UNIT 12 : ELLIPTIC DIFFERENTIAL EQUATIONS

UNIT STRUCTURE

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12.1. LEARNING OBJECTIVES

On completion of the course, learners will be able to –

- classify the Partial Differential Equation.
- understanding the properties of Elliptic Differential Equation.
- understanding how to solve Elliptic Differential Equation.
- understanding Laplace and Poisson Equation.
- know about Maximum Principle and Green's functions.

12.2 INTRODUCTION

Consider a general partial differential equation of second order of a function of two independent variables x and y in the form

$$Rr+Ss+Tt+F(x,y,z,p,q) = 0$$

where R,S,T are continous functions of x and y only possessing partial derivatives defined in some domain D on the xy-plane. Then the equation is said to be

(i) Elliptic at a point (x,y) in D if $S^2-4RT<0$

(ii) Parabolic at a point (x,y) in D if $S^2-4RT=0$

(iii) Hyperbolic at a point (x,y) in D if S2-4RT>0

Observe that the type of the differential equation is determined solely by its principal part (Rr + Ss + Tt, which involves the highest order derivatives of Z) and that the type will generally change with position in the xy- plane unless R, S and T are constant.

Also, we have

$$\mathrm{r} = \frac{\partial^2}{\partial \mathrm{x}^2}, \mathrm{s} = \frac{\partial^2}{\partial \mathrm{x}\,\partial \mathrm{y}} \text{ and } \mathrm{t} = \frac{\partial^2}{\partial \mathrm{t}^2}$$

Example. Consier two dimensional Laplace's equation

$$\frac{\partial^2 \mathrm{z}}{\partial \mathrm{x}^2} + \frac{\partial^2 \mathrm{z}}{\partial \mathrm{y}^2} = 0 \quad \mathrm{i.e.}\ \mathrm{r} + \mathrm{t} = 0$$

Here R = 1, S = 0, T = 1 and

S2 - 4RT = 0 - 4.1.1 < 0.

So the given equation is elliptic.

A linear partial differential equation of second order in 3 independent variables x1, x2, x3 is given by

$$\sum_{i=1}^{3} \sum_{j=1}^{3} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{\mathrm{i}=1}^{3} \mathrm{b}_i \frac{\partial \mathrm{u}}{\partial \mathrm{x}_i} + \mathrm{cu} = 0 \quad \text{.....(1)}$$

where $a_{ij} (= a_{ji})$, b_i and c are constants or some functions of the independent variables x1, x2, x3 and u is the dependent variable.

Since $a_{ij} = a_{ji}$, $A = [a_{ij}]_{3\times3}$ is real symmetric matrix of order 3×3. The eigenvalues of matrix A are roots of the characteristic of A.

With the help of matrix A, (1) is classified as follows :

I. If all the eigenvalues of A are non-zero and and of the same sign, then (1) is known as elliptic type of equation.

II. If |A| = 0 i.e. any one of the eigenvalues of A is zero, then (1) is known as parabolic type of equation.

III. If all the eigenavalues of A are now zero and have the same sign, except precisely one of them, then (1) is known as hyperbolic type of equation.

12.3 ELLIPTIC DIFFERENTIAL EQUATION

12.3.1 Laplace transformation. Reduction to Canonical (or normal) forms

Consider the partial differential equation of the type

Rr+Ss+Tt+f(x,y,z,p,q) = 0 ..... (1)

Where R,S,T, are continnous functions of x and y possessing continuous partial derivatives of as high an order as necessary. Laplace transformation on (1) consists of changing the independent variables x,y to new set of continnously differentiable independent variables u,v where

u = u(x,y) and v = v(x,y) ..... (2)

are to be chosen so that the resulting equation in independent variables u,v is transformed into one of there conocial forms, which are easily integrable. From (2),

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} \text{ and } q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} \quad \text{.....(3)}$$

$$(3) \Longrightarrow \frac{\partial}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v} \text{ and } \frac{\partial}{\partial y} = \frac{\partial u}{\partial y}\frac{\partial}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial}{\partial v} \quad \text{.....(4)}$$

$$r = \frac{\partial^2 z}{\partial x^2}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$$

$$= \left(\frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial u}{\partial x}\frac{\partial}{\partial v}\right)\left(\frac{\partial u}{\partial x}\frac{\partial z}{\partial u} + \frac{\partial v}{\partial x}\cdot\frac{\partial z}{\partial u}\right) \quad \text{[From (3) and (4)]}$$

$$= \frac{\partial u}{\partial x}\frac{\partial}{\partial u}\left(\frac{\partial u}{\partial x}\frac{\partial z}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial z}{\partial u}\right) + \frac{\partial v}{\partial x}\frac{\partial}{\partial v}\left(\frac{\partial u}{\partial x}\frac{\partial z}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial z}{\partial u}\right)$$

$$= \frac{\partial^2 z}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2 + 2\frac{\partial^2 z}{\partial u\,\partial v}\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial^2 z}{\partial v^2}\left(\frac{\partial v}{\partial x}\right)^2 + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial x^2} + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial x^2}$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

$$= \left(\frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}\right) \quad \text{[From (3) and (4)]}$$

$$= \frac{\partial^2 z}{\partial u^2}\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial^2 z}{\partial u\,\partial v}\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) + \frac{\partial^2 z}{\partial v^2}\frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial y\,\partial x} + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial y\,\partial x}$$

$$\text{and } t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial u}{\partial y}\cdot\frac{\partial}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial}{\partial v}\right)\left(\frac{\partial u}{\partial y}\frac{\partial z}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial^2 z}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2 + 2\frac{\partial^2 z}{\partial u\,\partial v}\frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + \frac{\partial^2 z}{\partial v^2}\left(\frac{\partial v}{\partial y}\right)^2 + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial y^2} + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial y^2}$$

Putting the above values of p, q, r, s, t in (1) and simplifying, we get

$$A\frac{\partial^2 z}{\partial u^2} + 2B\frac{\partial^2 z}{\partial u\,\partial v} + C\frac{\partial^2 z}{\partial v^2} + F\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right) = 0 \quad \ldots\ldots\ldots (5)$$

$$\text{Where } A = R\left(\frac{\partial u}{\partial x}\right)^2 + S\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + T\left(\frac{\partial u}{\partial y}\right)^2 \quad \ldots\ldots\ldots (6)$$

$$B = R\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{1}{2}S\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) + T\frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \quad \ldots\ldots\ldots (7)$$

$$C = R\left(\frac{\partial v}{\partial x}\right)^2 + S\frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + T\left(\frac{\partial v}{\partial y}\right)^2 \quad \ldots\ldots\ldots (8)$$

Now we shall find out u and v so that (5) reduces to simplest possible form. The method of evaluation of desired values of u and v becomes easy when the discriminant $S^2–4RT$ of the quadratic equation

$$R\lambda^2 + S\lambda + T = 0 \quad \ldots\ldots\ldots (9)$$

is everywhere either positive, negative or zero.

Case I. Let S2–4RT>0. When this condition is satisfied, then the roots λ_1, λ_2 of the equation (9) are real and distinct. The coefficients of $\frac{\partial^2 z}{\partial u^2}$ and $\frac{\partial^2 z}{\partial v^2}$ in the equation (5) will be vanish if we choose u and v such that

$$\frac{\partial u}{\partial x} = \lambda\frac{\partial u}{\partial y} \quad \ldots\ldots\ldots\ldots (10)$$

and

$$\frac{\partial v}{\partial x} = \lambda\frac{\partial v}{\partial y} \quad \ldots\ldots\ldots\ldots (11)$$

Since λ_1 is a root of (9), we have

$$R\lambda_1^2 + S\lambda_1 + T = 0 \quad \ldots\ldots\ldots (12)$$

Using (10), (6) gives $A = (R\lambda_1^2 + S\lambda_1 + T)\left(\frac{\partial u}{\partial y}\right)^2 = 0$, by(12) …….. (13)

Again λ_2 is a root of (9), so $R\lambda_2^2 + S\lambda_2 + T = 0$ ……..…. (14)

Using (11), (8) gives $C = (R\lambda_2^2 + S\lambda_2 + T)\left(\frac{\partial v}{\partial y}\right)^2 = 0$, [by 14] ……..(15)

Re-writing (10), we obtain $\frac{\partial u}{\partial x} - \lambda_1\left(\frac{\partial u}{\partial y}\right) = 0$ …… (16)

Lagrange's auxiliary equation for (16) are

$$\frac{dx}{1} = \frac{dy}{-\lambda_1} = \frac{du}{0} \quad \text{…… (17)}$$

From (17), du = 0 so that u = c_1 ..... (18)

c_1 being arbitrary constant.

Taking first and 2nd fractions of (17), we get

$$\frac{dy}{dx} + \lambda_1 = 0 \quad \text{..... (19)}$$

Let the solution of (19) be $f_1(x,y) = c_2$..... (20)

where c_2 being asbitrary constant.

From (18) and (20), the general solution of (10) is

$u = f_1(x,y)$ ..... (21)

Similarly, the general solution of (11) can be taken as

$v = f_2(x,y)$ ..... (22)

We can verify that

$$AC - B^2 = \frac{1}{4}(4RT - S^2)\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)^2$$

$$\Longrightarrow B^2 = \frac{1}{4}(S^2 - 4RT)\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)^2 \quad \text{..... (23)}$$

$$\text{Let } J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \neq 0 \quad [\because A = C = 0]$$

Since $S^2 - 4RT > 0$, (23) shows that $B^2 > 0$. Hence we may divide both sides of (5) by B^2. Then noting that A=C=0, (5) transform to the form

$$\frac{\partial^2 z}{\partial u\,\partial v} = \phi\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right) \quad \text{..... (24)}$$

which is the canonical form of (1) in this case.

Case II. Let $S^2 - 4RT = 0$. When this condition is satisfied, the roots λ_1, λ_2 of (9) are real and equal. Now we take u exactly as in case I and take v to be

any function of x,y which is independent of u. We have as in case (i), A=0. Also since S^2 - 4RT = 0, (23) shows that B^2=0 so that B=0.

Moreover in this case $C \neq 0$, otherwise v would be a function of u and consequently v would not be independant of u as already assumed.

Putting A = 0, B = 0 and dividing by C, (5) transforms to the form

$$\frac{\partial^2 z}{\partial v^2} = \phi\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right) \quad \text{..... (25)}$$

which is the cononial form of (1) in this case.

Case III. Let S^2 - 4RT < 0. Then the roots λ_1, λ_2 of (9) are complex. Hence case III is formally the same as case I. Therefore, proceeding as in case I, we find that (1) reduces to (24) but that the variables u,v instead of being real are now complex conjugates. To obtain a real canonical form we make further transformation u = a+iβ and v = a–iβ so that

$$\alpha = \frac{u+v}{2} \; and \; \beta = \frac{i\,(v-u)}{2} \quad \text{..... (26)}$$

Now $$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial u} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial u}$$

$$= \frac{1}{2}\left(\frac{\partial z}{\partial \alpha} - i\frac{\partial z}{\partial \beta}\right) \quad \text{..... (27)}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial v} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial v}$$

$$= \frac{1}{2}\left(\frac{\partial z}{\partial \alpha} + i\frac{\partial z}{\partial \beta}\right) \quad \text{..... (28)}$$

$$\therefore \frac{\partial^2 z}{\partial u \partial v} = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)$$

$$= \frac{1}{2}\left(\frac{\partial}{\partial \alpha} - i\frac{\partial}{\partial \beta}\right)\frac{1}{2}\left(\frac{\partial z}{\partial \alpha} + i\frac{\partial z}{\partial \beta}\right)$$

$$= \frac{1}{4}\left[\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha} + i\frac{\partial z}{\partial \beta}\right) - i\,\frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \alpha} + i\frac{\partial z}{\partial \beta}\right)\right]$$

$$= \frac{1}{4}\left[\frac{\partial^2 z}{\partial \alpha^2} + i\frac{\partial^2 z}{\partial \alpha \partial \beta} - i\,\frac{\partial^2 z}{\partial \beta \partial \alpha} + \frac{\partial^2 z}{\partial \beta^2}\right]$$

$$= \frac{1}{4}\left(\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2}\right) \; as \; \frac{\partial^2 z}{\partial \alpha \partial \beta} = \frac{\partial^2 z}{\partial \beta \partial \alpha} \quad \text{..... (29)}$$

Putting u = a + ib and v = α - iβ and using (27), (28) and (29), (24) reduces to

$$\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = \psi\left(\alpha, \beta, z, \frac{\partial z}{\partial \alpha}, \frac{\partial z}{\partial \beta}\right)$$

which is the canonical form of (1) in this case.

12.3.2 Working rule for reducing an elliptic equation to its canonical form

Step 1. Let the given equation

Rr + Ss + Tt + f (x, y, z, p, q) = 0 ..... (1)

be elliptic so that

$S^2 - 4RT < 0$

Step 2. Write l quadratic equation

$$R\lambda^2 + S\lambda + T = 0 \quad \text{..... (2)}$$

Let roots λ_1, λ_2 of (2) be complex conjugates.

Step 3. Then the corresponding characteristic equations are

$$\frac{dy}{dx} + \lambda_1 = 0 \text{ and } \frac{dy}{dx} + \lambda_2 = 0$$

solving these, we obtain solutions of the form

$f_1(x,y) + i f_2(x,y) = C_1$ and $f_1(x,y) - i f_2(x,y) = c_2$ ..... (3)

Step 4. Choose $u = f_1(x,y) + i f_2(x,y)$

$v = f_1(x,y) - i f_2(x,y)$

Let α and β be two new real independent variables such that u= α +iβ and v= α -iβ so that

$\alpha = f_1(x,y)$ and $\beta = f_2(x,y)$ ..... (4)

Step 5. Using relation (4), find p, q, r, s and t in terms of α and β.

Step 6. Substituting the values of p, q, r, s and t and relations (4) in (1) and simplifying we shall get the following canonical form of (1)

$$\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = \phi\left(\alpha, \beta. z, \frac{\partial z}{\partial \alpha}, \frac{\partial z}{\partial \beta}\right)$$

Example 1. Consider the partial differential equation

$$\frac{\partial^2 z}{\partial x^2} + x^2 \frac{\partial^2 z}{\partial y^2} = 0$$

i.e. $r + x2t = 0$ ..... (1)

Here $R = 1, S = 0, T = x^2$ so that

$S^2 - 4RT = 0 - 4.1.x2 < 0, x \neq 0.$

Therefore (1) is elliptic.

The λ quadratic $R\lambda^2 + S\lambda + T = 0$ reduces to

$1.\ \lambda^2 + 0.\lambda + x^2 = 0$

$\Rightarrow \lambda = ix, -ix$

The corresponding characteristic equations are given by

$$\frac{dy}{dx} + ix = 0 \quad \text{and} \quad \frac{dy}{dx} - ix = 0$$

Integrating, $y + i\left(\frac{x^2}{2}\right) = c_1$ and $y - i\frac{x^2}{2} = c_2$

Choose $u = y + i\left(\frac{x^2}{2}\right) = \alpha + i\beta$ and $v - i\frac{x^2}{2} = \alpha - i\beta$

Where $\alpha = y$ and $\beta = \frac{x^2}{2}$ ..... (2)

are two new independent variales.

Now $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial x}$

$= \frac{\partial z}{\partial \alpha} \cdot 0 + \frac{\partial z}{\partial \beta} \cdot x$

$= x\frac{\partial z}{\partial \beta}$ ..... (3)

$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial z}{\partial \beta} \cdot \frac{\partial \beta}{\partial y}$

$= \frac{\partial z}{\partial \alpha} \cdot 1 + \frac{\partial z}{\partial \beta} \cdot 0$

$= \frac{\partial z}{\partial \alpha}$ ..... (4)

$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$

$= \frac{\partial}{\partial x}\left(x\frac{\partial z}{\partial \beta}\right)$

$$= \frac{\partial z}{\partial \beta} + x \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial \beta}\right)$$

$$= \frac{\partial z}{\partial \beta} + x\left[\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \alpha}{\partial x} + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \beta}{\partial x}\right]$$

$$= \frac{\partial z}{\partial \beta} + x\left[\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \beta}\right)\cdot 0 + \frac{\partial^2 z}{\partial \beta^2}\cdot x\right]$$

$$= \frac{\partial z}{\partial \beta} + x^2\frac{\partial^2 z}{\partial \beta^2} \qquad \ldots\ldots\ldots\ldots (5)$$

and $t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$

$$= \frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha}\right)[\text{ using (2)}]$$

$$= \frac{\partial^2 z}{\partial \alpha^2} \qquad \ldots\ldots\ldots (6)$$

Using (5) and (6) in (1), the required canonical form is

$$\frac{\partial z}{\partial \beta} + x^2\frac{\partial^2 z}{\partial \beta^2} + x^2\frac{\partial^2 z}{\partial \alpha^2} = 0$$

$$\Rightarrow \frac{\partial^2 x}{\partial \beta^2} + \frac{\partial^2 z}{\partial \alpha^2} = -\frac{1}{2\beta}\frac{\partial z}{\partial \beta} \quad \text{where } \beta = \frac{x^2}{2}$$



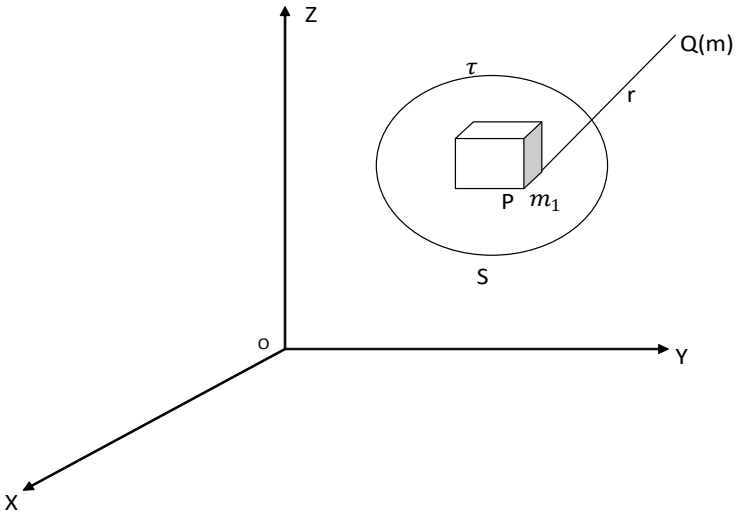
CHECK YOUR PROGRESS

Q.1: Reduce $\frac{\partial^2 z}{\partial x^2} + y^2\frac{\partial^2 z}{\partial y^2} = y$ to canonical form.

12.4. LAPLACE AND POISSON EQUATIONS

12.4.1 Derivation of Laplace Equations

Consider two particles of masses m and m1 at Q and P separated by a distance r as shown in Fig. 12.1.



According to Newton's universal law of gravitation, the magnitude of the force, proportional to the product of their masses and inversely proportional to the square of the distance, between then is given by

$$F = G\frac{mm_1}{r^2} \qquad \ldots\ldots\ldots (1)$$

where G is the gravitational constant. If $\vec{r}$ represents the vector $\overrightarrow{PQ}$ assuming unit mass at Q and G=1, the force at Q due to the mass at P is given by

$$\vec{F} = -\frac{m_1\vec{r}}{r^3} = \nabla\left(\frac{m_1}{r}\right) \qquad \ldots\ldots\ldots\ldots (2)$$

which is called the intensity of the gravitational force. Suppose a particle of unit mass moves under the attaction of a particle of mass m1 at P from infinity up to Q; then the work done by the force F is

$$\int_{\alpha}^{r} \vec{F}\cdot d\vec{r} = \int_{\alpha}^{r} \nabla\left(\frac{m_1}{r}\right)\cdot d\vec{r}$$

$$= \frac{m_1}{r} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

This is defined as the potential V at Q due to a particle at P and is denoted by

$$V = -\frac{m_1}{r} \qquad \ldots\ldots\ldots\ldots (4)$$

From equation (2), the intensity of the force at P is

$$\vec{F} = -\nabla V \qquad \ldots\ldots\ldots\ldots (5)$$

Now, if we consider a system of particles of masses m_1, m_2, ..., m_n which are at distances r_1, r_2, ..., r_n respectively, then the force of attraction per unit mass at Q due to the system is

$$\vec{F} = \sum_{i=1}^{n} \nabla \frac{m_i}{r_i}$$

$$= \nabla \sum_{i=1}^{n} \frac{m_i}{r_i} \qquad \ldots\ldots\ldots\ldots (6)$$

The work done by the force acting on the particle is

$$\int_{\alpha}^{r} \vec{F} \cdot d\vec{r} = \sum_{i=1}^{n} \frac{m_i}{r_i} = -V \qquad \ldots\ldots\ldots\ldots (7)$$

Therefore,

$$\nabla^2 v = -\nabla^2 \sum_{i=1}^{n} \frac{m_i}{r_i}$$

$$= -\sum_{i=1}^{n} \nabla^2 \frac{m_i}{r_i}, r_1 \neq 0$$

$$= 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

Where $\nabla^2 = div\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called the Laplace operator.

In the case of continuous distribution of matter of density P in a volume τ we have

$$V(x, y, z) = \iiint_{\tau} \frac{P(\xi, \eta, \zeta)}{r} \, d\tau$$

Where $r = \{(x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2\}^{1/2}$ and Q is outside the body. It can be verified that

$$\nabla^2 v = 0$$

Which is called the Laplace equation.

12.4.2 Derivation of Poisson Equation

Consider a closed surface S consisting of particles of masses m_1, m_2, ..., m_n. Let Q be any point on S. Let $\sum_{i=1}^{n} m_i = M$ be the total mass insides and let g_1, g_2, ..., g_n be the gravity field at Q due to the presence of m_1, m_2, ..., m_n respectively within S. Also, let $\sum_{i=1}^{n} g_i = g$, the entire gravity field at Q. Then, according to Gauss law, we have

$$\iint_S \vec{g} \cdot \overrightarrow{ds} = -4\pi GM \qquad \text{..... (1)}$$

where $M = \iiint_\tau \rho d\tau$, ρ is the mass density function and t is the volume in which the masses are distributed throughout since the gravity field is conservative, we have

$$\vec{g} = \nabla V \qquad \text{..... (2)}$$

Where V is a scolar potential. But the Gauss divergence theorem states that

$$\iint_S \vec{g} \cdot d\vec{s} = \iiint_\tau \nabla \cdot \vec{g} d\tau \qquad \text{..... (3)}$$

Also equation (1) gives

$$\iint_S \vec{g} \cdot d\vec{s} = -4\pi G \iiint_\tau \rho d\tau \qquad \text{..... (4)}$$

Combinining (3) and (4), we get

$$\iiint_\tau (\nabla \cdot \vec{g} + 4\pi G\rho)\, d\tau = 0$$

implying $\nabla \cdot \vec{g} = -4\pi G\rho = \nabla \cdot \nabla v$

Therefore $\nabla^2 v = -4\pi G\rho$

This equation is known as Poisson's equation.

12.5 PROPERTIES OF LAPLACE'S AND POISSON'S EQUATIONS

Theorem 13.5.1. (Mean Value Theorem for Harmonic Functions). Let u be harmonic in a region R. Also, let P(x, y, z) be a given point in R and S(P,r) be a sphere with centre at P and radious r such that S(P,r) is completely centained in the domain of harmonicity of u. Then

$$\mathrm{u}(P) = \overline{u}(r) = \frac{1}{4\pi r^2} \iint\limits_{S\,(p,r)} u(Q)ds$$

Proof: Since u is harmonic in R, its spherical mean $\overline{u}(r)$ is continuous in R and is given by

$$\overline{u}(r) = \frac{1}{4\pi r^2} \iint\limits_{S\,(p,r)} u(Q)ds$$

$$= \frac{1}{4\pi \mathrm{r}^2} \int_0^{2\pi} \int_0^{\pi} \mathrm{u}(\xi, \eta, \zeta)\mathrm{r}^2 \sin\theta \, \mathrm{d}\theta \mathrm{d}\phi$$

Therefore

$$\frac{\mathrm{d}\overline{\mathrm{u}}(\mathrm{r})}{\mathrm{dr}} = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \left(\mathrm{u}_\xi \, \xi_\mathrm{r} + \mathrm{u}_\eta \, \eta_\mathrm{r} + \mathrm{u}_\zeta \zeta_\mathrm{r}\right) \sin\theta \, \mathrm{d}\theta \mathrm{d}\phi$$

$$= \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \left(u_\xi \sin\theta \ cos\phi + u_\eta \sin\theta \ sin\phi + u_\zeta \, cos\theta\right)$$

$$\sin\theta \quad d\theta d\phi \quad \text{..... (1)}$$

Noting that sinq cosf, sinq sinf and cosq are the direction cosines of the normal $\hat{n}$ on S (P,r),

$$\nabla u = \hat{\imath} u_\xi + \hat{\jmath} u_\eta + \hat{k} u_\zeta, \hat{n} = \left(\hat{\imath} n_1, \hat{\jmath} n_2, \hat{k} n_3\right)$$

The expression within the parentheses of the integrand of equation (1) can be written as $\nabla u.\hat{n}$. Thus

$$\frac{\mathrm{d}\overline{\mathrm{u}}(\mathrm{r})}{\mathrm{dr}} = \frac{1}{4\pi \mathrm{r}^2} \iint\limits_{\mathrm{S(P,r))}} \nabla \mathrm{u} \cdot \hat{\mathrm{n}} \, r^2 \sin\theta \ \mathrm{d}\theta \mathrm{d}\emptyset$$

$$= \frac{1}{4\pi r^2} \iint\limits_{S(P,r)} \nabla u \cdot \hat{n} \, ds$$

$$= \frac{1}{4\pi r^2} \iiint_{V(P,r)} \nabla \cdot \nabla u \, dv \qquad \text{(by divergence theorem)}$$

$$= \frac{1}{4\pi r^2} \iiint_{V(P,r)} \nabla^2 u \, dv = 0 \qquad [\because u \text{ is harmomic}]$$

Therefore $\frac{d\bar{u}(r)}{dr} = 0$ implying $\bar{u}$ is a constant.

Now the continuity of $\bar{u}$ at r = 0 gives the relation

$$\bar{u}(r) = u(P) = \frac{1}{4\pi r^2} \iint_{S(P,r)} u(Q) ds$$

Definition 13.5.2. A continuous function satisfying Laplace's equation in an open region Ω, with continous first and second order derivatives, is called harmonic function. Functions u in $C^2(\Omega)$ with $\nabla^2 \Omega \geq 0$ (respectively $\nabla^2 \Omega \leq 0$) are called subharmonic (respectively superharmonic).

Theorem 13.5.3. Suppose that the subharmonic function u satisfies $\nabla^2 u = F$ in W, with F > 0 in Ω. Then u(x,y) attains its maximum on $\partial\Omega$, where W is an open region in R^n with boundary $\partial\Omega$.

Prof: Suppose for a contradiction that u attains its maximum at an interior point (x0, y0) of Ω. The at (x_0, y_0),

$$\frac{\partial u}{\partial x} = 0, \frac{\partial u}{\partial y} = 0, \frac{\partial^2 u}{\partial x^2} \leq 0 \; and \; \frac{\partial^2 u}{\partial y^2} \leq 0$$

since it is a maximum. So

$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2} \leq 0$ which contradicts F > 0 in Ω. Hence u must attain its maximum on $\partial\Omega$ i.e if $u \leq M$ on $\partial\Omega$, $u < M$ in Ω.

Theorem 13.5.4. (The weak Maximum-Minimum Principle for Laplace's equation). Suppose that u satisfies

$\nabla^2 u = 0$ in a bounded region W; if $m \leq u \leq M$ on $\partial\Omega$, then $m \leq u \leq M$ in Ω.

Proof: Consider the function

$v = u + \varepsilon (x^2+y^2)$, for any $\varepsilon > 0$.

Then $\nabla^2 v = 4\varepsilon > 0$ in Ω and using the previous theorem,

$$v \leq M + \varepsilon R^2 \text{ in } \Omega,$$

where $u \leq M$ on $\partial\Omega$ and R is the radius of the circle containing Ω. As this hold for any ε, let $\varepsilon \to 0$ to obtain $u \leq M$ in Ω,

i.e., if u satisfies $\nabla^2 u = 0$ in Ω, then u cannot exceed M, the maximum value of u on $\partial\Omega$. Also, if u is a solution of $\nabla^2 u = 0$, $u \geq m$ an $\partial\Omega$, then $u \geq m$ in Ω.

This theorem does not say that harmonic function cannot also attain m and M inside Ω though. We shall now progress into the strong Maximum - Minimum principle.

Theorem 12.5.5. Suppose that u has the mean value property in a bounded region Ω and that u is continuous in $\Omega = \Omega \cup \delta\Omega$. If u is not constant in Ω, then u attains its maximum value on the boundary $\partial\Omega$ of Ω, not in the interior of Ω.

Proof: Since u is continnous in the closed, bounded domain then Ω it attains its maximum M some where in Ω.

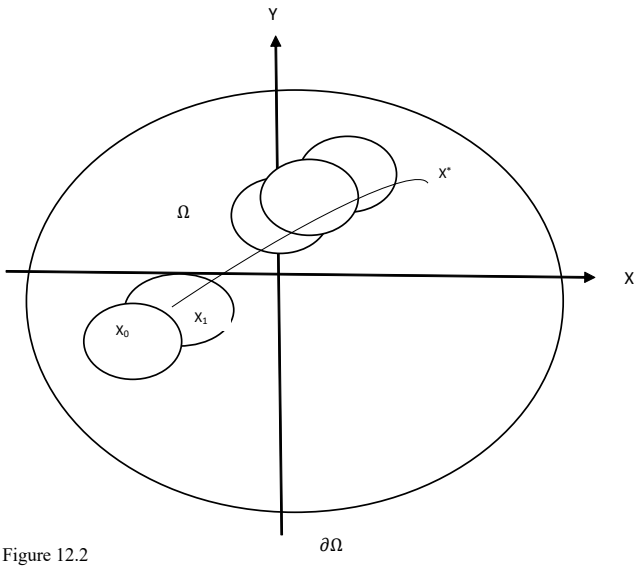


Figure 12.2

Our aim is to show that, if u attains its maximum at an interior point of Ω, then u is constant in Ω.

Suppose $u(x_0) = M$ and let x* be some other point of W. Join these with a path covered by a sequence of over lapping balls, B_r.

Consider the ball with x_0 at its centre. Since u has the mean value proper, therefore

$$M = u(x_0) = \frac{1}{4\pi r^2} \iint_{B_r} u(x)ds \leq M$$

This equality must hold throughout this statement and u = M throughout the sphere surrounding x_0. Since the balls overlap, there is x_1, centre of the next ball such that u (x_1) = M; the mean value property implies that u = M in this sphere also. Continuing like this gives u(x*) = M. Since x* is arbitrary, we conclude that u=M throught Ω and by continuity throughout Ω. Thus if u is not a constant in W it can attain its maximum value only on the boundary $\partial\Omega$.

Corollary 12.5.6. Applying the above theorem to –u establishes that if u is non constant it can attain its minimum only on $\partial\Omega$.

Also as a simple corollary, we can state the following theorem. The proof follows immediately the previous theorem and the weak Maximum - Minimum principle.

Theorem 12.5.7. (The strong Maximum - Minimum Principle for Laplace's equation). Let u be harmonic in W, i.e. solution of $\nabla^2 U=0$ in Ω and continuous in Ω, with M and m the maximum and minimum values respectively of u on the boundary $\partial\Omega$. Then either m < u < M in Ω or else m = u = M in Ω.

Example 1. Consider u(x,y) = e^x siny with $\Omega = \{(x,y) : -\alpha < x < \alpha, 0 < y < 2\pi\}$. Then $\nabla^2 u = 0$ and on the boundary of Ω we have u = 0 so that m = M = 0. But of course u is not identically zero in Ω.

12.6 GREEN FUNCTIONS AND PROPERTIES

In a region W with boundary S, let

L [u] = f(x) in Ω ..... (1)*

B [u] = 0 on S ..... (2)*

represent, respectively, a linear, second order PDE and linear boundary - initial conditions, such that for each continuous f, the problem (1)* – (2)* has a unique solution. Then G(x,ξ) is the Green's function for the problem if

this unique solution is given by

$$u(x) = \int_{\Omega} G(x;\ \xi)\, f(\xi) d_{\xi}\Omega$$

(we attach a subscript to the volume element to emphasize that the integration is with respect to the ξ- variables.)

Example 1. The initial value problem for the heat equation,

$$v_t(x,t) - v_{xx}(x,t) = 0, \quad -\alpha < x < \alpha,\ t > 0$$

$$v(x,0) = f(x) \quad -\alpha < x < \alpha$$

has the unique solution

$$v(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\alpha}^{\alpha} \exp\left[-\frac{(x-\xi)^2}{4t}\right] f(\xi) d\xi$$

It follows from Duhamel's principle that the problem

$$v_t(x,t) - v_{xx}(x,t) = 0, \quad -\alpha < x < \alpha,\ t > 0 \quad \text{..... (1)}$$

$$v(x,0) = f(x) \qquad -\alpha < x < \alpha \qquad \text{.....(2)}$$

has the unique solution

$$u(x,t) = \int_0^t v(x, t-\tau)\, d\tau = \int_0^t \int_{-\alpha}^{\alpha} \frac{1}{\sqrt{4\pi\,(t-\tau)}} \exp\left[-\frac{(x-\xi)^2}{4\,(t-\tau)}\right] \cdot$$

$$f(\xi) d\xi d\tau \text{..... (3)}$$

From (3) we infer that the Green's function for the problem (1) – (2) is

$$G(x,t;\ \xi,\tau) = \frac{1}{\sqrt{4\pi\,(t-\tau)}} \exp\left[-\frac{(x-\xi)^2}{4\,(t-\tau)}\right] \ (t > \tau > 0) \quad \text{..... (4)}$$

It is seen that the Green function (4) exhibits singular behavior as x = (x,t) approaches $\xi = (\xi, \tau)$. This holds true for Green's functions in general and it reflected in the fact that G (x, ξ) for (1)* – (2)* satisfies, as the function of x, the PDE

$$L[\sigma] = \delta(x-\xi) \quad \text{..... (5)}$$

which is (1)* with f(x) replaced by the function $\delta(x-\xi)$. We call a solution of (5) a singularity solution for L [ ]. The essence, then, of the Green's function method is to represent f(x) in (1)* as a sum of delta functions, thereby obtaining u(x) as the sum of the corresponding singularity solutions.

The function

$$\sigma(x;\,\xi) = \begin{cases} \dfrac{1}{2}|x-\xi| & \text{for n} = 1 \\ \dfrac{1}{2\pi}\log\ |x-\xi| & \text{for n} = 2 \\ \dfrac{1}{(2-\text{n})\text{A}_\text{n}(1)}|x-\xi|^{2-n} & \text{for n} \geq 3 \end{cases}$$

where x and ξ represent distinct points and A_n (1) denotes the area of the unit sphere, is the singularity solution for the Laplacian operator ∇^2 [ ] in IR^n.

Example 2. For Laplace's equation in three space, $u_{xx} + u_{yy} + u_{zz} = 0$, with x = (x,y,z) and $\xi = (\phi, \eta, \psi)$, the singularity solution is

$$\sigma\,(\text{x},\text{y},\text{z};\ \phi,\eta,\psi) = -\ \frac{1}{4\pi[(\text{x}-\phi)^2 + (\text{y}-\eta)^2 + (\text{z}-\psi)^2]^{1/2}}$$

Theorem 12.6.1. If ξ is a fixed point of R^n, then

(i) for n≥1, s (x;ξ) is dependent only on the distance r = |x–ξ|;

(ii) for n≥1, $\nabla^2_x\sigma = 0$ for all $x \neq \xi$;

(iii) for n > 1, the integral of the normal derivative of s over any sphere centred at x = ξ is since σ (x; ξ) =σ (ξ; x), Theorem 13.6.1 is valid with the roles of x and ξ reversed.

Because Poisson equation, $\nabla^2 u = \rho(x)$, is solved in unbounded R^n by

$$\text{u}(\text{x}) = \int_{\text{R}^\text{n}} \sigma(\text{x};\ \xi)\rho(\xi)\text{d}_\xi\Omega$$

the singularity solution serves as the Green's function for ∇^2 [ ] when no boundaries are present; we called it the free space Green's function. We now show how to modify this function so that it gives the Green's function when boundary conditions have to be satisfied.

Let Ω be a region with boundary S and take ξ to be a fixed point inside Ω. If as a function of x, $\nabla^2 h = 0$ in Ω, then the function

$$\phi(x,\ \xi) = \sigma\ (x;\ \xi) + h$$

where h may depend on ξ as well on x, is called a fundamental solution of Laplace's equation in Ω.

Example 3. Let Ω be the upper half of the xy - plane and let $\xi = (\varepsilon, \eta)$ be a fixed point in Ω. Both

$$\phi_1(x,y;\ \varepsilon,\eta) = \sigma\ (x,y;\ \varepsilon,\eta) + x^2 - y^2$$

and $\phi_2(x,y;\ \varepsilon,\eta) = \sigma\ (x,y;\ \varepsilon,\eta) + \sigma\ (x,y;\ \varepsilon, -\eta)$ are fundamental solutions of Laplaces equation in Ω.

Theorem 12.6.2. Let Ω be a bounded region to which the divergence theorem applies. If $\phi(x, \xi)$ is a fundamental solution of Laplaces's equation in Ω and if $\nabla^2 u = f(x)$ in Ω, then

$$u(\xi) = \int_\Omega\ \phi\ (x;\ \xi)\ f(x)\ d\Omega + \int_s\ \left(u\frac{\partial \Phi}{\partial n} - \frac{\partial u}{\partial n}\right)\ ds \ \text{ where } \frac{\partial}{\partial n} \equiv n.\nabla_x$$

Proof: Let $\xi = (\varepsilon, \eta)$ be any point in the two dimensional region Ω and let $r = |\ x - \xi\ |$ be the distance from ξ to any other point x = (x,y) in Ω. Let s be a circle of radius $\in$ centered at x and let Ω' be the portion of Ω exterior to σ. From Green's second identity,

$$\int_{\Omega'} (u\nabla^2\Phi - \Phi\nabla^2 u)d\Omega' = \int_s \left(u\frac{\partial \Phi}{\partial n} - \Phi\frac{\partial u}{\partial n}\right)ds + \int_s \left(u\frac{\partial \Phi}{\partial n} - \Phi\frac{\partial u}{\partial n}\right)ds \quad ....(1)$$

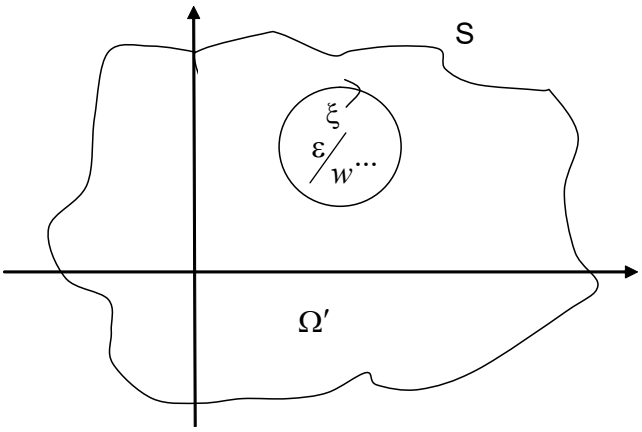


Figure 12.3

In both boundary integrats of (1) the center is described so that Ω' is kept to the left (see Fig 12.3).

Since f is a fundamental solution with singularity at ξ, which is not in Ω',

$$\nabla^2 f = 0 = \nabla^2 u - f \text{ in } \Omega'. \qquad \text{..... (2)}$$

On s the outward normal derivative of ϕ relative to Ω' is

$$\frac{\partial \Phi}{\partial n} = -\frac{\partial \Phi}{\partial r}\Big]_{r=\in} = -\frac{1}{2\pi \in} - \frac{\partial h}{\partial r}\Big]_{r=\epsilon}$$

Therefore (see figure 12.3; ds > o)

$$\int_s \left(u\frac{\partial \emptyset}{\partial n} - \Phi\frac{\partial u}{\partial n}\right) ds = \int_0^{2\pi} \left(\frac{-u}{2\pi \in} - u\frac{\partial h}{\partial r} + \Phi\frac{\partial u}{\partial r}\right) \in dw$$

$$= \frac{1}{2\pi}\int_0^{2\pi} (-u)dw - \epsilon \int_0^{2\pi} \left(u\frac{\partial h}{\partial r} - \Phi\frac{\partial u}{\partial r}\right) dw \quad \text{..... (3)}$$

Now, the first term on the right in (3) is just the mean value of –u over s; so it approaches –u (x) as $\in \to 0$. As for the second term, since

$$\Phi|_{r=\epsilon} = \frac{1}{2\pi}\log \epsilon + h|_{r=\epsilon}$$

it vanishes as $\in$ log $\in$, as $\in \to 0$. Consequently, if we let $\in \to 0$ in (1), we obtain

$$-\int_\Omega \Phi f \, d\Omega = \int_s \left(u\frac{\partial \Phi}{\partial n} - \Phi\frac{\partial u}{\partial n}\right) ds - u(\xi)$$

$$\Rightarrow u(\xi) = \int_\Omega \Phi f d\Omega + \int_s \left(u\frac{\partial \Phi}{\partial n} - \Phi\frac{\partial u}{\partial n}\right) ds$$

Thus the proof of Theorem 12.6.2 is completed.

If f and u are sufficiently well-behaved at infinity, then theorem 12.6.2. is valid in unbounded regions. In (*) the roles of x and ξ can be interchanged to obtain an expression for u(x).

Suppose that for each continuous f and g, the mixed problem

$$\nabla^2 u = f(x) \quad in\ \Omega$$

$$\alpha u + \beta\frac{\partial u}{\partial n} = g(x) \quad on\ S$$

has a unique solution. The Green's function, G (x,ξ), for this problem is the fundamental solution of Laplace's equation in Ω that satisfies

$$\propto G + \beta\frac{\partial G}{\partial n} = 0 \quad on\ S$$

This homogeneous boundary condition ensures that, for ϕ = G, the boundary integral in (*) depends only on the known functions G, α, β and g and not on u.

The remainder of this section assumes β = 0, i.e. it treats the Dirichlet problem

$\nabla^2 u = f(x)$ in Ω .....(*)^2

$u = g(x)$ in S .....(*)^3

Theorem 12.6.3. The Green's function for (*)^2 – (*)^3 is unique and is given by

$G(x; \xi) = \sigma(x; \xi) + h(x; \xi)$

where, for each fixed ξ in Ω, h satisfies

$\nabla^2 h = 0$ in Ω

$h = -\sigma$ on S

Theorem 12.6.4. For each fixed x in Ω, the Green's function for (*)^2 – (*)^3 satisfies

$\nabla^2_x G(x; \xi) = \delta(x - \xi)$ in Ω

$G = 0$ on S

where d (x –ξ) is the n-dimensional Dirac delta function.

Theorem 12.6.5. The Green's function for (*)^2 – (*)^3 is symmetric, G (x; ξ) = G (x; ξ) and it is negative for all distinet x and ξ in Ω.

Theorem 12.6.6. If W is bounded, the Green's function for (*)^2 – (*)^3 has the eigen function expansion

$$G(x; \xi) = \sum_{r=1}^{\alpha} \frac{u_r(x) u_r(\xi)}{\lambda_r}$$

Where $\nabla^2 u_r(x) = \lambda_r U_r(x)$ in Ω, $U_r(x) = 0$ on S, Ur $\neq$ 0.

Theorem 13.6.7. The solution of (*)^2 – (*)^3 is

$$u(x) = \int_{\Omega} G(x; \xi) f(\xi) d_\xi \Omega + \int_{s} g(\xi) \frac{\partial G}{\partial n}(x; \xi) \cdot d_\xi S$$

where $\frac{\partial G}{\partial n}(x; \xi) \equiv n . \nabla_\xi G(x; \xi)$

Theorem 13.6.8 (i) Let w = F (z) be an analytic function which maps the region Ω in the z-plane onto the upper half of the w-plane, with F′(z) $\neq$ 0

in Ω. Then, if z = x+iy and ζ=ξ+iη are any two points in Ω, the Green's function for $(*)^2 - (*)^3$ in R^2 is given by

$$G(x, y; \xi, \eta) = \frac{1}{2\pi} \log \left| \frac{F(z) - F(\zeta)}{F(z) - F(\zeta)} \right|$$

where the overbar denotes the complex conjugate.

(ii) Let w = f(z) be analytic function which maps the region Ω in the z-plane onto the unit circle in the w-plane, with f(z) = 0 and f′ (z) ≠ 0 in Ω. Then the Green's function for $(*)^2 - (*)^3$ in R^2 is

$$G(x, y; \xi, \eta) = \frac{1}{2\pi} \log |f(z)|$$

12.6.1 Image method of finding Green's functions

In order to solve Boundary value problems for Poisson's equation, such as $\nabla^2 u = F$ in an open region Ω Greem's function G such that, in Ω

$G(x; \xi) = \vartheta(x, \xi) + \omega(x, \xi) \text{where} \nabla^2 \vartheta = -\delta(x - \xi)\ and\ \nabla^2 w = 0 \text{ or } {}^1\!/_{v}\,(\Omega).$

Having found the free space Green's function v-which does not depend on the boundary conditions and so is the same for all problems - we still need to find the function w, solution of Laplace's equation and regular in x = ξ, which fixes the boundary conditions (v does not satisfy the boundary conditions required for G by itself). So we look for the function which satisfies

$$\nabla^2 \mathrm{w} = 0 \text{ or } {}^1\!/_{\mathrm{v}}\,(\Omega) \text{ in } \Omega, (\text{ensuring w is regular at } (\xi, \eta))$$

With $\mathrm{w} = -\mathrm{v}\ (\mathrm{i.e.}\, \mathrm{G} = 0) \text{on S for Dirichlet boundary conditions,}$

Or $\frac{\partial \mathrm{w}}{\partial \mathrm{n}} = -\frac{\partial \mathrm{v}}{\partial \mathrm{n}} \left(\mathrm{i.e.} \frac{\partial \mathrm{G}}{\partial \mathrm{n}} = 0 \right) \text{ on S for Neumann boundary condition.}$

To obtain such a function we superpose functions with singularities at the image points of (ξ, η). This may regarded as adding appropriate point sources and sinks to satisfy the boundary conditions. Note also that, since G and v are symmetric then w must be symmetric too.

Example 4. Suppose we wish to solve the Dirichlet BVP for Laplace's equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \ \text{in} y > 0 \ \text{with} u = f(x) \text{on} y = 0.$$

We know that in 2–D the free space function is

$$v = -\frac{1}{4\pi}\ln[(x-\xi)^2 + (y-\eta)^2]$$

If we superpose to v the function

$$w = \frac{1}{4\pi} ln[(x-\xi)^2 + (y-\eta)^2]$$

solution of $\nabla^2 w = 0$ in Ω and regular at ($x=\xi$, $y=\eta$), then

$$G(x,y;\xi,\eta) = v+w$$

$$= -\frac{1}{4\pi}\ln\left[\frac{(x-\xi)^2 + (y-\eta)^2}{(x-\xi)^2 + (y+\eta)^2}\right]$$

Note that, setting y=0, we get

$$G(x,0;\xi,\eta) = -\frac{1}{4\pi}\ln\left[\frac{(x-\xi)^2 + \eta^2}{(x-\xi)^2 + \eta^2}\right] = 0.$$



CHECK YOUR PROGRESS

Q.2. Consider the Green's function G such that G $(x, \xi) = v(x, \xi) + w(x, \xi)$ where $\nabla^2 v = -\delta(x-\xi)$ in Ω.

Find the free space Green's function v such that

$\nabla^2 v = -\delta(x-\xi)$ in Ω.



12.7 LET US SUM UP

- Consider a general partual differential equation of second order of a function of two independent variables x and y in the form

  $$Rr + Ss + Tt + f(x, y, z, p, q) = 0$$

  where R, S, T are continuous functions of x and y only possessing partial derivatives defined in some domain D on the xy-plane. Then the equation is said to be

  (i) Elliptic at a point (x,y) in D if $S^2 - 4RT < 0$

  (ii) Parabolic at a point (x,y) in D if $S^2 - 4RT = 0$

(iii) Hyperbolic at a point (x,y) in D if $S^2 - 4RT > 0$

where $r = \frac{\partial^2 z}{\partial x^2}$, $s = \frac{\partial^2 z}{\partial x \partial y}$, $and \quad t = \frac{\partial^2 z}{\partial y^2}$

- For potential V,

  $\nabla^2 v = 0$

  is called the Laplace equation and $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called the Laplace operator.

- $\nabla^2 v = -4\pi GP$

  where G is gravitational constant, P is the mass density function, is known as Poisson equation.

- Let u be harmonic in a region R. Also, let P(x, y, z) be a given point in R and S (P,r) be a sphere with centre at P and radious r such that S (P,r) in completely contained in the domain of harmonicity of u. Then

$$u(P) = \bar{u}(r) = \frac{1}{4\pi r^2} \iint_{S(P,r)} u(Q)ds$$

- A continuous function satisfying Laplace's equation in an open region W, with continuous first and second order derivatives is called harmonic function. Function u in $C^2(\Omega)$ with $\nabla 2u \geq 0$ (respectively $\nabla^2 u \leq 0$) are called subharmonic (respectively superharmonic).

- Suppose that the subharmonic function u satisfies $\nabla^2 u = F$ in Ω, with $F > 0$ in Ω. Then u(x,y) attains its maximum on $\partial\Omega$, where Ω is an open region in IRn with boundary $\partial\Omega$.

- The weak maximum - minimum principle of Laplaces's equation : Suppose that u satisfies $\nabla^2 u = 0$ in a bounded region Ω; if $m \leq u \leq M$ on $\partial\Omega$, then $m \leq u \leq M$ in Ω.

- Suppose that u has the mean value property in a bounded region Ω and that u is continuous in $\Omega = \Omega u \partial\Omega$. If u is not constant in Ω, then u attains its maximum value of the boundary $\partial\Omega$ of Ω, not in the interior of Ω.

- The strong maximum - minimum principle of Laplace's equation : Let u be harmonic in Ω and continuous in $\partial\Omega u\Omega = \Omega$, with M and m the

values respectively of u on the boundary $\partial\Omega$. Then either m < u < M in Ω or else m = u = M in Ω.

- In a region Ω with boundary S, let

  L[u] = f(x) in Ω ..... (1)

  B [u] = 0 on S ..... (2)

  represent; respectively, a linear, second order PDE and linear boundary - initial conditions, such that for each continuous f, the problem (1) – (2) has a unique solution. Then G (x; ξ) is the green's function for the problem if this unique solution is given by

$$u(x) = \int_{\Omega} G(x;\ \xi)\, f(\xi) d_{\xi}\Omega$$

- Let Ω be a bounded region to which divergence theorem applies. If ϕ (x; ξ) is a fundamental solution of Laplace's equation in Ω and if $\nabla^2 u = f(x)$ in Ω, then

$$u(\xi) = \int_{\Omega} \Phi(x;\ \xi) f(x) d\Omega + \int_{S} \left(u \frac{\partial \Phi}{\partial n} - \Phi \frac{\partial u}{\partial n} \right) ds$$

Where $\frac{\partial}{\partial n} \equiv n.\nabla_x$.



12.8 ANSWER TO CHECK YOUR PROGRESS

Ans to Q. No. 1.

The given differential equation is

$$\frac{\partial^2 z}{\partial x^2} + y^2 \frac{\partial^2 z}{\partial y^2} = y$$

$$\Rightarrow\ r + y^2 t - y = 0 \qquad \text{..... (1)}$$

Comparing (1) with Rr + Ss + Tt+ f (x, y, z, p, q) = 0, we have R = 1, S= 0, T = y^2 so that $S^2 – 4RT = – 4y^2 < 0$ for $y \neq 0$. showing that (1) is elliptic.

The λ – quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to

$\lambda^2 + y^2 = 0$

$\Rightarrow\ \lambda = iy, – iy$

The corresponding characteristic equations are given by

$$\frac{dy}{dx} + iy = 0 \quad and \quad \frac{dy}{dx} - iy = 0$$

Integrating, logy + ix = c_1 and logy – ix = c_2

Choose u = logy + ix = $\alpha + i\beta$ and V = logy – ix = $\alpha - i\beta$

where α = logy and β = x are the new independent variables.

Now

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \alpha} \cdot \frac{\partial \alpha}{\partial x} + \frac{\partial z}{\partial \beta} \cdot \frac{\partial \beta}{\partial x} = \frac{\partial z}{\partial \alpha} \cdot 0 + \frac{\partial z}{\partial \beta} \cdot 1 = \frac{\partial z}{\partial \beta}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial \alpha} \cdot \frac{\partial \alpha}{\partial y} + \frac{\partial z}{\partial \beta} \cdot \frac{\partial \beta}{\partial y} = \frac{\partial z}{\partial \alpha} \cdot \frac{1}{y} + \frac{\partial z}{\partial \beta} \cdot 0 = \frac{1}{y}\frac{\partial z}{\partial \alpha}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \beta}\right) = \frac{\partial^2 z}{\partial \beta^2} \qquad \text{..... (2)}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{1}{\partial y}\left(\frac{1}{y}\frac{\partial z}{\partial \alpha}\right)$$

$$= -\frac{1}{y^2}\frac{\partial z}{\partial \alpha} + \frac{1}{y}\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial \alpha}\right)$$

$$= -\frac{1}{y^2}\frac{\partial z}{\partial \alpha} + \frac{1}{y}\left\{\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha}\right)\frac{\partial \alpha}{\partial y} + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \alpha}\right)\frac{\partial \beta}{\partial y}\right\}$$

$$= -\frac{1}{y^2}\frac{\partial z}{\partial \alpha} + \frac{1}{y}\frac{\partial^2 z}{\partial \alpha^2} \cdot \frac{1}{y}$$

$$= \frac{1}{y^2}\left(\frac{\partial^2 z}{\partial \alpha^2} - \frac{\partial z}{\partial \alpha}\right) \qquad \text{..... (3)}$$

Using (2), (3) in (1) , the required canonical form is

$$\frac{\partial^2 z}{\partial \beta^2} + \frac{\partial^2 z}{\partial \alpha^2} - \frac{\partial z}{\partial \alpha} - y = 0$$

$$\Rightarrow \frac{\partial^2 z}{\partial \beta^2} + \frac{\partial^2 z}{\partial \alpha^2} = \frac{\partial z}{\partial \alpha} + e^{\alpha}$$

Ans to Q. No. 2

Consider the polar coordinates of around (ξ, η),

$$x - \xi = r\cos\theta \text{ and } y - \eta = r\sin\theta$$

and look for a solution of Laplace's equation which is independent of θ and which is sigular as $r \to 0$

Laplace's equation in polar coordinates is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial v}{\partial r}\right) + \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} = 0$$

which has solution

$v = B \ln r + A$

with A and B constant.

Put A = 0 and to determine the constant B, apply Green's theorem to v and 1 in a small disc Dr (with boundary C_r), of radius r around the origin (η, ξ),

$$\int_{C_r} \frac{\partial v}{\partial n} ds = \int_{D_r} \nabla^2 dv = -\int_{D_r} \delta\,(x-\xi)\,dv = -1$$

so we choose B to make

$$\int_{C_r} \frac{\partial v}{\partial n}\,ds = -1$$

Now, in polar coordinates, $\frac{\partial v}{\partial n} = \frac{\partial v}{\partial r} = \frac{B}{r}$ and ds = rdq

$$\text{So}, -1 = \int_0^{2\pi} \frac{B}{r} r d\theta = B\int_0^{2\pi} d\theta = 2\pi B$$

$$\Rightarrow B = -\frac{1}{2\pi}$$

$$\text{Hence } v = -\frac{1}{2\pi}\ln r$$

$$= -\frac{1}{4\pi}\ln r^2$$

$$= -\frac{1}{4\pi}\ln\{(x-\xi)^2 + (y-\eta)^2\}$$



12.9 FURTHER READINGS

1. Simmons, G.F., Differential Equations with applications and Historical Notes. 2nd edition, Tata Mc Fraw Hill, New delhi, 2016.
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3. Sneddon, I.N., Elements of Partial Differential Eqation, Dover Publication, 2004.
4. Raisinghania, M.D., Ordinary and Partial Differential Equation, 19th Edition, S. Chand and Company Ltd., New Delhi, 2018.
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6. Evans, L.C., Partial Differential Equations, American Mathematical Society, 1997.



12.10 MODEL QUESTIONS

1. Reduce the following partial differential equations to canonical forms :
   (i) $y^2t + r = 0$
   (ii) $xr + t - x^2 = 0$
   (iii) $r + 2s + 5t + p - 2q - 3z = 0$
2. Find the Green's function for
   $$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = F \text{ in the quadrant } x > 0, y > 0.$$
3. Find the Green's function for
   $$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ in } y > 0 \text{ with } u = f(x) \text{ on } y = 0$$

4. If a harmonic function vanishes everywhere on the boundary, then show that it is identically zero everywhere.

5. If f is harmonic function in Ω and $\frac{\partial \phi}{\partial n} = 0$ on $\partial\Omega$, then show that ϕ is constant in Ω.

6. Show that if the two- dimensional Laplace's equation $\nabla^2 u = o$ is transformed by introducing plane polar coordinators r, q defined by the relations $x = r\cos\theta$, $y = r\sin\theta$, it takes form

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

7. Assume that a Green's funtion, G (x;x), exists for the mixed boundary value problem

$$\nabla^2 u = f(x) \text{ in } \Omega$$

$$\alpha u + \beta \frac{\partial u}{\partial n} = 0 \quad \text{on S}$$

with $a \neq 0$. Formally show that

(a) G (x; ξ) as a function of x satisfies

$$\nabla^2 x\, G = \delta (x; \xi) \text{ in } \Omega$$

$$\alpha G + \beta \frac{\partial G}{\partial n} = 0 \quad \text{on S}$$

(b) G (x; ξ) is symmetric.

(c) The solution to given mixed boundary value problem subject to the non-hoogeneous boundary condition

$$\alpha u + \beta \frac{\partial u}{\partial n} = g (x) \quad \text{on S}$$

is given by

$$u(x) = \int_{\Omega} G (x; \xi) f(\xi) d_{\xi}\Omega + \int_{s} \frac{1}{\alpha(\xi)} \frac{\partial G}{\partial n}(x; \xi)\, d_{\xi} s$$

$$\text{where } \frac{\partial}{\partial n} \equiv n \cdot \nabla_{\xi}$$

8. If Ω is the rectangle $0 < x < a$, $0 < y < b$, final the Green's function's function for the boundary value problem,

$$\nabla^2 u = f \quad \text{in } \Omega$$
$$u = g \quad \text{on } S.$$

9. Use the method of images to find the Green's function for

$$\nabla^2 u = f \quad \text{in } \Omega$$
$$u = g \quad \text{on } S.$$

if Ω is (a) $-\alpha < x < \alpha$, $y > 0$

(b) $0 < x < \alpha$, $y > 0$

(c) $-\alpha < x < \alpha$, $0 < y < b$

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