UNIT 13 : PARABOLIC DIFFERENTIAL EQUATIONS

UNIT STRUCTURE

13.1 Learning objectives
13.2 Introduction
13.3 Parabolic Differential Equations
13.4 Heat Equation
13.5 Occurence of heat equation, Maximum principle
13.6 Green's function for heat equation
13.7 The Resolution of Boundary Value Problems for the Heat equation, Duhamels Theorem.
13.8 Let us sum up
13.9 Answer to check your progress
13.10 Further reading
13.11 Model questions

13.1 LEARNING OBJECTIVES

On completion of the unit learners will be able to –

- solve parabolic PDE;
- understanding properties of heat equation;
- understanding the role of Green's function for heat equation;
- know about Duhamel's principle.

13.2 INTRODUCTION

Consider a general form of partial differential equation of second order of a function of two indendent variables x and y –

$Rr + Ss + Tt + f(x,y,z,p,q) = 0$

where R,S,T are contimuous functions of x and y only possessing partial derivatives defined in some domain D on the xy- plane and $r = \frac{\partial^2 z}{\partial x^2}, s = \frac{\partial^2 z}{\partial x\, \partial y}, t = \frac{\partial^2 z}{\partial y^2}$. Then the equation is said to be parabolic at a point (x,y) in D if $S^2 - 4RT = 0$.

Now consider the typical parabolic equation

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

and its generalization to two and three dimensions. Because of its occurrence in the analysis of diffusion phenomena we shall refer to this equation as the one - dimensional heat (diffussion equation and to its generalization

$$k\nabla^2 u = \frac{\partial u}{\partial t}$$

(where k is a constant) as the heat equation.

Two dimensional heat equation : $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{k}\frac{\partial u}{\partial t}$

Three dimensional heat equation : $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{k}\frac{\partial u}{\partial t}$

Remark. ∇^2 is called the Laplacian operator and is defined as $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. A function u which satisfies Laplace's equation, is called the harmonic function.

13.3 PARABOLIC DIFFERENTIAL EQUATION

14.3.1. Working rule for reducing a parabolic equation to its canonical (or normal) form.

Step 1. Let the given equation

$Rr + Ss + Tt + f(x,y,z,p,q) = 0$ ..... (1)

be parabolic so that $S^2 - 4RT = 0$.

Step 2. Write λ– quadratic equation

$R\lambda^2 + S\lambda + T = 0$ ..... (2)

Let λ_1, λ_2 be two equal roots of (2).

Step 3. Write the characteristic equatin corresponding to

$\lambda = \lambda_1$ i.e. $\frac{dy}{dx} + \lambda_1 = 0$

Solving it, we get $f_1(x, y) = C_1$ ..... (3)

Step 4. Choose $u = f_1(x, y)$ and $v = f_2(x, y)$ ..... (4)

where f_2 (x, y) is an arbitrary function of x and y and is independent of f_1 (x, y). For this verify that Jacobian J of u and v given by (4) is non zero, i.e.

$$J = \frac{\partial (u, v)}{\partial (x, y)} \neq 0 \qquad \text{.....(5)}$$

Step 5. Using relation (4), find p, q, r, s and t in terms of u and v.

Step 6. Substituting the values of p, q, r, s, t in (1) and simplifying we get the the following canonical forms of (1)

$$\frac{\partial^2 z}{\partial u^2} = \Phi\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right) \quad \text{or} \quad \frac{\partial^2 z}{\partial v^2} = \Phi\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right)$$

Example 1. Consider the PDE,

$$y^2 \frac{\partial^2 z}{\partial x^2} - 2xy \frac{\partial^2 z}{\partial x \partial y} + x^2 \frac{\partial^2 z}{\partial y^2} = \frac{y^2}{x} \frac{\partial z}{\partial x} + \frac{x^2}{y} \frac{\partial z}{\partial y}$$

Re-writing,

$$y^2 r - 2xys + x^2 t - \frac{y^2}{x} p - \frac{x^2}{y} q = 0$$

Comparing (1) with Rr + Ss + Tt + f (x,y,z,p,q) = 0,

we get

R = y^2, S = –2xy, T = x^2 so that $S^2 - 4RT = 4x^2y^2 - 4x^2y^2 = 0$, showing that (1) is parabolic.

The λ– quadratic equation

$R\lambda^2 + S\lambda + T = 0$ reduces to

$y^2\lambda^2 - 2xy\lambda + x^2 = 0$

$$\Rightarrow \lambda = \frac{x}{y}, \frac{x}{y}$$

The corresponding characteristic equation is

$$\frac{dy}{dx} + \frac{x}{y} = 0$$
$$\Rightarrow xdx + ydy = 0$$

$$\text{Integrating,} \quad \frac{x^2}{2} + \frac{y^2}{2} = C_1$$

$$\text{Choose} \quad u = \frac{x^2}{2} + \frac{y^2}{2} \quad \text{and} \quad v = \frac{x^2}{2} - \frac{y^2}{2} \qquad \text{..... (2)}$$

where we have chosen $v = \frac{x^2}{2} - \frac{y^2}{2}$ in such a manner that u and v are independent function as verified below

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

$$= x(-y) - yx$$

$$= -2xy \neq 0$$

Now

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u}x + \frac{\partial z}{\partial v}\cdot x$$

$$= x\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) \quad \text{..... (3)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u}y + \frac{\partial z}{\partial v}(-y)$$

$$= y\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) \quad \text{..... (4)}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left\{x\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\right\}$$

$$= \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} + x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} + x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]$$

$$= \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} + x^2\left(\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u\,\partial v} + \frac{\partial^2 z}{\partial v^2}\right) \quad \text{..... (5)}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left[y\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\right] = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} + y\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} + y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right]$$

$$= \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} + y^2\left(\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u\,\partial v} + \frac{\partial^2 z}{\partial v^2}\right) \quad \text{..... (6)}$$

and

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left\{y\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\right\}$$

$$= y\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right\}$$

$$= xy\left(\frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial v^2}\right) \quad \text{..... (7)}$$

Using (3), (4), (5), (6) and (7) in (1), we finally get

$$4x^2y^2\,\frac{\partial^2 z}{\partial v^2} = 0$$

$$\Rightarrow \frac{\partial^2 z}{\partial v^2} = 0 \quad \text{..... (8)}$$

which is required canonical form.

Integrating (8) partially w. r. to v, $\frac{\partial z}{\partial n} = \phi(u)$ ..... (9)

Integrating (9) partially w. r. to v, $z = v\,\phi(u) + \psi(u)$

or $z = \left(\frac{x^2 - y^2}{2}\right)\Phi\left(\frac{x^2 + y^2}{2}\right) + \Psi\left(\frac{x^2 + y^2}{2}\right)$

or $z = (x^2 - y^2)\,F(x^2 + y^2) + G(x^2 + y^2)$, F, G being arbitrary function.



CHECK YOUR PROGRESS

Q.1. Reduce the PDE

$r - 2s + t + p - q = e^x(2y - 3) - e^y$

in canonical form and solve it.

13.4 HEAT EQUATION

General form of heat equation:

$$\nabla^2 u = \frac{1}{k}\frac{\partial u}{\partial t}$$

One - dimensional heat equation : $\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}$

Two-dimensional heat equation : $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{k}\frac{\partial u}{\partial t}$

Three-dimensional heat equation : $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{k}\frac{\partial u}{\partial t}$

13.4.1 Solution of one - dimensional heat (diffusion) equation

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad \text{.....(1)}$$

Let solution of (1) be of the form

$$u(x, t) = X(x)\, T(t) \quad \text{.....(2)}$$

Substituting in (1), we get

$$kX''(x)\, T(t) = X(x)\, T'(t)$$

$$\Rightarrow \frac{X''}{X} = \frac{T'}{kT} \quad \text{.....(3)}$$

Clearly the LHS of (3) is a function of x alone and the RHS is a function of t alone. Since x and t are independent variables, (3) can hold good if each side is equal to a constant, say u. Then (3) leads to

$$x'' - mx = 0 \quad \text{.....(4)}$$

and $$T' - mkT = 0 \quad \text{.....(5)}$$

Three cases arise according as u is zero, +ve or –ve.

Case I. Let $\mu = 0$. The solutions of (4) and (5) are

$$x = a_1 x + a_2 \text{ and } T = a_3 \quad \text{.....(6)}$$

Case II. Let μ be +ve, say λ^2. Then (4) and (5) become respectively

$x'' - \lambda^2 x = 0$ and $T' = \lambda^2 kT$. By soluings we get

$$x = b_1 e^{\lambda x} + b_2 e^{-\lambda x} \text{ and } T = b_3 e^{\lambda^2 kt} \quad \text{.....(7)}$$

Case III. Let μ be –ve, say $-\lambda^2$. The (4) and (5) become respectively,

$x'' + \lambda^2 x = 0$ and $T' = \lambda^2 kT$. Solving these, we get

$$X = C_1 \cos \lambda x + C_2 \sin \lambda x \text{ and } T = C_3 e^{-\lambda^2 kt} \quad \text{.....(8)}$$

Thus the various possible solutions are

$$u(x,t) = A_1 x + A_2 \quad \text{.....(9)}$$

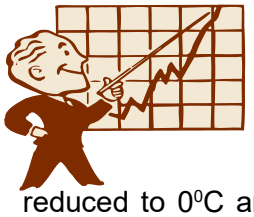
$$u(x,t) = (B_1 e^{-\lambda x} + B_2 e^{-\lambda x}) e^{\lambda^2 kt} \quad \text{.....(10)}$$

and $$u(x,t) = (C_1 \cos \lambda x + C_2 \sin \lambda x)\, e^{\lambda^2 kt} \quad \text{.....(11)}$$

where $A_1 = a_1 a_3$, $A_2 = a_2 a_3$, $B_1 = b_1 b_3$, $B_2 = b_2 b_3$, $C_1 = c_1 c_3$ and $C_1 = c_2 c_3$ are new arbitrary constants.

Now we have to choose that solution which is consistent with the physical nature of the problem. Since we are dealing with problem of heat conduction, temperature u (x,t) must decrease with the increase of time. According the solution given by (11) is the only suitable solution.

CHECK YOUR PROGRESS



Q.2. (a) An insulated rod of length l has its ends A and B maintained at 0^0C and 100^0C respectively until steady state conditions prevail. If B is suddenly reduced to 0^0C and maintained at 0^0C, find the tempertature at a distance x from A at time t.

(b) Find also the temperature if the change consists of raising the temperture of A to 20^0C and reducing that of B to 80^0C.

13.4.2 Solution of two dimensional heat equation :

Consider the two dimensional heat equation

$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}=\frac{1}{k}\frac{\partial\theta}{\partial t} \quad \text{.....(1)}$$

suppose that (1) has solutions of the form

$$\theta\,(x,y,t) = X\,(x)\,Y\,(y)\,T\,(t) \quad \text{.....(2)}$$

Substituting this value of θ in (1), we have

$$X''YT + XY''T = \frac{1}{k}XYT'$$

$$\Rightarrow \frac{X''}{X}+\frac{Y''}{Y}=\frac{1}{k}\frac{T'}{T} \quad \text{.....(3)}$$

Since x, y and t are independent variables, (3) is true if each term on each side is a constant, such that

$$\frac{X''}{X}=-n^2, \frac{Y''}{Y}=-m^2 \text{ and } \frac{T'}{kT}=-p^2 \quad \text{.....(4)}$$

with $n^2 + m^2 = p^2$.

We have chosen constants in (4) in such a way so that the solution θ has the property that $\theta \to 0$ as $t \to \alpha$. This is generally satisfied due to physical conditions of a real problem.

Solving differential equation given by (4), we have

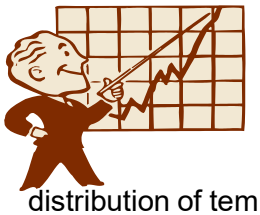
$$X_n(x) = A_n \cos nx + B_n \sin nx$$

$$Y_m(y) = C_m \cos my + D_m \sin my$$

$$T_p(t) = E_p e^{-p^2kt} = F_{nm} e^{-(n^2+m^2)\,kt}$$

Hence a suitable solution of (1) is

$$\theta_{nm}(x,y,t) = F_{nm}(A_n \cos nx + B_n \sin nx)(C_m \cos my + D_m \sin my) \cdot e^{-(n^2+m2)\,kt}$$



CHECK YOUR PROGRESS

Q.3. A thin rectangular plate whose surface is impervious to heat flow has at t = 0 an arbitrary distribution of temperature f (x,y). Its four edges x =0, y = 0, y = b are kept zero temperature. Determine the temperature at a point of the plate as t increases.

13.4.3 Solution of three dimensionel heat equations :

Consider the three dimensional heat equation –

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{k}\frac{\partial u}{\partial t} \quad \text{.....(1)}$$

Suppose (1) has solutions of the form

$$u(x,y,z,t) = X(x).\,Y(y).\,Z(z)\,T(t) \quad \text{.....(2)}$$

where X, Y, Z and T are respectively the functions of x, y, z and t alone. Substituting these value of u in (1), we have

$$X'' YZT + XY'' ZT + XYZ'' T = \frac{1}{k} XYZT'$$

$$\Rightarrow \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = \frac{1}{k}\left(\frac{T'}{T}\right) \quad \text{.....(3)}$$

Since x, y, z and t are independent variables, (3) is true only if each term on each side is a constant such that

$$\frac{X''}{X} = -n^2, \frac{Y''}{Y} = -m^2, \frac{Z''}{Z} = -l^2 \text{ and } \frac{T'}{kT} = p^2 \quad \text{.....(4)}$$

where $n^2 + m^2 + l^2 = p^2$ .....(5)

We have chosen constants in (4) in such a manner so that the solution u (x,y,z) has the property that u $\to$ 0 when t $\to$ a. This is generally satisfied due to physical conditions of actual physical problem.

Solving equations in (4), we have

$X_n(x) = A_n \cos nx + B_n \sin nx$

$Y_m(y) = C_m \cos my + D_m \sin my$

$Z_l(z) = E_l \cos lz + F_l \sin lz$

and $T_p(t) = G_p e^{-p^2kt} = H_{nml} e^{-(n^2+m^2+l^2)\,kt}$

$U_{nml}(x,y,z,t) = H_{nml}(A_n \cos nx + B_n \sin nx).$

$(C_m \cos my + D_m \sin my)(E_l \cos lz + F_l \sin lz)$

$. e^{-(n^2+m^2+l^2)\,kt}$ ..... (6)

are solution of (1).

13.5 OCCURENCE OF HEAT EQUATION, MAXIMUM-MINIMUM PRINCIPLES

13.5. The occurence of Heat equation :

(a) The conduction of Heat in Solids. If we denote by q the temperature at a point in a homogeneous isotropic solid, then it is readily shown that the rate of flow of heat per unit area accross any plane is

$$q = -k\frac{\partial \theta}{\partial n} \text{ .....(1)}$$

where k is the termal conductivity of the solid and the operator $\frac{\partial}{\partial n}$ denotes differentiation along the normal. Considering the flow of heat through a small element of volume, we can show that the variation of q is governed by the equation

$$\rho C\frac{\partial \theta}{\partial t} = \text{div}\,(k\,\text{grad}\,\theta) + H(r,\theta,t) \text{ ..... (2)}$$

where ρ is the density and C the specific heat of the solid, and H ($\vec{r}$, θ,t) dt is the amount of heat generated per unit time in the element dt situated at the point with position vector r .

The heat function H ($\vec{r}$, θ,t) may arise because the solid is undergoing radio active decay or is absorbing radiation. A term of this kind exists also when there is generation or absorption of heat in the solid as a result of a chemical reaction, e.g. hydration of cement.

If the conductivity k is a constant throughout the body and if we write

$$K = \frac{k}{\rho C} \ , \qquad Q(\vec{r}, \theta, t) = \frac{H(\vec{r}, \theta, t)}{\rho C}$$

euqation (2) reduces to the form

$$\frac{\partial \theta}{\partial t} = k\nabla^2\theta + Q(\vec{r}, \theta, t) \quad \text{.....(3)}$$

The fundamental problem of the mathematical theory of the conduction of heat is the solution of equation (2) when it is known that the boundary surfaces of the solid are treated in a preseribed manner. The boundary conditions are usually of three main types :

(i) The tempertaure is prescribed all over the boundary, i.e. the temperature θ ($\vec{r}$, t) is a prescribed function of t for every point r of the boundary surface ;

(ii) The flux of heat accross the boundary is prescribed; i.e. $\frac{\partial \theta}{\partial n}$ is prescribed.

(iii) There is radiation from the surface into a medium of fixed temperature θ_0 ; i.e.

$$\frac{\partial \theta}{\partial n} + h(\theta - \theta_0) = 0 \quad \text{..... (4)}$$

where h is a constant.

If we introduce the differential operator

$$\lambda = C_0 + C_1\frac{\partial}{\partial x} + C_2\frac{\partial}{\partial y} + C_3\frac{\partial}{\partial z} \quad \text{.....(5)}$$

where c_0, c_1, c_2, c_3 are functions of x, y, z only, we see that the general boundary condition

$$\lambda\theta(\vec{r}, t) = G(\vec{r}, t), \vec{r}\epsilon S \quad \text{.....(6)}$$

embraces all three cases.

(b) Diffusion in Isotropic Substances : Another example of the occurence of diffusion equation arises in the analysis of the process of

diffusion in physical chemistry. This is a process leading to the equalization of concentrations within a single phase and it is governed by laws connecting the rate of flow of the diffusing substance with the concentration gradient causing the flow. If C is the concentration of the diffusion substance, then the diffusion current vector $\vec{J}$ is given by

$$\vec{J} = -D \text{ grad } C \quad \text{.....(7)}$$

where D is the coefficient of diffusion for the substance under consideration. The equation of continuity for the diffusiong substance takes the form

$$\frac{dC}{dt} + \text{div } J = 0 \quad \text{..... (8)}$$

Substituting from equation (7) into equation (8), we find that the variation of the concentration is governed by the equation

$$\frac{\partial c}{\partial t} = \text{div } (D \text{ grad } C) \quad \text{.....(9)}$$

In the most general case the coefficient of diffusion D will depend on the concentration and the coordinates of the point in question. If, however, D does happen to be a constant, then equation (9) reduces to the form

$$\frac{\partial C}{\partial t} = D\nabla^2 C \quad \text{.....(10)}$$

(c) The diffusion of vorticity : In the case of viscous fluid of density ρ and coefficeient of viscosity m shich is started into motion from rest the vorticity ζ, which is related to the velocity q in the fluid by the equation

$$\zeta = \text{curl } q \quad \text{..... (11)}$$

is governed by the diffusion equation

$$\frac{\partial \zeta}{dt} = \upsilon\nabla^2\zeta \quad \text{..... (12)}$$

where $\upsilon = \frac{\mu}{\rho}$ is the kinematic viscosity.

(d) Conducting Media : Maxwell's equations for the electromagnetic field in a medium of conductivity σ, permeability μ and dielectric constant k may be written in the form

$$\begin{aligned}
&\text{div}\,(k\,E) = 0\\
&\text{div}\,(\mu\,H) = 0\\
&\text{curl}\,H = \frac{4\pi\sigma}{C}\,E + \frac{1}{C}\,\frac{\partial}{\partial t}\,(kE)\\
&\text{curl}\,E = -\frac{1}{C}\,\frac{\partial}{\partial t}\,(\mu H)
\end{aligned}$$

If we make use of the identity

$$\text{curl curl} \equiv \text{grad div} - \nabla^2$$

then it follows from this equations that when s, m, k are constant throughout the medium

$$\nabla^2 E = \frac{ku}{C^2}\,\frac{\partial^2 E}{\partial t^2} + \frac{4\pi\sigma\mu}{C^2}\,\frac{\partial E}{\partial t}$$

If we dealing with problems concerning the propagation of long waves in a good conductor, the first term on the right-hand side of this equation may be neglected in comparison with the second. We therefore find that the components of the vector E satisfy the equation

$$\nabla^2\theta = \frac{1}{\nu}\,\frac{\partial\theta}{\partial t}$$

where $\nu = c^2 / 4\pi\mu\rho$.

13.5.2 Maximum-Minimum Principle

Let Ω denoted the bounded region in IR^3 whose boundary is a smooth closed surface S. Suppose u (x, y, z, t) to be continuous for (x, y, z) in $\overline{\Omega}$ and $0 \le t \le T$; for short, in $\overline{\Omega} \times [0, T]$. Let

$M_s \equiv \max \{u (x, y, z,t) : (x, y, z)$ on S and $0 \le t \le T\}$

$M_o \equiv \max \{u (x, y, z,t) : (x, y, z)$ on Ω and $t = 0\}$

$M = \max \{M_s, M_o\}$

and let m_s, m_0 and m denote the corresponding minimum values of u.

Theorem 13.5.2.1. (Maximum-Minimum principle for the Heat Equation) : Given that u (x, y, z, t) is continuous in $\overline{\Omega} \times [0, T]$. Then

(i) If $u_t - \nabla 2u \le 0$ in $\Omega \times (0, T)$, then $u \le M$ in $\overline{\Omega} \times [0, T]$.

(ii) If $u_t - \nabla^2 u \geq 0$ in $\Omega \times (0, T)$, then $u \geq m$ in $\overline{\Omega} \times [0, T]$.

(iii) If $u_t - \nabla^2 u = 0$ in $\Omega \times (0, T)$, then $m \leq u \leq M$ in $\overline{\Omega} \times [0, T]$.

Proof: Suppose that u (x, y, z, t) is continuous in $\overline{\Omega} \times [0, T]$ and satisfies $u_t - \nabla^2 u \leq 0$ in $\Omega \times (0, T)$

For $\in > 0$, let $v(x,y,z,t) = U(x,y,z,t) + \in (x^2 + y^2 + z^2)$. Then

$$v_t - \nabla^2 v = - \in < 0 \text{ in } \Omega \times (0, T) \quad \ldots\ldots\ldots\ldots (1)$$

Suppose now that v assumes its maximum values at (x_o, y_o, z_o, t_o), where (x_o, y_o, z_o) is an interior point of Ω and $0 < t_o \leq T$. Then, at (x_o, y_o, z_o, t_o),

$$v_t \geq 0 \text{ and } \nabla^2 v \leq 0$$

which contradicts (1). Hence v must assume it maximum value for (x, y, z) an S and $0 \leq t \leq T$, or else for t = 0 and (x, y, z) in Ω.

The definition of v implies that

$$u \leq v \leq \max v = M + \in R^2 \text{ in } \overline{\Omega} \times [0, T]$$

where $R^2 = \max (x^2 + y^2 + z^2)$. SInce $\in > 0$ is arbitrary, it follows that $u \leq M$ in $\overline{\Omega} \times [0, T]$. Thus (i) is proved.

Theorem 13.5.2.1 (ii) is proved by applying Theorem 13.5.2.1 (i) to the function –u; then Theorem 13.5.2.1 (i) and (ii) together implying Theorem 13.5.2.1 (iii).

According to Theorem 13.5.2.1 (iii), the temperatures inside a heat conductor are bounded by the extreme temperatures attained either inside initially or on the boundary subsequently.

CHECK YOUR PROGRESS



Q.4. Let Ω denote the bounded region in IR^3 whose boundary is a smooth closed surface S. Suppose u (x, y, z, t) to be continuous for (x, y, z) in Ω and $0 \leq t \leq T$. Consider the problem

$$u_t - \nabla^2 u = \phi \quad (x, y, z, t) \text{ in } \Omega \times (0, T)$$

$$u (x, y, z, 0) = F (x, y, z) \text{ in } \Omega$$

$$u (x, y, z, t) = f \quad (x, y, z, t) \quad \text{in } S \times [0, T]$$

Show that if this problem has a solution u that is continuous in $\overline{\Omega} \times [0,T]$, then this solution is unique.

13.6 GREEN FUNCTIONS FOR HEAT EQUATION

Suppose we are considering the solution q (r, t) of the heat equation

$$\frac{\partial\theta}{\partial t} = k\nabla^2\theta \quad \text{..... (1)}$$

in the volume Ω, which is bounded by the simple surface S, subject to the boundary condition

$$\theta\,(r, t) = \phi\,(r, t) \text{ if } r \in S \quad \text{..... (2)}$$

and the initial condition

$$\theta\,(r, 0) = \phi\,(r) \text{ if } r \in \Omega \quad \text{..... (3)}$$

We define the Green's function G (r, r′, t – t′) (t > t′) of our problem as the function which satisfies the equation

$$\frac{\partial G}{\partial t} = k\nabla^2 G \quad \text{.....(4)}$$

the boundary condition

$$G\,(r, r', t - t') = 0 \text{ if } r' \in S \quad \text{.....(5)}$$

and the initial condition that $Lt_{t\to t'}$, G is zero at all points of Ω ezcept at the point r where G takes the form

$$\frac{1}{8[\pi k(t - t')]^{\frac{3}{2}}} \exp\left[-\frac{|r - r'|^2}{4k(t - t')}\right] \quad \text{.....(6)}$$

Because G depends on t only in that it is a function of t – t′, it follows that equation (4) is equivalent to

$$\frac{\partial^2 G}{\partial t'} + k\nabla^2 G = 0 \quad \text{.....(7)}$$

The physical interpretation of the Green's function G is obvious from these equation : G (r, r′, t – t′) is the temperature at r′ at time t due to an instantaneous point source of unit strength generated at time t′ at the point r, the solid being initially at zero temperature, and its surface being maintained at zero temperature.

Since the time t¢ lies within the interval of t for which equations (1) and (2) are valid, we may rewrite these equations in the form

$$\frac{\partial\theta}{\partial t'} = k\nabla^2\theta \quad t' < t \quad \text{..... (8)}$$

$$\theta(r', t') = \phi(r', t') \text{ if } r' \in S \quad \text{..... (9)}$$

It follows immediately from equations (7) and (8) that

$$\frac{\partial}{\partial t'}(\theta G) = \theta\frac{\partial G}{\partial t'} + G\frac{\partial\theta}{\partial t'} = k[G\nabla^2\theta - \theta\nabla^2 G]$$

so that if $\in$ is an arbitrary small positive constant,

$$\int_0^{t-\varepsilon}\left\{\int_\Omega \frac{\partial}{\partial t'}(\theta G)d\tau'\right\}dt' = k\int_0^{t-\varepsilon}\{[G\nabla^2\theta - \theta\nabla^2 G]d\tau'\}\cdot dt' \quad \text{.....(10)}$$

If we interchange the order in which we take the integrations on the left hand side, we find that it takes the form

$$\int_\Omega (\theta G)_{t'=t-\in}\,d\tau' - \int_\Omega (\theta G)_{t'=0}\,d\tau' = \theta(r,t)\int_\Omega [G(r, r', t-t')]_{t'=t-\in}\cdot d\tau'$$

$$-\int_\Omega G(r, r', t)f(r')d\tau'$$

Now from the expression (6) for G (r, r′, t – t′) we cam readily show that

$$\int_\Omega [G(r, r', t-t')]_{t'=t-\in}\,d\tau' = 1$$

so that if we let $\in \to 0$, the left hand side of equation (10) becomes

$$\theta(r,t) - \int_\Omega f(r')G(r, r', t)d\tau'$$

On the other hand if we apply Green's theorem to the right-hand side of equation (10) and make use of equations (2) and (5), we find that it reduces to

$$-k\int_0^t dt' \int_S \Phi(r', t)\frac{\partial G}{\partial n}ds$$

in the limit as $\in \to 0$. It will be recalled that $\frac{\partial}{\partial n}$ denotes differentiation along the outward -dtawn normal to S. We therefore obtain finally

$$\theta(r,t) = \int_{\Omega} f(r')G(r,r',t)d\tau' - k\int_{0}^{t} dt' \int_{S} \Phi(r',t)\frac{\partial G}{\partial n}ds \quad \text{..... (11)}$$

as the solution of the boundary value problem formulated in equations (1), (2) and (3).

Example 2. If surface z = 0 of the semi-infinite solid $z \geq 0$ is maintained at temperature ϕ (x, y, t) for t > 0, and if the initial temperature of the solid is f (x, y, z), determine distributiion of temperature in the solid.

It is readily shown that appropriate Green's function for this problem is

$$G(r,r',t-t') = \frac{1}{8[(\pi K(t-t'))]^{\frac{3}{2}}}\left\{\exp\left[-\frac{|r-r'|^2}{4K(t-t')}\right] - \exp\left[-\frac{|r-\rho'|^2}{4K(t-t')}\right]\right\}$$

where $\rho' = (x', y', -z')$ is the position vector of the image of the point r′ in the place z=0. For this function

$$\frac{\partial G}{\partial n} = -\left(\frac{\partial G}{\partial Z'}\right)_{z'=0}$$
$$= -\frac{z}{8\pi^{\frac{3}{2}}K^{\frac{5}{2}}(t-t')^{\frac{5}{2}}}\exp\left[-\frac{(x-x')^2+(y-y')^2+z^2}{4K(t-t')}\right]$$

so that from equation (11), we obtain the solution

$$r,t) = \frac{1}{8(\pi kt)^{3/2}}\int_{\Omega} f(r')\left[\exp\left\{-\frac{|r-r'|^2}{4kt}\right\} - \exp\left\{-\frac{|r-\rho'|^2}{4kt}\right\}\right]\cdot d\tau'$$

$$\frac{z}{\tau k)^{3/2}}\int_{0}^{t}\int_{IR^2} \frac{\Phi(x',y',t')}{(t-t')^{3/2}}\exp\left[-\frac{(x-x')^2+(y-y')^2+z^2}{4k(t-t')}\right]\cdot dx'dy'dz'$$

Where Ω denotes the half space $z \geq 0$ and IR^2 the entire xy-plane.

In this problem we have been able to guess readily the form of the Green's function. For more complicated types of boundary this may not be

possible, and so it is desirable to have avaible a tool for the determination of the Green's function. The most powerful analytical tool for this purpose is theory of Laplace transforms. We shall illustrate its use by considering :

Example 3 : Determine the Green's function for the thick plate of infinite radius bounded by the paralled planes z = 0 and z = a.

From equation (4), (5) and (6) we see that we have to determine a function G which vanishes on the planes z = 0, z = a and has a singularity of the type (6), we write

$$G(r', r', t) = \frac{1}{8(\pi kt)^{3/2}} \exp\left[-\frac{|r - r'|^2}{4kt}\right] + G_1(r, r', t) \quad \text{..... (12)}$$

where, by virtue of equation (4)

$$k\nabla^2 G_1 = \frac{\partial G_1}{\partial t} \quad \text{.....(13)}$$

If we multiply both sides of equation (12) and (13) by e^{-st}, integrate w. r. to t from 0 to a and make use of the fact that the laplace transform of

$$\frac{1}{8\pi K t^{\frac{3}{2}}} \exp\left[-\frac{|r - r'|^2}{4Kt}\right]$$

can be written in the form

$$\frac{1}{4\pi K} \int_0^{\alpha} e^{-\mu|z-z'|} \frac{J_O(\lambda R)}{\mu} \lambda d\lambda$$

where $R^2 = (x - x')2 + (y - y')2$ and $\mu^2 = \lambda^2 + {}^s/_k$, we find that these equations are equivalent to

$$\overline{G} = \frac{1}{2\pi^2 ik} \int_{-i\alpha}^{i\alpha} \frac{\xi J_0(\xi R) \sin h\,[\eta(a - z')] \sin h\,(\eta z)}{\eta \sin h\,(\eta a)} d\xi \quad \text{.....(14)}$$

$$\frac{\partial^2 \overline{G}_1^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial^2 \overline{G}_1}{\partial \rho} + \frac{\partial^2 \overline{G}_1}{\partial z^2} + \frac{1}{\rho^2}\frac{\partial^2 \overline{G}_1}{\partial \Phi^2} = \frac{s}{K}\overline{G}_1 \quad \text{.....(15)}$$

where $\overline{G}, \overline{G}_1$ are the Laplace transforms of G, G_1 and as usual, r, z, ϕ denote cyliindrical coordinates. Equation (15) has a solution of the form

$$\frac{1}{4\pi k} \int_0^{\alpha} \frac{\lambda}{\mu} J_0(\lambda R) \{F \sin h\,(\mu z) + H \sin h\,[\mu(a - z)]\} d\lambda$$

where the funtions F (λ) and H (λ) must be chosen so that $\overline{G}$ vanishes on the planes z = 0, z = a.

Therefore we must have

$$F = -e^{-\mu(a-z')}\,\text{cosec h}(\mu a), \quad H = -e^{-\mu z'}\,\text{cosec h}(\mu a)$$

Thus if 0 < z < z′, we find that

$$\overline{G} = \frac{1}{2\pi k}\int_0^{\alpha'} \frac{\lambda J_0(\lambda R)\sin h\,[\mu(a-z')]\sin h\,(\mu z)}{\mu \sin h\,(\mu a)}\,d\lambda$$

If we make the substitution λ = iξ in this integral, we obtain the form

$$\overline{G} = \frac{1}{2\pi^2 ik}\int_{-i\alpha}^{i\alpha} \frac{\xi J_0(\xi R)\sinh(n(a-z')\sinh/nz)_\xi}{n\sinh(na)}$$

where $\eta^2 = {}^{s}/_{k} - \xi^2$. Now it is readily shown by the calculus of residues that

$$\overline{G} = \frac{1}{\pi ka}\sum_{n=1}^{\alpha} \sin\left(\frac{n\pi z}{a}\right)\sin\left(\frac{n\pi z'}{a}\right)K_0(\xi_n R)$$

Where $\xi_n = \sqrt{{}^{n^2\pi^2}/_{a^2} + {}^{s}/_{K}}$. using the fact that $K_0\left[x\sqrt{{}^{s}/_{K}}\right]$ is the Laplace transform of $(2t)^{-1}\,e^{-x^2/4kt}$ and that the Laplace transform of e^{-at} f(τ) is f (s + a), we find that

$$G = \frac{e^{-R^2/4kt}}{2\pi kta}\sum_{n=1}^{\alpha} \sin\left(\frac{n\pi z}{a}\right)\sin\left(\frac{n\pi z'}{a}\right)e^{-n^2\pi^2 kt/a^2} \quad \text{..... (16)}$$

This expression could have been obtained by the method of separation of variables if we had been prepared to assume the possibility of the expansion of an arbitrary function in the form (16).



CHECK YOUR PROGRESS

Q.5 Solve the IBVP

$V_t - k\,V_{xx} = 0,\ 0 < x < \alpha,\ 0 < t < \alpha$

$V(x, 0) = \phi(x),\ x > 0$

$V(0, t) = 0,\ t > 0$

with the initial data $\phi(x) = e^x$.

13.7 THE RESOLUTION OF BVPS FOR THE HEAT EQUATION, DUHAMEL'S THEOREM

We shall now describe a method due to Bartels and Churchill for the resolution of complicated boundary problems for the generalized heat equation.

If we assume that the function H (r,θ,t), occuring in equation (2) in section 13.5 is a linear function of the temperture q of the form

$$H(r, \theta, t) = \rho C[C_0(r)\,\theta + F(r, t)] \quad \ldots\ldots\ldots\ldots\ldots (1)$$

where C0 is a function of r only, intruducing the linear differential operator

$$A = \frac{1}{\rho C}\,\mathrm{div}\,(k\,\mathrm{grad}) + C_o(r) \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

and denoting by r the position vector of a point in the solid and by r′ that of a point in its boundary, it follows from equations (2) and (6) of the section 13.5 that the boundary value problem for the temperature θ (r,t) in the solid can be writtenm in the form

$$\frac{\partial}{\partial t}\theta(r,t) = \Lambda\,\theta + F(r,t) \quad t > 0$$

$$\lambda\theta(r', t) = G(r', t) \qquad t > 0 \qquad\qquad (A)$$

$$\theta(r, 0) = J(r)$$

The third equation of this set merely expresses the fact that at the instant t = 0 the distribution of temperature throughout the solid is prescribed.

We shall now show that the complicated boundary value problem (A) may be resolved in to simpler problems.

Suppose that the function f (r, t, t′) depending on the fixed parameter t′ is a solution of the boundary value problem (A) in the case in which the source function F and the surface temperature G are functions of the space variables and of the parameter t′ but not of the time t, so that ϕ (r, t, t′) satisfies the equation

$$\frac{\partial}{\partial t}\Phi(r, t, t') = \Lambda\,\Phi + F(r, t')$$

$$\lambda\phi(r, t, t') = G(r', t') \qquad\qquad (B)$$

$$\phi(r, 0, t') = J(r)$$

Then it is readily shown that once the solution of the boundary value problem (B) is known, the solution of the boundary value (A) can be drived by a simple calculation. The method is contained in :

Theorem 13.7.1 (Duhamel's Theorem). The solution θ (r, t) of the boundary problem (A) with time dependent source and surface conditions is given in terms of the solution ϕ (r, t, t′) of the boundary problem (B) with constant source and surface conditions by the formula

$$\theta(r,t) = \frac{\partial}{\partial t}\int_o^t \Phi\,(r, t-t', t')dt'$$

If the boundary condition is

$$\lambda\theta(r', t) = \begin{cases} 0, & t > 0 \\ G(r', t'), & t < 0 \end{cases}$$

it follows that the corresponding solution of (A) is

$$\theta = \phi\,(r, t, t') \quad t > 0$$

Similarly if the boundary condition is

$$\lambda\theta(r', t) = \begin{cases} 0, & t < t' \\ G(r', t'), & t > t' \end{cases}$$

the corresponding temperature is

$$\theta = \phi\,(r, t' - t, t') \quad t > t'$$

Further if

$$\lambda\theta(r', t) = \begin{cases} 0, & t < t' + dt' \\ G(r, t'), & t > t' + dt' \end{cases}$$

Then $\theta = \phi\,(r', t - t' - dt', t') \quad t > t' + dt'$

and it follows that if the boundary condition is

$$\lambda\theta(r', t) = \begin{cases} 0, & t < t' \\ G(r, t'), & t' < t < t' + dt' \\ 0, & t > t' + dt' \end{cases}$$

the solution of the boundary value problem is

$$\theta = \phi(r, t - t', t') - \phi(r, t - t' - dt', t')$$

$$= dt' \frac{d\phi(r, t - t', t')}{\partial t}$$

By breaking up the internet t = 0 to t = t into small intervals in this way and integrating the results obtained we find that the solution of the boundary value problem (A) is

$$\theta(r,t) = \frac{\partial}{\partial t}\int_0^t \phi\,(r, t-t', t')dt' \text{ .....(3)}$$

This theorem is of great value in the solution of boundary value problems in the theory of the condition of heat, since it is often easier to derive the solution in the case of constant source and boundary consitions.

It can further be shown that solution of the boundary value problem (B) can be written in the form

$$\phi(r,t,t') = \phi_1(r,t') + \phi_2(r,t,t') + \int_0^t \phi_3\,(r,\tau,t')d\tau \text{ .....(4)}$$

where the funtions ϕ_1, ϕ_2 and ϕ_3 are solutions of the boundary value problems

$$\wedge\phi_1\,(r,t') = 0, \quad \lambda\phi_1\,(r',t') = G\,(r',t') \text{ .....(B1)}$$

$$\left(\frac{\partial}{\partial t} - \wedge\right)\phi_2(r,t,t') = 0, \lambda\phi_2(r',t.t') = 0$$

$$\phi_2(r,\theta,t') = J\,(r) - \phi_1\,(r,t') \text{ .....(B2)}$$

$$\left(\frac{\partial}{\partial t} - \wedge\right)\phi_3(r,t,t') = 0, \quad \lambda\phi_3(r',t,t') = 0, \qquad \phi_3(r,0,t') = F(r,t') \text{ .....(B3)}$$

From Duhamel's theorem it follows that the solution of the boundary value problem is

$$\theta(r,t) = \phi_1(r,t) + \frac{\partial}{\partial t}\int_0^t \phi_2\,(r,t-t',t')dt' + \int_0^t \phi_3\,(r,t-t')dt' \text{ ..... (5)}$$

The solutions of the three simpler boundary value problems (B1), (B2) and (B3) of which the first is a steady state problem, may therefore to used to derive the solution of the general boundary value problem (A).



13.8 LET US SUM UP

- General form of partial differential equation of second order of a function of two variables (independent) x and y is

$$Rr + Ss + Tt + f(x,y,z,p,q) = 0$$

where R, S, T are continuous functions of x and y only possessing partial derivatives defined in some domain D on the xy-plane and $r = \frac{\partial^2 z}{\partial x^2}, s = \frac{\partial^2 z}{\partial x \partial y}, t = \frac{\partial^2 z}{\partial y^2}$. The equation is said to be parabolic at a point (x,y) in D if $s^2–4RT=0$

- One - dimensional heat equation is

$$K\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

Two - dimensionel heat equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{k}\frac{\partial u}{\partial t}$$

Three - dimensional heat equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{k}\frac{\partial u}{\partial t}$$

- Maximum - minimum principle:

Let Ω denote the bounded region in IR^3 whose boundary is a smooth closed surface S. Suppose u(x, y, z, t) to be continuous for (x, y, z) in $\overline{\Omega}$ and $0 \le t \le T$; for short, n $\overline{\Omega}$ × [0, T].

Let

M_s = max {u (x,y,z,t) : (x,y,z) on S and $0 \le t \le T$}

M_o = max {u (x,y,z,t) : (x,y,z) on Ω and t = 0}

M = max {M_s, M_o}

and let ms, mo and m denote the corresponding minimum values of u. Then

(i) If $u_t - \nabla^2 u \le 0$ in $\Omega \times (0, T)$, then

$$u \le M \text{ in } \overline{\Omega} \times (0, T).$$

(ii) If $u_t - \nabla^2 u \ge 0$ in $\Omega \times (0, T)$, then

$$u \ge m \text{ in } \overline{\Omega} \times (0, T).$$

(iii) If $u_t - \nabla^2 u = 0$ in $\Omega \times (0, T)$, then

$$m \le u \le M \text{ in } \Omega \times (0, T).$$

- Duhamel's principle :

The solution q (r,t) of the BVP

$$\frac{\partial}{\partial t}\theta(r,t) = \wedge\theta + F(r,t), t > 0$$

$$\lambda\theta(r', t) = G(r', t), t > 0$$

$$\theta(r, 0) = J(r)$$

with time dependent source and surface condition is given in terms of the solution ϕ (r, t, t′) of the boundary problem

$$\frac{\partial}{\partial t}\phi(r,t,t') = \wedge\phi + F(r,t')$$

$$\lambda\phi(r,t,t') = G(r't')$$

$$\phi(r,0,t') = J(r)$$

with constant source and surface conditions by the formula

$$\theta(r,t,) = \frac{\partial}{\partial t}\int_0^t \phi\,(r, t-t', t')dt'$$

13.9 ANSWER TO CHECK YOUR PROGRESS

Ans to Q.No. 1:

Given equation is

$$r - 2s + t + p - q - e^x(2y - 3) + e^y = 0 \text{ .....(1)}$$

Comparing (1) with Rr + Ss + Tt + f (x,y,z,p,q) = 0, we get R = 1, S = –2, T= 1 so that

$S^2 - 4RT = (-2)^2 - 4.1.1 = 0$, showing that (1) is parabolic.

The λ – quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to

$$\lambda^2 - 2\lambda + 1 = 0$$

$$\Rightarrow \lambda = 1, 1$$

So, the corresponding characteristic equation is

$$\frac{dy}{dx} + 1 = 0$$

Intergrating $x + y = C_1$

Choose $u = x+y$ and $v = y$ ..... (2)

where we have chosen $v = y$ in such a manner that $u = v$ are independent functions as verified below

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = 1 \neq 0$$

Now $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u}$ [using (2)] .....(3)

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \quad \text{.....(4)}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} = \frac{\partial^2 z}{\partial u^2} \cdot 1 \quad \text{..... (5)}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v} + \frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}$$

$$= \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u\,\partial v} + \frac{\partial^2 z}{\partial v\,\partial u} + \frac{\partial^2 z}{\partial v^2}$$

$$= \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u\,\partial v} + \frac{\partial^2 z}{\partial v^2} \quad \text{..... (6)}$$

$$s = \frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} = \frac{\partial^2 x}{\partial u^2} + \frac{\partial^2 z}{\partial u\,\partial v} \quad \text{.....(7)}$$

Using (2), (3), (4), (5), (6) amd (7) in (1), we get

$$\frac{\partial^2 z}{\partial u^2} - 2\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u\,\partial v}\right) + \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u\,\partial v} + \frac{\partial^2 z}{\partial v^2} + \frac{\partial z}{\partial u} - \left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = e^{u-v}(2v-3) - e^v$$

$$\Rightarrow \frac{\partial^2 z}{\partial v^2} - \frac{\partial z}{\partial v} = e^{u-v}(2v-3) - e^v \quad \text{..... (8)}$$

which is the required canonical form of (1). Let

$D^o \frac{\partial}{\partial x}$, $D^o \frac{\partial}{\partial y}$. Then (8) can e written as

$$D'(D'-1)z = e^{u-v}(2v-3) - e^v \quad \text{..... (9)}$$

C.F. $= \phi(u) + e^v\psi(u) = \phi(x+y) + e^y\psi(x+y)$

P.I. $= \frac{1}{D'(D'-1)}\, e^{u+(-1)^v}(2v-3)$ which is corresponding to $e^{u-v}(2v-3)$.

$$= e^{u+(-1)v}\frac{1}{(D'-1)(D'-1-1)}(2v-3)$$

$$= \frac{1}{2}e^{u-v}(1-D')^{-1}\left(1-\frac{D'}{2}\right)^{-1}(2v-3)$$

$$= \frac{1}{2}e^{u-v}(1+D'+\cdots)\left(1+\frac{D'}{2}+\cdots\right)(2v-3)$$

$$= \frac{1}{2}e^{u-v}\left(1+\frac{3D'}{2}+\cdots\right)(2v-3)$$

$$= \frac{1}{2}e^{u-v}(2v-3+3)$$

$$= ve^{u-v} = ye^{x+y-y} = ye^x \quad \text{using (2)}$$

P.I. Corresponding to –ev is

$$\frac{1}{D'(D'-1)}(-e^v) = -\frac{1}{D'-1}\cdot\frac{1}{D'}e^v = -\frac{1}{D'-1}e^v \cdot 1$$

$$= -e^v \frac{1}{D'+1-1}\cdot 1$$

$$= -e^v \cdot \frac{1}{D'}\cdot 1$$

$$= -ve^v = -ye^y, \qquad \text{using (2)}$$

Hence the required general solution is given y

$$y = \phi(x+y) + e^y \psi(x+y) + ye^x - ye^y$$

$$= \phi(x+y) + e^y \psi(x+y) + ye^x - (x+y)e^y + xe^y$$

$$= \phi(x+y) + e^y [\psi(x+y) - (x+y)] + ye^x + xe^y$$

$$= \phi(x+y) + e^y F(x+y) + ye^x + xe^y$$

Where ϕ and F are arbitrary functions and

$F(x+y) = \psi(x+y) - (x+y)$.

Ans to Q. No. 2:

(a) The equation for conduction of heat is

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad \text{.....(1)}$$

Prior to temperature, change at the end when t = 0, the heat flow was independent of time (steady state condition, for which $\frac{\partial u}{\partial t}$ =0). When u depends only on x, (1) reduces to

$$\frac{\partial^2 u}{\partial x^2} = 0, \text{ so that } u = c_1 x + c_2 \text{ .....(2)}$$

Given u = 0 for x = 0 and u = 100 for x = l,
(2) gives 0 = c_2 and 100 = lc_1 + c_2
(3) becomes u = $(\frac{100}{l})$ x

Hence the coundary conditions for the susequent flow are

u (0, t) = 0 for all t ..... (3)

u (l, t) = 0 for all t ..... (4)

and the initial condition is

$u(x, 0) = \left(\frac{100}{l}\right) x$ ..... (5)

Suppose that (1) has solutions of the form

$u(x, t) = X(x)\, T(t)$ ..... (6)

Sustituting this value of u in (1), we get

$$KX'' T = XT' \Rightarrow \frac{X''}{X} = \frac{T'}{KT} \quad \text{.....(7)}$$

Since x and t are independent variales, (T) can only be true if each side is equal to the same constant, say u.

$$\therefore x'' - \mu X = 0 \quad \text{.....(8)}$$

and $T' = \mu KT$ .....(9)

Using (3), (4), (6) gives

$X(0)\, T(t) = 0$ and $X(l)\, T(t) = 0$ .....(10)

Since $T(t) = 0$ leads to $u = 0$, so suppose that $T(t) \neq 0$

From (10), $X(0) = 0$, $X(l) = 0$ .....(11)

Now solve (8) under boundary condition (11). Three cases arise.

Case I. Let $\mu = 0$. Then solution of (8) is

$X(x) = Ax + B$ .....(12)

Using (11), (12) gives $0 = B$, $0 = Al + B$ so that $A=B=0$.

Hence $X(x) = 0$ so that $u = 0$ which does not satisfies (5).

Case II. Let $\mu = \lambda^2$, $\lambda \neq 0$. The solution of (8) is

$$X(x) = Ae^{\lambda x} + Be^{-\lambda x} \quad \text{.....(13)}$$

Using (11), (13) gives $0 = A+B$ and $0 = Ae^{l\lambda} + Be^{-l\lambda}$ .....(14)

Showing (14), $A = B = 0$ so that $X(x) = 0$ and hence $u = 0$, which does not satisfy (5). So we also reject $u = \lambda^2$.

Case III. Let λ^2, $\lambda \neq 0$. Then solution of (8) is

$$X(x) = A\cos \lambda x + B \sin \lambda x \quad \text{.....(15)}$$

Using (11), (15) gives $0 = A$ and $0 = A \cos \lambda l + B \sin \lambda l$

so $\sin \lambda a = 0$. We have take $B \neq 0$, since otherwise $x = 0$ so that $u = 0$ which does not satisfies (5). Thus

$\sin la = 0 \quad \Rightarrow la = hp$ where $h = 1, 2, ...$

$$\Rightarrow \lambda = \frac{\eta\pi}{l} \qquad \text{.....(16)}$$

Hence non-zero solutions Xn(x) of (8) are given by

$$X_n(x) = B_n \sin\left(\frac{n\pi x}{l}\right) \qquad \text{.....(17)}$$

Using (16), (9) reduces to

$$\frac{dT}{T} = -\frac{n^2\pi^2 k}{l^2}dt \quad \left[\because \mu = \lambda^2 = -\frac{n^2\pi^2}{l^2}\right]$$

Integrating, $\log T + \log D_n = -\frac{n^2\pi^2 k}{l^2}t$

$$\Rightarrow T = D_n e^{\frac{-n^2\pi^2 k}{l^2}t}$$

Hence general solution is $T_n(t) = D_n e^{\frac{-n^2\pi^2 k}{l^2}t}$ .....(18)

$$\therefore u_n(x,t) = X_n(x)T_n(t) = E_n \sin\left(\frac{n\pi x}{l}\right) e^{-\frac{n^2\pi^2 k}{l^2}t} \qquad \text{.....(19)}$$

are solution of (1), satisfying (3), (4). Here $E_\eta = B_\eta D_\eta$ is another constant. In order to obtain solution also satisfying (5), we consider more general solution

$$u(x,t) = \sum_{n=1}^{\alpha} U_n(x,t)$$
$$= \sum_{n=1}^{\alpha} E_n \sin\left(\frac{n\pi x}{l}\right) e^{\frac{-n^2\pi^2 k}{l^2}t} \qquad \text{.....(20)}$$

Substituting t = 0 in (20) and using (5) we get

$$\frac{100x}{l} = \sum_{n=1}^{\alpha} E_n \sin\left(\frac{n\pi x}{l}\right)$$

which is fourier sine series. So E_n is given by

$$E_n = \frac{2}{l}\int_0^1 \frac{100x}{l}\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{200}{l^2}\left[x\left\{-\frac{\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)}\right\} - \left\{\frac{-\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2}\right\}\right]_0^1$$

$$= \frac{200}{l^2}\left(-\frac{l^2}{n\pi}\cos n\pi\right) = \frac{200}{n\pi}(-1)^{n+1}$$

From (20), the required solution is

$$u(x,t) = \frac{200}{\pi} \sum_{n=1}^{\alpha} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{l} e^{\frac{-kn^2\pi^2 t}{l^2}}$$

(b) Here the initial condition is (5) as before. The new boundary conditions are

$$u(x,t) = 20 \quad \text{for all } t \qquad \text{.....(21)}$$

and $$u(l,t) = 80 \quad \text{for all } t \qquad \text{.....(22)}$$

Note that in part (a), the two boundary values were both zero and the required solution was easily obtained. In the present part (b), the two boundary values are non-zero, so we modify the procedure as follows :

We split the temperature function u (x,t) into two parts as

$$u(x,t) = u_1(x) + u_2(x,t) \qquad (23)$$

where $u_1(x)$ is a solution of (1) involving x only and satisfying (21) and (22); $u_2(x,t)$ is then a function defined by (23). Hence $u_1(x)$ is a steady state solution of the form (2) and u2 (x,t) may be treat as a transient part of the solution, which decreases with increase of t.

Since $u_1(x) = 20$ for $x = 0$ and $u_1(x) = 80$ for $x = l$, (2) gives

$$u_1(x) = 20 + \left(\frac{60}{l}\right) x \qquad \text{.....(24)}$$

Putting x = 0 in (23) and using (21), we get

$$u_2(0,t) = u(0,t) - u_1(0) = 20 - 20 = 0 \qquad \text{.....(25)}$$

Next putting x = l in (23) and using (22), we get

$$u_2(l,t) = u(l,t) - u_1(l) = 80 - 80 = 0 \qquad \text{.....(26)}$$

Again

$$\begin{aligned} u_2(x,0) &= u(x,0) - u_1(x) \\ &= \left(\frac{100}{l}\right)x - \left[\left(\frac{60}{l}\right)x + 20\right] \qquad \text{..... (27)} \\ &= \left(\frac{40}{l}\right)x - 20 \end{aligned}$$

Hence the boundary conditions and initial condition to the transient solution $u_2(x,t)$ are given by (25), (26) and (27).

So, we now solve

$$\frac{\partial u_2}{\partial t} = k\,\frac{\partial^2 u_2}{\partial x^2}$$

subject to boundary condition (25) and (26) and initial condition (27) and obtain

$$u_2(x,t) = \sum_{n=1}^{\alpha} E_n \sin\left(\frac{n\pi x}{t}\right) e^{\frac{-kn^2\pi^2}{l^2}t} \quad \text{.....(28)}$$

where En is given by

$$E_n = \frac{2}{l}\int_0^1 \left(\frac{40x}{l} - 20\right)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= -\frac{40}{n\pi}(1+\cos n\pi) = -\frac{40}{n\pi}[1+(-1)^n]$$

$$= \begin{cases} 0, \text{when n is odd} \\ -\dfrac{80}{n\pi}, \text{when n is even} \end{cases}$$

Taking n = 2m, $E_n = -\frac{40}{m\pi}$. From (28), we have

$$u_2(x,t) = -\frac{40}{\pi}\sum_{m=1}^{\alpha}\frac{1}{m}\sin\left(\frac{2m\pi x}{l}\right)e^{-\frac{4km^2\pi^2}{l^2}t} \quad \text{.....(29)}$$

Combining (24) and (29), the required solution is

$$u(x,t) = 20 + \left(\frac{40}{l}\right)x - \frac{40}{\pi}\sum_{m=1}^{\alpha}\frac{1}{m}\sin\left(\frac{2m\pi x}{l}\right)e^{\frac{-4km^2\pi^2}{l^2}t}$$

Ans to Q. No. 3 :

Here the temperature u (x,y,t) in the plate is governed by the two-dimensional heat equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{k}\,\frac{\partial u}{\partial t} \quad \text{.....(1)}$$

Since the edges x=0, x=a, y=0,
y=b are kept at zero temperature,
so the boundary conditions are

u (0, y, t) = 0 ..... 2(x)

u (a, y, t) = 0 ..... $2(x^1)$

u (x, 0, t) = 0 ..... 2(y)

u (x, b, t) = 0 ..... $2(y^1)$

The initial condition is

u (x,y,0) = f (x,y) ..... (3)

Suppose (1) has solution of the form

u (x,y,t) = X(x) Y(y) T(t) ..... (4)

Substituting in (1), we obtain

$$X''YT + XY''T = \frac{1}{k} XYT'$$

$$\text{Þ} \quad \frac{X''}{X} + \frac{Y''}{Y} = \frac{T'}{KT} \qquad \text{..... (5)}$$

where the d ashes denote derivatives with respect to relevant variable. Since x, y and t are independent variables, (5) can only true if each term on each side is equal to a constant.

Let $\frac{X''}{X} = \mu_1 \Longrightarrow X'' - \mu_1 X = 0$ .....(6)

Using 2(x) and 2(xi), (4) gives

X(0) Y(y) T(t) = 0 and X(a) Y(y) T(t) = 0 .....(7)

Since y = 0 or T = 0 leads to u = 0, so we suppose that $y \neq 0$ and $T \neq 0$. Then (7) gives

X (0) = 0 and X (a) = 0 .....(8)

We now solve (6) under boundary condition (8). Three case arise.

Case (I). Let $\mu_1 = 0$. Then solution of (6) is

X(x) = Ax+B .....(9)

using boundary condition (8), (9) gives 0=B and 0 = Aa+B. This gives A=B=0 and have hence X(x)=0. This leads to u=0. Which does not satisfy (3). So we reject m 1=0.

Case II. Let $u_1 = +\lambda_1^2, \lambda_1 \neq 0$. Then solution (6) is

$$X(x) = Ae^{\lambda_1 x} + Be^{-\lambda_1 x} \text{ .....(10)}$$

using boundary condition (8), (10) gives

$$0 = A + B \text{ and } 0 = Ae^{a\lambda_1} + Be^{-a\lambda_1} \text{ .....(11)}$$

Solving (11), A=B=0 so that X(x)=0. This lead to u=0, which does satisfy (3).

So we reject $\mu_1 = \lambda_1^2$..

Case III. Let $\mu_1 = -\lambda_1^2, \lambda_1 \neq 0$. Then solution (6) is

$$X(x) = A \operatorname{ces} \lambda_1 x + B \sin \lambda_1 x \qquad \text{.....(12)}$$

Using boundary condition (8), (12) giives

$$0 = A \text{ and } 0 = A \text{ ces } \lambda_1 a + B \sin \lambda_1 x$$

Hence A = 0 and sin $\lambda_1 a = 0$

Where we have taken B≠0, since otherwise X=0

so that u=0. Which does not satisfy (3).

Now sin $\lambda_1 a = 0$ gives

$$\lambda_1 = \frac{m\pi}{a}, \; m = 1,2, \ldots\ldots\ldots \text{ (13)}$$

Hence nonzero solutions $X_m(x))$ of (6) are given by

$$X_m(x) = B_m \sin\left(\frac{m\pi x}{a}\right), m = 1, 2, 3, \ldots\ldots \ldots\ldots\text{(14)}$$

Next, let $\frac{Y''}{Y} = \mu_2$ so that $Y'' - \mu_2 Y = 0$ .....(15)

Using (2) (y), 2 (y′), (4) gives as before

Y(0) = 0, Y (b) = 0 .....(16)

Solving (15) inder Boundary condition (16) as before, we get

$$Y_n(y) = D_n \sin\left(\frac{n\pi y}{b}\right), n = 1, 2, 3, \ldots\ldots\text{(17)}$$

where $\mu_2 = -\lambda_2^2 \text{ and } \lambda_2 = \frac{n\pi}{b}, n = 1,2,3, \ldots\ldots$ (18)

In view of (6) and (15), (5) reduces to

$$T'/_{kT} = \mu_1 + \mu_2 = -\lambda_1^2 - \lambda_2^2$$

$$= -\pi^2\left(m^2/_{a^2} + n^2/_{b^2}\right)$$

$$\Rightarrow T' = -\lambda_{mn}^2 T$$

$$\Rightarrow \frac{dT}{T} = -\lambda_{mn}^2 dt \qquad \text{.....(19)}$$

Where $\lambda_{mn}^2 = \pi^2 k\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right), m = 1,2,3, \ldots\ldots$ (20)

$n = 1,2,3, \ldots\ldots$

Solving (19), $T_{mn}(t) = E_{mn} e^{-\lambda^2 mnt}$, where m, n, = 1,2,3,.....

$$\therefore u_{mn}(x, y, t) = X_m(x) Y_n(y) T_{mn}(t)$$

$$= F_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi t}{b}\right) e^{-\lambda^2 \; mnt}$$

are solution of (1) satisfying u(x), u(x′), u(y), u(y′), Here $F_{mn}=B_mD_n$ is another arbitrary constant. In order to obtain a solution also satisfying (3). we consider more general solution

$$U(x,y,t)=\sum_{m=1}^{\alpha}\sum_{n=1}^{\alpha}U_{mn}(x,y,t)$$
$$=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}F_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-\lambda_{mn}^2 t} \quad \text{.....(21)}$$

Putting t = 0 in (21) and using (3), we get

$$f(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}F_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{n}\right) \quad \text{.....(22)}$$

Which is double Fourier sine series. Accordingly we get

$$F_{mn}=\frac{4}{ab}\int_{x=0}^{a}\int_{y=0}^{b}f(x,y)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dx\,dy \quad \text{.....(23)}$$

Hence (21) is the solution where F_{mn} is given by (23).

Ans to Q. No. 5:

Given boundary value problem is

$$V_t - k\,V_{xx}=0,\ o<x<\infty,\ 0<t<\infty,$$
$$V(x,0)=\phi(x),\ x>0 \quad \text{..... (1)}$$
$$V(0,t)=0,\ t>0$$

with the initial data $\phi(x)=e^x$

We can write the solution formula for the IBVP (1) as follows

$$V(x,t)=\frac{1}{\sqrt{4\pi kt}}\int_0^{\infty}\left[e^{-(x-y)^2/4kt}-e^{-(x+y)^2/4kt}\right]d(y)dy \quad \text{.....(2)}$$

Using the solution formula (2), we have

$$V(x,t)=\frac{1}{\sqrt{4\pi kt}}\int_0^{\infty}\left[e^{-(x-y)^2/4kt}.e^{y}-e^{-(x+y)^2/4kt}.e^{y}\right]dy \quad \text{.....(3)}$$

Combining the exponential factors of the first product under the integral, we will get an exponential with the following exponent

$$-\frac{[y^2-2(x+2kt)y+x^2]}{4kt}=-\left[\frac{y-(x+2kt)}{\sqrt{4kt}}\right]^2+kt+x$$

$$=-\mathrm{p}^2+\mathrm{kt}+\mathrm{x}$$

where $\mathrm{p}=\dfrac{\mathrm{y}-\mathrm{x}-2\mathrm{kt}}{\sqrt{4\mathrm{kt}}}$

Similarly, the exponent of the combined expential from the second product under integral (3) is

$$-\frac{[y^2+2(x-2kt)y+x^2]}{4kt}=-\left(\frac{y+x-2kt}{\sqrt{4kt}}\right)^2+kt-x$$

$$=-\mathrm{q}^2+\mathrm{kt}-\mathrm{x}$$

where $\mathrm{q}=\dfrac{\mathrm{y}+\mathrm{x}-2\mathrm{kt}}{\sqrt{4\mathrm{kt}}}$

Braking integral (3) into a difference of two integrals and making the changes of variables $\mathrm{y}\mapsto\mathrm{p}$ and $\mathrm{y}\mapsto\mathrm{q}$ in the respective integrals, we will get

$$\mathrm{v}(\mathrm{x},\mathrm{t})=\mathrm{e}^{\mathrm{kt}+\mathrm{x}}\frac{1}{\sqrt{\pi}}\int_{\frac{-\mathrm{x}-2\mathrm{kt}}{\sqrt{4\mathrm{kt}}}}^{\infty}\mathrm{e}^{-\mathrm{p}^2}\mathrm{dp}-\mathrm{e}^{\mathrm{kt}-\mathrm{x}}\cdot\frac{1}{\sqrt{\pi}}\int_{\frac{\mathrm{x}-2\mathrm{kt}}{\sqrt{4\mathrm{kt}}}}^{\infty}\mathrm{e}^{-\mathrm{q}}\mathrm{dq}\quad\text{.....(4)}$$

Notice that

$$\frac{1}{\sqrt{\pi}}\int_{\frac{-\mathrm{x}-2\mathrm{kt}}{\sqrt{4\mathrm{kt}}}}^{\infty}\mathrm{e}^{-\mathrm{p}^2}\,\mathrm{dp}=\frac{1}{\sqrt{\pi}}\int_{\frac{-\mathrm{x}-2\mathrm{kt}}{\sqrt{4\mathrm{kt}}}}^{0}\mathrm{e}^{-\mathrm{p}^2}\,\mathrm{dp}+\frac{1}{\sqrt{\pi}}\int_{0}^{\infty}\mathrm{e}^{-\mathrm{p}^2}\,\mathrm{dp}$$

$$=\frac{1}{2}+\frac{1}{2}\mathrm{Erf}\left(\frac{\mathrm{x}+2\mathrm{kt}}{\sqrt{4\mathrm{kt}}}\right)$$

and similarly for the second integral. Putting these back into (4), we will arrive at the solution

$$v(x,t)=e^{kt+x}\left[\frac{1}{2}+\frac{1}{2}Erf\left(\frac{x+2kt}{\sqrt{4kt}}\right)\right]-e^{kt-x}\left[\frac{1}{2}-\frac{1}{2}Erf\left(\frac{x-2kt}{\sqrt{4kt}}\right)\right]$$



13.10 FURTHER READING

1. Sneddon, I.N., Elements of Partial Differential Equations, Dover Publication, 2006.
2. Raisinghania, M.D., Advanced Differential Equations, S. Chand and Company Ltd., New Delhi, 2001.
3. Duchateau, P. and Zachmann, D.W., Partial Differential Equations, Schaum's outline series, Mc Graw- Hill Book Company, 1988.



13.11 MODEL QUESTIONS

Q.1. Reduce the following partial differential equations into canonical form and solve them :

(I) $r + 2s + t = 0$ i.e. $\frac{\partial^2 z}{\partial x^2} + 2\left(\frac{\partial^2 z}{\partial x\,\partial y}\right) + \frac{\partial^2 z}{\partial y^2} = 0$

(II) $\frac{\partial^2 z}{\partial x^2} + 2x\frac{\partial^2 z}{\partial x\,\partial y} + x^2\frac{\partial^2 z}{\partial y^2} = 0$

(III) $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x\,\partial y} + \frac{\partial^2 z}{\partial y^2} + p - q = 0$

Q.2. If both the ends of a bar of length a are at temperature zero and the initial temperature is to be prescribed function f(x) in the bar, then find the temperature at a subsequent time t.

Q.3. A rod of length l with insulated sides, is initially at a uniform temperature u_0. Its ends are suddenly cooled to 0^0C and are kept at that temperature. Find the temperation u (x,t).

Q.4. Solve : $\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$, $t > 0, o \le x \le 1$

u (0,t) = 2

u (1,t) = 3

and u (x,0) = x (1–x)

Q.5. A rectangular plate bounded by the line x = 0, y = 0, x = a, y = b has an initial distribution given by $V = A \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{b}\right)$. The edges are kept at zero temperature and the plane faces are impervious to heat. Find the temperature at any point.

*** ***** ***