right)$

$= 1 - P(E_1 \cup E_2)$

$= 1 - [P(E_1) + P(E_2) - P(E_1 \cap E_2)]$

$= 1 - \left(\frac{1}{2} + \frac{1}{3} - \frac{1}{6}\right) = 1 - \frac{2}{3} = \frac{1}{3}$

CHECK YOUR PROGRESS

Q 1: A number is chosen at random from the first 150 natural numbers. Find the probability that the number so chosen is a multiple of 4 or a multiple of 5.

Q 2: A card is drawn at random from a well-shuffled pack of 52 cards. Find the probability that the card so chosen is a spade or a king.

Q 3: Let E_1, E_2 be two events associated with a random experiment. If $P(E_1) = 0.3$, $P(E_2) = 0.2$, and $P(E_1 \cap E_2) = 0.1$, Find

(i) $P\left(\overline{E_1 \cap E_2}\right)$, (ii) $P\left(\overline{E_1 \cup E_2}\right)$

Q 4: Let E_1, E_2 be two mutually exclusive events associated with a random experiment. If P(not E_1) = 0.45 and P(E_1 or E_2) = 0.65, find P(E_2).

Q 5: Let E_1, E_2 be two events associated with a random experiment. If $P(E_1) = \frac{1}{2}$, $P(E_2) = \frac{1}{3}$, P(E_1 and E_2) $= \frac{1}{6}$, Find (i) P(E_1 or E_2),

(ii) P (neither E_1 nor E_2)

Q 6: Two dice are thrown together. Find the probability that the sum of the numbers on the two faces is neither divisible by 3 nor by 4.

10.7 LET US SUM UP

- A random experiment is an action or an experiment which occurs under identical conditions. The result of a random experiment is called an outcome. 'Tossing of a coin', 'Throwing a die', 'Drawing a card from a well-shuffled pack of cards' are examples of random experiment.
- The set of all possible outcomes in a random experiment is called a sample space.
- A subset of a sample space is called an event.
- The probability of the occurrence of the event E, denoted by P(E), is defined by

  $$P(E) = \frac{n(E)}{n(S)}$$

  Where n(E) denotes the number of elements in the event E, and n(S) denotes the number of elements in the sample space S.
- Let S be a sample space in a random experiment. Let E_1, E_2 be two events. Then

  $$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap_2)$$

10.8 FURTHER READING

1) Statistical Methods by N.G. Das, McGraw Hill Education (India) Private Ltd. 2009

2) Probability and Statistics by G.Shanker Rao, University Press (India) private Ltd. 2011

3) Introduction to probability and Statistics by Seymour Lipschutz, John J.Schiller, Tata McGraw Hill Education private Ltd. 2005

4) Probability, Random Process, Queueing theory and Statistics by Dr. S.Rajasekaran, Prof.A.Chandrasekaran & G.Kavitha, Dhanam Publications, Chennai-60042

5) Probability and Statistics by G.Balaji, G.Balaji Publishers, Chennai-33

✓10.9 ANSWERS TO CHECK YOUR PROGRESS

Check Your Progress 1

Ans to Q No 1: Let S be the sample space.

One card can be drawn from a well-shuffled pack of 52 cards in ${}^{52}C_1$ ways i.e. 52 ways.

$\therefore$ n(S) = 52

Let E be the event of getting a club.

One club can be drawn in 13_{C_1} ways, i.e., 13 ways.

$\therefore$ n(E) = 13

$$\therefore P(E) = \frac{n(E)}{n(S)} = \frac{13}{52} = \frac{1}{4}$$

Ans to Q No 2: Total number of balls = 6 + 8 + 10 = 24. One ball can be drawn in ${}^{24}C_1$ ways, i.e., 24 ways.

Let S be the sample space.

$\therefore$ n(S) = 24

(i) Let E_1 be the event of drawing a red ball.

One red ball can be drawn in 6_{C_1} ways, i.e., 6 ways.

$\therefore$ n(E_1) = 6

$$\therefore P(E_1) = \frac{n(E_1)}{n(S)} = \frac{6}{24} = \frac{1}{4}$$

(ii) Let E_2 be the event of drawing a black ball.

One black ball can be drawn in 8_{C_1} ways, i.e., 8 ways.

$\therefore$ n(E_2) = 8

$$\therefore P(E_2) = \frac{n(E_2)}{n(S)} = \frac{8}{24} = \frac{1}{3}$$

(iii) Let E_3 be the event of drawing a white ball.

There is no white ball in the bag.

$\therefore E_3 = \phi$

$\therefore P(E_3) = P(\phi) = 0$

Ans to Q No 3: Let S be the sample space.

$\therefore$ S = {HHH, HTH, HHT, THH, HTT, TTH, THT, TTT}

(i) Let E_1 be the event of getting 2 heads.

$\therefore$ E_1 = { HTH, HHT, THH }

$$\therefore P(E_1) = \frac{n(E_1)}{n(S)} = \frac{3}{8}$$

(ii) Let E_2 be the event of getting at least 2 heads.

$\therefore$ E_2 = { HTH, HHT, THH, HHH }

$$\therefore P(E_2) = \frac{n(E_2)}{n(S)} = \frac{4}{8} = \frac{1}{2}$$

Ans to Q No 4: Let S be the sample space.

One number can be chosen from the first 150 natural numbers in ${}^{150}C_1$ ways, i.e., 150 ways.

$\therefore$ n(S) = 150

S = { 1, 2, 3, 4, ..., 149, 150 }

Let E be the event of choosing a multiple of 5.

$\therefore$ E = { 5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, 60, 65, 70, 75, 80, 85, 90, 95, 100, 105, 110, 115, 120, 125, 130, 135, 140, 145, 150 }

$\therefore$ n(E) = 30

$$\therefore P(E) = \frac{n(E)}{n(S)} = \frac{30}{150} = \frac{1}{5}$$

Ans to Q No 5: Let S be the sample space

$\therefore$ n(S) = 11

Let E be the event of getting a vowel

$\therefore$ n(E) = 4

$$\therefore P(E) = \frac{n(E)}{n(S)} = \frac{4}{11}$$

Ans to Q No 6: Let S be the sample space

$\therefore$ n(S) = 6 x 6 = 36

(i) Let E_1 be the event of getting a total of 6.

$\therefore$ E_1 = { (1, 5), (2, 4), (3, 3), (4, 2), (5, 1) }

$n(E_1) = 5$

$$\therefore P(E_1) = \frac{n(E_1)}{n(S)} = \frac{5}{36}$$

(ii) Let E_2 be the event of getting a total of 12.

$\therefore$ E_2 {(6, 6)}

$$\therefore P(E_2) = \frac{n(E_2)}{n(S)} = \frac{1}{36}$$

(iii) Let E_3 be the event of getting a total of 18.

Clearly $E_3 = \phi$, there cannot be total of 18.

$$\therefore P(E_3) = P(\phi) = 0$$

Ans to Q No 7: Two cards can be drawn in ${}^{52}C_2$ ways.

Let S be the sample space.

$$\therefore n(S) = {}^{52}C_2 = \frac{52!}{2!50!} = \frac{52 \times 51 \times 50!}{2 \times 1 \times 50!} = 26 \times 51$$

There are 4 kings, and 4 queens.

One king can be drawn in ${}^4C_1$, i.e., 4 ways.

One queen can be drawn in ${}^4C_1$, i.e., 4 ways.

Let E be the event of getting a king and a queen.

$\therefore$ n(E) = 4 x 4 = 16

$$\therefore P(E) = \frac{n(E)}{n(S)} = \frac{16}{26 \times 51} = \frac{8}{13 \times 51} = \frac{8}{663}$$

Ans to Q No 8: Let S be the sample space

$$\therefore n(S) = {}^{52}C_2 = 26 \times 51$$

Let E be event of getting 2 hearts.

$$\therefore n(S) = {}^{13}C_2 = \frac{13!}{2!11!} = \frac{13 \times 12 \times 11!}{2 \times 1 \times 11!} = 13 \times 6$$

$$\therefore P(E) = \frac{n(E)}{n(S)} = \frac{13 \times 6}{26 \times 51} = \frac{1}{17}$$

Ans to Q No 9: Total number of balls = 4 + 5 + 6 = 15

2 balls can be chosen from 15 balls in ${}^{15}C_1$ ways.

Let S be the sample space.

$$\therefore n(S) = {}^{15}C_2 = \frac{15!}{2!13!} = \frac{15 \times 14}{2 \times 1} = 15 \times 7 = 105$$

If none of the balls is black, then the two balls must be from white and green balls.

Total number of white and green balls = 4 + 6 =10

Let E be the event of getting 2 balls, none of which is black.

$$\therefore n(E) = {}^{10}C_2 = \frac{10!}{2!\,8!} = \frac{10\times 9}{2\times 1} = 45$$

$$\therefore P(E) = \frac{n(E)}{n(S)} = \frac{45}{105} = \frac{3}{7}$$

Ans to Q No 10: Let S be the sample space.

$\therefore$ n(S) = 26

Let E be the event of getting a letter that precedes 'h' and is a vowel.

$\therefore$ E = { a, e }

$$\therefore P(E) = \frac{n(E)}{n(S)} = \frac{2}{26} = \frac{1}{13}$$

Ans to Q No 11: A leap year contains 366 days

366 days = 52 weeks + 2 days.

52 weeks contain 52 Mondays. So the 53rd Monday will be from the remaining 2 days.

The remaining 2 days will be any one of the following, – Sun and Mon, Mon and Tues, Tues and Wed, Wed and Thurs, Thurs and Fri, Fri and Sat, Sat and Sun. Out of these 7 possible outcomes, two outcomes are favomable (Sun and Mon, Mon and Tues)

$\therefore$ the required probability $= \frac{2}{7}$

Check Your Progress 2

Ans to Q No 1: Let S be the sample space.

$\therefore$ n(S) = 150

Let E_1 be the event of getting a multiple of 4.

$\therefore$ n(E_1) = 37

Let E_2 be the event of getting a multiple of 5.

$\therefore$ n(E_2) = 30

$\therefore E_1 \cap E_2$ is the event of getting a multiple of 4 and of 5, i.e., of 20.

$\therefore n(E_1 \cap E_2) = 7$

Now, $P(E_1 \cup E_2)$

$$= P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

$$= \frac{n(E_1)}{n(S)} + \frac{n(E_2)}{n(S)} - \frac{n(E_1 \cap E_2)}{n(S)}$$

$$= \frac{37}{150} + \frac{30}{150} - \frac{7}{150}$$

$$= \frac{60}{150} = \frac{2}{5}$$

Ans to Q No 2: Let S be the sample space

$\therefore n(S) = {}^{52}C_1 = 52$

Let E_1 be the event of getting a spade.

$\therefore n(E_1) = {}^{13}C_1 = 13$

Let E_2 be the event of getting a king.

$\therefore n(E_2) = {}^{4}C_1 = 4$

$E_1 \cap E_2$ is the event of getting a spade and a king.

$\therefore n(E_1 \cap E_2) = 1$

$\therefore$ P (getting a spade or a king)

$= P(E_1 \cup E_2)$

$= P(E_1) + P(E_2) - P(E_1 \cap E_2)$

$$= \frac{n(E_1)}{n(S)} + \frac{n(E_2)}{n(S)} - \frac{n(E_1 \cap E_2)}{n(S)}$$

$$= \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$$

Ans to Q No 3: We have $P(E_1) = 0.3$, $P(E_2) = 0.2$, $P(E_1 \cap E_2) = 0.1$

(i) $P\left(\overline{E_1 \cap E_2}\right) = 1 - P(E_1 \cap E_2) = 1 - 0.1 = 0.9$

(ii) $P\left(\overline{E_1 \cup E_2}\right) = 1 - P(E_1 \cup E_2)$

$= 1 - [P(E_1) + P(E_2) - PE_1 \cap E_2]$

$= 1 - 0.3 - 0.2 + 0.1 = 0.6$

Ans to Q No 4: We have P(not E_1) = 0.45

or $P(\overline{E_1}) = 0.45$

or $1 - P(E_1) = 0.45$

$\therefore P(E_1) = 1 - 0.45 = 0.55$

E_1, E_2 are two mutually exclusive events.

$\therefore P(E_1 \text{ or } E_2) = P(E_1 \cup E_2) = P(E_1) + P(E_2)$

$\therefore P(E_1) + P(E_2) = 0.65$

or, 0.55 + P(E_2) = 0.65

$\therefore$ P(E_2) = 0.65 – 0.55 = 0.1

Ans to Q No 5: We have $P(E_1)=\frac{1}{2}, P(E_2)=\frac{1}{3}$

P (E_1 and E_2) = $P(E_1 \cap E_2)=\frac{1}{6}$

(i) P(E_1 or E_2) = $P(E_1 \cup E_2)$

$$= P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

$$= \frac{1}{2}+\frac{1}{3}-\frac{1}{6}=\frac{4}{6}=\frac{2}{3}$$

(ii) P(neither E_1 nor E_2)

= P($\overline{E}_1$ and $\overline{E}_2$)

$= P(\overline{E}_1 \cap \overline{E}_2)$

$= P\left(\overline{E_1 \cup E_2}\right)$

$= 1 - P(E_1 \cup E_2)$

$= 1-\frac{2}{3}=\frac{1}{3}$

Ans to Q No 6: Let S be the sample space.

$\therefore n(S) = {}^6C_1 \times {}^6C_1 = 6 \times 6 = 36$

Let E_1 be the event of getting a sum that is divisible by 3.

$\therefore$ E_1 = { (1, 2), (1, 5), (2, 1), (2, 4), (3, 3), (3, 6), (4, 2), (4, 5), (5, 1), (5, 4), (6, 3), 6, 6) }

$\therefore$ n (E_1) = 12.

Let E_2 be the event of getting a sum that is divisible by 4.

$\therefore$ E_2 = {(1, 3), (2, 2), (2, 6), (3, 1), (3, 5), (4,4), (5, 3), (6, 2), (6, 6)}

$\therefore$ n(E_2) = 9

$E_1 \cap E_2$ is the event of getting a sum that is divisible by 3 and 4.

$\therefore E_1 \cap E_2 = \{(6, 6)\}, \quad \therefore n(E_1 \cap E_2) = 1$

P(neither E_1 nor E_2)

= P($\overline{E}_1$ and $\overline{E}_2$)

$= P(\overline{E}_1 \cap \overline{E}_2)$

$= P\left(\overline{E_1 \cup E_2}\right)$

$= 1 - P(E_1 \cup E_2)$

$$= 1 - [P(E_1) + P(E_2) - P(E_1 \cap E_2)]$$

$$= 1 - \left[\frac{12}{36} + \frac{9}{36} - \frac{1}{36}\right]$$

$$= 1 - \frac{5}{9}$$

$$= \frac{4}{9}$$

10.10 MODEL QUESTIONS

Q 1: In a simultaneous toss of a coin and a die, what is the sample space?

Q 2: Find the probability of getting a tail in a throw of a coin.

Q 3: Three coins are tossed together, what is the probability of getting (i) one head, (ii) at least one head, (iii) two heads, (iv) at least two heads, (v) all heads.

Q 4: In a simultaneous throw of two dice, find the probability of getting a total of 9 or 11 ?

Q 5: What is the probability that an ordinary year (not leap year) has 53 Mondays ?

Q 6: An urn contains 15 balls numbering from 1 to 15 (no two balls are of the same number). What is the probability that a ball selected at random is a ball with number that is multiple of 3 or 4 ?

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UNIT 11: CONDITIONAL PROBABILITY

UNIT STRUCTURE

11.1 Learning Objectives

11.2 Introduction

11.3 Conditional probability

11.4 Multiplication theorem of probability

11.5 Law of probability

11.6 Baye's theorem

11.7 Let Us Sum Up

11.8 Further Readings

11.9 Answers to check your progress

11.10 Model Questions

11.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- define conditional probability
- know multiplication theorem of probability
- define law of probability
- know Baye's theorem

11.2 INTRODUCTION

In unit 10, we discussed about probbility and some important theorems related to probability. In this unit, we will discuss the concept of conditional probability. We will also discuss dependent and independent events. Multiplication theorems for dependent and independent events will also be discussed in this unit. Finally, we will disuss the law of probability and Baye's theorem.

11.3 CONDITIONAL PROBABILITY

In unit 10, we have discussed the methods of finding the probability of events. Supose, we have two events from the same sample space in a random experiment, does the occurrence of one of the events affect the probability of the other event? Let us see

this situation with the following examples

1. Consider a random experiment of throwing a die.

Let S be the sample space and A be the event of of getting an odd numbers.

$\therefore\ S=\{1,2,3,4,5,6\}$ and $A=\{1,3,5\}$

$$P(A)=\frac{3}{6}=\frac{1}{2}$$

Suppose we consider another event B such that B be the event of getting the numbers greater than 3. Then $B=\{4,5,6\}$.

When we say that B has already occurred,the event of getting ' numbers is less than or equal to 3'is not possible. Now, the number of elements of of the sample space is reduced to 3 i.e $\{4,5,6\}$. Out of these three points, two points are even number $\{4,6\}$.

Thus, given that if the event B has already occurred, then $P(A)=\frac{2}{3}$.

Hence, the probability of A given that B has already occurred is denoted by $P\left(\frac{A}{B}\right)$.

2. Consider the experiment of tossing three fair coins. The sample space of the experiment is $S=\{HHH,HHT,HTH,THH,HTT,THT,TTH,TTT\}$.

Let E be the event 'at least two tails appear' and F be the event 'first coin shows head'.

Now, suppose we are given that the first coin shows head, i.e. F occurs, then we have to find the probability of occurrence of E. With the information of occurrence of F, we are sure that the cases in which first coin does not result into a head should not be considered while finding the probability of E. This information reduces our sample space from the set S to its subset F for the event E. In other words, the additional information really amounts to telling us that the situation may be considered as being that of a new random experiment for which the sample space consists of all those outcomes only which are favourable to the occurrence of the event F. Now, the sample point of F which is favourable to event E is HTT.

Thus, Probability of E considering F as the sample space $=\frac{1}{4}$.

Probability of E given that the event F has occurred $=\frac{1}{4}$

This probability of the event E is called the *conditional probability of E given that F has already occurred*, and is denoted by P (E|F).

3. Consider the random experiment of drawing a card from a deck of 52 cards. Suppose we draw a queen in successive draws. In first draw the probability of drawing a queen is

$\frac{4}{52}$. If the card is not replaced before the second draw, then the event of getting queen in the second draw depends on the first event. Now, there will be only 3 queens. The probability of getting a queen in second draw will also be $\frac{3}{52}$.

In the above examples,we observe that the occurrence of one event depends on another event which has already happend. Hence,an event E is said to be a dependent event if the occurnce of A depends on another event F which has already happend. It is denoted by $P(E \mid F)$.

The above examples leads to the definition of conditional probability as :

Let E and F be two events defined on a sample space S. Let $P(F) > 0$,then the conditional probability of E given that F has already occurred denoted by $P(E \mid F)$ and mathematically written as :

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)}, P(F) > 0$$

Similarly, $P(F \mid E) = \frac{P(E \cap F)}{P(E)}, P(E) > 0$

Note : (1) $P(E \mid F)$ and $P(F \mid E)$ can also be expressed as follows :

$$P(E \mid F) = \frac{n(E \cap F)}{n(F)} \quad \text{and} \quad P(F \mid E) = \frac{n(E \cap F)}{n(E)}.$$

(2) $P(E \mid F)$ is read as probability of event E given event F.

Laws of Conditional probability :

(a) If E_1 and E_2 be any two events of a sample space S and A be any event of S such that $P(A) \neq 0$, then

$$P(E_1 \cup E_2 \mid A) = P(E_1 \mid A) + P(E_2 \mid A) - P(E_1 \cap E_2 \mid A)$$

We have $P(E_1 \cup E_2 \mid A) = \frac{P((E_1 \cup E_2) \cap A)}{P(A)}$

$$= \frac{P(E_1 \cap A) \cup P(E_2 \cap A)}{P(A)}$$

$$= \frac{P(E_1 \cap A) + P(E_2 \cap A) - P(E_1 \cap E_2 \cap A)}{P(A)}$$

$$= \frac{P(E_1 \cap A)}{P(A)} + \frac{P(E_2 \cap A)}{P(A)} - \frac{P(E_1 \cap E_2 \cap A)}{P(A)}$$

$$= P(E_1 \mid A) + P(E_2 \mid A) - P(E_1 \cap E_2 \mid A)$$

(b) $P(E' \mid F) = 1 - P(E \mid F)$

We have $P(S \mid F) = 1$

$$\Rightarrow P(E' \cup E \mid F) = 1$$

$$\Rightarrow P(E' \mid F) + P(E \mid F) = 1$$

$$\Rightarrow P(E' \mid F) = 1 - P(E \mid F)$$

Independent events

Events are said to be independent if the occurrence of one does not affect the others. In the experiment of tossing a fair coin, the occurrence of the event 'head' in the first toss is independent of the occurrence of the event ' head' in the second toss and third toss.

Example 1 : A card is chosen at random from an ordinary deck of 52 cards.

Let E be the event that the cards is an ace and F the event that the card is a club. Then E and F are independent.

Dependent events:

If the occurrence of one event influences the occurrence of the other, then the second event is said to be dependent on the first. In the above example, if we do not replace the first ball drawn, this will change the composition of balls in the bag while making the second draw and therefore the event of 'drawing a red ball' in the second will depend on event (first ball is red or white) occurring in first draw. Similarly, if a person draw a card from a full pack and does not replace it, the result of the draw made afterwards will be dependent on the first draw.

Example : Two fair dice are tossed. Let E be the event that the first die is a 3, F the event that the sum is 6 and G the event that sum is 7. Then E and F are dependent, but E and G are independent.

Note: If the events A and B are independent, that is the probability of occurrence of any one of them $P(E \mid F) = P(E)$ and $P(F \mid E) = P(F)$.

11.4 MULTIPLICATION THEOREM ON PROBABILITIES

Now, we discuss multiplication theorem on probabilities for both independent and dependent events.

Multiplication theorem on probabilities for independent events :

If two events E and F are independent, the probability that both of them occur is

equal to the product of their individual probabilities. i.e $P(E \cap F) = P(E)P(F)$.

Note: (1)The theorem can be extended to three or more independent events. If E, F,G……. be independent events, then

$$P(E \cap F \cap G) = P(E)P(F)P(G).$$

(2) If E and F are independent then the complements of E and F are also independent.

i.e $P\left(\bar{E} \cap \bar{F}\right) = P\left(\bar{E}\right)P\left(\bar{F}\right)$

Multiplication theorem for dependent events: If E and F be two dependent events, i.e the occurrence of one event is affected by the occurrence of the other event, then the probability that both A and B will occur is

$$P(E \cap F) = P(E)P(F \mid E).$$

Note: In the case of three events E, F, G

$P(E \cap F \cap G) = P(E)P(F \mid E)P(G \mid E \cap F)$ i.e., the probability of occurrence of E, F and G is equal to the probability of E times the probability of F given that E has occurred, times the probability of G given that both E and F have occurred.

Ilustrative Examples :

Example1: Let E and F are two events,such that $P(E) = 0.8$, $P(F) = 0.6$, $P(E \cap F) = 0.5$. Find the value of (i) $P(E \cup F)$ (ii) $P(F \mid E)$ (iii) $P(E \mid F)$.

Solution : (i) $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

$$= 0.8 + 0.6 - 0.5$$
$$= 0.9$$

(ii) $P(F \mid E) = \dfrac{P(E \cap F)}{P(E)} = \dfrac{0.5}{0.8} = \dfrac{5}{8}$

(iii) $P(E \mid F) = \dfrac{P(E \cap F)}{P(F)} = \dfrac{0.5}{0.6} = \dfrac{5}{6}$.

Example 2 : If $P(E) = 0.5, P(F) = 0.3$ and $P(E \cap F) = 0.15$. Find $P(E \mid F)$.

Solution : By definition of conditional probability

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)}$$

$$= \frac{0.15}{0.3}$$

$$= 0.5$$

Example 3 : If $P(E) = 0.9; P(F \mid E) = 0.8,$ find $P(E \cap F)$.

Solution : By definition of conditional probability

$$P(F \mid E) = \frac{P(F \cap E)}{P(E)}$$

$$\Rightarrow P(F \cap E) = P(E).P(F \mid E)$$

$$= 0.9 \times 0.8$$

$$= 0.72 .$$

Example 4 : Find the chance of drawing 2 white balls in succession from a bag containing 5 red and 7 white balls, the balls drawn not being replaced.

Solution : Let E be the event that ball drawn is white in the first draw and F be the event that ball drawn is white in the second draw.

$$\therefore P(F \cap E) = P(E).P(F \mid E)$$

Here $P(E) = \frac{7}{12}$, $P(F \mid E) = \frac{6}{11}$

$$\therefore P(F \cap E) = \frac{7}{12} . \frac{6}{11} = \frac{7}{22} .$$

Example 5 : In an examination 30% of the students have failed in Mathematics, 20% of the students have failed in chemistry and 10 % have failed in both Mathematics and Chemistry. A student is selected at random (i) What is the probability that the students has failed in Mathematics if it is known that he has failed in Chemistry ?

Solution : Let E be the event of failing in Mathematics.

$$\therefore P(E) = 30\% = 0.3$$

Let F be the event of failing in Chemistry

$$\therefore P(F) = 20\% = 0.2$$

Let $E \cap F$ be the event of failing in both subjects.

$$P(E \cap F) = 10\% = 0.1$$

The probability that the student has failed in Mathematics given that he has failed in Chemistry is

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)} = \frac{0.1}{0.2}$$

$$= 0.5 .$$

Example 5 : A bag contains 5 white and 3 black balls.Two balls are drawn at random one after the other without replacement.Find the probability that both balls are black.

Solution : Probability of drawing a black ball in the first attempt is

$$P(E) = \frac{3}{5+3} = \frac{3}{8}$$

Probability of drawing the second black ball given that the first ball drawn is black

$$P(F \mid E) = \frac{2}{5+2} = \frac{2}{7}$$

$\therefore$ The probability that both balls drawn are black is given by

$$P(E \cap F) = P(A) \times P(F \mid E) = \frac{3}{8} \times \frac{2}{7} = \frac{3}{28}.$$

CHECK YOUR PROGRESS

Q 1: Two fair coins are tossed. Given that at least one of them is heads, what is the probability that both of them are heads?

Q 2: Suppose that a box contains 8 red balls and 4 white balls. We randomly draw 3 balls from the box without replacement. Find the probability that all 3 are red.

Q 3: A die is rolled, If the outcome is an odd number. What is the probability that it is prime?

Q 4: A family has two children.Find the probability that both children are girls given that at least one of them is a girl.

11.5 TOTAL PROBABILITY

Theorem of total probability : If E_1 and E_2 are mutually exclusive and exhaustive events and $P(E_1) \neq 0$, $P(E_2) \neq 0$, then for any event A of S,

$$P(A) = P(E_1)P(A \mid E_1) + P(E_2)P(A \mid E_2)$$

Proof : Since E_1 and E_2 are mutually exclusive and exhaustive events, we have $E_1 \cup E_2 = S$ and $E_1 \cap E_2 = \phi$.

$\therefore A = A \cap S$

$= A \cap (E_1 \cup E_2)$

$= (A \cap E_1) \cup (A \cap E_2)$ .... (1)

Now, $(A \cap E_1) \cap (A \cap E_2) = A \cap (E_1 \cap E_2)$

$= A \cap \phi$

$= \phi$

$\therefore$ $A \cap E_1$ and $A \cap E_2$ are mutually exclusive and exhaustive events.

$\therefore$ From (1),we have

$P(A) = P(A \cap E_1) + P(A \cap E_2)$

$= P(E_1)P(A \mid E_1) + P(E_2)P(A \mid E_2)$ [multiplication theorem of probability]

If E_1, E_2, E_3 are mutually exclusive and exhaustive events and $P(E_1) \neq 0, P(E_2) \neq 0, P(E_3) \neq 0$ then for any event A of S,

$$P(A) = P(E_1)P(A|E_1) + P(E_2)P(A|E_2) + P(E_3)P(A|E_3)$$

11.6BAYE'S THEOREM

The concept of conditional probability discussed earlier takes into account information about the occurrence of one event to predict the probability of another event. This concept can be extended to revise probabilities based on new information and to determine the probability that a particular effect was due to specific cause. The procedure for revising these probabilities is known as Bayes theorem. The Principle was given by Thomas Bayes in 1763. By this principle, assuming certain prior probabilities, the posteriori probabilities are obtained.

Bayes' Theorem or Rule

Statement : Let $E_1, E_2, E_3, \cdots, E_n$ be a set of n mutually exclusive and collectively exhaustive events and $P(E_1), P(E_2), P(E_3), \cdots, P(E_n)$ are their corresponding probabilities. If A is another event such that P(A) is not zero and the priori probabilities $P(A \mid E_i)$ i =1,2…, n are also known. Then

$$P(E_i \mid A) = \frac{P(E_i)P(A \mid E_i)}{P(E_1)P(A \mid E_1) + P(E_2)P(A \mid E_2) + \cdots + P(E_n)P(A \mid E_n)}$$

i.e., $$P(E_i \mid A) = \frac{P(E_i)P(A \mid E_i)}{\sum_{i=1}^{n} P(E_i)P(A \mid E_i)}$$

Proof : Let S be the sample space of the random experiment.

The events $E_1, E_2, E_3, \cdots, E_n$ being exhaustive.

$$\therefore S = E_1 \cup E_2 \cup \cdots \cup E_n$$

$$A = A \cap S \qquad [A \subset S]$$

$$= A \cap (E_1 \cup E_2 \cup \cdots \cup E_n)$$

$$= (A \cap E_1) \cup (A \cap E_2) \cup \cdots (A \cap E_n) \quad \text{[Distributive law]}$$

$$\Rightarrow P(A) = P(A \cap E_1) + P(A \cap E_2) + \cdots + P(A \cap E_n)$$

$$= P(E_1)P(A \mid E_1) + P(E_2)P(A \mid E_2) + \cdots + P(E_n)P(A \mid E_n)$$

$$= \sum_{i=1}^{n} P(E_i)P(A \mid E_i) \quad \text{.... (1)}$$

Now, $P(A \cap E_i) = P(A)P(E_i \mid A)$

$$\Rightarrow P(E_i \mid A) = \frac{P(A \cap E_i)}{P(A)}$$

$$= \frac{P(E_i)P(A \mid E_i)}{\sum_{i=1}^{n} P(E_i)P(A \mid E_i)} \qquad \text{[using (1)]}$$

Note : (1) The probabilities $P(E_1), P(E_2), \cdots, P(E_n)$ are called as the 'a prori probabilites'.

(2) The probabilities $P(E_1 \mid A), P(E_2 \mid A), \cdots, P(E_n \mid A)$ are called'posteriori probabilities'.

Example 1 : A purse contains 6 silver coins and 3 gold coins.Another purse contains 4 silver coins and 5 gold coins.A purse is selected at random and a coin is drawn from it.What is the probability that it is a silver coin.

Solution : Let Let E = Event of selecting first purse

F = Event of selecting the second purse

A= Event of getting a silver coin

$\therefore P(E) = \frac{1}{2}$ and $P(F) = \frac{1}{2}$

Also, $P(E \mid A) = \frac{6}{9} = \frac{2}{3}$ and $P(F \mid A) = \frac{4}{9}$

$\therefore$ Required probability $P(A) = P(E)P(A \mid E) + P(F)P(A \mid F)$

$$= \frac{1}{2} \times \frac{6}{9} + \frac{1}{2} \times \frac{4}{9}$$

$$= \frac{10}{18}$$

$$= \frac{5}{9}$$

Example 2 : A box contains two coins. One coin is heads on both sides and the other is heads on one side and tails on the other. One coin is selected from the box at random and the face of one side is observed. If the face is heads, what is the probability that the other side is heads ?

Solution : Let E = Event of selecting two headed coin

F = Event of selecting the other coin (one head and one tail)

A= Event of getting the second side also head.

Now, we have to find $P(E \mid A)$.

Here, $P(E) = P(F) = \frac{1}{2}$

$$P(A \mid E) = 1, P(A \mid F) = \frac{1}{2}$$

By Baye's theorem, we have

$$P(E \mid A) = \frac{P(E)P(A \mid E)}{P(E)P(A \mid E) + P(F)P(A \mid F)}$$

$$= \frac{\frac{1}{2} \times 1}{\frac{1}{2} \times 1 + \frac{1}{2} \times \frac{1}{2}}$$

$$= \frac{2}{3}.$$

Example 3 : Assume that a factory has two machines. Past records show that machine I produces 20% of the items and machine II produces 80% of the items. Further, it shows that 6% of the items produced by machine I are defective and only 1 % of the items produced by machine II were defective. If a defective item is drawn at random,what is the probability that it was produced by machine I ?

Solution : Let E_1, E_2 be the events of drawing an item produced by machines I and II repectively. Let A denote the event of drawing a defective item.

We have $P(E_1) = \frac{20}{100} = \frac{1}{5}$, $P(E_2) = \frac{80}{100} = \frac{4}{5}$

Also, we have $P(A \mid E_1) = \frac{6}{100}$, $P(A \mid E_2) = \frac{1}{100}$

From Bayes' theorem, the required probability is

$$P(E_1 \mid A) = \frac{P(E_1).P(A \mid E_1)}{P(E_1)P(A \mid E_1) + P(E_2)P(A \mid E_2)}$$

$$= \frac{\frac{1}{5} \cdot \frac{6}{100}}{\frac{1}{5} \cdot \frac{6}{100} + \frac{4}{5} \cdot \frac{1}{100}} = \frac{6}{6+4} = 0.6.$$

Example4 : A bolt is manufactured by 3 machines A,B and C.A turns out twice as many items as B and machines B and C produce equal number of items. 2 % of bolts produced by A and B are defective and 4 % of bolts produced by C are defective.All bolts are put into one stock pile and 1 is chosen from this pile.What is the probability that it is defective ?

Solution : Let E_1, E_2, E_3 be the events of bolts manufactured by machine A,B,C repectively.

Here we have to apply total theorem on probability.

B and C produce equal no. of bolts and A produces two times that of B.

$$\therefore \quad P(E_1) = 50\% = \frac{50}{100} = 0.50$$

$$P(E_2) = 25\% = \frac{25}{100} = 0.25$$

$$P(E_3) = 25\% = \frac{25}{100} = 0.25$$

Let D be the event of getting defective bolt.

$$P(D \mid E_1) = \frac{2}{100} = 0.02$$

$$P(D \mid E_2) = \frac{2}{100} = 0.02$$

$$P(D \mid E_3) = \frac{4}{100} = 0.04$$

By total theorem on probability.

$$\begin{aligned} P(D) &= P(E_1)P(D \mid E_1) + P(E_2)P(D \mid E_2) + P(E_3)P(D \mid E_3) \\ &= 0.5 \times 0.02 + 0.25 \times 0.02 + 0.25 \times 0.04 \\ &= 0.0250 \end{aligned}$$

Example 5 : A bag A contains 2 white and 3 red balls and bag B contains 4 white and 5 red balls.One ball is drawn at random at random from one of the bags and is found to be red,find the probability that it is drawn from bag B.

Solution : Here given that the ball drawn is red and we have to find the probability of the ball coming from bag B.

Let E_1 be the event of selecting bag A.

$$\therefore \quad P(E_1) = \frac{1}{2}$$

Let E_2 be the event of selecting bag B.

$$\therefore \quad P(E_2) = \frac{1}{2}$$

Let D be the event of selecting red ball.

Probability of drawing red ball from bag A is

$$P(D \mid E_1) = \frac{3}{5}$$

Similarly, probability of drawing red ball from bag B is

$$P(D \mid E_2) = \frac{5}{9}$$

We have to find the red ball was drawn from bag B i.e., $P(E_2 \mid D)$

By Baye's Theorem

$$P(E_2 \mid D) = \frac{P(E_2)P(D \mid E_2)}{P(E_1)P(D \mid E_1) + P(E_2)P(D \mid E_2)}$$

$$= \frac{\frac{1}{2}\cdot\frac{5}{9}}{\frac{1}{2}\cdot\frac{3}{5} + \frac{1}{2}\cdot\frac{5}{9}}$$

$$= \frac{\frac{5}{18}}{\frac{3}{10} + \frac{5}{18}}$$

$$= \frac{\frac{5}{18}}{\frac{27+25}{90}}$$

$$= \frac{\frac{5}{18}}{\frac{52}{90}}$$

$$= \frac{25}{52}$$

Example 6 : In a bolt factory machines A, B, C manufacture repectively 25%, 35%, 40% of the total. Of their output 5%, 4%, 2% are defective bolts. A bolt is drawn at random and found to be defective. What is the probability that it was manufactured by machine B ?

Solution : Let E_1, E_2, E_3 be the events of bolts manufactured by machine A, B, C repectively.

$$P(E_1) = 25\% = \frac{25}{100} = 0.25$$

$$P(E_2) = 35\% = \frac{35}{100} = 0.35$$

$$P(E_3) = 40\% = \frac{40}{100} = 0.40$$

Let D be the event of drawing a defective bolt.

$$\therefore\ P(D \mid E_1) = \frac{5}{100} = 0.05 \text{ [defective from 'A']}$$

$$P(D \mid E_2) = \frac{4}{100} = 0.04 \text{ [defective from 'B']}$$

$$P(D \mid E_3) = \frac{2}{100} = 0.02 \text{ [defective from 'C']}$$

Now, we have to find the probability of defective bolt manufactured by machine B. i.e., $P(E_2 \mid D)$.

By Baye's Theorem :

$$P(E_2 \mid D) = \frac{P(E_2)P(D \mid E_2)}{P(E_1)P(D \mid E_1) + P(E_2)P(D \mid E_2) + P(E_3)P(D \mid E_3)}$$

$$= \frac{0.35 \times 0.04}{0.25 \times 0.05 + 0.35 \times 0.04 + 0.4 \times 0.02}$$

CHECK YOUR PROGRESS

Q 1: Assume that a factory has two machines. Past records show that machine I produces 20% of the items and machine II produces 80% of the items. Further, it shows that 6% of the items produced by machine I are defective and 1% of the items produced by machine II are defective. If a defective item is drawn at random, what is the probability that it was produced by machine I ?

Q 2: Three machines A, B, C produce repectively 60%, 30%,10% of the total number of items of a factory. The percentage of respective outputs of these machines are repectively 2%, 3% and 4%. An item is selected at random and is found to be defective.Find the probability that the item was produced by machine C.

Q 3: A bag X contains 2 white and 3 red balls and a bag Y contains 4 white and 5 red balls. One ball is drawn at random from one of the bags and is found to be red. Find the probability that it was drawn from bag Y.

Q 4: Urn A contains 3 red and 4 white balls and Urn B has 4 red and 5 white balls. One urn is selected at random and one ball is taken out of it. Thrown ball is red. What is the probability that it was taken from urn B ?

Q 5: A factory has three production lines I, II and III contributing 20%, 30 % and 50 %respectively, to its total output. The percentages of substandard items produced by lines I, II and III are, respectively, 15, 10 and 2. If an item chosen

at random from the total output is found to be substandard, what is the probability that the item is from line I ?.

11.7 LET US SUM UP

- For two events E and F, the probability of **an** event **E** under the condition that F has already occumh is the **conditional probability P(E|F)** of **E** and is given by $P(E \mid F) = \frac{P(E \cap F)}{P(F)}$.
- Two events for which the occurrence of one has no influence on the occurrence or non-occurrence of the other are **independent** events, otherwise, they are **dependent.**
- If the events E and F are independent and $P(E) > 0, P(F) > 0$, then $P(E \cap F) = P(E)P(F)$ and $P(E \mid F) = P(E), P(F \mid E) = P(F)$.

11.8 FURTHER READING

1) Rao. G. Shanker, Probability and Statistics, Universitis Press.

2) Statistics, R.S.N. Pillai and Bagavathi S. Chand and Company Ltd. New Delhi-110055.

11.9 ANSWERS TO CHECK YOUR PROGRESS

CHECK YOUR PROGRESS -1

Ans to Q No 1: Let E be the event that both of them are heads and F the event that at least one of them is heads.

Then, $P(E) = \frac{3}{4}, P(E \cap F) = \frac{1}{4}, P(F) = \frac{1}{4}$

$$\therefore P(E \mid F) = \frac{P(E \cap F)}{P(F)}$$

$$= \frac{\frac{1}{4}}{\frac{3}{4}}$$

$$= \frac{1}{3}.$$

By using the definition of conditional probability, we can easily get that

$$P(E \cap F) = P(F)P(E \mid F) = \frac{1}{4} \times \frac{1}{3} = \frac{1}{12}.$$

Ans to Q No 2: For i= 1;2;3, let E_i be the event that thei-th ball is red.

Then, $P(E_1 \cap E_2 \cap E_3) = P(E_1)P(E_2 \mid E_1)P(E_3 \mid E_1E_2)$

$$= \frac{8}{12}.\frac{7}{11}.\frac{6}{10} = \frac{14}{55}.$$

Ans to Q No 3: When a die is rolled, the sample is $S = \{1,2,3,4,5,6\}$.

Let $E =$ Event of getting an odd number

$= \{1,3,5\}$

F = Event of getting a prime number

$= \{2,3,5\}$

Then, $E \cap F = \{3,5\}$

$\therefore P(E) = \frac{3}{6} = \frac{1}{2}, P(F) = \frac{3}{6} = \frac{1}{2}$ and $P(E \cap F) = \frac{1}{3}$.

P(getting a prime already when an odd number)

= P(getting a prime number | getting an odd number)

$= P(E \mid F)$

$= P(E \mid F)$

$= \frac{P(E \cap F)}{P(F)}$

$= \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}.$

Ans to Q No 4: The sample space S is given by

$S = \{(b,b),(b,g),(g,b),(g,g)\}$

Let E : Event that both children are girls

F : Event that at least one of the child is a girl.

Then E $= \{(g,g)\}$ and F $= \{(b,g),(g,b),(g,g)\}$

$E \cap F = \{(g,g)\}$

$\therefore P(E) = \frac{3}{4}, P(E \cap F) = \frac{1}{4}$

$\therefore$ Required probability is $P(E \mid F) = \frac{P(E \cap F)}{P(F)}$

$$= \frac{\frac{1}{4}}{\frac{3}{4}}$$

$$= \frac{1}{3}$$

CHECK YOU PROGRESS-2

Ans to Q No 1: Let E_1, E_2 be the events of drawing an item produced by machines I and II repectively.Let A denote the event of drawing a defective item.

We have $P(E_1) = \frac{20}{100} = \frac{1}{5}, P(E_2) = \frac{80}{100} = \frac{4}{5}$

Also, we have $P(A \mid E_1) = \frac{6}{100}, P(A \mid E_2) = \frac{1}{100}$

From Bayes's Theorem, the required probability is

$$P(E_1 \mid A) = \frac{P(E_1)P(A \mid E_1)}{P(E_1)P(A \mid E_1) + P(E_2)P(A \mid E_2)}$$

$$= \frac{\frac{1}{5}.\frac{6}{100}}{\frac{1}{5}.\frac{6}{100} + \frac{4}{5}.\frac{1}{100}} = \frac{6}{6+4} = 0.6.$$

Ans to Q No 2: Let E, F, G denote the events in which the items are selected by machines A, B, C repectively.

Then, we have $P(E) = \frac{60}{100} = 0.6, P(F) = \frac{30}{100} = 0.3, P(G) = \frac{10}{100} = 0.1$

Let D denote the event in which the item selected at random is a defective item.Then

$$P(D \mid E) = \frac{2}{100} = 0.02, P(D \mid F) = \frac{3}{100} = 0.03, P(D \mid G) = \frac{4}{100} = 0.04$$

Applying Bayes's theorem, we obtain

$$P(G \mid D) = \frac{P(G)P(D \mid G)}{P(E)P(D \mid E) + P(F)P(D \mid F) + P(G)P(D \mid G)}$$

$$= \frac{(0.1)(0.04)}{(0.6)(0.02) + (0.3)(0.03) + (0.1)(0.04)}$$

$$= \frac{0.004}{0.012 + 0.009 + 0.004}$$

$$= \frac{0.004}{0.025}$$
$$= 0.16.$$

Ans to Q No 3: Let E_1 : the ball is drawn from bag X

E_2 : the ball is drawn from bag Y

and A ; the ball is red.

Here $P(E_1) = P(E_2) = \frac{1}{2}$

Also, $P(A \mid E_1)$ = P(a red ball is drawn from bag X)

$$= \frac{3}{5}$$

$P(A \mid E_2)$ = P(a red ball is drawn from bag Y)

$$= \frac{5}{9}$$

By Baye's Theorem ,we have

$$P(E_2 \mid A) = \frac{P(E_1)P(A \mid E_1)}{P(E_1)P(A \mid E_1) + P(E_2)P(A \mid E_2)}$$

$$= \frac{\frac{1}{2} \times \frac{5}{9}}{\frac{1}{2} \times \frac{3}{5} + \frac{1}{2} \times \frac{5}{9}} = \frac{25}{52}.$$

Ans to Q No 4: Required probability $= \frac{28}{45}$ [By Bayes's Theorem]

Ans to Q No 5: By Bayes formula

$$P(E_1 \mid A) = \frac{P(E_1)P(A \mid E_1)}{P(E_1)P(A \mid E_1) + P(E_2)P(A \mid E_2)}$$

$$= \frac{0.2 \times 0.15}{0.2 \times 0.15 + 0.3 \times 0.10 + 0.5 \times 0.02}$$
$$= 0.069.$$

? 11.10 MODEL QUESTIONS

Q 1: E and F are two events for which $P(E) = \frac{1}{2}$, $P(F) = \frac{1}{3}$ and $P(E \cap F) = \frac{1}{4}$.Find (i) $P(E \mid F)$,(ii) $P(E \cup F)$ and (iii) $P(E' \mid F')$.

Q 2: Three persons A, B, C supply raw mterials to a factory in the proportion $\frac{1}{2},\frac{1}{3},\frac{1}{6}$ and $5\%,6\%,8\%$ of the materials supplied by them repectively are found to be defective. A material supplied is selected at random from the total supply. What are the probabilities that material was supplied by A, or by B or by C ?

Q 3: One of three identical coins is unbiased while the remaining two are biased with probabilities of getting head in a toss are $\frac{1}{3}$ and $\frac{2}{3}$. One of the coins selected randomly shows head when tossed. Find the probability of the coin to be unbiased.

Q 4: Boys from the districts of Kamrup, Nagaon and Jorhat come to a college in Assam in the proportions 50%, 30% and 20% repectively. 80%, 60% and 40% of the boys comingrepectively from the districts wear Dhooti. A boy selected randomly from all the boys of the college is found to wear Dhooti. What is the probability that he comes kamrup?

Q 5: Three trains X, Y, Z carry repectively 25%, 35%, 40% of the total passengers. Of the passengers carried by the tains 5%, 4% and 3% respectively are ladies. If a pssengers chosen at random out of all the passengers of the trains found to be a lady. Prove that the probability that she was carried by X, or Y or Z are repectively $\frac{25}{69},\frac{28}{69},\frac{16}{69}$.

Q 6: Three are three groups of children. The first group consists of 3 girls and 1 boy, the second consists of 2 girls and 2 boys and the third consists of 1 girl and 3 boys. One child is picked up at random from each group. Find the probability of selecting 1 girl and 2 boys.

Q 7: A bag contains 5 white and 7 red balls. If a ball is drawn twice at random successively, find the probability that one ball is white and the other is red.

Q 8: In a running competition, the probabilities of the two competitiors of winning the race is $\frac{1}{3}$ and $\frac{1}{5}$. What is the probability that both of them will not win ?

Q 9: One card is drawn from a well shuffled pack of 52 cards.Deduce the probability of drawing.

(i) a red card or an ace or both

(ii) a diamond or an ace or both.

(iii) king of diamond or king of heart or both.

Q 10: 4 men play with a pack of cards containing 52 cards. What is the probability that the 4 kings are held by only one of the players ?

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UNIT 12 : RANDOM VARIABLES AND PROBABILITY DISTRIBUTION

UNIT STRUCTURE

12.1 Learning Objectives

12.2 Introduction

12.3 Random Variables

12.3.1 Discrete Random Variables

12.3.2 Continuous Random Variables

12.4 Probability Distribution of Random Variables

12.4.1 Mean and Variance of Random Variables.

12.5 Let Us Sum Up

12.6 Further Readings

12.7 Answers To Check Your Progress

12.8 Model Questions

12.1 LEARNING OBJECTIVES

After going through this unit, you will able to learn

- random variables
- discrete random variables
- continuous random variables
- probability distribution of random variables.

12.2 INTRODUCTION

Let 3 coins be tossed. One may be interested to know the number of heads obtained. Similarly, Let a pair of dice be thrown. One may be interested to know the sum of points. In each of the cases a real number as associated with the outcome. A rule that assigns a real number to each outcome is called a random variable.

12.3 RANDOM VARIABLES

Let S be a sample space (Collection of outcomes) associated with a given random experiment.

Let $\omega \in S$ i.e ω is an outcome.

A rule X that associates **each** $\omega \in S$ **to a unique** real number X(ω) is called a random variable. Thus, a random variable is a real-valued function X which assigns a **unique** real number X(ω) to **each** $\omega \in S$ **.**

Let 3 coins be flipped.

Let S be the sample space associated with this random experiment.

Then.

S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

Let us consider the function

X : $S \to IR$, where IR is the set of real numbers such that.

X(w) = number of heads in w, for all $\omega \in S$

$\therefore$ X (HHH) = 3, X(HHT) = 2, X(HTH) = 2, X(THH) = 2

X(HTT) = 1, X(THT) = 1, X(TTH) = 1, X(TTT) = 0

Here, X is a random variables

Clearly, range (X) = {0, 1, 2, 3}

12.3.1 Discrete Random Variable

A random variable which can take only finite or countable infinite number of values is called a discrete rancom variable.

The example given above is discrete random variable. In this case the random variable X takes four values, viz, 0, 1, 2, 3.

12.3.2 Continuours Random Variable

A random variable which can take any value between two given limits in called a continuous random variable.

Let the maximum life of electric bulbs be 2000 hours. The continuous random variable is

$X(w) = \{x : 0 \le x \le 2000\}$

12.4 PROBABILITY DISTRIBUTION OF RANDOM VARIABLES

If the values of a random variable together with the corresponding probabities are given, then this description is called a probability distribution of the random variable.

Let a random variable X take the values x_1, x_2, ..., x_n with probabilities p_1, p_2, . .

. . . , p_n respectively. Then the probability distribution of X is given by

X	x_1	x_2	–	– – – –	x_n
P(x)	p_1	p_2	–	– – – – –	p_n

Note : The above probability distribution of X is defined only when

(a) each $p_i \geq 0$

(b) $\sum_{i=1}^{n} p_i = 1$

Example 1: There coins are tossed simultaneously. Let X be the random variable such that.

X(w) = number of heads in w.

Write down the probability distribution of X.

Solution : Let S be the sample space.

Then

S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

We have n(S) = 8

Here X = number of heads in tossing 3 coins.

Since the number of heads may be 0, 1, 2 or 3, therefore the possible values of X are 0, 1, 2, 3.

$\therefore$ P(X = 0) = P (getting no head)

= P(TTT) = $\frac{1}{8}$

P(X=1) = P (getting 1 head)

= P(HTT or THT or TTH) = $\frac{3}{8}$

P(X=2) = P(getting 2 heads)

= P(HHT or HTH or THH} = $\frac{3}{8}$

P(X=3) = P(getting 3 heads)

= P(HHH) = $\frac{1}{8}$

$\therefore$ the probability distribution is

X	0	1	2	3
P(X)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Note : Here each $P_i > 0$

Also $P_1 + P_2 + P_3 + P_4 = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = 1$

Example 2 : A die is tossed. Let X be the random variable defined by

$$X = \begin{cases} 1, \text{ if the outcome is an even number} \\ 0, \text{if the outcome is an odd number} \end{cases}$$

Write down the probability distribution of X.

Solution: Let S be the sample space.

$\therefore$ S = {1, 2, 3, 4, 5, 6}

Here X takes the values 0, 1.

P(X = 0) = P (getting an odd number)

$$= \frac{3}{6} = \frac{1}{2}$$

P(X = 1) = P (getting an even number)

$$= \frac{3}{6} = \frac{1}{2}$$

$\therefore$ the probability distribution is

X	0	1
P(X)	$\frac{1}{2}$	$\frac{1}{2}$

CHECK YOUR PROGRESS

Q 1: Given an example of a discrete random variable.

Q 2: Give an example of a continuous random variable.

Q 3: Two coins are tossed simultaneously. Let X be the random variable such that

X(w) = number of heads in w. Write down the probability distribution of X.

Q 4: A die is tossed twice. 'Getting an even number on a toss' is considered a success. Find the probability distribution of number of successes.

Q 5: A die is tossed twice. 'Getting a number greater than 4' is considered a success. Find the probability distribution of number of successes.

Q 6: Two cards are drawn simultaneously from a well-shuffled pack of 52 cards. Find the probability distribution of number of queens.

Q 7: Two cards are drawn without replacement from a well-shuffled pack of 52

cards. Find the probability distribution of the number of face cards.

Q 8: 20% of the bulbs produced by a factory are defective. Find the probability distribution of the number of defective bulbs in a sample of 4 bulbs chosen at random.

12.4.1 Mean and Variance of Random Variables

Let a random variable X take the values $x_1, x_2, \ldots, x_n$ with probabilities $p_1, p_2, \ldots, p_n$ respectively such that each $p_i \geq 0$, and $\sum_{i=1}^{n} p_i = 1$

The mean of x, denoted by μ, is defined by

$$\mu = \frac{\sum_{i=1}^{n} x_i p_i}{\sum_{i=1}^{n} p_i} = \sum_{i=1}^{n} x_i p_i, \because \sum_{i=1}^{n} p_i = 1$$

The mean of x is also called expected value of X. It is denoted by E(X). Thus,

$$\mu = E(X) = \sum_{i=1}^{n} x_i p_i$$

The variance of X, denoted by σ^2, is given by $\sigma^2 = \Sigma x_i^2 p_i - \mu^2$

The standard deviation of X, denoted by σ, is the positive square root of variance.

Example 3: Find the mean and the standard defiation for the following probability distribution.

X	1	3	5
P(X)	0.3	0.4	0.3

Solution : Mean μ = 1 x 0.3 + 2 x 0.4 + 5 x 0.3

= 0.3 + 1.2 + 1.5

$$SD = \sigma = \sqrt{\Sigma x_i^2 p_i - \mu^2}$$

Now $\Sigma x_i^2 p_i = 1 \times 0.3 + 9 \times 0.4 + 25 \times 0.3$

= 0.3 + 3.6 + 7.5

= 11.4

$$\therefore SD = \sigma = \sqrt{11.4 - 9} = \sqrt{2.4}$$

Example 4 : A random variable X has the following probability distribution

X	0	1	2	3	
P(X)	a	3a	3a	a	

Find the value of a. Also find the mean.

Solution : We know that $\Sigma p_i = 1$

$\therefore$ a + 3a + 3a + a = 1

or 8a = 1

$\therefore\ a = \frac{1}{8}$

2nd part : Mean $\mu = \Sigma x_i p_i$

$$= 0 \times \frac{1}{8} + 1 \times \frac{3}{8} + 2 \times \frac{3}{8} + 3 \times \frac{1}{8}$$

$$= \frac{3}{8} + \frac{6}{8} + \frac{3}{8} = \frac{12}{8} = \frac{3}{2}$$

Example 5 : A random variable X has the following probability distribution.

X	0	1	2	3	4
P(X)	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$

Find the mean of X. Also find (i) P (X > 3), (ii) P(0 < X < 3)

Solution : Mean $\mu = 0 \times \frac{1}{16} + 1 \times \frac{1}{4} + 2 \times \frac{3}{8} + 3 \times \frac{1}{4} + 4 \times \frac{1}{16}$

$$= \frac{1}{4} + \frac{3}{4} + \frac{3}{4} + \frac{1}{4} = 2$$

(i) $P(X > 3) = P(X = 4) = \frac{1}{16}$

(ii) $P(0 < X < 3) = P(X = 1) + P(X = 2)$

$$= \frac{1}{4} + \frac{3}{8} = \frac{2+3}{8} = \frac{5}{8}$$

CHECK YOUR PROGRESS

Q 1: A random variable X has the following probability distribution.

X	0	1	2	3
P(X)	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{30}$

Find the mean and the standard deviation.

Q 2: Two cards are drawn simultaneously from a well-shuffled pack of 52 cards. Find the mean and the standard deviation of the number of aces.

12.5 LET US SUM UP

- Let S be a sample space associated with a given random experiment. A rule X that associates each $\omega \in S$ to a unique real number X(w) is called a random variable.
- A random variable which can take only finite or countably infinite number of values is called a discrete random variable.
- A random variable which can take any value between two given limits is called a continuous random variable.
- If the values of a random variable together with the corresponding probabilities are given, then this description is called a probability distribution of the random variable.
- Let a random variable X take the values $x_1, x_2, \ldots\ldots, x_n$ with probabilities $p_1, p_2, \ldots\ldots, p_n$ respectively. The mean of x, denoted by μ, is defined by $\mu = \sum_{i=1}^{n} x_i p_i$

  The variance of X, denoted by σ^2, is given by $\sigma^2 = \Sigma x_i^2 p_i - \mu^2$

  The standard deviation of X, denoted by σ, is the positive square root of variance.

12.6 FURTHER READING

1) Statistical Methods by N.G. Das, McGraw Hill Education (India) Private Ltd. 2009
2) Probability and Statistics by G.Shanker Rao, University Press (India) private Ltd. 2011
3) Introduction to probability and Statistics by Seymour Lipschutz, John J.Schiller, Tata McGraw Hill Education private Ltd. 2005
4) Probability, Random Process, Queueing theory and Statistics by Dr. S.Rajasekaran, Prof.A.Chandrasekaran & G.Kavitha, Dhanam Publications, Chennai-60042
5) Probability and Statistics by G.Balaji, G.Balaji Publishers, Chennai-33

12.7 ANSWERS TO CHECK YOUR PROGRESS

Check Your Progress 1

Ans to Q No 1: An example of a discrete random variable :

Three coins are tossed simultaneously. Let X be the the random variable such that X(w) = number of heads in w.

In this case the random variable X takes from values, viz, 0, 1, 2, 3. So, it is a discrete random variable.

Ans to Q No 2: An example of a continuous random variable :

Let the maximum life of electric bulbs be 2000 hours. The continuous random variable is $x(w) = \{x : 0 \le x \le 2000\}$

Ans to Q No 3: Let S be the sample space.

$\therefore$ S = {HH, HT, TH, TT}

We have n(S) = 4

Here X = number of heads in tossing 2 coins.

Since the number of heads may be 0, 1, or 2, therefore the possible values of X are 0, 1, 2.

$\therefore$ P(X = 0) = P (getting no head)

= P (TT) = $\frac{1}{4}$

P (x = 1) = P (getting 1 head)

= P (HT or TH) $= \frac{2}{4} = \frac{1}{2}$

P (x = 2) = P (getting 2 heads)

= P (HH) = $\frac{1}{4}$

$\therefore$ the probability distribution is

X	0	1	2
P(X)	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Ans to Q No 4: Let the random variable, i.e., the number of successes be X.

On flipping a die twice, the number of successes can be 0, 1, or 2. Thus, X can take values 0, 1, 2.

P(X = 0) = P (no success in both flips)

= P (no success in 1st flip). P(no success in 2nd flip)

= $\frac{3}{6} \cdot \frac{3}{6} = \frac{1}{4}$

P(X = 1) = P(success in 1st toss and no-success in 2nd toss) or P(no-success in 1st toss and success in 2nd toss)

$= \frac{3}{6} \cdot \frac{3}{6} + \frac{3}{6} \cdot \frac{3}{6} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

P(X = 2) = P (success in both the flips)

=P(success in 1st flip). P(success in 2nd flip)

$= \frac{3}{6} \cdot \frac{3}{6} = \frac{1}{4}$

$\therefore$ the probability distribution is

X	0	1	2
P(X)	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Ans to Q No 5: Let the random variable, i.e., the number of successes be X.

One flipping a die twice, the number of successes can be 0, 1, or 2.

$\therefore$ X can take values 0, 1, 2.

P(X = 0) = P (no success in both flips)

= P(no success in 1st flip). P(no success in 2nd flip)

= P(getting a number ≤ 4). P (getting a number ≤ 4) $= \frac{4}{\sigma} \cdot \frac{4}{\sigma} = \frac{4}{9}$

P(X = 1) = P(success in 1st flip and no-success in 2nd slip) or P (no-success in 1st flip and success in 2nd slip)

= P(getting a number ≤ 4 and getting a number > 4) + P(getting a number > 4, and getting a number ≤ 4)

$= \frac{4}{6} \cdot \frac{2}{6} + \frac{2}{6} \cdot \frac{4}{6} = \frac{4}{9}$

P(X = 2) = P (success in both the flips)

= P(success in 1st flip). P(success in 2nd flip)

$= \frac{2}{6} \cdot \frac{2}{6} = \frac{1}{9}$

$\therefore$ the probability distribution is

X	0	1	2
P(X)	$\frac{4}{9}$	$\frac{4}{9}$	$\frac{1}{9}$

Ans to Q No 6: [There are 4 queens in a pack of 52 cards.]

Let the random variable, i.e., the number of queens be X.

On drawing 2 cards, the number of queens can be 0, 1, or 2.

$\therefore$ X can take values 0, 1, 2.

P(X = 0) = P(getting no queen)

= P(2 non-queens out of 48)

$$= \frac{^{48}C_2}{^{52}C_2} = \frac{188}{221}$$

P(X = 1) = P(getting 1 queen and 1 non-queen)

$$= \frac{^{4}C_1 . {}^{48}C_1}{^{52}C_2} = \frac{32}{221}$$

P(X = 2) = P(getting 2 queens)

$$= \frac{^{4}C_2}{^{52}C_2} = \frac{1}{221}$$

$\therefore$ the probability distribution is

X	0	1	2
P(X)	$\frac{188}{221}$	$\frac{32}{221}$	$\frac{1}{221}$

Ans to Q No 7: [The jack, queen, and the king are the face cards. There are 4 jacks, 4 queens, and 4 kings. Thus, there are altogether 12 face cards.]

Let the random variable, i.e., the number of faces be X.

On drawing 2 cards, the number of faces can be 0, 1, or 2.

$\therefore$ X can take values 0, 1, 2.

P(X = 0) = P (getting no face cards)

= P (2 non-face cards out of 40)

$$= \frac{^{40}C_2}{^{52}C_2} = \frac{10}{17}$$

P(X = 1) = P(getting 1 face card and 1 non-face card)

$$= \frac{^{12}C_1 . {}^{40}C_1}{^{52}C_2} = \frac{80}{221}$$

P(X = 2) = P (getting 2 face cards)

$$= \frac{^{12}C_2}{^{52}C_2} = \frac{11}{221}$$

$\therefore$ the probability distribution is

X	0	1	2
P(X)	$\frac{10}{17}$	$\frac{80}{221}$	$\frac{11}{221}$

Ans to Q No 8: Let the number of bulbs be 100.

$\therefore$ number of defective bulbs is 20, and the number of non-defective bulbs is 80.

Let the random variable, i.e., the number of defective bulbs be X.

On choosing 4 bulbs, the number of defective bulbs can be 0, 1, 2, 3 or 4.

$\therefore$ X can take values 0, 1, 2, 3, 4.

P(X = 0) = P(Choosing no defective bulbs)

$$= \frac{^{80}C_4}{^{100}C_4}$$

P(X = 1) = P(Choosing 1 defective bulb and 3 non-defective bulbs)

$$= \frac{^{20}C_1 \times ^{80}C_1}{^{100}C_4}$$

P(X = 2) = P(Choosing 2 defective bulbs and 2 non-defective bulbs)

$$= \frac{^{20}C_2 \times ^{80}C_2}{^{100}C_4}$$

P(X = 3) = P(Choosing 1 defective bulb and 3 non-defective bulbs)

$$= \frac{^{20}C_3 \times ^{80}C_1}{^{100}C_4}$$

P(X = 4) = P(Choosing 4 defective bulbs)

$$= \frac{^{20}C_4}{^{100}C_4}$$

$\therefore$ the probability distribution is

X	0	1	2	3	4
P(X)	$\frac{^{80}C_4}{^{100}C_4}$	$\frac{^{20}C_1 \times ^{80}C_3}{^{100}C_4}$	$\frac{^{20}C_2 \times ^{80}C_2}{^{100}C_4}$	$\frac{^{20}C_3 \times ^{80}C_2}{^{100}C_4}$	$\frac{^{20}C_4}{^{100}C_4}$

Check Your Progress 2

Ans to Q No 1: The probability distribution is

X	0	1	2	3
P(X)	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{10}$

Mean $\mu = \Sigma x_i p_i$

$$= \mathrm{O} \times \frac{1}{6} + 1 \times \frac{1}{2} + 2 \times \frac{3}{10} + 3 \times \frac{1}{30}$$

$$= \frac{1}{2} + \frac{3}{5} + \frac{1}{10} = \frac{5+6+1}{10}$$

$$= \frac{12}{10} = 1.2$$

$$SD = \sigma = \sqrt{\Sigma x_i^2 p_i - \mu^2}$$

Now $\Sigma x_i^2 p_i$

$$= 0 \times \frac{1}{6} + 1 \times \frac{1}{2} + 4 \times \frac{3}{10} + 9 \times \frac{1}{30}$$

$$= \frac{1}{2} + \frac{6}{5} + \frac{3}{10} = \frac{5+12+3}{10} = \frac{18}{10} = 1.8$$

$$\therefore\ SD = \alpha = \sqrt{1.8 - 1.44} = \sqrt{0.36} = 0.6$$

Ans to Q No 2: [There are 4 aces in a pack of 52 cards]

Let the random variable, i.e., the number of aces be X.

On drawing 2 cards, the number of aces can be 0, 1, or 2.

$\therefore$ X can take values 0, 1, 2

P(X = 0) = P(getting no ace)

= P (2 non-aces out of 48)

$$= \frac{{}^{48}C_4}{{}^{52}C_2} = \frac{188}{221}$$

P(X = 1) = P(getting 1 ace and 1 non-ace)

$$= \frac{{}^{4}C_1 \cdot {}^{48}C_1}{{}^{52}C_2} = \frac{32}{221}$$

P(X = 2) = P(getting 2 aces)

$$= \frac{{}^{4}C_1}{{}^{52}C_2} = \frac{1}{221}$$

$\therefore$ the probability distribution is

X	0	1	2
P(X)	$\frac{188}{221}$	$\frac{32}{221}$	$\frac{1}{221}$

Mean $\mu = \Sigma x_i p_i = 0 \times \frac{188}{221} + 1 \times \frac{32}{221} + 2 \times \frac{1}{221}$

$$= \frac{32}{221} + \frac{2}{221} = \frac{34}{221} = \frac{2}{13}$$

$$SD = \sigma = \sqrt{\Sigma x_i p_i - \mu^2}$$

Now $\Sigma x_i^2 p_i$

$$= 0 \times \frac{188}{221} + 1 \times \frac{32}{221} + 4 \times \frac{1}{221} = \frac{32}{221} + \frac{4}{221}$$

$$= \frac{36}{221}$$

$$\therefore\ SD = \sigma = \sqrt{\frac{36}{221} - \frac{4}{169}} = \sqrt{\frac{400}{2873}}$$

12.8 MODEL QUESTIONS

Q 1: A die is thrown. Let X be the number of points thrown. Write down the probability distribution and find the mean.

Q 2: Find the mean, and the standard deviation of the number of heads in two tosses of a coin.

Q 3: Find the mean, and the standard deviation of the number of heads in three tosses of a coin.

Q 4: A die is tossed twice. 'Getting a number greater than 3' is considered a success. Find the probability distribution of number of successes. Also find the mean and the standard deviation of number of successes.

Q 5: Find the probability distribution of the number of doublets in four throws of a pair of dice.

[Meaning of the word 'doublet' in this context is as follows.

'Either of two dice, each of which, when thrown, has the same number of spots on the face lying uppermost.']

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UNIT 13 : THEORETICAL DISTRIBUTION I

UNIT STRUCTURE

13.1 Learning Objectives

13.2 Introduction

13.3 Bernoulli Trial

13.4 Binomial (and Bernoulli) DIstribution

13.5 Poisson DIstribution

13.6 Let Us Sum Up

13.7 Further Readings

13.8 Answers To Check Your Progress

13.9 Model Questions

13.1 LEARNING OBJECTIVES

After going through this unit, you will able to-

- bernoulli trial
- binomial (and bernoulli) distribution
- possion distribution.

13.2 INTRODUCTION

In this unit, we will study theoretical discrete univariate (of only one variable) distribution. It gives us a law according to which different values of the random variable are distributed with specified probabilities. We will discuss two main distributions, viz., Binomial distribution, and poisson distribution.

13.3 BERNOULLI TRIALS

A Bernoulli trial (named after Jacob Bernoulli 1654-1705) is an experiment satisfying the following conditions.

1. There are only two possible outcomes, viz., success or failure (occurrence of an event is ‘success’. non-occurrence is ‘failure’.)
2. The prabability of a success (or failure) remains same in each trial.
3. Trails must be independent.
4. Number of trails must be finite.

5. The sum of probabilities of success and failure is 1.
6. The probability of succes vis-a-vis failure is not very low.

13.4 BINOMIAL (AND BERNOULLI) DISTRIBUTION

Let there be n Bernoulli trials. Let the random variable X denote the number of successes in these trials. Let the probability of getting a success in a single trial be p, and that of getting a failure be q so that p + q = 1. Then $P(X=r) = {}^nC_r\, p^r\, q^{n-r}$

Proof : Let a success be denoted by S, and a failure be denoted by F.

Number of ways of getting r successes in n trials = ${}^nC_r$

$\therefore$ $P(X = r) = {}^nC_r$. P(SSS . . . . . S and FFF . . . . . F)

(r times, (n - r) times)

$= {}^nC_r$. P (S) P (S) . . . . P(S) . P(F) P(F) . . . . P(F)

(r times, (n - r) times)

$= {}^nC_r\, p^r\, q^{n-r}$ (P(S) = p, P(F) = q)

Note :

1. $P(x = 0) = {}^nC_0\, p^0\, q^{n-0} = q^n$

   $P(x = 1) = {}^nC_1\, p\, q^{n-1}$

   $P(x = 2) = {}^nC_2\, p^2\, q^{n-2}$

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   $P(x = n) = {}^nC_n\, p^n\, q^{n-n} = p^n$

   $\therefore$ the distribution can be expressed as

X	0	1	2	. . . . .	n
P(X)	q^n	${}^nC_1\, pq^{n-1}$	${}^nC_2\, p^2q^{n-2}$	. . . . .	p^n

2. The distribution is called Binomial distribution since the probabilities of 0, 1, 2, ..., n successes, viz. q^n, ${}^nC_1\, pq^{n-1}$, ${}^nC_2\, p^2q^{n-2}$, ..., p^n, are the successive terms of the binomial expression $(q + p)^n$.
3. Sum of the probabilities

   $= q^n + {}^nC_1\, pq^{n-1} + {}^nC_2\, p^2\, q^{n-2} + ... + p^n$

   $= (q + p)^n = 1$
4. X can take only the integral values, viz,. 0, 1, 2, ... , n. Therefore, Binomial distribution

is a discrete distribution.

5. Any random variable which follows Binomial distribution is called a binomial variate.

6. We have

$P(X = r) = {}^{n}C_r\, p^r\, q^{n-r}$

Thus, P(X = r) depends only on n, and p(or q). These constants are called parameters. Thus, Binomial distribution has two parameters.

Remark : The Bernoulli distribution is a special case of the binomial distribution where n = 1.

Example 1 : Find the probability distribution of the number of heads in 6 tosses of a coin.

Solution : If a coin is tossed, S = { H, T}

$\therefore$ P (getting a head) $= \frac{1}{2}$,

P (not getting a head) $= 1 - \frac{1}{2} = \frac{1}{2}$

Let X be the random variable denoting the number of heads.

In 6 trials, there may be 0 or 1 or 2 or 3 or 4 or 5 or 6 heads.

$$P(X=0) = {}^{6}C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^{6-0} = \frac{1}{64}$$

$$P(X=1) = {}^{6}C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^{6-1} = \frac{3}{32}$$

$$P(X=2) = {}^{6}C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{6-2} = \frac{15}{64}$$

$$P(X=3) = {}^{6}C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^{6-3} = \frac{5}{16}$$

$$P(X=4) = {}^{6}C_4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^{6-4} = \frac{15}{64}$$

$$P(X=5) = {}^{6}C_5 \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^{6-5} = \frac{3}{32}$$

$$P(X=6) = {}^{6}C_6 \left(\frac{1}{2}\right)^6 \left(\frac{1}{2}\right)^{6-6} = \frac{1}{64}$$

$\therefore$ the probability distribution is

X	0	1	2	3	4	5	6
P(X)	$\frac{1}{64}$	$\frac{3}{32}$	$\frac{15}{64}$	$\frac{5}{16}$	$\frac{15}{64}$	$\frac{3}{32}$	$\frac{1}{64}$

Note : Sum of the probabilities

$$= \frac{1}{64} + \frac{3}{32} + \frac{15}{64} + \frac{5}{16} + \frac{15}{64} + \frac{3}{32} + \frac{1}{64} = 1$$

Example 2 : A die is thrown 3 times. Find the probability of getting at least one six.

Solution : If a die is thrown, S = { 1, 2, 3, 4, 5, 6}

$\therefore$ P (getting a six) $= \frac{1}{6}$

P (not getting a six) $= 1 - \frac{1}{6} = \frac{5}{6}$

Let X be the random variable denoting the number of sixes.

In 3 trials, there may be 0, or 1, or 2, or 3 sixes.

$\therefore$ the probability of getting at least 1 six

= P(X = 1) + P(X = 2) + P(X = 3)

$$= {}^3C_1\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^{3-1} + {}^3C_2\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^{3-2} + {}^3C_3\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^{3-3}$$

$$= \left(3 \times \frac{1}{6} \times \frac{25}{36}\right) + \left(3 \times \frac{1}{36} \times \frac{5}{6}\right) + \left(1 \times \frac{1}{216} \times 1\right)$$

$$= \frac{25}{72} + \frac{5}{72} + \frac{1}{216} = \frac{75+15+1}{216} = \frac{91}{216}$$

Mean of a Binomial Distribution :

Let a random variable X take the values $x_1, x_2, x_3, \ldots, x_n$ with probabilities $p_1, p_2, \ldots, p_n$ respectively. We know that the mean μ is defined by

$$\mu = \sum_{i=1}^{n} x_i p_i$$

Now, let us consider the binomial distribution.

$P(X = r) = {}^nC_r\, p^r\, q^{n-r}$, r = 0, 1, 2, 3, ..., n.

$$\therefore \mu = \sum_{r=0}^{n} r \,.\, P(X = r)$$

$$= \sum_{r=0}^{n} r \,.\, {}^nC_r\, p^r q^{n-r}$$

$$= 0 + 1.{}^nC_1 pq^{n-1} + 2{}^nC_2 p^2 q^{n-2} + 3{}^nC_3 p^3 q^{n-3} + \ldots\ldots + {}^n n.\,{}^nC_n p^n q^{n-n}$$

$$= 1.np.q^{n-1} + n(n-1)p^2q^{n-2} + \frac{n(n-1)(n-2)}{2!}p^3q^{n-3} + ... + np^n$$

$$= np\left[1.q^{n-1} + (n-1)pq^{n-2} + \frac{n(n-1)(n-2)}{2!}p^2q^{n-3} + ... + p^{n-1}\right]$$

$$= np\left[{}^{n-1}C_0\, q^{n-1} + {}^{n-1}C_1\, q^{(n-1)-1}\ p + {}^{n-1}C_2\, q^{(n-1)-2}p2 + ... + {}^{n-1}C_{n-1}\, p^{n-1}\right]$$

$$= np\ (q + p)^{n-1}$$

$$= np, \quad \therefore q + p = 1$$

Variance of a Binomial Distribution

We know that the variance σ^2 is defined by

$$\sigma^2 = \sum_{i=1}^{n} r_i^2 p^i - \mu^2$$

So for the above binomial distribution,

Variance = $\sigma^2 = \sum_{r=0}^{n} r^2 p(X = r) - \mu^2$

$$= \sum_{r=0}^{n} r^2\ {}^nC_r p^r q^{n-r} - n^2p^2$$

$$= \sum_{r=0}^{n} \left[r + r(r-1)\right]{}^nC_r p^r q^{n-r} - n^2p^2$$

$$= \sum_{r=0}^{n} r\ .\ {}^nC_r p^r q^{n-r} + \sum_{r=0}^{n} r(r-1){}^nC_r p^r q^{n-r} - n^2p^2$$

$$= np + \sum_{r=2}^{n} r(r-1)\frac{n(n-1)}{r(r-1)}\ {}^{n-2}C_{r-2}p^2p^{r-2}q^{n-r} - n^2p^2$$

$$= np + n(n-1)p^2 \sum_{r=2}^{n}\ {}^{n-2}C_{r-2}\ p^{r-2}q^{n-r} - n^2p^2$$

$$= np + n(n-1)p^2 - n^2p^2$$

$$= np + n^2p^2 - np^2 - n^2p^2$$

$$= np - np^2$$

$$= np(1-p)$$

$$= npr, \quad \because\ p + q = 1$$

Thus, variance $= \sigma^2 = npq$

$\therefore\ SD = \sigma = \sqrt{npq}$

Recurrence relation for a binomial distribution :

We know that

$$P(X=r)={}^{n}C_{r}\,p^{r}q^{n-r}$$

$$\therefore\ P(X=r+1)={}^{n}C_{r+1}\,p^{r+1}q^{n-r-1}$$

$$\therefore \frac{P(X=r+1)}{P(X=r)}=\frac{{}^{n}C_{r+1}\,p^{r+1}q^{n-r-1}}{{}^{n}C_{r}\,p^{r}q^{n-r}}$$

$$=\frac{n!}{(r+1)!\,(n-r-1)!}\,\frac{r!(n-r)!}{n!}\cdot\frac{p}{q}$$

$$=\frac{n-r}{r+1}\cdot\frac{p}{q}$$

$$\therefore\ P(X=r+1)=\frac{(n-r}{(r+1)}\cdot\frac{p}{q}\cdot P(X=r)$$

Example 3 : The mean of a binomial distribution is 5, and the variance is 2.5. Find the distribution.

Solution : Let X be a binomial distribution with parameters n, and p.

We know that the mean is np, and the variance is npq, q = 1 – p.

$\therefore$ np = 5, and npq = 2.5

$\therefore$ 5q = 2.5 $\quad\Rightarrow q=\frac{2.5}{5}=\frac{1}{2}$

$\therefore$ p = 1 – q = 1 – $\frac{1}{2}=\frac{1}{2}$

$\therefore$ n = 10

$\therefore$ the distribution is

$${}^{10}C_{r}\left(\frac{1}{2}\right)^{r}\cdot\left(\frac{1}{2}\right)^{10-r},\qquad 0\ \le\ r\ \le 10\,.$$

Example 4 : The mean of a binomial distribution is 4 and the variance is $\frac{4}{3}$. Find $P(X\ge 1)$.

Solution : We have np = 4, and npq = $\frac{4}{3}$

$$\therefore\ 4q=\frac{4}{3}\Rightarrow q=\frac{1}{3}$$

$$\therefore\ p=1-q=1-\frac{1}{3}=\frac{2}{3}$$

$$\therefore\ \frac{2n}{3}=4\ \ \Rightarrow n=6$$

Now $P(X\ge 1)=1-P(X=0)$

$$= 1 - {}^{6}C_0 p^0 q^{6-0} = 1 - q^6 = 1 - \left(\frac{1}{3}\right)^6$$

$$= 1 - \frac{1}{729} = \frac{728}{729}$$

Property : For a binomial distribution, mean is greater than variance.

Proof : Let X be a binomial distribution with parameters n, and p.

Now, mean = np

$> npq, \ \because \ 0 < q < 1$

= variance.

Example 5 : Can there be a binomial distribution with mean 5, and variance 7.5 ?

Solution : We know that in a binomial distribution, mean > variance.

Now, 5 < 7.5

$\therefore$ there cannot be binomial distribution with mean 5, and variance 7.5.

CHECK YOUR PROGRESS

Q 1: Find the probability distribution of the number of tails in 4 tosses of a coin.

Q 2: Find the probability distribution of the number of doublets in three throws of a pair of dice.

Q 3: A coin is flipped 8 times. Find the probability of getting at least 2 heads.

Q 4: Find the binomial distribution whose mean is 3, and the variance is 2.

Q 5: Find P for a binomial random variable X, if n = 6, and 9p(X = 4) = P(X = 2). Also, find the mean and the variance of the distribution.

Q 6: Let X be a binomial variate such that the mean is double the standard deviation. If P(X = 0) = $\frac{1}{16}$, find the probability that X is at most 1.

Q 7: A coin is flipped 8 times. Find the probability of getting at least 7 heads.

Q 8: For a binomial distribution, the mean is 6 and the standard deviation is $\sqrt{2}$. Write down the distribution.

Q 9: The mean and variance of a binomial variate X with parameters n and p are 16 and 8. Find (i) P (X = 0), (ii) P (X = 1), (iii) $P(X \geq 2)$

Q 10: Are the following statements true ? Justify your answer.

(i) The mean of a binomial distribution is 6, and its standard deviation is 3.

(ii) The mean of a binomial distribution is 5, and its variance is 2.

13.5 POISSON DISTRIBUTION

(After the name of French mathematician simeon Devis Poisson, 1781-1840)

Poisson distribution is a limiting case of the binomial distribution under the following three conditions.

1. The number of trials n is indefinitely large, i.e., $n \to \infty$
2. The constant probability p of success for each trial is indefinitely small, i.e., $p \to 0$
3. np = m (say) is finite.

We have

$$P(r) = P(X = r) = {}^{n}C_{r}\, p^{r} q^{n-r}$$

We know that $P(0) = q^n$

$$\therefore \lim_{n\to\infty} p(0) = \lim_{n\to\infty} q^{n}$$

$$= \lim_{n\to\infty} (1-p)^{n}$$

$$= \lim_{n\to\infty} \left(1 - \frac{m}{n}\right)^{n}, \qquad \text{from np = m}$$

$$= e^{-m}$$

Again, we know that $\frac{P(r+1)}{P(r)} = \frac{n-r}{r+1} \cdot \frac{p}{q}$

$$\therefore \lim_{n\to\infty} \frac{P(r+1)}{P(r)} = \lim_{n\to\infty} \frac{n-r}{r+1} \cdot \frac{p}{q}$$

$$= \lim_{n\to\infty} \frac{n-r}{r+1} \cdot \frac{\frac{m}{n}}{1-\frac{m}{n}}$$

$$= \lim_{n\to\infty} \frac{1-\frac{r}{n}}{1-\frac{m}{n}} \cdot \frac{m}{r+1} = \frac{m}{r+1}$$

Now, $\lim_{n\to\infty} P(r) = \lim_{n\to\infty} \frac{P(r)}{P(r-1)} \cdot \lim_{n\to\infty} \frac{P(r-1)}{P(r-2)} \cdot \ldots \cdot \lim_{n\to\infty} \frac{P(1)}{P(0)} \cdot \lim_{n\to\infty} P(0)$

$$= \frac{m}{r} \cdot \frac{m}{r-1} \cdot \ldots \cdot \frac{m}{1}\, e^{-m}$$

$$= \frac{m^{r}\, e^{-m}}{r!}$$

Definition : A random variable X is said to follow a poisson distribution if it takes non-negative values with the probability given by

$$P(X = x)\begin{cases} = \dfrac{e^{-m}m^{x}}{x!}, x = 0,1,2.....,m > o \\ = 0 \text{ otherwise} \end{cases}$$

Note :

1. X can take only the integral values, viz., 0, 1, 2, ... Therefore, poisson distribution is a discrete distribution.
2. Poisson distribution has only one parameter, viz, m.

Some examples where poisson distribution may be applied :

1. Number of defective blades in a packet of 100 blades.
2. Number of printing mistakes at each page of a book.
3. Number of deaths from heart attack or snake bite.
4. Number of air accidents in India in one year.
5. Number of cars passing through a street-crossing during the busy hours of a day.

Mean of a poisson distribution :

Mean = μ

$$= \sum_{x=0}^{\infty} x P(X = x)$$

$$= \sum_{x=0}^{\infty} x \frac{e^{-m}\ m^{x}}{x!}$$

$$= \sum_{x=0}^{\infty} x \frac{e^{-m}\ m m^{x-1}}{x(x-1)!}$$

$$= m e^{-m} \sum_{x=1}^{\infty} \frac{m^{x-1}}{(x-1)!}$$

$$= m e^{-m}\left[1 + m + \frac{m^2}{2!} + \frac{m^3}{3!} + \ldots . \infty\right]$$

$$= m e^{-m} e^{m}$$

$$= m$$

Variance of a poisson distribution :

Variance $= \sigma^2$

$$= \sum_{x=0}^{\infty} x^2 P(X = x) - \mu^2$$

$$= \sum_{x=0}^{\infty} x^2 \frac{e^{-m}\, m^x}{x!} - m^2$$

$$= \sum_{x=0}^{\infty} \left(x(x-1)\frac{e^{-m}\, m^x}{x!} + \frac{x e^{-m}\, m^x}{x!} \right) - m^2$$

$$= \sum_{x=0}^{\infty} x(x-1)\frac{e^{-m}\, m^x}{x(x-1)(x-2)!} + \sum_{x=0}^{\infty} \frac{x e^{-m}\, m^x}{x!} - m^2$$

$$= \sum_{x=0}^{\infty} \frac{e^{-m}\, m^2 m^{x-2}}{(x-2)!} + m - m^2$$

$$= e^{-m} m^2 \sum_{x=2}^{\infty} \frac{m^{x-2}}{(m-2)!} + m - m^2$$

$$= e^{-m} m^2 \left[1 + m + \frac{m^2}{2!} + \frac{m^3}{3!} + \ldots\ldots\infty \right] + m - m^2$$

$$= e^{-m} m^2 e^m + m - m^2$$

$$= m^2 + m - m^2 = m$$

Note : Mean of a poisson distribution

= Variance

= m.

Recurrence relation for a poisson distribution

We have $P(X = x) = \frac{e^{-m} m^x}{x!}, \; x = 0,1,2, \ldots.\infty$

$$\therefore P(X = x+1) = \frac{e^{-m} m^{x+1}}{(x+1)!}$$

$$\therefore \frac{P(X = x+1)}{P(X = x)} = \frac{e^{-m} m^{x+1}}{(x+1)!} \frac{x!}{e^{-m} m^x}$$

$$= \frac{m}{x+1}$$

$$\therefore P(X = x+1) = \frac{m}{x+1} P(X = x)$$

Example 6 : A random variable X follows poisson distribution such that P(X = 0) = P(X = 1) = a. Find the mean of the distribution. Also find the value of a.

Solution : We have $P(X = x) = \frac{e^{-m} m^x}{x!}$

Now, P(X = 0) = P (X = 1), (given)

$$\therefore \frac{e^{-m}m^0}{0!} = \frac{e^{-m}m^1}{1!}$$

$$\therefore m = 1$$

i.e., mean = 1

Again P (X = 0) = a

or, $e^{-m} = a$

or, $e^{-1} = a$

i.e. $a = \frac{1}{e}$

Example 7 : A random variable X follows poisson distribution such that P(X = 1) = 2P(X = 2). Find P(X = 0), $P(X \geq 0)$.

Solution : P(X = 1) = 2P(X = 2), given

$$\therefore \frac{e^{-m}m^1}{1!} = 2\,\frac{e^{-m}m^2}{2!}$$

or, m = 1

$$\therefore P(X = 0) = \frac{e^{-1}1^0}{0!} = e^{-1} = \frac{1}{e}$$

$$P(X \geq 0)$$

$$= P(X = 0) + P(X = 1) + P(X = 2) + \ldots.$$

$$= e^{-1} + \frac{e^{-1}}{1!} + \frac{e^{-1}}{2!} + \frac{e^{-1}}{3!} + \frac{e^{-1}}{4!} + \ldots..$$

$$= e^{-1}\left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{3!} + \frac{1}{4!} + \ldots..\right)$$

$$= e^{-1}\ e$$

$$= e^0 = 1$$

Example 8 : 6 coins are tossed 6400 times. Use poisson distribution to find the approximate probability of getting 6 heads r times.

Solution : Here n = 6400

p = the probability of getting 6 heads in one throw of 6 coins

$$\therefore\ m = np = \frac{6400}{2^6} = \frac{6400}{64} = 100$$

$$\therefore P(X = r) = \frac{e^{-m}m^r}{r!}$$

$$= \frac{e^{-100}\,100^{r}}{r!}$$

CHECK YOUR PROGRESS

Q 1: A random variable X follows Poisson distribution such that P(X = 1) = P(X = 2). Find P(X = 3)

Q 2: A random variable X follows poisson distribution such that P(X = 2) = 3 P(X = 3). Find the probability that X is at least one.

Q 3: Examine if the following statement is right.
The mean of a poisson distribution is 3, and its standard deviation is 4.

Q 4: A random variable X follows poisson distribution such that P(X = 2) = 9P(X = 4). Find the mean of X.

Q 5: A random variable X follows poisson distribution such that P(X = 2) = P(X = 3). Find the variance of X.

13.6 LET US SUM UP

- A Bernoulli trial is an experiment satisfying the following conditions.
  1. There are only two possible outcomes, viz,. success or failure (occurrence of an event is 'success', non-occurrence is 'failure').
  2. The probability of a success (or failure) remains same in each trial.
  3. Trials must be independent.
  4. Number of trials must be finite.
  5. The sum of probabilities of success and failure is 1.
  6. The probability of success vis-a-vis failure is not very low.
- Let there be n Bernoulli trials. Let the random variable X denote the number of successes in these trials. Let the probability of getting a success in a single trial be p, and that of getting a failure be q so that p + q = 1. Then $P(X = r) = {}^{n}C_{r}\, p^{r}\, q^{n-r}$
  This is known as Binomial distribution.
- The Bernoulli distribution is a special case of the binomial distribution where n = 1.
- Mean of a binomial distribution is np, and the variance is npq.
- Poisson distribution is a limiting case of the binomial distribution under the following there conditions.
  1. The number of trials n is indefinitely large, i.e., $n \to \infty$.
  2. The constant probability p of success for each trial is indefinitely small, i.e.,

$p \to 0$.

3. np = m, (say), is finite.

- A random variable X is said to follow a poisson distribution if it takes non-negative values with the probability given by

$$P(X = x)\begin{cases} = \dfrac{e^{-m}m^{x}}{x!}, x = 0,1,2.....,m > o \\ = 0 \text{ otherwise} \end{cases}$$

- Mean of a poisson distribution
  = variance
  = m

13.7 FURTHER READINGS

1) Statistical Methods by N.G. Das, McGraw Hill Education (India) Private Ltd. 2009
2) Probability and Statistics by G.Shanker Rao, University Press (India) private Ltd. 2011
3) Introduction to probability and Statistics by Seymour Lipschutz, John J.Schiller, Tata McGraw Hill Education private Ltd. 2005
4) Probability, Random Process, Queueing theory and Statistics by Dr. S.Rajasekaran, Prof.A.Chandrasekaran & G.Kavitha, Dhanam Publications, Chennai-60042
5) Probability and Statistics by G.Balaji, G.Balaji Publishers, Chennai-33

13.8 ANSWERS TO CHECK YOUR PROGRESS

Check Your Progress 1

Ans to Q No 1: If a coin is tossed, S = {H, T}

$\therefore$ P (getting a head) $\frac{1}{2}$

P (not getting a head) $= 1 - \frac{1}{2} = \frac{1}{2}$

Let X be the random variable denoting the number of tails.

In 4 trials there may be 0 or 1 or 2 or 3 or 4 tails.

$$P(X = 0) = {}^{4}C_{0}\left(\frac{1}{2}\right)^{0}\left(\frac{1}{2}\right)^{4-0} = \frac{1}{16}$$

$$P(X=1)={}^4C_1\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^{4-1}=\frac{1}{4}$$

$$P(X=2)={}^4C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{4-2}=\frac{3}{8}$$

$$P(X=3)={}^4C_3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^{4-3}=\frac{1}{4}$$

$$P(X=4)={}^4C_4\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^{4-4}=\frac{1}{16}$$

$\therefore$ the probability distribution is

X	0	1	2	3	4
P(X)	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$

Ans to Q No 2: [One of the meanings of the word doublet is as follows.

'Either of two dice, each of which, when thrown, has the same number of spots on the face lying uppermost.']

If a pair of dice is thrown, there will be 36 outcomes.

S = { (1, 2), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), ..., (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)}

Thus n(s) = 36

Possible doublets are (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6).

i.e., the number of doublets is 6.

$\therefore$ P(getting a doublet) $=\frac{6}{36}=\frac{1}{6}$

$\therefore$ P(not getting a doublet) $=1-\frac{1}{6}=\frac{5}{6}$

In 3 trials, there may be 0 or 1 or 2 or 3 doublets.

$$P(X=0)={}^3C_0\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^3=\frac{125}{216}$$

$$P(X=1)={}^3C_1\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^2=\frac{25}{72}$$

$$P(X=2)={}^3C_2\left(\frac{1}{6}\right)^2\frac{5}{6}=\frac{5}{72}$$

$$P(X=3)={}^3C_3\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^0=\frac{1}{216}$$

$\therefore$ the probability distribution is

X	0	1	2	3
P(X)	$\frac{125}{216}$	$\frac{25}{72}$	$\frac{5}{72}$	$\frac{1}{216}$

Ans to Q No 3: If a coin is flipped, S = {H, T}

$\therefore$ P(getting a head) $=\frac{1}{2}$

P (not getting a head) $=1-\frac{1}{2}=\frac{1}{2}$

Let X be the random variable denoting the number of heads.

$\therefore\ P(X=r)={}^nC_r p^r q^{n-r}$

$$={}^8C_r\left(\frac{1}{2}\right)^r\left(\frac{1}{2}\right)^{8-r}={}^8C_r\left(\frac{1}{2}\right)^8$$

$\therefore$ P (getting at least 2 heads)

$$=P(X\geq 2)$$

$$=1-[P(X=0)+P(X=1)]$$

$$=1-\left[{}^8C_0\left(\frac{1}{2}\right)^8+{}^8C_1\left(\frac{1}{2}\right)^8\right]$$

$$=1-\left[\frac{1}{256}+\frac{8}{256}\right]$$

$$=1-\frac{9}{256}$$

$$=\frac{247}{256}$$

Ans to Q No 4: Let X be the binomial variate.

We have mean = np = 3 (1)

variance = npq = 2 (2)

From (1) and (2), 3q = 2

$$\therefore\ q=\frac{2}{3}$$

$$\therefore\ p = 1 - q = 1 - \frac{2}{3} = \frac{1}{3}$$

Again, from (1) $\frac{n}{3} = 3$

$$\therefore\ n = 9$$

$\therefore$ the distribution is

$P(X = r) = {}^{9}C_r \left(\frac{1}{3}\right)^r \left(\frac{2}{3}\right)^{9-r}$, r = 0, 1, 2, 3, ... , 9.

Ans to Q No 5: We have

9p (x = 4) = p(x = 2)

or, 9 ${}^6C_4\ p^4\ q^{6-4}$ = ${}^6C_2\ p^2\ q^{6-2}$, (n = 6 given)

or, $9p^2 = q^2$, (${}^6C_4 = {}^6C_2$)

or, 3p = q

or, 3p = 1 – p

or, 4p = 1

$\therefore\ p = \frac{1}{4}$ $\therefore\ q = 1 - \frac{1}{4} = \frac{3}{4}$

$\therefore$ mean $= np = 6 \times \frac{1}{4} = \frac{3}{2}$

variance $= npq = 6 \times \frac{1}{4} \times \frac{3}{2} = \frac{9}{4}$

Ans to Q No 6: We have

mean = 2 (standard deviation), (given)

or, $np = 2\sqrt{npq}$

or, $n^2 p^2 = 4npq$

or, np = 4q (1)

Again, $P(X = 0) = \frac{1}{16}$ (given)

$$\therefore q^n = \frac{1}{16}$$

or $q^n = \left(\frac{1}{2}\right)^4$ or $q^n = \left(\frac{1}{4}\right)^2$

$\therefore$ either $q = \frac{1}{2},\ n = 4$ or $q = \frac{1}{4},\ n = 2$

If $q = \frac{1}{2},\ n = 4$ then from (1)

$4p = 4 \times \frac{1}{2}$ or $p = \frac{1}{2}$. $\therefore q = 1 - p = \frac{1}{2}$

If $q = \frac{1}{4},\ n = 2$ then from (1)

$2p = 4 \times \frac{1}{4}$ or 2p = 1 or $p = \frac{1}{2}$

$\therefore\ q = 1 - \frac{1}{2} = \frac{1}{2}$ which is not true.

$\therefore\ q = \frac{1}{4},\ n = 2$ is not admissible

$\therefore n = 4,\ q = \frac{1}{4},\ p = \frac{1}{2}$

P (X is at most 1)

= P(X = 0) + P(X = 1)

$$= {}^4C_0\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^{4-0} + {}^4C_1\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^{4-1}$$

$$= \left(\frac{1}{2}\right)^4 + 4 \times \left(\frac{1}{2}\right)^4$$

$$= \frac{1}{16} + \frac{4}{16}$$

$$= \frac{5}{16}$$

Ans to Q No 7: If a coin is flipped, S = {H, T}

$\therefore$ P(getting a head) $= \frac{1}{2}$

P (not getting a head) $= 1 - \frac{1}{2} = \frac{1}{2}$

$\therefore\ p = \frac{1}{2},\ q = \frac{1}{2}$

Let X be the random variable denoting the number of heads.

In 8 trials there may be 0 or 1 or 2 or 3 or 4 or 5 or 6 or 7 or 8 heads.

$\therefore$ P (getting at least 7 heads)

$= P(X = 7) + P(X = 8)$

$$= {}^8C_7\left(\frac{1}{2}\right)^7\left(\frac{1}{2}\right)^{8-7} + {}^8C_8\left(\frac{1}{2}\right)^8\left(\frac{1}{2}\right)^{8-8}$$

$$= 8\times\frac{1}{2^8}+\frac{1}{2^8}$$

$$= \frac{8}{256}+\frac{1}{256}$$

$$= \frac{9}{256}$$

Ans to Q No 8: We have np = 6 (1)

$$\sqrt{npq} = \sqrt{2}$$

$$\therefore\ npq = 2$$

or 6q = 2

$$\therefore\ q = \frac{1}{3}$$

$$\therefore\ p = 1-q = 1-\frac{1}{3} = \frac{2}{3}$$

$\therefore$ from (1), $\frac{2n}{3} = 6$ or n = 9

$$\therefore\ P(X=r) = {}^9C_r\left(\frac{2}{3}\right)^r\left(\frac{1}{q}\right)^{q-r},$$ r = 0, 1, 2, 3, 4, 5, 6,7, 8, 9

Ans to Q No 9: We have np = 16, (1)

npq = 8 (2)

$$\therefore 16q = 8 \quad \text{or} \quad q = \frac{1}{2}$$

$$\therefore\ p = 1-q = 1-\frac{1}{2} = \frac{1}{2}$$

$\therefore$ from (1), $\frac{n}{2} = 16$ or n = 32

(i) $P(X=0) = {}^{32}C_0\left(\frac{1}{2}\right)^6\left(\frac{1}{2}\right)^{32-0}$

$$= \frac{1}{2^{32}}$$

(ii) $P(X=1) = {}^{32}C_1\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^{32-1}$

$$= 32 \cdot \frac{1}{2^{32}} = \frac{2^5}{2^{32}} = \frac{1}{2^{27}}$$

(iii) $P(X \geq 2)$

$= 1 - [P(X = 0) + P(X = 1)]$

$= 1 - \left(\frac{1}{2^{32}} + \frac{1}{2^{27}} \right)$

Ans to Q No 10: (i) Mean = 6, SD = 3

$\therefore$ variance = 9

We know that for a binomial distribution mean > variance

In this case mean = 6, variance = 9

$\therefore$ the statement is not true.

(ii) Mean = 5

Variance = 2

We know that for a binomial distribution, mean > variance.

In this case mean = 5, variance = 2.

$\therefore$ the statement is true.

Check Your Progress 2

Ans to Q No 1: We know that, for a poisson distribution,

$$P(X = x) = \frac{e^{-m} m^x}{x!}$$

Now P(X=1) = P(X=2), (given)

$$\frac{e^{-m} m^1}{1!} = \frac{e^{-m} m^2}{2!}$$

$\therefore$ m = 2

$$\therefore P(X = 3) = \frac{e^{-2} 2^3}{3!} = \frac{4e^{-2}}{3} = \frac{4}{3e^2}$$

Ans to Q No 2: We have

$$P(X = x) = \frac{e^{-m} m^x}{x!}$$

P (x = 2) = 3 P (x = 3)

or, $\frac{e^{-m} m^2}{2!} = 3 \cdot \frac{e^{-m} m^3}{3!}$

or, m = 1

$\therefore\ P(X \geq 1)$

$= 1 - P(X = 0)$

$= 1 - \frac{e^{-1} 1^0}{0!}$

$= 1 - e^{-1} = 1 - \frac{1}{e} = \frac{e-1}{e} = 0.632$

Ans to Q No 3: We know that the mean of a poisson distribution = variance.

$\therefore$ the given statement is not true.

Ans to Q No 4: We have $P(X = x) = \frac{e^{-m} m^x}{x!}$

Now P(x = 2) = 9 P(x = 4)

or, $\frac{e^{-m} m^2}{2!} = 9 \cdot \frac{e^{-m} m^4}{4!}$

or, $\frac{m^2}{2} = \frac{3m^4}{8}$

or, $\frac{3m^2}{4} = 1$

or, $m^2 = \frac{4}{3}$

$\therefore\ m = \frac{2}{\sqrt{3}}$

$\therefore$ mean = $m = \frac{2}{\sqrt{3}}$

Ans to Q No 5: We have

P(x = 2) = P(x = 3)

or, $\frac{e^{-m} m^2}{2!} = \frac{e^{-m} m^3}{3!}$

or, m = 3

$\therefore$ variance = m = 3

?13.9 MODEL QUESTIONS

Q 1: Name the discrete probability distribution for which

(i) mean is greater than the variance,

(ii) mean is equal to the variance.

Q 2: Derive poisson distribution as a limiting form of a binomial distribution.

Q 3: 8 coins are thrown simultaneously. Find the probability of obtaining at least 6 heads.

Q 4: Let X be a binomial variate such that mean = 1, variance = 0.75.
Find (i) P(x = 0), (ii) P (x > 0).

Q 5: Write down the conditions under which a binomial distribution tends to the poisson distribution.

Q 6: Let X be a poisson variate such that $P(X=2)=\frac{2}{3}P(X=1)$.
Find P(x = 0)

Q 7: If the sum of the mean and variance of a binomial distribution for 5 trials is 1.8, find the distribution.

Q 8: The mean and variance of a binomial distribution are 4 and $\frac{4}{3}$ respectively. FInd $P(X \geq 1)$.

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UNIT 14 : THEORETICAL DISTRIBUTION II

UNIT STRUCTURE

14.1 Learning Objectives

14.2 Introduction

14.3 ContinuousProbability Distribution

14.4 Normal Distribution

14.4.1 Properties of Normal Distribution

14.4.2 Standard Normal Variate

14.5 Let Us Sum Up

14.6 Further Readings

14.7 Answers To Check Your Progress

14.8 Model Questions

14.1 LEARNING OBJECTIVES

After going through this unit, you will able to learn

- continuous probability distribution and probability density function
- normal distribution
- standard normal variate
- properties of normal distribution.

14.2 INTRODUCTION

Normal distribution is a continuous distribution. It is a limiting case of the binomial distribution. It has immense application. It is the most popular and commonly used distribution. It was discovered by Abraham de Moivre (1667–1754) in 1733.

14.3 CONTINUOUS PROBABILITY DISTRIBUTION

If a random variable is a continuous variable, its probability distribution is called a continuous probability distribution.

A continuous probability distribution cannot be expresed in tabular form. An equation or formula is used to describe a continuous probability distribution. The equation used to describe a continuous probability distribution is called a probability. density function (p.d.f.)

14.4 NORMAL DISTRIBUTION

Normal distribution is derived from the binomial distribution under two conditions, viz.

(i) n, the number of trials, is sufficiently large, i.e. $n \to \infty$

and (ii) neither p nor q is very small.

A random variable X is said to follow normal distribution, if and only if, its probability density function is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Where x is the real value of X, i.e. $-\infty < X < \infty$

The variable X is said to be distributed normally with mean μ and variance σ^2 symbolically we write $X \sim N(\mu, \sigma^2)$. It is read as 'X is distributed as $N(\mu, \sigma^2)$'.

Note :

1. The probability density function given by (1) has two parameters, viz., μ and σ. μ is any real number, i.e. $-\infty < \mu < \infty$ whereas σ is a positive real number, i.e., $\sigma > 0$.
2. We know that probability can never be negative.
   $\therefore f(x) \geq 0, \forall x$
3. If $\mu = 0,\ \sigma = 1$ the p.d.f is given by

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

In this case $X \sim N(0, 1)$. In this case X is called the standard normal variate, and the p.d.f given by (2) is called standard normal distribution.

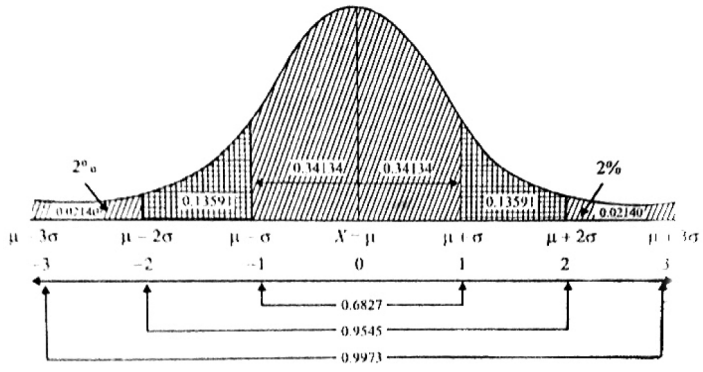
14.4.1 Properties of Normal Distribution

Normal distribution has many properties, some of which are given below.

1. The equation of the normal probability curve is

$$Y = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

It is of the shape given below. It is bell-shaped.



[Source : **Biostatistics** by Dr. P N Arora and Dr. P K Malhan (Himalya Publishing House, 2010)]

2. The curve is symmetrical about the line $x = \infty$, and x lies between $-\infty$ and $+\infty$

3. At $x = \mu$, the ordinate is maximum, and it is $Y = \frac{1}{\sigma\sqrt{2\pi}}$

4. The total area under the normal curve within the limits $-\infty$ to ∞ is 1, i.e.,

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = 1$$

5. The percentage distribution of area under the normal curve given below, and is shown in the figure given above.
   (i) About 68% of the area falls between $\mu - \sigma$ and $\mu + \sigma$
   (ii) About 95.5% of the area falls between $\mu - 2\alpha$ and $\mu + 2\sigma$.
   (iii) About 99.7% of the area falls between $\mu - 3\sigma$ and $\mu + 3\sigma$.
6. The curve has a single peak, i.e., it is unimodal.
7. The distribution is symmetrical. Mean = median = mode = μ.
8. The curve is asymptotic, i.e., it meets the x-axis at both the ends at infinity. Theoretically the curve never touches the x-axis.
9. The points of inflexion are $x = \mu - \sigma$ and $x = \mu + \sigma$. [Inflexion points are where the function changes concavity.]
10. Coefficient of skewness = 0, and kurtosis = 3.
11. Mean deviation about mean $\simeq \frac{4}{5}\sigma$
12. $Q.D. \simeq \frac{2}{3}\sigma$

14.4.2 Standard Normal Variate

Let a random variable X be distributed as $N(\mu, \sigma^2)$. The location and shape of the normal curve depends on μ, and σ, Where $-\infty < \mu < \infty$, and $\sigma > 0$. Therefore, no master table for the area can be prepared. This hindrance can be removed if we introduce a new variable Z defined by.

$$Z = \frac{X - \mu}{\sigma}$$

In this case Z is called the standard normal variate.

Now we prove that the mean of Z is 0, and the standard deviation is 1. Before that we state some related properties.

We know that mean of a random variable X is given by

mean $= E(X) = \Sigma x_i p_i$

We prove that

(i) E(a) = a; a is a constant

(ii) E (aX + b) = aE(x) + b; a, b are constants

(iii) $E\left(X - \overline{X}\right) = 0$

Proof : (i) $E(a) = \Sigma a p_i$

$= a\ \Sigma p_i$

= a, $\because\ \Sigma p_i = 1$

(ii) $\Sigma(aX + b) = \Sigma(ax_i + b)p_i$

$= a\Sigma x_i p_i + b\Sigma p_i$

$= a\,E(X) + b,\ \because\ \Sigma p_i = 1$

(iii) $E\left(X - \overline{X}\right)$

$= E(X) - E(\overline{X})$

$= \overline{X} - \overline{X},\quad \because\ \overline{X}$ is constant

= 0

Also, we note that

(iv) Var (aX) = a^2 Var (X), where a is a constant

(v) Var (aX $\pm b$) = a^2 Var (X), where a, b are constants.

Now we prove that the mean of Z is 0, and the standard deviation is 1.

Mean = E (Z)

$$= E\left(\frac{X-\mu}{\sigma}\right)$$

$$= \frac{1}{\sigma} E(X-\mu), \qquad \text{property (ii)}$$

$$= \frac{1}{\sigma} \left[E(X)-\mu\right], \qquad \text{property (ii)}$$

$$= \frac{1}{\sigma}[\mu-\mu]$$

= 0

Variance = Var (Z)

$$= Var\left(\frac{X-\mu}{\sigma}\right)$$

$$= \frac{1}{\sigma^2} \text{var}(X-\mu), \qquad \text{property (v)}$$

$$= \frac{1}{\sigma^2} \text{var } X, \qquad \text{property (iv)}$$

$$= \frac{1}{\sigma^2} . \sigma^2$$

= 1

Thus, $Z = \frac{X-\mu}{\sigma}$ is distributed as N (0, 1), i.e., Z ~ N(0, 1).

All the properties for a normal variable hold good for standard normal variable $(\mu = 0,\ \sigma = 1)$

Let Z be a standard normal variate N(0, 1). The area under the standard normal curve is given by a table which is addended at the end of this chapter.

We know that the total area under the normal curve within the limits $-\infty$ to ∞ is 1. The standard normal curve is symmetrical about Z = 0. Therefore, the area to the left of z = 0 is 0.5, and the area to the right of z = 0 is 0.5.

Suppose, we are to find the area between z = 0, and z = 1.89, i.e, we are to find out $P(0 \le Z \le 1.89)$.

First we find 1.8 at the left of the table and single out that row.

Next we find 0.09 at the top of the table and single out that column.

Then we obtain the entry where the row for 1.8 and the column for 0.09 meet. the entry is 0.47062.

Thus $P(0 \le Z \le 1.89)$

= (the are a from z = 0 to z = 1.89)

= 0.47062.

Example 1 : Find the probability of the standard normal variate z ehen z is greater than 1.09.

Solution : We know that the total area under the normal curve within the limits $-\infty$ to ∞ is 1. Therefore, the area to the right of z = 0 is 0.5, as the curve is symmetrical about z = 0.

$\therefore$ P(Z > 1.09) = 0.5 – 0.36214 = 0.13786

Example 2 : Find the probability of the standard normal variate z when z is less than 1.09.

Solution : We know that the total area under the normal curve within the limits $-\infty$ to $+\infty$ is 1. Therefore, the area to the left of z = 0 is 0.5, and the area to the right or z = 0 is 0.5.

Again, from the table, we get that the area from z = 0 to z = 1.09 is 0.36214.

$\therefore$ P(z < 1.09)

= (area to the left of z = 0) + (area from z = 0 to z = 1.09)

= 0.5 + 0.36214

= 0.86214

Example 3 : Find P(0.5 < z < 1.09)

Solution : We know that the area to the right of z = 0 is 0.5. From the table, the area from z = 0 to z = 0.5 is 0.19146. Again, the area from z = 0 to z = 1.09 is 0.36214.

$\therefore$ P(0.5 < z < 1.09)

= (Area from z = 0 to z = 1.09) – (area from z = 0 to x = 0.5)

= 0.36214 – 0.19146

= 0.17068

Example 4 : A large number of measurements is normally distributed with mean 65.5 cm and standard deviation 6.2 cm. Find the percentage of measurements that fall between 54.8 cm and 68.8 cm.

Solution : Let X be the normal variate showing the large number of measurements.

We have $\mu = 65.5,\ \sigma = 6.2$

Now, $Z = \dfrac{X-\mu}{\sigma} = \dfrac{X-65.5}{6.2}$

When, X = 54.8 $Z = \frac{54.8 - 65.5}{6.2} = \frac{-10.7}{6.2} = -1.7$

When, X = 68.8, $Z = \frac{68.8 - 65.5}{6.2} = 0.5$

Thus, when X lies between 54.8 and 68.8, Z lies between –1.7 and 0.5

Now P(–1.7 < X < 0.5)

= P(–1.7 < Z < 0) + P(0 < Z < 0.5)

= 0.45543 + 0.19146

= 0.64689 = 64.69%

$\therefore$ the percentage of measurements is 64.69%

Example 5 : For the distribution of marks of candidates in an examination, mean is 14 and standard deviation is 2.5. Assuming the normality of the distribution, find the probability that a randomly selected student will score above 15.

Solution : Let X be the normal variate showing the scores of candidates.

We have $\mu = 14,\ \sigma = 2.5$

Now, $Z = \frac{X - \mu}{\sigma} = \frac{X - 14}{2.5}$

When X=15 $Z = \frac{15 - 14}{2.5} = \frac{1}{2.5} = 0.4$

$\therefore$ P(x > 15) = P(z > 0.4)

= 0.5 – (Area between z = 0 and Z = 0.4)

= 0.5 – 0.15542

= 0.34458.

CHECK YOUR PROGRESS

Q 1: Find (i) P (Z > 2.58), (ii) P(Z < 1.96),
(iii) P (Z > –3)

Q 2: For a certain normal distribution, mean is 50, and standard deviation is 5. Find $P(45 \le X \le 55)$, X being a normal variate.

Q 3: A normal variate X has mean 100 and standard deviation 4. Find the probability that X lies between 92 and 10.8

Q 4: The distribution of scores of students in an examination is normal. If the mean is 45, and the standard deviation is 5, find the percentage of students who scored between 40 and 50.

Q 5: The marks obtained in a certain examination follow normal distribution with mean 45, and standard deviation 10. If 1000 students appear at the examination, calculate the number of students scoring (i) less than 30 marks, and (ii) more than 80 marks.

14.5 LET US SUM UP

- Normal distribution is a continous distribution. It is a limiting case of the binomial distribution.
- Normal distribution is derived from the binomial distribution under two conditions, viz.,
  (i) n, the number of trials, is sufficiently large, i.e., $n \to \infty$ and
  (ii) neither p nor q is very small.

  A random variable X is said to follow normal distribution, if and only if, its probability density function is given by

$$f(x) = \frac{1}{\alpha\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\alpha}\right)^2}$$

  Where x is the real value of X, i.e., $-\infty < X < \infty$

  The variable X is said to be distributed normally with mean μ and variance σ^2. Symbolically, we write $X \sim N(\mu, \sigma^2)$. It is read as 'X is distributed as $N(\mu, \sigma^2)$.
- The normal curve is bell-shaped, and the curve is symmetrical about the line $x = \mu$, and x lies between $-\infty$, and $+\infty$.
- The total area under the normal curve within the limits $-\infty$ to ∞ is 1.
- If a new variable Z is defined by $Z = \frac{X - \mu}{\sigma}$, Z is called the standard normal variate.

  The mean of Z is 0, and the standard deviation is 1.

14.6 FURTHER READING

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✓14.7 ANSWERS TO CHECK YOUR PROGRESS

Check Your Progress 1

Ans to Q No 1: (i) We know that tha total area under the normal curve within the limits $-\infty$ to $+\infty$ is 1. Therefore, the area to the left of z = 0 is 0.5, and the area to the night of z = 0 is 0.5. Again, from the table, we get that the area from z = 0 to z = 2.58 is 0.49506

$\therefore$ P (z > 2.58)

= (area to the right of z = 0) – (area from z = 0 to z = 2.58)

= 0.5 – 0.49506

= 0.00494

(ii) We know that the total area under the normal curve within the limits $-\infty$ to $+\infty$ is 1. Therefore, the area to the left of z = 0 is 0.5, and the area to the right of z = 0 is 0.5. Again from the table, we get that the area from z = 0 to z = 1.96 is 0.47500.

$\therefore$ P (z < 1.96)

= (area to the left of z = 0) + (area from z = 0 to z = 1.96)

= 0.5 + 0.47500

= 0.97500

(iii) We know that the total area under the normal curve within the limits $-\infty$ to ∞ is 1. Therefore, the area to the left of z = 0 is 0.5, and the area to the right of z = 0 is 0.5.

Now, P(Z > –3)

= area from z = –3 to z = 0) + (area to the right of z = 0)

= (area from z = 0 to z = 3) + (area to the right of z = 0)

= 0.49865 + 0.5

= 0.99865

Ans to Q No 2: X is a normal variate.

We have $\mu = 50,\ \sigma = 5$

Now, $Z = \dfrac{X - \mu}{\sigma} = \dfrac{X - 50}{5}$

When X = 45 $Z = \frac{45-50}{5} = -1$

When X = 55, $Z = \frac{55-50}{5} = 1$

Thus, when X lies between 45 and 55, Z lies between –1 and 1.

Now $P(-1 \le Z \le 1)$

$= P(-1 \le Z \le 0) + P(0 \le Z \le 1)$

= 0.34134 + 0.34134

= 0.68268

Ans to Q No 3: Do yourself.

Ans to Q No 4: Do yourself.

Ans to Q No 5: Let X be the normal variate showing marks.

We have $\mu = 45,\ \sigma = 10$

Now $Z = \frac{X-\mu}{\sigma} = \frac{X-45}{10}$

When X = 30, $Z = \frac{30-45}{10} = \frac{-15}{10} = -1.5$

When X = 80, $Z = \frac{80-45}{10} = \frac{35}{10} = 3.5$

(i) Now $P(X \le 30)$

$= P(z \le .1.5)$

= (area to the left of z = 0) – (area from z = – 1.5 to z = 0)

(area to the left of z = 0) – (area from z = 0) to z = 1.5)

= 0.5 – 0.43319

= 0.06681

$\therefore$ the required number of students

= 1000 x 0.06681

= 67 (approximately)

(ii) $P(Z \ge 80)$

$= P(Z \ge 3.5)$

= (area to the right of z = 0) – (area from z = 0 to z = 3.5)

= 0.5 – 0.49977

= 0.00023

Now 1000 x 0.00023 = 0.23

$\therefore$ there are no students scoring more than 80 marks.

14.8 MODEL QUESTIONS

Q 1: Define standard normal variate and write down the probability density function of this variate.

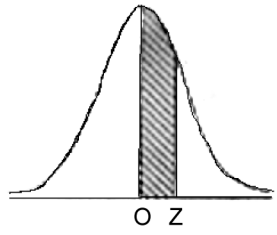
Q 2: State the conditions under which the normal distribution is obtained as a limiting case of the binomial distribution.

Q 3: Find (i) $P(Z \leq -1.96)$ (ii) $P(Z \leq 1.96$ (iii) $P(Z > 2.58)$

Q 4: A normal variate X has mean 50 and standard deviation 5. Find $P(40 \leq X \leq 60)$.

Q 5: The mean of a normal distribution is 50 and the standard deviation is 4. What percentage of total frequency lies between 42.16 and 57.84?

Q 6: Let a continuous variable X follow a normal distribution with mean 12 and standard deviation 2. What is the probability that the value of X selected at random lies between 11 and 14 ?



Area under the Standard Normal Curve

Z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.00000	.00399	.00798	.01197	.01595	.01994	.02392	.02790	.03188	.03586
0.1	.03983	.04383	.04776	.05172	.05567	.05962	.06356	.06749	.07142	.07535
0.2	.07926	.08317	.08706	.09095	.09483	.09871	.10257	.10642	.11026	.11409
0.3	.11791	.12172	.12552	.12930	.13307	.13683	.14058	.14431	.14803	.15173
0.4	.15542	.15910-	.16276	.16640	.17003	.17364	.17724	.18082	.18439	.18793
0.5	.19146	.19497	.19847	.20194	.20540	.20884	.21226	.21566	.21904	.22240
0.6	.22575	.22907	.23237	.23565	.23891	.24215	.24537	.24857	.25175	.25490
0.7	.25804	.26115	.26424	.26730	.27035	.27337	.27637	.27935	.28230	.28524
0.8	.28814	.29103	.29389	.29673	.29955	.30234	.30511	.30785	.31057	.31372
0.9	.31594	.31859	.32121	.32381	.32639	.32894	.33147	.33398	.33646	.33891
1.0	.34134	.34375	.34614	.34849	.35083	.35314	.35543	.35769	.35993	.36214
1.1	.36433	.36650	.36864	.37076	.37286	.37493	.37698	:37900	.38100	.38298
1.2	.38493	.38686	.38877	.39065	.39251'	.39435	.39617	.39796	.39973	.40147
1.3	.40320	.40490	.40658	.40824	.40988	.41198	.41309	.41466	.41621	.41774
1.4	.41924	.42073-	.42220	.42364	.42507	.42647	.42785	.42922	.43056	.43189
1.5	.43319	.43448	.43574	.43699	.43822	.43943	.44062	.44179	.44295	.44408
1.6	.44520	.44630	.44738	.44845	.44950	.45053	.45154	.45254	.45352	.45449
1.7	.45543	.45637	.45728	.45818	.45907	.45994	.46080	.46164	.46246	.46327
1.8	.46407	.46485	.46562	.46638	.46712	.46784	.46856	.46926	.46995	.47062
1.9	.47128	.47193	.47257	.47320	.47381	.47441	.47500	.47558	.47615	.47670
2.0	.47725	.47784	.47831	.47882	.47932	.47982	.48030	.48077	.48124	.48169
2.1	.48214	.48257	.48300	.48341	.48382	.48422	.48461	.48500	.48537	.48574
2.2	.48610	.48645	.48679	.48713	.48745	.48778	.48809	.48840	.48870	.48899

2.3	.48928	.48956	.48983	.49010	.49036	.49061	.49086	.49111	.49134	.49158
2.4	.49180	.49202	.49224	.49245	.49266	.49286	.49305	.49324	.49343	.49361
2.5	.49379	.49396	.49413	.49430	.49446	.49461'	.49477	.49492	.49506	.49520
2.6	.49534	.49547	.49560	.49573	.49585	.49598	.49609	.49621	.49632.	.49643
2.7	.49653	.49664	.49674	.49683	.49693	.49702	.49711	.49720	.49728	.49736
2.8	.49744	.49752	.49760	.49767	.49774	.49781	.49788	.49795	.49801	.49807
2.9	.49813	.49819	.49825	.49831	.49836	.49841	.49846	.49851	.49856	.49861
3.0	.49865	.49869	.49874	.49878	.49882	.49886	.49889	.49893	.49896	.49900
3.1	.49903	.49906	.49910	.49913	.49916	.49918	.49921	.49924	.49926	.49929
3.2	.49931	.49934	.49936	.49938	.49940	.49942	.49944	.49946	.49948	.49950
3.3	.49952	.49953	.49955	.49957	.49958	.49960	.49961	.49962	.49964	.49965
3.4	.49966	.49968	.49969	.49970	.49971	.49972	.49973	.49974	.49975	.49976
3.5	.49977	.49978	.49978	.49979	.49980	.49981	.49981	.49982	.49983	.49983
3.6	.49984	.49985	.49985	.49986	.49986	.49987	.49987	.49988	.49988	.49989
3.7	.49989	.49990	.49990	.49990	.49991	.49991	.49992	.49992	.49992	.49992
3.8	.49993	.49993	.49993	.49994	.49994	.49994	.49994	.49995	.49995	.49995
3.9	.49995	.49995	.49996	.49996	.49996	.49996	.49996	.49996	.49997	.49997
4.0	.49997	-	-	-	-	-	-	-	-	-
4.5	.499997	-	-	-	-	-			-	-
5.0	.4999997	-	-	-	-	-	-	-	-	-

[Source : Basic statisties by B. L. Agarwal (New Age International Publishers, 2009)]

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