UNIT 10 : FIRST ORDER NONLINEAR PDE I

UNIT STRUCTURE

10.1 Learning objectives

10.2 Introduction

10.3 Complete Integrals, Singular Integral and General Integerals.

10.4 General method of solving nonlinear partial differential equation of first orders :Charpit's Method

10.5 Characteristics and Envelopes.

10.6 Let Us Sum Up

10.7 Answer to Check Your Progress

10.8 Further Reading

10.9 Model Question

10.1 LEARNING OBJECTIVES

After reading this unit a student will be able to

- solve different types of linear PDE by using Charpit's Method
- find characteristics equation of nonlinear PDE and also find integers surface by eliminating parameter form characteristics equation.

10.2 INTRODUCTION

Partial differential equation arise in Geometry and physics when the numbers of independent variables in the problem under discussion is two or more.

When such is the case, any dependent variable is likely to be a function of more then one variable, so that it posses not ordinary derivates with respect to a single variable but partial derivatives with respect to several variable. For instance, in the study of thermal effects in a solid body the temperature may vary from point to point in the solid as well as from time to time, and, as a consequence, the derivatives

$$\frac{\partial\theta}{\partial x},\frac{\partial\theta}{\partial y},\frac{\partial\theta}{\partial z},\frac{\partial\theta}{\partial t}$$

Will in general, be non zero. Furthermore in any particular problem it may happen that higher derivatives of the types

$$\frac{\partial^2\theta}{2x^2},\frac{\partial^2\theta}{2y},\frac{\partial^3\theta}{2x^2\partial t},\text{etc}$$

May be of physical significance.

When the laws of physics are applied to a problem of this kind, we some times obtain a relation between the derivates of the kind

$$F\left(\frac{\partial\theta}{\partial x},...,\frac{\partial^2\theta}{\partial x^2},...,\frac{\partial^2\theta}{\partial x^2\partial t},...\right)=0$$

Such an equation relating partial derivatives is called a partial differential equation.

A partial differential equation of the form $F(x,y,z,p,q)=0$,where $p=\frac{\partial z}{\partial z},q=\frac{\partial z}{\partial y}$ and F is not linear in p and q, is called non-linear partial differential equation of order one.

10.3 COMPLETE INTEGRAL, SINGULAR INTEGRAL AND GENERAL INTEGRAL.

10.3.1 Complete Integral (C.L) or Complete Solution (C.S).

Let us consider a relation

$\phi(x,y,z,a,b)=0$ ..... (1)

In which x, y, z are variables such that z is dependent on x and y. Differentiating (1) partially w.r to x and y respectively, we obtain

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial z}\frac{\partial z}{\partial x}=0 \text{ and } \quad \frac{\partial\phi}{\partial y}+\frac{\partial\phi}{\partial z}\frac{\partial z}{\partial y}=0$$

$$\Rightarrow\frac{\partial\phi}{\partial x}+\frac{\partial z}{\partial z}p=0 \text{ and } \frac{\partial\phi}{\partial y}+\frac{\partial\phi}{\partial z}\frac{\partial z}{\partial y}=0--(2)$$

Since there are two arbitrary constants (namely a and b) connected by the above three equations, these can be eliminated and there will appear a relation of the from

$f(x,y,z,p,q) = 0$ ..... (3)

Which is a partial differential equation of order one. Now ,suppose that (1) has been derived from (3), by using some method ; then the integral (1), which has as many arbitrary constants as there are independent variables, is called the complete integral of (3).

Particular Integral: A particular integral of (3) is obtained by giving particular values to a and b in (1) which is the complete integral of (3).

10.3.2 Singular Integral (S.I) or Singular Solution (S.S).

We know that the locus of all the points whose co-ordinates along with the values of p and q satisfy (3), represent the doubly infinite system of surfaces given by (1). The system is doubly infinite, since there are two constants a and b and each of these can take an infinite number of values. Since the envelope of all the surfaces given by (1) is touched at each of its points by some one of these surfaces, the coordinates of any point on the envelope along with the values of p and q belonging the envelope at that point must also satisfy (3). Hence we conclude that the equation of the envelope is a solution of (3). The envelope of the surfaces given by (2) is obtained by eliminating a and b between equation

$$\phi(x,y,z,a,b)=0, \frac{\partial\phi}{\partial a}=0 \text{ and } \frac{\partial\phi}{\partial b}=0. \quad \text{..... (4)}$$

The relation between x, y and z so obtained is called the singular integral. In general, it is distinct from the complete integral. However, in exceptional cases it may be contained in the complete integral, that is , singular integral may be obtained by giving particular values to the constants in the complete integral. Since other relation may appear in the process of getting the singular integral, it is necessary to test that the equation.

10.3.3 General Integral (G.I) or General Solution (G.S). Assume that in (1), one of the constants in a function of the other, say *b* = F (a), then (1) becomes

$$\phi(x,y,z,a,F(a))=0 \quad \text{..... (5)}$$

Now (5) represents one of the families of surfaces given by the system (1). As before, the equation of the envelope of the family of surfaces given by (5) must also satisfy (3). Again the equation so obtained will be

distinct from that of the envelope of the surfaces, and it is not a particular integral. It is know as the general integral and is obtained by eliminating a between

$$\phi\left(x,y,z,a,F(a)\right)=0 \text{ and } \frac{\partial\phi}{\partial a}=0 \quad \text{..... (6)}$$

Since other relation any appear in the process of getting the singular integral, it is necessary to test that the equation of general integral satisfies the given differential equation.

Example 1. Consider the nonlinear partial differential equation

$$z^2\left(1+p^2+q^2\right)=1 \quad \text{..... (1)}$$

Then $\left(x-a\right)^2+\left(y-b\right)^2+z^2=1$ ..... (2)

Is a solution of this equation with arbitrary constants a and b. Since it contains two arbitrary constants, therefore the solution (2) is a complete integral of the equation (1).

Putting $b=F(a)=a$ in equation (2), we obtain the one-parameter subsystem

$$\left(x-y\right)^2+\left(y-a\right)^2+z^2=1$$

Whose envelope is obtained by eliminating a between this equation and $x+y-2a=0$

So that it has equation

$$\left(x-y\right)^2+2z^2=2 \quad \text{..... (3)}$$

Differentiating both side of this equation with respect to x and y respectively, we obtain the relations

$$2zp=y-x$$
$$2zq=x-y$$

From which it follows immediately that (3) is an integral surface of the equation (1). It is a general solution of the equation (1).

The envelope of the two parameter system (1) is obtained by eliminating a and b from (2) and the two equations

$x=a$ and $y=b$

i.e the envelope consists of the pair of planes $z=\pm1$

It is readily verified that these planes are integral surfaces of the equation (1). So this is the singular integral of the equation.

12.3.4 Geometrical interpretation of three types of the equation of $f(x,y,z,p,q)=0$

(i) Complete Integral: A complete integral being a relation between x, y and z representing equation of surface. Since it involves two arbitrary parameters, it belongs to a double infinite system of family of surface.

(ii) General integral: let a complete solution of

$$f(x,y,z,p,q)=0$$

Be $\phi(x,y,z,a,b)=0$

A general integral us obtained by eliminating 'a' between (1) and the equations

$$b=\psi(a) \quad \text{..... (2)}$$

$$\frac{\partial\phi}{\partial a}+\frac{\partial\phi}{\partial b}\psi'(a)=0 \quad \text{..... (3)}$$

Where ψ is an arbitrary function.

The operation of elimination is equivalent to selecting from the system of families of surfaces a representative family and finding the envelope. Equations (1), (2) and (3) together represent a curve drawn on the surface of the family whose parameter is 'a' between them is the envelope of the family. It follows that the envelope touches the surface represented by (1) and (2) along the curve represented by (1) , (2) and (3). This curve is known as characteristic of the envelope and the general integral thus represents the envelope of a family of surface considered as composed of its characteristics.

(iii) Singular Integral: the singular integral is obtained by eliminating 'a' and 'b' between (1) and

$$\frac{\partial\phi}{\partial a}=0 \quad \text{..... (4)}$$

$$\frac{\partial\phi}{\partial b}=0 \quad \text{..... (5)}$$

The operation of eliminating is equivalent to finding the envelope of all the surfaces included in the complete integral.(1),(4)and(5) give the point of contact of the particular surfaces represented by (1) with the general envelope. It follows that the singular integral represents the general envelope of all surfaces included in the complete integral.

10.3.5 Compatible system of first order equations.

Consider first order differential equations

f(x, y, z, p,q) = 0 ..... (1)

g(x, y, z, p,q) = 0.....(2)

Equation (1) and (2) are known as compatible when every solution of one is also a solution of the other.

To find condition for (1) and (2) to be compatible :

If $J = \frac{\partial(f,g)}{\partial(p,q)} \neq 0$ .....(3)

We can solve equations (1) and (2) to obtain the explicit expressions

$p = \phi(x,y,z), q = \psi(x,y,z)$ .....(4)

For p and q. The condition that the pair of equation (1) and (2) should be compatible integrable, i.e. the equation dx = pdx+qdy

$$\Rightarrow \phi dx + \psi dy - dz = 0$$

Should be integrable, i.e.

$$\phi(\psi_z - 0) + \psi(0 - \phi_z) + (-1)(\phi_y - \psi_x) = 0$$

$$\Rightarrow \psi_x + \phi\psi_z = \phi_y + \psi\phi_z \text{ .....(5)}$$

Substitution from equation (4) into equation (1) and differentiating w.r. to x and z respectively, we obtain the equations.

$$f_x + f_p\phi_x + f_q\psi_x = 0$$

$$f_z + f_p\phi_z + f_q\psi_z = 0$$

From which it is reading deduced that

$$f_x + \phi f_z + f_p(\phi_x - \phi\phi_z) + f_q(\psi_x + \phi\psi_z) = 0$$

Similarly, we may deduce from equation (2) that

$$g_x + \phi g_z + g_p(\phi_x - \phi\phi_z) + g_q(\psi_x + \phi\psi_z) = 0$$

Solving these equations, we find that

$$\psi_x + \phi\psi_z = \frac{1}{J}\left\{\frac{\partial(f,g)}{\partial(x,p)} + \phi\frac{\partial(f,q)}{\partial(z,p)}\right\} \text{ ..... (6)}$$

Where J is defined as equation (3)

If we had differentiated the given pair of equations with respect to y and z, we should have obtained

$$\phi_y + \phi\psi_z = -\frac{1}{J}\left\{\frac{\partial(f,g)}{\partial(y,q)} + \psi\frac{\partial(f,g)}{\partial(z,q)}\right\} \text{ ..... (7)}$$

So that, substituting from equations (6) and (7) into equation (5) and replacing f, y by p, q respectively, we see that the condition that the two conditions should be compatible is that

[f, g] = 0

Where $[f,g]=\frac{\partial(f,g)}{\partial(x,p)}+p\frac{\partial(f,g)}{\partial(z,p)}+\frac{\partial(f,g)}{\partial(y,q)}+q\frac{\partial(f,g)}{\partial(z,q)}$

Example 1. Consider the equations

f(x, y, z, p, q) = xp - yq = 0 ..... (1)

and g(x, y, z, p, q) = z(xp + yq) – 2xy .....(2)

$$\frac{\partial(f,g)}{\partial(x,p)}=\begin{vmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial p}\\ \frac{\partial g}{\partial x} & \frac{\partial q}{\partial p}\end{vmatrix}=\begin{vmatrix} p & x\\ zp-2y & xz\end{vmatrix}=2xy$$

$$\frac{\partial(f,g)}{\partial(z,p)}=\begin{vmatrix}\frac{\partial f}{\partial z} & \frac{\partial f}{\partial p}\\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial p}\end{vmatrix}=\begin{vmatrix} 0 & x\\ xp+yq & xz\end{vmatrix}=-x^2p-xyq$$

and $$\frac{\partial(f,g)}{\partial(y,q)}=\begin{vmatrix}\frac{\partial f}{\partial y} & \frac{\partial f}{\partial q}\\ \frac{\partial g}{\partial y} & \frac{\partial g}{\partial q}\end{vmatrix}=\begin{vmatrix} -q & -y\\ zq-2x & zy\end{vmatrix}=-2xy$$

$$\frac{\partial(f,g)}{\partial(z,q)}=\begin{vmatrix}\frac{\partial f}{\partial z} & \frac{\partial f}{\partial q}\\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial q}\end{vmatrix}=\begin{vmatrix} 0 & -y\\ xp+yq & zy\end{vmatrix}=xyp+y^2q$$

Now $$[f,g]=\frac{\partial(f,g)}{\partial(x,p)}+\frac{\partial(f,g)}{\partial(z,p)}+\frac{\partial(f,g)}{\partial(y,q)}+\frac{\partial(f,g)}{\partial(x,q)}$$

$$=2xy-x^2p^2-xypq-2xy+xypq+y^2q^2$$

$$=y^2q^2-x^2p^2$$
$$=(yq+xp)(yq-xp)$$
$$=0[\because xp=yq]$$

Hence equation (1) and (2) are compatible.

Solving (1) and (2) for p and q, $p=\frac{y}{z}$ and $q=\frac{x}{z}$ .....(3)

Using (3) in dz = pdx + qdy

$$\Rightarrow dz = \frac{y}{z}dx + \frac{x}{z}dy$$
$$\Rightarrow zdz = ydx + xdy$$
$$\Rightarrow zdz = d(xy)$$

Integrating, $\frac{z^2}{2} = xy + \frac{c}{2}$

$\Rightarrow z^2 = 2xy + c,$ where c is an arbitrary constant.

10.4 GENERAL METHOD OF SOLVING NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF ORDER ONE (CHARPIT'S METHOD)

Let us consider the equation

f(x, y, z, p, q) = 0..... (1)

we know that

dz = pdx + qdy .....(2)

Now, we have to find another relation

F(x, y, z, p, q) = 0 .....(3)

Such that when the values of p and q obtained by solving (1) and (3), are substituted is (2), it becomes integrable. The integration of (2) will give the complete integral of (1).

To obtain (3), differentiate (1) and (3) w. R. To x and y, we get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}p + \frac{\partial f}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial x} = 0$$

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}p + \frac{\partial F}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial F}{\partial q}.\frac{\partial q}{\partial x} = 0,$$

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}q + \frac{\partial f}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial y} = 0$$

$$\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}q + \frac{\partial F}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial F}{\partial q}.\frac{\partial q}{\partial y} = 0$$

Eliminating $\frac{\partial p}{\partial x}$ from the first pair of these equations,

We get

$$\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}p + \frac{\partial f}{\partial q}\frac{\partial q}{\partial x}\right)\frac{\partial F}{\partial p} - \left(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}p + \frac{\partial F}{\partial q}\frac{\partial q}{\partial x}\right)\frac{\partial f}{\partial p} = 0$$

$$\Rightarrow \left(\frac{\partial f}{\partial x}.\frac{\partial F}{\partial p} - \frac{\partial F}{\partial x}\frac{\partial f}{\partial p}\right) + \left(\frac{\partial f}{\partial z}.\frac{\partial F}{\partial p} - \frac{\partial F}{\partial z}\frac{\partial f}{\partial p}\right)\frac{\partial f}{\partial p}p +$$

$$\Rightarrow \left(\frac{\partial f}{\partial q} - \frac{\partial F}{\partial p} - \frac{\partial F}{\partial q}\frac{\partial f}{\partial p}\right)\frac{\partial q}{\partial x} = 0 \quad \text{.....(4)}$$

Similarly, the elimination of $\frac{\partial q}{\partial y}$ between the second pair gives

$$\left(\frac{\partial f}{\partial y}\frac{\partial F}{\partial q} - \frac{\partial F}{\partial y}\frac{\partial f}{\partial q}\right) + \left(\frac{\partial f}{\partial z}\frac{\partial F}{\partial q} - .\frac{\partial F}{\partial z}\frac{\partial f}{\partial q}\right)q +$$

$$\left(\frac{\partial f}{\partial p}\frac{\partial F}{\partial q} - \frac{\partial F}{\partial p}\frac{\partial f}{\partial q} - \frac{\partial F}{\partial p}\frac{\partial f}{\partial q}\right)\frac{\partial b}{\partial y} = 0 \quad \text{..... (5)}$$

Since $\frac{\partial q}{\partial x} = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial p}{\partial y}$, so the last term in (4) is the same as that in (5), except for a minus sign and hence they cancel on adding (4) and (5). Therefore, adding (4) and (5) and rearranging the terms, we obtain

$$\left(\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}\right)\frac{\partial F}{\partial p} + \left(\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}q\right)\frac{\partial F}{\partial q} + \left(-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}\right)\frac{\partial F}{\partial z}$$

$$+\left(-\frac{\partial f}{\partial p}\right)\frac{\partial F}{\partial x} + \left(-\frac{\partial f}{\partial q}\right)\frac{\partial F}{\partial y} = 0 \quad \text{.....(6)}$$

This is a linear equation of the first order to obtain the derived function F. Integral of (6) is obtained by solving the auxiliary equations

$$\frac{\partial p}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{\partial q}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{\partial z}{-p\left(\frac{\partial f}{\partial p}\right) - q\left(\frac{\partial f}{\partial q}\right)}$$

$$= \frac{\partial x}{-\frac{\partial f}{\partial p}} = \frac{\partial y}{-\frac{\partial f}{\partial q}} = \frac{\partial F}{0} \quad \text{..... (7)}$$

Since any of the integrals of (7) will satisfy (6) an integral of (7) which involves p or q (or both) will serve along with the given equation to find p and q.

Working Rule of Charpit's method :

Step 1. Transfer all terms of the given equation to LHS and denote the entire expression by f.

Step 2. Write done the Charpit's auxiliary equations (7).

Step 3. Using the value of f in step 1 write down the values of

$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \cdots\cdots\cdots$ etc. Occuring in step 2 and put these in Charpit's auxiliary equation (7).

Step 4 After simplifying the step 3, select two proper fractions so that the resulting integral may come out to be the simplest relation involving at least one of p and q.

Step 5. The simplest relation of step 4 is solved along with the given equation to determine p and q Put these values of p and q in dz = pdx + qdy which on integration gives the complete integral of the given equation.

The singular and General integrals may be obtained in the usual manner.

Example 1. Consider the equation

Px + qy = pq

Let f(x, y, z, p, q) = px + qy – pq ..... (1)

The Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

$$\Rightarrow \frac{dp}{p+p.0} = \frac{dq}{q+q.0} = \frac{dz}{-p.(x-q)-q(y-p)} = \frac{dx}{-(x-q)} = \frac{dy}{-(y-p)}$$

$$\Rightarrow \frac{dp}{p} = \frac{dq}{q} = \frac{dz}{-px-qy+2pq} = \frac{dx}{q-x} = \frac{dy}{p-y}$$

Taking the first two fractions, we get

$$\frac{1}{p}dp = \frac{1}{q}dq$$

Integration, log p = log q + log a

$\Rightarrow$ p = aq..... (2)

Substituting this value of p in (1), we get

aqx + qy – ag^2 = 0

$\Rightarrow$ aq = ax + y as q $\neq$ 0

$\Rightarrow$ q = $\frac{1}{a}$(ax + y) .....(3)

(2) $\Rightarrow$ P = ax + y .....(4)

Putting the values of p and q from equation (3) and (4) in

dz = pdx + qdy, we get

dz = (ax + y)dx + $\frac{1}{a}$(ax + y) dy

$\Rightarrow$ dz = (ax + y) (dx + $\frac{1}{a}$ dy)

$\Rightarrow$ adz = (ax + y) (adx + dy)

$\Rightarrow$ adz = (ax + y) d(ax + y)

Integratingm az = $\frac{1}{2}$ (ax + y)^2 + b

Which in complete integral, where a and b being arbitrary constants.

Example 2. Consider the nonlinear partial differential equation

Z = pq ..... (1)

Let f(x, y, z, p, q) $\equiv$ z – pq = 0

The Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}$$

$$\Rightarrow \frac{dp}{0+p.1}=\frac{dq}{0+q.1}=\frac{dz}{-p.(-q)-q(-p)}=\frac{dx}{-(-q)}=\frac{dy}{-p}$$

$$\Rightarrow \frac{dp}{p}=\frac{dq}{q}=\frac{dz}{2pq}=\frac{dx}{q}=\frac{dy}{p}$$

Taking first and middle fractions,

$$\frac{dp}{p}=\frac{dz}{2pq}$$

$$\Rightarrow \frac{dp}{p}=\frac{dz}{2z} \qquad [\because pq=z]$$

Integrating log p = $\frac{1}{2}$ log z + $\frac{1}{2}$ log a

$\Rightarrow$ p2 = az

$\Rightarrow$ p = $\sqrt{az}$

Substituting the value of p in (1), we get

$$z=\sqrt{az}q$$

$$\Rightarrow q=\sqrt{\frac{z}{a}}$$

Putting these values of p and q in dz = pdx + qdy,

We get dz = $\sqrt{az}dx+\sqrt{\frac{z}{a}}dy$

$$\Rightarrow \frac{1}{\sqrt{z}}dz = \sqrt{a}dx+\frac{1}{\sqrt{a}}dy$$

Integrating,

$$2\sqrt{z} = \sqrt{a}x + \frac{1}{\sqrt{a}}y + b$$

Which is a complete integral, a and b being arbitrary constants.

Example 3. Consider the differential equation

$F(x, y, z, p, q) = (p^2 + q^2)\, y - qz = 0$ ..... (1)

The Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

$$\Rightarrow \frac{dp}{0 + p.(-q)} = \frac{dq}{p^2 + q^2 + q(-q)} = \frac{dz}{-p.2py - q(2qy - z)} = \frac{dx}{-2pq} = \frac{dy}{-2qy + z}$$

$$\Rightarrow \frac{dp}{-pq} = \frac{dq}{p^2} = \frac{dz}{qz - 2p^2y - 2q^2y} = \frac{dx}{-2py} = \frac{dy}{-2qy + z}$$

Taking the first two fractions, we get

$pdp + qdq = 0$

Inregrating, $p^2 + q^2 = a^2$, where a is arbitrary constant.

Fron (1), $a^2y = qz \Rightarrow q = \frac{a^2y}{z}$

Putting the value of q in $p^2 + q^2 = a^2$, we get

$$p^2 + \left(\frac{a^2y}{z}\right)^2 = a^2$$

$$\Rightarrow p^2 = a^2 - \frac{a^4y^2}{z^2}$$

$$\Rightarrow p^2 = \frac{a}{z}\sqrt{z^2 - a^2y^2}$$

Now putting these values of p and q in $dz = pdx + qdy$,

$$dz = \frac{a}{z}\sqrt{z^2 - a^2y^2}\,dx + \frac{a^2y}{z}dy$$

$$\Rightarrow zdz = a\sqrt{z^2 - a^2y^2}\,dx + a^2y\,dy$$

$$\Rightarrow \frac{1}{2}\frac{2zdz - 2a^2ydy}{\sqrt{z^2 - a^2y^2}} = adx$$

$$\Rightarrow \frac{1}{2}\frac{2zdz - 2a^2dy}{\sqrt{z^2 - a^2y^2}} = adx$$

$$\Rightarrow \frac{1}{2}\frac{d\left(z^2 - a^2y^2\right)}{\sqrt{z^2 - a^2y^2}} = adx$$

Integrating, we get

$$\frac{1}{2}\sqrt{z^2 - a^2y^2} = ax + b$$

$\Rightarrow z^2 - a^2y^2 = (ax + b)^2$

$\Rightarrow z^2 = a^2y^2 + (ax + b)^2$ ..... (2)

Which is requived complete integral.

Singular Integral : Differentiating (2) partially w.r. to a and b, we have

$0 = 2ay^2 + 2(ax + b)x \Rightarrow ay^2 + ax^2 + bx = 0$ ..... (3)

$0 = 0 + 2(ax + b) \Rightarrow ax + b = 0$ .....(4)

From (3) and (4), $a^2y^2 = 0$ .....(5)

From (2), (4) and (5), we get

$Z = 0$

Which is clearly satisfies (1) and hence it is the singular integral.

General Integral : Replacing b by f(a) in (2), we get

$Z^2 - a^2y^2 = [ax + f(a)]^2$ .....(6)

Differeentiating (6) partially w.r. to a, we get

$-2ay^2 = 2[ax + f(a)].[x+p(a)]$ .....(7)

Eliminating a from (6) and (7), we get the general solution.



CHECK YOUR PROGRESS

Q.1: Find the complete integral of the following non linear partial differential equations :

(i) $zpq = p + q$

(ii) $(p^2 + q^2)x = pz$

(iii) $q = px + p^2$

(iv) $z = px + qy + p^2 + q^2$

(v) $p^2x + q^2y = z$

(vi) $pq = px + qy$

10.4.1 Standard From 1. Only p and q present. Under this standard form, we consider equation of the form $f(p, q) = 0$.....(1)

The Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}$$

$$\Rightarrow \frac{dp}{0}=\frac{dq}{0}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}$$

Taking first ratio, $dp=0$

Integrating, $p=a$, constant .....(2)

Substituting in (1), we get $f(a,q)=0$

$\Rightarrow q=b$ (say).....(3)

Where b is such that $f(a,b)=0$

$\therefore dz=pdx+qdy$

$=adx+bdy$

Integrating, we get

$z=ax+by+c.....(5)$

Where c is a arbitrary constant. (4) and (5) give the required solution.

Now solving (4) for b, suppose we obtain $b=F(a)$. Putting this value of b in (5), we get

$z=ax+F(a)y+c.....(6)$

Which is complete integral of (1) contains two arbitrary constants which is equal to the number of independent variables.

Differentiating (6) partially with respect to a and c, we get $0=x+yF'(a)$ and $0=1$

Which is impossible. Therefore we conclude that the equations of this form have no singular solution.

In order to find the general integral of (1), we first take $c=\phi(a)$ in (6), we get

$z=ax+F(a)y+\phi(a).....(8)$

We differentiate (8) partially w.r. to a and get

$0=x+F'(a)y+\phi'(a).....(9)$

Eliminating (a) between (8) and (9), we get the general solution of (1).

Example 4. Consider the equation

$pq=k$,where k is a constant.....(1)

Since (1) is of the form $f(p,q)=0$,it solution is

$$z=ax+by+c.....(2)$$

Where $ab=k\Rightarrow b=\frac{k}{a}$,putting a for p and b for q in (1).

Form (2), the complete integral is

$$z=ax+\frac{k}{a}y+c.....(3)$$

Which contains two arbitrary constants a and c.

For singular solution differentiating (3) partially with respect to a and c, we get

$$0=x-\frac{k}{a^2}y \text{ and } 0=1$$

But $0=1$ is absurd. Hence there is no singular solution of (1).

To find the general solution, put $c=\phi(a)$ in (3). Then , we get

$$z=ax+\left(\frac{k}{a}\right)y+\phi(a).....(4)$$

Differentiating (4) partially w.r. to 'a', we get

$$0=x-\left(\frac{k}{a^2}\right)y+\phi'(a).....(5)$$

Eliminating a from (4) and (5), we get the requited general solution.



CHECK YOUR PROGRESS

Q.2. Solve the following nonlinear partial differential equations :

(i) $p^2 + q^2 = m^2$ where m is a contant.

(ii) p + q = pq

10.4.2. Equations reducible to standard form 1.

Example 5. Consider the equation.

$X^2p^2 + y^2q^2 = z$

The equation can be rewrite as

$$\frac{x^2}{x}\left(\frac{\partial z}{\partial x}\right)^2+\frac{y^2}{z}\left(\frac{\partial z}{\partial x}\right)^2=1$$

$$\Rightarrow\left(\frac{x}{\sqrt{z}}\frac{\partial z}{\partial x}\right)^2+\left(\frac{y}{\sqrt{z}}\frac{\partial z}{\partial x}\right)^2=1 \text{ ..... (1)}$$

Put $\frac{1}{x}dx = dx, \frac{1}{y}dy = dy$ and $\frac{1}{\sqrt{z}}dz = dZ$ .....(2)

So that log x = X, log y = Y and $2\sqrt{z} = Z$ ..... (3)

Using (2), (1) becomes $\left(\frac{\partial Z}{\partial X}\right)^2 + \left(\frac{\partial Z}{\partial Y}\right)^2 = 1$

$\Rightarrow$ $P^2 + Q^2 = 1$ .....(4)

Where $P = \frac{\partial Z}{\partial X}$ and $Q = \frac{\partial Z}{\partial Y}$. So (4) is of the form f(P, Q) = 0

Thus solution of (4) is Z = aX + bY + c .....(5)

Where $a^2 + b^2 = 1$ i.e. $b = \sqrt{1 - a^2}$.

Therefore the required complete solution is

$$Z = aX + \sqrt{1 - a^2}Y + c$$

$\Rightarrow 2\sqrt{z} = a \log x + \sqrt{1 - a^2} \log y + c$

$\Rightarrow 2\sqrt{z} = \log xa + \log y^{\sqrt{1-a^2}} - \log c'$ where c = - log c'

$$\Rightarrow 2\sqrt{z} = \log\left(\frac{x^a y^{\sqrt{1-a^2}}}{c'}\right)$$

$$\Rightarrow x^a y^{\sqrt{1-a^2}} = c' e^{2\sqrt{z}}$$

Where a and c' are two arbitrary constant.



CHECK YOUR PROGRESS

Q. 3: Find the complete integral by reducing to standard form I,

$$x^2p^2 + y^2q^2 = z^2$$

10.4.3. Standard form II. Clairaut equation. A first order partial differential equation is said to be of clairaut form if it can be written in the form

Z = px + qy + f(p,q) ..... (i)

Let F(x, y, z, p, q) = px + qy + f(p, q) –z ..... (ii)

The charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z}} = \frac{dq}{\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z}} = \frac{dz}{-p\frac{\partial F}{\partial p} - q\frac{\partial F}{\partial q}} = \frac{dx}{-\frac{\partial F}{\partial p}} = \frac{dy}{-\frac{\partial F}{\partial q}}$$

$$\Rightarrow \frac{dp}{p+p(-1)} = \frac{dq}{q+q(-1)} = \frac{dz}{-p\left(x+\frac{\partial f}{\partial p}\right)-q\left(y+\frac{\partial f}{\partial q}\right)}$$

$$= \frac{dx}{-x-\frac{\partial f}{\partial p}} = \frac{dy}{-y-\frac{\partial f}{\partial q}}$$

Then first and 2nd Fraction imples

dp = 0 and dq = 0

Integrating, p = a and q = b, where a and b are arbitrary constant.

Substituting these values in (1), the complete integral is

Z = ax + by + f (a, b)

The singular and general integrals can be obtained by usual method.

Example 6. Consider the Clairaut form of equation

Z = px + qy + pq

The complete integral is

Z = ax + by + ab..... (i)

Singular Integral : Differentiating (i) partially w.r. to a and b, we get

0 = x + b and 0 = y + a .....(ii)

Eliminating a and b between (i) and (ii), we get

$z = (-y)x + (-x)y + (-x)(-4)$

$\Rightarrow$ z + xy= 0

General Integral : Ptting b = f(a)in equation (i), we get

Z= ax + ϕ (a)y + aϕ (a)..... (iii)

Differentiating (iii) partially w. r. to a, we get

z = x + ϕ' (a) y + aϕ' (a) + ϕ (a) ..... (iv)

The general integral is obtained by eliminating a between (iii) and (iv)



CHECK YOUR PROGRESS

Q. 4: Solve (i) $z = px + qy + c\sqrt{1+p^2+q^2}$

(ii) $z = px + qy + \log(pq)$

(iii) $z = px + qy - 2\sqrt{pq}$

10.4.4 Standard form III. Only p, q and z present.

We consider the differential equation of the form

F(p, q, z) = 0.....(1)

The first two ratios of charpit's auxiliary equation gives

$$\frac{1}{p}dp = \frac{1}{q}dq$$

Integrating, log p = log q by a

$\Rightarrow$ ap = q where a is arbitrary constant.

Then dz = pdx + qdy becomes

dz = pdx + apdy

= p (dx + ady)

= pd(x + ay)

= pdu ..... (2)

Where u = x + ay .....(3)

(2) $\Rightarrow$ p = $= \frac{dz}{du}$ and then

q = ap = a $\frac{dz}{du}$

Substituting these values of p and q in (1), we get

$$f\left(\frac{dz}{du}, a\frac{dz}{du}, z\right) = 0 \text{ ..... (4)}$$

which is an ordinary differential equation of order one. Solving (4), we get z as a function of u.

Complete integral then obtained by replacing u by x + ay.

Example 7. Consider the equation

$$q(p^2z + q^2) = 4$$

Which is of the form f(p, q, z) = 0, Let u = x + ay, where a is an arbitrary constant. Replacing p and q by $\frac{dz}{du}$ and $a\frac{dz}{du}$ respectively, we get

$$q\left[\left(\frac{dz}{du}\right)^2 z + \left(a\frac{dz}{du}\right)^2\right] = 4$$

$$\Rightarrow \left(\frac{dz}{du}\right)^2 (z + a^2) = \frac{4}{q}$$

$$\Rightarrow \frac{dz}{du} = \pm\frac{2}{3}\frac{1}{\sqrt{z + a^2}}$$

$$\Rightarrow \pm\left(z+a^2\right)^{\frac{1}{2}} dz = \frac{2}{3} du$$

Integrating, $\pm\frac{2}{3}\left(z+a^2\right)^{\frac{3}{2}}$ where b is arbitrary constant

$$\Rightarrow u + b = \pm(z+a^2)^{3/2}$$

$$\Rightarrow (u + b)^2 = (z + a^2)^3$$

$$\Rightarrow (x + ay + b)^2 = (z + a^2)^3 \qquad [\because u = a + ay]$$

Which is a complete integral.



CHECK YOUR PROGRESS

Q.5: Find complete integrals :

(i) $z = pq$

(ii) $pz = 1 + q^2$

Q.6: Find a complete and singular integral of

$z^2(p^2 + q^2 + 1) = k^2$

10.4.5 Standard form IV. Equation of the form $f_1(x, p) = f_2(y, q)$.

Let $F(x, y, z, p, q) = f_1(x, p) - f_2(y, q) = 0$ ..... (1)

The Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z}} = \frac{dq}{\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z}} = \frac{dz}{-p\frac{\partial F}{\partial p} - q\frac{\partial F}{\partial q}} = \frac{dx}{-\frac{\partial F}{\partial p}} = \frac{dy}{-\frac{\partial F}{\partial q}}$$

$$\Rightarrow \frac{dp}{\frac{\partial f_1}{\partial x}} = \frac{dq}{-\frac{\partial f_2}{\partial y}} = \frac{dz}{-p\frac{\partial f_1}{\partial p} + q\frac{\partial f_2}{\partial q}} = \frac{dx}{-\frac{\partial f_1}{\partial p}} = \frac{dy}{-\frac{\partial f_2}{\partial q}}$$

Taking the first and the fourth ratios, we have

$$\frac{\partial f_1}{\partial p} dp + \frac{\partial f_1}{\partial x} dx = 0$$

$$\Rightarrow df_1 = 0$$

Integrating $f_1 = a$, a being an arbitrary constant.

(1) $\Rightarrow f_1(x, p) = f_2(y, q) = a$

$\Rightarrow f_1(x, p) = a$ and $f_2(y, q) = a$ .....(2)

From (2), an solving for p and q, we get

$P = f_1(x, a)$ and $q = f_2(y, a)$ .....(3)

Substituting these values in dz = pdx + qdy, we get

$$dz = F_1(x,a)dx + F_2(y,a)dy$$

Integrating, $z = \int F_1(x,a)dx + \int F_2(y,a)dy + b$

Which is a complete integral containing two arbitrary constants a and b.

Example 8. Consider the differential equation

Which is of the standard form $f_1(x,p) = f_2(y,q)$. Therefore equation each side to an arbitrary constants a, we get

$p - 3x^2 = a$ and $q^2 - y = a$

$\Rightarrow p = a + 3x^2$ and $q = (a+y)^{\frac{1}{2}}$

Putting these values of p and q in dz=pdx+qdy, we get

$$dz = (a + 3x^2)dx + (a+y)^{\frac{1}{2}}dy$$

Integrating, $z = ax + x^3 + \frac{2}{3}(a+y)^{\frac{3}{2}} + b$

Which is complete integral containing two arbitrary constants a and b.



CHECK YOUR PROGRESS:

Q.7: Find complete integral of the following partial differential equation:

(i) $q = px + p^2$

(ii) $py + qx + pq = 0$

(iii) $z^2(p^2 + q^2) = x^2 + y^2$

(iv) $p^2 + q^2 = z^2(x+y)$

(v) $p^2 + q^2 = (x^2 + y^2)z$

10.5 CHARACTERISTICS AND ENVELOPES

Consider the non-linear partial differential equation

$$F\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{y}\right) = 0 \quad \text{..... (1)}$$

The plane passing through the point P(x_0, y_0, z_0) with its normal parallel to the direction n defined by the direction ratios (p_0, q_0, -1) is uniquely specified by the set of numbers D(x_0, y_0, z_0, p_0, q_0), Conversely any such

set of five real number defines a plane in three dimensional space. For this reason a set of five numbers D(x, y, z, p, q) is called a plane element of the space. In particular, a plane element (x_0, y_0, z_0, p_0, q_0) whose components satisfy an equation

$$F(x, y, z, p, q) = 0 \quad \ldots\ldots (2)$$

is called an integral element of the equation (2) at the point (x_0, y_0, z_0). It is theoretically possible to solve an equation of the type (2) obtain an expression.

$$Q = G(x, y, z, p) \quad \ldots\ldots (3)$$

from which to calculate q when x, y, z and p are known. Keeping x_0, y_0, z_0 fixed and varying p, we obtain a set of plane elements { x_0, y_0, z_0,p, G(x_0, y_0, z_0,p)} which depend on the single parameter p. As p varies, we obtain a set of plane elements all of which pass through the point P and which therefore envelope a cone with vertex P; the cove so generated is called the elementary cove of equation (2) at the point P.

Consider now a surface S whose equation is

$$Z = g(x, y) \quad \ldots\ldots (4)$$

If the function g(x, y) and its partial derivatives g_x and g_y are continuous in a certain region R of the xy-plane, then the tangent plane at each point of S determines a plane elements of the type

$$\{ x_0, y_0, g(x_0, y_0), g_x(x_0, y_0), g_y(x_0, y_0)\} \quad \ldots\ldots (5)$$

Which we shall call the tangent element of the surface S at the point (x_0, y_0, g(x_0, y_0))

Therefore 10.5.1. A necessary and sufficient condition that a surface to be an integral surface of a partial differential equation is that at each point its tangent element should touch the elementary cone of the equation.

A curve C with parametric equations

$$a = x(t), y=y(t), z=z(t) \quad \ldots\ldots (6)$$

lies on the surface (4) if

z(t) = g(x/t), y(t))

For all value of t in the appropriate interval I. If P_0 is a point on this curve determines by the parameters t_0, then the direction ratios of the tangent tine P_0P_1 are $x'(t_o), y'(t_o).z'(t_0)$, where $x'(t_0) = \frac{dx}{dt}\Big]_{t=t_0}$ etc. This direction will

be perpendicular to the direction (p_0, q_0, -1) if

$$x'(t_0) = p_0 + y'(t_0).q_0 + z'(t_0).(-1) = 0$$

$$\Rightarrow z'(t_0) = p_0 x'(t) + q_0 y'(t_0)$$

For this reason we say that any {x(t), y(t), z(t), p(t), q(t)} .....(7)

Of five real functions satisfying the condition.

$$z'(t) = p(t)x'(t) + q(t)y'(t) \text{ .....(8)}$$

Defines a strip at the point (x, y, z) of the curve C. If such a strip is also an integral element of equation (2), we say that it is an integral strip of equation (2); i.e., the set functions (7) is an integral strip of equation (2); provided they satisfy condition (8) and the further condition

F(x|t), y(t), z(t), p(t), q(t)) = 0 ..... (9)

For all t in I.

If at each point the curve (6) touches a generator of the elementary cone, we saw that the corresponding strip is a characteristic strip. We shall now derive the equations determining a characteristic strip. The point (x + dx, y + dy, z+dz) lies in the tangent plane to the elementary cone at P if.

cz = pdx + qdy .....(10)

Where p, q satisfy the relation (2). Differentiating (10) with respect to p, we obtain

$$0 = dx + \frac{dq}{dp}dy \text{ ..... (11)}$$

From (2), $\frac{\partial F}{\partial p} + \frac{\partial F}{\partial q}.\frac{dq}{dp} = 0$ ..... (12)

From equations (10), (11) and (12),

$$\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{dz}{pF_p + qF_q} \text{ ..... (13)}$$

That along a characteristic strip $x'(t), y'(t), z'(t)$ must be proportional to Fp, Fq, pFp + qFq respectively. If we choose the parameter t in such a way that

$$x'(t) = F_p, y'(t) = F_q \text{ ..... (14)}$$

Then $z'(t) = pF_p + qF_q$ .....(15)

Along a characteristic strip, p is a function of t so that

$$p'(t) = \frac{\partial p}{\partial x}x'(t) + \frac{\partial p}{\partial y}y'(t)$$

$$= \frac{\partial p}{\partial x}\frac{\partial F}{\partial p} + \frac{\partial p}{\partial y}\frac{\partial F}{\partial q}$$

$$= \frac{\partial p}{\partial x}\frac{\partial F}{\partial p} + \frac{\partial q}{\partial x}\frac{\partial F}{\partial q} \qquad \left[\because \frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}\right]$$

Differentiating equation (2) partially with to x, we find that

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}p + \frac{\partial F}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial F}{\partial q}\frac{\partial q}{\partial x} = 0$$

So that on a characteristic strip

$$p'(t) = -(F_x + pF_z) \text{ .....(16)}$$

And it can be shown similarly that

$$q'(t) = -(F_y + qF_z) \text{ .....(17)}$$

Collecting equations (14) to (17) together, we see that we have that following ordinary differential equations for the determination of the characteristic strip

$$\left.\begin{aligned} x'(t) &= F_p, y'(t) = F_q, z'(t) = pF_p + qF_q \\ p'(t) &= -F_x - pF_z, q'(t) = -F_y - qF_z \end{aligned}\right\} \text{ ..... (18)}$$

This equations are known as the characteristic equation of the differential equation (2). If the functions which appear in equations (18) satisfy a Lipschitz condition, then there is a unique solution of the equations for each prescribed set of initial value of the variables, Therefore the characteristic strip is determined uniquely by any initial element $(x_0, y_0, z_0, p_0, q_0)$ and any initial value t_0 of t.

Theoram 10.5.2. Along every characteristic strip of the equation F(x, y, z, p, q)=0 the function F(x, y, z, p, q) is a constant.

Along a characteristic strip, we have

$$\frac{dx}{dt}F(x(t), y(t), z(t), z(t), p(t), q(t)$$

$$= F_x.x'(t) + F_y.y'(t) + F_z.z'(t) + F_p p'(t) + F_q.q'(t)$$

$$= F_x.F_p + F_y.F_g + F_z(PF_p + qF_q.q) + F_p(-F_x - pF_z) + F_q(-F_y - qF_z)$$

= 0

So that F(x, y, z, p, q) = k, a constant along the strip.

Theorem 10.5.3. If a characteristic strip contains at least one integral element of F(x, y, z, p, q) = 0 it is an integral strip of the equation F(x, y, z, p, q) = 0.

Suppose we wish t find the partial differential equation (1) which passes through a curve Γ whose freedom equations are

$$x = \theta(\nu), y = \phi(\nu), z = \lambda(\nu) \quad \text{..... (19)}$$

Then in the solution

$$x = x(p_0, q_0, x_0, y_0, z_0, t_0, t) \quad \text{.....(20)}$$

In the characteristic equation (18) we may take

$$x_0 = \theta(\nu), y_0 = \phi(\nu), z_0 = \lambda(\nu)$$

As the initial values of x,y,z. The corresponding initial values of p_0, q_0 are determined by the relations

$$\lambda(\nu) = p_0\theta'(\nu) + q_0(\phi'(\nu)$$

$$F(\theta(\nu), \phi(\nu), \lambda(\nu), P_0, q_0) = 0$$

We substitute these values of x_0, y_0, z_0, p_0, q_0 and the appropriate value of t0 in equation (20), we find that x, y, z can be express in terms of two parameters t,ν to give

$$x = x_1(\nu, t), y = y_1(\nu, t), z = z_1(\nu, t), \quad \text{..... (21)}$$

Eliminating ν, t from these equations, we get a reelation

ψ (x, y, z) = 0

This is the equation of the integral surface of equation (1) through the curve Γ.

Example 1. Consider the equation

$$z = \frac{1}{2}(p^2 - q^2) + (p - x)(q - y) \quad \text{.....(1)}$$

We have to find its integral surface which passes through x-axis which is given by the equation

Y = 0 and z = 0.....(2)

Re-writing (2) in parametric form

X = λ, y = 0, z = 0 .....(3)

λ being the parameter.

Let the initial values x_0, y_0, z_0, p_0, q_0 of x, y, z, p, q be taken as

$$x_0 = x_0(\lambda) = \lambda, y_0 = y_0(\lambda) = 0, z = z_0(\lambda) = 0 \quad \text{..... (4)}$$

Since initial values x_0, y_0, z_0, p_0, q_0 satisfy (1), we have

$$z_0 = \frac{1}{2}\left(p_0^2 + q_0^2\right) + (p_0 - x_0)(q_0 - y_0)$$

$$\Rightarrow\ 0 = \frac{1}{2}\left(p_0^2 + q_0^2\right) + (p_0 - \lambda)(q_0 - 0)$$

$$\Rightarrow\ p_0^2 + q_0^2 + 2p_0q_0 - 2q_0\lambda = 0 \quad \text{.....(5)}$$

Also, we have $z_0'(\lambda) = p_0x_0'(\lambda) + q_0y_0'(\lambda)$

$$\Rightarrow 0 = p_0.1 + q_0.0$$

$$\Rightarrow p_0 = 0 \quad \text{.....(6)}$$

Solving (5) and (6), $p_0 = 0$ and $q_0 = 2\lambda$ ..... (4)*

Collecting relations (4) and (4)* together, we get

$x_0 = \lambda, y_0 = 0, z_0 = 0, p_0 = 0, q_0 = 2\lambda$ where $t = t_0 = 0$ .....(7)

Let $f(x,y,z,p,q) = \frac{1}{2}\left(p^2 + q^2\right) + pq - py - qx + xy - z$ .....(8)

The usual characteristics equations of (8) are given by

$$\frac{dx}{dt} = \frac{\partial f}{\partial p} = p + q - y \quad \text{.....(9)}$$

$$\frac{dy}{dt} = \frac{\partial f}{\partial q} = q + p - x \quad \text{.....(10)}$$

$$\frac{dz}{dt} = p\left(\frac{\partial f}{\partial p}\right) + q\left(\frac{\partial f}{\partial q}\right) = p(p + q - y) + q(q + p - x) \quad \text{.....(11)}$$

$$\frac{dp}{dt} = -\frac{\partial f}{\partial x} - p\frac{\partial f}{\partial z} = p + q - y \quad \text{.....(12)}$$

$$\frac{dq}{dt} = -\frac{\partial f}{\partial x} - q\frac{\partial f}{\partial z} = p + q - x \quad \text{.....(13)}$$

From (9) and (12), $\frac{dx}{dt} - \frac{dp}{dt} = 0$ so that

$x - p = c_1$ .....(14)

Where c_1 is an arbitrary constant. Using initial condition (7), (14) gives

$\lambda - 0 = c_1 \Rightarrow c_1 = \lambda$

Hence (14) reduces to $x - p = \lambda \Rightarrow x = p + \lambda$ .....(15)

From (10) and (13), $\frac{dy}{dt} - \frac{dq}{dt} = 0$ so that $y - q = c_2$ .....(16)

Where c_2 is an arbitrary constant.

Using initial condition (7), (16) gives

$0 - 2\lambda = c_2 \Rightarrow c_2 = -2\lambda$

Hence (16) reduces to $y - q = -2\lambda \Rightarrow y = q - 2\lambda$ .....(17)

From (9),(12) and (13),

$$\frac{d(p+q-x)}{dt}=\frac{dp}{dt}+\frac{dq}{dt}-\frac{dx}{dt}=p+q-y+p+q-x-(p+q-y)$$

$$\Rightarrow \frac{d(p+q-x)}{dt}=p+q-x$$

$$\Rightarrow \frac{d(p+q-x)}{p+q-x}=dt$$

Integrating, $\log(p+q-x)$-$\log c_3 = t$

$\Rightarrow p+q-x=c_3e^t$ .....(18)

Where c_3 is an arbitrary constant. Using initial conditions (7),(18)gives $0+2\lambda-\lambda=c_3 \Rightarrow c_3=\lambda$

Hence (18) reduces to $p+q-x=\lambda e^t$ .....(19)

From (10),(12)and (13),

$$\frac{d(p+q-y)}{dt}=\frac{dp}{dt}+\frac{dq}{dt}-\frac{dy}{dt}=p+q-y+p+q-x-(q+p-x)$$

$$\Rightarrow \frac{d(p+q-y)}{dt}=p+q-y$$

$$\Rightarrow \frac{d(p+q-y)}{p+q-y}=dt$$

Integrating, $\log(p+q-y)-\log c_4 = t$

$\Rightarrow p+q-y=c_4e^t$ .....(20)

Where c_4 is an arbitrary constant. Using initial condition (7),(20) gives $0+2\lambda-0=c_4 \Rightarrow 2\lambda$.

Hence (20) reduces to $p+q-y=2\lambda e^t$ .....(21)

From (9)and(21), $\frac{dx}{dt}=2\lambda e^t$ so that $x=2\lambda e^t+c_5$ .....(22)

Where c_5 is an arbitrary constant. Using initial condition (7), (22) gives $\lambda=2\lambda+c_5 \Rightarrow c_5=-\lambda$

Hence (22) reduces to $x=2\lambda e^t = \lambda$

$$=\lambda(2e^t-1)\text{.....(23)}$$

From (10) and (19), $\frac{dy}{dt}=\lambda e^t \Rightarrow y=\lambda e^t+c_6$ .....(24)

Where c_6 is an arbitrary constant. Using initial condition (7), (24) gives $0=\lambda+c_6 \Rightarrow -\lambda=c_6$

Hence (24) reduces to $y = \lambda e^t - \lambda$

$$= \lambda\left(e^t - 1\right) \text{.....(25)}$$

Substituting value of y from (17) in (12), we get

$$\frac{dp}{dt} = p + q - (q - 2\lambda)$$

$$\Rightarrow \frac{dp}{dt} - p = 2\lambda \text{ .....(26)}$$

Which is linear whose $I.F = e^{\int(-1)dt} = e^{-t}$ and solution is

$$pe^{-t} = \int 2\lambda . e^{-t} dt + c_7$$

$$\Rightarrow p = -2\lambda + c_7 e^t \text{ .....(27)}$$

Where c_7 is an arbitrary constant. Using initial conditions (7),(27) gives

$$0 = -2\lambda + c_7 \Rightarrow c_7 = 2\lambda$$

Hence (27) reduces to $p = -2\lambda + 2\lambda e^t$

$$= 2\lambda\left(e^t - 1\right) \text{.....(28)}$$

Substituting value of x from (15) in (13), we get

$$\frac{dq}{dt} = p + q - (p + \lambda)$$

$$\Rightarrow \frac{dq}{dt} - q = -\lambda \text{ .....(29)}$$

Which is a linear equation whose integrating factor $= e^{\int(-1)dt} = e^{-t}$ and solution is

$$qe^{-t} = \int (-\lambda) e^{-t} dt + c_8$$

$$\Rightarrow q = \lambda + c_8 e^t \text{ .....(30)}$$

Where c_8 is an arbitrary constant. Using initial conditions (7), (30) gives

$$2\lambda = \lambda + c_8 \Rightarrow c_8 = \lambda$$

Hence (30) reduces to $q = \lambda + \lambda e^t$

$$= \lambda\left(e^t + 1\right) \text{.....(31)}$$

Substituting the values of p+q-x and p+q-y from (13) and (24) respectively in (1) gives

$$\frac{dz}{dt} = p\left(2\lambda e^t\right) + q\left(\lambda e^t\right)$$

$= 2\lambda\left(e^t - 1\right)2\lambda e^t + \lambda\left(1 + e^t\right)\lambda e^t$ [on putting the values of p anf q with the help of (28) and (31)]

$$\Rightarrow \frac{dz}{dt} = 5\lambda^2 e^{2t} - 3\lambda^2 e^t$$

Integrating,

$$z = \frac{5}{2}\lambda^2 e^{2t} - 3\lambda^2 e^t + C_9 \quad \text{..... (32)}$$

Where Cq is an arbitrary constant. Using initial conditions (7), namely $z = 0$ where $t = 0$, (32) gives

$$0 = \frac{5}{2}\lambda^2 - 3\lambda^2 + cq \Rightarrow cq = 3\lambda^2 - \frac{5}{2}\lambda^2 = \frac{1}{2}\lambda^2$$

Hence (32 reduces to

$$z = \frac{5}{2}\lambda^2\left(e^{2t} - 1\right) - 3\lambda^2\left(e^t - 1\right) \quad \text{.....(33)}$$

Solving (23) and (25) for λ and e^t, we get

$$\lambda = x - 2y \text{ and } e^t = \frac{x - y}{x - 2y}$$

Putting the values of λ and e^t in (33), we get

$$z = \frac{5}{2}(x - 2y)^2\left\{\left(\frac{x - y}{x - 2y}\right)^2 - 1\right\} - 3(x - 2y)^2\left\{\frac{x - y}{x - 2y} - 1\right\}$$

$$\Rightarrow z = \frac{1}{2}y(4x - 3y)$$

Which is the required solution.



CHECK YOUR PROGRESS

Q.8: Find the characteristic of the equation pq=z and determine the integral surface which passes though the parabola $x = 0, y^2 = z$.



10.6 LET US SUM UP

- A partial differential equation of the form F(x, y, z, p, q) = 0 where $p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}$ and F is not linear in p and q is called non-linear partial differential equation of order one.
- Let us consider a relation

$$\phi(x, y, z, a, b) = 0 \quad \text{..... (1)}$$

Where a and b are arbitrary constant. Suppose (1) has been derived from $f(x, y, z, p, q) = 0$, by using some method, then the integral (1), which has many arbitrary constants as there are independent variables is called the complete integrable.

- The Charpit's auxiliary equations of solving non-linear partial differential equation f(x, y, z, p, q) are

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}$$

- The characteristic equations of the non-linear partial differential equation F(x, y, z, p, q) = 0 are

$x'(t)=F_p, y'(f)=F_q, z'(t)=pF_p+qF_q$

$p'(t)=-Fx-pF_z$ and $q'(t)=-Fy-qF_z$ where t is a parameter.

10.7 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q No 1:

(i) Solution: let $f(x,y,z,p,q)=zpq-p-q=0$ .....(1)
the Charpit's auxiliary equation are

$$\frac{dp}{f_x+pf_z}=\frac{dq}{f_y+qf_z}=\frac{dz}{-pf_p-qf_q}=\frac{dx}{-f_p}=\frac{dy}{-f_q}$$

$$\Rightarrow \frac{dp}{0+p.pq}=\frac{dq}{0+q.p.q}=\frac{dz}{-p(zq-1)-q(zp-1)}=\frac{dx}{-(zq-1)}=\frac{dy}{-(zp-1)}$$

$$\Rightarrow \frac{dp}{p^2q}=\frac{dq}{pq^2}=\frac{dz}{-2pqz-p+q}=\frac{dx}{1-zq}=\frac{dy}{1-zp}$$

From the first two ratios,

$\frac{dp}{p}=\frac{dq}{q}$

$\Rightarrow$ log p = log q + log a, where a is a positive arbitrary constant

$\Rightarrow p=aq$ .....(2)

From (1) and (2), $zaq.q-aq-q=0$

$$\Rightarrow zaq-a-1=0$$

$$\Rightarrow q=\frac{1+a}{az}$$

And $$p=\frac{1+a}{z}$$

Putting the values of p and q in dz=pdx+qdy, we get

$$dz = \left(\frac{1+a}{z}\right)dx + \left(\frac{1+a}{az}\right)dy$$

$$\Rightarrow zdz = (1+a)\left[dx + \frac{1}{a}dy\right]$$

$$\Rightarrow 2zdz = 2(1+a)\left[dx + \frac{1}{a}dy\right]$$

Integrating, $z^2 = 2(1+a)\left(x + \frac{1}{a}y\right) + b$

Which is required complete integral.

(ii) Solution. Let f(x,y,z,p,q) $= \left(p^2 + q^2\right)x - pz = 0$ .....(1)

The Charpit's auxiliary equation are

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

$$\Rightarrow \frac{dp}{p^2 + q^2 + p(-q)} = \frac{dp}{0 + q.(-p)} = \frac{dz}{-p(2px - z) - q(zqx)} = \frac{dx}{-(zpx - z)} = \frac{dy}{2qz}$$

From first two ratios,

$$\frac{dp}{q^2} = -\frac{dq}{pq}$$

$$\Rightarrow pdp = qdq$$

Integrating, $p^2 + q^2 = a^2$, where a is arbitrary constant.

From (1), $a^2x = pz \Rightarrow p = \frac{a^2x}{z}$

And $\left(\frac{a^2x}{z}\right)^2 + q^2 = a^2$

$$\Rightarrow q^2 = a^2 - \frac{a^4x^2}{z^2} = a^2\left(1 - \frac{a^2x^2}{z^2}\right)$$

$$\Rightarrow q = \frac{a}{z}\sqrt{z^2 - a^2x^2}$$

Putting the values of p and q in dz=pdx+qdy, we get

$$dz = \frac{a^2x}{z}dx + \frac{a}{z}\sqrt{z^2 - a^2x^2}dy$$

$$\Rightarrow zdz - a^2xdx = a\sqrt{z^2 - a^2x^2}dy$$

$$\Rightarrow 2zdz - 2a^2xdx = 2a\sqrt{z^2 - a^2x^2}dy$$

$$\Rightarrow d\left(z^2 - a^2x^2\right) = 2a\sqrt{z^2 - a^2x^2}\,dy$$

$$\Rightarrow \frac{d\left(z^2 - a^2x^2\right)}{\sqrt{z^2 - a^2x^2}} = 2a\,dy$$

Integrating, $2\sqrt{z^2 - a^2x^2} = 2ay + 2b$

$$\Rightarrow \sqrt{z^2 - a^2x^2} = ay + b$$

$$\Rightarrow z^2 - a^2x^2 = (ay + b)^2$$

$$\Rightarrow z^2 = a^2x^2 + (ay + b)^2$$

Which is a complete integral.

(iii) Solution: let f(x,y,z,p,q)=q-px-p^2 =0.....(1)

The Charpit's auxiliary equations are

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-fp} = \frac{dy}{-f_q}$$

$$\Rightarrow \frac{dp}{-p} = \frac{dq}{0} = \frac{dz}{-p(-x - 2p) - q} = \frac{dx}{-(-x - 2p)} = \frac{dy}{-1}$$

The second fraction gives dq=0

Integrating q=a

Putting q=a in (1), we get $a - px - p^2 = 0$

$$\Rightarrow p^2 + px - a = 0$$

$$\Rightarrow p = \frac{1}{2}\left[-x \pm \left(x^2 + 4a\right)^{\frac{1}{2}}\right]$$

Putting these values of p and q in $dz = pdx + qdy$,we get

$$dz = -\frac{1}{2}xdx \pm \frac{1}{2}\left(x^2 + 4a\right)^{\frac{1}{2}}dx + ady$$

Integrating,

$$z = -\frac{x^2}{4} \pm \frac{1}{2}\left[\frac{x}{2}\left(x^2 + 4a\right)^{\frac{1}{2}} + 2a\log\left\{x + \left(x^2 + 4a\right)^{\frac{1}{2}}\right\}\right] + ay + b$$

Which is a complete solution of the given equation.

(iv) Complete integral is $z = ax + by + a^2 + b^2$, where a and b being arbitrary constant.

(v) Complete integral is $(1 + a)^{\frac{1}{2}}\sqrt{z} = \sqrt{a}\sqrt{x} + \sqrt{y} + b$, where a,b being arbitrary constants.

(vi) Complete integrals are

$z = \frac{1}{2a}(ax + y)^2 + b$;a,b being arbitrary constants.

$z = xy + y\left(x^2 - a\right)^{\frac{1}{2}} + b$;a,b being arbitrary constants.

Ans to Q No 2:

(i) Soln: The given partial differential equation is $p^2 + q^2 = m^2$, where m is a constant.

Which is of the form $f(p,q) = 0$, so its solution is $z = ax + by + c$, where

$a^2 + b^2 = m^2$

$\Rightarrow b = \left(m^2 - a^2\right)^{\frac{1}{2}}$

Hence complete integral is

$$z = ax + \left(m^2 - a^2\right)\frac{1}{2}y + c$$

(ii) The given partial differential equation is $p + q = pq$

Which is of the form f(p,q)=0, so its solution is

$z = ax + by + c$

Where $a + b = ab$

$\Rightarrow a = b(a - 1)$

$\Rightarrow a = \frac{a}{a-1}$

Hencc complete integral is

$$z = ax + \left(\frac{a}{a-1}\right)y + c$$

Ans to Q No 3:

The given differential equation is

$x^2p^2 + y^2q^2 = z$

$\Rightarrow x^2\left(\frac{\partial z}{\partial x}\right)^2 + y^2\left(\frac{\partial z}{\partial y}\right)^2 = z$

$\Rightarrow \left(\frac{x}{\sqrt{z}}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{y}{\sqrt{z}}\frac{\partial z}{\partial y}\right)^2 = 1$ .....(1)

Put $\frac{1}{x}dx = dx$, $\frac{1}{y}dy = dy$ and $\frac{1}{\sqrt{z}}dz = dz$ .....(2)

So that $\log x = x$,$\log y - y$ and $z\sqrt{z} = z$ .....(3)

Using (2), (1) becomes

$$\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial x}\right)^2=1$$

$$\Rightarrow p^2+b^2=1 \text{ .....(4)}$$

Where $p=\frac{\partial z}{\partial x}$ and $Q=\frac{\partial z}{\partial y}$. Equation (4) is of the form $f(P,Q)=0$, so its solution is

$$Z=aX+bY+c$$

Where $a^2+b^2=1\Rightarrow b=\sqrt{1-a^2}$

Hence a complete solution is

$$z=aX+\sqrt{1-a^2}\,y+c$$

$$\Rightarrow 2\sqrt{2}=a\log x+\sqrt{1-a^2}\log y+c$$

Ans to Q No 4:

(i).Solution. the given differential equation is $z=px+qy+c\sqrt{1+p^2+q^2}$

Which is of the form $z=px+qy+f(p,q)$. So that complete integral is

$z=ax+by+c\sqrt{1+a^2+b^2}$ .....(1); a and b being arbitrary constants.

Singular Integral . Differentiating (1) partially w.r. to a and b, we get

$$0=x+\frac{ac}{\sqrt{1+a^2+b^2}} \text{ and } 0=y+\frac{bc}{\sqrt{1+a^2+b^2}}$$

$$\Rightarrow x^2+y^2=\frac{a^2c^2+b^2c^2}{1+a^2+b^2}$$

$$\Rightarrow c^2-x^2-y^2=c^2-\frac{a^2c^2+b^2c^2}{1+a^2+b^2}$$

$$\Rightarrow 1+a^2+b^2=\frac{c^2}{c^2-x^2-y^2}$$

Hence $a=-\frac{x\sqrt{1+a^2+b^2}}{c}=-\frac{x}{\sqrt{c^2-x^2-y^2}}$

$$b=-\frac{y}{\sqrt{c^2-x^2-y^2}}$$

Putting these values of a and b in (1), the singular solution is

$$z = -\frac{x^2}{\sqrt{c^2 - x^2 - y^2}} - \frac{y^2}{\sqrt{c^2 - x^2 - y^2}} + \frac{c^2}{\sqrt{c^2 - x^2 - y^2}}$$

$$= \frac{c^2 - x^2 - y^2}{\sqrt{c^2 - x^2 - y^2}}$$

$$\Rightarrow z = \sqrt{c^2 - x^2 - y^2}$$

$$\Rightarrow x^2 + y^2 + z^2 = c^2$$

(ii) Complete integral is $z = ax + by + leg(ab)$; a,b being arbitrary constants.

Singular integral is

z+z+log(xy)=0

(iii) Complete integral is $z = ax + by - 2\sqrt{ab}$

Where a and b are arbitrary constants.

Singular integral is $(x - z)(y - z) = 1$

Ans to Q No 5:

(i) Solution. Given partial differential equation is

$z = pq$ .....(1)

Which is of the form $f(p, qz) = 0$. Let $u = x + ay$, where a is an arbitrary constant. Replacing p and q by $\frac{dz}{du}$ and $a\left(\frac{dz}{du}\right)$ respectively in (1), we get

$$z = a\left(\frac{dz}{du}\right)^2$$

$$\Rightarrow \frac{dz}{du} = \pm\frac{\sqrt{z}}{\sqrt{a}}$$

$$\Rightarrow \pm\sqrt{a}.\frac{1}{\sqrt{z}}dz = du$$

Integrating, $\pm\sqrt{a}.2\sqrt{z} = u + b$, where b is an arbitrary constant.

$$\Rightarrow 4az = (u + b)^2$$

$$\Rightarrow 4az = (x + ay + b)^2$$

Which is a complete integral .

(ii) Complete integral is

$$z^2 \mp \left[z\left(z^2 - 4a^2\right)^{\frac{1}{2}} - 4a^2 \log \right]\left\{ z + (z^2 - 4a^2)^{\frac{1}{2}} \right\} = 4(x + ay) + b$$

Where a and b are arbitrary constant.

Ans to Q No 6:

The given differential equation is

$z^2\left(p^2+q^2+1\right)=k^2$ .....(1)

Where k is a constant .

The given equation is of the form $f(p,q,z)=0$. Let $u=x+ay$, where a is an arbitrary constant.

Replacing p by $\frac{dz}{du}$ and q by $a\frac{dz}{du}$ in (1), we get

$$z^2\left\{\left(\frac{dz}{du}\right)^2+\left(a\frac{dz}{du}\right)^2+1\right\}=k^2$$

$$\Rightarrow\left(\frac{dz}{du}\right)^2+a^2\left(\frac{dz}{du}\right)^2=\frac{k^2}{z^2}-1$$

$$\Rightarrow\left(\frac{dz}{du}\right)^2\left(1+a^2\right)=\frac{k^2-z^2}{z^2}$$

$$\Rightarrow\left(\frac{dz}{du}\right)^2=\frac{1}{1+a^2}\left(\frac{k^2-z^2}{z^2}\right)$$

$$\Rightarrow\frac{dz}{du}=\pm\frac{1}{\sqrt{1+a^2}}\sqrt{\frac{k^2-z^2}{z^2}}$$

$$\Rightarrow\pm\sqrt{1+a^2}\,\frac{z}{\sqrt{k^2-z^2}}dz=du$$

Integrating,

$$\pm\sqrt{1+a^2}\int\frac{z}{\sqrt{k^2-z^2}}dz=\int du+b$$

$$\Rightarrow\pm\frac{\sqrt{1+a^2}}{-2}\int\frac{-2z}{\sqrt{k^2-z^2}}dz=u+b$$

$$\Rightarrow\mp\frac{\sqrt{1+a^2}}{2}.2\sqrt{k^2-z^2}=u+b$$

$$\Rightarrow\left(1+a^2\right)\left(k^2-z^2\right)=(u+b)^2$$

$\Rightarrow\left(1+a^2\right)\left(k^2-z^2\right)=(x+ay+b)^2$ which is a complete integral. Differentiating partially w.r. to a and b we get

$2a\left(k^2 - z^2\right) = 2y\left(x + ay + b\right)$.....(2)

And $0 = 2\left(x + ay + b\right)$.....(3)

From (2) and (3), we get $x + ay + b = 0$ and $a = 0$. With this complete integral reduces to

$k^2 - z^2 = 0$

$\Rightarrow z^2 = k^2$

Which is the singular solution of the given equation.

Ans to Q No 7:

(i) The given partial differential equation is

$q = px + p^2$

Which is of the form $f_1(x,p) = f_2(y,q)$. So equating each side to an arbitrary constant a, we get

$q = a$ and $px + p^2 = a$

$\Rightarrow q = a$ and $p^2 + px - a = 0$

$\Rightarrow q = a$ and $p = \frac{1}{2}\left[-x \pm \sqrt{x^2 + 4a}\right]$

Putting these values of p and q in dz = pdx + qdy,
We get

$dz = \frac{1}{2}\left[-x \pm \sqrt{x^2 + 4a^2}\right]dx + ady$

Integrating,

$z = -\frac{1}{2}\left[-\frac{x^2}{2} \pm \left\{\frac{x\sqrt{x^2 + 4a^2}}{2} + \frac{4a^2}{2}\log\left(x + \sqrt{x^2+4a^2}\right)\right]\right] + ay + b$

$\Rightarrow z = \frac{x^2}{4} \pm \frac{1}{2}\left[\frac{x\sqrt{x^2 + 4a^2}}{2} + 2a^2\log\left(x + \sqrt{x^2+4a^2}\right)\right] + ay + b,$

Which is complete integral with two arbitrary constants a and b.

(ii) The given equation is
py + qx +pq = 0
$\Rightarrow$ py + q(x + p) = 0

$\Rightarrow \frac{p}{x + p} = -\frac{q}{y}$

Which is of the form $f_1(x, p) = f_2(y, q)$. So equation each sides to an arbitrary constant a,
We get

$\frac{p}{x+p} = a$ and $-\frac{q}{y} = a$

$\Rightarrow p = ax + ap \quad \text{and} \quad q = -ay$

$\Rightarrow p = \frac{ax}{1-a} \quad \text{and} \quad q = -ay$

Putting these values of p and q in dz = pdx + qdy
We obtain

$dz = \frac{ax}{1-a}dx - aydy$

Integrating, $\Rightarrow z = \left(\frac{a}{1-a}\right)\frac{x^2}{2} - a\frac{y^2}{2} + \frac{b}{2}$

$\Rightarrow 2z = \left(\frac{a}{1-a}\right)x^2 - ay^2 + b$

Which is a complete integral with arbitrary constants a and b.

(iii) The given equation is
$z^2(p^2 + q^2) = x^2 + y^2$

$\Rightarrow z^2\left(\frac{\partial z}{\partial x}\right)^2 + z^2\left(\frac{\partial z}{\partial y}\right)^2 = x^2 + y^2$

$\Rightarrow \left(z\frac{\partial z}{\partial x}\right)^2 + \left(z\frac{\partial z}{\partial y}\right)^2 = x^2 + y^2$ .....(1)

Let zdz = dz so that $\frac{z^2}{2} = Z$. Then (1) becomes

$\left(\frac{\partial Z}{\partial x}\right)^2 + \left(\frac{\partial Z}{\partial y}\right)^2 = x^2 + y^2$

$\Rightarrow p^2 + Q^2 = x^2 + y^2$ where $P = \frac{\partial z}{\partial x}$ and $Q = \frac{\partial Z}{\partial y}$

$\Rightarrow P^2 - x^2 = y^2 - Q^2$

Which is of the form $f_1(x,p) = f_2(y,q)$. So equation each side of the above equation to an arbitrary constant a^2, we get

$p^2 - x^2 = a^2$ and $y^2 - Q^2 = a^2$

$\Rightarrow p^2 = a^2 + x^2$ and $y^2 - a^2 = Q^2$

$\Rightarrow P = \sqrt{a^2 + x^2}$ and $Q = \sqrt{y^2 - a^2}$

Putting these values of P and Q in dZ = Pdx + Qdy, we obtain

$$dZ = \sqrt{a^2 + x^2}\,dx + \sqrt{y^2 + a^2}\,dy$$

Integrating

$$Z = \frac{x\sqrt{a^2 + x^2}}{2} + \frac{a^2}{2}\log\left(x + \sqrt{a^2 + x^2}\right) + \frac{y\sqrt{y^2 - a^2}}{2} - \frac{a^2}{2}\log\left(y + \sqrt{y^2 - a^2}\right) + \frac{b}{2}$$

$$\Rightarrow \frac{z^2}{2} = \frac{1}{2}\left[x\sqrt{a^2 + x^2} + a^2\log\left(x + \sqrt{a^2 + x^2}\right) + y\sqrt{y^2 - a^2} - a^2\log\left(y + \sqrt{y^2 - a^2}\right) + b\right]$$

$$\Rightarrow z^2 = x\sqrt{a^2 + x^2} + a^2\log\left(x + \sqrt{a^2 + x^2}\right) + y\sqrt{y^2 - a^2} - a^2\log\left(y + \sqrt{y^2 + a^2}\right) + b$$

Which is a complete integral with arbitrary a and b.

(iv) Hint : Put $\frac{1}{\sqrt{z}}dz = dZ$ so that $2\sqrt{z} = Z$

Ans. : Complete integral is

$$4\sqrt{z} = x\sqrt{x^2 + a^2}\,a^2\log\left(x + \sqrt{x^2 + a^2}\right) + y\sqrt{y^2 - a^2} - a^2\log\left(y + \sqrt{y^2 - a^2}\right) + b$$

Where a and b are arbitrary constant.

(v) Hint. Putting $\frac{1}{z}dz = dZ$ so that leg z = Z.

Ans. : Complete integral is

$$\log z = \frac{2}{3}\left[(a + x)^{3/2} + (y - a)^{3/2}\right] + b$$

Where a and b are arbitrary constants.

Ans to Q No 8:

Given equation is

$pq = z$ ..... (1)

We have to find its integral surface which passes through

the parabola $x = 0$ and $y^2 = z$ .....(2)

Rewriting (2) in parametric form, we have

$x = 0, y = \lambda, z = \lambda^2$ ..... (3)

Where λ is a parameter.

Let the initial values x_0, y_0, z_0, p_0, q_0 of x, y, x, p, q be taken as

$x_0 = x_0(\lambda) = 0, y_0 = y_0(\lambda) = \lambda, z_0 = z_0(\lambda) = \lambda^2$ ..... (4)

Let $p_0 q_0$ be the initial values of p, q corresponding to the initial values x_0, y_0, z_0. Corresponding to the initial values x_0, y_0, z_0. Since the initial values x_0, y_0, z_0, p_0, q_0 satisfy (1), we have

$p_0 q_0 = z_0$

$\Rightarrow p_0 q_0 = \lambda_0^2$ .....(5)

Also, we have

$$z'_0(\lambda) = p_0 x_0'(\lambda) + q_0 y_0'(\lambda)$$
$$\Rightarrow 2\lambda = p_0 . 0 + q_0 \times 1$$
$$\Rightarrow q_0 = 2\lambda \quad \text{.....(6)}$$

From (5) and (6), we get

$$p_0 \frac{\lambda}{2} \text{ and } q_0 = 2\lambda \quad \text{..... (4)*}$$

Collecting (4) and (4)* together, we obtain

$x_0 = 0, y_0 = \lambda, z_0 = \lambda^2, p_0 = \frac{\lambda}{2}, q_0 = 2\lambda$ ..... (7)

When $t = t_0 = 0$

Re-writing (1), let $f(x, y, z, p, q) = pq - z = 0$ ..... (8)

The usual characteristic equations of (8) are given by

$$\frac{dx}{dt} = \frac{\partial f}{\partial p} = q \quad \text{..... (9)}$$

$$\frac{dy}{dt} = \frac{\partial f}{\partial q} = p \quad \text{..... (10)}$$

$$\frac{dx}{dt} = p\frac{\partial f}{\partial q} + q\frac{\partial f}{\partial q} = 2pq \quad \text{..... (11)}$$

$$\frac{dp}{dt} = \frac{\partial f}{\partial x} - p\frac{\partial f}{\partial z} = p \quad \text{..... (12)}$$

$$\frac{dq}{dt} = \frac{\partial f}{\partial y} - q\frac{\partial f}{\partial z} = q \quad \text{.....(13)}$$

$(9) - (13) \Rightarrow \frac{dx}{dt} - \frac{dq}{dt} = 0$ so that $x - q = c_1$ ..... (14)

Where C_1 is an arbitrary constant. Using initial values

(7) in (14) gives $x_0 - q_0 = c_1$

$$\Rightarrow 0 - 2\lambda = c_1$$
$$\Rightarrow c_1 = -2\lambda$$

Then (14) reduces to

$x - q = -2\lambda$

$\Rightarrow x = q - 2\lambda$ ..... (15)

$(10) - (12) \Rightarrow \frac{dy}{dt} - \frac{dp}{dt} = 0$ so that $y - p = c_2$ ..... (16)

Where C_2 is an arbitrary constant. Using initial Values (7), (16) gives

$$y_0 - p_0 = c_2$$
$$\Rightarrow \lambda - \frac{\lambda}{2} = c_2$$
$$\Rightarrow c_2 = \frac{\lambda}{2}$$

There (16) becomes $y - p = \frac{\lambda}{2}$

$$\Rightarrow y = p + \frac{\lambda}{2} \quad \text{..... (17)}$$

From $(12), \frac{1}{p}dp = dt$ so that logp – logc_3 = t

$$\Rightarrow p = c_3 e^t \quad \text{..... (18)}$$

Using initial conditions (7), (18) gives

$$p_0 = c_3 e^0 \Rightarrow \frac{\lambda}{2} = c_3$$

(18) becomes $p = \frac{\lambda}{2} e^t$ ..... (19)

Frpm $(13), \frac{1}{q}dq = dt$ so that logq – logc_4 = t

$$\Rightarrow q = c_4 e^t \quad \text{..... (20)}$$

Using initial conditions (7), (20) gives

$$q_0 = c_4 e^{t_0} \Rightarrow 2\lambda = c_4$$

(20) becomes $q = 2\lambda e^t$ ..... (21)

From (15) and (21), $x = 2\lambda e^t - 2\lambda$

$$= 2\lambda\left(c^t - 1\right) \quad \text{..... (22)}$$

From (19) and (17), $y = \frac{\lambda}{2} e^t + \frac{\lambda}{2}$

$$= \frac{\lambda}{2}\left(e^t + 1\right) \quad \text{.....(23)}$$

Substituting values of p and q from (19) and (21) in (11),

We get $$\frac{dz}{dt} = z.\frac{\lambda}{2} e^t . 2\lambda e^t$$

$$\Rightarrow dz = 2\lambda^2 e^{2t} dt$$

Integrating, $z = \lambda^2 e^{2t} + c_5$ ..... (24)

Where C5 being arbitrary constant.

Using initial values (7), (24) gives

$$z^0 = \lambda^2 e^{2.0} + c_5$$

$$\Rightarrow \lambda^2 = \lambda^2 + c_5$$

$$\Rightarrow c_5 = 0$$

Then (24) becomes $z = \lambda^2 e^{2t}$ ..... (25)

The required characteristics of (1) are given by (22), (23) and (25).

Now, we eliminate two parameter t and λ from three equation (22), (23) and (25). Solving (22) and (23) for et and λ, we get

$$e^t = \frac{x+4y}{4y-x} \text{ and } \lambda = \frac{4y-x}{4}$$

Substituting these values of e^t and λ in (25), we obtain

$$z = \frac{(4y-x)^2}{16}\left\{\frac{x+4y}{4y-x}\right\}^2$$

$$\Rightarrow 16z = (4y+x)^2$$

Which is the required integral surface of (1) passing through (2).



10.8 FURTHER READINGS

1) Simmons, G.F., Differential Equations with Applications and Historical Notes, 2nd edition, Tata Mc Graw Hill, New Delhi, 2016.
2) Raisinghania, M.D., Ordinary and Partial Differential Equations, S.Chand and Company Ltd, New Delhi,2001.
3) Sneddan, I.N., Elements of Partial Differential Equations, Dover Publications, 2006.
4) Ross, S.L., Differential Equations , 3rd Edition, Wiley India, 2007.



10.9 MODEL QUESTIONS

Q 1: Find the complete integral of the following equations

(i) $(p^2+q^2)y = qz$

(ii) $p = (z+qy)^2$

(iii) $z^2 = pqxy$

(iv) $2(y - zq) = q(xp + yq)$

(v) $z = p^2 - q^2$

(vi) $zpq = p = q$

Q 2: Find the complete integral, singular integral and general integral of the equation $p^2 + q^2 = 1$.

Q 3: Show that the complete integral of the equation $z = px + qy + \left(p^2 + q^2 + 1\right)^{\frac{1}{2}}$ represents all planes at unit distance from the origin.

Q 4: Find the complete integral of the following partial differential equations:

(i) $p^2 = zq$

(ii) $p^3 + q^3 - 3pqz = 0$

(iii) $z^2\left(p^2 + q^2 + 1\right) = 1$

(iv) $z^2 = 1 + p^2 + q^2$

(v) $z\left(p^2 - q^2\right) = x - y$

(vi) $p_1 + p_2 + p_3 - p_1p_2p_3 = 0$

(vii) $x_1p^2{}_1 + x_2p^2{}_2 - x_3p^2{}_3 = 0$

Q 5: Determine the characteristics of the equation $z = p^2 - q^2$ and find the integral surface which passes through the parabola $4z + x^2 = 0, y = 0$

Q 6: Integrate the equations for the characteristics of the equation $p^2 + q^2 = 4z$

Expressing x,y,z and p in terms of q and then find the solutions of the equations which reduce to $z = x^2 + 1$ when $y = 0$

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