UNIT 8 : GROUP ACTION AND SYLOW THEOREM

UNIT STRUCTURE

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8.1 LEARNING OBJECTIVES

After going through this unit, you will able to-

- understanding what group actions are and how groups can act on other sets.
- studying the properties of group actions, such as transitivity, freeness and faithfulness.
- exploring applications of group actions in various mathematical contexts, such as combinatorics, geometry and number theory.
- exploring applications of the Sylow theorems in group theory and related areas, such as cryptography and Galois theory.

8.2 INTRODUCTION

Symmetry is everywhere around us. It underlies beauty and structure. Its study is therefore important. The theory of group actions, which we will be concerned with in this unit, is indeed the study of symmetry.

Group actions provide a powerful framework for studying symmetries and structures of mathematical objects, including groups themselves. They have applications in various branches of mathematics, including algebra, combinatorics, geometry and number theory, making them a fundamental concept in modern mathematics.

The symmetric group S_n behaves, by its very definition, as permutations of the set {1, 2, ...n}. Dihedral groups D_n for $n \geq 3$, while interpreted geometrically as certain motions of the plane preserving a regular n-gon, can be considered as a group of permutations just of the n vertices; rigid motions of the vertices determine where the rest of the n-gon goes.

If we label the n vertices in a definite manner by the numbers from 1 to n then we can view D_n as a subgroup of S_n.

The basic idea in a group action is that the elements of a group are viewed as permutations of a set in such a way that composition of the corresponding permutations matches multiplication in the original group.

This section introduces the main definition of this unit, namely, what is means for a group to act on a set.

8.3 GROUP ACTION

Definition 8.3.1 : G-sets. Let G be a group and X a set. We say that G acts on X or that there is an action of G on X or simply that X is a G-set if there exists a function $G \times X \rightarrow X$ satisfying the following axioms. Denoting by g.x the image of (g; x) under this fuction (for g an element of G and x an element of X), the axioms are :

i) e.x = x for every x in X, where e denotes the identity element of the group G.

ii) g.(h.x) = (gh).x for every x in X, every g in G, and every h in G; here gh denotes the product in G of the elements g and h (in that order).

Terminology and notation : Let G be a group and X a G-set. We refer to the function $G \times X \rightarrow X$, defining the action as the ‘‘action map’’. It is often convenient to omit the ‘‘dot” in the notation g.x AND and write just gs instead.

Definition 8.3.2: (G-maps between G-sets) Let G be a group and let X and Y be G-sets. When should we consider X and Y to be ‘‘the same’’? Recall that we consider two groups to be the same when they are isomorphic. Do we have an analogous notion of ‘‘sameness’’ for G-sets?

In order to introduce such a notion, we first introduce the notion of a G-map. A map (or function) $\phi X \rightarrow Y$ between G-sets is said to be a G-

equivariant map or simply a G-map of g ($\phi(x) = \phi(gx)$ for all x in X and for all g in G. We sometimes use the phrase ϕ commutes with the action of G to mean that ϕ is a G-map. A composition of G-maps is a G-map, as can be readily checked.

A G-map that is bijective (as a map) is said to be a G-isomorphism.

We say that X is isomorphic (as a G-set) to Y if there exists a G-isomorphism from X to Y. For a G-isomorphism $\phi : X \to Y$, its inverse $\phi^{-1} : Y \to X$ is also a G-map, as can be readily checked. Thus the relation ''X is isomorphic to Y'' defines an equivalence relation on G-sets.

We consider two G-sets to be the same if they are isomorphic (as G-sets). This makes sense, for any aspect of a action of G on one of them is mirrored in the action of G on the other (via a G-isomorphism between them).

Definition 8.3.3: An action of a group G on a set X is the choice, for each $g \in G$, of a permutation $\pi_g : X \to X$ such that the following two conditions hold :

(i) π_e is the identiy : $\pi_e(x) = x$ for each $x \in X$

(ii) for every g_1 and g_2 in G, $\pi_{g_1}{}^{0}\pi_{g_2} = \pi_{g_1 g_2}$

Example 8.3.4 : Each group G acts on itself (X = G) by left multiplication functions.

That is, we set $\pi_g : G \to G$ by $\pi_g(h) = gh$ for all $g \in G$ and h ∈ G. Then the conditions for being a group action are eh = h for all h ∈ G and g_1 (g_2) h = (g_1g_2) h for all g_1, g_2, h ∈ G, which are both true since e is an identity and multiplication in G is associative.

Theorem 8.3.5: Let a group G act on the set X. If x ∈ X, g ∈ G, and y = g.x, then x = g^{-1}.y. If $x \neq x'$ and g.x ≠ g.x′.

Proof : From y = g.x we get, g^{-1}, y = g^{-1}. (g.x) = (g^{-1}g).x = ex = x.

To show $x \neq x' \Rightarrow$ g.x′, we show the contrapositive; if g.x = g.x′, then applying both side g^{-1} both sides gives g^{-1}. (g.x) = g^{-1} (g.x′)

$\Rightarrow$ (g^{-1}g) .x = (g^{-1}g).x′

$\Rightarrow$ x = x′.

Theorem 8.3.5 : Actions of the group G on the set X are the same as group homomorphism from G to Sym (X), the group of permutations of X.

Proof : Suppose we have an action of G on X. We view g.x as a function of x (with g fixed). That is, for each g $\in$ G,,

we have a function $\pi_g : X \to X$ defined by $\pi_g(X) = g.x.$

The axiom e.x = x says that π_e is the identity function on X. The axiom g_1 (g_2.x) = (g_1g_2).x says $\pi_{g_1}{}^{0}\pi_{g_2} = \pi_{g_1g_2}$ so compositino of functions on X corresponds to multiplication in G. Moreover, π_g is an invertible function since π_{g-1} is an inverse. The composition of π_g and π_{g-1} is π_e, which is the identity function on G. Therefore $\pi_g \in$ Sym (X) and g $\to \pi_g$ is a homomorphism G to Sym (X).

Conversely suppose that we have a homomorphism f : G $\to$ Sym (X). For each, $g \in G$, we have a permutation f(g) on X and f (g_1g_2) = f (g_1)$_0$f (g_2).

Now, setting g.x = f (g) x defines a group action of G on X. since the homomorphism properties of f yield the defining properties of a group action.

Example 8.3.6 : Let G be the group of Rubik's cube : all sequences of motions on the cube (Keeping center colors in fixed locations). This group acts on two different sets : the 12 edge cubelets and the 8 corner cubelets. Or we could let G act on the set of all 20 non-centerface cubelets together.

Example 8.3.7 : (The symmetric group action on a set) The group of symmetries of a set X consists of all bijective maps from X to itself. This group is called the symmetric group of X and denoted by S_x. There is thus a defining action of S_x on X $\phi : S_x \times X \to X$ is given by (ϕ, x) $\to \phi$(x), where ϕ (x) denotes the image of x under ϕ .

Definition 8.3.8 : (The regular action of G on itself) Let G be a group. Put X = G, and consider the map G × X $\to$ X defined by (g, x) $\to$ gx, where gx denotes the product in the group G of its elements g and x (in the specified order). It is readily checked that this defines an action of G on itself. We call this action the regular action (of G on itself).

Definition 8.3.9 : (The conjugation action of G on itself) let X = G. This time define G × G $\to$ G by (g, x) = g x g^{-1}.

8.4 SYLOW'S THEOREM

The reader may have noticed the frequent application of Lagrange's theorem for finite groups or its consequences. However, it is well-known that the converse of Lagrange's theorem does not generally hold. One of the primary objectives of this section is to establish theorems that provide sufficient conditions on a divisor k of the order of a finite group G such that G contains a subgroup of order k.

Theorem 7.2.2 Demonstrates that the converse of Lagrange's theorem fails for a finite abelian group. Additionally, we introduce theorems that elucidate the relationships between various subgroups of special types within a finite group. The most significant theorems in this context are attributed to the Norwegian Mathematician Sylow.

In this section, we will use the notions and results about group actions developed in the previous unit to derive some basis theorems about finite groups, providing thereby yet another motivation for the study of group actions.

Throughtout this unit, G denotes a finite group and | G | the order of G.

Recall the statement of Lagrange's theorem from the previous unit : the order of any subgroup of G divides |G|.

The question naturally arises whether the converse holds, to wit : given a factor k of |G|, does there exist a subgroup of G of order k? This scetion is centred around this question.

As an easy example shows, the answer is ''no'' in general. Nevertheless, as we shall presently see, there are important special cases when it is ''yes''.

(i) If G is an abelian group, then, for any factor k of |G|, there exists is subgroup of order k in G.

(ii) If p is a prime dividing |G|, then G admits a subgroup of order p. This assertion is known as

CAUCHY'S THEOREM already discussed in Chapter previous chapter.

(iii) More generally (than the previous item), if q is a prime power dividing |G|, then G admits a subgroup of order q. This assertion is known as SYLOW'S FIRST THEOREM.

(iv) As a particular case of the previous item, we have : if G is a p-group for some prime p, then, for any factor k of |G|, there exists a subgroup of order k in G.

As already mentioned, this unit is centred around the following question: given a factor k of the order |G| of a finite group G, does there exist a subgroup in G of order k? Let us begin with an example to show that the answer is ''no'' in general:

For example, the alternating group A_4 has no subgroup of order 6.

We prove it by contradiction.

Let H be a subgroup of order 6 of the alternating group A_4.

Being a group of order 6, H contains precisely two elements of order 3, each the square of the other. An element of the symmetric group S_4 (and so also of A_4 or H) has order 3 if and only if it is a 3-cycle.

Since A_4 is normal in S_4 (being a subgroup of index 2), the later acts by conjugation on subgroups of order 6 of A_4, so we may assume that the two 3- cycles in H are (123) and (132). Being a subgroup of index 2 in A_t, H is normal in A_4.

But (123) when conjugated by the clement (12) (34) of A_4 gives (142), which is not in H, a contradiction.

Thus the alternating group A_4 has no subgroup of order 6.

Proposition 8.4.1: Let N be a finite normal subgroup of order n of a group G and let $\pi : G \to {}^{G}/_{N}$ be the natural epimorphism. If K is a subgroup of order k of ${}^{G}/_{N}$, then the preimage $\pi^{-1}(K)$ of K under π is a subgroup of order kn of G.

Proof : It is readily verified that the inverse image of a subgroup under a homomorphism is a subgroup. Let $g_1, g_2, g_3, ...g_k$ be clements of G such that

$K = \{g_1N, g_2N, g_3N, ...G_kN\}$.

Then $\pi^{-1}(K)$ equals the union $g_1N \cup g_2N \cup .....g_kN$ of cosets. Since each coset g_iN has n clements and any two of them cosets are disjoint,

it follows that $\pi^{-1}(K)$ consists of nk elements.

Theorem 8.4.2 (Sylow's first Theorem)

Let G be a finite group of order $n = p^k q$, for some $k \geq 1$, where p is a prime number and q is any positive integer such that (p, q) = 1, then for i, $(1 \leq i \leq k)$, G has at least one subgroup of order p^i.

Proof : Let $o(G)=p^k q$, if o (G) = p, a prime number, then the group G is cyclic and the theorem holds trivially.

We now proceed for induction on the order of the group G.

Suppose the result holds for all group T of order less than order of the group G such that p|o (T). Let Z(G) be the centre of the group G. Then the following cases arises.

Case 1: When p|o(Z(G)), in this case by the Cauchy's theorem for Abelian groups, $\exists a \in Z(G)$ of order p. Since 'a' commute with all elements of G, H = < a > is a normal subgroup of G of order p. If k = 1, then H is the desired subgroup of G of order p. Let k > 1, then $o\left(G/H\right)=p^{k-1}q$. By induction hypothesis, $\overline{G}$ contains subgroup $\overline{H}$, of order p^i, (i = 1, 2, 3,.....k–1). Now, for each i, $\overline{H}_i = H_i/H$, for some subgroup H_i of G containing H. Then $o(\overline{H}_i)=o(H_i)=p^{i+1}$, $(i=1,2,3,.....k-1)$

Writing $K_1, K_2, ...$ K for H. $H_1, H_2, ...H_{k-1}$, we see that o (K_i) = p^i, $\forall$ i = 1, 2, ...k.

Case II: When $p \mid o(Z(G))$.

We have $o(G)=o\left(Z(G)\right)+\sum_{N(a)\neq G}\frac{o(G)}{o\left[N(a)\right]}$

Where the summation runs over the clement a taken one each from cach conjugate class containing more than one element.

Since p | o (G), therefore if $p \mid \frac{o(G)}{o\left[N(a)\right]}$ for all a for which $N(a) \neq G$ we will get from the above equality; p | o(Z (G)), which is against the hypothesis of this case. Hence there exists an $a \in G$ such that $N(a) \neq G$ and $p \mid \frac{o(G)}{o\left[N(a)\right]}$

Now, $o(G) = \frac{o(G)}{o[N(a)]}.o[N(a)]$ and $p \mid \frac{o(G)}{o[N(a)]}$, then $p^k \mid o(G)$ implies that $p^k \mid o[N(a)]$.

But $o[N(a)] < o(G)$, so by the induction hypothesis N(a) contains subgroups H_i's of order $p^i, 1 \le i \le k$, which are also subgroup of G.
This establishes the theorem.

Corollary 8.4.4: (Cauchy's Theorem for finite groups): If a prime number p divides the order of a finite group i. then G contains at least one element of order p.

Proof : By Sylow's first theorem, G contains a subgroup H of order p. Then H is cyclic and so, there exists a $\in$ G such that H = < a>. Hence o(a) = o(H) = p.

Corollary 8.4.5: If $p^k \mid o(G)$ and $p^{k+1} \mid o(G)$, then G has a subgroup of order p^k.

Proof: Directly follows from the Sylow's first theorem.

Lemma 8.4.6: Let P be a Sylow p-subgroup of a finite group G. Then P contains all elements in N(P), which are of order, a power of p.

Proof: Let X $\in$ N (P) be such that $x^{p^k} = e$, for some k $\ge$ 0. Since P is normal in $N(P)$, $N(P)/P$ is a group and $xP \in N(P)/P$.

Then $(xP)^{p^k} = x^{p^k}P = P$ gives $o(xP) \mid p^k \Rightarrow o(xP) = p^t$ for some t $\ge$ 0.

Let $\overline{K}$ be a subgroup of $N(P)/P$ generated by $\bar{x} = xP$.

Then $o(\bar{x}) = o(\overline{K}) = p^t$

Now, $\overline{K} = K/P$ for some subgroup K of N(P) containing P. Let $o(P) = p^\alpha$, then $o(K) = o(\overline{K}).o(P) = p^{t+\alpha}$. Thus, K is a p-subgroup of G containing P. Since no p-subgroup of G can contain Sylow p-subgroups of G properly, we get K = P. Hence, xP = P and x $\in$ P.

This proves the lemma.

Theorem 8.4.7. : Let G be a finite group and p, a prime number such that p | o(G). Then

(i) All Sylow p-subgroup of G are conjugates of one another. (Sylow's second Theorem)

(ii) Number of Sylow p-subgroups is of the form 1+mp, where m is some non- negative integer such that (1 + mp) |o(G). (Sylow's Third Theorem)

Proof : (i) Let $o(G) = p^k q$, where $(p, q) = 1$ and $k \geq 1$. Let S be any Sylow p- subgroup of G and M, the set of all conjugates of S.

We shall show that every Sylow p-subgroup P of G is a conjugate of S, i.e, $P \in M$.

Suppose on the contrary that there exists a Sylow p-subgroup of P of G which is not conjugate of S. Let Q be any Sylow p-subgroup of G, different from P.

Since $o(Q) = o(P) = p^k$, we get $P \subseteq Q$.

This yield that there exists $x \in P$ such that $x \notin Q$

Then $x^{-1}Qx \neq Q$, for otherwise $x^{-1}Qx = Q \Rightarrow x \in N(Q)$;

Now, x being a member of P, is of order, a prime power of p, then by Lemma 7.2.6, $x \in Q$, which leads to a contradiction.

Thus, the number of conjugates of Q determined by the elements of P is more than one.

Since this number is equal to $[P: N_p(Q)]$, $N_p(Q) \neq P$.

As $o(P) = p^k$, and $[P: N_p(Q)] \mid p^k$, we get that the number of conjugates is p^l, for some $l \geq 1$, hence it is a multiple of p.

On M define a relation '~' as follows:

$S_1 \sim S_2$ if and only if $x^{-1}S_2x = S_1$ for some $x \in P$.

This is an equivalence relation and it divides M into disjoint equivalence classes. Since for any $S' \in M$, $S' \neq P$, number of conjugates of S' determind by elements of P is a multiple of p, so the equivalence class to which S', belongs has the number of members equals to a multiple of p. Since this holds for every equivalence class, we get the number t of members of M is a multiple of p.

However, $t = [G: N(S)]$. Since $N(S) \supseteq S$ and $[G:S] = q$, we get $[G: N(S)] \mid q$. This gives $p \mid t$, as $p \mid q$.

This contradicts the fact, proved above, that t is a multiple of p.

Hence P must be a conjugate of S.

(ii) Let M be the set of all Sylow p-subgroup of G. Take any $P \in M$,

As in (i) define an equivalence relation '~' as follows:

$S_1 \sim S_2$ if and only if $x^{-1}S_2x = S_1$ for some $x \in P$.

Now, if S is any Sylow p-subgroup of G different from P, as seen in (i), the number of conjugates of S determined by the elements of P is a multiple of p.

This shows that the number of members in any equivalence class to which P does not belong is a multiple of p.

Since $x^{-1}Px = P$, for all $x \in P$

So, P is the only member of the equivalence class in which P belongs.

Thus, the number of members of M is of the form 1 + mp, where m is some non- negative integer.

Since number of conjugates of a subgroup divides the order of group, 1 + mp divides the order of the group.

This completes the proof.

Theorem 8.4.8: Let G be a finite group and p, a prime number such that p | o (G). Then, every p-subgroup B of G is contained in some Sylow p-subgroup of G.

Proof: Let M be the set of all Sylow p-subgroup of G. By Sylow's Third Theorem, the number of members of M is equal to 1 + mp, where m is some non- negative integer.

On M, define a relation '~' as follows:

For any, $P_1, P_2 \in M$.

$P_1 \sim P_2$ if and only if $x^{-1}P_2x = P_1$ for some $x \in B$.

This is an equivalence relation, and for any $P \in M$, the number of members of M in the conjugate class of P is same as the number of conjugates of P determined by elements of B. If for some $P \in M$, $B \subseteq P$, then there exists $x \in B$, such that $x \notin P$.

If $x^{-1}Px = P$

As the order of x is a power of P (B is a p-subgroup) and $x \in N(P)$, we get $x \in P$. (By Lemna 7.2.6). Which leads to a contradiction.

So, the number of conjugates of P is greater than 1.

However, $t = [B: N_B(P)]$ and $o(B)\ p^l$, for some non-negative integer *l*, we get $t = p^s$, for some $s \geq 1$. Thus, t is a multiple of p. Consequently, if no

P in M had contained B, then the number of members of in each equivalence class would have been multiple of p. This would contradict the fact that the number of members of M is 1+ mP.

Hence there exists a Sylow p-subgroup P of G such that $B \subseteq P$.

This proves the theorem.



CHECK YOUR PROGRESS

Q.1: Explain how the symmetric group S_n acts on the set $\{1,2,3,4,...n)$.

Q.2: Let G be a group and S be a G-set. Then the action of G on S induces a homomorphism from G into the group A(S), of all permutations of S.

[A G-set is a set S together with an action of a group G on that set.]

Q.3: Show that every group is isomorphic to a group of permutations.



8.5 LET US SUM UP

- A group action describes how a group can operate on a set. The action satisfies specific properties, such as preserving identity and compatibility with group multiplication.
- Group actions are used to study symmetries in mathematics and have applications in various fields like geometry and combinatory.
- Sylow's theorems provide conditions for the existence of subgroups (Sylow p-subgroups) in finite groups. They are important for analyzing the structure of finite groups. The theorems outline the number and existance of these subgroups, often used in conjunction with Lagrange's theorem and are crucial in group theory applications like cryptography and Galois Theory in abstract algebra.



8.6 FURTHER READING

1) Fraleigh, Johu B : A First Course in Abetract Algebra (2003), Pearson Education (Singapure) Pvt. Ltd, Indian Branch, Delhi 110092,

2) Mapa, S.K.: Higher Algebra (Abstract and Linear) (1994), Asoke Prakasan, Calcutta 700007.

3) Khanna, V. K. and Bhambri, S. K.: A Course in Abstract Algebra (2006), Vikas Publishing House Pvt. Ltd. New Delhi 110014.

4) Singh Surjeet and Zameeruddin Qazi; Modern Algebra, VIKASH PUBLISHING HOUSE PVT LTD (Seventh revised Edition-2000).



8.7 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q.No.1: The symmetric group S_n acts on the set $\{1,2,3,...,n\}$ by permutations. That is each element of S_n can be viewed as a function that rearranges the elements of the set.

Consider an element $\sigma \in S_n$, and consider an element $x \in \{1,2,3,...,n\}$

The action of σ on x in defined as follows:

$\sigma .x = \sigma(x)$, the image of x under the permutation σ.

Clearly, the identity permutation, say 'e', acts as an identity element of $\{1,2,3,...,n\}$ because $e.x = e(x) = e$.

Also, for any two permutations σ and τ of S_n and for any element x of $\{1,2,3,...,n\}$, we have $(\sigma \circ \tau)\cdot x = \sigma(\tau(x))$.

Thus the above definition defines a group action.

Hence, the symmetric group S_n acts on the set $\{1,2,3,...,n\}$ by permutations.

Ans to Q.No.2: Let $g \in G$. Define a mapping $\tau_g : S \to S$ by $\tau_g(a) = ga$, for all $a \in S$.

We shall show that τ_g is onto:

Then for every $a \in S$

$g(g^{-1}a) = (gg^{-1})a = ea = a$.

Hence τ_g is onto.

τ_g **is one-one :**

Now, for $a,b \in S$,

$\tau_g(a) = \tau_g(b) \Rightarrow ga = gb$

$\Rightarrow g^{-1}(ga) = g^{-1}(gb)$

$\Rightarrow (g^{-1}g)a = (g^{-1}g)b$

$\Rightarrow a = b$

Hence, τ_g is one-one.

Hence, $\tau_g \in A(S)$

Also for every $g_1, g_2 \in G$ and $a \in S$

$$\begin{aligned}\tau_{g_1g_2}(a) &= (g_1g_2)(a)\\ &= (g_1(g_2a)\\ &= \tau_{g_1}(g_2a)\\ &= \tau_{g_1}[\tau_{g_2}(a)] = \tau_{g_1}.\tau_{g_2}(a)\end{aligned}$$

Hence, $\tau_{g_1g_2} = \tau_{g_1}.\tau_{g_2}$

Hence the mapping $\psi : G \to A(S)$ defined by $\psi(g) = \tau_g$ for all $g \in G$ is a homomorphism from G into A(S).

Ans to Q.No.3: Let G be a group. Define an action of G on G by left translation, that is, $(g, a) \to ga$.

Then $\psi : G \to A(G)$ as defined in the previous result **8.6.2** is a homomorphism of G into A (G).

Thus it remains to show only that ψ is one-to-one.

Now for $g_1, g_2 \in G$,

$\psi(g_1) = \psi(g_2) \Rightarrow \tau_{g_1} = \tau_{g2}$

$\Rightarrow \tau_{g_1}(x) = \tau_{g_2}(x), \text{for all } x \in G$

$\Rightarrow \tau_{g_1}(e) = \tau_{g_2}(e) \text{ for all identity of } G$

$\Rightarrow g_1 = g_2$

Hence ψ is a monomorphism.

Thus, ψ is an isomrphism.



8.8 MODEL QUESTIONS

Q 1: State Sylow's Theorem and its significance in the theory of finite groups.

Q 2: Discuss the existence and uniqueness parts of Sylow's theorem. What do they entail?

Q 3: Investigate the consequences of Sylow's theorem for the structure of finite groups.

Q 4: Provide examples illustrating the application of Sylow's theorem in analyzing the structure of finite groups.

Q 5: Discuss the relationship between Sylow subgroups and normalizers in a group.

Q 6: Investigate the role of Sylow's theorem in characterizing the solvability and simplicity of finite groups.

Q 7: Discuss the Cayley's theorem and its connection to group actions. How does Cayley's theorem relate group actions to the structure of groups?

Q 8: Explain the concept of the regular group action. What is its significance in group theory?

Q 9: Discuss the connection between group actions and permutation groups. How can permutation groups be viewed as acting on sets?

Q 10: Define the notion of the fixed point set under a group action.

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