UNIT 11 : FIRST ORDER NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS II

UNIT STRUCTURE

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11.1 LEARNING OBJECTIVES

After going through this unit, you will able to-

- solve non-linear PDE using Jacobi's method.
- find characteristics equations for non linear PDE
- solve special types of first order PDE.

11.2 INTRODUCTION

A solution or integral of a differential equation is a relation between the variables by means of which and the derivatives obtained there from the equation is satisfied.

11.3 JACOBI'S METHOD OF SOLVING NONLINEAR PARTIAL DIFFERENTIAL EQUATION OF FIRST ORDER

This method is used for solving PDEs involving three or more independent variables. The central idea of Jacobi's method is almost the same as that of Charpit's method for two independent variables.

Let us consider the PDE with three independent variables x_1, x_2, x_3 as

$f(x_1, x_2, x_3, p_1, p_2, p_3) = 0$ — (1)

z is the dependent variable not occurring in the equation except in its partial derivatives p_1, p_2, p_3 w.r.t. x_1, x_2, x_3 respectively.

The idea of Jacobi's method is to find two additional equations.

$F_1(x_1, x_2, x_3, p_1, p_2, p_3) = a_1$ — (2)

& $F_2(x_1, x_2, x_3, p_1, p_2, p_3) = a_2$ — (3)

where a_1, a_2 are arbitrary constants such that (1), (2), (3) may be solved for p_1, p_2, p_3 as functions of x_1, x_2, x_3 which when substituted in

$dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3$ — (4)

it becomes integrable, for which conditions are

$$\frac{\partial p_2}{\partial x_1} = \frac{\partial^2 z}{\partial x_1 \partial x_2} = \frac{\partial p_1}{\partial x_2}$$

$$\frac{\partial p_3}{\partial x_2} = \frac{\partial p_2}{\partial x_3}$$

$$\frac{\partial p_1}{\partial x_3} = \frac{\partial p_3}{\partial x_4}$$

Now differentiating (1) and (2), partially w.r.t. x_1, we have

$$\frac{df}{dx_1} + \frac{df}{dp_1}.\frac{dp_1}{dx_1} + \frac{df}{dp_2}.\frac{dp_2}{dx_1} + \frac{df}{dp_3}.\frac{dp_3}{dx_1} = 0 \quad \text{— (5)}$$

$$\frac{dF_1}{dx_1} + \frac{dF_2}{dp_1}.\frac{dp_1}{dx_1} + \frac{dF_1}{dp_2}.\frac{dp_2}{dx_1} + \frac{dF_1}{dp_3}.\frac{dp_3}{dx_1} = 0 \quad \text{— (6)}$$

Eliminating $\frac{dp_1}{dx_1}$ from (5) & (6), we have

$$\left(\frac{\partial f}{\partial x_1}\frac{\partial F_1}{\partial p_1} - \frac{\partial f}{\partial p_1}\frac{\partial F_1}{\partial x_1}\right) + \left(\frac{\partial f}{\partial p_2}\frac{\partial F_1}{\partial p_1} - \frac{\partial f}{\partial p_1}\frac{\partial F_1}{\partial p_2}\right)\frac{\partial p_2}{\partial x_1} + \left(\frac{\partial f}{\partial p_3}\frac{\partial F_1}{\partial p_1} - \frac{\partial f}{\partial p_1}\frac{\partial F_1}{\partial p_3}\right)\frac{\partial p_3}{\partial x_1} = 0 \quad \text{—(7)}$$

Similarly, we have

$$\left(\frac{\partial f}{\partial x_2}\frac{\partial F_1}{\partial p_2} - \frac{\partial f}{\partial p_2}\frac{\partial F_1}{\partial x_2}\right) + \left(\frac{\partial f}{\partial p_1}\frac{\partial F_1}{\partial p_2} - \frac{\partial f}{\partial p_2}\frac{\partial F_1}{\partial p_1}\right)\frac{\partial p_2}{\partial x_2} + \left(\frac{\partial f}{\partial p_3}\frac{\partial F_1}{\partial p_2} - \frac{\partial f}{\partial p_2}\frac{\partial F_1}{\partial p_3}\right)\frac{\partial p_3}{\partial x_2} = 0 \quad \text{—(8)}$$

and

$$\left(\frac{\partial f}{\partial x_3}\frac{\partial F_1}{\partial p_3} - \frac{\partial f}{\partial p_3}\frac{\partial F_1}{\partial x_3}\right) + \left(\frac{\partial f}{\partial p_1}\frac{\partial F_1}{\partial p_3} - \frac{\partial f}{\partial p_3}\frac{\partial F_1}{\partial p_1}\right)\frac{\partial p_1}{\partial x_3} + \left(\frac{\partial f}{\partial p_2}\frac{\partial F_1}{\partial p_3} - \frac{\partial f}{\partial p_2}\frac{\partial F_1}{\partial p_2}\right)\frac{\partial p_2}{\partial x_3} = 0 \quad \text{—(9)}$$

Adding (7), (8) and using $\frac{\partial p_2}{\partial x_1}=\frac{\partial p_1}{\partial x_2}$, $\frac{\partial p_3}{\partial x_2}=\frac{\partial p_2}{\partial x_3}$ and $\frac{\partial p_1}{\partial x_3}=\frac{\partial p_3}{\partial x_1}$, we have

$$\left(\frac{\partial f}{\partial x_1}\frac{\partial F_1}{\partial p_1}-\frac{\partial f}{\partial p_1}\frac{\partial F_1}{\partial x_1}\right)+\left(\frac{\partial f}{\partial x_2}\frac{\partial F_1}{\partial p_3}-\frac{\partial f}{\partial p_2}\frac{\partial F_1}{\partial p_2}\right)+\left(\frac{\partial f}{\partial x_3}\frac{\partial F_1}{\partial p_3}-\frac{\partial f}{\partial p_3}\frac{\partial F_1}{\partial x_3}\right)=0 \quad \text{..... (10)}$$

i.e. $$[f_1\, F_1]=\sum_{r=1}^{3}\left(\frac{\partial f}{\partial x_r}\frac{\partial F_1}{\partial p_r}-\frac{\partial f}{\partial p_r}\frac{\partial F_1}{\partial x_r}\right)=0$$

Similarly, we have

$$[f_1\, F_2]=\sum_{r=1}^{3}\left(\frac{\partial f}{\partial x_r}\frac{\partial F_2}{\partial p_r}-\frac{\partial f}{\partial p_r}\frac{\partial F_2}{\partial x_1}\right)=0$$

and $$[f_1, F_2]=\sum_{r=1}^{3}\left(\frac{\partial F_1}{\partial x_r}\frac{\partial F_2}{\partial p_r}-\frac{\partial F_1}{\partial p_r}\frac{\partial F_2}{\partial x_r}\right)=0$$

Since (10) is a linear equation of the first order with $x_1, x_2, x_3, p_1, p_2, p_3$ as independent variables and F_1 as dependent variable.

$\therefore$ The auxiliary equations are

$$\frac{dx_1}{-\frac{\partial f}{\partial p_1}}=\frac{dp_1}{\frac{\partial f}{\partial x_1}}=\frac{dx_2}{-\frac{\partial f}{\partial p_2}}=\frac{dp_2}{\frac{\partial f}{\partial x_2}}=\frac{dx_3}{-\frac{\partial f}{\partial p_3}}=\frac{dp_3}{\frac{\partial f}{\partial x_3}} \qquad \text{— (4)}$$

Any integral of (11) wqill satisfy (10).

Working Method :

Find two independent integrals $F_1=a_1$ and $F_2=a_2$ of the subsidiary equation (11).

If these satisfy the condition

$$(F_1, F_2)=\sum_{r=1}^{3}\left(\frac{\partial F_1}{\partial x_r}\frac{\partial F_2}{\partial p_r}-\frac{\partial F_1}{\partial p_r}\cdot\frac{\partial F_2}{\partial x_r}\right)=0 \qquad \text{— (12)}$$

and if p_1, p_2, p_3 can be found as functions of x_1, x_2, x_3 from $F=0, F_1=a_1$ and $F_2=a_2$ then substitute these values in

$$dz=p_1dx_1+p_2dx_2+p_3dx_3$$

and integrate to get the complete integral.

Ex.8. Apply Jacobi's method to find complete integral of $p_1^3+p_2^3+p_3-1=0$

Soln: Here $f\equiv p_1^3+p_2^2+p_3-1=0$ — (1)

The subsidiary equations are

$$\frac{dx_1}{-fp_1}=\frac{dp_1}{fx_1}=\frac{dx_2}{-fp_2}=\frac{dp_2}{fx_2}=\frac{dx_3}{-fp_3}=\frac{dp_3}{fx_3}$$

$$\Rightarrow \frac{dx_1}{3p_1^2}=\frac{dp_1}{0}=\frac{dx_2}{-2p_2}=\frac{dp_2}{0}=\frac{dx_3}{-1}=\frac{dp_3}{0}$$

$dp_1 = 0$ and $dp_2 = 0$

or $F_1 \equiv p_1 = a_1$ and $F_2 \equiv p_2 = a_2$

Now $(F_1, F_2) = 0$

Therefore, these may be taken as additional equations.

$\therefore$ From (1), $p_3 = 1 - a_1^3 - a_2^2$

Substituting these values in

$$dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3$$

$$\Rightarrow dz = a_1 dx_1 + a_2 dx_2 + \left(1 - a_1{}^3 - a_2{}^2\right) dx_3$$

$$\Rightarrow z = a_1 x_1 + a_2 x_2 + \left(1 - a_1{}^3 - a_2{}^2\right) x_3 + a_3$$

which is the complete integral.

Ex.9. Find the complete integral of

$$p_3 x_3 (p_1 + p_2) + x_1 + x_2 = 0$$

Soln: Here $f \equiv p_3 x_3 (p_1 + p_2) + x_1 + x_2 = 0$ .....(1)

The subsidiary equations are

$$\frac{dx_1}{-fp_1} = \frac{dp_1}{fx_1} = \frac{dx_2}{-fp_2} = \frac{dp_2}{fx_2} = \frac{dx_3}{-fp_3} = \frac{dp_3}{fx_3}$$

$$\Rightarrow \frac{dx_1}{-p_3 x_3} = \frac{dp_1}{1} = \frac{dx_2}{-p_3 x_3} = \frac{dp_2}{1} = \frac{dx_3}{-x_3(p_1 + p_2)} = \frac{dp_3}{p_3(p_1 + p_2)} \quad .....(2)$$

Taking the second and fourth fraction, we have

$$dp_1 = dp_2$$

$$\Rightarrow p_1 - p_2 = a_1$$

$$F_1 \equiv p_1 - p_2 = a_1 \quad ..... (3)$$

Again taking 5th & 6th fraction we have

$$\frac{dx_3}{-x_3(p_1 + p_2)} = \frac{dp_3}{p_3(p_1 + p_2)}$$

$$\Rightarrow \frac{dx_3}{x_3} + \frac{dp_3}{p_3} = 0$$

Integrating $\log x_3 + \log p_3 = \log a_2$

$$\Rightarrow x_3 p_3 = a_2$$

$$F_2 \equiv p_3 x_3 = a_2 \quad .....(4)$$

Now, $(\mathrm{F_1}, \mathrm{F_2}) = \sum_{r=1}^{3} \left(\frac{\partial \mathrm{F_1}}{\partial \mathrm{x_r}} \frac{\partial \mathrm{F_2}}{\partial \mathrm{p_r}} - \frac{\partial \mathrm{F_1}}{\partial \mathrm{p_r}} . \frac{\partial \mathrm{F_2}}{\partial \mathrm{x_r}} \right) = 0$

$\therefore$ (2) and (3) may be taken as additional equations

From (1) & (3), we have

$$a_2(p_1 + p_2) + x_1 + x_2 = 0$$

$$\Rightarrow p_1 + p_2 = -\frac{x_1 + x_2}{a_2} \quad \text{..... (4)}$$

(2) & (4) $\Rightarrow p_1 = \frac{\alpha_1}{2} - \frac{x_1 + x_2}{2a_2}$ and $p_2 = \frac{-a_1}{2} - \frac{x_1 + x_2}{2a_2}$

and (3) $\Rightarrow p_3 = \frac{a_2}{x_3}$

Substituting these values in

$$dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3$$

$$\Rightarrow dz = \left(\frac{a_1}{2} - \frac{a_1 + x_2}{2a_2}\right)dx_1 + \left(\frac{a_1}{2} - \frac{x_1 + x_2}{2a_2}\right)dx_2 + \frac{a_2}{x_3}dx_3$$

$$= \frac{a_1}{2}(dx_1 - dx_2) - \frac{x_1 + x_2}{2a_2}(dx_1 + dx_2) + \frac{a_2}{x_3}dx_3$$

Integrating we have

$$z = \frac{a_1}{2}(x_1 - x_2) - \frac{1}{4a_2}(x_1 + x_2)^2 + a_2 \log x_3 + a_3$$

$$\Rightarrow 4a_2 z = 2a_1 a_2 (x_1 - x_2) - (x_1 + x_2)^2 + 4{a_2}^2 \log x_3 + 4a_2 a_3$$

Ex.10. Find the complete integral of

$$2p_1 x_1 x_3 + 3p_2 {x_3}^2 + {p_2}^2 p_3 = 0$$

Soln: Here $f \equiv 2p_1 x_1 x_3 + 3p_2 {x_3}^2 + {p_2}^2 p_3 = 0$ ..... (1)

The subsidiary equations are

$$\frac{dx_1}{-2x_1 x_3} = \frac{dp_1}{2px_3} = \frac{dx_2}{-3{x_3}^2 + 2p_2 p_3} = \frac{dp_2}{0} = \frac{dx_3}{-{p_2}^3} = \frac{dp_3}{2p_1 x_1 + 6x_3}$$

Taking the first two fractions, we have

$$\frac{dx_1}{x_1} + \frac{dp_1}{p_1} = 0$$

$\therefore F_1 \equiv p_1 x_1 = a_1$ ..... (2)

Again taking 4th fractions, we have

$$dp_2 = 0$$

$\therefore F_2 \equiv p_2 = a_2$ ..... (3)

Now, $(F_1, F_2) = \sum_{r=1}^{3}\left(\frac{\partial F_1}{\partial x_r}\frac{\partial F_2}{\partial p_r} - \frac{\partial F_1}{\partial p_r}.\frac{\partial F_2}{\partial x_r}\right) = 0$

Therefore (2) and (3) may be taken as additional equations

$$\therefore p_1 = \frac{a_1}{x_1}, \quad p_2 = a_2$$

and from (1), $2a_1x_3 + 3a_2{x_3}^2 + {a_2}^2 p_3 = 0$

$$\therefore\ p_3 = \frac{-1}{{a_2}^2}\left(2a_1x_3 + 3a_2{x_3}^2\right)$$

Substituting in $dz = p_1dx_1 + p_2dx_2 + p_3dx_3$, we have

$$dz = \frac{a_1}{x_1}dx_1 + a_2dx_2 - \frac{1}{{a_2}^2}\left(2a_1x_3 + 3a_2{x_3}^2\right)dx_3$$

Integrating, we have

$$z = a_1\log x_1 + a_2x_2 - \frac{1}{{a_2}^2}\left(a_1{x_3}^2 + a_2{x_3}^3\right) + a_3$$

which is the complete integral.

Ex.11. Find the complete integral of

$$p_1p_2p_3 + {p_4}^3x_1x_2x_3{x_4}^3 = 0$$

Soln: Here $f \equiv p_1p_2p_3 + {p_4}^3x_1x_2x_3{x_4}^3 = 0$ ..... (1)

The subsidiary equations are

$$\frac{dx_1}{-p_2p_3} = \frac{dp_1}{{p_4}^3x_2x_3{x_4}^3} = \frac{dx_2}{-p_1p_3} = \frac{dp_2}{{p_4}^3x_1x_3{x_4}^3} = \frac{dx_3}{-p_1p_2}$$

$$= \frac{dp_3}{{p_4}^3x_1x_2{x_4}^3} = \frac{dx_4}{-3{p_4}^2x_1x_2x_3{x_4}^3} = \frac{dp_4}{3{p_4}^2x_1x_2x_3{x_4}^2}$$

Taking the first two fractions, we have

$$\frac{p_1dx_1}{-p_1p_2p_3} = \frac{x_1dp_1}{{p_4}^3x_1x_2x_3{x_4}^3}$$

$$\Rightarrow p_1dx_1 = x_1dp_1 \quad \text{[using (1)]}$$

$$\Rightarrow \frac{dp_1}{p_1} = \frac{dx_1}{x_1}$$

$$\Rightarrow \log p_1 = \log x_1 + \log a_1$$

$$\therefore\ F_1 \equiv p_1 - a_1x_1 = 0 \quad \text{..... (2)}$$

Similarly, $F_2 \equiv p_2 - a_2x_2 = 0$ ..... (3)

and $F_3 \equiv p_3 - a_3x_3 = 0$ ..... (4)

Now, $(\mathrm{F_1}, \mathrm{F_2}) = \sum_{\mathrm{r=1}}^{4}\left(\frac{\partial \mathrm{F_1}}{\partial \mathrm{x_r}}\frac{\partial \mathrm{F_2}}{\partial \mathrm{p_r}} - \frac{\partial \mathrm{F_1}}{\partial \mathrm{p_r}}\frac{\partial \mathrm{F_2}}{\partial \mathrm{x_r}}\right) = 0$

Similarly $(F_2, F_3) = 0$ and $(F_1, F_3) = 0$

Therefore (2), (3) and (4) may be taken as three additional equations

Solving (1), (2), (3) and (4), we have

$$p_1 = a_1 x_1,\ p_2 = a_2 x_2,\ \ p_3 = a_3 x_3$$

$$p_4 = -\frac{(a_1 a_2 a_3)^{1/3}}{x_4}$$

Substituting these values in

$dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3 + p_4 dx_4$, we have

$$dz = a_1 x_1 dx_1 + a_2 x_2 dx_2 + a_3 x_3 dx_3 - \frac{(a_1 a_2 a_3)^{1/3}}{x_4} dx_4$$

Integrating

$$2z = a_1 x_1^{\,2} + a_2 x_2^{\,2} + a_3 x_3^{\,2} - 2\,(a_1 a_2 a_3)^{1/3} \log x_4 + a_4$$

which is the complete integral.

Ex.12. Solve $p_1 p_2 p_3 = z^3 x_1 x_2 x_3$ ..... (1)

Soln: This equation is not of the form considered, as it involves the dependent variable *z* also.

The equation can be written as

$$\left(\frac{1}{z}\frac{dz}{dx_1}\right)\left(\frac{1}{z}\frac{dz}{dx_2}\right)\left(\frac{1}{z}\frac{dz}{dx_3}\right) = x_1 x_2 x_3$$

Putting, $\frac{dz}{z} = dz$ so that $\log z = z$, we have

$$\frac{dz}{dx_1}.\frac{dz}{dx_2}.\frac{dz}{dx_3} = x_1 x_2 x_3$$

Or $p_1 p_2 p_3 = x_1 x_2 x_3$

$\therefore$ Now, $f \equiv p_1 p_2 p_3 - x_1 x_2 x_3 = 0$ ..... (2)

$\therefore$ The subsidiary equations are

$$\frac{dx_1}{-p_2 p_2} = \frac{dp_1}{-x_2 x_3} = \frac{dx_2}{-p_1 p_3} = \frac{dp_2}{-x_1 x_3} = \frac{dx_3}{-p_1 p_2} = \frac{dp_3}{-x_1 x_2}$$

Taking the first two fractions, we have

$$\frac{p_1 dx_1}{-p_1 p_2 p_3} = \frac{x_1 dp_1}{-x_1 x_2 x_3}$$

$$\Rightarrow \frac{dx_1}{x_1} = \frac{dp_1}{p_1}$$

Or $\log p_1 = \log x_1 + \log a_1$

$\therefore\ F_1 \equiv p_1 - a_1 x_1 = 0$ — (3)

Similarly $F_2 \equiv p_2 - a_2 x_2 = 0$ — (4)

Now, $(F_1, F_2) = \sum_{r=1}^{3}\left(\frac{\partial F_1}{\partial x_r}\cdot\frac{\partial F_2}{\partial p_r} - \frac{\partial F_1}{\partial p_r}\frac{\partial F_2}{\partial x_r}\right) = 0$

Therefore (3), (4) may be taken as two additional equations.

Substituting $p_1 = a_1x_1,\ p_2 = a_2x_2,\ p_3 = \frac{x_3}{a_1a_2}$ in

$dz = p_1dx_1 + p_2dx_2 + p_3dx_3$, we have

$$dz = a_1x_1dx_1 + a_2x_2dx_2 + \frac{x_3}{a_1a_2}dx_3$$

Integrating, $2z = a_1{x_1}^2 + a_2{x_2}^2 + \frac{{x_3}^2}{a_1a_2} + a_3$

Or $\frac{2}{a_3}\log z = \frac{a_1}{a_3}{x_1}^2 + \frac{a_2}{a_3}{x_2}^2 + \frac{1}{a_1a_2a_3}{x_3}^2 + 1$

Which is the complete integral.

Note: If $\frac{a_1}{a_3} = A_1,\ \frac{a_2}{a_3} = A_2,\ \frac{1}{a_1a_2a_3} = A_3,$ then

$$A_1A_2A_3 = \frac{1}{{a_3}^3}$$

$$\therefore\ \frac{1}{a_3} = (A_1A_2A_3)^{1/3}$$

$\therefore$ The complete integral is

$$2(A_1A_2A_3)^{1/3}\log z = A_1{x_1}^2 + A_2{x_2}^2 + A_3{x_3}^3 + 1$$

Where A_1, A_2, A_3 are arbitrary constants.

Ex.13. Solve $(x_2 + x_3)(p_2 + p_3)^2 + zp_1 = 0$ ..... (1)

Soln: This equation is not of the form considered, as it involves the dependent variable ***z***.

Putting $z = x_4$

So that $p_1 = \frac{\partial z}{\partial x_1} = \frac{\partial z}{\partial x_4}\cdot\frac{\partial x_4}{\partial x_1} = \frac{-\frac{\partial u}{\partial x_1}}{\frac{\partial u}{\partial x_4}} = -\frac{P_1}{P_4}$ (say)

where $u = 0$ is on integral of (1)

Similarly $p_2 = -\frac{P_2}{P_4}$ and $p_3 = -\frac{P_3}{P_4}$

The given equation (1) reduces to

$$(x_2 + x_3)\left(-\frac{P_2}{P_4} - \frac{P_3}{P_4}\right)^2 + x_4\left(-\frac{P_1}{P_4}\right) = 0$$

$$\Rightarrow (x_2 + x_3)(P_2 + P_3)^2 - x_4 P_1 P_4 = 0 \qquad \text{..... (2)}$$

i.e., $f \equiv (x_2 + x_3)(P_2 + P_3)^2 - x_4 P_1 P_4 = 0$

is an equation in four independent variables x_1, x_2, x_3, x_4 and not involving the dependent variable i.e. except by its partial derivatives p_1, p_2, p_3, p_4

The subsidiary equations are

$$\frac{dx_1}{x_4 P_4} = \frac{dP_1}{0} = \frac{dx_2}{2(x_1 + x_3)(P_2 + P_3)} = \frac{dP_2}{(P_2 + P_3)^2}$$

$$= \frac{dx_3}{-2(x_2 + x_3)(P_2 + P_3)} = \frac{dP_3}{(P_2 + P_3)^2} = \frac{dx_4}{x_4 P_1} = \frac{dP_4}{-P_1 P_4}$$

$\therefore \quad dP_1 = 0$

Or $F_1 \equiv P_1 = a_1$ ..... (3)

Similarly taking 4th and 6th fractions, we have

$$dP_2 - dP_3 = 0$$

$\therefore \quad F_2 = P_2 - P_3 = a_2$ ..... (4)

Taking last two fractions, we have

$$\frac{dx_4}{x_4} + \frac{dP_4}{P_4} = 0$$

$\therefore \quad F_3 = P_4 x_4 = a_3$

Now $(F_1, F_2) = \sum_{r=1}^{4}\left(\frac{\partial F_1}{\partial x_r}\frac{\partial F_2}{\partial p_r} - \frac{\partial F_1}{\partial p_r}\frac{\partial F_2}{\partial x_r}\right) = 0$

Similarly, $(F_2, F_3) = 0$ and $(F_1, F_3) = 0$

Therefore (3), (4) and (5) may be taken as three additional equations.

Solving (3) and (5), we have

$P_1 = a_1, \quad P_4 = \frac{a_3}{x_4}$, putting in (2), we have

$$(P_2 + P_3)^2 = a_1 a_3 \Big/ (x_2 + x_3)$$

$$\therefore \quad P_2 + P_3 = \pm\sqrt{\frac{a_1 a_3}{x_2 + x_3}} \qquad \text{..... (6)}$$

Solving (4) and (6), we have

$$P_2 = \frac{a_2}{2} \pm \frac{1}{2}\sqrt{\frac{a_1 a_3}{(x_2 + x_3)}}$$

and $$P_3 = \frac{a_2}{2} \pm \frac{1}{2}\sqrt{\frac{a_1 a_3}{x_2 + x_3}}$$

Substituting these values in

$$du = p_1 dx_1 + p_2 dx_2 + p_3 dx_3 + p_4 dx_4$$

We have

$$du = a_1 dx_1 + \left[\frac{a_2}{2} \pm \frac{1}{2}\sqrt{\frac{a_1 a_3}{x_2 + x_3}}\right] dx_2 + \left[-\frac{a_2}{2} \pm \frac{1}{2}\sqrt{\frac{a_1 a_3}{x_2 + x_3}}\right] dx_3 + \frac{a_3}{x_4}\, dx_4$$

Or $$du = a_4 dx_1 + \frac{a_2}{2}(dx_2 - dx_3) \pm \sqrt{(a_1 a_3)}.\frac{dx_2 + dx_3}{\sqrt{(x_2 + x_3)}} + \frac{a_3}{x_4}\, dx_4$$

Integrating,

$$u = a_1 x_1 + \frac{a_2}{2}(x_2 - x_3) \pm \sqrt{\{a_1 a_3 (x_2 + x_3)\}} + a_3 \log x_4 + a_4$$

$\therefore\ u = 0$ gives

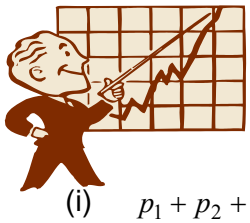
$$a_1 x_1 + \frac{a_2}{2}(x_2 - x_3) \pm \sqrt{\{a_1 a_3 (x_2 + x_3)\}} + a_3 \log z + a_4 = 0$$

Or $$\log z + \frac{a_1}{a_3} x_1 + \frac{a_2}{2a_3}(x_2 - x_3) \pm \sqrt{\left\{\frac{a_1}{a_3}(x_2 + x_3)\right\}} + \frac{a_4}{a_3} = 0$$

If $$\frac{a_1}{a_3} = A_1,\quad \frac{a_2}{2a_3} = A_2,\quad \frac{a_4}{a_3} = A_3$$

The complete integral of (1) is

$$\log z + A_1 x_1 + A_2 (x_2 - x_3) \pm \sqrt{\{A_1 (x_2 + x_3)\}} + A_3 = 0$$



CHEK YOUR PROGRESS 1

Q 1: Find the complete integral of the following equations using Jacobi's method:

(i) $p_1 + p_2 + p_3 - p_1 p_2 p_3 = 0$

(ii) $p_1 x_1^{\,2} - p_2^{\,2} - a p_3^{\,2} = 0$

(iii) $x_3(x_3 + p_3) = p_1^{\,2} + p_2^{\,2}$

(iv) $x_3 + 2p_3 - (p_1 + p_3^{\,2}) = 0$

(v) $p_1 x_1 + p_2 x_2 = p_3^{\,2}$

11.4 CAUCHY'S METHOD OF CHARACTERISTICS

Let us consider a non-linear PDE

$$f(x, y, z, p, q) = 0 \quad \text{..... (1)}$$

We know that the plane passing through the point $p(x_\circ, y_\circ, z_\circ)$ with its normal parallel to the direction *n* whose direction ratio's are $p_\circ, q_\circ, -1$ is uniquely given by the set of five numbers $D(x_\circ, y_\circ, z_\circ, p_\circ, q_\circ)$ and conversely any such set of five numbers defines a plane in three dimensional space.

In view of this fact a set of five numbers $D(x_\circ, y_\circ, z_\circ, p_\circ, q_\circ)$ is known as a plane element of a three dimensional space.

As a special case a plane element $(x_\circ, y_\circ, z_\circ, p_\circ, q_\circ)$ whose components satisfy (1) is known as an integral element of (1) at P. Solving (1) for q, suppose, we get

$$q = F(x, y, z, p)$$

Which gives a value of q corresponding to known values of *z, y, z* and *p*. Then keeping $x_\circ, y_\circ$ and $z_\circ$ fixed and varying *p*, we shall arrive at a set of plane elements $\{x_\circ, y_\circ, z_\circ, p, G(x_\circ, y_\circ, z_\circ, p)\}$ which depend on the single parameter *p*, as *p* varies, we get a set of plane elements all of which pass through the point p. Hence the above mentioned set of plane elements, envelope a cone with vertex p. The cone thus obtained is known as the elementary cone of (1) at the point p.

Consider a surface S with equation *z* = *g* (*x, y*) ..... (2).

If the function *g* (*x, y*) and its first partial derivatives g_x (*x,y*) and g_y(*x,y*) are continuous in a certain region R of the *xy*-plane, then the target plane at each point of S determines a plane element of the form $\{x_\circ, y_\circ, g(x_\circ, y_\circ), g_x(x_\circ, y_\circ), g_y(x_\circ, y_\circ)\}$ which will be referred as the tangent element of the surface S at the point $\{x_\circ, y_\circ, g(x_\circ, y_\circ)\}$.

Consider the curve C with parametric equations.

$x = x(t),\ y = y(t),\ z = z(t)$, *t* being the parameter (2). Then the curve C lies on (2) provided $z = g\{x(t), y(t)\}$ (4) holds good for all values of t in the appropriate interval I. Let P_0 be a point on curve C corresponding to $t = t_0$. Now, the direction ratios of the tangent line $P_\circ$, P_1 are $x'(t_0), y'(t_0), z'(t_0)$,

where $(x'(t_0), y'(t_0), z'(t_0)$ denote the values of $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ respectively at $t = t_0$.

This direction will be perpendicular to direction of normal n (with direction ratios $p_0, q_0, -1)$ if $p_0 x'(t_0) + q_0 y'(t_0) + (-1)z'(t_0) = 0$ or $z'(t_0) = p_0 x'(t_0) + q_0 y'(t_0)$.

It follows that any set $\{x(t), y(t), z(t), p(t), q(t)\}$ .....(5) of five real functions satisfying the condition that $z(t) = p(t)x'(t) + q(t)y'(t)$ .....(6) defines a strip at the point (*x,y,z*) of the curve C when such strip is also an integral element of (1), then the strip under consideration is known as an integral strip of (1). In other words, the set of functions (5) is known as an integral strip of (1) provided these satisfy (6) and the following additional condition.

$f\{x(t), y(t), z(t), p(t), q(t)\} = 0$, for all *t* in I.

If at each point of the curve (3) touches a generator of the elementary cone, then the corresponding strip is known as a characteristic strip.

Derivastion of the equations determining a characteristic strip :

Clearly, the point $(x + dx, y + dy, z + dz)$ lies in the tangent plane to the elementary cone at P if

$$dz = pdx + qdy \quad \text{..... (7)}$$

where *p, q* satisfy (1). Differentiating (7) w.r.t. p, we have

$$0 = dx + \left(\frac{dq}{dp}\right) dy \quad \text{..... (8)}$$

Again, differentiating (1) partially w.r.t. p, we get

$$\frac{\partial \mathrm{f}}{\partial \mathrm{p}} + \left(\frac{\partial \mathrm{f}}{\partial \mathrm{q}}\right)\left(\frac{\partial \mathrm{q}}{\partial \mathrm{p}}\right) = 0$$

i.e., $\mathrm{f_p} + \mathrm{f_q}\left(\frac{\mathrm{dq}}{\mathrm{dp}}\right) = 0$ ..... (9)

where $\mathrm{f_p} = \frac{\partial \mathrm{f}}{\partial \mathrm{p}}$, and $\mathrm{f_q} = \frac{\partial \mathrm{f}}{\partial \mathrm{q}}$

Solving (7), (8) and (9) for the ratios of *dy, dz* to *dz*, we get

$$\frac{\mathrm{dx}}{\mathrm{f_p}} = \frac{\mathrm{dy}}{\mathrm{f_q}} = \frac{\mathrm{dz}}{\mathrm{pf_p} + \mathrm{qf_q}} \quad \text{..... (10)}$$

Hence, along a characteristic strip $x'(t), y'(t), z'(t)$ will be proportional to f_p, fq, $pfp + qfq$ respectively. If the parameter t be selected satisfying the relation

$$x'(t) = f_p \text{ and } y'(t) = f_q \quad \text{..... (11)}$$

then, we have $z'(t) = pfp + qfq$

Since along a characteristic strip p is a function of t, hence

$$p'(t) = \left(\frac{\partial p}{\partial x}\right)\left(\frac{\partial x}{\partial t}\right) + \left(\frac{\partial p}{\partial y}.\frac{\partial y}{\partial t}\right) = \frac{\partial p}{\partial x}.\frac{\partial t}{\partial p} + \frac{\partial p}{\partial y}.\frac{\partial f}{\partial q} \text{ [Using (11)]}$$

Thus, $p'(t) = \frac{\partial p}{\partial x}.\frac{\partial f}{\partial p} + \frac{\partial q}{\partial x}.\frac{\partial f}{\partial q}$ ..... (12)

$$\left[\because \frac{\partial p}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial q}{\partial x}\right]$$

Now, differentiating (1), partially w.r.t. 'x' gives

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}p + \frac{\partial f}{\partial p}.\frac{\partial p}{\partial x} + \frac{\partial f}{\partial q}.\frac{\partial q}{\partial x} = 0$$

Or $f_x + pf_z + p'(t) = 0$ [Using (12)]

Hence on a characteristic strip, $p'(t) = -f_x - pf_z$ ..... (13)

Similarly, we have $q'(t) = -f_y - qf_z$ ..... (14)

Here, $f_x = \frac{df}{dx}$, $f_y = \frac{df}{dy}$, $f_z = \frac{df}{dz}$

From (11), (13) and (14), we get the following system of five ordinary differential equations for the determination of the characteristic strip.

$$x'(t) = f_p,\ y(t) = f_q,\ z'(t) = pf_p + qf_q,\ p'(t) = -f_x - pf_z$$

and $q'(t) = -f_y - qf_z$ ..... (15)

The above equations are called the characteristic equations of (1). In view of a well known result of the functions which are involved in (15) satisfy a Lipschitz condition, there exists a unique solution of (15) for given set of initial values of the variables. It follows that the characteristic strip is determined uniquely by any initial element $(x_0, y_0, z_0, p_0, q_0)$ and any initial value t_0 of t.

Theorem : Along every characteristic strip of the partial differential equation $f(x, y, z, p, q) = 0$, the function $f(x, y, z, p, q)$ is a constant.

Proof : Along a characteristic strip, we have

$$\frac{d}{dt} f\{x(t), y(t), z(t), p(t), q(t)\} = f_x x'(t) + f_y y'(t) + f_z z'(t) + f_p p'(t) + f_q q'(t)$$

$= f_x f_p + f_y f_q + f_z(pf_p + qf_q) - f_p(f_x + pf_z) - f_q(f_y + qf_y) = 0$ (using the characteristic equation)

Showing that $f(x, y, z, p, q) = k$, a constant along the strip.

Corollary : If a characteristic strip contains at least one integral element of $f(x, y, z, p, q) = 0$ it is an integral strip of the equation $f\left(x, y, z, \frac{dz}{dx}, \frac{dz}{dy}\right) = 0$.

Ex. 14: Find the characteristics of the equation $pq = z$, and determine the integral surface which passes through the parabola $x = 0,\ y^2 = z.$

Soln: Given $pq = z$ ..... (1)

We are to find its integral surface which passes through the given parabola given by

$x = 0$ and $y^2 = z$ ..... (2)

Rewriting (2) in parametric form, we have

$x = 0,\ y = \lambda,\quad z = \lambda^2\quad \lambda \rightarrow$ being a parameter ..... (3)

Let the initial values x_0, y_0, z_0, p_0, q_0 of x, y, z, p, q be taken as

$x_0 = x_0(\lambda) = 0,\quad y_0 = y_0(\lambda) = \lambda,\ \ z_0 = z_0(\lambda) = \lambda^2$ — (4)

Let p_0, q_0 be the initial values of p, q corresponding to the initial values x_0, y_0, z_0. Since initial values $(x_0, y_0, z_0, p_0, q_0)$ satisfy (1), we have

$p_0 q_0 = z_0,$ or $p_0 q_0 = \lambda^2$ by (4) ..... (5)

Also, we have $z_0'(\lambda) p_0 x_0'(\lambda) + q_0 y_0'(\lambda)$

so that $2\lambda = p_0 \times 0 + q_0 \times 1$ or $q_0 = 2\lambda$ ..... (6)

Solving (5) & (6), $p_0 = \lambda/2$ and $q_0 = 2\lambda$ ..... (7)

Collecting relation (4) & (7) together, initial values of x_0, y_0, z_0, p_0, q_0 are given by

$x_0 - 0,\ y_0 = \lambda,\ z_0 = \lambda^2,\ p_0 = \lambda/2,\quad q_0 = 2\lambda$ when $t = t_0 = 0$ ..... (8)

Rewriting (1) $ut\ f(x, y, z, p, q) = pq - z = 0$ ..... (9)

The usual characteristic equation of (9) are given by

$$\frac{dx}{dt} = \frac{\partial f}{\partial p} = q \quad \text{..... (10)}$$

$$\frac{dy}{dt} = \frac{\partial f}{\partial q} = p \quad \text{..... (11)}$$

$$\frac{dz}{dt} = p\left(\frac{\partial f}{\partial p}\right) + q\left(\frac{\partial f}{\partial q}\right) = 2pq \quad \text{..... (12)}$$

$$\frac{dp}{dt} = -\left(\frac{\partial f}{\partial x}\right) = p\left(\frac{\partial f}{\partial z}\right) = p \quad \text{..... (13)}$$

$$\frac{dq}{dt} = -\left(\frac{\partial f}{\partial y}\right) - q\left(\frac{\partial f}{\partial z}\right) = q \quad \text{..... (14)}$$

From (10) and (14) $\frac{dx}{dt} - \frac{dq}{dt} = 0$

$\Rightarrow x - q = c_1$ ..... (15), $c_1 \rightarrow$ being arbitrary constant.

Using initial values of (8) & (15) gives

$$x_0 - q_0 = c_1$$
$$\Rightarrow 0 - 2\lambda = c_1$$
$$\Rightarrow c_1 = -2\lambda$$
$$\therefore \ (15) \Rightarrow \quad x - q = -2\lambda$$
$$\Rightarrow x = q - 2\lambda \quad \text{..... (16)}$$

From (11) & (13)

$$\frac{dy}{dt} - \frac{dp}{dt} = 0$$
$$\Rightarrow y - p = c_2 \quad \text{..... (17)} \qquad c_2 \text{ being arbitrary constants.}$$

Using initial values (8) and (17) gives

$$y_0 - p_0 = c_2$$
$$\Rightarrow \lambda - \frac{\lambda}{2} = c_2$$
$$\Rightarrow c_2 = \lambda/2$$
$$\therefore \ (17) \Rightarrow \ y - p = \lambda/2$$
$$y = p + \lambda/2 \quad \text{..... (18)}$$

From (13), $\frac{1}{p}dp = dt$

$$\Rightarrow \log p = t + \log c_3$$
$$\Rightarrow p = c_3 e^t \quad \text{..... (19)}$$

Using initial values (8) & (18) gives

$$p_0 = c_3 e^0$$
$$\Rightarrow \lambda/2 = c_3$$

Hence (19) $\Rightarrow p = \frac{\lambda}{2} e^t$ ..... (20)

Similarly, from (14), $\frac{1}{q}dq = dt$

$$\Rightarrow \log q = t + \log c_4$$

$$\Rightarrow q = c_4 e^t \quad \text{..... (21)}$$

Using initial values (8), (2) gives $q_0 = c_4 e^0$

$$\Rightarrow c_4 = 2\lambda$$

Hence (21) $\Rightarrow q = 2\lambda e^t$ ..... (22)

Using (22), (18) gives $x = 2\lambda e^t - 2\lambda$

$$\Rightarrow x = 2\lambda\left(e^t - 1\right) \quad \text{..... (23)}$$

Using (20), (18) gives $y = \frac{\lambda}{2}e^t - \lambda/2 = \frac{\lambda}{2}\left(e^t + 1\right)$ ..... (24)

Substituting values of p and q from (20) and (22) in (12) we get

$$\frac{dz}{dt} = 2\left\{\frac{\lambda}{2} \times e^t\right\} \times \left\{2\lambda e^t\right\}$$

$$\Rightarrow dz = 2\lambda^2 e^{2t} dt$$

Integrating

$$z = \lambda^2 e^{2t} + c_5 \quad \text{..... (25)}$$

Using initial values (8), (25) gives

$$z_0 = \lambda^2 e^0 + c_5$$

$$\Rightarrow \lambda^2 = \lambda^2 + c_5$$

$$\Rightarrow c_5 = 0$$

Then (25) gives $\zeta = \lambda^2 e^{2t}$

$$\Rightarrow z = \lambda^2\left(e^t\right)^2 \quad \text{— (26)}$$

The required characteristics of (1) are given by (23), (24) and (26)

To find the required integral surface of (1), we now proceed to eliminate two parameters *t* and λ from three equations (23), (24) and (26). Solving (23) and (24) for e^t and λ, we have $e^t = (x+4y)/(4y-x)$ and $\lambda = (4y-x)/4$

Substituting these values of e^t and λ in (26), we have

$$z = \left\{(4y-x)^2/16\right\} \times \left\{(x+4y)/(4y-x)\right\}^2$$

$$\Rightarrow 16z = (4y+x)^2$$

Which is the required integral surfaces of (1) passing through (2).

Ex. 15: Find the solution of the equation $z = \frac{p^2+q^2}{2} + (p-x)(q-y)$ which passes through the *x*-axis.

Soln: Given $z = \frac{p^2+q^2}{2} + (p-x)(q-y)$ ..... (1)

We have to find its integrall surface which passes through *x*-axis which is given by equations

$y = 0$ and $z = 0$ ..... (2)

Rewriting (2) in parametric form

$x = \lambda, \quad y = 0, \ z = 0, \ \lambda$ being the parameter ..... (3)

Let the initial values z_0, y_0, z_0, p_0, q_0 of *x, y, z, p, q* be taken as $x_0 = x_0(\lambda) = \lambda$, $y_0 = y_0(\lambda) = 0, \ z_0 = z_0(\lambda) = 0$ ..... (4)

Let p_0, q_0 be the initial values of p, q corresponding to the initial values x_0, y_0, z_0. Since initial values $(x_0, y_0, z_0, p_0, q_0)$ satisfy (1), we have

$$z_0 = \frac{\left(p_0^{\,2} + q_0^{\,2}\right)}{2} + (p_0 - x_0)\,(q_0 - y_0)$$

$$\Rightarrow 0 = \frac{\left(p_0^{\,2} + q_0^{\,2}\right)}{2} + q_0\,(p_0 - \lambda)$$

$$\Rightarrow p_0^{\,2} + q_0^{\,2} + 2q_0p_0 - 2q_0\lambda = 0 \quad \text{..... (5)}$$

Also, we have

$$z_0'(\lambda) = p_0 x_0'(\lambda) + q_0 y_0'(\lambda)$$

$$\Rightarrow 0 = p_0 \times 1 + q_0 \times 0$$

$$\Rightarrow p_0 = 0 \quad \text{..... (6)}$$

Solving (5) and (6), $p_0 = 0, \ q_0 = 2\lambda$ ..... (7)

Collecting result (4) and (7) together, initial values of $x_0, y_0, z_0, \ p_0, \ q_0$ are given by

$x_0 = \lambda, y_0 = 0, z_0 = 0, \ p_0 = 0, \ q_0 = 2\lambda$ when $t = t_0 = 0$ ..... (8)

Let $f(x, y, z, p, q) = \frac{p^2+q^2}{2} + pq - py - qx + xy - z = 0$ ..... (9)

The usual characteristic equations of (9) are given by

$$\frac{dx}{dt} = \frac{\partial f}{\partial p} = p + q - y \quad \text{..... (10)}$$

$$\frac{dy}{dt} = \frac{\partial f}{\partial q} = q + p - x \quad \text{..... (11)}$$

$$\frac{dz}{dt} = p\left(\frac{\partial f}{\partial p}\right) + q\left(\frac{\partial f}{\partial q}\right) = p(p+q-y) + q(q+p-x) \quad \text{..... (12)}$$

$$\frac{dp}{dt} = -\frac{\partial f}{\partial x} - p\frac{\partial f}{\partial z} = p+q-y \quad \text{..... (13)}$$

$$\frac{dq}{dt} = -\frac{\partial f}{\partial y} - q\frac{\partial f}{\partial z} = p+q-x \quad \text{..... (14)}$$

From (10) and (13) $\frac{dx}{dt} - \frac{dp}{dt} = 0$

$$\Rightarrow x - p = c_1 \quad \text{..... (15)}$$

Where c_1 is an arbitrary constant. Using initial conditions (8), (15) gives

$\lambda - 0 = c_1 \Rightarrow c_1 = \lambda$

Hence (15) reduces to $x - p = \lambda$

$$\Rightarrow x = p + \lambda \quad \text{..... (16)}$$

From (11) and (14), $\frac{dy}{dt} - \frac{dq}{dt} = 0$

$$\Rightarrow y - q = c_2 \quad \text{..... (17)}$$

Where c_2 is an arbitrary constants.

Using initial conditions (8), (16) gives

$$0 = 2\lambda = c_2$$

$$\Rightarrow c_2 = -2\lambda$$

Hence (17) reduces to $y = q - 2\lambda$ ..... (18)

$$\therefore \frac{d(p+q-x)}{dt} = \frac{dp}{dt} + \frac{dq}{dt} - \frac{dx}{dt}$$

$$= p+q-y+p+q-x-(p+q-y) \quad \text{[Using (10), (13) &}$$

(14)

$$\Rightarrow \frac{d(p+q-x)}{dt} = p+q-x$$

$$\Rightarrow \frac{d(p+q-x)}{p+q-x} = dt$$

Integrating $\log(p+q-x) = \log c_3 + t$

$$\Rightarrow p+q-x = c_3 e^t \quad \text{..... (19)}$$

Where c_3 is an arbitrary constant. Using initial conditions (8), (19) gives

$0 + 2\lambda - \lambda = c_3$

$$\Rightarrow c_3 = \lambda$$

Hence (19) reduces to $p+q-x=\lambda e^t$ ..... (20)

Now $\frac{d(p+q-y)}{dt}=\frac{dp}{dt}+\frac{dq}{dt}-\frac{dy}{dt}$

$$=p+q-y+p+q-x-(q+p-x)$$

$$\Rightarrow \frac{d(p+q-y)}{dt}=p+q-y$$

$$\Rightarrow \frac{d(p+q-y)}{p+q-y}=dt$$

Integrating $\log(p+q-y)=t+\log c_4$

$$\Rightarrow p+q-y=c_4e^t \qquad \text{..... (21)}$$

Where c_4 is an arbitrary constant. Using initial conditions (8), (21) gives

$$0+2\lambda-0=c_4$$

$$\Rightarrow c_4=2\lambda$$

Hence (21) reduces to $p+q-y=2\lambda e^t$ ..... (22)

From (10) and (22), $\frac{dx}{dt}=2\lambda e^t$

$$\Rightarrow x=2\lambda e^t+c_5 \quad \text{..... (23)}$$

Where c_5 is an arbitrary constant. Using initial conditions (8), (23) gives

$$\lambda=2\lambda+c_5$$

$$\Rightarrow c_5=-\lambda$$

Hence (23) reduces to $x=\lambda\left(2e^t-1\right)$ ..... (24)

From (11) and (20), $\frac{dy}{dt}=\succ e^t$

$$\Rightarrow y=\lambda e^t+c_6 \quad \text{..... (25)}$$

Where c_6 is an arbitrary constant. Using initial conditions (8), (25) gives

$$0=\lambda+c_6$$

$$\Rightarrow c_6=\lambda$$

Hence (25) reduces to $y=\lambda e^t-\lambda$

$$\Rightarrow y=\lambda\left(e^t-1\right) \qquad \text{..... (26)}$$

Substituting value of y from (18) in (12), we get

$$\frac{dp}{dt}=p+q-(q-2\lambda)$$

$$\Rightarrow \frac{dp}{dt}-p=2\lambda \qquad \text{..... (27)}$$

Which is a linear equation whose integrating factor, (I.F.) $= e^{\int(-1)dt}$

$= e^{-t}$

$\therefore$ Solution is, $pe^{-t} = \int (2\lambda)e^{-t}dt + c_7$

$\Rightarrow pe^{-t} = -2\lambda e^{-t} + c_7$

$\Rightarrow p = -2\lambda + c_7 e^{t}$ ..... (28)

Where c_7 is an arbitrary constant. Using initial condition (8), (28) gives

$0 = -2\lambda + c_7$

$\Rightarrow c_7 = 2\lambda$

Hence (28) reduces to $p = -2\lambda + 2\lambda e^{t}$

$\Rightarrow p = 2\succ\left(e^{t} - 1\right)$ ..... (29)

Substituting value of *x* from (16) in (14), we get

$$\frac{dq}{dt} = p + q - (p + \lambda)$$

$$\Rightarrow \frac{dq}{dt} = q = -\lambda \quad \text{..... (30)}$$

Which is a linear equation whose I.F. $= e^{\int(-1)dt} = e^{-t}$ and solution is

$$qe^{-t} = \int(-\lambda)e^{-t}dt + c_8$$

$$qe^{-t} = \lambda e^{-t} + c_8$$

$$\Rightarrow q = \lambda + c_8 e^{t} \quad \text{..... (31)}$$

Where c_8 is an arbitrary constant. Using initial condition (8), (31) gives

$2\lambda = \lambda + c_8$

$\Rightarrow c_8 = \lambda$

Hence (31) reduces to $q = \lambda + \lambda e^{t}$

$\Rightarrow q = \lambda\left(e^{t} + 1\right)$ ..... (32)

Substituting the values of $p + q - z$ and $p + q - y$ from (14) and (25) respectively in (1), gives

$$\frac{dz}{dt} = p\left(2\lambda e^{t}\right) + q\left(\lambda e^{t}\right)$$

$$= 2\lambda\left(e^{t} - 1\right)\left(2\lambda e^{t}\right) + \lambda\left(1 + e^{t}\right)\left(\lambda e^{t}\right)$$

$$\Rightarrow \frac{dz}{dt} = 5\lambda^2 e^{2t} - 3\lambda^2 e^{t}$$

$$\Rightarrow dz = \left(5\succ^2 e^{2t} - 3\succ^2 e^{t}\right)dt$$

Integrating $z = \left(\frac{5}{2}\right) \times \lambda^2 e^{2t} - 3\lambda^2 e^{t} + c_9$ ..... (33)

Where c_9 is an arbitrary constant. Using initial conditions (8), namely $z = 0$ where $t = 0$, (33) gives

$$0 = \frac{5}{2} \times \lambda^2 - 3\lambda^2 + c_9$$

$$\Rightarrow c_9 = 3\lambda^2 - \left(\frac{5}{2}\right)\lambda^2$$

Hence (33) reduces to

$$z = \left(\frac{5}{2}\right) \times \lambda^2 \left(e^{2t} - 1\right) - 3\lambda^2 \left(e^{t} - 1\right) \quad \text{..... (34)}$$

Solving (24) and (26) for λ and e^t, $\lambda = x - 2y$ and $e^t = {(x-y)}/{(x-2y)}$ .....(35)

Eliminating λ and e^t from (34) and (35), we have

$$z = \frac{5}{2}(x-2y)^2 \left\{ \left(\frac{x-y}{x-2y}\right)^2 - 1 \right\} - 3(x-2y)^2 \left(\frac{x-y}{x-2y} - 1\right)$$

$$= \frac{5}{2} \times \left\{ (x-y)^2 - (x-2y)^2 \right\} - 3\left\{ (x-2y)(x-y) - (x-2y)^2 \right\}$$

$$z = \left({y}/{2}\right) \times (4x - 3y)$$



CHECK YOUR PROGRESS 2

Q 1: Determine the characteristics of the equation $z = p^2 - q^2$ and find the integral surface which passes through the parabola $4z + x^2 = 0,\ y = 0$

Q 2: Prove that for the equation $z + px + qy - 1 - pqx^2y^2 = 0$ the characteristic strips are given by $x = (B + Ce^{-t})^{-1},$ $y = (A + De^{-t})^{-1}$, $z = E - (AC + BD)e^{-t}$, $p = A(B + Ce^{-t})^2$, $q = B(A + Ee^{-t})^2$ where A, B, C, D and E are arbitary constants. Hence find the integral surface which passes through the line $z = 0$, $x = y$.

Q 3: Find the characteristics of the equation $pq = z$ and determine the integral surface which passes through the straight line $x = 1$, $z = y$.

Q 4: Find the characteristics of the PDE $p^2 + q^2 = 2$ and determine the integral surface which passes through $x = 0$, $z = y$.

11.5 SPECIAL TYPES OF FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

We now consider equations in which p and q occur other than in the first degree, that is non-linear equations. We have already discussed the general method (i.e. Charpit's method). We now discuss four standard forms to which may equations can be reduced and for which a complete integral can be obtained by inspection or by other shorter methods.

Standard Form I : Equations involving band q only. That is, equation of type $f(p,q)=0$ ..... (1)

Let $z=ax+by+c$ is a solution of the fiven PDE, describe by $f(p,q)=0$, then

$$p=\frac{\partial z}{\partial x}=a, \quad q=\frac{\partial z}{\partial y}=b$$

Substituting these values of p and q in the given PDE, we get

$f(a,b)=0$ ..... (2)

Solving for b, we get $b=\phi(a)$, say then

$z=ax+\phi(a)+c$ ..... (3)

is the complete integral of the given PDE.

Ex. 16 Solve : $p^2+q^2=1$

Solution : The given PDE is of the form $f(p,q)=0$

$\therefore$ The solution is given by –

$$z=ax+by+c$$

where $a^2+b^2=1$

$\Rightarrow b=\sqrt{1-a^2}$

$\therefore$ The complete integral is

$z=ax+\sqrt{(1-a^2)}y+c$

Equations reducible to standard form I :

Ex. 17: Solve $x^2p^2+y^2q^2=z^2$

Solution : The given equation can be written as

$$\left(\frac{x}{z}\frac{\partial z}{\partial x}\right)^2+\left(\frac{y}{z}\frac{\partial z}{\partial y}\right)^2=1 \quad \text{..... (1)}$$

Let $\frac{1}{z}dx=dz$ i.e., $z=e^z$

$\frac{1}{x}dx = dx$ i.e., $x = e^z$

& $\frac{1}{y}dy = dy$ i.e., $y = e^y$

$\therefore$ Equation (1) becomes

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 1$$

which is of the form of standard I.

$\therefore$ The complete integral is given by

$z = ax + by + c_1$

where $x^2 + b^2 = 1$

$\Rightarrow b = \sqrt{(1-a^2)}$

$\therefore Z = ax + Y\sqrt{(1-a^2)} + c_1$

$\Rightarrow \log z = a\log x + \sqrt{(1-a^2)}\log y + c_1$ ..... (2)

which is the required complete integral.

If we put $a = \cos\alpha$

$\log z = \cos\alpha.\log x + \sin\alpha \log y + \log c$

$\therefore z = cx^{\cos\alpha}.y^{\sin\alpha}$ ..... (3)

General Integral : It is obtained by eliminating α from

$z = \phi(\alpha)x^{\cos\alpha}.y^{\sin\alpha}$ [Taking $c = \phi(\alpha)$]

and $0 = \phi(\alpha)x^{\cos\alpha}.y^{\sin\alpha} + \phi(\alpha)x^{\cos\alpha}.y^{\sin\alpha}(-\sin\alpha).\log x$

$+ \phi'(\alpha)x^{\cos\alpha}.\cos\alpha \log y$

Singular Integral : It is obtained by eliminating α and c, from

$z = cx^{\cos\alpha}.y^{\sin\alpha}$

$$\frac{\partial z}{\partial \alpha} = -c\sin\alpha.\, x^{\cos\alpha}.y^{\sin\alpha}\log x + c\cos x^{\cos\alpha}.y^{\sin\alpha}\log y = 0$$

and $\frac{\partial z}{\partial c} = x^{\cos\alpha}.y^{\sin\alpha} = 0$

$\therefore$ Singular integral is z = 0

Ex. 18: Find the complete interal of

$(y-x)(9y-px) = (p-q)^2$

Solution : Let x = x + y and y = xy

So that $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial x} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial x} = \frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$

and $q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial y} = \frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y}$

$$q = \frac{dz}{dy} = \frac{dz}{dX}\frac{dX}{dy} + \frac{dz}{dY}.\frac{dY}{dy} = \frac{dz}{dX} + x\frac{dz}{dY}$$

Substituting in the given equation, we have

$$(y-x)\left[y\left(\frac{dz}{dX} + x\frac{dz}{dY}\right) - x\left(\frac{dz}{dX} + y\frac{dz}{dY}\right)\right] = \left[\left(\frac{dz}{dX} + y\frac{dz}{dY}\right) - \left(\frac{dz}{dX} + x\frac{dz}{dY}\right)\right]^2$$

$$\Rightarrow (y-x)(y-x)\frac{dZ}{dX} = (y-x)^2\left(\frac{dZ}{dY}\right)^2$$

$$\Rightarrow \frac{dZ}{dX} = \left(\frac{dZ}{dY}\right)^2$$

which is of othe form of standard I,

$\therefore$ The complete integral is given by $Z = aX + bY + C$ where $a = b^2$

$\therefore$ The complete integral is

$$Z = b^2(x+y) + bxy + c$$

Ex. 19: Solve $\left(x^2 + y^2\right)\left(p^2 + q^2\right) = 1$

Soln: Let us put

$x = rCos\theta,\ y = rSin\theta$, i.e. $x^2 + y^2 = r^2,\ \theta = \tan^{-1} y/x$ so that

$$p = \frac{dz}{dx} = \frac{dz}{dr}\frac{dr}{dx} + \frac{dz}{d\theta}.\frac{d\theta}{dx} = \frac{dz}{dr}\cos\theta - \frac{Sin\theta}{r}\frac{dz}{d\theta}$$

and $q = \frac{dz}{dy} = \frac{dz}{dr}.\frac{dr}{dy} + \frac{dz}{d\theta}\frac{d\theta}{dy} = \frac{dz}{dr}Sin\theta + \frac{Cos\theta}{r}\frac{dz}{d\theta}$

$\therefore$ Given equation reduces to

$$r^2\left[\left(\frac{dz}{dr}\right)^2 + \frac{1}{r^2}\left(\frac{dz}{d\theta}\right)^2\right] = 1$$

$$r^2\left(\frac{dz}{dr}\right)^2 + \left(\frac{dz}{d\theta}\right)^2 = 1 \qquad \text{— (1)}$$

Again putting $du = \frac{dr}{r}$ i.e. $u = \log r$ in (1), we have

$$\left(\frac{dz}{du}\right)^2 + \left(\frac{dz}{d\theta}\right)^2 = 1$$

Which is of the form of standard I.

$\therefore$ The complete integral is given by

$Z = au + b\theta + C$ where $a^2 + b^2 = 1$ or $b = \sqrt{(1 - a^2)}$

$\therefore$ The complete integral

$$Z = a \log r + \sqrt{(1 - a^2)}\,\theta + C$$

$$= \frac{a}{2}\log\left(x^2 + y^2\right) + \sqrt{(1 - a^2)}\,\tan^{-1}\left(\frac{y}{x}\right) + C$$

Standard form II : Equations not involving the independent variables :

That is, equations of the type

$$f(z, p, q) = 0 \quad \text{— (1)}$$

As a trial solution, let us assume that z is a function of $u = x + ay$, where *a* is an arbitrary constant.

Then, $z = f(u) = f(x + ay)$ — (2)

$$p = \frac{dz}{dx} = \frac{dz}{du}.\frac{du}{dx} = \frac{dz}{du}$$

$$q = \frac{dz}{dy} = \frac{dz}{du}.\frac{du}{dy} = a\frac{dz}{du}$$

Substituting these values of *p* and *q* in the given PDE, we get

$f\left(z, \frac{dz}{du}, a\frac{dz}{du}\right) = 0$ — (3)

which is an ordinary differential equation of first order.

Solving equation (3) for $\frac{dz}{du}$, we obtain

$$\frac{dz}{du} = p\,(z, a) \quad \text{(say)}$$

$$\Rightarrow \frac{dz}{\phi(z, a)} = du$$

On integration, we find

$$\int \frac{dz}{\phi(z, a)} = u + c$$

That is F(*z, a*) = *u* + *c* = *x* + *ay* + *c* — (4)

Which is the complete integral of the given PDE.

Ex. 5: Find the complete integral of $p^2 = zq$

Soln: Putting $z = f(x+ay) = f(x)$ in the given equation, we have

$$\left(\frac{dz}{dX}\right)^2 = za\frac{dz}{dX}$$

$$\Rightarrow \frac{dz}{dX} = Za$$

$$\Rightarrow \frac{dz}{Z} = a\,dX$$

Integrating $\log z = aX + \log b$

$$\therefore\ z = be^{aX}$$

$$\Rightarrow z = be^{a(x+ay)}$$

$$\Rightarrow z = be^{ax+a^2y}$$

Which is the complete integral.

Ex. 20 : Solve $9\left(p^2z + q^2\right) = 4$

Soln: Putting $z = f(x+ay) = f(x),$ where $X = x + ay$ so that

$$p = \frac{dz}{dx} = \frac{dz}{dX} \text{ and } q = \frac{dz}{dy} = a\frac{dz}{dX}$$

The equation reduces to

$$9\left[\left(\frac{dz}{dX}\right)^2 z + a\left(\frac{dz}{dX}\right)^2\right] = 4$$

$$\Rightarrow 9\left(\frac{dZ}{dX}\right)^2\left(Z + a^2\right) = 4$$

$$\Rightarrow 3\,\frac{dZ}{dX}\sqrt{\left(Z + a^2\right)} = 2$$

$$\Rightarrow 3\,\sqrt{Z + a^2}\ \ dZ = 2dX$$

Integrating

$$\left(Z + a^2\right)^{3/2} = X + b$$

$$\Rightarrow \left(Z + a^2\right)^{3/2} = (x + ay + b)$$

$$\Rightarrow \left(Z + a^2\right)^3 = (x + ay + b)^2$$

Which is a complete integral containing two arbitrary constants *a* and *b*.

Ex. 21 : Solve $p\left(1 + q^2\right) = q\left(Z - a\right)$

Soln: Given equation

$$p\left(1+q^2\right)=q\left(Z-a\right) \quad \text{— (1)}$$

Putting $Z=f(x+ay)=f(x)$, where $X=x+ay$ so that

$$p=\frac{dZ}{dx}=\frac{dZ}{dX} \text{ and } q=\frac{dZ}{dy}=\frac{dZ}{dX}$$

$\therefore$ (1) Can be rewritten as

$$\frac{dZ}{dX}\left[1+a^2\left(\frac{dZ}{dX}\right)^2\right]=a\frac{dZ}{dX}(Z-a)$$

$$\Rightarrow a^2\left(\frac{dZ}{dX}\right)^2=a(Z-a)-1$$

$$\Rightarrow a\frac{dZ}{dX}=\sqrt{a(Z-a)-1}$$

$$\Rightarrow \frac{adZ}{\sqrt{a(Z-a)-1}}=dX$$

Integrating, we have

$$2.\sqrt{(a(z-a)-1}=X+b$$

$$\Rightarrow 4a(Z-a)-4=\left(x+ay+b\right)^2$$

$$\Rightarrow 4a(Z-a)=\left(x+ay+b\right)^2+4$$

Which is the required complete integral.

Ex. 22 : Solve $pq=x^m y^n z^l$

Soln: Given $pq=x^m y^n z^l$ — (1)

Putting $\frac{x^{m+1}}{m+1}=X,\ \frac{y^{n+1}}{n+1}=Y$

so that $p=\frac{dz}{dx}=\frac{dz}{dX}.\frac{dX}{dx}=x^m\frac{dz}{dX}$

and $q=\frac{dz}{dy}=\frac{dz}{dy}.\frac{dY}{dy}=y^n\frac{dz}{dY}$

The given equation (1) reduces to

$$x^m\frac{dz}{dX}.y^n\frac{dx}{dY}=x^m y^n z^l$$

$$\Rightarrow \frac{dz}{dX}\frac{dz}{dY}=z^l \quad \text{— (2)}$$

Which is of the form of standard II.

$\therefore$ Putting $z=f(x+aY)=f(u),$ where $u=X+aY$

$$\frac{dZ}{dX}=\frac{dz}{du}.\frac{du}{dX}=\frac{dz}{du}$$

and $$\frac{dZ}{dY}=\frac{dz}{du}.\frac{du}{dY}=a\frac{dz}{du}$$

Equation (2) becomes

$$a\left(\frac{dz}{du}\right)^2=z^l$$

$$\Rightarrow \frac{dz}{du}=\sqrt{\frac{z^l}{a}}$$

$$\Rightarrow Z^{-l/2}dz=\frac{du}{\sqrt{a}}$$

Integrating, we have

$$\frac{Z^{-l/2+1}}{-\frac{l}{2}+1}=\frac{u}{\sqrt{a}}+b$$

$$\Rightarrow \frac{Z^{-l/2+1}}{-\frac{l}{2}+1}=\frac{1}{\sqrt{a}}\left(\frac{x^{m+1}}{m+1}+a\frac{y^{n+1}}{n+1}\right)+b$$

Which is the required solution.

Standard form III Separable Equations

An equation in which *z* is absent and the terms containing *x* and *p* can be separated from those containing *y* and *q* is called a separable equation.

That is, equation of type

$$f(x,p)=F(y,q) \quad \text{— (1)}$$

As a trial solution, let us assume that

$$f(x,p)=F(y,q)=a \text{ (say)} \quad \text{— (2)}$$

Now solving them for *p* and *q*, we obtain

$$p=\phi(x),\ \ q=\psi(y)$$

Since $dz=\frac{dz}{dx}dx+\frac{dz}{dy}dy$

$$dz=pdx+qdy$$

On integration, we get the complete integral in the form

$$z=\int\phi(x)dx+\int\psi(y)dy+b \quad \text{— (3)}$$

Ex.23: Find a complete integral of

$$p^2 + q^2 = x + y$$

Soln: The given PDE is of separable type and can be rewritten as $p^2 - x = y - q^2 = a$ (say)

$$\therefore\ p = \sqrt{x+a} \quad \text{and} \quad q = \sqrt{y-a}$$

Putting in $dz = pdx = qdy,$ we have

$$dz = \sqrt{(x+a)}\,dx + \sqrt{(y-a)}\,dy$$

Integrating $z = \frac{2}{3}(x+a)^{3/2} + \frac{2}{3}(y-a)^{3/2} + b$

which is the complete integral of the given PDE.

Ex.24: Find a complete integral of $z^2\left(p^2+q^2\right) = x^2 + y^2$

Soln: The given PDE is

$$z^2\left(p^2+q^2\right) = x^2 + y^2 \qquad \text{— (1)}$$

Putting $zdz = dZ$

so that $\frac{z^2}{2} = Z$

The given equation (1) reduces to

$$P^2 PQ^2 = x^2 + y^2 \text{ where } P = \frac{dZ}{dx} \text{ and } Q = \frac{dZ}{dy}$$

$$\Rightarrow P^2 - x^2 = y^2 - Q^2$$

Which is of the form of Standard III.

$$\therefore\ \text{Let } P^2 - x^2 = y^2 - Q^2 = a$$

$$\therefore\ P = \sqrt{\left(x^2+a\right)} \text{ and } Q = \sqrt{\left(y^2-a\right)}$$

Putting in $dZ = Pdx + Qdy,$ we have

$$\Rightarrow dZ = \sqrt{\left(x^2+a\right)}\,dx + \sqrt{\left(y^2-a\right)}dy$$

Integrating $Z = \frac{x}{2}\sqrt{\left(x^2+a\right)} + \frac{a}{2}\log\left\{x + \sqrt{\left(x^2+a\right)}\right\}$

$$+ \frac{y}{2}\sqrt{\left(y^2-a\right)} - \frac{a}{2}\log\left\{y + \sqrt{\left(y^2-a\right)}\right\}$$

$$\Rightarrow Z^2 = x\sqrt{\left(x^2+a\right)} + a\log\left\{x + \sqrt{\left(x^2+a\right)}\right\}$$

$$+ y\sqrt{\left(y^2-a\right)} - a\log\left\{y + \sqrt{\left(y^2-a\right)}\right\} + C$$

which is the complete integral of the given PDE.

Ex.25: Solve $z\left(p^2 - q^2\right) = x - y$

Soln: The given equation can be written as

$$\left(\sqrt{z}\,\frac{dz}{dx}\right)^2 - \left(\sqrt{z}\,\frac{dz}{dy}\right)^2 = x - y \quad \text{— (1)}$$

Putting $\sqrt{z}\,dz - dZ$ so that $Z = \frac{2}{3}\,z^{3/2}$

The equation (1) becomes

$$\left(\frac{dZ}{dx}\right)^2 - \left(\frac{d'Z}{dy}\right)^2 = x - y$$

$$\Rightarrow P^2 - Q^2 = x - y \text{ where } P = \frac{d'Z}{dx},\ Q = \frac{d'Z}{dy}$$

$$\Rightarrow P^2 - x = Q^2 - y$$

Which is of the form of Standard III.

$\therefore$ Let $P^2 - x = Q^2 - y = a$

$\therefore$ $P = \sqrt{(x+a)}$ and $Q = \sqrt{(y+a)}$

Putting in $dZ = Pdx + Qdy,$ we have

$$dZ = \sqrt{(x+a)}\ dx + \sqrt{(y+a)}\,dy$$

Integrating, $Z = \frac{2}{3}(x+a)^{3/2} + \frac{2}{3}(y+a)^{3/2} + b$

$$\Rightarrow z^{3/2} = (x+a)^{3/2} + (y+a)^{3/2} + C$$

Standard Form IV. Clairaut's Form

A first order PDE is said to be of Clairaut's form if it can be written as

$$z = px + qy + f(p,q) \quad \text{— (1)}$$

The corresponding Charpit's equations are

$$\frac{dp}{p-p} = \frac{dq}{q-q} = \frac{dz}{px+qy+pfp+qfq} = \frac{dx}{x+fp} = \frac{dy}{y+fq} \quad \text{— (2)}$$

The integration of the first two fractions of (2) gives us $p = a,\ q = b$

Substituting these values of p and q in the given PDE, we get the required complete integral in the form

$$z = ax + by + f(a,b) \quad \text{— (3)}$$

Ex.26: Find the complete integral of the equation

$$z = px + qy + c\sqrt{\left(1 + p^2 + q^2\right)}$$

Soln: This is of the form of Standard IV.

$\therefore$ The complete integral is

$$z = ax + by + c\sqrt{\left(1 + a^2 + b^2\right)} \quad \text{— (1)}$$

Singular Integral. Differentiating (1) partially w.r.t. *a* & *b*, we have

$$0 = x + \frac{ac}{\sqrt{\left(1 + a^2 + b^2\right)}} \quad \text{— (2)}$$

and $$0 = y + \frac{cb}{\sqrt{\left(1 + a^2 + b^2\right)}} \quad \text{— (3)}$$

$$\therefore \quad x^2 + y^2 = \frac{\left(a^2 + b^2\right)c^2}{1 + a^2 + b^2}$$

$$\Rightarrow \quad x^2 + y^2 = \frac{\left(1 + a^2 + b^2\right)c^2 - c^2}{\left(1 + a^2 + b^2\right)}$$

$$\Rightarrow \quad c^2 - x^2 - y^2 = \frac{c^2}{1 + a^2 + b^2}$$

$$\Rightarrow \quad 1 + a^2 + b^2 = \frac{c^2}{c^2 - x^2 - y^2}$$

$\therefore$ From (2) and (3),

$$a = -\frac{x\left(1 + a^2 + b^2\right)}{c} = \frac{-x}{\sqrt{\left(c^2 - x^2 - y^2\right)}}$$

and $$b = -\frac{y\left(1 + a^2 + b^2\right)}{c} = \frac{-y}{\sqrt{\left(c^2 - x^2 - y^2\right)}}$$

Putting these values of *a* and *b* in (1), the singular integral is

$$z = \frac{-x^2}{\left(c^2 - x^2 - y^2\right)} - \frac{y^2}{\sqrt{\left(c^2 - x^2 - y^2\right)}} + \frac{c^2}{\sqrt{\left(c^2 - x^2 - y^2\right)}}$$

$$\Rightarrow \quad z = \frac{\left(c^2 - x^2 - y^2\right)}{\left(c^2 - x^2 - y^2\right)}$$

$$\Rightarrow \quad z = \sqrt{\left(c^2 - x^2 - y^2\right)}$$

$$\Rightarrow \quad z^2 = c^2 - x^2 - y^2$$

$$\Rightarrow \quad x^2 + y^2 + z^2 = c^2$$

Ex.27: Find the singular integral of

$$z = px + qy + \log pq$$

Soln: The complete integral is

$$z = ax + by + \log ab \quad \text{— (1)}$$

Differentiating (1) partially w.r.t. *a* and *b*, we have

$$0 = x + \frac{1}{a} \text{ and } 0 = y + \frac{1}{b}$$

$$\therefore \ a = -\frac{1}{x} \text{ and } b = -\frac{1}{y}$$

Putting in (1), the singular integral is

$$z = -1 - 1 + \log \frac{1}{xy}$$

$$\Rightarrow z = -2 - \log xy$$

Ex.28: Solve $z = px + qy - z\sqrt{(pq)}$

Soln: The complete integral is

$$z = ax + by - z\sqrt{ab} \quad \text{— (1)}$$

Singular Integral.

Differentiating (1) partially w.r.t. *a* and *b*, we have

$$0 = x - \frac{2}{2\sqrt{ab}}.b$$

$$\Rightarrow x = \sqrt{\left(\frac{b}{a}\right)} \quad \text{— (2)}$$

and $$0 = y - \frac{2}{2\sqrt{ab}}\, a$$

$$\Rightarrow y = \sqrt{\left(\frac{a}{b}\right)} \quad \text{— (3)}$$

Now, using (1) $$x - z = x - \left(ax + by - 2\sqrt{ab}\right) = \sqrt{\frac{b}{a}} - a\sqrt{\frac{b}{a}} - b\sqrt{\frac{a}{b}} + 2\sqrt{ab}$$

[using (2)]

$$\Rightarrow x - z = \sqrt{b/a} \quad \text{— (4)}$$

Similarly, using (1),

$$y - z = y - \left(ax + by - 2\sqrt{ab}\right) = \sqrt{\frac{a}{b}} - a\sqrt{\frac{b}{a}} - b\sqrt{\frac{a}{b}} + 2\sqrt{ab}$$

$$\Rightarrow y - z = \sqrt{\frac{a}{b}} \quad \text{— (5)}$$

From (4) & (5), $(x-z)\,(y-z)=1$

which is singular integral as it satisfies the given equation.



CHECK YOUR PROGRESS 3

1. Solve the following :

   (i) $p+q=pq$

   (ii) $xy^2p=x\left(y^2+z^2q^2\right)$

   (iii) $\left(1-y^2\right)xq^2-y^2p=0$

   (iv) $p^2+q^2=z$

2. Find the complete integral of the following :

   (i) $zpq=p+q$

   (ii) $p^2z^2+q^2=1$

   (iii) $p^3=qz$

   (iv) $pz=1+q^2$

3. Find a complete integral of the following :

   (i) $p^2=q+x$

   (ii) $p^2+q^2=x+y$

   (iii) $q=2yp^2$

   (iv) $p+q=px+qy$

4. Solve the following PDE :

   (i) $z=px+qy-2p-3q$

   (ii) $z=px+qy+5pq$

   (iii) $z=px+qy+p^2-q^2$

   (iv) $z=px+qy+2\sqrt{pq}$



11.6 LET US SUM UP

The Jacobi's auxiliary equations of solving non-linear partial differential equation
$F(x_1, x_2, x_3, p_2, p_3) = 0$ of more two independent variables are

$$\frac{dp_1}{\frac{\partial f}{\partial x_1}} = \frac{dx_1}{\frac{-\partial f}{\partial p_1}} = \frac{dp_2}{\frac{\partial f}{\partial x_2}} = \frac{dx_2}{-\frac{\partial f}{\partial p_2}} = \frac{dp_3}{\frac{\partial f}{\partial x_3}} = \frac{dx_3}{-\frac{\partial f}{\partial p_3}}$$

11.7 ANSWERS TO CHECK YOUR PROGRESS

Answer to check your progress 1

Ans to Q No 1: (i) $z = a_1x_1 + a_2x_2 + a_3x_3 + a_4$

where $a_1 + a_2 + a_3 - a_1a_2a_3 = 0$

(ii) $z = -\left(a_1^{\,2} + a_2a_2^{\,2}\right)x_1^{-1} + a_1x_2 + a_2a_3x_3 + a_3$

(iii) $z = a_1x_1 + a_2x_2 + \left(a_1^{\,2} + a_2^{\,2}\right)\log x_3 - \frac{x_3^{\,2}}{2} + a_3$

(iv) $z = a_1x_1 + a_2x_2 + (a_1 + a_2)^2 \times \left(\frac{x_3^{\,2}}{4}\right) + a_3$

(v) $z = a_1 \log x_1 + a_2 \log x_2 + x_3\sqrt{(a_1 + a_2)} + a_3$

Answer to check your progress 2

Ans to Q No 1: $4z + \left(x + y\sqrt{2}\right)^2 = 0$

Ans to Q No 2: $z = xy$

Ans to Q No 3: $z' = y \pm x$

Answer to check your progress 3

Ans to Q No 1: (i) $c_1 - z = ax + \left\{\frac{a}{(a-1)}\right\}y + c$

(ii) $z^2 = ax^2 \pm y^2(-1)^{1/2} + c$

(iii) $\left(2z - ax^2 - b\right)^2 = a\left(1 - y^2\right)$

(iv) $2z^{1/2} = ax \pm \left(1 - a^2\right)^{1/2} y + c$

Ans to Q No 2: (i) $z = a\tan(x + ay + b)$

(ii) $x + ay + b = \pm\left[\left(z/2\right) \times \left(z^2 + a^2\right)^{1/2} + \frac{a^2}{2} \times \sin h^{y-1}\left(z/a\right)\right]$

(iii) $4z = (x + ay + b)^2$

(iv) $z^2 \mp \left[z\left(z^2 - 4a^2\right)^{1/2} - 4a^2 \log\left\{ z + \left(z^2 - 4a^2\right)^{1/2} \right\} \right] = 4(x + ay) + b$

Ans to Q. No.3: (i) $z = \frac{2}{3} \times (a + x)^{3/2} + ay + b$

(ii) $3z = 2(x + a)^{3/2} + 2\,(y - a)^{3/2} + b$

(iii) $z = ax + a^2 y^2 + b$

(iv) $z = -a \log(1 - x) + a \log(y - 1) + b$

Ans to Q. No.4:

(i) C.I, $z = ax + by - 2a - 36$

S.S, $z = 0$

G.S, $z - ax - \psi(a)\, y + 2a + 3\psi(a) = 0;$

$x + (y - 3)\psi'(a) - 2 = 0$

(ii) S.I, $z = ax + by + 5ab,$

S.S, $5z + xy = 0,$

G.S, $z - ax - \psi(a) - 5a\,\psi(a) = 0;$

$x + 5\psi(a)\ (y + 5a)\psi'(a) = 0$

(iii) C.I, $z = ax + by - a^2 - b^2$

S.S, $x^2 - y^2 + 4z = 0$

G.S, $z - ax - \psi(a) y - a^2 + \{\psi(a)\}^2 = 0;$

$x + 2a + \{y - 2\psi(a)\}\psi'(a) = 0$

(iv) C.I, $z = ax + by + 2\sqrt{ab}$

S.S, $(x - z)(y - z) = 0$

G.S, $z - ax - \psi(a)\, y - 2\sqrt{a\,\psi(a)} = 0$

$x + \psi'(a) + \{\psi(a) + a\,\psi'(a)\} / 2\sqrt{a\,\psi(a)} = 0$



11.8 FURTHER READINGS

1) Ross, S.L, 'Differential Equations' 3rd edition, Wiley India, 2007.

2) Sneddon, I.N. 'Elements of partial differential equations', Dover Publications, 2006.

3) Raisinghania, M.D., 'Advanced Differential Equations', S. Chand & Company Ltd., New Delhi, 2001.

4) Gupta R.K., and Sharma, J.N., 'Differential Equations', Krishna Prakashan Media (P) Ltd., 2017.



11.9 MODEL QUESTIONS

1. Find the complete integral of the following equations using Jacobi's method

   (i) $2x_1x_3zp_1p_3 + x_2p_2 = 0$

   (ii) $p_1^2 + p_2p_3 - z(p_2 + p_3) = 0$

2. Find the complete integral of the following equations

   (i) $z(p^2 - q^2) = x - y$

   (ii) $p - 3x^2 = q^2 - y$

   (iii) $(1 - x^2)yp^2 + x^2q = 0$

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