UNIT 4 : EXISTENCE AND UNIQUENESS

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4.1 LEARNING OBJECTIVES

After going through this unit, you will able to-

- well posed problems and examples
- Lipschitz condition and lipschtz constants with examples
- formula of picard's method with example
- existence and uniqueness theorem for first order differential equations
- definition and examples of linealy dependent and linearly independent fuction
- definitions and theorem related to wronskian.

4.2 INTRODUCTION

An equation involving derivatives on diffentials of one as more dependent variables with respect to one or more independent variables is called a differential equation. A differential equation is called linear if every dependent variable and every derivative involved occurs in the first degree only. However, to exist solution of a differential equation, it must satisfy some conditions. In this chapter, we will discuss about the existence and uniqueness of solution of a differential equation of first and second order. The Wronskian of solution of differential equation enables us to confirm whether the solutions are linearly dependent or independent.

4.3 WELL POSED PROBLEMS

In mathematics, a problem is called well posed if it possesses the following properties –

(i) The problem has a solution

(ii) The solution is unique

(iii) The solution's behaviour changes continuously with the initial conditions.

Some examples of well-posed problems are Dirichlet problem for Laplace's equation, heat equation with specified initial conditions etc.

A problem which is not well-posed is called as ill-posed problem. Inverse problems are often ill-posed. For example, the inverse heat equation, deducing a previous distribution of temperature from final data is ill-posed

4.4 LIPSCHITZ CONDITION AND LIPSCHITZ CONSTANT

Let f(x, y) be a function defined for all (x, y) in a domain D. We say f (x, y) satisties a Lipschitz condition on D, if there exists a constant k >0 such that.

$$|f(x, y_1) - f(x, y_2)| \leq k |y_1 - y_2|$$

for all (x, y_1), (x, y_2) in D. The constant k is known as Lipschitz

constant.

As a consequence of the definition, a function f(x, y) satisfies Lipschitz condition if and only if there exists a constant k>0 such that

$\frac{|f(x, y_1)-f(x, y_2)|}{|y_1-y_2|} \le k,\ y_1 \neq y_2$ whenever $(x, y_1), (x, y_2) \in D$.

We now state a general criterion which world ensure the Lipschitz condition :

Let f(x, y) be a continuous function defined over a rectangle

$D = \{(x, y) : |x-x_0| \le a, |y-y_0| \le b\}$

Here a, b are some positive real numbers.

Let $\frac{\partial f}{\partial y}$ be defined and continuous on D and

$\left|\frac{\partial f}{\partial y}\right| \le K$ for each $(x, y) \in D$ for some k>0.

Then f satisfies a Lipchitz condition on D with Lipschitz constant k.

Again $\frac{|f(x,y_1)-f(x,y_2)|}{|y_1-y_2|} = \frac{||y_1|-|y_2||}{|y_1-y_2|} \le 1$, as $|y_1-y_2| \ge \big||y_1|-|y_2|\big|$

Example 1

If S is defined by the rectangle $|x| \le a$, $|y| \le b$, show that the function f(x, y) = x sin y+cos x, satisfy the Lipschitz condition. Find the Lipchitz constant.

Solution: Given function is

$f(x, y) = x \sin y + y \cos x \quad \rightarrow (1)$

$\therefore \frac{\partial f}{\partial y} = x\cos y + \cos x$

$\therefore \left|\frac{\partial f}{\partial y}\right| = |x\cos y + |\cos x|$

$\le |x\cos y| + |\cos x|$

$\le |x|\ |\cos y| + |\cos x|$

$\le |x|.1+1$

$\le a+1$, for each $(x, y) \in s \quad [\because |x|<a]$

$\therefore \frac{\partial f}{\partial y} \le a+1$

$\therefore$ f(x, y) satisfies Lipchitz condition and Lipchitz constant is a+1.

Example 2

Give and example to show that there exist functions which do not satisfy the Lipchitz condition.

Solution: Let $f(x,y) = y^{1/2}$ be defind on the rectangle

D = {(x, y) : |x| $\leq$ 2, |y| $\leq$ 2}

Let $y_1 > 0$

Then $$\frac{|f(x, y_1) - f(x,0)|}{|y_1 - 0|}$$

$$= \frac{|y_1^{1/2} - 0^{1/2}|}{|y_1|}$$

$$= \frac{y_1^{1/2}}{y_1} \qquad (\because y_1 > 0)$$

$$= \frac{1}{y_1^{1/2}}$$ which is unbounded as $y_1 \to 0$

Hence f(x, y) does not satisfy Lipchitz condition on D.

Example 3

Give an example to show that the existence of partial derivative of f(x, y) is not necessary for f(x, y) to be a Lipschitz function.

Solution: Let f(x, y) = |y| be defined on the square D{(x, y) : |x| $\leq$ 1, |y| $\leq$ 1}

Now, partial derivatives of f(x, y) wrt. y at the point (x_1, y_1) is defined as

$$\left(\frac{\partial f}{\partial y}\right)_{(x_1, y_1)} = \underset{K\to 0}{Lt} \frac{f(x_1, y_1 + k) - f(x_1, y_1)}{k}$$

$$\therefore \left(\frac{\partial f}{\partial y}\right)(x, 0) = \underset{k\to 0}{Lt} \frac{f(x, 0 + k) - f(x, 0)}{k}$$

$$= \underset{k\to 0}{Lt} \frac{f(x, k) - f(x, 0)}{k}$$

$$= \underset{k\to 0}{Lt} \frac{|k| - |0|}{k}$$

$$= \underset{k\to 0}{Lt} \frac{|k|}{k}$$

Now $$\underset{k\to 0}{Lt} -\left(\frac{|k|}{k}\right) = \underset{k\to 0}{Lt} -\left(\frac{-k}{k}\right) = -1$$

$$\lim_{k\to 0^+}\left(\frac{|k|}{k}\right)=\lim_{k\to 0^+}\left(\frac{k}{k}\right)=1$$

$$\therefore \lim_{k\to 0}\left(\frac{|k|}{k}\right)\neq \lim_{k\to 0^+}\left(\frac{|k|}{k}\right)$$

$\therefore \lim_{k\to 0}\left(\frac{|k|}{k}\right)$ does not exist

$\therefore \left(\frac{\partial f}{\partial y}\right)$ does not exist on (x, 0)

$\therefore$ f(x, y) satisfies Lipschitz condition in x on D with Lipschitz constant k = 1



CHECK YOUR PROGRESS –1

Q 1: Show that the function f(x, y) = $\sqrt{x^2+y^2}$ satisfy Lipschitz condition in y. Find the Lipschitz constant.

Q 2: Examine if the function f(x, y) = $\sqrt{|y|}$ satisfy Lipschitz condition.

Q 3: Show that the function f(x, y) = $y^{2/3}$ does not satisfy the Lipschitz condition on the rectangle D = {(x, y) : $|x|\le 1$, $|y|\le 1$}.

4.5 PICARD'S METHOD

Consider the initial value problem of the form

$$\frac{dy}{dx} = f(x, y),\ y(x_0) = y_0 \qquad (1)$$

By integrating over the interval (x_0, x), (1) gives

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x, y)\,dx$$

$$\Rightarrow y(x) - y_0 = \int_{x_0}^{x} f(x, y)\,dx$$

$$\Rightarrow y(x) = y_0 + \int_{x_0}^{x} f(x, y)\,dx \qquad (2)$$

Thus, the solving of initial value problem (1) is equalent to finding a function y(x) which satisfies equation (2).

By differentiating (2), we get $\frac{dy}{dx} = f(x, y)$ and putting $x = x_0$ in (2) yields $y(x_0) = y_0+0$, ie, $y(x_0) = y_0$. conversely (2) has been obtained from (1) by integration over the interval (x_0, x) and employing the intial condition $y(x_0) = y_0$.

Since the information concerning the expression of y in terms of x is absent, the integral on the R.H.S. of (2) cannot be evaluated. Hence, the exact value of y cannot be obtained. Therefore, we detarmine a sequence of approximations to the solution (2) as follows. As a crude approximation, we put $y = y_0$ in the integral on the right of (2) and obtain.

$$y_1(x) = y_0 + \int_{x_0}^{x} f(x, y_0)dx \qquad (3)$$

Where $y_1(x)$ is the corresponding value of y(x) and is called first approximation and is better approximation of y(x) at any x. To determine still better approximation, we replace y by y_1 in the integral on R.H.S. in (2) and obtain the second approximation y_2 as

$$y_2(x) = y_0 + \int_{x_0}^{x} f(x, y_1)dx$$

Proceeding in this way, the n^{th} approximation y_n is given by

$$y_n(x) = y_0 + \int_{x_0}^{x} f(x, y_{n-1})dx$$

Thus, we arrive at a sequence of approximate solutions

$y_1(x), y_2(x), y_3(x), \ldots, y_n(x) \ldots$

Example 4

Apply Picard's method to solve the initial value problem upto third approximation–

$$\frac{dy}{dx} = 2y–2x^2–3, \; y(0) = 2$$

Solution:

Given problem is

$$\frac{dy}{dx} = 2y–2x^2–3, \; y(0) = 2 \qquad (1)$$

By Picard's method, the n^{th} approximation y_n of the initial value problem

$$\frac{dy}{dx} = f(x, y),\ y(x_0) = y_0 \qquad (2)$$

is given by

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})dx \qquad (3)$$

comparing (1) & (2),

$$f(x, y) = 2y - 2x^2 - 3,\ x_0 = 0,\ y_0 = 2 \qquad (4)$$

$$\therefore \text{ from (3), } y_n = 2 + \int_0^x (2y_{n-1} - 2x^2 - 3)dx \qquad (5)$$

First approximation

Putting n = 1 in (5), we have

$$y_1 = 2 + \int_0^x (2y_0 - 2x^2 - 3)dx$$

$$= 2 + \int_0^x (4 - 2x^2 - 3)dx \text{ using (4)}$$

$$= 2 + \int_0^x (1 - 2x^2)dx$$

$$= 2 + \left[x - 2\left(\frac{x^3}{3}\right)\right]_0^x$$

$$\Rightarrow y_1 = 2 + x - \frac{2x^3}{3} \qquad (6)$$

Second approximation

Putting n = 2 in (5), we have

$$y_2 = 2 + \int_0^x (2y_1 - 2x^2 - 3)dx$$

$$= 2 + \int_0^{x-} \left\{2\left(2 + x - \frac{2x^3}{3}\right) - 2x^2 - 3\right\}dx \quad \text{using (6)}$$

$$y_2 = 2 + \int_0^x \left(1 + 2x - 2x^2 - \frac{4x^3}{3}\right)dx$$

$$\Rightarrow y_2 = 2 + x + x^2 - \frac{2x^3}{3} - \frac{x^4}{3} \qquad (7)$$

Third approximation

Putting n = 3 in (5), we have

$$y_3 = 2 + \int_0^x (2y_2 - 2x^2 - 3)\,dx$$

$$= 2 + \int_0^x \left\{2\left(2 + x + x^2 - \frac{2x^3}{3} - \frac{x^4}{3}\right) - 2x^2 - 3\right\}dx$$

$$= 2 + \int_0^x \left(1 + 2x - \frac{4x^3}{3} - \frac{2x^4}{3}\right)dx$$

$$\Rightarrow y_3 = 2 + x + x^2 - \frac{x^4}{3} - \frac{2x^5}{15}$$

Example 5

Apply Picard's method to solve the initial value problem upto fourth approximation–

$$\frac{dy}{dx} = 1+xy, \quad y(0) = 1$$

Solution:

Given problem is

$$\frac{dy}{dx} = 1+xy, \quad y(0) = 1 \qquad (1)$$

By Picard's method, the n^{th} approximation y_n of the initial value problem

$$\frac{dy}{dx} = f(x, y), \; y(x_0) = y_0 \qquad (2)$$

is given by

$$y_n = y_0 + \int_{x_0}^x f(x, y_{n-1})\,dx \qquad (3)$$

comparing (1) & (2),

$$f(x, y) = 1+xy, \; x_0 = 0, \; y_0 = 1 \qquad (4)$$

$$\therefore \text{ from (3), } \quad y_n = 1 + \int_0^x (1 + xy_{n-1})\,dx \qquad (5)$$

First approximation

Putting n = 1 in (5), we have

$$y_1 = 1 + \int_0^x (1 + xy_0)\,dx$$

$$= 1 + \int_0^x (1 + x)\,dx$$

$$= 1 + x + \frac{x^2}{2} \qquad (6)$$

Second approximation

Putting n = 2 in (5), we have

$$y_2 = 1 + \int_0^x (1 + xy_1)\,dx$$

$$= 1 + \int_0^x \left\{1 + x\left(1 + x + \frac{x^2}{2}\right)\right\} dx, \text{ using (6)}$$

$$= 1 + \int_0^x \left(1 + x + x^2 + \frac{x^3}{2}\right) dx$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} \qquad (7)$$

Third approximation

Putting n = 3 in (5), we have

$$y_3 = 1 + \int_0^x (1 + xy_2)\,dx$$

$$= 1 + \int_0^x \left\{1 + x\left(1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8}\right)\right\} dx$$

$$= 1 + \int_0^x \left(1 + x + x^2 + \frac{x^3}{2} + \frac{x^4}{3} + \frac{x^5}{8}\right) dx$$

$$\Rightarrow y_3 = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{15} + \frac{x^6}{48}$$

Fourth approximation

Putting n = 4 in (5), we have

$$y_4 = 1+\int_0^x (1+xy_3)\,dx$$

$$= 1+\int_0^x \left\{1+x\left(1+x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{8}++\frac{x^5}{15}++\frac{x^6}{48}\right)\right\}dx$$

$$= 1+\int_0^x \left(1+x+x^2+\frac{x^3}{2}+\frac{x^4}{3}+\frac{x^5}{8}+\frac{x^6}{15}+\frac{x^7}{48}\right)dx$$

$$= 1+x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{8}+\frac{x^5}{15}+\frac{x^6}{48}+\frac{x^7}{105}+\frac{x^8}{348}$$



CHECK YOUR PROGRESS –2

Q 1: Using the Picard's method of successive approximations, find the third approximation of the solution of the equation $\frac{dy}{dx} = x+y^2$ where y = 0, when x = 0

Q 2: Find the third approximation of the solution of the equation $\frac{dy}{dx} = 2-\frac{2}{x}$ by Picard's method, where y = 2 when x = 1.

4.6 EXISTENCE AND UNIQUENESS THEOREM FOR FIRST ORDER EQUATIONS (PICARD'S THEOREM)

Statement

Let f(x, y) be continuous in a domain D of the (x, y) plane and let M be a constant such that

$|f(x, y)| \le M$ in D (1)

Let f(x, y) satisfy in D the Lipschitz condition in y namely

$|f(x, y_1) - f(x, y_2)| \le k|y_1 - y_2|$ (2)

Where the constant k is independent of x, y_1, y_2

Let the rectangle R, defined by $|x-x_0| \le h$, $|y-y_0| \le k$ (3)

lie in D, where Mh < k. Then, for $|x-x_0| \le h$, the differential equation $\frac{dy}{dx} = f(x, y)$ has a unique solution y = y(x) for which $y(x_0) = y_0$.

Proof :

We shall prove this theorem by the method of successive approximations. Let x be such that $|x-x_0| \leq h$. We now define a sequence of functions $y_1(x)$, $y_2(x)$, . . ., $y_n(x)$, . . ., called the successive approximations as follows:

$$\left.\begin{aligned} y_1(x) &= y_0 + \int_{x_0}^{x} f(x, y_0)\,dx \\ y_2(x) &= y_0 + \int_{x_0}^{x} f(x, y_1)\,dx \\ &\text{- - - - - - - - - - - -} \\ &\text{- - - - - - - - - - - -} \\ &\text{- - - - - - - - - - - -} \\ y_{n-1}(x) &= y_0 + \int_{x_0}^{x} f(x, y_{n-2})\,dx \\ y_n(x) &= y_0 + \int_{x_0}^{x} f(x, y_{n-1})\,dx \end{aligned}\right\} \quad (4)$$

We shall divide the proof into five main steps.

First step :

We prove that, for $x_0 - h \leq x < x_0 + h$, the curve $y = y_n(x)$ lies in the rectangle R, i.e, to say $y_0 - k < y < y_0 + k$

Now $$|y_1 - y_0| = \left| \int_{x_0}^{x} f(x, y_0)\,dx \right| \leq \int_{x_0}^{x} |f(x, y_0)|\,|dx|$$

$\Rightarrow |y_1 - y_0| \leq M h < k$, using (1), (3) and the given result $Mh < k$

This proves the desired result for n = 1.

Assume that $y = y_{n-1}(x)$ lies in R and so $f(x, y_{n-1})$ is defined and continuous and satisfies

$|f(x, y_{n-1})| \leq M$ on $[x_0 - h, x_0 + h]$

From (4), we have

$$|y_n - y_0| = \left| \int_{x_0}^{x} f(x, y_{n-1})\,dx \right|$$

$$\leq \int_{x_0}^{x} |f(x, y_{n-1})| \, |dx|$$

$$\leq M |x - x_0|$$

$$\leq Mh$$

$$\leq k$$

as before which shows that $y_n(x)$ lies in R and hence $f(x, y_n)$ is defined and continuous on $[x_0–h, x_0+h]$. The above arguments show that the desired result holds for all n by induction.

Second step :

We prove again by induction, that

$$|y_n - y_{n-1}| \leq \frac{Mk^{n-1}}{n!} |x - x_0|^n \qquad (5)$$

We have already verified (5) for n = 1 in first step where we have shown that $|y_1 - y_0| \leq M |x–x_0|$ Assume that this inequality (5) holds for n–1 in place of n, that is let,

$$|y_{n-1} - y_{n-1}| \leq \frac{Mk^{n-2}}{(n-1)!} |x - x_0|^{n-1} \qquad (6)$$

Then, we have $|y_n - y_{n-1}| = \left| \int_{x_0}^{x} \{f(x, y_{n-1}) - f(x, y_{n-2})\} \, dx \right|$ by (4)

$$\Rightarrow |y_n - y_{n-1}| \leq \int_{x_0}^{x} |f(x, y_{n-1}) - f(x, y_{n-2})| \, |dx| \qquad (7)$$

Lipschitz condition (2) gives

$$|f(x, y_{n-1}) - f(x, y_{n-2})| \leq k \, | y_{n-1} - y_{n-2}| \qquad (8)$$

$\therefore$ From (7) and (8), we get

$$|y_n - y_{n-1}| \leq \int_{x_0}^{x} k |y_{n-1} - y_{n-2}| \, |dx|$$

$$\leq k \cdot \frac{Mk^{n-1}}{\lfloor n-1} \cdot \frac{|x - x_0|^n}{n} \quad \text{by (6)}$$

Hence by mathematical induction, we conclude that (5) is true for each natural number n.

Third step :

We shall now prove that the sequence y_n converges uniformly to a

limit for $x_0-h \le x \le x_0+h$. For the interval under consideration $|x-x_0| \le h$

Hence from second step, we get

$$|y_n - y_{n-1}| \le \frac{Mk^{n-1}h^n}{\lfloor n} \text{ is true for all n}$$

Using this, the infinite series

$$y_0 + (y_1-y_0) + (y_2-y_1) + \ldots + (y_n-y_{n-1}) + \ldots$$

$$\le y_0 + Mh + \frac{1}{2}Mkh^2 + \ldots + \frac{1}{n}Mk^{n-1}h^n + \ldots$$

$$\le y_0 + \frac{M}{k}\left[e^{kh} - 1\right] \qquad (9)$$

Which is known to be convergent for all values of k, h and M. Consequatly, the series (9) surely convegent. Thus by weierstrass M-test, the series (9) convenges unifromly on $[x_0-h, x_0+h]$.

Now, since the terms of (9) are continuous functions of x, therefore, its sum = $\lim_{n \to \infty} y_n(x) = y(x)$, say

$$\text{as } y_n = y_0 + \sum_{n-1}^{n}(y_n - y_{n-1}) \qquad (10)$$

must be continuous.

Fourth step :

We now show that y = y(x) satisfies the differential equation

$$\frac{dy}{dx} = f(x, y)$$

Since $y_n(x)$ tends uniformly to y(x) in $[x_0-h, x_0+h]$ and by Lipschitz condition.

$$|f(x, y) - f(x, y_n)| \le k|y-y_n|$$

it follow that $f[x, y_n(x)]$ tends uniformly to f[x, y(x)]. Again from (4) we have

$$y_n(x) = y_0 + \int_{x_0}^{x} f\left[x, y_{n-1}(x)\right]dx$$

$$\lim_{n\to\infty} y_n(x) = y_0 + \lim_{n\to\infty} \int_{x_0}^{x} f\left[x, y_{n-1}(x)\right]dx \text{ letting } n \to \infty$$

Since the sequence $f[x, y_n(x)]$, consisting of continuous functions

on the given interval, converges uniformly to f[x, y(x)] on the same interval, the interchanges of limiting operations given below are valid. Thus, using (10) we have

$$y_n(x) = y_0 + \int_{x_0}^{x} \underset{n\to\infty}{\text{Lt}}\ f[x, y_{n-1}(x)]dx$$

$$\Rightarrow y_n(x) = y_0 + \int_{x_0}^{x} f[x, y(x)]dx \qquad (11)$$

The integrand on the right-hand side of (11) being a continuous function of x, we conclude that the integral has the derivative. Thus, the limit function y(x) satisfies the differential equation $\frac{dy}{dx} = f(x, y)$ on $[x_0-h, x_0+h]$ and is such that $y(x_0) = y_0$.

(In the above four steps, we have this proved the existence of a solution of the given initial value problem. The next step will show that the solution y(x) is unique)

Fifth step :

We now prove that the solution y = Y(x) just found is the only solution for which $y(x) = y_0$.

Assume, if possible y = y(x), say is another solution of the given initial value proble.

Let $|Y(x) - y(x)| \le B$, when $x_0-h \le x \le x_0+h$ (12)

It may be noted have that we can surely take B = 2k. From (11), we get

$$|Y(x) - y(x)| = \int_{x_0}^{x} \{f\{x, y(x) - f(x, y(x)\}]$$

$$\Rightarrow |Y(x) - y(x)| \le \int_{x_0}^{x} |\{f\{x, y(x) - f(x, y(x)\}|\,|dx| \qquad (13)$$

$$\Rightarrow |Y(x) - y(x)| \le k\int_{x_0}^{x} |y(x) - y(x)|\,|dx|$$

$\Rightarrow |Y(x) - y(x)| \le kB|x-x_0|$, using (12)

Now substituting (14) for integrand in (13), we get

$$\left|Y(x)-y(x)\right| \le k^2 B\int_{x_0}^{x}\left|x-x_0\right|dx \le \frac{k^2 B\left|x-x_0\right|^2}{\lfloor 2}$$

Again substituting (15) for the integrand in (13), we get

$$\left|Y(x)-y(x)\right| \le \frac{k^2 B}{\lfloor 2}\int_{x_0}^{x}\left|x-x_0\right|^2 dx \le \frac{k^3 B\left|x-x_0\right|^3}{\lfloor 3}$$

Continuing in this way, we get

$$\left|Y(x)-y(x)\right| \le \frac{k^n B\left|x-x_0\right|^n}{\lfloor n}\int_{x_0}^{x} B \le \frac{(kh)^n B}{\lfloor n}$$

Now the saies $\sum_{n=0}^{n} B\frac{(kh)^n}{\lfloor n}$ convages end so

$$\underset{n\to\infty}{Lt}\ B\frac{(kh)^n}{\lfloor n} = 0$$

Thus $|Y(x) - y(x)|$ can be made less than any number however small and consequently we conclude that

$Y(x) - y(x) = 0$

$\Rightarrow Y(x) = y(x)$

This shows that the solution $y = y(x)$ is always unique and the proof of the theorem is complete.

Remark

If $f(x, y)$ satisfies the condition

$$\left|\frac{\partial f}{\partial y}\right| \le k \qquad \text{(i)}$$

for all values of x, y in the given range then for the same constant k the Lipschitz condition is also satisfied.

By the mean value theorem of differential calculus, we get.

$f(x, y_2) - f(x, y_1) = (y_2 - y_1)\left(\frac{\partial f}{\partial y}\right)_{y=\bar{y}}$ when $y_1 < \bar{y} < y_2$ (11)

Where (x, y_1) and (x, y_2) are assumed in the given range.

Now from (i) and (ii)

$|f(x, y_2) - f(x, y_1)| \le k\,|y_2 - y_1|$ (ii)

Which is Lipschitz condition. If follows that the Lipschitz condition

(iii) can be replaced by the stronger condition (1)

Theorem :

If S in either a rectangle $|x-x_0| \le h$, $|y-y_0| \le k$ $(k, k > 0)$ on a step $|x-x_0| \le h$, $|y| < \infty$ $(h>0)$, and if f(x, y) is a real valued function defined on S such that $\frac{\partial f}{\partial y}$ exists, is continuous on S, and $\left|\frac{\partial f}{\partial y} f(x,y)\right| \le k$, $(x, y) \in S$ for a positive constant k, then (x, y) satisfies a Lipschitz condition on S with Lipschitz constant k.

Proof :

$$|f(x, y_1) - f(x, y_2)| = \left|\int_{y_1}^{y_2} \frac{\partial}{\partial y} f(x,y)\, dy\right|$$

$$= \int_{y_1}^{y_2} \left|\frac{\partial}{\partial y} f(x,y)\, dy\right| |dx|$$

$$\le k \int_{y_1}^{y_2} |dy|$$

Thus $|f(x, y_1) - f(x, y_2)| = k|y_1 - y_2|$ for $(x, y_1), (x, y_2) \in S$

This shows that f(x, y) satisfies Lipschitz condition on S with Lipschitz constant k.

Example 6

Show that $f(x, y)= xy^2$ satisfies the Lipschitz condition on the rectangle R : $|x| \le 1$, $|y| \le 1$ but does not satisfy a Lipschitz condition on the strip S : $|x| \le 1$, $|y| < \infty$

Method I Given, $f(x, y) = xy^2$

We have

$|f(x, y_2) - f(x, y_1)| = |xy_2^2 - xy_1^2|$

$= |x(y_2^2 - y_1^2)|$

$= |x\,(y_2+y_1)\,(y_2-y_1)|$

$\Rightarrow$ If $(x, y_2) - f(x, y_1) = |x|\,|y_2+y_1|\,|y_2-y_1|$ (1)

Hence in the rectangle $|x| \le 1$, $|y| \le 1$,

(1) $\Rightarrow$ $|f(x, y_2) - f(x, y_1)| \le 1 \times 2 \times |y_2 - y_1|$

Showing that Lipschitz condition is satisfied. Now,

$$\left|\frac{f(x, y_2) - f(x, 0)}{y_2 - 0}\right|$$

$$= \left|\frac{x y_2^{\,2} - x 0^2}{y^2}\right|$$

$= |xy_2|$

$= |x|\,|y_2| \to \infty$ when $|y_2| \to \infty$ if $|x| \neq 0$

This show that the Lipschitz condition is not satisfied on the strip $|x| \leq 1$, $|y| \to \infty$

Method II

Given $f(x, y) = xy^2$

$$\frac{\partial f}{\partial y} = 2xy$$

$$\left|\frac{\partial f}{\partial y}\right| = 2|xy| = 2|x|\,|y| \quad \to (2)$$

$\therefore$ In the rectangle $|x| \leq 1$, $|y| \leq 1$,

$$\left|\frac{\partial f}{\partial y}\right| \leq 2 \text{ for each } (x, y) \in R$$

This show that f(x, y) satisfies Lipschitz condition on R, with Lipschitz constant 2.

Again, on the strip S : $|x| < 1$, $|y| < \infty$, (2)

Shows that $\left|\frac{\partial f}{\partial y}\right|$ is unbounded on the strip S as $|y| \to \infty$. Hence Lipschitz conditions is not satisfied on the strip S.

Example 7

Illustrate by an example that a continuous function may not satisfy a Lipschitz condition on a rectangle.

Solution:

Consider the function

$f(x, y) = y^{2/3}$ on the rectangle S : $|x| \leq 1$, $|y| \leq 1$ (1)

Clearly, f(x, y) is continuous in the rectangle S.

Now $\frac{\partial}{\partial y} f(x, y) = \frac{2}{3} y^{-1/3}$

$$\left|\frac{\partial}{\partial y} f(x, y)\right| = \frac{2}{3y^{-1/3}} \to \infty \quad \text{when } y \to 0 \qquad (2)$$

Since y = 0 is a point of the rectangle S, (2) shows that the Lipschitz condition is not satisfied by the fuction f(x, y) = $y^{2/3}$ on the rectangle S.



CHECK YOUR PROGRESS – 3

Q 1: If S is defined by the rectangle $|x| \leq a$, $|y| \leq b$, show that $f(x, y) = x^2+y^2$ satisfies the Lipschitz condition. Find the Lipschitz constant.

Q. 2: Show that $f(x, y) = xy^2$

(i) Satisfies a Lipschitz condition on any rectangle $a \leq x \leq b$, $c \leq y \leq d$

(ii) Does not satisfy a Lipschitz condition on any strip $a \leq x \leq b$, $-\infty < y \leq \infty$.

4.7 EXISTENCE AND UNIQUENESS THEOREM FOR SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

Consider a second order linear differential equation of the form.

$a_0(x)y'' + a_1(x)y'+a_2(x)y = r(x)$ (1)

Where $a_0(x)$, $a_1(x)$, $a_2(x)$ and r(x) are continuous functions on an interval (a, b) and $a_0(x) \neq 0$ for each $x \in (a, b)$. Let e_1 and e_2 be orbitrary real numbers and $x_0 \in (a, b)$. Then there exists a unique solution y(x) of (1) satisfying $y(x_0) = e_1$ and $y'(x_0) = e_2$. Morever, this solution y(x) is defined over the interval (a, b).

Example 8

Show that the function $y = ex^2+x+3$ is a solution though not unique of the initial value problem.

$x^2y'' - 2xy'+2y = 6$ with y(0)=3, $y'(0) = 1$ on $(-\infty, \infty)$

Solution:

Given equation is

$x^2y'' -2xy'+2y = 6$ (1)

and given function is

$$y(x) = ex^2+x+3 \qquad (2)$$

differentiating (2), we get

$$y' = 2ex + 1$$

$$y'' = 2e \qquad (3)$$

$\therefore$ LHS of (1) $= x^2(2e) - 2x(2ex+1) + 2(ex^2+x+3)$ {by (2) & (3)}

$= 2ex^2 - 4ex^2 - 2x + 2ex^2 + 2x + 6$

$= 6$

= R.H.S. of (1)

$\therefore$ (2) is a solution of (1).

Again from (2) & (3)

$y(0) = e \times 0 + 0 + 3 = 3$

and $y'(0) = 2e \times 0 + 1 = 1$

Now companing (1) with $a_0(x)y'' + a_1(x)y' + a_2(x)y = r(x)$, we get

$a_0(x) = x^2$, $a_1(x) = -2x$, $a_2(x) = 2$ and $r(x) = 6$

which are all continuous fuctions on $(-\infty, \infty)$

Since $a_0(x) = x^2 = 0$ for $x = 0 \in (-\infty, \infty)$, therefore the solution $y = ex^2+x+3$ is not unique.

4.8 LINEARLY DEPENDENT AND INDEPENDENT SET OF FUNCTIONS

n functions $y_1(x), y_2(x), \ldots y_n(x)$ are called linearly dependent if there exists constants $e_1, e_2, \ldots e$ (not all zero) such that

$$e_1y_1 + e_2y_2 + \ldots + e_ny_n = 0 \qquad (1)$$

On the other hand, if identity (1) implies that $e_1 = e_2 = \ldots = e_n = 0$,

Then $y_1, y_2, \ldots, y_n$ are said to be linearly independent.

For example, the identity $e_1e^{-x} + e_2e^x + e_3e^{2x} = 0$ implies $e_1 = e_2 = e_3 = 0$. Hence, the functions e^{-x}, e^x and e^{2x} are linearly independent.

4.9 WRONSKIAN

The wronskian of n functions $y_1(x), y_2(x), \ldots y_n(x)$ is denoted by $W(x)$ or $W(y_1, y_2, \ldots y_n)(x)$ and is defined to be the determinant.

$$w(y_1, y_2, \ldots y_n)(x) = w(x) = \begin{vmatrix} y_1 & y_2 & .... & y_n \\ y_1' & y_2' & .... & y_n' \\ y_1^{(n-1)} & y_2^{(n-1)} & .... & y_n^{(n-1)} \end{vmatrix}$$

Theorem

If $y_1(x)$ and $y_2(x)$ are any two solutions of

$a_0(x)y'' (x) + a_1(x)\, y'(x)+a_2(x)\, y(x) = 0$

then the linear combination $e_1y_1(x) + e_2y_2(x)$, where e_1 and e_2 are constants, is also a solution of the given equation.

Proof :

Given equation is

$a_0(x)\, y''(x) + a_1(x)\, y'(x) + a_2(x)\, y(x) = 0$ (1)

Since $y_1(x)$ and $y_2(x)$ are solutions of (1), so.

$a_0(x)\, y_1''(x) + a_1(x)\, y_1'(x) + a_2(x)\, y_1(x) = 0$ (2)

$a_0(x)\, y_2''(x) + a_1(x)\, y_2'(x) + a_2(x)\, y_2(x) = 0$ (3)

Let $u(x) = e_1y_1(x) + e_2y_2(x)$ (4)

differentiating (4) w.r.t.x_1

$u'(x) = e_1y'_1(x)+e_2y'_2(x)$ (5)

$u''(x) = e_1y''_1(x)+e_2y''_2(x)$ (6)

Now $a_0(x)u''(x)+a_1(x)\, u'(x)+a_2(x)\, u(x)$

$= a_0(x)[e_1y_1''(x)+e_1y_2''(x)] +a_1(x)[e_1y_1'(x)+e_2y'_2(x)]$

$+a_2(x)\, [e_1y_1(x) + e_2y_2(x)]$ [using (4), (5) & 6]

$= e_1[a_0(x)y_2''(x)+ a_1(x) + y_1(x)+a_2(x)y(x)]$

$+e_2[a_0(x)y_2''(x)+a_1(x)\, y_2'(x)+a_2(x)y_2(x)]$

$= e_1.0 +e_2.0$ [using (2) & 3]

$= 0$

Thus, $a_0(x)u''(x) + a_1(x)\, u'(x) + a_2(x)\, u(x) = 0$

Hence u(x), i.e., $e_1y_1(x) + e_2y_2(x)$ is also solution of (1).

Theorem

Two solutions $y_1(x)$ and $y_2(x)$ of the equation

$a_0(x)y''+a_1(x)y'+a_2(x)y=0$, $a_0(x) \neq 0$, $x \in (a, b)$ are linearly dependent if and only if their wronskian is identically zero.

Let $y_1(x)$ and $y_2(x)$ be linearly dependent, then there exists to constants e_1 and e_2, not both zero such that

$e_1y_1(x)+e_2y_2(x) = 0$ for each $x \in (a, b)$ (1)

$\therefore$ From (1) $e_1y_1'(x) + e_2y_2'(x) = 0$ for each $x \in (a, b)$ (2)

Since e_1 and e_2 cannot be zero simultaneously, the system of simultaneous equation (1) and (2) passess non-zero solutions for which the condition is

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \text{ for each } x \in (a, b)$$

$\Rightarrow w(x) \equiv 0$ on (a, b)

$\therefore$ Wronskian is identically zero.

Conversely

Suppose wronskian of $y_1(x)$ and $y_2(x)$ is identically zero on (a, b), i.e., let

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \equiv \text{ on } (a, b) \qquad (3)$$

Let $x = x_0 \in (a, b)$. Then from (3), we have

$$\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = 0 \qquad (4)$$

Now (4) is the condition for existence of two constant k_1 and k_2, both not zero, such that

$k_1y_1(x_0) + k_2y_2(x_0) = 0$ (5)

and $k_1y_1'(x_0) + k_2y_2'(x_0) = 0$ (6)

Let $y(x) = k_1y_1(x) + k_2y_2(x)$ (7)

Then y(x) being a linear combination of solutions $y_1(x)$ and $y_2(x)$ is also a solution of the given equation.

Again, from (7), $y'(x) = k_1y_1'(x) + h_2y_2'(x)$ (8)

Now (7) $\Rightarrow y(x_0) = k_1y_1(x_0)+k_2y_2(x_0) = 0$ {using (5)}

(8) $\Rightarrow y'(x_0) = k_1y_1'(x_0) + k_2y_2'(x_0) = 0$, {using (6)}

Thus, we find that y(x) is a solution of the given equation such that $y(x_0) = 0$ and $y'(x_0) = 0$

Hence $y(x) \equiv 0$ on (a, b)

and y_0 by (7), we have

$k_1y_1(x)+ k_2y_2(x) = 0$ for each $x \in (a, b)$

Where k_1 and k_2 are constants, both not zero.

Hence, by definition $y_1(x)$ and $y_2(x)$ linearly dependent.

Theorem

The Wronskian of two solutions of the equation

$a_0(x)y''+a_1(x)y'+a_2(x)y = 0$, $a_0(x) \neq 0$ $x \in (a, b)$ is either identically zero on never zero on (a, b)

Given differential equation is

$$a_0(x)y''+a_1(x)y'+a_2(x)y=0,\ a_0(x) \neq 0 \qquad x \in (a, b) \qquad (1)$$

Let $y_1(x)$ and $y_2(x)$ be two solutions of (1).

Then their Wronskian is given by

$$W'(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)y_2'(x) - y_2(x)y_1'(x) \qquad (2)$$

Differentiating both sides of (2) w.r.t. x, we get

$$W'(x) = \frac{d}{dx}\left(y_1(x)y_2'(x)\right) - \frac{d}{dx}\left(y_2(x)y_1'(x)\right)$$

$$\Rightarrow W'(x) = \{y_1'(x)y_2'(x)+y_1(x)y_2''(x)\} - \{y_2'(x)y_1'(x)+y_2(x)y_1''(x)\}$$

$$\Rightarrow W'(x) = y_1(x)-y_2''(x)-y_2(x)y_1''(x) \qquad (3)$$

Since $a_0(x) \neq 0$, dividing by $a_0(x)$ and rewritting, (1) becomes

$$y''(x) = -\frac{a_1}{a_0}y'(x) - \frac{a_2}{a_0}y(x) \qquad (4)$$

Since $y_1(x)$ and $y_2(x)$ are solutions of (4), we have

$$y_1''(x) = -\frac{a_1}{a_0}y_1'(x) - \frac{a_2}{a_0}y_1(x) \qquad (5)$$

$$y_1''(x) = -\frac{a_1}{a_0}y_2'(x) - \frac{a_2}{a_0}y_2(x) \qquad (6)$$

Substituting the values of $y_1''(x)$ and $y_2''(x)$ given by (5) and (6) in (3), we get

$$W'(x) = y_1(x)\left(-\frac{a_1}{a_0}y_2'(x) - \frac{a_2}{a_0}y_2(x)\right) - y_2(x)\left(-\frac{a_1}{a_0}y_1'(x) - \frac{a_2}{a_0}y_1(x)\right)$$

$$\Rightarrow W'(x) = \frac{a_1}{a_0}\left(y_1(x)y_2'(x) - y_2(x)y_1'(x)\right)$$

$$\Rightarrow W'(x) = \frac{a_1}{a_0}w(x), \text{ using (2)} \qquad (7)$$

$$\Rightarrow a_0(x)\, w'(x)+a_1(x)w(x) = 0 \qquad (8)$$

From (8), it follows that w(x) is a solution of (8). Now, the following two cases arise :

Case I

Let $W(x) \neq 0$ on (a, b)

Then the second part of the theorem is proved.

Case II

If possible, let $w(x_0) = 0$ for some $x_0 \in (a, b)$

Then (7) $\Rightarrow W'(x) = -\frac{a_1}{a_0} w(x_0) = 0$

Thus w(x) is a solution of (8), such that $w(x_0) = 0$ and $w'(x_0)$. Hence, $w(x) \equiv 0$ on (a, b)

i.e., Wronskian is identically zero on (a, b)

This proves the first part of the theorem.

Theorem (Abel's Formula)

Let functions $p_1(x)$ and $p_2(x)$ in

$L(y)(x) = y''(x)+p_1(x)y'(x)+p_2(x)y(x) = 0, x \in I$ (1)

be defined and continuous on an interval I. Let ϕ_1 and ϕ_2 be any two linearly independent solutions of (1) existing on I containing a point x_0. Then

$$W(\phi_1, \phi_2)(x) = e^{-\int_{x_0}^{x} p_1(x)dx} - W(\phi_1, \phi_2)(x_0) \qquad (2)$$

Given $y''(x)+p_1(x)y'(x)+p_2(x)y(x) = 0, x \in I$

Here $W(\varphi_1, \varphi_2) = \varphi_1.\varphi_2' - \varphi_1'.\varphi_2$ (3)

$\therefore$ From (3), $W'(\varphi_1, \varphi_2) = \varphi_1'.\varphi_2' + \varphi_1'.\varphi_2'' - (\varphi_1''.\varphi_2 + \varphi_1'.\varphi_2')$

$\Rightarrow W'(\varphi_1, \varphi_2) = \varphi_1\varphi_2'' - \varphi_1''\varphi_2$ (4)

Since φ_1 and φ_2 satisfy (1), we have

$\varphi_1''+p_1\varphi_1'+p_2\varphi_1' = 0$

$\Rightarrow \varphi_1'' = -p_1\varphi_1' - p_2\varphi_1$

and $\varphi_2''+p_1\varphi_2'+p_2\varphi_2' = 0$

$\Rightarrow \varphi_2'' = -p_1\varphi_2' - p_2\varphi_2$

Substituting above values of φ_1'' and φ_2'' in (4) we get

$W'(\varphi_1\varphi_2) = \varphi_1(-p_1\varphi_1' - p_2\varphi_1) - \varphi_2(-p_1\varphi_2' - p_2\varphi_2)$

$\Rightarrow W'(\varphi_1\varphi_2) = -p_1(\varphi_1\varphi_2' - \varphi_2\varphi_1')$

$\Rightarrow W'(\varphi_1\varphi_2) = -p_1 w(\varphi_1\varphi_2)$, using (3)

$\Rightarrow W'(\varphi_1 \varphi_2) + p_1 w(\varphi_1 \varphi_2) = 0$

Thus, $w(\varphi_1, \varphi_2)$ satisfies a first order linear homogeneous equation $w'+p_1 w = 0$, $x \in I$.

$$\Rightarrow \frac{dw}{dx} = -p_1 w$$

$$\Rightarrow \frac{dw}{w} = -p_1 dx$$

Integrating,

$$\log w - \log e = -\int_{x_0}^{x} p_1 dx$$

$$\Rightarrow \log\left(\frac{w}{e}\right) = -\int_{x_0}^{x} p_1 dx$$

$$\Rightarrow \frac{w}{e} = e^{-\int_{x_0}^{x} p_1 dx}$$

$$\Rightarrow w(\varphi_1, \varphi_2)(x) = c\, e^{-\int_{x_0}^{x} p_1(x) dx}$$

Where c is a constant.

By putting $x = x_0$ in (5), we get

$e = w(\varphi_1, \varphi_2)(x_0)$

Substituting this value of c in (5), we get

$$w(\varphi_1, \varphi_2)(x) = e^{-\int_{x_0}^{x} p_1(x) dx}\, w(\varphi_1, \varphi_2)(x_0)$$

Example 9

Show that linearly independent solutions of $y''-2y'+2y = 0$ are $e^x \sin x$ and $e^x \cos x$. What is the general solution? Find the solution y(x) with the property $y(0) = 2$, $y'(0) = -3$

Given equation is

$$y'' - 2y'+2y = 0 \qquad (1)$$

Let $y_1(x) = e^x \sin x$

$$y_2(x) = e^x \cos x \qquad (2)$$

$$\therefore\ y_1'(x) = \frac{d}{dx}\left(e^x \sin x\right)$$

$$= e^x \frac{d}{dx}\sin x + \sin x \frac{d}{dx} e^x$$

$= e^x\cos x + \sin x\, e^x$

$y_1'(x) = e^x(\sin x + \cos x)$ (3)

$\therefore\ y_2''(x) = \frac{d}{dx}e^x(\sin x + \cos x)$

$= e^x\frac{d}{dx}(\sin x + \cos x) + (\sin x + \cos x)\frac{d}{dx}e^x$

$= e^x(\cos x - \sin x) + (\sin x + \cos x)e^x$

$y_2''(x) = 2e^x\cos x$

(4)

Now $y_1''(x) - 2y_1'(x) + 2y_1(x)$

$= 2e^x\cos x - 2e^x(\sin x + \cos x) + 2e^x\sin x$

$= 0$

$\therefore\ y_1(x) = e^x\sin x$ is a solution of (1)

$y_2'(x) = \frac{d}{dx}e^x\cos x$

$= e^x\frac{d}{dx}\cos x + \cos x\frac{d}{dx}e^x$

$= e^x(-\sin x) + \cos x\, e^x$

$y_2'(x) = e^x(\cos x - \sin x)$ (5)

$y_2''(x) = \frac{d}{dx}e^x(\cos x - \sin x)$

$= e^x\frac{d}{dx}(\cos x - \sin x) + (\cos x - \sin x)\frac{d}{dx}e^x$

$= e^x(-\sin x - \cos x) + (\cos x - \sin x)e^x$

$= e^x(-2\sin x)$

$y_2''(x) = -2e^x\sin x \quad \rightarrow (6)$

Now $y_2''(x) - 2y_2'(x) + 2y_2(x)$

$= -2e^x\sin x - 2e^x(\cos - \sin x) + 2e^x(\cos x - \sin x)$

$= 0$

$\therefore\ y_2(x) = e^x\cos x$ is also a solution of (1)

Now, the wronskian W(x) of $y_1(x)$ and $y_2(x)$ is given by

$$w(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

$$= \begin{vmatrix} e^x\sin x & e^x\cos x \\ e^x(\sin x + \cos x) & e^x(\cos x - \sin x) \end{vmatrix}$$

$= e^x \sin x . e^x(\cos x - \sin x) - e^x \cos x . e^x (\sin x + \cos x)$

$= e^{2x}(\sin x \cos x - \sin^2 x) - e^{2x}(\cos x \sin x + \cos^2 x)$

$= - e^{2x}(\sin^2 x + \cos^2 x)$

$= - e^{2x}$

$\neq 0$

$w(x) \neq 0$

$\therefore$ $y_1(x)$ and $y_2(x)$ are linearly independent solution of (1).

$\therefore$ The general solution of (1) is

$y(x) = c_1 y_1(x) + c_2 y_2(x)$

$\Rightarrow y(x) = c_1 e^x \sin x + c_2 e^x \cos x$

$\Rightarrow y(x) = e^x(c_1 \sin x + c_2 \cos x)$ (7)

Where c_1 and c_2 are arbitrary constants.

Now from (7)

$$y'(x) = \frac{d}{dx} e^x (c_1 \sin x + c_2 \cos x)$$

$$\Rightarrow y'(x) = e^x \frac{d}{dx}(c_1 \sin x + c_2 \cos x) + (c_1 \sin x + c_2 \cos x)\frac{d}{dx} e^x$$

$$\Rightarrow y'(x) = e^x (c_1 \cos x + c_2 \sin x) + (c_1 \sin x + c_2 \cos x) e^x$$

$$\Rightarrow y'(x) = e^x ((c_1 + c_2)\cos x + (c_1 - c_2)\sin x) \quad (8)$$

Now putting x = 0 in (7)

$y(0) = e^0(c_1 \sin 0 + c_2 \cos 0)$

$\Rightarrow 2 = c_2 x 1$ (given $y(0) = 2$)

$\Rightarrow c_2 = 2$

Putting x = 0 in (8)

$y'(0) = e^0\{(c_1 + c_2)\cos 0 + (c_1 - c_2) \sin 0\}$

$\Rightarrow - 3 = c_1 + c_2$ (given $y'(0) = -3$)

$\Rightarrow - 3 = c_1 + 2$

$\Rightarrow c_1 = -5$

$\therefore$ Putting the values of c_1 and c_2 in (7), we get

$y(x) = e^x(-5 \sin x + 2 \cos x)$

$\Rightarrow y(x) = e^x(2 \cos x - 5 \sin x)$

This is the solution of (1) satisfying the given properties.

Example 10

Show that e^x and e^{-x} are linearly independent solutions of $y''-y = 0$ on any interval.

Given equation is

$$y''-y = 0 \qquad \rightarrow (1)$$

Let $y_1(x) = e^x$, $y_2(x) = e^{-x}$ $\rightarrow (2)$

$$\therefore\ y_1'(x) = \frac{d}{dx}e^x = e^x$$

$$y_1''(x) = \frac{d}{dx}\left(e^x\right) = e^x$$

Now $y_1''(x) - y(x)$

$= e^x - e^x$

$= 0$

$\therefore\ y_1(x) = e^x$ is a solution of (1)

Now $y_2(x) = e^{-x}$

$$y_2'(x) = \frac{d}{dx}\left(e^{-x}\right) = -e^{-x}$$

$$y_2''(x) = \frac{d}{dx}\left(-e^{-x}\right) = -\left(-e^{-x}\right) = e^{-x}$$

Now $y_2''(x) - y_2(x)$

$= e^{-x} - e^{-x}$

$= 0$

$\therefore\ y_2(x) = e^{-x}$ is also a solution of (1)

Now, the wronskian $w(x)$ of $y_2(x)$ and $y_2(x)$ is given by

$$w(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

$$= \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}$$

$= e^x(-e^{-x}) - e^{-x}.e^x$

$= -1 - 1$

$= -2 \neq 0$

$\therefore\ y_1(x)$ and $y_2(x)$ are linearly independent solution of (1)

The general solution of (1) is given by

$y(x) = c_1y_1(x) + c_2y_2(x)$

$\Rightarrow y(x) = c_1e^x + c_2e^{-x}$

Where c_1 and c_2 are arbitrary constants.



CHECK YOUR PROGRESS – 4

Q 1: Show that $y_1(x) = e^{-x/2}\sin\left(\frac{\sqrt{3}}{2}x\right)$ and $y_2(x) = e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right)$ are linearly independent solution of the differential equation $y''+y'+y = 0$.

Q. 2: Prove that sin(2x) and cos(2x) are solutions of $y''+4y = 0$ and these solutions are linearly independent.

Q. 3: Show that e^{2x} and e^{3x} are linearly independent solutions of $y''-5y'+6y = 0$. Find the solution y(x) with the property $y(0) = 0$ and $y'(0) = 1$.



4.10 LET US SUM UP

- If f(x, y) is a function defined for all (x, y) in a domain D, then we say that f(x, y) satisfy Lipschitz condition on D, if there exists a constant k such that.

  $|f(x, y_1) - f(x, y_2)| \le k\,|y_1 - y_2|$

  = $(x, y_1), (x, y_2) \in D$.

  The constant k is called Lipschitz constant.

- Picard's formula is

  $$y_n(x) = y_0 + \int_{x_0}^{x} f(x, y_{n-1})dx$$

  if we given I.V.P. is of the form

  $\frac{dy}{dx} = f(x, y)$ with intial condition $y(x_0) = y_0$

- The functions $y_1(x), y_2(x), \ldots, y_n(x)$ are called linearly dependent if

there exists constants $e_1, e_2, \ldots e_n$ (not all zero) such that $e_1y_1+e_2y_2+\ldots+e_ny_n = 0$

If $e_1 = e_2 = \ldots = e_n = 0$, Then $y_1, y_2, \ldots y_n$ are said to be linearly independent.

- The wronskian of n functions $y_1(x), y_2(x), \ldots y_n(x)$ is

$$w(x) = \begin{vmatrix} y_1 & y_2 & \ldots. & y_n \\ y_1' & y_2' & \ldots. & y_n' \\ y_1^{(n-1)} & y_2^{(n-1)} & \ldots. & y_n^{(n-1)} \end{vmatrix}$$

- Two solution $y_1(x)$ and $y_2(x)$ of the equation $a_0(x)\, y'' + a_1(x)y'+a_2(x)y = 0$, $a_0(x) \neq 0$ are linearly dependent if and only if their wronskian is identically zero.
- The wronskian of two solution of the equation $a_0(x)\, y''+a_1(x)\, y'+a_2(x)y = 0$, $a_0(x) \neq 0$, $x \in (a, b)$ is either identically zero on never zero on (a, b)



4.11 FURTHER READINGS

1) Ross, Sheply L. (2004) : Differential Equations (3rd Ed), John Wiley & Sons, India.
2) Raisinghania, M. D. : Ordinary and Partial Differential Equations, S. Chand & Co.
3) Raisinghania, M. D. : Advanced Differential Equations S. Chand & Co.



4.12 ANSWERS TO CHECK YOUR PROGRESS

Check Your Progress-1

Ans to Q No 1: Given $f(x, y) = \sqrt{x^2+y^2}$

Now $|f(x, y_1)| = \left|\sqrt{x^2+y_1^2} - \sqrt{x^2+y_2^2}\right|$

$$= \left|\frac{\left(\sqrt{x_1^2+y_1^2} - \sqrt{x^2+y_2^2}\right)\left(\sqrt{x+y_1^2} + \sqrt{x+y_2^2}\right)}{\sqrt{x_1^2+y_1^2} + \sqrt{x^2+y_2^2}}\right|$$

$$= \left| \frac{\left(x^2 + {y_1}^2\right) - \left(x^2 + {y_2}^2\right)}{\sqrt{x^2 + {y_1}^2} + \sqrt{x^2 + {y_2}^2}} \right|$$

$$= \left| \frac{{y_1}^2 - {y_2}^2}{\sqrt{x^2 + {y_1}^2} + \sqrt{x^2 + {y_2}^2}} \right|$$

$$= \frac{|y_1 - y_2|\,|y_1 + y_2|}{\sqrt{x^2 + {y_1}^2} + \sqrt{x^2 + {y_2}^2}}$$

$$\leq \frac{|y_1| + |y_2|}{\sqrt{x^2 + {y_1}^2} + \sqrt{x^2 + {y_2}^2}} |y_1 - y_2|$$

$$\leq |y_1 - y_2|$$

$|f(x, y_1) - f(x, y_2)| \leq |y_1 - y_2|$

$\therefore$ f(x, y) satisfies Lipschitz condition in y.

The Lipschitz constant is 1

Ans to Q No 2: Left as exercise.

Ans to Q No 3: Left as exercise.

Check Your Progress-2

Ans to Q No 1: Left as exercise

Ans to Q No 2: Given problem is

$\frac{dy}{dx} = 2 - \frac{y}{x}$, where y = 2 when x = 1 $\rightarrow (1)$

We known that the n^{th} approximation y_n of the initial value problem.

$\frac{dy}{dx} = f(x, y)$ where $y = y_0$ when $x = x_0$ $\rightarrow (2)$

is given by

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\,dx \qquad \rightarrow (3)$$

Compaing (1) & (2)

$f(x,y) = 2 - \frac{y}{x}$, $x_0 = 1$ and $y_0 = 2$ $\rightarrow (4)$

$\therefore$ From (3), $y_n = 2 + \int_1^2 \left(2 - \frac{1}{x} y_{n-1}\right) dx$ $\rightarrow (5)$

First approximation

Putting n = 1 in (5), we get

$$y_1 = 2 + \int_1^x \left(2 - \frac{1}{x} y_0\right) dx$$

$$= 2 + \int_1^x \left(2 - \frac{2}{x}\right) dx$$

$$= 2 + [2x - 2\log x]_1^x$$

$$= 2 + 2x - 2\log x - (2x1 - 2\log 1)$$

$$= 2 + 2x - 2\log x - 2 + 2 \times 0$$

$$\Rightarrow y_1 = 2x - 2\log x \qquad \rightarrow (6)$$

Secong approximation

Putting n = 2 in (5) we get

$$y_2 = 2 + \int_1^x \left(2 - \frac{y_1}{x}\right) dx$$

$$= 2 + \int_1^x \left\{2 - \frac{1}{x}(2x - 2\log x)\right\} dx$$

$$= 2 + \int_1^x \log x \cdot \frac{1}{x} dx$$

$$= 2 + \int_1^x \left[(\log x)^2\right]_1^x$$

$$= 2 + (\log x)^2 - (\log 1)^2$$

$$\Rightarrow y_2 = 2 + (\log x)^2 \qquad \rightarrow (7)$$

Third approximation

Putting n = 3 in (5) we get

$$y_3 = 2 + \int_1^x \left(2 - \frac{1}{x} y_2\right) dx$$

$$= 2 + \int_1^x \left\{2 - \frac{1}{x}\left(2 + (\log x)^2\right)\right\} dx \text{ , by (7)}$$

$$= 2 + \int_1^x \left\{2 - \frac{2}{x} - (\log x)^2 \frac{1}{x}\right\} dx$$

$$= 2 + \left[2x - 2\log x - \frac{(\log x)^3}{3}\right]_1^x$$

$$= 2 + 2x - 2\log x - \frac{1}{3}(\log x)^3 - \left\{2x0 - 2\log 1 - \frac{(\log 1)^3}{3}\right\}$$

$$= 2 + 2x - 2\log x - \frac{1}{3}(\log x)^3 - 2$$

$$\Rightarrow y_3 = 2x - 2\log x - \frac{1}{3}(\log x)^3$$

Check Your Progress-3

Ans to Q No 1: Left as exercise

Ans to Q No 2: Left as exercise

Check Your Progress-4

Ans to Q No 1: Here given equation as

$y'' + y + y = 0$

Let $y_1(x) = e^{-x/2}\sin\left(\frac{\sqrt{3}}{2}x\right)$

$$y_2(x) = e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right)$$

$$\therefore y_1'(x) = \frac{d}{dx}e^{-x/2}\sin\left(\frac{\sqrt{3}}{2}x\right)$$

$$= e^{-x/2}\frac{d}{dx}\sin\left(\frac{\sqrt{3}}{2}x\right) + \sin\left(\frac{\sqrt{3}}{2}x\right)\frac{d}{dx}e^{-x/2}$$

$$= e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right)\frac{d}{dx}\left(\frac{\sqrt{3}}{2}x\right) + \sin\left(\frac{\sqrt{3}}{2}x\right)e^{-x/2}\frac{d}{dx}\left(-x/2\right)$$

$$\therefore y_1'(x) = -\frac{1}{2}e^{-x/2}\sin\left(\frac{\sqrt{3}}{2}x\right) + e^{-x/2}\frac{\sqrt{3}}{2}\cos\left(\frac{\sqrt{3}}{2}x\right)$$

$$y_2''(x) = \frac{d}{dx}\left(-\frac{1}{2}e^{-x/2}\sin\left(\frac{\sqrt{3}}{2}x\right) + \frac{\sqrt{3}}{2}e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right)\right)$$

$$= -\frac{1}{2}\left(e^{-x/2}\frac{d}{dx}\sin\left(\frac{\sqrt{3}}{2}x\right) + \sin\left(\frac{\sqrt{3}}{2}x\right)\frac{d}{dx}e^{-x/2}\right)$$

$$+ \frac{\sqrt{3}}{2}\left(e^{-x/2}\frac{d}{dx}\cos\left(\frac{\sqrt{3}}{2}x\right) + \cos\left(\frac{\sqrt{3}}{2}x\right)\frac{d}{dx}e^{-x/2}\right)$$

$$= -\frac{1}{2}\left(e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right) + \frac{d}{dx}\left(\frac{\sqrt{3}}{2}x\right) + \sin\left(\frac{\sqrt{3}}{2}x\right)e^{-x/2}\frac{d}{dx}\left(-x/2\right)\right)$$

$$+ \frac{\sqrt{3}}{2}\left(e^{-x/2} - \left(\sin\left(\frac{\sqrt{3}}{2}x\right)\frac{d}{dx}\left(\frac{\sqrt{3}}{2}x\right)\right) + \cos\left(\frac{\sqrt{3}}{2}x\right)e^{-x/2}\frac{d}{dx}\left(-x/2\right)\right)$$

$$= -\frac{\sqrt{3}}{2\sqrt{2}}e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right) + \frac{1}{4}e^{-x/2}\sin\left(\frac{\sqrt{3}}{2}x\right)$$

$$-\frac{3}{4}e^{-x/2}\sin\left(\frac{\sqrt{3}}{2}x\right) - \frac{\sqrt{3}}{2\sqrt{2}}e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right)$$

$$= -\frac{\sqrt{3}}{2}e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right) - \frac{1}{2}e^{-x/2}\sin\left(\frac{\sqrt{3}}{2}x\right)$$

$\therefore\ y_1''(x) + y_1'(x) + y(x)$

$$= -\frac{\sqrt{3}}{2}e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right) - \frac{1}{2}e^{-x/2}\sin\left(\frac{\sqrt{3}}{2}x\right) - \frac{1}{2}e^{-x/2}\sin\left(\frac{\sqrt{3}}{2}x\right)$$

$$+ \frac{\sqrt{3}}{2}e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right) + e^{-x/2}\sin\left(\frac{\sqrt{3}}{2}x\right)$$

$= 0$

$\therefore\ y_1(x) = e^{-x/2}\sin\left(\frac{\sqrt{3}}{2}x\right)$ is a solution of (1)

Similarly, we can show that

$y_2(x) = e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right)$ is also a solution of (1)

Now, the Wronskian W(x) of $y_1(x)$ and $y_2(x)$ is given by

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

$$W(x) = \begin{vmatrix} e^{-x/2}\sin\left(\frac{\sqrt{3}}{2}x\right) & e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right) \\ -\frac{1}{2}e^{-x/2}\sin\left(\frac{\sqrt{3}}{2}x\right) + \frac{\sqrt{3}}{2}e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right) & -\frac{1}{2}e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right) + \frac{\sqrt{3}}{2}e^{-x/2}\sin\left(\frac{\sqrt{3}}{2}x\right) \end{vmatrix}$$

$$= -\frac{1}{2}e^{-x}\sin\left(\frac{\sqrt{3}}{2}x\right)\cos\left(\frac{\sqrt{3}}{2}x\right) - \frac{\sqrt{3}}{2}e^{-x}\sin^2\left(\frac{\sqrt{3}}{2}x\right)$$

$$+ \frac{1}{2}e^{-x}\sin\left(\frac{\sqrt{3}}{2}x\right)\cos\left(\frac{\sqrt{3}}{2}x\right) - \frac{\sqrt{3}}{2}e^{-x}\cos^2\left(\frac{\sqrt{3}}{2}x\right)$$

$$= -\frac{\sqrt{3}}{2}e^{-x}\left(\sin^2\left(\frac{\sqrt{3}}{2}x\right)+\cos^2\left(\frac{\sqrt{3}}{2}x\right)\right)$$

$$= -\frac{\sqrt{3}}{2}e^{-x}$$

$\neq 0$

$\therefore$ $y_1(x)$ and $y_2(x)$ are linerarly independent solution of (1)

Ans to Q No 2: Left as exercise

Ans to Q No 3: Left as exercise



4.13 MODEL QUESTIONS

Q 1: Apply Picard's method and find the first three successive approximation of $\frac{dy}{dx} = 2xy$, $y(0) = 1$.

Q 2: Use Picard's method to approximate the solution of $\frac{dy}{dx} + 2xy^2 = 0$, $y(0) = 1$ (use three approximations)

Q 3: Show that $f(x, y) = xy$ satisfies Lipschitz condition on any rectangle $a \le x \le b$, $c \le y \le d$

Q 4: Prove that two solutions $y_1(x)$ and $y_2(x)$ of the equation $a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$, $a_0(x) \neq 0$, $x \in (a, b)$ are linearly independent if and only if their Wronskian is not zero at some point $x_0 \in (a, b)$.

Q 5: Find the Wronskian of x and xe^x.

Q 6: Show that linearly independent solution of $y'' - 3y' + 2y = 0$ are e^x and e^{2x}. Find the solution with the property $y(0) = 0$, $y'(0) = 1$.

*** ***** ***