UNIT 14 : HYPERBOLIC DIFFERENTIAL EQUATIONS

UNIT STRUCTURE

14.1 Learning Structure

14.2 Introduction

14.3 Canonical form and General solution of Hyperbolic differential Equations

14.4 Occurence of the wave equations.

14.5 Solution of One-dimensional, Two-dimensional and Three-dimensional wave equations.

14.6 Non homageneous wave equations, Duhamel's Principle and Method Descent.

14.7 Let us sum up

14.8 Answer to check your progress

14.9 Further reading

14.10 Model Questions

14.1 LEARNING OBJECTIVES

On completion of the unit learners will be able to –

- solve hyperbolic PDE;
- know about occurrence of the wave equation;
- solve wave equations;
- know about non homogeneous wave equation and Duhamel's principle.

14.2 INTRODUCTION

Consider the general form of partial differential equation of second order of a function of two independent variables x and y -

Rs + Ss + Tt + f (x,y,z,p,q) =0

where R, S, T are continuous functions of x and y only possessing partial derivatives defined in some domain D on the xy-plane and $r=\frac{\partial^2 x}{\partial x^2}$, $s=\frac{\partial^2 z}{\partial x\,\partial y}$, $t=\frac{\partial^2 z}{\partial y^2}$. Then the equation is said to be hyperbolic at a point (x,y) if S^2–4RT > 0.

In this unit we shall consider the wave equation

$$\nabla^2 u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

which is a typical hyperbolic equation. This equation is sometimes written in the form

$$\nabla^2 u = 0$$

where ∇^2 denotes the operator

$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

One-dimensional wave equation : $\frac{\partial^2 u}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$

Two-dimensional wave equation : $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$

Three-dimensional wave equation : $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$

14.3. CANONICAL FORM AND GENERAL SOLUTION OF HYPERBOLIC DIFFERENTIAL EQUATIONS

Working rule for reducing a hyperbolic equation to its canonical (or normal) form :

Step 1. Let the given equation

Rs + Ss + Tt + f (x,y,z,p,q) = 0 .....(1)

be parabolic so that S^2 – 4RT > 0.

Step 2. Write λ– quadratic equation

$$R\lambda^2 + S\lambda + T = 0 \qquad \text{.....(2)}$$

Let λ_1, λ_2 be two real and distinet roots of (2).

Step 3. Write the characteristic equation corresponding to

$\lambda = \lambda_1, \lambda_2$ i.e.

$$\frac{dy}{dx} + \lambda_1 = 0 \quad \text{and} \quad \frac{dy}{dc} + \lambda_2 = 0$$

Solving these equations, we get

$f_1(x,y) = c_1$ and $f_2(x,y) = c_2$

Step 4. Choose $u = f_1(x,y)$ and $v = f_2(x,y)$ .....(4)

Step 5. Using relation (4), find p, q, r, s and t in terms of u and v.

Step 6. Substituting the values of p,q,r,s,t in (1) and simplifying, we get the following canonical forms of (1),

$$\frac{\partial^2 z}{\partial u\, \partial v} = \phi\left(u, v, z\, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right)$$

Example 1. Consider the differential equation

$$(y-1)\,r - (y^2-1)\,s + y\,(y-1)\,t + p - q - 2ye^{2x}\,(1-y)^2 = 0 \text{ .....(1)}$$

Comparing (1) with $Rs + Ss + Tt + f(x,y,z,p,q) = 0$, we get

$$R = y - 1,\ S = -(y^2 - 1),\ T = y\,(y-1)$$

So that
$$\begin{aligned} S^2 - 4RT &= (y^2-1)^2 - 4\,(y-1)\,y\,(y-1) \\ &= (y-1)^2\,[(y+1)^2 - 4y] \\ &= (y-1)^2\,(y-1)^2 \\ &= (y-1)^4 > 0 \qquad [\because y \neq 1] \end{aligned}$$

So the given differential equation is hyperbolic.

The λ– quadratic $R\lambda^2 + S\lambda + T = 0$ gives

$$(y-1)\,\lambda^2 - (y^2-1)\,\lambda + y\,(y-1) = 0$$

$$\Rightarrow \lambda_1 = 1 \text{ or } \lambda_2 = y$$

Characteristic equations are

$$\frac{dy}{dx} + \lambda_1 = 0 \quad \text{and} \quad \frac{dy}{dx} + \lambda_2 = 0$$

$$\Rightarrow \frac{dy}{dx} + 1 = 0 \quad \text{and} \quad \frac{dy}{dx} + y = 0$$

Integrating, $x + y = c_1$ and $ye^x = c_2$

To reduce (1) to canonical form, we change the independent variables x, y,

to new independent variables u and v by taking

$u = x+y$ and $v = ye^x$ .....(2)

Now

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u}\cdot 1 + \frac{\partial z}{\partial v}ye^x$$

$$= \frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v} \qquad \text{.....(3)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

$$= \frac{\partial z}{\partial u} + e^x\frac{\partial z}{\partial v} \qquad \text{.....(4)}$$

$$r = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial^2 z}{\partial u^2} + 2v\frac{\partial^2 z}{\partial u\,\partial v} + v^2\frac{\partial^2 z}{\partial v^2} + v\frac{\partial z}{\partial v} \qquad \text{..... (5)}$$

$$s = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u} + e^x\frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + e^x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right) + e^x\frac{\partial z}{\partial v}$$

$$= \left(\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}\right)\frac{\partial z}{\partial u} + e^x\left(\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}\right)\frac{\partial z}{\partial v} + e^x\frac{\partial z}{\partial v}$$

$$= \frac{\partial^2 z}{\partial u^2} + (e^x + v)\frac{\partial^2 z}{\partial u\,\partial v} + ve^x\frac{\partial^2 z}{\partial v^2} + e^x\frac{\partial z}{\partial v} \qquad \text{..... (6)}$$

and

$$t = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u} + e^x \frac{\partial z}{\partial v}\right)$$

$$= \left(\frac{\partial}{\partial u} + e^x \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} + e^x \frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial^2 z}{\partial u^2} + 2e^x \frac{\partial^2 z}{\partial u\,\partial v} + e^{2x} \frac{\partial^2 z}{\partial v^2} \quad \text{..... (7)}$$

Using (3), (4), (5), (6) and (7) in (1), we get

$$(1-y)^3 e^x \frac{\partial^2 z}{\partial u\,\partial v} = 2ye^{2x}(1-y)^3$$

$$\Longrightarrow \frac{\partial^2 z}{\partial u\,\partial v} = 2v \quad \text{.....(8)}$$

which in the canonical form of (1)

Intergrating (6) w. r. to v, $\frac{\partial z}{\partial u} = v^2 + f_1(u)$ (9)

Intergrating (9) w. r. to u, $z = uv^2 + f_1(u) + f_2(v)$

$= (x+y)y^2e^{2x} + f_1(x+y) + f_2(ye^x)$

Which is the general solution of (1), where ϕ_1 and ϕ_2 are arbitrary functions.



CHECK YOUR PROGRESS

Q.1. Reduce the following partial diffential equation in canonical formand solve :

$$xy\frac{\partial^2 z}{\partial x^2} - xy\frac{\partial^2 z}{\partial y^2} - (x^2 - y^2)\frac{\partial^2 z}{\partial x\,\partial y} + py - qx = 2(x^2 - y^2)$$

14.4 OCCURRENCE OF THE WAVE EQUATIONS

(a) Transverse Vibrations of a string : If a string of uniform linear density ρ is stretched to a uniform tension T and if, in the equilibrium positin, the string coincides with the x-axis, then when the string is disturbed slightly from its equilibrium position, the trabsverse displacement y (x,t) satisfies the one dimensional wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \quad \text{.....(1)}$$

where c2 = $T/_P$. At any point x=a of the string which is fixed y (a,t) = 0 for all values of t.

(b) Longitudinal vibrations in a bar : If a uniform bar of elastic material of uniform cross section whose axis coinsides with ox is stressed in such a way that each point of a typical cross section of the bar take the same displacement u(x,t), then

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \quad \text{.....(2)}$$

where c^2 = $E/_P$, E being the Young's modulus and ρ the density of the material of the bar. The stress at any point of the bar is

$$\sigma = E\frac{\partial u}{\partial x} \quad \text{.....(3)}$$

For instance, suppose that the velority of the end x=0 of the bar $0 \le x \le a$ is prescribed to be v(t), say, and that the other end x=a is free from stress.Suppose further that at time t=0 the bar is at rest. Then the longitudinal displacement of sections of the bar are determined by the partial differential equation (2) and the boundary and initial conditions

$$\text{(i)} \quad \frac{\partial u}{\partial t} = v(t) \qquad \text{for } x = 0$$

$$\text{(ii)} \quad \frac{\partial u}{\partial x} = 0 \qquad \text{for } x = a$$

$$\text{(iii)} \quad u = \frac{\partial u}{\partial t} = 0 \quad \text{at} \quad t = 0 \text{ for } 0 \le x \le a$$

(C)Longitudinal Sound Waves : If plane waves of sound are being propagated in a horn whose cross section for the section with abscissa x is A(x) in such a way that every point of that section has the same longitudinal displacement u(x,t), then u satisfies the partial differential equation

$$\frac{\partial}{\partial x}\left\{\frac{1}{A}\frac{\partial}{\partial x}(Au)\right\} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \quad \text{.....(4)}$$

which reduces to the one-dimensional wave equation (2) in the case in which the cross section is uniform. In equation (4),

$$c^2 = \left(\frac{dp}{dP}\right)_0 \quad \text{.....(5)}$$

where the suffix 0 denotes that we take the value of $\frac{dp}{dp}$ in the equilibrium state. The change in pressure in the gas from the euilibrium value p_0 is given by the formula

$$p - p_0 = -c^2\rho_0\frac{\partial u}{\partial x} \quad \text{.....(6)}$$

where $\rho 0$ is the density of the gas in the equilibrium state. For instance, if we are considering the motion of the gas when a sound wave passes along a tube which is free at each of the ends x=0, x=a, then we must determine solutions of equation (4) which are such that

$\frac{\partial u}{\partial x}$ = 0 at x = 0 and at x = a.

(d) Electric signals in cables : If the resistance per unit length, R and the leakage parameter G are both zero, the voltage V(x,t) and the current z(x,t) both satisfy the one-dimensional wave equation with wave velocity c defined by the equation

$$c^2 = \frac{1}{Lc} \quad \text{.....(7)}$$

where L is the inductance and C the capacity per unit length.

(e) Transverse Vibrations of a Membrane : If a thin elastic membrance of uniform areal density σ is stretched to a uniform tension T, and if, in the equilibrium position, the membrane coincides with the xy plane, then the small transverse vibration of the membrane are governed by the wave equation

$$\nabla^2 z = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2} \quad \text{.....(8)}$$

where z(x,y,t) is the transverse displacement (assumed small) at time t of the point (x,y) of the membrance. The wave velocity c is defined by the

equation

$$c^2 = \frac{T}{\sigma} \quad \text{..... (9)}$$

If the membrane is held fixed at its boundary ⌈, the we must have z=0 on ⌈ for all values of t.

(f) Sound Waves in space : Suppose that because of the passage of a sound wave the gas at the point (x,y) at time t has velocity v=(u,v,w) and that the pressure and density there and then are p, p respectively; then if p_0, p_0 are the corresponding values in the equilibrium state we may write

$$\rho = \rho_0 (1+s),\ p = p_0 + c^2 p_0 s \quad \text{.....(10)}$$

where s is called the condensation of the gas and c2 is given by equation (5). If we substitute these expressions in the equations of motion

$$\rho \frac{Dv}{dt} = -\text{grad}\, \rho \quad \text{.....(11)}$$

where $\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$

and restrict ourselves to small oscillations of the gas, we find that

$$\rho \frac{\partial v}{\partial t} = -c^2 \rho_0 \text{ grad } S \quad \text{.....(12)}$$

Similarly, the continuity equation

$$\frac{D\rho}{Dt} + \rho \text{ div } v = 0$$

is equivalent, in this approximation, to the equation

$$\frac{\partial s}{dt} \rho_0 + \rho \text{ div } v = 0 \quad \text{.....(13)}$$

If the motion of the gas is irrotational, then there exists a sealar function ϕ with the property

$$v = -\text{grad } \phi \quad \text{.....(14)}$$

Substituting from equation (14) into equation (12), we find that for small oscillations

$$\text{grad}\left(\frac{\partial \phi}{\partial t} - c^2 s\right) = 0 \quad \text{.....(15)}$$

Similarly, equation (13) is equivalent to

$$\frac{\partial s}{\partial t} = \nabla^2 \phi \quad \text{.....(16)}$$

Eliminating s between equations (15) and (16), we find that f satisfies the wave equation

$$\nabla^2\phi = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} \quad \text{.....(17)}$$

14.5 SOLUTIONS OF ONE-DIMENSIONAL, TWO-DIMENSIONAL AND THREE-DIMENSIONAL WAVE EQUATIONS

14.5.1. Solution of one-dimensional wave equation : Consider the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\cdot\frac{\partial^2 u}{\partial t^2} \quad \text{.....(1)}$$

Let solution of (1) be of the form

$$u(x,t) = X(x)\,T(t) \quad \text{.....(2)}$$

From (1) and (2), $X'' T = \left(\frac{1}{c^2}\right) XT''$

$$\Rightarrow \frac{X''}{X} = \frac{T''}{c^2 T}$$

As usual $X'' - \mu x = 0$ .....(3)

$$T'' - c^2 \mu\, T = 0 \quad \text{.....(4)}$$

Solving (3) and (4), we obtain

(i) when μ = 0, then $X = a_1 x + a_2$, $T = a_3 t + a_4$

(ii) when μ is –ve and $\mu = +\lambda^2$ (say), then

$$X = b_1 e^{\lambda x} + b_2 e^{-\lambda x},\ T = b_3 e^{c\lambda t} + b_4 e^{-c\lambda t}$$

(iii) when μ is positive and $\mu = -\lambda^2$ (say), then

$$X = c_1 \cos \lambda x + c_2 \sin \lambda x,\ T = c_3 \cos cpt + c_4 \sin cpt$$

Thus the various possible solutions are

$$u(x,t) = (a_1 x + a_2)(a_3 t + a_4) \quad \text{.....(5)}$$

$$u(x,t) = (b_1 e^{\lambda x} + b_2 e^{-\lambda x})(b_3 e^{c\lambda t} + b_4 e^{-c\lambda t}) \quad \text{.....(6)}$$

$$u(x,t) = (c_1 \cos \lambda x + c_2 \sin \lambda x)(c_3 \cos cpt + c_4 \sin cpt) \quad \text{.....(7)}$$

Now we have to choose that solution which is constant with the physical nature of the problem. As we will be dealing with problems on vibrations, u(x,t) must be a periodic function of t. Hence u(x,t) must involve trigonometric terms. Accordingly, the solution given by (7) is the only suitable solution.



CHECK YOUR PROGRESS

Q. 2. (a) Obtained the general solution of the one-dimension wave equation $\frac{\partial^2 u}{\partial x^2} = \left(\frac{1}{c^2}\right)\frac{\partial^2 u}{\partial t^2}$ and obtained the particular solution for which u = f(x), $\frac{\partial u}{\partial t}$ = g(x) at t = 0.

(b) Find the deflaction u(x,t) of vibrating string (length = π and c2 = 1) corresponding to zero initial velocity and initial deflection f(x) = K (sin x – sin 2x).

14.5.2 Solution of two-dimension wave equation

Consider the equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] \quad \text{.....(1)}$$

subject to boundary conditions

u(0,y,t) = 0	2(x)
u(a,y,t) = 0	2(x′)
u(x,0,t) = 0	2(y)
u(x,b,t) = 0	2(y′)

and subject to initial conditions

u (x,y,0) = f(x,y)	3(t)
$\left[\frac{\partial u}{\partial t}\right] = g(x,y)$	3(t′)

Suppose that (1) has solutions of the form

u (x,y,t) = X(x) Y(y) T(t) .....(4)

Substituting in (1), we obtain

$$X'' YT + XY'' T = \frac{1}{c^2} XYT''$$

$$\Rightarrow \frac{X''}{X} + \frac{Y''}{Y} = \frac{T''}{CT} \qquad \text{.....(5)}$$

Since x, y and t are independent variables, (5) can only be true if each term on each side equal to a constant.

Let $\frac{X''}{X} = \mu_1$ so that $X'' - \mu_1 x = 0$

using 2(x) and 2(x1), (4) gives

$$X(0) Y(y) T(t) = 0 \text{ and } X(a) Y(y) T(t) = 0 \qquad \text{.....(7)}$$

We assume that $Y(y) \neq 0$ and $T(t) \neq 0$. So by (7), we get

$$X(0) = 0 \text{ and } X(a) = 0 \qquad \text{.....(8)}$$

Now we solve (6) under boundary condition (8). Three cases arise:

Case I. Let $\mu_1 = 0$. Then solution of (6) is

$$X(x) = Ax + B \qquad \text{.....(9)}$$

Using boundary condition (8), (9) gives 0=B and 0= Aa+B. These gives A = B = 0 and hence X(x) = 0. This leads u = 0 which does not satisfy 3(t) and 3(t′). So we reject $\mu_1 \neq 0$

Case II. Let $\mu_1 = \lambda_1^2$, $\lambda_1 \neq 0$. Then solution of (6) is

$$X(x) = Ae^{\lambda_1 x} + Be^{-\lambda_1 x} \qquad \text{.....(10)}$$

Using boundary condition (8), (10) gives

$$0 = A + B \text{ and } 0 = Ae^{a\lambda_1} + Be^{-a\lambda_1} \qquad \text{.....(11)}$$

Solving (11), A = B = 0 so that X(x) = 0. This leads to u = 0. So we reject $\mu_1 = \lambda_1^2$.

Case III. Let $\mu_1 = -\lambda_1^2$, $\lambda_1 \neq 0$. Then solution of (6) is

$$X(x) = A \cos \lambda_1 x + B \sin \lambda_1 x \qquad \text{.....(12)}$$

Using boundary condition (8), (12) gives

$$0 = A \text{ and } 0 = A \cos \lambda_1 a + B \sin \lambda_1 a$$

$$\Rightarrow A = 0 \text{ and } \sin \lambda_1 a = 0$$

where we have taken $B \neq 0$, since otherwise X(x) = 0 so that u = 0 which does not satisfy 3(t) and 3(t′).

Now $\sin \lambda_1 a = 0 \Rightarrow \lambda = \frac{m\pi}{a}, m = 1,2,$ ..... (13)

Hence non zero solutions $X_m(x)$ of (6) are given by

$$X_m(x) = B_m \sin \frac{m\pi x}{a}, m = 1,2, \quad \text{..... (14)}$$

Next, let $\frac{Y''}{Y} = u_2$ so that $Y'' - u_2 Y = 0$ ..... (15)

Using 2 (y) and 2 (y^1), (4) gives as before

Y (0) = 0 and Y (b) = 0 ..... (16)

Solving (15) under boundary condition (16) as before, we get

$$Y_n(y) = D_n \sin\left(\frac{n\pi y}{b}\right) n = 1,2, \ldots \ldots \text{ (17)}$$

where $\mu_2 = -\lambda_2^2$ and $\lambda_2 = \frac{n\pi}{b}, n = 1,2,3, \ldots \ldots$ (18)

In view of (6) and (15), (5) reduces to

$$\frac{T''}{c^2 T} = u_1 + u_2 = -\lambda_1^2 - \lambda_2^2 = -\pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right) \quad \text{.....(19)}$$

$$\Rightarrow T'' + \lambda_{mn}^2 T = 0$$

where $\lambda_{mn}^2 = c^2 \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)$ .....(20)

Solving (19), $T_{mn}(t) = E_{mn} \cos \lambda_{mn} t + F_{mn} \sin \lambda_{mn} t$ .....(21)

Therefore

$$U_{mn}(x, y, t) = X_m(x) Y_n(y) T_{mn}(t)$$

$$= (A_{mn} \cos \lambda_{mn} t + B_m \sin \lambda_{mn} t) \sin\left(\frac{m\pi x}{a}\right) \cdot \cos\left(\frac{n\pi y}{b}\right) \quad \text{....(22)}$$

are solutions of (1) satisfying 2 (x), 2 (x1), 2 (y) and 2(y^1). Here $A_{mn} (= B_m D_n E_{mn})$ and $B_{mn} (= B_m D_n F_{mn})$ are new arbitrary constants. In order to obtain a solution also satisfying 3(t) and 3(t¢), we consider more general solution, i.e.,

$$u(x,y,t)=\sum_{m=1}^{\alpha}\sum_{n=1}^{\varepsilon}(A_{mn}\cos\lambda_{mn}{}^{t}+B_{mn}\sin\lambda_{mn}{}^{t})\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right).....(23)$$

Differentiating (23) partially w. r. to t, we get

$$\frac{\partial u}{\partial t}=\sum_{m=1}^{\alpha}\sum_{n=1}^{\alpha}(-A_{mn}\lambda_{mn}\sin\lambda_{mn}t+B_{mn}\lambda_{mn}\cos\lambda_{mn}t)\cdot\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\quad .....(24)$$

Putting t = 0 in (23) and (24) and using 3(t) and 3(t′), we get

$$f(x,y)=\sum_{m=1}^{\alpha}\sum_{n=1}^{\alpha}A_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\quad .....(25)$$

and $g(x,y)=\sum_{m=1}^{\alpha}\sum_{n=1}^{\alpha}B_{mn}\lambda_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$ .....(26)

which are double Fourier sine series. According, we get

$$A_{mn}=\frac{4}{ab}\int_{x=0}^{a}\int_{y=0}^{b}f(x,y)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dxdy\quad ..... (27)$$

and $B_{mn}\lambda_{mn}=\frac{4}{ab}\int_{x=0}^{a}\int_{y=0}^{b}g(x,y)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dxdy$

i.e. $B_{mn}=\frac{4}{ab\lambda_{mn}}\int_{x=0}^{a}\int_{y=0}^{b}g(x,y)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dxdy.....(28)$

Thus the desired deflection u (x, y, t) of the given membrane is given by (23) wherein Amn and Bmn are given by (27) and (28) respectively.

If g (x, y) = $\left[\frac{du}{dt}\right]_{t=0}$ = 0, then from (28), Bmn = 0. So from (23), the required solution is

$$u(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\cos\lambda_{mn}t\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\quad ..... (29)$$

where Amn are given by (27).

From equation (20), $\lambda_{mn}=c\pi\sqrt{m^2/a_2+n^2/b_2}$ for m, n = 1, 2, 3, ... are called eigenvalues and the function umn (x, y, t) given by (22) are (22) are called the corresponding eigen functions.



CHECK YOUR PROGRESS

Q.3. Find the deflection u (x,y,t) of the square membrane of each side unity and c=1, if the initial velocity is zero and initial deflection is f(x,y) = A sin πx sin 2πy.

14.5.3. Solution of three-dimension wave equation :

Consider the three-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = \rho^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \quad \text{.....(1)}$$

Let u (x, y, z, t) = X(x) Y(y) Z(z) T(y) .....(2)

be a solution of (1).

From (1),

$$\frac{1}{\rho^2} XYZT'' = X'' YZT + XY'' ZT + XYZ''T$$

$$\Rightarrow \frac{1}{\rho^2} \frac{T''}{T} = \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} \quad \text{.....(3)}$$

This is true only when each member is a constant,

$$\frac{d^2X}{dx^2} + k_1^2 X = 0, \frac{d^2Y}{dy^2} + k_2^2 Y = 0, \frac{d^2Z}{dz^2} + k_3^2 Z = 0$$

and $$\frac{d^2T}{dt^2} + (k_1^2 + k_2^2 + k_3^2)T = 0 \quad \text{..... (4)}$$

The solutions of equation (4) are

$$X = c_1 \cos k_1 x + c_2 \sin k_1 x$$

$$Y = c_3 \cos k_2 x + c_4 \sin k_2 x \quad \text{.....(5)}$$

$$Z = c_5 \cos k_3 x + c_6 \sin k_3 x$$

and $T = c_7 \cos \sqrt{k_1 + k_2 + k_3}\, \rho t + c_8 \sin \sqrt{k_1 + k_2 + k_3} \rho t$

The solutions of equation (1)

$$U(x, y, z, t) = (c_1 \cos k_1 x + c_2 \sin k_1 x)(c_3 \cos k_2 x + c_4 \sin k_2 x)(c_5 \cos k_3 x + c_6 \cdot$$

$$\sin k_3 x \left(c_7 \cos \sqrt{k_1 + k_2 + k_3}\, \rho t + c_8 \sin \sqrt{k_1 + k_2 + k_3}\, \rho t \right) \quad \text{.... (6)}$$

Now let us suppose that membrane is cuboids and stretched among the line x = 0, x = a, y = 0, y = b, z =0, z = c.

Then the boundary conditions are

(i) u(0,y,z,t) = 0 (ii) u(a,y,z,t) = 0 (iii) u(x,0,z,t) = 0

(iv) u(x,b,z,t) = 0 (v) u(x,y,0,t) = 0 (vi) u(x,y,c,t) = 0

Applying condition (i) and (ii), we have

$$c_1 = 0 \text{ and } k_1 = \frac{l\pi}{a}, l = 1, 2, ...$$

Applying conditions (iii) and (iv),

we have $c_3 = 0 \text{ and } k_2 = \frac{m\pi}{b}$

Appying conditions (v) and (vi),

we have $c_5 = 0 \text{ and } k_3 = \frac{m\pi}{c}$,

Where n = 1, 2, 3, ...

Thus

$$u(x,y,z,t) = c_2c_4c_6 \sin\left(\frac{l\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \cdot \sin\left(\frac{n\pi z}{c}\right) \left(c_7 \cos\sqrt{k_1 + k_2 + k_3}\cdot \rho t + c_8 \sin\sqrt{k_1 + k_2 + k_3}\cdot \rho t\right) \quad .....(7)$$

$$\Rightarrow U(x,y,z,t) = c_2c_4c_6 \sin\left(\frac{l\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \sin\left(\frac{n\pi c}{z}\right) \cdot \left(c_7 \cos\sqrt{\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}}\,\pi\rho t + c_8 \sin\sqrt{\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}}\,\pi\rho t\right)$$

$$= c_2c_4c_6 \sin\left(\frac{l\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \sin\left(\frac{n\pi c}{z}\right) (c_7 \cos\rho t + c_8 \sin\rho t) \quad .....(8)$$

$$\text{where } p = \sqrt{\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}}\,\pi\rho$$

Therefore the more general solution is

$$u = \sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{n\pi z}{c}\right)$$

$$(A_{lmn}\cos\rho t + B_{lmn}\sin\rho t) \qquad \text{..... (9)}$$

Suppose the membrances start from the rest from the initial position

u (x,y,z,0) = f (x,y,z)

If t = 0, then

$$u(x,y,z,0) = f(x,y,z) = \sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{lmn}\sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{n\pi z}{c}\right)\cdot\cos\rho t \text{.....(10)}$$

Using Fourier series

$$A_{lmn} = \frac{8}{abc}\int_0^a\int_0^b\int_0^c f(x,y,z)\sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{n\pi c}{z}\right)\,dxdydz$$

Thus the required solution of 3-D wave equation is

$$u(x,y,z,t) = \sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{lmn}\sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{n\pi z}{c}\right)$$

$$\cdot\cos\sqrt{\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}}\,\pi\rho t$$

where

$$A_{lmn} = \frac{8}{abc}\int_0^a\int_0^b\int_0^c f(x,y,z)\sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{n\pi z}{c}\right)$$

$$\cdot\,dxdydz$$

CHECK YOUR PROGRESS



Q.4: Solve the three dimensional wave equation in spherical polar coordinates.

14.6 HOMOGENEOUS WAVE EQUATIONS, DUHAMEL'S PRINCIPLE AND METHOD OF DESCENT

14.6.1. Non homogeneous wave equations and Duhamel's Principle.

The second-order hyperbolic equation

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \nabla^2 u = f(x,t)$$

which arises in electromagnetic theory and other branches of mathematical physics is called the non homogeneous wave equation.

Now we consider the non homogeneous wave equation on the real line

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = f(x,t)$$

subject to the initial conditions :

u (x,0) = g (x), ut (x, 0) = h (x)

Solution of the non homogeneous equarion can be represented as a sum of two other solutions :

Problem I. The non homogeneous wave equation

$$\frac{\partial^2 v}{\partial t^2} - c^2\frac{\partial^2 v}{\partial x^2} = f$$

with homogeneous initial conditions

v(x,0) = 0 and vt(x,0) = 0

Problem II. The homogeneous wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0$$

with non homogeneous initial conditions

u(x,0) = g(x) and ut(x,0) = h(x)

Thus, it suffices onto to consider the first problem. We apply the method due to Duhamel.

Consider an auxiliary problem

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0 \qquad \text{for} \quad x \in \mathbb{R}, t > 0$$

$$U(x,0,s) = 0,\ U_t(x,0,s) = f(x,s) \text{ for } x \in IR,\ s > 0.$$

where f (x,s) is the right hand side in our equation given aove.

Duhamel's principle. Assume that U(x,t,s) is a c^2– function of $x \in IR$ and $t > 0$, continnous in s, s>0. If U solves the above auxiliary problem, then solution of the problem I is given by

$$u(x,t) = \int_0^t U(x, t-s, s)\,ds$$

Proof: Apply differentiation with respect to parameter,

$$\frac{\partial}{\partial t}\int_{a(t)}^{b(t)} F(x,t)dt = F(b(t),t)b'(t) - F(a(t),t)a'(t) + \int_{a(t)}^{b(t)} \frac{\partial}{\partial t}F(x,t)dt$$

This formula holds if F and Ft are continuous. In our case

F(x,t) = U(x, t–s, s), thus

$$U_t(x,t) = U(x,t-t,t)\cdot 1 - U(x,t-0,0)\cdot 0 + \int_0^t \frac{\partial}{\partial t}U(x,t-s,s)ds$$

and applying U (x,0,s) = 0, we find

$$U_t(x,t) = \int_0^t U_t(x,t-s,s)\,ds$$

Differentiate again and apply the second initial condition :

$$U_{tt}(x,t) = U_t(x,0,t) + \int_0^t U_{tt}(x,t-s,s)ds$$

$$= f(x,t) + \int_0^t U_{tt}(x,t-s,s)ds$$

Differentiation w. r. to x yields

$$U_{xx}(x,t) = \int_0^t U_{xx}(x,t-s,s)ds$$

$$U_{tt} - c^2U_{xx} = f(x,t) + \int_0^t [U_{tt}(x,t-s,s) - c^2U_{xx}(x,t-s,s)]ds$$

$$= f(x,t)$$

Corollarly : The solution of problem I is given by the following explicit formula

$$U(x,t) = \frac{1}{2c}\int_0^t ds \int_{x-c(t-s)}^{x+c(t-s)} f(r,s)dr$$

Example. Find the solution of

$$u_{tt} - u_{xx} = x+t$$

with the initial condition u(x,0) = x, u_t (x,0) = 1.

Solution. We have u = u_1 + u_2 where

u_1 is solution to $u_{tt} - u_{xx}$ = x+t, u(x,0) = 0, u_t (x,0) = 0

u_2 is solution to $u_{tt} - u_{xx}$ = 0, u(x,0) = x, u_t (x,0) = 1

We find u_1 by Duhamel's principle,

$$U_1(x,t) = \frac{1}{2}\int_0^t ds \int_{x-(t-s)}^{x+(t-s)} (r+s)dr$$

$$= \frac{1}{2}\int_0^t (t-s)(s+x)ds$$

$$= \frac{t^2(t+3x)}{6}$$

Similarly, applying D' Alembert's formula to g = x, h = 1, we find u_2 :

$$u_2(x,t) = \frac{x+t+x-t}{2} + \frac{1}{2}\int_{x-t}^{x+t} 1ds = x+t$$

Thus the solution is

$$U = U_1 + U_2 = x + t + \frac{t^2(t+3x)}{6}$$

D'Alembert's formula : Consider the one-dimensional wave equation

$u_{tt} = c^2 u_{xx}$ for $-\alpha < x < \alpha$ and $t > 0$

$$u(x,0) = f(x)$$

$$u_t(x,0) = g(x)$$

Then its general solution is

$$U(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds$$



CHECK YOUR PROGRESS

Q.5. Find the solution of the following problem

$$u_{tt} - u_{xx} = te^x,\ (x,t) \in IR \times (0,x),$$

$$u(x,0) = 0$$

$$u_t(x,0) = 0$$

14.6.2. Method of Descent

In this section, we use Kirchoff's formula for the solution of the wave equation in three dimensions to derive the solution of the wave equation in two dimension. This technique is known as method of Descent. This technique is used in general to find the solution of the wave equation in even dimensions, using the solution of the wave equation in odd dimensions.

Suppose u is a solution of the initial value problem for the wave equation in two dimenssions,

$$U_{tt} - \Delta u = 0\ ,\ u \in IR^2\ ,\ t \geq 0$$

$$u(x,0) = \phi(x)$$

$$u_t(x,0) = \psi(x)$$

We will find a solution in the 2 - D case, by using the solution to the 3 - D problem. Let u (x_1, x_2, t) be the solution to the 2 - D problem. Define

$$\tilde{u}(x_1, x_2, x_3, t) \equiv u(x_1, x_2, t)$$

$$\tilde{u}(x_1, x_2, x_3, 0) \equiv u(x_1, x_2, 0) = \phi(x_1, x_2)$$

Therefore, $\tilde{u}_t(x_1, x_2, x_3, 0) \equiv u(x_1, x_2, 0) = \psi(x_1, x_2)$

Clearly, $\tilde{u}(x_1, x_2, t)$ is a solution of 3D wave equation with initial data $\phi(x_1, x_2)$ and $\psi(x_1, x_2)$.

$$\tilde{u}_{tt} - \tilde{u}_{x_1x_1} - \tilde{u}_{x_2x_2} - \tilde{u}_{x_3x_3} = 0$$

$$\tilde{u}(x_1, x_2, x_3, 0) = \tilde{\phi}(x_1, x_2, x_3) = \phi(x_1, x_2)$$

$$\tilde{u}_t(x_1, x_2, x_3, 0) = \tilde{\Psi}(x_1, x_2, x_3) = \Psi(x_1, x_2)$$

Now we can solve the 3D wave equation using Kirchoff's formula. In particular, our solution is given by

$$\tilde{u}(x_1, x_2, 0, t) = \int_{\partial \overline{B}(\overline{x},t)} \left[\tilde{\phi}(y) + \nabla\tilde{\phi}(y) \cdot (y - x) + t\tilde{\psi}(y)\right] ds(y)$$

where $\partial\overline{B}(\overline{x},t)$ is the boundary of $\overline{B}(\overline{x},t)$ is the ball of radius t in IR^3 about the point $\overline{x} = (x_1, x_2, 0)$. Now we note that

$$\int_{\partial \overline{C}(\overline{x},t)} \tilde{\Phi}(y) ds(y) = \frac{1}{2\pi t^2} \int_{B(x,t)} \Phi(y)[1 + |\nabla\gamma(y)|^2]^{\frac{1}{2}} dy$$

where B (x,t) is the ball in IR^2 of radious t about x = (x1, x2) and $\gamma(y) = \sqrt{t^2 - |y - x|^2}$. Therefore

$$\nabla\gamma(y) = -\frac{y - x}{\sqrt{t^2 - |y - x|^2}}$$

Which impliles

$$\sqrt{1 + |\nabla\gamma(y)|^2} = \left[\frac{t^2}{t^2 - |y - x|^2}\right]^{\frac{1}{2}}$$

Therefore

$$\int_{\partial \overline{B}(\overline{x},t)} \tilde{\phi}(y)\, ds(y) = \frac{1}{2\pi t^2} \int_{B(x,t)} \frac{t\phi(y)}{[t^2 - |y - x|^2]^{\frac{1}{2}}} dy$$

Similarly,

$$\int_{\partial \overline{B}(\overline{x},t)} t\tilde{\psi}(y) ds(y) = \frac{1}{2\pi t^2} \int_{B(x,t)} \frac{t^2\psi(y)}{[t^2 - |y - x|^2]^{1/2}} dy$$

and

$$\int_{\partial \overline{B}(\overline{x},t)} \nabla\tilde{\phi}(y)\cdot(y-x)ds(y) = \frac{1}{2\pi t^2}\int_{B(x,t)} \frac{t\nabla\phi(y)\cdot(y-x)}{[t^2-|y-x|^2]^{1/2}}\,dy$$

Therefore, the solution of the initial value problem for the wave equation in IR2 (with c = 1) is given by

$$u(x,t) = \frac{1}{2\pi t^2}\int_{B(x,t)} \frac{t\phi(y)+t^2\psi(y)+t\nabla\phi(y)\cdot(y-x)}{[t^2-|y-x|^2]^{1/2}}\,dy$$

Again, by making a change of variables, we see that the solution of the wave equation in two dimension is given by

$$u(x,t) = \frac{1}{2\pi c^2 t^2}\int_{B(x,c,t)} \frac{ct\phi(y)+ct^2\psi(y)+ct\nabla\phi(y)\cdot(y-x)}{[c^2t^2-|y-x|^2]^{1/2}}\,dy$$

Kirchoff's formula : Consider the initial value problem for the wave equation in IR3

$$U_{tt} - c^2\Delta u = 0\ ,\ x \in IR^3$$

$$u(x,0) = \phi(x)$$

$$u_t(x,0) = \psi(x)$$

For c = 1 and ϕ smooth, the solution of the IVP for the wave equation in IR^3 is given by

$$u(x,t) = \frac{1}{4\pi t^2}\int_{\partial B(x,t)} [\phi(y)+\nabla\phi(y)\cdot(y-x)+t\psi(y)]ds(y)$$

where ∂B (x,t) is boundary of ball of radious t about x.

This is known as Kirchoff's formula for the solution of the IVP for the wave equiation in IR3.



14.7 LET US SUM UP

- Hyperolic partial differential equation : The partial differential equation of 2nd degree of two independent variales x and y,

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0$$

where R, S and T are conrinous functions of x and y only possessing partial derivaties in some domain D on the xy-plane and $r = \frac{\partial^2 z}{\partial x^2}, s = \frac{\partial^2 z}{\partial x \partial y}$ and $t = \frac{\partial^2 z}{\partial y^2}$; is called hyperolic differential equation if $S^2 - 4RT > 0$. One-dimensional wave equation :

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

Two -dimensional wave equation :

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

Three-dimensional wave equation :

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

The solution of the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

is $u(x,t) = (c_1 \cos \lambda x + c_2 \sin \lambda x)(c_3 \cos cpt + c_4 \sin cpt)$

- The solution of two-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y}\right)$$

subject to boundary contitions

u(a,y,t) = 0, u(a,y,t) = 0, u(x,o,t) = 0, u(x,b,t) = 0

and subject to initial conditions

$u(x,y,0) = f(x,y)$, $u_t(x,y,0) = g(x,y)$

is

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(A_{mn}\cos\lambda_{mn}t + B_{mn}\sin\lambda_{mn}t)\sin\left(\frac{m\pi x}{a}\right)\cdot\sin\left(\frac{n\pi y}{b}\right)$$

where

$$A_{mn} = \frac{4}{ab}\int_0^a\int_0^b f(x,y)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dxdy$$

$$B_{mn} = \frac{4}{ab\lambda_{mn}}\int_0^a\int_0^b g(x,y)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dxdy$$

- The solution of three-dimentional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

where the boundary condition

$$u(0,y,z,t) = 0,\ u(a,y,z,t) = 0,\ u(x,0,z,t) = 0$$

$$u(x,b,z,t) = 0,\ u(x,y,0,t) = 0,\ u(x,y,c,t) = 0$$

and under the initial conditin

$u(x,y,z,0) = f(x,y,z)$

is

$$u(x,y,z,t) = \sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{lmn}\sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{n\pi z}{c}\right)\cos\sqrt{\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}}\,\pi pt$$

where

$$A_{lmn} = \frac{8}{abc}\int_o^a \int_o^b \int_o^c f(x,y,z)\sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{n\pi z}{c}\right) \cdot dxdydz$$

- The second order hyperbolic equation

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \nabla^2 u = f(x,t)$$

which arises in electromagnetic theory and the other branch of mathematical Physics is called non homogenencens wave equation.

- Duhamel's Principle : Consider the non homogeneous wave equation on the real line

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = f(x,y)$$

subject to the initial conditions

$u(x,0) = g(x)$, $u_t(x,0) = h(x)$

Then solution of the nonhomogeneous equation represented as a sum of the following problems :

(I) The nonhomogeneous wave equation

$$\frac{\partial^2 v}{\partial t^2} - c^2\frac{\partial^2 v}{\partial x^2} = f(x)$$

with the homogeneous initial condition

$v(x,o) = 0$ and $v_t(x,o) = 0$

(II) The homogeneous wave eqution

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0$$

with non homogeneous initial conditions

$u(x,0) = g(x)$ and $u_t(x,0) = h(x)$

consider an auxiliary problem

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{for} \quad x \in \mathbb{R}, t > 0$$

u(x,0,s) = 0, u_t(x,0,s) = f(x,s) for all $x \in IR$, s > o and f(x,s) is the right hand side in our equation given above.

Assume that u(x,t,s) is a C^2 - function of $x \in IR$ and t > o, continuous in s, s > o. If u solves the above auxiliary prolem, then solution of the problem (I) is given by $u(x,t) = \int_0^t U(x,t-s,s)\, ds$

14.8 ANSWER TO CHECK YOUR PROGRESS

Ans to Q. No. 1:

Given partial differential Equation is

$$xy\frac{\partial^2 z}{\partial x^2} - (x^2 - y^2)\frac{\partial^2 z}{\partial x \partial y} - xy\frac{\partial^2 z}{\partial y^2} + py - qx = 2(x^2 - y^2) \quad .....(1)$$

comparing equation (1) with Rs + Ss + Tt + f (x,y,z,p,q) = 0 we get R = xy, S = – (x2 – y2), T = – xy

Now $S^2 - 4RT = (x^2-y^2)^2+4x^2y^2 = (x^2+y^2)^2 > 0$ for $x \neq 0, y \neq 0$

Therefore given equation is hyperbolic. The λ– quadratic is

$$R\lambda^2 + S\lambda + T = 0$$

$$\Rightarrow \quad xy\lambda^2 - (x^2-y^2)\lambda - xy = 0$$

$$\Rightarrow \quad \lambda = -\frac{y}{x}, \frac{y}{x}$$

Hence

$$\frac{dy}{dx} + \lambda_1 = 0 \quad \text{and} \quad \frac{dy}{dx} + \lambda_2 = 0 \quad \text{becomes}$$

$$\frac{dy}{dx} - \frac{y}{x} = 0 \quad \text{and} \quad \frac{dy}{dx} + \frac{y}{x} = 0$$

Integrating,

$$y/x = c_1 \quad \text{and} \quad x^2 + y^2 = c_2$$

So, we let $u = \frac{y}{x}$ and $v = x^2 + y^2$

Then

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x} = -\frac{y}{x^2} \frac{\partial z}{\partial u} + 2x \frac{\partial z}{\partial v}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y} = \frac{1}{x} \frac{\partial z}{\partial u} + 2y \frac{\partial z}{\partial v}$$

$$r = \left(-\frac{y}{x^2}\right)^2 \cdot \frac{\partial^2 z}{\partial u^2} + 2.2x. \left(-\frac{y}{x^2}\right) \frac{\partial^2 z}{\partial v \, \partial u} + 4x^2 \frac{\partial^2 z}{\partial v^2} + \frac{2y}{x^3} \frac{\partial z}{\partial u} + 2 \frac{\partial z}{\partial v}$$

$$s = \left(-\frac{y}{x^2}\right) \frac{1}{x} \frac{\partial^2 z}{\partial u^2} + \left\{2y\left(-\frac{y}{x^2}\right) + 2x \cdot \frac{1}{x}\right\} \frac{\partial^2 z}{\partial u \, \partial v} + 4xy \frac{\partial^2 z}{\partial v^2} - \frac{1}{x^2} \frac{\partial z}{\partial u}$$

$$t = \left(\frac{1}{x}\right)^2 \frac{\partial^2 z}{\partial u^2} + 2 \cdot \frac{1}{x} \cdot 2y \frac{\partial^2 z}{\partial u \, \partial v} + 4y^2 \frac{\partial^2 z}{\partial v^2} + 2 \frac{\partial z}{\partial v}$$

Substituting in the given equation, we get

$$(x^2 + y^2)^2 \frac{\partial^2 z}{\partial u \, \partial v} = (y^2 - x^2)x^2$$

$$\Rightarrow \frac{\partial^2 z}{\partial u \, \partial v} = \frac{(y^2 - x^2)x^2}{(x^2 + y^2)^2} = \frac{u^2 - 1}{(u^2 + 1)^2}$$

Intergrating w. r. to u, we get

$$\frac{\partial z}{\partial v} = \int \frac{u^2 - 1}{(u^2 + 1)^2} du + \phi_1(v)$$

$$= \int \frac{du}{u^2 + 1} - 2 \int \frac{du}{(u^2 + 1)^2} + \phi_1(v)$$

$$= \int \frac{du}{u^2 + 1} - 2\left\{\frac{u}{2.1.1(u^2 + 1)} + \frac{1}{2.1.1} \int \frac{du}{u^2 + 1}\right\} + \phi_1(v)$$

$$= -\frac{u}{u^2 + 1} + \phi_1(v)$$

Again, integrating w. r. to v, we get

$$z = -\frac{uv}{u^2+1} + \phi_1(v) + \phi_2(u)$$

$$\Rightarrow z = -\frac{\frac{y}{x}(x^2+y^2)}{\left(\frac{y}{x}\right)^2+1} + \phi_1(x^2+y^2) + \phi_2\left(\frac{y}{x}\right)$$

$$\Rightarrow z = -xy + \phi_1(x^2+y^2) + \phi_2\left(\frac{y}{x}\right)$$

which is the required general solution.

Ans to Q. No. 2:

(a) Given $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$ .....(1)

where u (x,t) is the deflection of the string. Let it be stretched between fixed points (0,0) and (a,0). Then we are to find u (x,t) under the following boundary conditions and initial conditions

B.C. $u(0,t) = 0, \quad u(a,t) = 0 \quad$ for all t ..... (2)

I.C. $u(x,0) = f(x), u_t(x,0) = g(x)$ ..... (3)

Let $u(x,t) = X(x).T(t)$ ..... (4)

From (1), we get

$$X''T = \frac{1}{c^2}XT''$$

$$\Rightarrow \frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} \quad \text{..... (5)}$$

Since x and t are independent, (5) can only true if each side is equal to the same constant, say $-\mu_2$. Then

$$X'' + \mu^2 X = 0 \quad \text{..... (6)}$$

$$T'' + \mu^2 c^2 T = 0 \quad \text{..... (7)}$$

The solution of equations (6) and (7) is

$$X = c_1 \cos \mu x + c_2 \sin \mu x$$

and $T = c_3 \cos \mu c\, t + c_4 \sin \mu c\, t$

The solution of equation (1) is

$u(x,t) = (c_1 \cos \mu x + c_2 \sin \mu x)(c_3 \cos \mu c\, t + c_4 \sin \mu c\, t)$ .....(8)

Applying B.C. (2), we get X (0) = 0, X (a) = 0

$$\Rightarrow c_1 = 0,\ 0 = c_1 \cos \mu a + c_2 \sin \mu a$$

$$\Rightarrow c_1 = 0,\ \sin \mu a = 0$$

$$\Rightarrow \mu a = n\pi,\ n = 1, 2, ...$$

$$\Rightarrow \mu = \frac{n\pi}{a},\ n = 1, 2, ...$$

Hence from (8), we get

$$u(x,t) = c_2 \sin\left(\frac{n\pi x}{a}\right)\left\{c_3 \cos\left(\frac{n\pi c}{a}\right)t + c_4 \sin\left(\frac{n\pi c}{a}\right)t\right\}$$

$$= c_2 \sin\frac{n\pi x}{a}(c_3 \cos pt + c_4 \sin pt) \quad ...(9)$$

where $p = \frac{n\pi\pi}{a}$

Hence General solution is

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{a}\right)(A_n \cos pt + B_n \sin pt) \quad .....(10)$$

Differentiating (10) partially w. r. to t, we get

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} (-A_n p \sin pt + B_n \cos t) \sin\left(\frac{n\pi x}{a}\right) \quad .....(11)$$

Apply initial condition (3) in (10) and (11), we get

$$f(x) = \sum_{n=1}^{\alpha} A_n \sin\left(\frac{n\pi x}{a}\right) \quad .....(12)$$

$$g(x) = \sum_{n=1}^{\infty} B_n p \sin\left(\frac{n\pi x}{a}\right)$$

$$= \sum_{n=1}^{\infty} \frac{n\pi c\, B_n}{a} \sin\left(\frac{n\pi x}{a}\right) \qquad \left[\because p = \frac{n\pi c}{a}\right] \quad ..... (13)$$

which are Fourier sine series expansion for f(x) and g(x) respectively.

According, we get

$$A_n = \frac{2}{a}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)dx \quad \text{.....(14)}$$

and $\frac{n\pi c\, B_n}{a} = \frac{2}{a}\int_0^a g(x)\sin\frac{n\pi x}{a}\,dx$

$$\Longrightarrow B_n = \frac{2}{n\pi c}\int_0^a g(x)\sin\frac{n\pi x}{a}\,dx \quad \text{.....(15)}$$

Hence the required solution is given by (10) where in A_n and B_n are given by (14) and (15) respectively.

(b) If initial velocity $u_t(x,o) = g(x) = 0$, then

$B_n = 0$ from (15). Thus, in this case the solution (10) reduces to

$$u(x,t) = \sum_{n=1}^{\infty} A_n\, pt \sin\left(\frac{n\pi x}{a}\right)$$

$$= \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi c}{a}\right) t \sin\left(\frac{n\pi x}{a}\right) \quad \text{.....(16)}$$

Here $a = \pi$, $c^2 = 1$, so from (16),

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos nt \sin nx$$

where

$$A_n = \frac{2}{\pi}\int_o^{\pi} f(x)\sin nx\,dx$$

$$= \frac{2k}{\pi}\int_o^{\pi}(\sin x - \sin 2x)\sin nx\,dx$$

$$= \frac{2k}{\pi}\left[\int_o^{\pi}\sin x\sin nx\,dx - \int_o^{\pi}\sin 2x\sin nx\,dx\right] \quad \text{.....(17)}$$

But

$$\int_0^{\pi} \sin px \sin qx \, dx = \begin{cases} 0, & \text{if } p \neq q \\ \frac{\pi}{2}, & \text{if } p = q \end{cases}$$

Therefore $A_1 = k$, $A_2 = -k$ and $An = 0$ for all $n \geq 3$.

Putting these values in (16), the required deflection is given by

$u(x,t) = k (\cos t \sin x - \cos 2t \sin 2x)$.

Ans to Q. No. 3:

Proceed as section 14.5.2. Here a = b = 1, c = 1.

From equation (27) in section 14.5.2

$$A_{mn} = \frac{4}{1.1} \int_0^1 \int_0^1 A \sin \pi x . \sin 2\pi y . \sin m\pi x . \sin n\pi y \, dxdy$$

$$= A \int_0^1 2 \sin \pi x \sin m\pi x \, dx \int_0^1 2 \sin 2\pi y . \sin n\pi y \, dy \quad ....(1)$$

But

$$\int_0^1 2 \sin \pi x . \sin m\pi x \, dx = \begin{cases} 0, & \text{if } m \neq 1 \\ 1, & \text{if } m = 1 \end{cases} \quad .....(2)$$

and

$$\int_0^1 2 \sin 2\pi y \sin \pi y \, dy = \begin{cases} 0, & \text{if } n \neq 2 \\ 1, & \text{if } n = 2 \end{cases} \quad .....(3)$$

Using (2) and (3), (1) gives

$A_{12} = A$ and $A_{mn} = 0$ otherwise .....(4)

Again

$$\lambda_{mn} = c\pi \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} \text{ gives } \lambda_{12} = \pi\sqrt{1^2 + 2^2} = \pi\sqrt{5} \ldots\ldots\ldots (5)$$

Now from equation (29) in section 14.5.2, we get

$$u(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \cos \lambda_{mn} t \sin \frac{m\pi x}{a} \cos \frac{n\pi x}{a} \quad .....(6)$$

Using (4) and (5), (6) reduces to

u (x,y,t) = A_{12} cos λ_{12}t sin πx sin 2πy = A cos $(\sqrt{5\pi}t)$ sin πx sin 2 πy.

Ans to Q. No. 4:

Wave equation in spherical polar coordinates (ρ, θ, ϕ) is

$$\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \quad .....(1)$$

let a solution of (1) be of the form

$$u(r,\theta,\phi,t)=R(r)\,\Theta(\theta)\,\Phi(\phi)\,T(t) \quad .....(2)$$

Substituting this value of u in (1) and dividing by R θ Φ T, we have

$$\frac{R''}{R}+\frac{2}{r}\frac{R'}{R}+\frac{1}{r^2}\frac{\Theta''}{\Theta}+\frac{\cot\theta}{r^2}\frac{\Theta}{\Theta}+\frac{1}{r^2\sin^2\theta}\frac{\Phi}{\Phi}=\frac{1}{c^2}\frac{T''}{T} \quad .....(3)$$

Since the right hand side depends only on t while the left hand side depends only on r, θ and ϕ, (3) can only be true if each side is equal to the same constant say $-k^2$. The (3) gives

$$T''+k^2c^2T=0 \quad .....(4)$$

and

$$\left[\frac{1}{R}\left(R''+\frac{1}{r}R'\right)+\frac{1}{r\theta}(\Theta''+\cot\theta\,\Theta')+k^2\right]r^2\sin^2\theta=\frac{\Phi''}{\Phi} \quad .....(5)$$

Since the right hand side depends only on f while the left hand side depends only on r and θ, (5) can only be true if each side is equal to the same constant, say m^2. Then (5) gives

$$\Phi''+m^2\Phi=0 \quad .....(6)$$

and

$$\frac{1}{r}(r^2R''+2rR')+k^2r^2+\frac{1}{\Theta}(\Theta''+\cot\theta\,\Theta'')=\frac{m^2}{\sin^2\theta}$$

$$\Rightarrow \frac{1}{\Theta}(\Theta''+\cot\theta\,\Theta')-\frac{m^2}{\sin^2\theta}=-\frac{1}{R}(r^2R''+2rR')-k^2r^2 \quad ..... (7)$$

Since r and q are independent, (7) can only be true if each side is equal to the same constant, say – n (n + 1). Then (7) gives

$$r^2R''+2rR''+\{k^2r^2-n(n+1)\}R=0 \quad .....(8)$$

and

$$\Theta'' + \cot\theta\,\Theta' + \left\{n(n+1) - \frac{m^2}{\sin^2\theta}\right\}\Theta = 0 \quad \text{..... (9)}$$

Putting u = cos q, (9) may be re- written as

$$(1-u^2)\frac{d^2\Theta}{du^2} - 2u\frac{d\Theta}{du} + \{n(n+1)\Theta = 0 \quad \text{.....(10)}$$

which is associated Legendre's equation.

Solutions of (4), (6), (8) and (10) are

$$T_k(t) = A_k \cos(ckt) + B_k \sin(ckt) \quad \text{..... (11)}$$

$$\Phi_m(\phi) = c_m \cos m\phi + D_m \sin m\phi \quad \text{..... (12)}$$

$$R_{kn}(r) = (kr)^{-\frac{1}{2}}\left[E_{kn} J_{n+\frac{1}{2}}(kr) + F_{kn} J_{-\left(n+\frac{1}{2}\right)}(kr)\right] \quad \text{.....(13)}$$

$$\Theta_{mn}(\theta) = G_{mn} P_n^m(\cos\theta) + H_{mn} Q_n^m(\cos\theta) \quad \text{..... (14)}$$

The more general solution is given by

$$u(r,\theta,\phi,t) = \sum_k \sum_m \sum_n T_k(t)\Phi_m(\phi)R_{kn}(r)\Theta_{mn}(\theta)$$

where T_k, Φ, R_{kn} and Θ_{mn} are given by (11), (12), (13) and (14) respectively.

Ans to Q. No. 5:

We have $u = u_1 + u_2$ is the solution of given problem, where

(i) u_1 is solution to $u_{tt} - u_{xx} = te^x$, $u(x,o) = 0$, $u_t(x,o) = 0$

(ii) u_2 is solution to $u_{tt} - u_{xx} = 0$, $u(x,o) = 0$, $u_t(x,o) = 0$

We find u_1 by Duhamel's principle,

$$u_1(x,t) = \frac{1}{2.1}\int_0^t ds \int_{x-(t-s)}^{x+(t-s)} se^r dr \quad [\because c = 1]$$

$$= \frac{1}{2}\cdot\int_0^t s\,[e^r]_{x-t+s}^{x+t-s}\,ds = \frac{1}{2}\int_0^t s\,(e^{x+t-s} - e^{x-t+s})ds$$

$$= \frac{1}{2}\int_0^t (e^{x+t}se^{-s} - e^{x-t}se^s)ds$$

we find u_2 be D' Alembert's principle,

$$= \frac{1}{2}e^{x+t}\int_0^t se^{-s}ds - \frac{1}{2}e^{x-t}\int_0^t se^{s}ds$$

$$= \frac{1}{2}[e^{x+t}(-te^{-t} - e^{-t} + 1) - e^{x-t}(te^{t} - e^{t} + 1)]$$

$$= \frac{1}{2}(-2te^{x} + e^{x+t} - e^{x-t})$$

Hence $u = u_1 + u_2 = \frac{1}{2}(-2t\ e^x + e^{x+t} - e^{x-t})$ is a solution of the given solution.



14.9 FURTHER READING

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14.10 MODEL QUESTIONS

Q. 1. Reduce the equation $(n-1)^2\left(\frac{\partial^2 z}{\partial x^2}\right) - y^{2n}\frac{\partial^2 z}{\partial y^2} = ny^{2n-1}\frac{\partial z}{\partial y}$ to canonical form and find its general solution.

Q. 2. Solve : (i) $y(x+y)(r-s) - xp - yq - z = 0$

(ii) $x^2r - y^2t + px - qy = x^2$

Q. 3. Reduce the equation $x(xy-1)r-(x^2y^2-1)s+y(xy-1)t+(x-1)p+(y-1)q=0$ to canonical form and hence solve it.

Q. 4. Solve the following :

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le 1, t \ge 0$$

Boundary conditions : $u(0,t) = 0, t > 0$

$$ux(1,t) = 0, t > 0$$

Initial conditions :

$$u(x,0) = \begin{cases} x, 0 < x < \frac{1}{4} \\ \frac{1}{2} - x, \frac{1}{4} < x < \frac{1}{2} \\ 0, \quad \frac{1}{2} < x < 1 \end{cases}$$

$u_t(x,0) = 0, 0 < x < 1.$

Q. 5. A tightly stretched string with fixed and points x = 0 and x = 1 is initialy in a position given by $u = u_0 \sin 3\left(\frac{\pi x}{1}\right)$. It is release from rest from this position. Find the displacement u (x,t).

Q. 6. Solve the initial value problems :

(i) $u_{tt} - u_{xx} = t \sin x$, when $-\infty < x < \infty, t > 0$

$u(x, o) = 0, u_t(x, 0) = \sin x$, when $-\infty < x < \infty$

(ii) $u_{tt} - u_{xx} = x - t$, when $-\infty < x < \infty, t > 0$

$u(x, o) = x^2, u_t(x, 0) = x$, when $-\infty < x < \infty$

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