UNIT 1 : INTRODUCTION TO DIFFERENTIAL EQUATIONS

UNIT STUCTURE

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1.1 LEARNING OBJECTIVES

After going through this unit, you will able to learn

- the concept of differential equation
- different types of differential equation
- order and degree of a differential equation and related problems
- concept of Solution of a differential equation
- formation of a differential equation and related problems
- initial and boundary value problems.

1.2 INTRODUCTION

Differential equations plays an important role in science, engineering economics, commerce and social sciences.

Differential equation is like a bridge linking Mathematics and Science with its applications.Many physical problems, when formulated in mathematical forms, lead to **differential equations.** Differential equations

arise in a variety of applications, may it be in Physics, Chemistry, Biology, Anthropology, Geology, Economics etc.

In this unit, we will study some basic concepts related to differential equation, general and particular solutions of a differential equation and formation of differentialequations.

1.3 DIFFERENTIAL EQUATION

By a differential equation, we mean an equation involving independent variable, dependent variable and differential coefficients of the dependent variable w.r.t independent variables.

Differential equations are of two types :

(a) Ordinary differential equation

(b) Partial differential equation

(a) Ordinary differential equation : A differential equation is called an ordinary differential equation (ode) if only ordinary differential coefficients w.r.t a single independent variable involve.

(b) Partial differential equation : A differential equation is called a partial differential equation (pde) if in it there are two or more independent variables and partial derivative w.r.t. any of these independent variables involve.

Examples : (i) $\frac{d^2y}{dx^2} + x = xe^{2x}$

(ii) $\frac{d^4x}{dt^4} + \frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^5 = e^t$

(iii) $\frac{d^3y}{dx^3} + 3\left(\frac{d^2y}{dx^2}\right)^2 + \frac{dy}{dx} = e^x$

(iv) $\frac{\partial^2 v}{\partial t^2} = k\left(\frac{\partial^3 v}{\partial x^3}\right)^2$

(v) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

(vi) $\frac{\partial^2 z}{\partial x^2} - a^2 \frac{\partial^2 z}{\partial y^2} = 0$

Equations (i), (ii) and (iii) are all ordinary differential equations. Equations (iv), (v) and (vi) are all partial differential equations.

1.4 ORDER AND DEGREE OF A DIFFERENTIAL EQUATION

The order of an ordinary differential equation is the order of the highest derivative present in the equation.

The degree of an ordinary differential equation is the degree of the highest order derivative, when the differential coefficients become free from fraction and radical sign.

Note : (1) The degree of a differential equation is the power of the highest order derivative occurring in a differential equation when it is written as a polynomial in differential coefficients.

(2) Order and degree (if defined) of a differential equation are always positive integers.

Examples :

(i) $\frac{d^3 y}{dx^3} - y = \sin x \cos x$ is an ordinary differential equation of order 3 and degree 1.

(ii) $\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2 y}{dx^2}} = r$ can be written as $\left(1+\left(\frac{dy}{dx}\right)^2\right)^3 = r^2\left(\frac{d^2 y}{dx^2}\right)^2$ after making free from radical sign and fraction.Its order is 2 and degree 2.

(iii) Consider the differential equation $\left(\frac{d^2 y}{dx^2}\right)^2 + \sin\left(\frac{dy}{dx}\right) = 0$

The highest order derivative present in the differential equation is $\frac{d^2 y}{dx^2}$. So, its order is 2. Since the differential equation cannot be expressed as a polynomial in differential coefficients, so its degree is not defined.

Example 1 : In each of the following differential equations indicate its degree, wherever possible. Also, give the order of each of them.

(i) $\frac{dy}{dx}+\sin\left(\frac{dy}{dx}\right)=0$ (ii) $\frac{d^5y}{dx^5}+e^{\frac{dy}{dx}}+y^2=0$

Solution : (i) The highest order derivative present in the differential equation is $\frac{dy}{dx}$. So, it is of order 1. Clearly, the differential equation cannot be expressed as a polynomial in $\frac{dy}{dx}$. So, its degree is not defined.

(ii) The highest order derivative present in the differential equation is $\frac{d^5y}{dx^5}$. So, it is of order 5. But the differential equation is not expressible as a polynomial in $\frac{dy}{dx}$ .So,its degree is not defined.

Example 2 : Find the order and degree of the following differential equations:

(i) $\frac{d^3y}{dx^3}+\left(\frac{d^2y}{dx^2}\right)^3+\left(\frac{dy}{dx}\right)^5+y=7$ (ii) $\frac{d^2y}{dx^2}=\left[4+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{4}}$

Solution : (i) The order of the highest derivative in this equation is 3. The degree of the highest order is 1.

$\therefore$ order = 3, degree=1

(ii) Given, $\frac{d^2y}{dx^2}=\left[4+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{4}}$

To eliminate the radical in the above equation, raising to the power 4 on both sides, we get $\left(\frac{d^2y}{dx^2}\right)^4=\left[4+\left(\frac{dy}{dx}\right)^2\right]^3$.

$\therefore$ order = 2, degree=4

Example 3 : Write down the order and degree of the following differential equations.

(i) $\left(\frac{dy}{dx}\right)^3-4\left(\frac{dy}{dx}\right)+y=3e^x$ (ii) $\left(\frac{d^2y}{dx^2}\right)^3+7\left(\frac{dy}{dx}\right)^4=3\sin x$

(iii) $\frac{d^2x}{dy^2}+a^2x=0$ (iv) $\left(\frac{dy}{dx}\right)^2-3\frac{d^3y}{dx^3}+7\frac{d^2y}{dx^2}+4\left(\frac{dy}{dx}\right)-\log x=0$

(v) $\frac{d^2y}{dx^2}-\sqrt{\frac{dy}{dx}}=0$ (vi) $\sqrt{1+x^2}=\frac{dy}{dx}$.

Solution :

The order and the degree respectively are

(i) 1 ; 3 (ii) 2 ; 3 (iii) 2 ; 1 (iv) 3 ; 1

(v) 2 ; 2 (vi) 1 ; 1

Remarks :

1. Ode $y\frac{dy}{dx}-x=0$ of first order and first degree can be written as $ydy=xdx$.

2. Similarly, $\cos\theta\frac{dr}{d\theta}+r\sin\theta=0$ is equivalent to $\cos\theta dr+r\sin\theta d\theta=0$.



CHECK YOUR PROGRESS - 1

Q 1 : Find the order and degree of the following ode.

(i) $\frac{d^2y}{dx^2}+4xy\frac{dy}{dx}=5x$ (ii) $\left(\frac{dy}{dx}\right)^2=\frac{1-y^2}{1+x^2}$

(iii) $\left(\frac{d^2y}{dx^2}\right)^3-x^2\left(\frac{dy}{dx}\right)^4=0$ (iv) $\frac{d^3y}{dx^3}-3\left(\frac{d^2y}{dx^2}\right)^2+\left(\frac{dy}{dx}\right)^3+4y^5=0$

(v) $x+\frac{d^2y}{dx^2}=\sqrt{1+\frac{dy}{dx}}$ (vi) $\left(\frac{d^3y}{dx^3}\right)^{1/3}=\left(\frac{dy}{dx}\right)^{1/2}$

(vii) $y=\sqrt{x}\left(\frac{dy}{dx}\right)+k\Big/\left(\frac{dy}{dx}\right)$ (viii) $y=x\left(\frac{dy}{dx}\right)+a\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}$

(ix) $dy=(y+\sin x)dx$ (x) $\left(\frac{d^2y}{dx^2}\right)^3+x\left(\frac{dy}{dx}\right)^5+y=x^2$

(xi) $\frac{d^4y}{dx^4} + \sin\left(\frac{d^3y}{dx^3}\right) = 0$ (xii) $\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^2 = x\sin\left(\frac{d^2y}{dx^2}\right)$

1.5 LINEAR AND NON LINEAR DIFFERENTIAL EQUATION

A differential equation is said to be linear if the unknown function (i.e. the dependent variable) and its derivatives be of first power and be free from product of them. An ode which is not linear is called a nonlinear ode.

Examples :

(i) $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = e^x$ is linear ode.

(ii) $x^2\frac{d^2y}{dx^2} + 2x\left(\frac{dy}{dx}\right)^2 + y^2 = 0$ is non linear.



CHECK YOUR PROGRESS - 2

Q 1 : Classify the following equations as linear and nonlinear equations and write down their orders.

(a) $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2}.\frac{dy}{dx} + y = x$ (b) $x\frac{d^4y}{dx^4} + \frac{d^2y}{dx^2} = e^x$ (c) $\frac{dy}{dx} + y^2 = x^2$

1.6 FORMATION OF ORDINARY DIFFERENTIAL EQUATION

$y = \phi(x)$ is said to be solution of an ode if on substituting $y = \phi(x)$, the equation reduces to an identity. $y = e^x$ is a solution of $\frac{d^2y}{dx^2} - y = 0$. It is easy to check that Ae^x, e^{-x}, Be^{-x} and $Ae^x + Be^{-x}$ are also solutions of this equation. The last solution is called the general solution of the equation, since former solutions are obtainable from it with values (1, 0), (A, 0),(0,1), (0, B) of (A, B). Observe that the number of arbitrary constants in general solution of an ode is equal to the order of the equation. Solutions reducible

from the general solution are called special solution.sometimes,a solution is obtained which is neither contains constant nor it is a special solution.Such a solution is called a singular solution.Thus,

(i) A solution of an ode is called a general solution if it contains as many constants as the order of the equation.

(ii) A solution of an ode is called a special solution if it is obtainable from the general equation.

(iii) A solution other than special solution and free from constant is called a singular solution.

Conversely, from the general solution $y = Ae^{x} + Be^{-x}$, we can obtain the differential equation. Since there are two arbitrary constants, we differentiate two times and obtain two equations

$$\frac{dy}{dx} = Ae^{x} - Be^{-x} \text{ and } \frac{d^2y}{dx^2} = Ae^{x} + Be^{-x}$$

Eliminating the arbitrary constants A and B,we obtain

$\frac{d^2y}{dx^2} = y$ or $\frac{d^2y}{dx^2} - y = 0$, which is the differential equation with general solution $y = Ae^{x} + Be^{-x}$.

This is general technique of formation of an ode from its general solution.

Example 1 : Form differential equation from

$x = a\cos nt + b\sin nt$ where a and b are arbitrary constants.

Solution : Given $x = a\cos nt + b\sin nt$ ... (1)

Differentiating (1) twice w.r.t t, we have

$$\frac{dx}{dt} = -an\sin nt + bn\cos nt$$

$$\frac{d^2x}{dt^2} = -an^2\cos nt - bn^2\sin nt$$

$$= -n^2\left(a\cos nt + b\sin nt\right)$$

$$= -n^2x \quad \text{using (1)}$$

Hence $\frac{d^2x}{dt^2} + n = 0$ is the required equation.

Example 2 : Eliminate the constants from the equation $y = e^{x}\left(A\cos x + B\sin x\right)$ and obtain the differential equation.

Solution : Given, $y = e^x\left(A\cos x + B\sin x\right)$ .... (1)

Differentiating (1) w.r.t. x, we have

$$\frac{dy}{dx} = e^x\left(A\cos x + B\sin x\right) + e^x\left(-A\sin x + B\cos x\right)$$

$$= y + e^x\left(-A\sin x + B\cos x\right) \quad \text{... (2)}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{dy}{dx} + e^x\left(-A\sin x + B\cos x\right) + e^{-x}\left(-A\cos x - B\sin x\right)$$

or $\frac{d^2y}{dx^2} = \frac{dy}{dx} + \frac{dy}{dx} - y - y$, from (1) and (2)

or $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$, which is the required differential equation.

Example 3 : Form a differential equation for $x^2 + \left(y-a\right)^2 = a^2$.

Solution : Given $x^2 + \left(y-a\right)^2 = a^2$

or $x^2 + y^2 - 2ya + a^2 = a^2$

or $x^2 + y^2 - 2ya = 0$ ... (1)

Differentiating both sides with respect to x, we get

$$2x + 2y\frac{dy}{dx} - 2a\frac{dy}{dx} = 0$$

or $x + y\frac{dy}{dx} - a\frac{dy}{dx} = 0$

or $x + y\frac{dy}{dx} - \frac{x^2+y^2}{2y}\frac{dy}{dx} = 0$, from (1)

or $2yx + 2y^2\frac{dy}{dx} - \left(x^2+y^2\right)\frac{dy}{dx} = 0$

or $2yx + \left(2y^2 - x^2 - y^2\right)\frac{dy}{dx} = 0$

or $2xy = \left(x^2 - y^2\right)\frac{dy}{dx}$

$\therefore \frac{dy}{dx} = \frac{2xy}{x^2-y^2}$, which is the required differential equation.

Example 4 : Form differential equation from

(i) $x = a\sin\left(y+b\right)$ (ii) $y = ax\cos\left(\frac{n}{x} + b\right)$

(iii) $y = a\left(\sin x + \frac{1}{x}\cos x\right) + b\left(\cos x - \frac{1}{x}\sin x\right)$

(iv) $y = ae^{x} + be^{3x} + ce^{5x}$

Solution : (i) Differentiating $x = a\sin(y+b)$ w.r.t. y, we have

$$\frac{dx}{dy} = a\cos(y+b)$$

so, $\frac{dy}{dx} = \frac{1}{a}\sec(y+b)$

Differentiating again w.r.t.x, we have

$$\frac{d^2y}{dx^2} = \frac{1}{a}\sec(y+b)\tan(y+b).\frac{dy}{dx}$$

or, $\frac{d^2y}{dx^2} = \tan(y+b)\frac{1}{a}\sec(y+b).\frac{dy}{dx}$

or, $\frac{d^2y}{dx^2} = \tan(y+b)\left(\frac{dy}{dx}\right)^2$

or, $\frac{d^2y}{dx^2} = \frac{a\sin(y+b)}{a\cos(y+b)}\left(\frac{dy}{dx}\right)^2$

$$\therefore \frac{d^2y}{dx^2} = \frac{x}{\frac{dx}{dy}}\left(\frac{dy}{dx}\right)^2 = x\left(\frac{dy}{dx}\right)^3$$

i.e., $\frac{d^2y}{dx^2} = x\left(\frac{dy}{dx}\right)^3$ is the required differential equation.

(ii) $y = ax\cos\left(\frac{n}{x} + b\right)$ is a given relation with two arbitrary constants a and

b. Differentiating w.r.t.x , we have

$$y_1 = a\cos\left(\frac{n}{x} + b\right) + ax.\left(-\sin(\frac{n}{x} + b)\right)(-\frac{n}{x^2})$$

$$= a\cos\left(\frac{n}{x} + b\right) + \frac{an}{x}\sin(\frac{n}{x} + b)$$

or $xy_1 = y + an\sin(\frac{n}{x} + b)$

Differentiating once again, we have

$$y_1 + xy_2 = y_1 + an\cos(\frac{n}{x} + b)\left(-\frac{n}{x^2}\right)$$

or $xy_2 = -\frac{n^2}{x^3}y$

Hence, $x^4y_2 + n^2y = 0$ is the required differential equation.

(iii) $y = a\left(\sin x + \frac{1}{x}\cos x\right) + b\left(\cos x - \frac{1}{x}\sin x\right)$

$$= a\sin x + b\cos x + \frac{1}{x}(a\cos x - b\sin x) \quad \text{... (1)}$$

is a given relation with arbitrary constants a and b.

Differentiating (1) w.r.t. x , we have

$$y_1 = a\cos x - b\sin x - \frac{1}{x^2}(a\cos x - b\sin x) + \frac{1}{x}(-a\sin x - b\cos x)$$

or $xy_1 = x(a\cos x - b\sin x) - \frac{1}{x}(a\cos x - b\sin x) - (a\sin x + b\cos x)$

$$= x(a\cos x - b\sin x) - y$$

or $xy_1 + y = x(a\cos x - b\sin x)$ .... (2)

Again, differentiating (2) w.r.t. x , we have

Then we have

$y_1 + xy_2 + y_1 = a\cos x - b\sin x + x(-a\sin x - b\cos x)$

or $x^2y_2 + 2xy_1 = x(a\cos x - b\sin x) - x^2(a\sin x - b\cos x)$

$$= xy_1 + y - x^2\left[y - \frac{1}{x}(a\cos x - b\sin x)\right], \text{ using (1) and (2)}$$

$$= xy_1 + y - \left[x^2y - (xy_1 + y)\right]$$

or $x^2y_2 - 2y + x^2y = 0$, which is the required differential equation.

(iv) $y = ae^x + be^{3x} + ce^{5x}$ with arbitrary constants a, b and c.

Differentiating successively, we have

$$y_1 = ae^x + 3be^{3x} + 5ce^{5x}$$

$$y_2 = ae^x + 9be^{3x} + 25ce^{5x}$$

$$y_3 = ae^x + 27be^{3x} + 125ce^{5x}$$

so, $y_1 - y = 2be^{3x} + 4ce^{5x}$

$$y_2 - y_1 = 6be^{3x} + 20ce^{5x}$$

$$y_3 - y_2 = 18be^{3x} + 100ce^{5x}$$

Eliminating b and c, we have

$$\begin{vmatrix} y_1 - y & 1 & 1 \\ y_2 - y_1 & 3 & 5 \\ y_3 - y_2 & 9 & 25 \end{vmatrix} = 0$$

or $y_3 - 9y_2 + 23y_1 - 15y = 0$ (on expanding)

This is the required differential equation.

Example 5 : (i) Find the differential equation of the family of coplanar straight lines passing through the origin.

(ii) Find the differential equation of the family of straight lines at unit distance from the origin.

Solution : (i) $y = mx$ is the equation of the family of straight lines passing through the origin. Here m is the only arbitrary constant.

Differentiating w.r.t.x, we have

$$\frac{dy}{dx} = m = \frac{y}{x}$$

or $x\frac{dy}{dx} - y = 0$ is the required equation.

(ii) $x\cos\alpha + y\sin\alpha = 1$ is the equation of family of straight lines at a unit distance from the origin.

Differentiating w.r.t.x, we have

$$\cos\alpha + \frac{dy}{dx}\sin\alpha = 0$$

From the above two equations, eliminating $\cos\alpha$, we have

$$\sin\alpha = \frac{1}{\left(y - x\frac{dy}{dx}\right)}$$

so, $\cos\alpha = -\dfrac{\dfrac{dy}{dx}}{\left(y - x\dfrac{dy}{dx}\right)}$

Squaring and adding, we have $1 = \dfrac{1}{\left(y - x\dfrac{dy}{dx}\right)^2}\left(1 + \left(\dfrac{dy}{dx}\right)^2\right)$

or $\left(y - x\dfrac{dy}{dx}\right)^2 = \left(\dfrac{dy}{dx}\right)^2 + 1$ is the required equation.

Example 6 : Find the differential equation of family of

(i) circles of radius r.

(ii) circles with centre on x-axis and touching y-axis.

Solution : (i) Equation of family of circles centred at (a, b) and radius are r are $(x-a)^2 + (y-b)^2 = r^2$ ... (1), where a and b are arbitrary constants and r is a fixed constant.Here we shall eliminate a and b to obtain the differential equation but not r.

Differentiating (1) w.r.t x, we have

$$(x-a) + (y-b)\frac{dy}{dx} = 0 \quad \text{... (2)}$$

Again, Differentiating (2) w.r.t x, we have

$$1 + \left(\frac{dy}{dx}\right)^2 + (y-b)\frac{d^2y}{dx^2} = 0 \quad \text{... (3)}$$

From (2), $(y-b) = -\dfrac{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]}{\dfrac{d^2y}{dx^2}}$ ... (4)

and from (3), $x - a = -(y-b)\dfrac{dy}{dx} = \dfrac{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]\dfrac{dy}{dx}}{\left(\dfrac{d^2y}{dx^2}\right)}$ ... (5)

squaring and adding (4) and (5), we have

$$r^2 = (x-a)^2 + (y-b)^2 = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^2}{\left(\frac{d^2y}{dx^2}\right)^2}\left(1+\left(\frac{dy}{dx}\right)^2\right)$$

or $r^2\left(\frac{d^2y}{dx^2}\right)^2 = \left[1+\left(\frac{dy}{dx}\right)^2\right]^3$, which is the required equation.

(ii) A circle with centre on x-axis and radius a must touch the y-axis at the origin. So, its centre is (a, 0) and radius a. The equation of family of such circles is $(x-a)^2 + y^2 = a^2$, where a is arbitrary constant.

or $x^2 - 2ax + y^2 = 0$ ... (1)

Differentiating (1) w.r.t. x, we have

$$2x - 2a + 2y\frac{dy}{dx} = 0$$

or $x - a + y\frac{dy}{dx} = 0$

or $a = x + y\frac{dy}{dx}$

Putting the value of a in (1), we get

$$x^2 - 2\left(x + y\frac{dy}{dx}\right)x + y^2 = 0$$

or $x^2 - 2x^2 - 2xy\frac{dy}{dx} + y^2 = 0$

or $(y^2 - x^2)dx - 2xydy = 0$,which is the required equation.



CHECK YOUR PROGRESS - 3

Q 1 : Form differential equation from

(i) $y^2 = Ax^2 + Bx + C$

(ii) $c(y+c)^2 = x^2$

(iii) $y = ae^{bx}$

(iv) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

(v) $y^2=a(b-x)(b+x)$

Q 2 : Form the differential equation of

(i) all parabolas each of which has a latus rectum 4a and whose axes are parallel to the x -axis.

(ii) all circles which pass through the origin and whose centres on the y-axis.

(iii) all parabolas whose axes are parallel to the axis of y.

1.7 SOLUTION OF A DIFFERENTIAL EQUATION

A function is said to be a solution of a differential equation if it satisfies the differential equation.

Thus if $y=f(x)$ be the solution, then by replacing y and its derivatives with respect to x, the given differential equation will reduce to an identity.

For example, (i) $y=c_1\cos x+c_2\sin x$ is the solution of the differential equation

$$\frac{d^2y}{dx^2}+y=0.$$

Since $\frac{dy}{dx}=-c_1\sin x+c_2\cos x$

$$\frac{d^2y}{dx^2}=-c_1\cos x-c_2\sin x=-y$$

(ii) $y=e^x$ is a solution of the differential equation $\frac{dy}{dx}=y$.

since $\frac{dy}{dx}=e^x=y$.

The general (or complete) solution of a differential equation is that in which the number of independent arbitrary constants is equal to the order of the differential equation.

Thus, $y=c_1\cos x+c_2\sin x$ (involving two arbitrary constants c_1,c_2)

is the general solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$ of second order.

A particular solution of a differential equation is that which is obtained from its general solution by giving particular values to the arbitrary constants.

For example, $y = c_1e^x + c_2e^{-x}$ is the general solution of the differential equation

$\frac{d^2y}{dx^2} - y = 0$, whereas $y = e^x - e^{-x}$ or $y = e^x$ are its particular solutions.

Example 1 : Verify that $y = \cos\alpha t$ satifies the differential equation $\frac{d^2y}{dt^2} = -\alpha^2 y$.

Solution : Given that $y = \cos\alpha t$ ... (1)

We differentiate (1) twice with respect to t

$$\frac{dy}{dt} = -\alpha \sin\alpha t$$

$$\frac{d^2y}{dt^2} = -\alpha^2 \cos\alpha t$$

However, we have $y = \cos\alpha t$, and so, substituting this into the equation, we have that $\frac{d^2y}{dx^2} = -\alpha^2 y$ as required.

Therefore, $y = \cos\alpha t$ is a solution of $\frac{d^2y}{dt^2} = -\alpha^2 y$.

Example 2 : Verify that the function $y = -e^{-x}$ is a solution of the differential equation $\frac{dy}{dx} = y^2e^x$.

Solution : Given that $y = -e^{-x}$ ... (1)

we differentiate (1) with respect to x

$$\frac{dy}{dx} = e^{-x}$$

Now, $y = -e^{-x} \Rightarrow y^2 = e^{-2x}$ and so, $y^2 = e^{-x} \times e^{-x} \Rightarrow e^{-x} = y^2e^x$.

we can now substitue this expression for e^{-x} into our differential equation.i.e.,

into $\frac{dy}{dx} = e^{-x}$.

This then gives, $\frac{dy}{dx} = y^2 e^x$, as required.

That is, $y = -e^{-x}$ satisfies the differential equation $\frac{dy}{dx} = y^2 e^x$.

Example3 : Show that $y = cx + \frac{a}{c}$ (where c is a constant) is a solution of the differential equation $y = x\frac{dy}{dx} + a\frac{dx}{dy}$.

Solution : Consider $y = cx + \frac{a}{c}$ ... (1)

We differentiate (1) with respect to t

$$\frac{dy}{dx} = c$$

or $$\frac{dx}{dy} = \frac{1}{c}$$

On substituting the values of $\frac{dy}{dx}$ and $\frac{dx}{dy}$ in R.H.S of the differential equation, we have $x(c) + a\left(\frac{1}{c}\right) = cx + \frac{a}{c} = y$

$\Rightarrow$ R.H.S = L.H.S.

Example 4 : If $y = 3x^2 + C$ is a general solution of the differential equation $\frac{dy}{dx} - 6x = 0$,then find the particular solution when $y = 3, x = 2$.

Solution : The general solution of the differential equation is $y = 3x^2 + C$.

Now on substituting $y = 3, x = 2$,we get $3 = 12 + C$

or $C = -9$.

By substituting the value of C in the general solution, we will have

$y = 3x^2 - 9$ which is the required particular solution.



CHECK YOUR PROGRESS - 4

Q 1: Verify that the function $y = a\cos x + b\sin x$, where, $a, b \in R$ is solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$.

Q 2 : Verify that the function $y - \cos y = x$ is a solution of the differential equation $(y \sin y + \cos y + x)y' = y$.

Q 3 : Verify that the function $x + y = \tan^{-1} y$ is a solution of the differential equation $y^2 y' + y^2 + 1 = 0$.

Q 4 : Show that $y = x^{-\frac{3}{2}}$ is a solution to $4x^2 y'' + 12xy' + 3y = 0$ for $x > 0$.

1.8 INITIAL AND BOUNDARY VALUE PROBLEMS

Initial Value Problem. An initial value problem is a differential equation with a set of constraints at the initial point. Initial value problems are ones in which we know the initial state of a system and wish to compute how the system changes as the value of the independent variable changes. The following are the examples of initial value problem

(i) $\frac{dy}{dx = 2x,} \quad y(1) = 4$

(ii) $\frac{dy}{dx} + y = 2xe^{-x}, \quad y(0) = 2$

(iii) $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 12y = 0, \; y(0) = 5, y'(0) = 6$

(iv) $x^3 \frac{d^3y}{dx^3} - 3x^2 \frac{d^2y}{dx^2} + 6x \frac{dy}{dx} - 6y = 0, y(2) = 0, y'(2) = 2, y''(2) = 6$

Boundary Value Problem. A boundary value problem for a given differential equation consists of finding a solution of the given differential equation subject to a given set of boundary conditions. A boundary condition is a prescription some combinations of values of the unknown solution and its derivatives at more than one point. The following are the examples of boundary value problem one point. The following are the examples of boundary value problem

(i) $\frac{d^2y}{dx^2} + y = 0, y(0) = 1, y(\pi) = 5$

(ii) $\frac{d^2y}{dx^2} + y = 0, y(0) = 1, y'\left(\frac{\pi}{2}\right) = -1$

Ex. (a) $\frac{d^2y}{dx^2} + y = 0, y(1) = 3, y'(1) = -4$

This problem consists in finding a solution of the given second order differential equation $\frac{d^2y}{dx^2} + y = 0$, which assumes the value 3 at x = 1 and whose first derivatives assumes the value –4 at . Both of these conditions relate to one x value, namely, . Thus, this is an initial value problem.

(b)

In this problem, we again looking a solution of the same differential equation, but this time the solution must assume the value 1 at and the value 5 at . That is, the condition relates to the two different x values, 0 and . Thus, this is a Boundary value problem.



1.8 LET US SUM UP

- An equation involving derivatives or differentials of one or more dependent variables with respect to one or more independent variables is called a differential equation.

  Differential equation are of two types :

  (i) Ordinary differential equation

  A differential equation involving derivatives with respect to a single independent variable is called an ordinary differential equation.

  (ii) partial differential equation

  A differential equation involving partial derivatives with respect to more than one independent variables is called a partial differential equation.
- The order of the highest order derivative involved in a differential equation is called the order of the differential equation.

- The degree of a differential equation is the degree of the highest derivatives which occur in it,after the differential equation has been made free from radicals and fractions.
- A differential equation is called linear if (i) every dependent variable and every derivative involved occurs in the first degree only, and (ii) no products of dependent variables and/or derivatives occur.
- Given an equation in the variables x and y containing n arbitrary constants,we differentiate it n times to get n additional equations invloving the n arbitrary constants.we eliminate the n arbitrary constants from the above (n+1) equations to obtain a differential equation of nth order.
- A general solution of a differential equation is that solution which contains the number of arbitrary constants equal to the order of the differential equation.
- A general solution will be a particular solution when partticular values of the arbitrary constants are determined satisfying the given conditions.



1.9 FURTHER READINGS

1) Tenenbaum, Morris and Pollard, Harry. Ordinary Differential Equations, Dover Publications, Inc, N. Y 11501.
2) Differential Equations : An Introduction with Mathematica. Clay C. Ross, second Edition, Springer.
3) Differential Equations :Linear, Nonlinear, ordinary, partial.
A. C. King, J. Billinggham and S.R. Otto, Cambridge-2003.
4) Differential equations with Application and Programs. S. Balachandra Rao and H. R. Anuradha, University Press (India) Limited 1996.

1.10 ANSWERS TO CHECK YOUR PROGRESS

CHECK YOUR PROGRESS – 1

Ans to Q No 1 : (i) 2,1 (ii) 1, 2 (iii) 2, 3 (iv) 3, 1 (v) 2, 2 (vi) 3, 2 (vii) 1, 2 (viii)1, 2 (ix) 1, 1 (x) 2, 3.

(xi) 4, not defined (xii) 2, not defined .

CHECK YOUR PROGRESS – 2

Ans to Q No 1 : (a) Non-linear,3 (b) Non-Linear,4 (c) Linear,1.

CHECK YOUR PROGRESS – 3

Ans to Q No 1: (i) Given $y^2 = Ax^2 + Bx + C$

On differentiating $2y\frac{dy}{dx} = 2Ax + B$

Again differentiating $2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = 2A$

On differentiating again $y\frac{d^3y}{dx^3} + \frac{dy}{dx}.\frac{d^2y}{dx^2} + 2\frac{dy}{dx}\frac{d^2y}{dx^2} = 0$

or $y\frac{d^3y}{dx^3} + 3\frac{dy}{dx}.\frac{d^2y}{dx^2} = 0$.

(ii) Given, $c(y+c)^2 = x^2$ ... (1)

Differentiating (1) w.r.t. 'x', we get $2c(y+c).\frac{dy}{dx} = 2x$

or $c(y+c).\frac{dy}{dx} = x$ ... (2)

or $\frac{c(y+c)^2}{y+c}.\frac{dy}{dx} = x$

or $\frac{x^2}{y+c}.\frac{dy}{dx} = x$ From (1)

or $x\frac{dy}{dx} = y + c$

or $c = x\frac{dy}{dx} - y$.

Substiturting this value of c in (1), we get

$$\left(x\frac{dy}{dx}-y\right)\left(y+x\frac{dy}{dx}-y\right)^2=x^2$$

or $\left(x\frac{dy}{dx}-y\right)x^2\left(\frac{dy}{dx}\right)^2=x^2$

or $\left(x\frac{dy}{dx}-y\right)\left(\frac{dy}{dx}\right)^2=1$.

which is the required differential equation.

(iii) Given, $y=ae^{bx}$ ... (1)

$$\therefore \frac{dy}{dx}=abe^{bx}=by \quad \text{... (2)}$$

$$\therefore \frac{d^2y}{dx^2}=b\frac{dy}{dx}=\frac{1}{y}.\frac{dy}{dx}.\frac{dy}{dx}=\frac{1}{y}\left(\frac{dy}{dx}\right)^2 \qquad [\text{ by (2) }]$$

or $y\frac{d^2y}{dx^2}=\left(\frac{dy}{dx}\right)^2$, which is the required differential equation.

(iv) Given, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ... (1)

Differentiating (1) w.r.t. 'x', we get $\frac{2x}{a^2}+\frac{2y}{b^2}.\frac{dy}{dx}=0$

or $\frac{y}{x}.\frac{dy}{dx}=\frac{-b^2}{a^2}$ ... (2)

Differentiating both sides with respect to x, we get

$$\left(\frac{y}{x}\right)\frac{d^2y}{dx^2}+\frac{\left(x\frac{dy}{dx}-y\right)}{x^2}\frac{dy}{dx}=0$$

or $xy\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2-y\frac{dy}{dx}=0$.

which is the required differential equation.

(v) Given, $y^2=a(b-x)(b+x)$ ... (1)

Differentiating (1) w.r.t. x,we get

$$2y\frac{dy}{dx}=-2ax$$

or $y\frac{dy}{dx} = -ax$ ... (2)

Differentiating (2) w.r.t. x, we get

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = -a \quad \text{... (3)}$$

From (2) and (3), we get

$$x\left\{y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right\} = y\frac{dy}{dx}$$

which is the required differential equation.

Ans to Q No 2: (i) The equation of the family of parabolas is $(y-k)^2 = 4a(x-h)$, where h and k are arbitrary constants.

Differentiating this relation twice and eliminating h and k, we get the required differential equation.

(ii) If a be the radius of a circle, then its centre is the point $(0, a)$ and its equation is $x^2 + (y-a)^2 = a^2$

or $x^2 + y^2 - 2ay = 0$ ... (1)

Differentiating (1) w.r.t. x, we get

$$2x + 2y\frac{dy}{dx} - 2a\frac{dy}{dx} = 0$$

or $x + y\frac{dy}{dx} = a\frac{dy}{dx}$ ... (2)

Eliminating a from (1) and (2), we get

$$x + y\frac{dy}{dx} = \frac{x^2 + y^2}{2y}\frac{dy}{dx}$$

or $2xy + 2y^2\frac{dy}{dx} = (x^2 + y^2)\frac{dy}{dx}$

or $2xy + (y^2 - x^2)\frac{dy}{dx} = 0$

which is the required differential equation.

(iii) Let the equation of a parabola whose axis is parallel to y-axis be $(x-h)^2 = 4a(y-k)$ ... (1) where h, k and a are arbitrary constants.

Differentiating w.r.t. x, we get

$$2(x-h)=4a\frac{dy}{dx}$$

Differentiating again w.r.t. x , we get

$$2=4a\frac{d^2y}{dx^2}$$

and again differentiating w.r.t. x ,we get

$$0=4a\frac{d^3y}{dx^3}$$

$$\Rightarrow \frac{d^3y}{dx^3}=0$$

Hence $\frac{d^3y}{dx^3}=0$ is the required differential equation.

CHECK YOUR PROGRESS- 4

Ans to Q No 1 : Given, $y=a\cos x+b\sin x$ ... (1)

Differentiating both sides of (1) with respect to *x* successively, we get

$$\frac{dy}{dx}=-a\sin x+b\cos x$$

$$\frac{d^2y}{dx^2}=-a\cos x-b\sin x$$

Substituting the values of $\frac{d^2y}{dx^2}$ and y in the given differential equation,

we get

L.H.S $=(-a\cos x-b\sin x)+(a\cos x+b\sin x)=0$ = R.H.S..

Therefore, the given function is a solution of the given differential equation.

Ans to Q No 2 : Given, $y-\cos y=x$ ... (1)

Differentiating both sides with respect to x, we get

$$\frac{dy}{dx}-\frac{d}{dx}(\cos y)=\frac{d}{dx}(x)$$

$$\Rightarrow y'+\sin y.y'=1$$

$$\Rightarrow y'(1+\sin y)=1$$

$$\Rightarrow y' = \frac{1}{1+\sin y}$$

Substituting the values of x and y' in the given differential equation, we get

L.H.S $= (y\sin y + \cos y + x)y'$

$$= (y\sin y + \cos y + y - \cos y) \times \frac{1}{1+\sin y}$$

$$= y(1+\sin y) \times \frac{1}{1+\sin y}$$

$= y$ =R.H.S..

Hence the given function is the solution of the corresponding differential equation.

Ans to Q No 3 : Given, $x + y = \tan^{-1} y$ ... (1)

Differentiating both sides with respect to x, we get

$$\frac{d}{dx}(x+y) = \frac{d}{dx}(\tan^{-1} y)$$

$$\Rightarrow 1 + y' = \left[\frac{1}{1+y^2}\right] y'$$

$$\Rightarrow y'\left[\frac{1}{1+y^2} - 1\right] = 1$$

$$\Rightarrow y'\left[\frac{1-(1+y^2)}{1+y^2}\right] = 1$$

$$\Rightarrow y'\left[\frac{-y^2}{1+y^2}\right] = 1$$

$$\Rightarrow y' = \frac{-(1+y^2)}{y^2}$$

Substituting the value of y' the given differential equation, we get :

$$\text{L.H.S} = y^2 y' + y^2 + 1 = y^2\left[\frac{-(1+y^2)}{y^2}\right] + y^2 + 1$$

$$= -1 - y^2 + y^2 + 1$$

$=0$ =R.H.S

Hence the given function is a solution of the given differential equation.

Ans to Q No 4 : Given $y = x^{-\frac{3}{2}}$

$\therefore y' = -\frac{3}{2}x^{-\frac{5}{2}}$ and $y'' = \frac{15}{4}x^{-\frac{7}{2}}$.

Substituting the values of y, y' and y'' in the given differential equation, we get :

$$\text{L.H.S} = 4x^2\left(\frac{15}{4}x^{-\frac{7}{2}}\right) + 12x\left(-\frac{3}{2}x^{-\frac{5}{2}}\right) + 3\left(x^{-\frac{3}{2}}\right)$$

$$= 15x^{-\frac{3}{2}} - 18x^{-\frac{3}{2}} + 3x^{-\frac{3}{2}}$$

$$= 0$$

So, $y = x^{-\frac{3}{2}}$ does satisfy the differential equation and hence is a solution.



1.11 MODEL QUESTIONS

Q 1 : Form the differential equations for the following

(a) $y = k\sin^{-1}x$, k parameter.

(b) $y = \alpha x + \beta x^2$, α, β parameter.

Q 2 : Find the differential equation of all the family of curves

$y = Ae^{3x} + Be^{5x}$ for different values of A and B.

Q 3 : Find the order and degree of the following ordinary differential equations.

(a) $y' = 3x^2 - 7$ (b) $\sqrt{y' + y} = \sin x$ (c) $y''' + xy'' + 2y(y')^2 + xy = 0$

(d) $(y''')^3 + (y'')^4 + y = 0$ (e) $y'' = \left[y + (y')^2\right]^{\frac{1}{4}}$ (f) $\left[1 + (y')^2\right]^3 = ky''$

Q 4 : Classify the following equations as linear and non-linear equations and write down their orders.

(a) $\frac{d^3y}{dx^3}+\frac{d^2y}{dx^2}.\frac{dy}{dx}+y=x$

(b) $x\frac{d^4y}{dx^4}+\frac{d^2y}{dx^2}=e^x$

(c) $\frac{dy}{dx}+y^2=x^2$

Q 5 : Find the differential equation of all

(a) tangent lines to the parabola $y=x^2$.

(b) ellipses centered at the origin.

(c) circles through the origin.

(d) circles tangent to y-axis.

(e) parabolas with axis parallel to the axis of y.

(f) parabolas with foci at the origin and axis along x-axis.s

Q 6 : Verify that

(a) $y=x^5+9$ satisfies the differential equation $\frac{dy}{dx}=5x^4$.

(b) $y=\cos 2x$ satisfies the differential equation $\frac{d^2y}{dx^2}+4y=0$.

(c) $y=\sqrt{x^2+1}$ satisfies the differential equation $\frac{dy}{dx}=\frac{x}{y}$.

(d) $y=x\log x$ satisfies the differential equation $x^2\frac{d^2y}{dx^2}+y=x\frac{dy}{dx}$.

(e) $y=xe^x$ satisfies the differential equation $\frac{dy}{dx}-y=e^x$.

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