UNIT 3 : GROUP OF PERMUTATIONS

UNIT STRUCTURE

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3.1 LEARNING OBJECTIVES

After going through this unit, you will able to know -

- permutation and its examples
- multiplication of permutations
- different kinds of permutation
- permutation group

3.2 INTRODUCTION

The word permutation is used in two different ways. Firstly, it is an ordered arrangement of the elements of a finite set of distinct elements. Secondly, a permutation is a one-one and onto mapping from a non-empty set to itself. In Group theory, permutation is taken as a one-one and onto mapping from a non-empty set to itself. In this unit, we will define a permutation, give examples and obtain the number of permutations from a finite set to itself. We will prove that the collection of permutations on a finite set is a group under multiplication of permutations. This group is called permutation group (or symmetric group).

3.3 DEFINITION OF PERMUTATION, EXAMPLES

A bijective mapping (i.e. a one-one and onto mapping) from a non-empty set to itself is called a permutation on the set.

Example 1 : Let us consider the set S = {1, 2, 3}. Let f ; S $\rightarrow$ S be a mapping such that f(1) = 2, f(2) = 3, f(3) = 1

Clearly, f is a one-one and onto mapping from S to itself.

$\therefore$ f is a permutation on S.

Notation for permutation

Let us consider the permutation given above.

S = {1, 2, 3}

f : S $\rightarrow$ S, f(1) = 2, f(2) = 3, f(3) = 1

The above permutation is, generally, written as

$$f = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

We write the elements of the set S in the first row. In the second row, we write their respective images. The image of 1 under f is 2. We write 2 just below 1. Similarly the image of 2 is 3. We write 3 just below 2. Similarly, we write 1 just below 3. In this way, a permutation is written.

Number of permutations on S = {1, 2, 3}.

Here S has 3 distinct elements.

Now 1 can be associated in 3 ways. It can be associated with 1 or with 2 or with 3.

2 can be associated in 2 ways.

3 can be associated in 1 way.

$\therefore$ the number of permutations = $3 \times 2 \times 1 = 3$!

Number of permutations on a finite set having n elements

Let us consider a set S having n distinct elements.

The first element can be associated in a ways.

The second element can be associated in (n-1) ways.

The third element can be associated in (n-2) ways.

In this way (n-1)th element can be associated in 2 ways, and the nth element can be associated in 1 way.

$\therefore$ the number of permutations

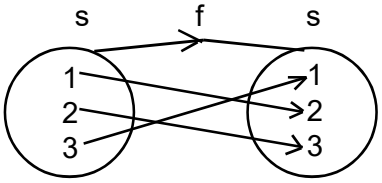
= n(n-1)(n-2) . . . . . 2.1 = n!

Example 2 : Let S = {1, 2}. Write down all the permutations on S.

Solution : The number of elements of S is 2.

$\therefore$ there are 2!, i.e. 2 permutations on S. They are

$$i = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \qquad f = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$



Note : The permutation i is called the identity permutation. Here the image of 1 is 1, the image of 2 is 2.

Example 3 : Let S = {1, 2, 3}. Write down all the permutations on S.

Solution : The number of elements of S is 3.

$\therefore$ there are 3!, i.e. 6 permutations on S.

They are

$$i = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, f_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, f_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$$f_3 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, f_4 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, f_5 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

Note : In this case $i = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$

is the identity permutation.

Identity permutation

Let S = {a_1, a_2, ..., a_n} where all the a_i's are distinct.

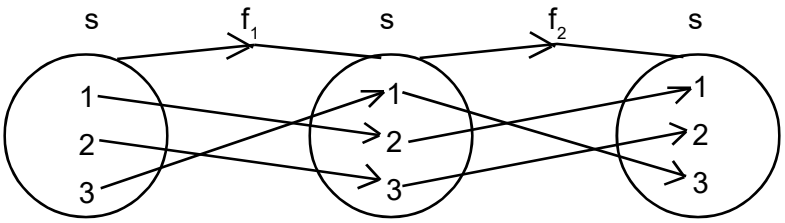
The identity permutation i on S is defined by

$$i = \begin{pmatrix} a_1 & a_2 & a_3 .....a_n \\ a_1 & a_2 & a_3 .....a_n \end{pmatrix}$$

3.4 MULTIPLICATION OF PERMUTATION

Let S = {1, 2, 3}. Let us consider the following two permutations on S

$$f_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, f_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$



Then we get the composite mapping

$f_2of_1 : S \to S$, such that

$(f_2of_1)(1) = f_2(f_1(1)) = f_2(2) = 1$

$(f_2of_1)(2) = f_2(f_1(2)) = f_2(3) = 2$

$(f_2of_1)(3) = f_2(f_1(3)) = f_2(1) = 3$

Clearly, f_2of_1 is a permulation on S, and $f_2of_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$

Similarly f_1of_2 is also a permutation on S.

The product f_2f_1 of permutations is defined by the composite mapping f_2of_1. Thus

$$f_2 f_1 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

Note : 1. We know that the composite mapping of two one-one and onto functions is also a one-one and onto function. Therefore, the product of two permutations is always a permutation.

2. We know that the composition of mappings is associative. Therefore, the multiplication of permutations is associative.

3. We know that the composition of mappings is not, in general, commutative. Therefore, the multiplication of permutations is not, in general, commutative.

Let $$f = \begin{pmatrix} a_1 & a_2 & a_3 & .....a_n \\ f(a_1) & f(a_2) & f(a_3).....f(a_n) \end{pmatrix}$$

$$g = \begin{pmatrix} a_1 & a_2 & a_3 & .....a_n \\ g(a_1) & g(a_2) & g(a_3).....g(a_n) \end{pmatrix}$$

be two permutations on S.

Then $$fg = \begin{pmatrix} a_1 & a_2 & a_3 & ..... a_n \\ f(g(a_1)) & f(g(a_2)) & f(g(a_3)).....f(g(a_n)) \end{pmatrix}$$

Similarly

$$gf = \begin{pmatrix} a_1 & a_2 & a_3 & \ldots\ldots\ldots\ldots & a_n \\ g(f(a_1)) & g(f(a_2)) & g(f(a_3)) & \ldots\ldots & g(f(a_n)) \end{pmatrix}$$

Example 4 : Let S ={1, 2, 3, 4}. Consider the following permutations f and g on S.

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix},\ g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}$$

Find fg and gf

Solution : $fg = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ f(g(1) & f(g(2) & f(g(3) & f(g(4) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ f(4) & f(2) & f(1) & f(3) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}$$

$$gf = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ g(f(1) & g(f(2) & g(f(3) & g(f(4) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ g(2) & g(3) & g(4) & g(1) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}$$

Clearly fg $\neq$ gf [Recall the definition of equality of two functions.]

Example 5 : Let S= {1, 2, 3, 4}. Consider the following permutations on S.

$$\phi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix},\ \psi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}$$

Find $\phi\psi$ and $\psi\phi$

Solution : $\phi\psi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

$$\psi\phi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

In this case $\phi\psi = \psi\phi$

Example 6: Let S = {1, 2, 3, 4}. Consider the following permutations on S.

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix},\ h = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$$

Find fh and hf

Solution : $fh = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = i$$

$$hf = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = i$$

Here fh = hf = i

Note : In the above example fh=hf = i, the identity permutation f is called the inverse of h, and vice versa, h is the inverse of f. We write $f = h^{-1}$ or $h = f^{-1}$

Inverse of a permutation

Let S = $\{a_1, a_2, \ldots, a_n\}$ where the a_i's are distinct.

Let $f = \begin{pmatrix} a_1 & a_2 & a_3 & \ldots & a_n \\ f(a_1) & f(a_2) & f(a_3) \ldots & & f(a_n) \end{pmatrix}$ be a permutation on f.

Then the inverse of f, denoted by f^{-1}, is given by

$$f^{-1} = \begin{pmatrix} f(a_1) & f(a_2) & f(a_2) \ldots & f(a_n) \\ a_1 & a_2 & a_3 \ldots & a_n \end{pmatrix}$$

We note that $ff^{-1} = f^{-1}f = i$



CHECK YOUR PROGRESS

Q 1: Let S = {a_1, a_2, a_3}, where all the a_i's are distinct. How many permutations are there on S? Write down all the permutations on S.

Q 2: Let S = {1, 2, 3, 4}. How many permutations are there on S ? Write down any five permutations on S.

Q 3: Let S = {1, 2, 3, 4, 5}. Consider the following permutations on S.

$$f=\begin{pmatrix}1&2&3&4&5\\2&3&4&5&1\end{pmatrix},\quad g=\begin{pmatrix}1&2&3&4&5\\2&4&3&1&5\end{pmatrix}$$

$$\phi=\begin{pmatrix}1&2&3&4&5\\1&2&5&4&3\end{pmatrix},\quad \psi=\begin{pmatrix}1&2&3&4&5\\5&3&4&2&1\end{pmatrix}$$

(i) Find $fg,\ f\phi,\ f\psi,\ g\psi$

(ii) Find $g\phi,\ \phi g$. Are they equal ?

(iii) Find $\phi\psi,\ \psi\phi$. Are they equal?

Q 4: Let S = {1, 2, 3, 4, 5, 6, 7}. Write down the inverse of each of the following permutations on S.

(i) $f=\begin{pmatrix}1&2&3&4&5&6&7\\2&3&4&5&6&7&1\end{pmatrix}$

(ii) $g=\begin{pmatrix}1&2&3&4&5&6&7\\7&6&5&4&3&2&1\end{pmatrix}$

(iii) $h=\begin{pmatrix}1&2&3&4&5&6&7\\3&4&5&6&7&2&1\end{pmatrix}$

3.5 DIFFERENT KINDS OF PERMUTATION

Cyclic Permutation

Let S be a finite set. Let f be a permutation on S. f is called a cyclic permutation if there exist elements $a_1, a_2, a_3, \ldots, a_{n-1}, a_n$ in S such that $f(a_1) = a_2$, $f(a_2) = a_3$, $f(a_3)=a_4$, ..., $f(a_{n-1}) = a_n$, $f(a_n) = a_1$, and all other elements, if any, remain fixed under f, i.e., f(a) = a, for all other $a \in S$.

In this case we denote f by $(a_1, a_2, a_3 \ldots . a_n)$.

f is called a cyclic permutation of length n.

Note : A cyclic permutation is also called a cycle. A cyclic permutation of length n is called an n-cycle.

Transposition

A cyclic permutation of length 2 is called a transposition.

Example 7: Let S ={1, 2, 3}

Let $f=\begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 1\end{pmatrix}$ be a permutation on S.

Here f(1) = 2, f(2) = 3, f(3) = 1

$1\to 2\to 3\to 1$

Thus, $f=\begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 1\end{pmatrix}=\begin{pmatrix}1 & 2 & 3\end{pmatrix}$

f is a cyclic permutation of length 3 or a 3-cycle.

Example 8 : Let S = {1, 2, 3, 4, 5}. Let

$f=\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 1 & 3 & 4 & 2 & 5\end{pmatrix}$ be a permutation on S.

Here f(2) = 3, f(3) = 4, f(4) = 2

$2\to 3\to 4\to 2$

And f(1) = 1, f(5) = 5

i.e, 1, 5 remain fixed under f, i.e. f(1) = 1, f(5) = 5

Thus, $f=\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 1 & 3 & 4 & 2 & 5\end{pmatrix}=\begin{pmatrix}2 & 3 & 4\end{pmatrix}$

f is a cyclic permutation of length 3 or a 3-cycle.

Example 9 : Let S = {1, 2, 3, 4}. Let

$f=\begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 3 & 2 & 4\end{pmatrix}$ be a permutation on S

Here f(2) = 3, f(3) = 2

$2\to 3\to 2$

And f(1) = 1, f(4) = 4

Thus, $f=\begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 3 & 2 & 4\end{pmatrix}=\begin{pmatrix}2 & 3\end{pmatrix}$

f is a cyclic permutation of length 2, i.e. f is a transposition.

Note : Let S = {1, 2, 3}

$$f = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (1 \quad 2 \quad 3) \qquad (1)$$

is a cyclic permutation on S = (1, 2, 3}.

Now, we consider

S = {1, 2, 3, 4, 5}

$$\phi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 4 & 5 \end{pmatrix} = (1 \quad 2 \quad 3) \qquad (2)$$

is a cyclic permutation on S = {1, 2, 3, 4, 5}.

We note that (1) and (2) are different permutations. In (2), (1 2 3) means the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 4 & 5 \end{pmatrix}$$

Cycles of a permutations

Let S = {1, 2, 3}

Let $f = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (1 \quad 2 \quad 3)$

be a permutation of S = {1, 2, 3}

Now $(1 \quad 3) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$, $(1 \quad 2) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$

$$(1 \quad 3)(1 \quad 2) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (1 \quad 2 \quad 3)$$

Thus, we see that

$$f = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (1 \quad 2 \quad 3) = (1 \quad 3)(1 \quad 2)$$

Again, we consider S = {1, 2, 3, 4, 5}

Let $g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix} = (1 \quad 2 \quad 3 \quad 4 \quad 5)$

be a permutation on S = {1, 2, 3, 4, 5}

Now, it can be shown that

$$g = (1 \quad 2 \quad 3 \quad 4 \quad 5)$$
$$= (1 \quad 5)(1 \quad 4)(1 \quad 3)(1 \quad 2)$$

Again, we consider S = {1, 2, 3, 4, 5, 6}

Let $h = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 6 & 4 \end{pmatrix}$

be a permutation on S = {1, 2, 3, 4, 5, 6}

Now h can be expressed as

$h = (1 \quad 2 \quad 3)(4 \quad 5 \quad 6)$

Here $(1 \quad 2 \quad 3), (4 \quad 5 \quad 6)$ are called the cycles of the permutation h. Here $(1 \quad 2 \quad 3)$ and $(4 \quad 5 \quad 6)$ have no common elements. They are called disjoint cycles.

Disjoint Permutations

Let S = {1, 2, 3, 4, 5, 6}

Let $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 4 & 5 & 6 \end{pmatrix}$

$$= (1 \quad 2 \quad 3)$$

and $g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 5 & 6 & 4 \end{pmatrix}$

$$= (4 \quad 5 \quad 6)$$

be two permutations on S = {1, 2, 3, 4, 5, 6}

We note that $(1, \quad 2, \quad 3)$, and $(4, \quad 5, \quad 6)$ have no common elements. In this case, the permutations f, and g are said to be disjoint.

Again, consider the permutations

$$\phi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 1 & 5 & 6 \end{pmatrix} = (1 \quad 2 \quad 3 \quad 4)$$

$$\psi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 4 & 5 & 6 & 3 \end{pmatrix} = (3 \quad 4 \quad 5 \quad 6)$$

on S = {1, 2, 3, 4, 5, 6}

Here $(1 \quad 2 \quad 3 \quad 4)$ and $(3 \quad 4 \quad 5 \quad 6)$ have common elements. So, the permutations ϕ and ψ are not disjoint.

Thus, we define disjoint permutations as follows.

Two permutations f and g on a finite set S are said to be disjoint if

(i) $f(a) \neq a \Rightarrow g(a) = a,$ for any $a \in S$.

and (ii) $g(a) \neq a \Rightarrow f(a) = a,$ for any $a \in S$.

It can be proved that multiplication of two disjoint cycles, as well as of two disjoint permutations is commutative.

Even and odd permutations

Let S = {1, 2, 3}

$f = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ is a permutation on S = {1, 2, 3}

Again $f = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (1 \quad 2 \quad 3) = (1 \quad 3) = (1 \quad 2)$

Again, consider the set S = {1, 2, 3, 4}

Let $g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$ be a permutation on S = {1, 2, 3, 4}

Now $g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = (1 \quad 2 \quad 3 \quad 4) = (1 \quad 4)(1 \quad 3)(1 \quad 2)$

Thus, we see that f has been expressed as the product of even number of transpositions (two), whereas g has been expressed as the product of odd number of transpositions (three).

If a permutation can be expressed as the product of even number of transpositions, then it is said to be even. On the other hand, if a permutation can be expressed as the product of odd number of transpositions, then it is said to be odd.

The permutation f, given above, is even, whereas the permutation g is odd.

We note the following results without proof.

1. The product of two even permutations is an even permutation.
2. The product of two odd permutation is an even permutation.
3. The product of an even permutation and an odd permutation is an odd permutation.
4. Identity permutation is an even permutation.
5. The inverse of an even permutation is even.
6. The inverse of an odd permutation is odd.



CHECK YOUR PROGRESS

Q 5: Write down all the permutations on the set S = {1, 2, 3}, and pick out the even permutations and the odd permutations.

Q 6: Let S = {1, 2, 3, 4, 5, 6}. Determine whether the following permutations on S are even or odd.

(i) $f_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 4 & 5 & 6 & 1 \end{pmatrix}$

(ii) $f_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 5 & 6 & 1 & 4 \end{pmatrix}$

(iii) $f_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 3 & 4 & 5 \end{pmatrix}$

(iv) $f_4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 4 & 5 & 6 & 3 \end{pmatrix}$

(v) $f_5 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 5 & 4 & 2 & 3 & 6 \end{pmatrix}$

Number of even permutations and odd permutations on S = {1, 2, 3, . . . , n}, $n \geq 2$

Let S = {1, 2, 3, ..., n}

Let S_n be the set of all permutations on S. Then number of elements of S_n is n!. Of these n! permutations, some are even and some are odd.

Let the number of even permutations be m, and the number of odd permutations be k.

Clearly, m+k = n!

Let the even permutations be $e_1, e_2, \ldots, e_m$; where the e_i's are distinct.

Let f be any transposition

Then $fe_1, fe_2, \ldots, fe_m$ are odd permutations, and are distinct, for if $fe_i = fe_j$, then $e_i = e_j$, which is a contradiction.

Since the number of odd permutations is k,

$\therefore\ m \leq k$

Interchanging the roles of m, k, we get

$k \le m$

$\therefore \ m = k$

$\because \ m + k = n!, \quad \therefore \ m = k = \frac{n!}{2}$

$\therefore$ Number of even permutations on $S = \frac{n!}{2}$, and number of odd permutations on $S = \frac{n!}{2}$, where S = {1, 2, 3, ..., n}, $m \ge 2$.

3.6 PERMUTATION GROUP

Let S = {1, 2, 3, 4, ... , n}, and let S_n be the set of all permutations on S.

(i) Let $f, g \in S_n$.

We know that the product fg is defined by the composite mapping fog.

Also we know that the composite mapping of two one-one and onto function is also one-one and onto.

Therefore, the product of two permutations is always a permutation .

$\therefore \ fg \in S_n$

(ii) We know that the composition of mappings is associative. Therefore, the multiplication of permutations is associative.

(iii) The identity permutation

$$i = \begin{pmatrix} 1 & 2 & 3 & ..... & n \\ 1 & 2 & 3 & ..... & n \end{pmatrix} \in S_n$$

Also if = f = fi, $\forall f \in S_n$

i is the identity element of S_n.

(iv) Let $f = \begin{pmatrix} 1 & 2 & 3 & ..... & n \\ f(1) & f(2) & f(3) & ..... & f(n) \end{pmatrix} \in S_n$

Now $g = \begin{pmatrix} f(1) & f(2) & f(3) & ..... & f(n) \\ 1 & 2 & 3 & ..... & n \end{pmatrix} \in S_n$

and fg = i = gf

g is the inverse of f.

We denote g by f^{-1}

i.e., $ff^{-1} = i = f^{-1} f$.

Thus S_n is a group under multiplication of permutations, called the permutation group (or symmetric group) of degree n.

Note : 1) The group S_n is not Abelian, since in general, non commutative.

2) The number of elements of S_n is n!, i.e, the order of S_n is n! .

Example 10 : Let S = {1, 2, 3}. Let S_3 be the set of all permutations on S. Show that S_3 is a group under product of permutations. Is the group Abelian?

Solution : Let $S_n = \{i, f_1, f_2, f_3, f_4, f_5\}$, where

$$i = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad f_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (1 \quad 2 \quad 3)$$

$$f_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = (1 \quad 3 \quad 2), \quad f_3 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = (2 \quad 3)$$

$$f_4 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = (1 \quad 3), \quad f_5 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = (1 \quad 2)$$

We construct the operation table as follows :

	i	f_1	f_2	f_3	f_4	f_5
i	i	f_1	f_2	f_3	f_4	f_5
f_1	f_1	f_2	i	f_5	f_3	f_4
f_2	f_2	i	f_1	f_4	f_5	f_3
f_3	f_3	f_4	f_5	i	f_1	f_2
f_4	f_4	f_5	f_3	f_2	i	f_1
f_5	f_5	f_3	f_4	f_1	f_2	i

(i) From the table we see that S_3 is closed under multiplication of permutations.

(ii) We know that multiplication of permutations is associative.

(iii) The identity permutations $i \in S_3$, and if = f = fi, $\forall f \in S_3$

(iv) Every element of S_3 possesses inverse in S_3.

The inverse of i is i.

The inverse of f_1 is f_2.

The inverse of f_2 is f_1.

The inverse of f_3 is f_3.

The inverse of f_4 is f_4.

The inverse of f_5 is f_5.

Thus, S_3 is a group under multiplication of permutations.

2nd Part : The group S_3 is not Abelian.

$$f_2 f_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = f_4$$

$$f_3 f_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = f_5$$

$\therefore \; f_2 f_3 \neq f_3 f_2$

Alternating Group

Let S = {1, 2, 3, ..., n}

Let A_n be the set of all even permutation on S. Then A_n is a group under multiplication of permutations, called the **alternating group** of degree n.

Number of elements of A_n is $\frac{n!}{2}$,i.e order of A_n is $\frac{n!}{2}$.

Clearly A_n is a subgroup of S_n.

Example 11 : Let S = {1, 2, 3}. Let A_3 be the set of all even permutations on S. Show that A_3 is a group under multiplication of permutations. Is the group Abelian ?

Solution : Let S = {1, 2, 3}.

Let A_3 be the set of all even permutations.

Let $A_3 = \{i, f_1, f_2\}$, where

$$i = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

$$f_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (1 \;\; 3 \;\; 2) = (1 \;\; 3)(1 \;\; 2)$$

$$f_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = (1 \;\; 3 \;\; 2) = (1 \;\; 2)(1 \;\; 3)$$

i, f_1, f_2 are even permutations.

We construct the operation table as follows.

(i) From the table, we see that A_3, is closed under multiplication of permutations.

(ii) We know that multiplications of permutations is associative.

(iii) The identity permutation $i \in A_3$, and if = f = fi, $\forall f \in A_3$

(iv) Every element of A_3 possesses inverse in A_3.

Th inverse of i is i.

The inverse of f_1 is f_2.

The inverse of f_2 is f_1.

Thus, A_3 is a group under multiplication of permutations.

2nd Part : From the table we see that multiplication is commutative.

$\therefore$ A_3 is an Abelian group

Note : We have seen that S_3 is a non-Abelian group, whereas A_3 is an Abelian group.

Moreover, A_3 is a subgroup of S_3.

Thus, we can conclude that a non-Abelian group can have an Abelian subgroup.



CHECK YOUR PROGRESS

Q 7: Let S = {1, 2, 3, ..., n}. Is the set of all odd permutations on S a group? Justify.

Q 8: Let S = {1, 2, 3, 4}

Let V = {1, f_1, f_2, f_3}, where

$$i = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \quad f_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}$$

$$f_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}, \quad f_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

Show that V is group under multiplication of permutations. Is the group Abelion ?

Q 9: Give an example of a finite non-Abelian group.



3.7 LET US SUM UP

- A bijiective mapping (i. e., a one-one and onto mapping) from a non-empty set to itself is called a permutation on the set.
- The number of permutations on a finite set S having n elements is n!.
- Let S be a finite set. Let f be a permutation on S. f is said to be a cyclic permutation of length n if there exist elements $a_1, a_2, a_3, \ldots . a_{n-1}, a_n$ in S such that $f(a_1) = a_2$, $f(a_2) = a_3$, ..., $f(a_{n-1}) = a_n$, $f(a_n) = a_1$, and all other elements, if any, remain fixed under f, i.e., f(a) = a, for all $a \in S$. In this case f is denoted by $(a_1\ a_2\ a_3 \ldots . a_n)$.
- A cyclic permutation of length 2 is called a transposition.
- If a permutation can be expressed as the product of even number of transpositions, then it is said to be even.
- If a permutation can be expressed as the product of odd number of transpositions, then it is said to be odd.
- Let S = {1, 2, 3, ..., n}, and let S_n be the set of all permutations on S. S_n is a group under multiplications of permutations called the permutation group (or symmetric group) of degree n, and of order n!. If A_n is the set of all even permutations on S, then A_n is a group under multiplications of permutations, called the alternating group of degree and of order $\frac{n!}{2}$.



3.8 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q No 1: Here S = $(a_1, a_2, a_3\}$, where all the a_i1s are distinct.

$\therefore$ there are 3!, i.e., $3 \times 2 \times 1$, i.e. 6 permutations on S. These are

$$i = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{pmatrix}, f_1 = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_1 \end{pmatrix}$$

$$f_2 = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_3 & a_1 & a_2 \end{pmatrix}, f_3 = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{pmatrix}$$

$$f_4 = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_3 & a_2 & a_1 \end{pmatrix}, f_5 = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_2 & a_1 & a_3 \end{pmatrix}$$

Ans to Q No 2: S = {1, 2, 3, 4}

The number of permutations on S is 4!, i.e., $4 \times 3 \times 2 \times 1$, i.e., 24.

Any five permutations on S :

$$i = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \quad f_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}$$

$$f_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}, \quad f_4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$$

Ans to Q No 3: (i) $fg = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 3 & 1 & 5 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 4 & 2 & 1 \end{pmatrix}$$

$$f\phi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 5 & 4 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}$$

$$f\psi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 4 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 5 & 3 & 2 \end{pmatrix}$$

$$g\phi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 3 & 1 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 4 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 1 & 4 & 2 \end{pmatrix}$$

(ii) $g\phi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 3 & 1 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 5 & 4 & 3 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 1 & 3 \end{pmatrix}$$

$$\phi g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 5 & 4 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 3 & 1 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 1 & 3 \end{pmatrix}$$

Clearly $g\phi = \phi g$

(iii) $\phi\psi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 5 & 4 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 4 & 2 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 4 & 2 & 1 \end{pmatrix}$$

$$\psi\phi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 4 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 5 & 4 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 1 & 2 & 4 \end{pmatrix}$$

Clearly $\phi\psi \neq \psi\phi$

Ans to Q No 4: (i) $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 5 & 6 & 7 & 1 \end{pmatrix}$

$$\therefore f^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$$

(ii) $g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}$

$$\therefore g^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}$$

(iii) $h = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 6 & 7 & 2 & 1 \end{pmatrix}$

$$\therefore h^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 1 & 2 & 3 & 4 & 5 \end{pmatrix}$$

Ans to Q No 5: Here S = {1, 2, 3}

The permutations on S are

$$i\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix},\ f_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (1 \quad 2 \quad 3) = (1 \quad 3)(1 \quad 2)$$

$$f_2\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = (1 \quad 3 \quad 2) = (1 \quad 2)(1 \quad 3)$$

$$f_3\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = (2 \quad 3)$$

$$f_4 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = (1 \quad 3)$$

$$f_5 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \end{pmatrix} 3 = (1 \quad 2)$$

Even permutations : i, f_1, f_2

Odd permutations : f_3, f_4, f_5.

Ans to Q No 6: S = {1, 2, 3, 4, 5, 6}

(i) $f_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 4 & 5 & 6 & 1 \end{pmatrix}$

$$= (1 \quad 3 \quad 4 \quad 5 \quad 6) = (1 \quad 6)(1 \quad 5)(1 \quad 4)(1 \quad 3)$$

$\therefore$ f_1 is an even permutations.

(ii) $f_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 5 & 6 & 1 & 4 \end{pmatrix}$

$$= (1 \quad 2 \quad 3 \quad 5)(4 \quad 6)$$

$$= (1 \quad 5)(1 \quad 3)(1 \quad 2)(4 \quad 6)$$

$\therefore$ f_2 is an even permutation.

(iii) $f_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 3 & 4 & 5 \end{pmatrix}$

$$= (1 \quad 6 \quad 5 \quad 4 \quad 3 \quad 2)$$

$$= (1 \quad 2)(1 \quad 3)(1 \quad 4)(1 \quad 5)(1 \quad 6)$$

$\therefore$ f_3 is an odd permutation.

(iv) $f_4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 4 & 5 & 6 & 3 \end{pmatrix}$

$$= (3 \quad 4 \quad 5 \quad 6) = (3 \quad 6)(3 \quad 5)(3 \quad 4)$$

$\therefore$ f_4 is an odd permutation.

(v) $f_5 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 5 & 4 & 2 & 3 & 6 \end{pmatrix}$

$$= (2 \quad 5 \quad 3 \quad 4) = (2 \quad 4)(2 \quad 3)(2 \quad 5)$$

$\therefore$ f_5 is an odd permutation.

Ans to Q No 7: Here S = {1, 2, 3, ..., n}

Let O_n be the set of odd permutations on S.

Let f. $g \in O_n$

We know that the product of two odd permutations is an even permutation.

$\therefore fg \notin O_n$

$\therefore$ O_n cannot be a group.

Ans to Q No 8: Here S = {1, 2, 3, 4}

V = {i, f_1, f_2, f_3}, where

$i = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$, $f_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}$

$f_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}$, $f_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$

(i) From the table we see that V is closed under multiplication of permutations.

(ii) We know that multiplications of permutations is associative.

(iii) We have $i = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \in V$

and if = f = f_1, $\forall f \in V$.

$\therefore$ i is the identity

(iv) Every element of V has inverse in V.

i is the inverse of i.

f_1 is the inverse of f_1.

f_2 is the inverse of f_2.

f_3 is the inverse of f_3.

[Here every element is its own inverse]

$\therefore$ V is a group under multiplication of permutations.

2nd Part: From the table we see that multiplication is commutative.

$\therefore$ V is an Abelian group

Ans to Q No 9: Let S = {1, 2, 3}

Let S_3 be the set of all permutations on S. Then S_3 is a group under multiplication of permutations.

Clearly S_3 is a finite group of order 3! i.e. 6.

S_3 is not an Abelian group.



3.9 FURTHER READINGS

1) Lanski, Charles : *Concepts in Abstract Algebra*, (First Indian edition, 2010) American Mathematical Society.
2) Vasistha, A. R. : *Modern Algebra (Abstract algebra),* Krishna Prakashan Media (p) Ltd. Meerut-250001 (U. P).
3) Vinberg, E. B. : *A course in Algebra* (First Indian edition, 2009), American Mathematical Society).
4) Vatsa, B. S. and Vatsa, Suchi : *Modern Algebra*, New Age International (p) Ltd., New Delhi-110002.
5) Sen, M. K, Ghosh, Shamik and Mukhopadhyay, Parthasarthi : *Topics in Abstract Algebra,* Universities Press (India) Pvt. Ltd., Hyderabad, 500029.



3.10 MODEL QUESTIONS

Q 1: Let S = {1, 2, 3}. Show that the set S_2 of all permutations on S is an Abelian group under multiplication of permutations.

Q 2: Let S = {1, 2, 3, 4, 5, 6, 7, 8}.
Express the following permutations on S as product of trenspositions.

(i) $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 5 & 2 & 4 & 3 & 1 & 7 & 8 \end{pmatrix}$

(ii) $g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 1 & 4 & 7 & 2 & 5 & 8 & 6 \end{pmatrix}$

Q 3: Let S = {1, 2, 3, 4}
Let V = {1, f_1, f_2, f_3}, where

$$i = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \; f_1 = (1 \quad 2)(3 \quad 4)$$

$$f_2 = (1 \quad 3)(2 \quad 4), \; f_3 = (1 \quad 4)(2 \quad 3)$$

are permutations on V.
Examine if V is an Abelian group under multiplication of permutations.

Q 4: Prove that the set S_n of all permutations on S = {1, 2, 3, 4, ..., n} is a group under multiplication of permutations.

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