UNIT 7 : CLASS EQUATION AND CAUCHY'S THEOREM

UNIT STRUCTURE

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7.1 LEARNING OBJECTIVES

After going through this unit, you will able to-

- define the class equation of a group and understand its components
- learn how to calculate the class equation for specific groups.
- explore the relationship between the class equation and the structure of a group
- understand the proof of Cauchy's theorem and its significance in group theory
- explore applications of Cauchy's theorem in various contexts, such as in the study of solvable groups and group actions.

7.2 INTRODUCTION

The class equation of a finite group is a useful tool in group theory that relates the order of the group to the sizes of its conjugacy classes. It is particularly important in understanding the structure of finite groups.

In this unit we shall discuss about class equation, Cauchy's theorem and Sylow p-subgroups and its applications.

7.3 CLASS EQUATION

Definition : Let G be a group and a,b $\in$ G, then a is said to be conjugate of b in G if there exists c $\in$ G such that a = c^{-1}bc.

Symbol a ~ b will denote a is a conjugate of b.

Lemma 7.3.1: The relation of conjugacy on a group G is an equivalence relation.

Proof : (i) Reflexive : Since a = a^{-1}aa, so a ~ a

(ii) Symmetry. We have a ~ b $\Rightarrow$ a = x^{-1}bx, for some x $\in$ G

$\Rightarrow b = x\,a\,x^{-1} = y^{-1}ay$ where $y = x^{-1}$

$\Rightarrow$ b ~ a

(iii) Transitivity : If a ~ b and b ~ c, then there exists x and y in G such that a = x^{-1}bx and b = y^{-1}cy

$$\Rightarrow a = x^{-1}(y^{-1}cy)x$$
$$\Rightarrow a = (x^{-1}y^{-1})\,c\,(yx)$$
$\Rightarrow$ a ~ c

That is, we can divide a group into equivalence classes under the relation of conjugacy. Each such equivalence class is called a conjugate class. For any a $\in$ G, C(a) denotes the conjugate class to which a belongs.

i.e C (a) = {b $\in$ G : a ~ b}. Thus C(a) consists of all elements of the types x^{-1}ax. Hence, a $\in$ Z (G), center of G, if and only if C (a) = { a }.

Let G be a finite group. Let C (a_1), C(a_2).....C (a_t) be the totality of all conjugate classes of G. Let $C_{a_1}, C_{a_2}, ...C_{a_t}$, be their respective orders. Since they are pairwise disjoint and their union is G and hence, we have, $o(G) = \sum_{i=1}^{t} C_{a_t}$, where the sum runs over elements a, taken one each from each conjugate class.

Definition : (Centralizer or Normalizer of an element)

For any element a of a group G, the set N(a) = {x $\in$ G : ax = xa} is called the Normalizer or Centralizer of a in G.

One can see clearly from the definition of centre of a group that a is in the centre of G if and only if N(a) G.

Definition : The conjugacy class of an element g in a group G is the set of all elements of G that are conjugate to g. Formally, the conjugacy

class of g is denoted by *cl* (g) and defined as *cl* (g) = $\{x\,g\,x^{-1} : x \in G\}$

Where, g is an element of the group G. xgx^{-1} denotes the conjugate of g by x.

Thus, two elements g and h are in the same conjugacy class if there exists an element x in the group G such that $xgx^{-1} = h$.

Note : Conjugacy classes provide important information about the structure of a group.

(i) Every element of G belongs to exactly one conjugacy class.

(ii) An element g is in the center of G if and only if its conjugacy class consists only of g itself.

(iii) A subgroup of G is normal if and only if it is a union of whole conjugacy classes.

(iv) The size of a conjugacy class *cl*(g) is given by the index of the centralizer of g in G, denoted by $[G : C_G (g)]$.

Definition : (Class equation) Let G be a finite group. The equation $o(G) = o[N(a)] + \sum_a \frac{o(G)}{o[N(a)]}$, where the sum runs over elements a, taken one from each of those distinct conjugates classes which contain more than one element, is called the class equation of the group G.

The class equation has played great role in the development of the structure theory of non-commulative finite groups.

Thus, the class equation essentially states that the order of the group is equal to the order of its center plus the sum of the indices of the centralizers of representatives of each conjugacy class.

This equation is particularly useful because it provides information about the structure of the group.

For instance, if Z (G) is non-trivial, then the group is not simple.

Theorem 7.3.1: The number of terms in the class equation equals the number of distinct normal subgroups of G plus one.

Left as an exercise.

Lemma 7.3.2 : For any $x \in G$, N(a) is a subgroup of G.

Proof : If e is the identity of g then ea = ae gives $x \in N(a)$. So that $N(a) \neq \phi$.

Let x, y $\in$ N (a)

Then (xy) a = x (ya) = x (ay) = (xa) y = (ax) y = a (xy)

Thus xy $\in$ N (a)

Again, ax = xa $x^{-1}a = ax^{-1} \in$ N (a)

Hence, N(a) is a subgroup of G.

Theorem 7.3.2: If G is a finite group and a $\in$ G, then

$$C_a = \frac{o(G)}{o[N(a)]}$$

Proof : Let o(G) = n, If N(a) has t-distinct right cosets, namely

$N(a)x_1, N(a)x_2, N(a)x_3, ...N(a)x_t$ then we have

$$t = \frac{o(G)}{o[N(a)]}$$

Now, for $1 \leq i, i, j \leq t$,

We have $x_i^{-1}ax_i = x_i^{-1}ax_j$

$$\Rightarrow x_i x_j^{-1}ax_j x_i^{-1} = a$$
$$\Rightarrow x_i x_j^{-1}a(x_i x_j^{-1})^{-1} = a$$
$$\Rightarrow x_i x_j^{-1} \in N(a)$$
$$\Rightarrow N(a)x_i = N(a)x_j$$
$$\Rightarrow i = j$$

Since $N(a)x_i's$ are all distinct.

Hence, $x_i^{-1}ax_i = x_j^{-1}ax_j \Rightarrow i = j$

So, $x_1^{-1}ax_1, x_2^{-1}ax_2, ax_3, ...x_t^{-1}ax_t$ are all disjoint conjugate of a.

If we show that these are the only conjugates of a then if follows that C(a) contains only t-elements.

That

$$C_a = t = \frac{o(G)}{o[N(a)]}$$

Consider for some x $\in$ G, x^{-1} ax = b.

But $G = U_{i=1}^{t} N(a)x_i$, so x = cx_i for some c $\in$ N(a) and some positive integer i.

Therefore, $x^{-1}ax = (cx_i)^{-1}a(cx_i) = x_i^{-1}(c^{-1}ac)x_i = x_i^{-1}ax_i$ (since

$c^{-1}ac = a$)

And so we can say that any conjugate b of a is equal to one of $x^{-1}ax_i$. Thus a has only t conjugates.

Hence the theorem is proved.

Exercise 7.3.8 : For any finite group G, $o(G) = \sum_a \frac{o(G)}{o[N(a)]}$ where the sum runs over elements a, taken one each from each conjugate class.

Exercise 7.3.9 : Let G be a finite group and Z(G), its centre, then $o(G) = o[N(a)] + \sum_a \frac{o(G)}{o[N(a)]}$, where the sum runs over elements a, taken one from each of those distinct conjugate classes which contains more that one element.

Theorem 7.9.3 : If o(G) = p^n, where p is a prime number, then Z (G) $\neq$ (e)

Proof : Let o [Z (G) = z. By the class equation we have

$$o(G) = o[N(a)] + \sum_a \frac{o(G)}{o[N(a)]}$$

So, $p^n = z + \sum_a \frac{o(G)}{o[N(a)]}$ ..... (1)

Where the sum runs over elements a, taken one from each of those distinct conjugates classes which contain more than one element.

Now for each $a \notin Z(G), p^n = o(G) > o[N(a)]$ and $o[N(a)] | o(G)$ which give for some $1 \le n_a < n$

$$\Rightarrow p | \frac{o(G)}{o[N(a)]}$$

Further, we have p | p^n.

Hence from (1) we get p | z

Which prove that $p | o[N(a)] \Rightarrow o[N(a)] > 1 \Rightarrow Z(G) \neq (e)$.

Corollary 7.3.11 : If o(G) = p^2, where p is a prime number, then G is Abelian.

Definition 7.3.12 : A group G is called a finite Conjugate group (FC-group) if and only if G be a finite group each x $\in$ G has only a finite number of

conjugates.

Exercise 7.3.13 : Show that (1 2 3 4 5), (1 2 3 5 4) are conjugate in S_5, but not in A_5

Solution : Two permutations are conjugate in a group G if they can be written in the form $g^{-1}\sigma g$ for some g in the group G.

In S_5, any permutation can be written in eyele notation. So, let's express the given permutations in eyele notation:

(1 2 3 4 5) can be expressed as the cycle (1 2 3 4 5)

And (1 2 3 5 4) can be expressed as the eyele (1 2 3) (4 5)

Now, we need to find a permutation g such that g^{-1} (1 2 3 4 5) g = (1 2 3) (4 5)

For this, let's try conjugating (1 2 3 4 5) by various permutations:

If we try conjugating by (1 2 3),

we get (1 2 3) (1 2 3 4 5) $(1\ 2\ 3)^{-1}$ = (1 2 3) (4 5).

Hence, So, indeed, in S_5. (1 2 3 4 5) and (1 2 3 5 4) are conjugate.

Now let's consider A_5, the alternating group of degree 5.

In A_5, every permutation is a product of an even number of transpositions.

Now, if two permutations are conjugate in A_5 then they must have the same cycle structure. However, (1 2 3 4 5) and (1 2 3 5 4) have different cycle structures, as one is a 5-cycle and the other is a 3-cycle and a 2-cycle.

Since the cycle structures are different, they cannot be conjugate in A_5.

Thus, we've shown that (1 2 3 4 5), (1 2 3 5 4) are conjugate in S_5 but not in A_5.

7.4 CAUCHY'S THEOREM

Cauchy's theorem is a fundamental result in group theory.

One important application of Cauchy's theorem is in the classification of finite abelian groups. Specifically, it helps in understanding the structure of finite abelian groups by providing information about their cyclic subgroups.

Another application of Cauchy's theorem is in Sylow theory. The Sylow theorems are a set of results that provide information about the number of certain kinds of subgroups in a finite group. Cauchy's theorem is used in the proof of one of the Sylow theorems, specifically in proving the existence of Sylow subgroups.

Cauchy's theorem is also used in various other areas of group theory and its applications, such as in the study of permutation groups, Galois theory, and in cryptography where understanding the structure of groups is essential for designing secure cryptographic systems.

Theorem 7.4.1: (Cauchy's Theorem for Abelian group)

Suppose G is a finite Abelian group and a positive integer p divides o(G), where p is a prime number. Then there exists $a(\neq e) \in G$ such that $a^p = e$.

Proof : Since p|o(G), so $o(G) = n_1 p$, for some $n_1 \geq 1$

We proceed by induction on n_1.

Let $n_1 = 1$, then o(G) = p. As every group of prime order is cyclic, so G is a cyclic group i.e. $\exists a \in G$ Gsuch that G = < a >, then o (a) = o(G) = p. Consequently $a \neq e$ and $a^p = e$. Hence the result holds for $n_1 = 1$.

To apply induction, let $n_1 > 1$ and assume that the result is true for every abelian group G′ for which o(G′) = mp where $m < n_1$ i.e. Suppose the result is true for all Abelian group G′ such that p|o (G′) and o(G′) < o(G).

Since $o(G) = n_1 p$ is not a prime number, G must contain a proper subgroup.

Suppose that G has a proper subgroup N such that p|o(N). As o(N) < o(G), by induction hypothesis, $\exists a \in N$ such that $a \neq e$ and $a^p = e$. This 'a' is also in G. So, the result holds if G has a proper subgroup N with p|o (N).

Now, suppose that p does not divide order of any proper subgroup of G. Let H be any proper subgroup of G.

Then $o(G) = o\left(G/H\right).o(H)$, gives p|o $\left(G/H\right)$.

since, p | o (H).

Furthe $o\left(G/H\right) < o(G)$ so by inductioon hypothesis G/H contains an

elements bH of order p.

Thus bH $\neq$ H and H = $(bH)^p$ = b^pH. This give $b^p \in H$. This further yield $\left(b^p\right)^{o(H)} = e$. Let $a = b^{o(H)}$, then $a^p = e$.

If a = e, then we get o (bH) | o(H) $\Rightarrow$ p | o(H) which leads to a contradiction. Hence a $(\neq e) \in G$ such that $a^p = e$.

Theorem 7.4.2 : If G is finite abelian group and a positive integer k divides o(G), then G contains a subgroup of order k.

Proof : We established the theorem by induction of o(G). If o(G) = 1, then $k \mid o(G) \Rightarrow k = 1$

In this case, G is its own subgroup of order k.

To apply induction on the order of the group, we assume that o(G) >1.

And assume that the theorem hold for all abelian group of order < o (G).

We can suppose that k > 1, because for k = 1, (e) is a subgroup of order 1.

Let p be a prime number such that p | k. Then p | o (G), and hence by Cauchy's theorem, there exists $a \in G$ such that o(a) = p. Let H = < a > be a cyclic subgroup generated by a then o(H) = p.

Now, G/H is an abelian group such that

$$o\left(G/H\right) = \frac{o(G)}{o(H)} = \frac{o(G)}{p} < o(G)$$

Since p | k, we have $k = k_1 p$ and $k_1 \mid o\left(G/H\right)$.

Hence by induction hypothesis, G/H has a subgroup say T, of order k_1. But every subgroup is of order K/H, where K is a subgroup of G containing H. So, in particular, $T = \left(K/H\right)$, where K is a subgroup of G containing H, then $o\left(K/H\right) o(H) = k_1 p = p$

This completes the proof.

Theorem 7.4.3: Any finite additive Abelian group is internal direct sum of its Sylow subgroup.

Proof: To prove that any finite additive Abelian group is the internal direct sum of its Sylow subgroups, we need to show two things:

(i) Every Sylow subgroup is a direct summand of the group.

(ii) The sum of all Sylow subgroups gives the original group.

First, we shall prove every Sylow subgroup is a direct summand of the group.

Let G be a finite additive Abelian group and P be a Sylow subgroup of G.

We want to show that there exists a subgroup H such that $G = P \oplus H$

Since G is Abelian, every subgroup of G is normal. Thus, P is a normal subgroup of G. By the Fundamental Theorem of Abelian Groups, any subgroup of an Abelian group is also Abelian. Therefore, P is Abelian.

Now, consider the quotient group, G / P,

Since P is a Sylow subgroup, order of G / P= O (G/P)= | G /P |, is coprime to the order of P.

This implies that G/P, is torsion-free [every element of the group, except for the identity element, has infinite order] as any torsion element [a torsion element is an element of a group that has finite order,] in G/P, would lead to an element of order dividing | P |. which contradicts Lagrange's Theorem.

Since G/P is torsion-free, it is isomorphic to a subgroup of Q, the additive group of rational numbers.

Let H be the preimage of G/P under the homomorphism and surjective (or canonical projection) $G \to$ G/P. Then, H is a subgroup of G containing P

By the Isomorphism Theorem, $G/H \cong G/P$.

Thus, every element in G can be uniquely expressed as the sum of an element in P and an element in H, showing that $G = P \oplus H$.

Now, we shall prove that the sum of all Sylow subgroups gives the original group.

Let G be a finite additive Abelian group and $P_1, P_2, ...P_k$ be the distinct Sylow subgroups of G.

We want to show that $G = P_1 \oplus P_2 \oplus ..... \oplus P_k$

By the previous step, each P_i is a direct summand of G.

We only need to show that this sum is direct.

Suppose g is an element in the intersection of P_i and P_j where $i \neq j$.

Then g has two representations : $g = p_i + p_j$ for some $p_i \in p_i, p_j \in p_j$

Since p_i and p_j are subgroups, $p_i + p_j$ must also belong to both p_i and p_j

Since p_i and p_j are distinct Sylow subgroups, $|p_i|$ and $|p_j|$, are co-prime, meaning $p_i \cap p_j = \{e\}$ by the second Sylow theorem.

Thus, g = e, showing that the sum is direct.

Therefore, $G = P_1 \oplus P_2 \oplus .....\oplus P_k$ as required.

Thus, any finite additive Ablian group is internal direct sum of its Sylow subgroup.

7.5 SYLOW P-SUBGROUP

Sylow p-subgroups play a central role in the study of finite groups, particularly due to Sylow's Theorem, which provides important information about their existence, conjugacy, and number within a group.

Understanding the properties and behaviour of Sylow p-subgroups is crucial for analysing the structure and properties of finite groups, and it has numerous applications in various areas of mathematics, including algebra, combinatorics, and number theory.

Definition 7.5.1: (p-subgroup)

Let p be a prime number. Then a subgroup H of a group G is called a p-subgroup if the order of each element of H is a power of p.

In, particular, if each element of G is of order, a power of some fixed prime number p, then G is called a p-group.

Definition 7.5.2: Let G be a group and p, a prime number such that $p^k \mid o(G)$ and $p^{k+1} \mid o(G)$. Then any group of order p^k is called Sylow's p-subgroup of G.

The above definition implies that all Sylow p-subgroup of G are of

same order. Since any p-subgroup is of order, a power of p, a prime number, we see that no p-subgroup H of G can contain properly a Sylow p-subgroup of G.

Example 7.5.3 : Let G = $\langle a \rangle$ be a cyclic group of order 8. Since, 8-23, a power of prime number, G is a p-group.

Example 7.5.4: Consider S_3, $o(S_3) = 6 = 3.2$, thus $3|0(S_3)$ but $3^2 \nmid o(S_3)$. So, any 3-subgroup of S_3 is of order 3.

Now, $A_3 = \{1, (123), (132)\}$ is the only subgroup of S_3 of order 3. Hence A_3 is the only Sylow 3-subgroup of order 3. Again, the highest power of 2 which divides $o(S_3)$, is 2 itself. In this way any Sylow 2-subgroup of S_3 is of order 2. The only subgroups of S_3 of order 2 are $H_1 = \{1, (12)\}$ and $H_2 = \{1, (13)\}$. Thus these three subgroups are the only Sylow subgroups of S_3.

7.6 APPLICATIONS OF SYLOW P- SUBGROUP

Exercise 7.6.1: If G is a finite group and p is a prime number dividing o(G), prove that a Sylow p-subgroup of G .

Solution : Suppose $o(G) = p^k q$ with (p, q) =1. Then, o(P) p^k. Let $g \in G$, gPg^{-1} is a subgroup of G. As $gpg^{-1} = gp'g^{-1} \Rightarrow p \Rightarrow = p'$, $p, p' \in P$, we get that

$$o(gPg^{-1}) = o(P) = p^k.$$

By the hypothesis, this means that $gPg^{-1} = P$, hence p is normal subgroup of G.

Conversely, let p be a normal Sylow p-subgroup of G and Q, any Sylow p -subgroup of G.

By Sylow's second theorem, Q and must be conjugate, i.e. $gpg^{-1} = Q$ for some $x \in G$. But p is a normal subgroup of G, so, $xPx^{-1} = P$.

Exercise 7.6.2 : Show that no subgroup of order 30 is simple.

Solution : Let G be a group of order of order 30. Since 30 = 2.3.5, so G has at least at Sylow 2-subgroup , a Sylow 3-subgroup, and a Sylow 5-subgroup.

Number of Sylow 5-subgroup of is 1 + 5k, such htat (1 + 5k) | 6 $\Rightarrow$ k = 0 or 1.

If k = 0, then Sylow 5-subgroup is a normal subgroup of G.

If k = 1, then number os Sylow 5-subgroups is 6

Let H_1, H_2, H_3, H_4, H_5, H_6 be the Sylow 5-subgroup of G.

Since $o(H_i \cap H_j) = 1$ for $i \neq j$ these 6 Sylow 5-subgroup contains 24 elements of order 5 and there is one element of order1. Thus only 5 remaining elements of G can be of order 2 or 3.

Now, the number of Sylow 3-subgroup of G is 1 + 3m such that 1 + 3m | 10.

Then, m = 0 or 3.

If m = 0, we have only one Sylow 3-subgroup of G which is the normal subgroup of G.

If m = 3, we have 10 Sylow 3-subgroups. But, these will contains 20 elements of order 10.

This is absurd in view of arguments given above. Hence G gas either a normal subgroup of order 5 or a normal subgroup of order 3. Thus G s not simple.

Exercise 7.6.3 : If a group G of order p^n, p is a prime number and n is an integer $n \geq 1$, has exactly one subgroup of order p^{n-1}, then prove that G is cyclic.

Solution : If a group o(G) = pn, p is a prime number and n is an integer $n \geq 1$.

Let H be the subgroup of G of order p^{n-1}.

Since $G \neq H$ there exists $a \in G$ but $a \notin H$

Now, $o(a) \mid p^n \Rightarrow o(a) = p^m$ where $o \leq m \leq n$

If ,$m \neq n$, the clyclic subgroup T of G generated by 'a' is of order p^m, $o \leq m \leq n-1$

So there exists a subgroup K of order p^{n-1} such that $K \subseteq T$.

Thus, by hypothesis, K = H and hence $\langle a \rangle = T \subseteq H$. But implies that $a \in H$, which leads to a contradiction.

Hence m = n and o (a) = p^n, consequently G is cyclic.



CHECK YOUR PROGRESS

Q.1: What is the class equation of a group, and how does it provide insights into the group's structure?

Q.2: Explain Sylow's First Theorem.

Q.3: Prove that the group of order 15 is cyclic.

Q.4: If a group *G* has order 36, prove that *G* has 1 or 4 Sylow 3-subgroups.

Q.5: Prove that a group of order 48 has a normal subgroup of order 8 or 16.

Q.6: What is Sylow p-subgroup?

Q.7: If G is a group of order p_n, where p is a prime, and G has exactly one subgroup of order p^{n-1}, prove that G is cyclic.



7.7 LET US SUM UP

- **Class Equations** : This defines conjugacy in groups and describes how to categorize elements into conjugate classes. It introduces the centralizer and the significance of conjugacy classes for understanding group structure, including the class equation itself, which connects the group's order to its center and centralizers of conjugacy classes.
- **Cauchy's Theorem** : The Cauchy's theorem asserts that if a prime divides the order of a finite group, then the group contains an element of that order. It has significant implications for classifying not only for finite abelian groups but also the foundation for the Sylow theorems, which analyze the existence and properties of Sylow subgroups, Sylow p-subgroups.
- Statement of Cauchy's theorem : Let G be a finite group. If a prime p|o(G), then G has an element of order p.
- Let G be a finite abelian group such that $|G| > 1$. Then G is simple if and only if G is a cyclic group of order p, for some prime p.

- Let p be a prime. A group G is called a p-group if the order of every element of G is p^n for some integer $n \geq 0$. A subgroup H of a group G is called a p-subgroup if H is a p-group.
- Let G be a finite non-trivial group. Then G is a p-group if and only if $|G| = p^n$ for some positive integer n.
- Let G be a finite p-group such that $|G| > 1$. Then $|Z(G)| > 1$.
- Let p be a prime number. Then every group of order p^2 is abelian.



7.8 FURTHER READING

1) Fraleigh, Johu B: A First Course in Abetract Algebra (2003), Pearson Education (Singapure) Pvt. Ltd, Indian Branch, Delhi 110092.
2) Mapa, S.K.: Higher Algebra (Abstract and Linear) (1994), Asoke Prakasan, Calcutta 700007.
3) Khanna, V. K. and Bhambri, S. K.: A Course in Abstract Algebra (2006), Vikas Publishing House Pvt. Ltd. New Delhi 110014.
4) Surjeet Singh, and Q. Zameeruddin: Modern Algebra, 7th Revised Edition, VIKAS PUBLISHING HOUSE PVT LTD.



7.9 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q.No.1: Class Equation : For a finite group G, the class equation is given by :

$$|G| = |Z(G)| + \Sigma |G : C_G(a)|$$

where $|Z(G)|$ is the order of the center of G,–$[G : C_G(a)]$ is the index of the centralizer $C_G(a)$ for representatives a from each conjugacy class that doesn't belong to the center.

Ans to Q.No.2: Sylow's First Theorm : If G is a finite group and p is a prime number such that p^k divides the order of G, then G has a subgroup of order p^k. This subgroup is called a Sylow p-subgroup.

Ans to Q.No.3: A group of order 15 has order $15 = 3 \times 5$, where 3 and 5 are primes. By Cauchy's theorem, there exist elements of order 3 and order 5 in the group. Let a be an element of order 3 and b an element of order 5. The group generated by a and b is cyclic with order 15, since the least common multiple (LCM) of 3 and 5 is 15. Hence, the group is cyclic.

Ans to Q.No.4: The order of the group is $|G| = 36 = 2^2 \times 3^2$. By Shylow's theorem, the number of Sylow 3-subgroups n_3 must divide $2^2 = 4$ and $n_3 \equiv (\mathrm{mod}\, 3)$. Hence, n_3 can be either 1 or 4. Therefore, G has either 1 or 4 Sylow 3-subgroups.

Ans to Q.No.5: A group of order 48 has $|G| = 2^4 \times 3$. By Sylow's theorem, the number of Sylow 2-subgroups must divide 3 and be congruent to 1 modulo 2, so there is either 1 or 3 Sylow 2-subgroups. If there is exactly 1 Sylow 2-subgroup, it is normal in G. Therefore, G must have a normal subgroup of order 8 or 16.

Ans to Q.No.6: A Sylow p-subgroup of a group G is a maximal subgroup whose order is a power of a prime p. Specifically, if $|G| = p^k.m$, where p does not divide m, then a Sylow p-subgroup has order p^k.

Ans to Q.No.7: Let H be the unique subgroup of order p^{n-1}. Since $G \neq H$, there exists an element $a \in G$ such that the order of a is p^n. Therefore, G is cyclic as it is generated by a.



7.10 MODEL QUESTIONS

Q 1: Prove that the group of order 15 and 35 are cyclic.

Q 2: If o(G) = 36, prove that G has a 1 or 4 Sylow 3-subgroups.

Q 3: Give an example of a group of order p^3, p is a prime number, which is not commutative.

Q 4: Prove that a group of order 48 must have a normal subgroup of order 8 or 16.

Q 5: Prove that a group G of order 28 has a normal subgroup of order 4, then G must be abelian.

Q 6: Show that a normal p-subgroup K of a finite group G is contained in every Sylow p-subgroup of G.

Q 7: Let G be an Abelian group and h and K are two finite subgroups of G of order m and n respectively. Prove that g has a subgroup of order LCM of m and n.

Q 8: Show that no group of order 108 is simple.

Q 9: If a group G of order p^n, p is a prime number and n is an integer $n \geq 1$, then prove that any subgroup of G of order p^{n-1}, is a normal subgroup of G.

Q 10: Prove that A_4 has 4 Sylow 3-subgroups and determine them.

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