UNIT 6 : GROUP SERIES AND STRUCTURE

UNIT STRUCTURE

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6.1 LEARNING OBJECTIVES

After going through this unit, you will able to-

- understand the definition of normal and subnormal series in group theory.
- recognize the importance of normal and subnormal series in analyzing the structure of groups.
- apply normal and subnormal series to solve problems related to group theory.
- understand the statement of the Jordan-Holder Theorem and its significance in group theory.
- apply the Jordan-Holder Theorem to analyze the structure of finite groups.
- recognize the importance of Schreier's theorem in constructing and analyzing group presentations.
- apply knowledge of solvable groups to solve problems and prove theorems in group theory.
- understand the definition of the commutator subgroup (or commutator group) of a group.
- investigate the connection between commutators and group

homomorphisms, as well as their applicattions in various algebraic structures.

6.2 INTRODUCTION

In this unit, two more important series of a group are introduced. These will be useful in the insolvability of the Qunitic. Here two main theorems namely Schreier Refinement Theorem, and Jordan-H older Theorem will be stated and proved. This result illustrates and usefulness of a study of finite simple groups.

In addition, we shall define and discuss about Commutator group, Solvable groups and Nilpotent groups.

6.3 NORMAL AND SUBNORMAL SERIES, COMPOSITE SERIES

Definition : Let G be a group. A subnormal series of subgroups of G is a sequence $G_0 = \{e\} < G_1 < G_2 < ..... < G_n = G$ such that $G_{i-1} \triangleleft G_i$ for i = 1, ..., n.

A normal series is a subnormal series such that each G_i is normal in G. The number n is called the length of the (sub) normal series.

A (sub) normal series is called proper if all the quotient groups G_i/G_{i-1} are nontrivial. These quotient groups are referred to as successive quotients of the (sub) normal series.

We say that two (sub) normal series are equivalent if they have the same length and there is a permutation $\pi \in S_n$ such that the groups G_i/G_{i-1} and $G_{\pi(i)}/G_{\pi(i)-1}$ are isomorphic for all i = 1, 2, ...n

A refinement of a (sub) normal series $G_0 = \{e\} < G_1 < G_2 < ..... < G_n = G$ is a (sub) normal series $G'_0 = \{e\} < G'_1 < G'_2 < ..... < G'_m = G$ such that there is a strictly increasing function.

$f : \{1, ..., n\} \to \{1, 2, ...m\}$ such that $G_i = G'_{f(i)}$ for all i.

Note that every normal series is a subnormal series, but a subnormal series may not be normal.

Example : Consider the alternating group A_4, which consists of every permutations of four elements. Let's construct a subnormal series for A_4.

$$\{e\} \triangleleft \langle (1\ 2), (3\ 4) \rangle \triangleleft \langle (1\ 2\ 3), (1\ 3\ 2) \rangle \triangleleft A_4$$

Here $\langle (1\ 2), (3\ 4) \rangle$ is the subgroup generated by the permutation (1 2) and (3 4), and $\langle (1\ 2\ 3), (1\ 3\ 2) \rangle$ is the subgroup generated by the permutations (1 2 3) and (1 3 2).

This is a subnormal series because each subgroup is subnormal in the next one. However, this series is not a normal series because the subgroups are not necessarily normal in the whole group.

In fact, $\langle (1\ 2\ 3) \rangle, (1\ 3\ 2)$ is not a normal subgroup of A_4.

So, in this example, we see that every normal series is a subnormal series, but the reverse is not true.

Definition : $N \neq G$ is a maximal normal subgroup of G if there is no normal subgroup $M \neq G$ with N/M and $N \neq M$.

Theorem

a) Every finite group G has a composition series.

b) Every refinement of a solvable series is a solvable series.

c) A subnormal series is a composition series if and only if it has no proper refinements.

Proof : (a) We will prove this by induction on the order of the group.

If |G| = 1 then G is the trivial group, and the series {e} = G is a composition series.

Assume that every group of order less than |G| has a composition series.

Now, consider a finite group G of order n.

Case I : If G is simple, then the series {e} = G is a composition series. Nothing to prove.

Case II : If G is not simple, then there exists a nontrivial normal subgroup N of G.

Consider the quotient group G/N, which has order strictly less than |G|.

By the induction hypothesis, G/N has a composition series :

$\{e\} \lhd H_0 / N \lhd H_1 / N \lhd ... \lhd H_m / N = G / N$

Now, if this series back to G by taking the preimages :

$\{e\} \lhd H_0 \lhd H_1 \lhd ... \lhd H_m = G$

We know that each quotient group H_{i+1}/H_i is isomorphic to (H_{i+1}/H_i) / (H_i / N) and therefore, is a quotient of G/N. Since G/N has no nontrivial normal subgroups, each H_{i+1}/H_i is a simple group.

Additionally, each H_i is normal in G because it is the preimage of a normal subgroup in G/N.

Therefore, we have constructed a composition series for G, completing the induction step. This esatablishes that every finite group has a composition series.

(b) Let $H_0 \le H_1 \le H_2 \le ..... \le H_m$ be a solvable series, where each quotient H_{i+1}/H_i is abelian.

Also, let $K_0 \le K_1 \le K_2 \le ..... \le K_n$ be a refinement of this series.

We need to show that each quotient K_{i+1}/K_i is abelian.

For each I, there exists j(i) such that $K_i / H_{j(i)}$ .Since $H_{j(i+1)} / H_{j(i)}$ is abelian (by the definition of a solvable series), it follows K_{i+1} / K_i is a quotient of an abelian group and, therefore, is itself abelian.

(c) Assume that a subnormal series $\{e\} = \lhd G_0 \lhd G_1 \lhd G_2 \lhd G_3 .... \lhd G_n = G$ is a composition series.

We want to show that it has no proper refinements.

Suppose, for the sake of contradiction, that there exists a proper refinement of this composition series.

$\{e\} = H_0 \le H_1 \le H_2 \le H_3 ..... \le H_m$

Now, since $G_0 \lhd G_1 \lhd G_2 \lhd G_3 ..... \lhd G_n$ is a composition series, each quotient is a simple group.

As $H_0 \le H_1 \le H_2 \le H_3 ..... \le H_m$ is a refinement, each H_i must contain some G_i as a normal subgroup.

This implies that the quotient H_{i+1} / H_i contains the quotient G_{j+1} / G_j as a normal subgroup.

However, since G_{j+1} / G_j is simple, the only normal subgroups of H_{i+1} / H_i are and H_{i+1}/H_i {e}.

This means that H_{i+1}/H_i is isomorphic to G_{j+1}/G_j violating the simplicity of G_{j+1}/G_j.

This contradiction implies that our assumption of the existence of a proper refinement was incorrect.

Conversely, assume that a subnormal series has no proper refinments. We want to show that it is a composition series.

Let $\{e\} = \triangleleft G_0 \triangleleft G_1 \triangleleft G_2 \triangleleft G_3 ..... \triangleleft G_n = G$ be a subnormal series with no proper refinements.

To prove that it is a composition series, we need to show that each quotient G_{i+1}/G_i is a simple group.

Suppose, for the sake of contradiction, that there exists an i such that G_{i+1}/G_i is not simple.

This means there is a notrivial normal subgroup N of G_{i+1}/G_i.

Consider the preimage of N in G_{i+1} denoted as N' .i.e. $N' = \pi^{-1}(N)$

where, $\pi : G_{i+1} \to G_{i+1}/G_i$ is the natural projection map.

Since N is normal in G_{i+1}/G_i and N/ is normal in G_{i+1}

Now, consider the subnormal series $\{e\} = G_0 \triangleleft G_1 \triangleleft G_2 \triangleleft ... G_i \triangleleft N \triangleleft G_{i+1} ..... \triangleleft G_n = G$ where N/ is inserted between G_i and G_{i+1}

This series is a proper refinement of the original series, which contradicts our assumption that the original series has no proper refinements.

Therefore, our assumption that there exists an i such that G_{i+1}/G_i is not simple must be false. Hence, each quotient each quotient G_{i+1}/G_i is a simple group.

i.e. If a subnormal series has no proper refinements, then it is a composition series.

Note : Due to the ability to refine a subnormal series within a given group, it is evident that multiple subnormal (or solvable) series may exist for a single group. Similarly, diverse composition series can also arise. Nonetheless, within this variability lies a form of uniqueness characterized by equivalence, as elucidated in the subsequent definition.

Definition : Two subnormal series S and T of a group G are quivalent if there is a one-to-one correspondence between the nontrivial

factors of S and the nontrivial factors of T such that corresponding factors are isomorphic groups.

Example 6.3.7 : The two series $\{\bar{0}\} < \langle\bar{5}\rangle < Z_{15}$ and $\{\bar{0}\} < \langle\bar{3}\rangle < Z_{15}$ are equivalent normal series since the set of factor groups for $\{\bar{0}\} < \langle\bar{5}\rangle < Z_{15}$ is $\left\{ Z_{15}/\langle\bar{5}\rangle \cong Z_5, \langle\bar{5}\rangle/\langle\bar{0}\rangle \cong Z_3 \right\}$ and the set of factor groups for $\{\bar{0}\} < \langle\bar{3}\rangle < Z_{15}$ is $\left\{ Z_{15}/\langle\bar{5}\rangle \cong Z_5, \langle\bar{5}\rangle/\langle\bar{0}\rangle \cong Z_5 \right\}$

Note : Two subnormal series of a given group do not have to have the same number of terms in order to be equivalent, but they do have to have the same ‘‘length’’ (that is, number of nontrivial factors).

6.4 ZASSENHAUS LEMMA, SEHREIER'S REFINEMENT THEOREM, JORDAN HOLDER THEOREM

Lemma 6.4.1 : If S is a composition series of a group G, then any refinement of S is equvalent to S.

Proof : Let S = $\{e\} = H_0 \trianglerighteq H_1 \trianglerighteq ..... \trianglerighteq H_n = G$ be a composition series of G.

Consider a refinement $S' = \{e\} = K_0 \trianglerighteq K_1 \trianglerighteq ..... \trianglerighteq K_m = G$ of S.

Since S/ is a refinement of S for each i = 0, 1, 2, 3, 4,..... m, there exists j such that K_i is a subgroup of H_j.

Now, let's define a map ϕ : {0, 1, 2, 3, ...n} $\rightarrow$ {0, 1, 2, 3,.....m} as follows :

ϕ (i) = the smallest index j such that $K_i \subseteq H_j$

This map is well-defined because for each i, there exists at least one j such that $K_i \subseteq H_j$ due to the fact that S/ is a refinement of S.

We claim that ϕ is a bijection.

(a) Injectivity : Suppose ϕ (i) = ϕ (t), for some i and t. Then $K_i \subseteq H_{\phi(j)} = H_{\phi(t)}$ and $K_t \subseteq H_{\phi(t)}$. Since K_i and K_t are subgroups of G in the refinement series S/ and Ht is a part of the composition series S, the only possibility is that . Hence ϕ is injective.

(b) Surjectivity : For each j, there exists an i in such that ϕ (i) = j,

since for each j, there exists at least one i such that $K_i \subseteq H_j$ (by definition of ϕ). Thus ϕ is surjective.

Since ϕ is both injective and surjective, it is a bijective.

Moreover, $K_{\phi(i)}$ for all i = 0, 1, 2, 3, ...n

because ϕ (i) is defined as the smallest index j such that $K_i \subseteq H_j$

Therefore, S/ and S are equivalent series, which completes the proof of the lemma.

Note : The following result, the Zassenhaus Lemma or Butterfly Lemma is important in the role it plays in the proof of Schrier's Theorem and ultimately in the proof of the Jordan-H older Theorem.

Lemma 6.4.2 : Zassenhaus's Lemma/The Butterfly Lemma.

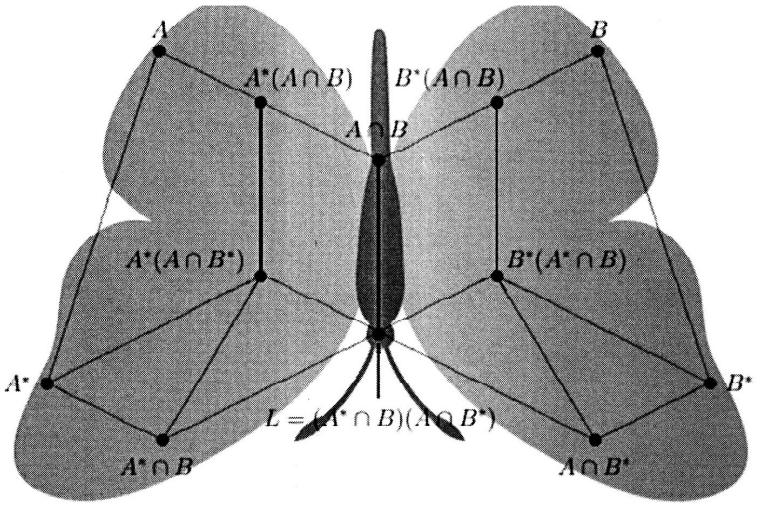
Let A^*, A, B^*, B be subgroups of a group G such that A^* is normal in A and B^* is normal in B.

Then

(i) A^* (A $\cap$ B^*) is a normal subgroup A^* (A$\cap$ B)

(ii) B^* (A$\cap$ B) is a normal subgroup of B^* (A$\cap$ B).

(iii) $\dfrac{A^*(A\cap B)}{A^*(A\cap B^*)} \cong \dfrac{B^*(A\cap B)}{B^*(A^*\cap B)}$



We prove now the following important theorem.

Theorem 6.4.2: (Schreier Refinement Theorem)

Any two (sub) normal series in G have equivalent refinements. If the series are proper; the refinements can be chosen proper as well.

Proof : Note first that each integer k can be uniquely written in the form k = (m + 1) i + j, where i is and integer and j is an integer in some range {0,....m}.

If $0 \leq k < (m+1)(n+1)$ then $0 \leq i \leq n$. Similarly, each integer $0 \leq k < (m+1)(n+1)$ can be uniquely written in the form $(n+1)j+i$, where $0 \leq i \leq n$ and $0 \leq j \leq m$.

For $1 \leq k < (m+1)(n+1)$ write $k = (m+1)i+j$ as above and set $f(k) = (n+1)j + i$. It is clear that f is a bijection of the set {1, ..., mn + m + n}.

Let $G_0 = \{e\} < G_1 < G_2 < .... < G_n = G$ and

$G'_0 = \{e\} < G'_1 < G'_2 < .... < G'_m = G$ (sub) normal series.

Now, we define $H_{(m+1)i+j} = \left(G_i \cap G'_j\right) G_{i-1}$ for i = 0, ...n and j = 0, ..., m (Note that groups with negative indices are, by definition, trivial).

Similarly, define $H_{(n+1)j+i} = \left(G_j \cap G_i\right) G'_{j-1}$ for i = 0, ...n and j = 0, ...m.

Clearly, $H_0 = \{e\} < H_1 < H_2 < .... < H_{mn+m+} = G$ and

$H'_0 = \{e\} < H'_1 < H'_2 < .... < H'_{mn+m+n} = G$ are (sub) normal series which are refinements of $G_0 = \{e\} < G_1 < G_2 < .... < G_n = G$ and $G'_0 = \{e\} < G'_1 < G'_2 < .... < G'_m = G$ respectively.

To check it, note that $H_{(m+1)i+m=G_i}$ and $H'_{(n+1)j+n} = G'_j$.

Now, we claim that these series are equivalent.

More precisely, the quotient groups H_k / H_{k-1} and $H'_{f(k)} / H'_{f(k-1)}$ are isomorphic for k = 1, ..., (mn + m + n)

Certainly, let k = (m + 1) i + j

If $j \neq 0$, then

$$H_k / H_{k-1} = \left(G_i \cap G'_j\right) / \left(G_i \cap G'_{j-1}\right) G_{i-1}$$

We apply the second homomorphism theorem to the group $A = G_i$, its subgroups $M = \left(G_i \cap G'_{j-1}\right) / \left(G_i \cap G'_j\right)$ and its normal subgroup $G_{i-1} = B$.

Recall that the second homomorphism theorem says that NB/MB and N/M ($N \cap B$) are isomorphic.

In our case this means that

$$H_k / H_{k-1} \approx \left(G_i \cap G'_j\right) / \left(G_i \cap G'_{j-1}\right)\left(G_{j-1} \cap G'_i\right)$$

If j = 0 the above formula still holds since in this case both sides are trivial groups.

For, f(k) = (n + 1) j + i and similarly we can show exactly that

$$H'_{f(k)} / H'_{f(k-1)} \approx \left(G'_j \cap G_i\right) / \left(G'_j \cap G_{i-1}\right)\left(G'_{j-1} \cap G_i\right)$$

Since both $G'_j \cap G_{i-1}$ and $G'_{j-1} \cap G_i$ are normal subgroups of $G'_j \cap G_i = G_i \cap G'_j$,

We have $\left(G'_j \cap G_{i-1}\right)\left(G'_{j-1} \cap G_i\right) = \left(G_i \cap G'_{j-1}\right)\left(G_{i-1} \cap G'_j\right)$. It follows that the groups H_k / H_{k-1} and $H'_{f(k)} / H'_{f(k-1)}$ are indeed isomorphic.

This shows that $H_0 = \{e\} < H_1 < H_2 < .... < H_{mn+m+n} = G$ and

$H'_0 = \{e\} < H'_1 < H'_2 < .... < H'_{mn+m+n} = G$ are (sub)normal series which are refinements of $G_0 = \{e\} < G_1 < G_2 < .... < G_n = G$ and $G'_0 = \{e\} < G'_1 < G'_2 < .... < G'_m = G$ respectively.

Theorem : Jordan-H" older Theorem

Any two composition series of a group G are equivalent. [Therefore, every group having a composition series determines a unique list of simple groups.]

Proof : Let $\{e\} = G_0 \trianglerighteq G_1 \trianglerighteq ..... \trianglerighteq G_n = G$ and $\{e\} = H_0 \trianglerighteq H_1 \trianglerighteq ..... \trianglerighteq H_m = G$ be two composition series of the group G.

Since both are composition series, each factor group G_{i+1} / G_i and H_{j+1} / H_j is simple.

Now, we want to show that any two composition series of G are equivalent.

First, note that the since both series are composition series, they have the same number of factors, i.e. m = n. Denote this common value by k. Hence, both G and H have the same number of subgroups in their series.

Since each factor group G_{i+1} / G_i and H_{j+1}/H_j is simple, there exists an isomorphism between them.

We can reorder the indices such that the factor groups are pairwise isomorphic. That is if we write G_i / G_{i-1} and H_j / H_{j-1}, where i and j values lies in between I to k. Then there exists a permutation σ the indices such that G_i / G_{i-1} is isomorphic to $H_{\sigma(i)} / H_{\sigma(i)-1}$, for all i = 1, 2, 3, ...k

Therefore, the two composition series of G are equivalent.

Thus, we can conclude that every group G having a composition series determines a unique list of simple groups (up to isomorphism)

6.5 COMMUTATOR GROUP, SOVABLE GROUP, NILOPTENT GROUPS

In this section, we define, commutator group, solvable group and nilpotent group and we discuss few properties on it.

Definition : (Center of a group) The center of a group G is denoted by Z(G) and is define as

Z (G) = {a $\in$ G | ax = xa for all x $\in$ G}.

Clearly, Z(G) is an abelian subgroup of G and we can easily proof that Z(G) is a normal subgroup of G.

Now if we take the canonical epimorphism: $G \to {}^{G}/Z(G)$ since $Z^{(G)}/Z(G) \rhd {}^{G}/Z(G)$, so we have

$$\pi^{-1}\left(Z^{(G)}/Z(G)\right) \lhd G$$

Define Z_1 (G) = Z(G) and $Z_i(G) = \pi_i^{-1}(z\left({}^{G}/Z_{i-1}(G)\right))$ fir i > 1

where $\pi_i : G \to {}^{G}/Z_{i-1}(G)$

We have $Z_i(G) \lhd G$ for all i.

Definition : (Upper Central Series) The upper central series of a group G is a sequence of normal subgroups of G :

$$\{e\} = Z_0(G) \subseteq Z_1(G) \subseteq Z_2(G).....$$

Definition : (Nilpotent Group) A group G is nilpotent if Z_i (G) = G, for some i.

If G is a nilpotent group then the nilpotentcy class of G is the smallest $n \geq 0$ such that Z_n (G) = G.

Proposition : 1) Every abelian group is nilpotent.

2) Every finite p-group is nilpotent

Proof : (i) Let G be an abelian group then obviusely Z_1(G) = G and hence G is nilpotent.

(ii) If G is a p-group then so is ${G}/{Z_i(G)}$ for every i.

Also, we know that if $G/Z_i(G)$ is non-trivial its center $\left(G/Z_i(G)\right)$ a non-trivial group. This means that if $Z_i(G) \neq G$ then $Z_i(G) \subseteq Z_{i+1}(G)$ and $Z_i(G) \neq Z_{i+1}(G)$.

But G is a finite group so there exists at least one n for which C_n (G) = G. Hence G is nilpotent.

Proposition 6.5.5 : Every nilpotent group is solvable.

Proof : If G is nilpotent group then the upper central series of G, $\{e\} = Z_0(G) \subseteq Z_1(G) \subseteq Z_2(G) ..... \subseteq Z_n(G) = G$ is a normal series.

Moreover, for every i we have, $Z_i(G)/Z_{i-1}(G) = Z\left(G/Z_{i-1}(G)\right)$

Hence the above series will be the solvable series for G. So G is solvable group. But the converse may not be true in general, i.e. Not every solvable group is nilpotent.

For example, consider the dihedral group D_8, also known as the symmetry group of a square. It consists of eight elements, and it is solvable but not nilpotent.

Definition : A central series of a group G is a normal series $\{e\} = G_0 \subseteq G_1 \subseteq G_2 \subseteq .... G_k = G$ such that $G_i \lhd G$ and $G_{i+1}/G_i \subseteq Z\left(G/G_i\right)$ for all i.

Proposition 6.5.7: If $\{e\} = G_0 \subseteq G_1 \subseteq G_2 \subseteq .... G_k = G$ is a central series of G then $G_i \subseteq Z(G_i)$

Proof : Let $x \in G_i$ We want to show that x commutes with every element of G_i.

By the definition of a central series, $G_i/G_{i-1} \subseteq Z\left(G/G_i - 1\right)$

Since $x \in G_i$ it follows that $xG_{i-1} \in G/G_{i-1}$

Now, consider any element $g \in G_i$. We need to show that xg = gx.

Since $G_i/G_{i-1} \subseteq Z\left(G/G_i - 1\right)$ so we have $(xg)G_{i-1} = (gx)G_{i-1}$ in G/G_{i-1}

This implies $xgG_{i-1} = gxG_{i-1}$ which means $(xg)(g^{-1}x^{-1}) \in G_i$

Since x and g is in G_i, also $g^{-1}x^{-1}$ is in G_i, so $(xg)(g^{-1}x^{-1}) \in G_i$

So, $(xg)(g^{-1}x^{-1}) \in G_{i-1}$ implies that xg = gx.

This shows that x commutes with every element of G_i and hence $x \in Z(G_i)$. Since this is true for arbitrary $x \in G_i$ we conclude that $G_i \subseteq Z(G_i)$

Thus if $\{e\} = G_0 \subseteq G_1 \subseteq G_2 \subseteq .... G_k = G$ is a central series of G then $G_i \subseteq Z(G_i)$.

Theorem : A group G is solvable if and only if it has solvable series.

Proof : Necessary Part (if Direction)

Let G be a solvable group. Then, there exists a subnormal series $\{e\} = G_0 \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq .... \trianglerighteq G_n = G$ such that each factor group G_{i+1}/G_i is abelian.

This chain of subgroups forms a solvable series for G.

Sufficient Part (Only if direction) :

Assume G has a solvable series, i.e., there exists a chain of subgroups

$\{e\} = G_0 \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq .... \trianglerighteq G_n = G$ such that each factor group G_{i+1}/G_i is abelian.

We want to show that G is solvable. We'll prove this by induction on , the length of the series.

Case I : For n = 1, $G_1 = G$ and $G_1/G_0 = G/\{e\}$ is trivially abelian since it's isomorphic of G, which is group. So for n = 1 it holds good.

Case II : Assume the statement holds for a solvable series of length (n – 1).

Now, consider a solvable series of length n :

$$\{e\} = G_0 \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq .... \trianglerighteq G_n = G$$

Since G_1/G_0 is abelian, it is solvable.

Now, consider the series :

$$\{e\} = G_0 \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq .... \trianglerighteq G_{n-1} = G_{n-1}$$

By the inductive hypothesis, G_{n-1} is solvable.

Now G/G_{n-1} is also solvable because it's a quotient of a solvable group. Thus, G is solvable.

Therefore, we have shown both directions, and hence proved that a group G is solvable if and only if it has a solvabe series.

Example 6.5.9 : The dihedral group D_n is solvable.

Let "a" be a generator of a subgroup of order n [say $a = \rho_1$, a "fundamental rotation" which generates the cyclic subgroup of D_n of order n consisting only of the rotations].

Then we have the series $D_n > \langle a \rangle > \{e\}$ and $D_n / \{a\} \cong Z_2$ is abelian, and $\langle a \rangle / \{e\} \cong Z_n$ is abelian. So, this is a solvable series and by Theorem 6.1.2, D_n is solvable.

Proposition 6.5.10 : A finite group G is solvable if and only if G has a composition series whose factors are cyclic and of prime order.

Proof : Necessary Condition (if direction)

Let us first assume that G is solvable.

By definition, there exists a chain of subgroups :

$$\{e\} = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq .... \trianglelefteq G_n = G$$

Such that each quotient group G_{i+1}/G_i is abelian, it is well-known (Jordan-Holder theorem) that G has a composition series.

Let's denote this series by $\{e\} = H_0 \trianglelefteq H_1 \trianglelefteq .... \trianglelefteq H_m = G$ where each factor H_{i+1}/H_i is abelian.

Now, since each quotient group G_{i+1}/G_i is abelian, each factor in the composition series Hi+1/Hi must also be abelian.

An abelian simple group is cyclic and of prime order. Thus, every factor in the composition series of G is cyclic and of prime order.

Sufficient Part (Only if part) :

Assume G has a composition series whose factors are cyclic and of prime order :

$$\{e\} = H_0 \trianglelefteq H_1 \trianglelefteq .... \trianglelefteq H_m = G$$

We want to show that G is solvable. We'll prove this by induction on m, the length of the composition series.

Base Case : For m = 1, G itself is cyclic and of prime order, hence abelian, so G is trivially solvable.

Inductive Case :

Assume the statement holds for a composition series of length (m – 1).

Now, consider a composition series of length m :

$\{e\} = H_0 \trianglelefteq H_1 \trianglelefteq .... \trianglelefteq H_m = G$

Since H_1/H_0 is cyclic and of prime order, it's abelian. Also H/H_1 is abelian because it's a quotient of a cyclic group, which is always abelian.

By the inductive hypothesis H/H_1 is solvable.

Since both H_1/H_0 and H/H_1 are solvable, their direct product is also solvable.

Using the fact that a group G is solvable if and only if it has a solvable series, we conclude that G is solvable.

Therefore, we have proved both directions, establishing the proposition.

Theorem 6.5.11 : A group G is nilpotent if it has a central series.

Proof : If G is nilpotent then G has a central series $\{e\} = G_0 \subseteq G_1 \subseteq G_2 \subseteq .... \subseteq G_k = G$

Conversely if $\{e\} = G_0 \subseteq G_1 \subseteq G_2 \subseteq .... \subseteq G_k = G$ is a central series, the by the Proposition 6.2.7, we have $G_k = G \subseteq C(G_k)$ and thus $G = C_n(G)$ and hence G is nilpotent.

Theorem 6.5.12 :

1. Every subgroup of a nilpotent group is nilpotent.
2. Every quotient group of a nilpotent group is nilpotent.

Proof : (1) Let G be a nilpotent group. We want to show that every subgroup H of G is also nilpotent.

Since G is nilpotent, there exists a series of subgroups as

$\{e\} = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq .... \trianglelefteq G_n = G$

Consider the subgroup H of G. By the Fundamental Theorem of Group Homomorphisms, for any normal subgroup N of G, the image of N under any homomorphisms is also a normal subgroup.

Since each G_i is a normal subgroup of G, then $H \cap G_i$ is a normal subgroup of H for each i = 0, 1, 2, 3, ...n.

Now, consider the series of subgroups

$$H \cap G_0 \trianglelefteq H \cap G_1 \trianglelefteq H \cap G_2 \trianglelefteq ..... \trianglelefteq H \cap G_n = H$$

Then each quotient $(H \cap G_{i+1}) / (H \cap G_i)$ is isomorphic to $(H \cap G_{i+1}) / (H \cap G_i \cap G_{i+1})$ which is a subgroup of G_{i+1} / G_i.

Since each G_{i+1} / G_i is a normal subgroup G / G_i then $(H \cap G_{i+1}) / (H \cap G_i)$ is a normal subgroup of $H / H \cap G_i$.

Hence H is also nilpotent. i.e. every subgroup of nilpotent group is nilpotent.

(ii) Let G is nilpotent, there exists a series of subgroups as

$$\{e\} = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq .... \trianglelefteq G_n = G$$

Consider an arbitrary quotient group G_i / G_j where $0 \le j < i \le n$

Since G_{j+1} is a normal subgroup of G_j and G_i is a normal subgroup of G their respective quotient groups G_j / G_{j+1} and G / G_i are well-defined.

But we know that from the homomorphism theorem states that if N is a normal subgroup of G, then there exists a natural isomorphism between G/N and $G_i / G_i \cap N$.

Applying this theorem, we have, $G_j / G_{j+1} \cong (G / G_i) / \left((G_i \cap G_j) / G_i\right)$.

Since G is nilpotent, every quotient G_{i+1} / G_i is cyclic, so G / G_i.

Also G_{j+1} is a normal subgroup of G_j, $G_{j+1} / G_i \cap G_j$ is normal subgroup of G / G_i.

Thus, G_j / G_{j+1} is a cyclic quotient of a cyclic group, which is cyclic.

Since this holds for an arbitrary quotient of G_i / G_j where $0 \le j < i \le n$

We conclude that every quotient group of G is cyclic.

Therefore, by definition, every quotient group of a nilpotent group is nilpotent.

Preposition 6.5.13 : If $G_1, G_2, ... G_k$ are nilpotent groups then the direct product

$G_1 \times G_2 \times .....\times G_k$ is also nilpotent.

Proof : Exercise

Theorem 6.5.14 : Let G be a finite group. The following conditions are equivalent.

1. G is nilpotent.
2. Every Sylow subgroup of G is a normal subgroup
3. G isomorphic to the direct product of its Sylow subgroups.

Left as an exercise



CHECK YOUR PROGRESS

Q.1. What is a solvable group? Give an example.

Q.2. Give an example that is solvable but not nilpotent.

Q.3. What is a commutator subgroup?



6.6 LET US SUM UP

- A *subnormal series* is a chain of subgroups where each subgroup is a normal subgroup of the next one in the sequence. If each subgroup is also normal in the entire group, it's called a *normal series*.
- The series length is determined by the number of steps, and it's called *proper* if each successive quotient is nontrivial.
- A *composition series* is a subnormal series where each quotient is a simple group (having no nontrivial normal subgroups).

- A *solvable series* is a subnormal series where each quotient is abelian.
- Theorems confirm that every finite group has a composition series, and refinements of solvable series are also solvable.
- Series can be refined by inserting additional subgroups while retaining the normality condition.
- Two series are equivalent if they can be matched such that their successive quotient groups are isomorphic.
- Center of a Group (Z(G)): The center is defined as elements in G that commute with all other elements. It is an abelian and normal subgroup.
- A group is nilpotent if its upper central series reaches G after a finite number of steps. Abelian groups and finite p-groups are examples of nilpotent groups.
- A group is solvable if there exists a series of subgroups where each quotient is abelian. While all nilpotent groups are solvable, not all solvable groups are nilpotent.
- A sequence of normal subgroups where each subgroup's quotient is included in the center of the quotient group. A group is nilpotent if it has a central series.
- Every subgroup and quotient group of a nilpotent group is nilpotent.
- The direct product of nilpotent groups is also nilpotent.
- A solvable series is a sequence of subgroups where each factor group is abelian.
- A composition series has factors that are cyclic of prime order, indicating solvability.



6.7 FURTHER READING

1) Fraleigh, Johu B: A First Course in Abetract Algebra (2003), Pearson Education (Singapure) Pvt. Ltd, Indian Branch, Delhi 110092.

2) Mapa, S.K.: Higher Algebra (Abstract and Linear) (1994), Asoke Prakasan, Calcutta 700007.

3) Khanna, V. K. and Bhambri, S. K.: A Course in Abstract Algebra (2006), Vikas Publishing House Pvt. Ltd. New Delhi 110014.

4) Surjeet Singh, and Q. Zameeruddin: Modern Algebra, 7th Revised Edition, VIKAS PUBLISHING HOUSE PVT LTD.



6.8 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q.No.1: A solvable group is a group with a subnormal series where each successive quotient is abelian.

Example : The dihedral group D_n is solvable because it has a series with abelian quotients.

Ans to Q.No.2: The dihedral group D_8 (symmetry group of a square) is solvable but not nilpotent.

Ans to Q.No.3: The commutator subgroup (or derived group) of a group G, denoted by G' is the subgroup generated by all commutators in G where a commutator is an element of the form

$$[x, y] = x^{-1}y^{-1}xy$$



6.9 MODEL QUESTIONS

Q 1: Explain the concept of a subnormal series in group theory. How does it differ from a normal series? Give an example of a group with a subnormal series.

Q 2: Define a composition series of a group. How does it relate to normal and subnormal series?

Q 3: Can a group have multiple composition series? Justify your answer with an example or counter example.

Q 4: Discuss the process of refining a series. How is it related to the concepts of normal, subnormal, and composition series?

Q 5: Provide examples of groups that are commutative but not nilpotent. Explain why they exhibit this property.

Q 6: Investigate the connection between the commutator subgroup, solvability, and nilpotency. How do these concepts interrelate in the context of group theory?

Q 7: Define a nilpotent group. How does it differ from a solvable group?

Q 8: Prove that any solvable group with a composition series is a finite group.

Q 9: Prove that any commutative group is nilpotent.

Q 10: Prove that every group of ofrer p^2q, p, q are prime number, is solvable.

Q 11: Let G = H × K, then show that G is solvable if and only if H and K are solvable.

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