UNIT 5 : LINEAR HOMOGENEOUS DIFFERENTIAL EQUATION OF HIGHER ORDER

UNIT STUCTURE

5.1 Learning Objectives

5.2 Introduction

5.3 Linear Homogeneous Equation

5.3.1 Equation Reducible to Homogeneous form

5.4 Method of undetermined coefficients

5.4 Method of Variation of Parameters

5.5 Let Us Sum Up

5.6 Further Reading

5.7 Answers To Check Your Progress

5.8 Model Questions

5.1 LEARNING OBJECTIVES

After going through this unit, you will able to know

- linear homogeneous equation
- equation reducible to homogeneous form
- method of variation of parameters.

5.2 INTRODUCTION

In this unit, we will discuss solution of linear homogeneous differential equations. We shall also discuss the method of variation of parametres for solving linear differential equation with constant coefficients.

5.3 LINEAR HOMOGENEOUS EQUATION

A differential equation of the form

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_2 x^2 \frac{d^2 y}{dx^2} + a_1 x \frac{dy}{dx} + a_0 y = X \quad \text{... (1)}$$

where $a_0, a_1, a_2, \cdots, a_{n-1}, a_n$ are constants and X is a function of x is called a homogeneous linear equation.

The equation is reducible to a linear equation with constants coefficients with the help of substitution $x = e^z$ which can be solved the methods described in the section 7.3.

Let $x = e^z$ i.e., $z = \log x$, then

$\frac{dy}{dx} = \frac{dy}{dz}.\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz}$ so that $x\frac{dy}{dx} = \frac{dy}{dz} = Dy$ where $D = \frac{d}{dz}$

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d^2 y}{dz^2}\frac{dz}{dx}$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2 y}{dz^2}$$

or $x^2 \frac{d^2 y}{dx^2} = -\frac{dy}{dz} + \frac{d^2 y}{dz^2} = \left(D^2 - D\right)y = D(D-1)y$

Similarly, $x^3 \frac{d^3 y}{dx^3} = \left(D^2 - D\right)y = D(D-1)(D-2)y$

and in general $x^n \frac{d^n y}{dx^n} = D(D-1)(D-2)\cdots(D-n+1)y$

Subtituting for $x\frac{dy}{dx}$, $x^2 \frac{d^2 y}{dx^2}$,... the equation (1) reduces to a linear equation with constants coefficients which is solvable by the methods already discussed.

Example 1 : Solve $x^2 \frac{d^2 y}{dx^2} - x\frac{dy}{dx} + y = \log x$

Solution : Let $x = e^z$ i.e., $z = \log x$

$\frac{dy}{dx} = \frac{dy}{dz}.\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz}$ so that $x\frac{dy}{dx} = \frac{dy}{dz} = Dy$ where $D = \frac{d}{dz}$

so, $x^2\frac{d^2y}{dx^2} = -\frac{dy}{dz} + \frac{d^2y}{dz^2} = \left(D^2 - D\right)y = D(D-1)y$

The given equation reduces

$$\left(D(D-1) - D + 1\right)y = z$$

or $(D-1)^2 = y$

C.F $= (Az + B)e^z = (A\log x + B)x$

P.I $= \frac{1}{(D-1)^2}z$

$= (1-D)^{-2}z$

$= (1+2D)z$

$= z + 2$

$= \log x + 2$

The general solution is $y = (A\log x + B)x + \log x + 2$.

Example 2 : Solve $x^3\frac{d^3y}{dx^3} + 3x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = x^3\log x$

Solution : Let $x = e^z$ i.e., $z = \log x$

$\frac{dy}{dx} = \frac{dy}{dz}.\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz}$ so that $x\frac{dy}{dx} = \frac{dy}{dz} = Dy$ where $D = \frac{d}{dz}$

so, $x^2\frac{d^2y}{dx^2} = -\frac{dy}{dz} + \frac{d^2y}{dz^2} = \left(D^2 - D\right)y = D(D-1)y$

The given equation reduces

$\left[D(D-1)(D-2) + 3D(D-1) + D\right]y = e^{3z}z$

or $D^3y = e^{3z}z$

C.F $= \left(A + Bz + Cz^2\right)e^{0.z} = A + B\log x + C(\log x)$

P.I $= \frac{1}{D^3}\left(e^{3z}z\right)$

$= e^{3z}\frac{1}{(D+3)^3}z$

$= e^{3z}\left(1 + \frac{D}{3}\right)^{-3}z$

$= \frac{e^{3z}}{27}(1-D)z$

$$= \frac{e^{3z}}{27}(z-1)$$

$$= \frac{x^3}{27}(\log x - 1)$$

The general solution is $y = A + B\log x + C(\log x)^2 + \frac{x^3}{27}(\log x - 1)$.

5.3.1 Equations Reducible to Homogeneous form

An equation of the form

$$(a+bx)^n \frac{d^n y}{dx^n} + P_1(a+bx)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + P_{n-1}(a+bx)\frac{dy}{dx} + P_n y = X$$

where $P_1, P_2, \cdots, P_n$ are all constants,can be reduced to the homogeneous linear form by putting $a+bx = e^z$ and can be solved by the method discussed in 7.4.

Example 3 : Solve $(1+x)^2\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 2\sin\log(1+x)$

Solution : Let $1+x = e^z$ i.e., $z = \log(1+x)$

$\frac{dy}{dx} = \frac{dy}{dz}.\frac{1}{1+x}$ or $(1+x)\frac{dy}{dx} = \frac{dy}{dz} = Dy$

similarly, $(1+x)^2\frac{d^2y}{dx^2} = D(D-1)y$

The given equation becomes

$$D(D-1)y + Dy + y = 2\sin z$$

or $(D^2+1)y = 2\sin z$

Auxiliary equation is $m^2 + 1 = 0$

or $m = \pm i$

C.F $= A\cos z + B\sin z = A\cos\log(1+x) + B\sin\log(1+x)$

P.I $= \frac{1`}{D^2+1}(2\sin z)$

$$= 2\mathrm{Im}\left(\frac{1`}{D^2+1}e^{iz}\right)$$

$$= 2\mathrm{Im}\left(e^{iz}\frac{1`}{(D+i)^2+1}1\right)$$

$$= 2\text{Im}\left(e^{iz} \frac{1`}{D^2 + 2iD} 1 \right)$$

$$= 2\text{Im}\left(e^{iz} \frac{1}{2iD}\left(1 + \frac{D}{2i}\right)^{-1} 1 \right)$$

$$= 2\text{Im}\left(e^{iz} \frac{1}{2iD} 1 \right)$$

$$= 2\text{Im}\left(e^{iz} \frac{z}{2i} \right)$$

$$= \text{Im}\left(\frac{z\cos z + 2i\sin z}{i} \right)$$

$$= -z\cos z$$

$$= -\log(1+x)\cos\log(1+x)$$

The general solution is

$$y = A\cos\log(1+x) + B\sin\log(1+x) - \log(1+x)\cos\log(1+x).$$



CHECK YOUR PROGRESS - 1

Q 1 : Solve the following differential equations :

(a) $x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right)$

(b) $\left(x^2D^2 - 3xD + 4\right)y = 2x^2$

(c) $\left(x^2D^2 - xD + 4\right)y = \cos(\log x) + x\sin(\log x)$

(d) $(x+a)^2 \frac{d^2y}{dx^2} - 4(x+a)\frac{dy}{dx} + 6y = x$



Hoene Wronsky (1776-1853) was a Polish mathematician and philospher

5.4 METHOD OF UNDETERMINED COEFFICIENTS

The non-homogeneous differential equation

$ay'' + by' + cy = g$ .....(i)

can be solved using method of undetermined coefficients.

Working rule.

Step 1: Solve the homogeneous equation $ay'' + by' + cy = 0$

i.e., find y_1, y_2 linearly independent of each other, and form the solution $Y_H = C_1y_1 + C_2y_2$.

Step 2: Find a particular/specific solution Y for (i), by method of undetermined coefficients;

Step 3: The general solution for (i) is then

$$y = Y_H + Y = C_1 y_1 + C_2 y_2 + Y$$

Find C_1, C_2 by initial conditions, if given.

The following table suggests the form of the trial solution y (for particular integral) to be used corresponding to a special form of x.

S.No.	Special form of x	Trial solution y for P.I.
1.	x^n or $a^n x^n$ or $a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$	$A_0 + A_1 x + ... + A_n x_n$
2.	e^{ax} or pe^{ax}	Ae^{ax}
3.	$a_n x^n e^{ax}$ or $e^{ax}\ (a_0 + a_1 x + a_2 x^2 + ... + a_n x^n)$	$e^{ax}(A_0 + A_1 x + A_2 x^2 + ... + A_n x^n)$
4.	$p \sin ax$ or $q \cos ax$ or $p \sin ax + q \cos ax$	$A \sin ax + B \cos ax$
5.	$pe^{bx} \sin ax$ or $qe^{bx} \cos ax$ or e^{bx} $(p \sin ax + q \cos ax)$	$e^{bx}(A \sin ax + B \cos ax)$
6.	$x^n \sin ax$ or $a_n x^n \sin ax$ or $(a_0 + a_1 x + ... + a_n x^n) \sin ax$ or $x^n \cos ax$ or $a_n x^n \cos ax$ or $(a_0 + a_1 x + ... + a_n x^n) \cos ax$	$(A_0 + A_1 x + ... + A_n x^n) \sin ax$ $+ (A_0' + A_1' x + ... + A_n' x^n) \cos ax$

In the above table, n is a positive integer and $a_0, a_1, ...a_n$, $p, q, a, b, A_0, A_1, ...A_n, A_0', A_1', ...A_n'$ are constants.

Ex 1: By the method of undetermined coefficients, solve $(D^2 + 4)y = x^2$

Solution: Given, $(D^2 + 4)y = x^2$ (1)

Auxiliary equation is

$m^2 + 4 = 0$ so that $m = \pm i$

$\therefore$ C.F. $= c_1 \cos 2x + c_2 \sin 2x$, c_1, c_2 being arbitrary constants.

Let the trial solution be $y^* = A_0 + A_1 x + A_2 x^2$ (2)

Since y* must satisfy (i), we have

$$(D^2 + 4)y^* = x^2$$

$$\Rightarrow D^2 y^* 4y^* = x^2$$

Now, (ii) $\Rightarrow Dy^* = A_1 + 2A_2x$ (3)

$\Rightarrow D^2y^* = 2A_2$ (4)

Using (ii) and (iv), (iii) reduces to

$$2A_2 + 4(A_0 + A_1x + A_2x^2) = x^2$$

$$\Rightarrow 2A_2 + 4A_0 + 4A_1x + 4A_1x + 4A_2x^2 = x^2$$

Comparing the coefficients of like terms, we get

$$2A_2 + 4A_0 = 0 \qquad 4A_1 = 0 \qquad 4A_2 = 0$$

We get, $A_1 = 0, A_2 = \frac{1}{4}, A_0 = -\frac{1}{8}$

from (ii), we have $y^* = -\frac{1}{8} + \frac{x^2}{4} = \frac{1}{8}(2x^2 - 1)$

Hence the required solution is Y = C.F + P.I

= CF + y*

$$\therefore y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{8}(2x^2 - 1)$$



CHECK YOUR PROGRESS - 2

Q 1 : Using the method of undetermined coefficient, solve

(i) $y'' - 2y' + y = x^2$

(ii) $(D^2 + 3D + 2)y = x + \cos x$

(iii) $(D^2 + 1)y = 12\cos^2 x$

5.4 METHOD OF VARIATION OF PARAMETERS

Theorem : For the equation $y'' + Py' + Qy = X$ where P, Q and X being functions of x, the P.I is given by $-y_1\int \frac{y_2X}{W}dx + y_2\int \frac{y_1X}{W}dx$. Here y_1 and y_2 are solutions of $y'' + Py' + Qy = 0$ and $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$, the Wronskian of y_1, y_2 named after the Polish mathematician and philospher Hoene Wronsky.

Proof : Let the complementary function of $y'' + Py' + Qy = X$ be $y = c_1y_1 + c_2y_2$. Here c_1 and c_2 can be regarded as parametric

functions $u(x)$, $v(x)$.

Let the P.I be $y = u(x)y_1 + v(x)y_2$

Differentiating w.r.t x, we get

$y' = uy_1' + u'y_1 + vy_2' + v'y_2$

Choose u & v such that $u'y_1 + v'y_2 = 0$ so that

$$y' = uy_1' + vy_2'$$

and $y'' = u'y_1' + uy_1'' + v'y_2' + vy_2''$

Since y_1, y_2 are roots of $y'' + Py' + Qy = 0$, we have

$$y_1'' + Py_1' + Qy_1 = 0$$

and $$y_2'' + Py_2' + Qy_2 = 0$$

The given equation becomes

$u'y_1' + uy_1'' + v'y_2' + vy_2'' + P\left(uy_1' + vy_2'\right) + Qy = X$

or $u(y_1'' + Py_1' + Qy_1) + v(y_2'' + Py_2' + Qy_2)$

$$- Q\left(uy_1 + vy_2\right) + u'y_1' + v'y_2' + Qy = X$$

or $u'y_1' + v'y_2' - Q\left(uy_1 + vy_2\right) + Qy = X$

or $u'y_1' + v'y_2' - Qy + Qy = X \quad (\ y = uy_1 + vy_2)$

or $u'y_1' + v'y_2' - X = 0$ ... (1)

Also, $u'y_1 + v'y_2 + 0.X = 0$ ... (2)

Solving (1) and (2), we get

$$\frac{u'}{y_2} = \frac{v'}{-y_1} = \frac{X}{y_1'y_2 - y_1y_2'} = \frac{X}{-W}$$

Hence $u' = \frac{X}{-W}y_2$ and $v' = \frac{X}{W}y_1$

i.e., $u = -\int \frac{y_2X}{W}dx$ and $v = \int \frac{y_1X}{W}dx$

and P. I $= -y_1\int \frac{y_2X}{W}dx + y_2\int \frac{y_1X}{W}dx$

Hence proved.

Example 4 : Solve $\frac{d^2y}{dx^2} + 4y = \tan 2x$

Solution : Auxiliary equation is $m^2 + 4 = 0$

or $m = \pm 2i$

C.F $= A\cos 2x + B\sin 2x$

$= Ay_1 + By_2$ where $y_1 = \cos 2x$ and $y_2 = \sin 2x$ and $X = \tan 2x$.

Wronskian, $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$.

P. I $= -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx$

$= -\cos 2x \int \frac{\sin 2x \tan 2x}{2} dx + \sin 2x \int \frac{\cos 2x \tan 2x}{2} dx$

$= -\frac{\cos 2x}{2} \int \frac{1-\cos^2 2x}{\cos 2x} dx + \sin 2x \int \frac{\sin 2x}{2} dx$

$= -\frac{1}{2}\cos 2x \int (\sec 2x - \cos 2x) dx - \frac{1}{4}\sin 2x \cos 2x$

$= -\frac{1}{2}\cos 2x \left\{ \frac{1}{2}\log(\sec 2x + \tan 2x) - \frac{1}{2}\sin 2x \right\} - \frac{1}{4}\sin 2x \cos 2x$

$= -\frac{1}{4}\cos 2x \left\{ \frac{1}{2}\log(\sec 2x + \tan 2x) - \sin 2x \right\} - \frac{1}{4}\sin 2x \cos 2x$

$= -\frac{1}{4}\cos 2x \log(\sec 2x + \tan 2x)$

The general solution is

$y = A\cos 2x + B\sin 2x - \frac{1}{4}\cos 2x \log(\sec 2x + \tan 2x)$.

Example 5 : Solve $\frac{d^2 y}{dx^2} + a^2 y = \sec ax$

Solution: Auxiliary equation is $m^2 + a^2 = 0$

or $m = \pm ia$

C.F $= A\cos ax + B\sin ax$

Let $y_1 = \cos ax$ and $y_2 = \sin ax$

Wronskian, $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix} = a$ and $X = \sec ax$.

P. I $= -y_1 \int \frac{y_2 \sec ax}{a} dx + y_2 \int \frac{y_1 \sec ax}{a} dx$

$= -\frac{1}{2}\cos ax \int \frac{\sin ax}{\cos ax} dx + \frac{\sin ax}{a} \int dx$

$$= \frac{1}{a^2}\cos ax \log \cos ax + \frac{x}{a}\sin ax$$

The general solution is

$$y = A\cos ax + B\sin ax + \frac{1}{a^2}\cos ax \log \cos ax + \frac{x}{a}\sin ax .$$

Example 6 : Solve by the method of variation of parameters :

$$y'' - 2y' + y = e^x \log x$$

Solution : The given equation can be written as

$$\left(D^2 - 2D + 1\right)y = e^x \log x$$

The auxiliary equation is $m^2 - 2m + 1 = 0$

or $(m-1)^2 = 0$

or $m = 1,1$

C. F $= (Ax + B)e^x$

Let $y_1 = e^x, y_2 = xe^x, X = e^x$ and $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix}$

$$= e^{2x}\begin{vmatrix} 1 & x \\ 1 & 1+x \end{vmatrix} = e^{2x}$$

P. I $= -y_1\int \frac{y_2 X}{W}dx + y_2\int \frac{y_1 X}{W}dx$

$$= -e^x\int \frac{xe^{2x}\log x}{e^{2x}}dx + xe^x\int \frac{e^x.e^x \log x}{e^{2x}}dx$$

$$= -e^x\int x\log x dx + xe^x\int \log x dx$$

$$= -e^x\left\{\log x.\frac{x^2}{2} - \int \frac{1}{x}.\frac{x^2}{2}dx\right\} + xe^x\left\{x\log x - \int dx\right\}$$

$$= -e^x\left\{\frac{x^2}{2}\log x - \frac{x^2}{4}\right\} + xe^x(\log x - 1)$$

$$= \frac{1}{4}x^2e^x\{2\log x + 1 + 4(\log x - 1)\}$$

$$= \frac{1}{4}x^2e^x(2\log x - 3)$$

The general solution is

$$y = (Ax + B)e^x + \frac{1}{4}x^2 e^x (2\log x - 3).$$

Example 7 : Solve by method of varation of parameters :

$$\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$$

Solution : Given, $\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$

A. E is $m^2 - 1 = 0$

or $m^2 = 1$

or $m = \pm 1$

C.F $= C_1 e^x + C_2 e^{-x}$

and $y^1 . y_2^{/} - y_1^{/} . y_2 = -e^x . e^{-x} - e^x . e^{-x} = -2$

$$u = \int \frac{-y^2 X}{y_1 . y_2^{/} - y_1^{/} . y_2} dx = -\int \frac{e^{-x}}{-2} \times \frac{2}{1+e^x} dx$$

$$= \int \frac{e^{-x}}{1+e^x} dx$$

$$= \int \frac{dx}{e^x (1+e^x)}$$

$$= \int \left(\frac{1}{e^x} - \frac{1}{1+e^x} \right) dx$$

$$= \int e^{-x} dx - \int \frac{e^{-x}}{e^{-x}+1} dx$$

$$= -e^{-x} + \log(e^{-x} + 1)$$

$$v = \int \frac{y_1 X}{y_1 . y_2^{/} - y_1^{/} . y_2} dx = \int \frac{e^{-x}}{-2} \times \frac{2}{1+e^x} dx$$

$$= \int \frac{e^x}{e^x + 1} dx$$

$$= -\log(e^x + 1)$$

P.I $= u . y_1 + v . y_2$

$= \left[-e^{-x} + \log(e^{-x} + 1) \right] e^x - e^{-x} \log(1 + e^x)$

$= -1 + e^x \log(e^{-x} + 1) - e^{-x} \log(1 + e^x)$

The general solution is

$y = C_1 e^x + C_2 e^{-x} - 1 + e^x \log\left(e^{-x}+1\right) - e^{-x}\log\left(1+e^x\right).$



CHECK YOUR PROGRESS - 3

Q 1 : Apply the method of variation of parameters to solve.

(i) $y_2 + a^2 y = \cot ax$

(ii) $y_2 - 2y_1 + y = xe^x \log x, x > 0$

(iii) $\left(D^2 - 2D + 2\right)y = e^x \tan x$

(iv) $y'' + 3y' + 2y = x + \cos x$



5.5 LET US SUM UP

- A differential equation of the form

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_2 x^2 \frac{d^2 y}{dx^2} + a_1 x \frac{dy}{dx} + a_0 y = X$$

...(1)

where $a_0, a_1, a_2, \cdots, a_{n-1}, a_n$ are constants and X is a function of x is called a homogeneous linear equation.

The equation is reducible to a linear equation with constants coefficients with the help of substitution $x = e^z$ which can be solved the methods described in the section 12.3.

Let $x = e^z$ i.e., $z = \log x$

$$x\frac{dy}{dx} = \frac{dy}{dz} = Dy$$

$$x^2 \frac{d^2 y}{dx^2} = -\frac{dy}{dz} + \frac{d^2 y}{dz^2} = \left(D^2 - D\right)y = D(D-1)y$$

Similarly, $x^3 \frac{d^3 y}{dx^3} = \left(D^2 - D\right)y = D(D-1)(D-2)y$

and in general $x^n \frac{d^n y}{dx^n} = D(D-1)(D-2)\cdots(D-n+1)y$

The substitution of these values in the given equation reduces the given homogeneous equation to a differential equation with constants coefficients.

- Method of variation of parameter to find particular integral of

$P\frac{d^2y}{dx^2}+Q\frac{dy}{dx}+Ry=X$ is as follows :

Let complementary function(C.F) $= Ay_1 + By_2$

then particular integral(P.I) $= uy_1 + vy_2$

where $u=\int\frac{-y_2X}{W}dx$ and $v=\int\frac{y_1X}{W}dx$, $W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$.

General solution =C.F +P.I.



5.6 FURTHER READING

1) Tenenbaum, Morris and Pollard, Harry. Ordinary Differential Equations, Dover Publications, Inc, N.Y 11501.
2) Differential Equations: An Introduction with Mathematica. Clay C.Ross, second Edition, Springer.
3) Differential Equations: Linear, Nonlinear, ordinary, partial. A.C. King, J.Billinggham and S. R. Otto, Cambridge-2003.
4) Differential equations with Application and Programs. S. Balachandra Rao and H.R. Anuradha, University Press (India) Limited 1996.



5.7 ANSWERS TO CHECK YOUR PROGRESS

CHECK YOUR PROGRESS – 1

Ans to Q No 1 : (b) Given that $\left(x^2D^2-3xD+4\right)y=2x^2$ ... (1)

Let $x = e^z$ or $z = \log x$ and $D_1 = \frac{d}{dz}$.

Then (1) becomes

$\{D_1(D_1 - 1) - 3D_1 + 4\}y = 2e^{2z}$

or $(D_1 - 2)^2 y = 2e^{2z}$

Auxiliary equation is $(m-2)^2 = 0$

or $m = 2,2$

C. F $= (C_1 + C_2 z)e^{2z} = (C_1 + C_2 z)(e^z)^2 = (C_1 + C_2 \log x)x^2$

P. I $= \frac{1}{(D_1 - 2)^2} 2e^{2z}$

$= 2\frac{z^2}{2!}e^{2z} = z^2(e^z)^2$

$= (\log x)^2 x^2$

Complete solution is

$y = (C_1 + C_2 \log x)x^2 + (\log x)^2 x^2$.

(c) Putting $x = e^z$ and denoting $\frac{d}{dz}$ by D_1, the given equation becomes

$[D_1(D_1 - 1) - D_1 + 4]y = \cos z + e^z \sin z$

or $[D_1^2 - 2D_1 + 4]y = \cos z + e^z \sin z$

Auxiliary equation is $m^2 - 2m + 4 = 0$

or $m = 1 \pm \sqrt{3}i$

C.F $= e^z[C_1 \cos \sqrt{3}z + C_2 \sin \sqrt{3}z]$

$= x[C_1 \cos \sqrt{3}(\log x) + C_2 \sin \sqrt{3}(\log x)]$

P.I $= \frac{1}{D_1^2 - 2D_1 + 4}\cos z + \frac{1}{D_1^2 - 2D_1 + 4}e^z \sin z$

$= \frac{1}{-1^2 - 2D_1 + 4}\cos z + e^z \frac{1}{(D_1 + 1)^2 - 2(D_1 + 1) + 4}\sin z$

$= \frac{1}{3 - 2D_1}\cos z + e^z \frac{1}{D_1^2 + 4}\sin z$

$= \frac{(3 + 2D_1)}{9 - 4D_1^2}\cos z + e^z \frac{1}{-1^2 + 3}\sin z$

$$= \frac{3\cos z - 2\sin z}{9+4} + \frac{e^z \sin z}{2}$$

$$= \frac{1}{13}\left[3\cos(\log x) - 2\sin(\log x)\right] + \frac{x}{2}\sin(\log x)$$

Hence the complete solution is

$$y = x\left[C_1 \cos\sqrt{3}(\log x) + C_2 \sin\sqrt{3}(\log x)\right] + \frac{1}{13}\left[3\cos(\log x) - 2\sin(\log x)\right] + \frac{x}{2}\sin(\log x)$$

(d) Putting $x + a = e^z$ and denoting $\frac{d}{dx}$ by D ,the equation becomes

$$\left[D(D-1) - 4D + 6\right]y = e^z - a$$

or $\left[D^2 - 5D + 6\right]y = e^z - a$

Auxiliary equation is $m^2 - 5m + 6 = 0$

or $m = 3,2$

C.F $= C_1 e^{3z} + C_2 e^{2z} = C_1(x+a)^3 + C_2(x+a)^2$

P.I $= \frac{1}{D^2 - 5D + 6}.e^z - \frac{1}{D^2 - 5D + 6}.a$

$$= \frac{1}{(D-3)(D-2)}.e^z - \frac{1}{(D-3)(D-2)}.a$$

$$= \frac{e^z}{(1-3)(1-2)} - a\frac{1}{(D-3)(D-2)}e^{0z}$$

$$= \frac{1}{3}e^z - a\frac{1}{(0-3)(0-2)}e^{0z}$$

$$= \frac{1}{3}e^z - a\frac{1}{6}e^{0z}$$

$$= \frac{1}{3}e^z - \frac{1}{6}a$$

$$= \frac{1}{3}(x+a) - \frac{1}{6}a$$

Hence, the complete solution is

$$y = C_1(x+a)^3 + C_2(x+a)^2 + \frac{1}{3}(x+a) - \frac{1}{6}a$$

CHECK YOUR PROGRESS – 2

Ans to Q No 1 : (i) Given, $y_2 + a^2 y = \cot ax$ ... (1)

Auxiliary equation is $m^2 + a^2 = 0$

or $m = \pm ia$

C.F $= C_1 \cos ax + C_2 \sin ax$

Here $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix} = a \neq 0$

Then P. I $= u.y_1 + v.y_2$

where $u = -\int \frac{y_2.X}{W} dx = -\int \frac{\sin ax \cot ax}{a} dx = -\frac{1}{a}\int \cos ax dx = -\frac{\sin ax}{a^2}$

and $v = \int \frac{y_1.X}{W} dx = \int \frac{\cos ax \cot ax}{a} dx = \frac{1}{a}\int \frac{1-\sin^2 ax}{\sin ax} dx$

$$= \frac{1}{a}\int (\operatorname{cosec} ax - \sin ax) dx$$

$$= \frac{1}{a}\int (\operatorname{cosec} ax - \sin ax) dx = \frac{1}{a^2}\left(\log \tan \frac{ax}{2} + \cos ax\right)$$

P. I $= \cos ax \times \left(-\frac{1}{a^2}\right) \times \frac{1}{a^2} \times \sin ax + \sin ax \times \frac{1}{a^2} \times \left(\log \tan \frac{ax}{2} + \cos ax\right)$

$$= \frac{1}{a^2} \times \log \tan\left(\frac{ax}{2}\right)$$

Hence, the general solution of (1) is y = C.F+P.I.

i.e., $y = C_1 \cos ax + C_2 \sin ax + \frac{1}{a^2} \times \log \tan\left(\frac{ax}{2}\right)$.

(ii) Given, $(D^2 - 2D + 1)y = xe^x \log x, x > 0$, $D = \frac{d}{dx}$

Auxiliary equation is $(m-1)^2 = 0$

or $m = 1,1$

C.F $= (C_1 + C_2 x)e^x = C_1 e^x + C_2 x e^x$

Let $y_1 = e^x$, $y_2 = xe^x$. Also, $X = xe^x \log x$

Here, $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^{2x} \neq 0$

Then, P. I $= u.y_1 + v.y_2$

where $u = -\int \frac{y_2.X}{W} dx = -\int \frac{xe^x.xe^x \log x}{e^{2x}} dx = -\int x^2 \log x dx$

$$= -\left[\log x.\frac{x^3}{3} - \int \frac{1}{x}.\frac{x^3}{3}dx\right]$$

$$= -\left[\log x.\frac{x^3}{3} - \frac{1}{9}x^3\right]$$

and $v = \int \frac{y_1.X}{W}dx = \int \frac{e^x.xe^x \log x}{e^{2x}}dx = \int x \log x dx$

$$= \frac{x^2}{2}\log x - \int \frac{x^2}{2}.\frac{1}{x}dx$$

$$= \frac{x^2}{2}\log x - \frac{x^2}{4}$$

P.I $= -e^x\left\{\frac{x^3}{3}\log x - \frac{x^3}{9}\right\} + xe^x\left\{\frac{x^2}{2}\log x - \frac{x^2}{4}\right\}$

$$= x^3e^x \log x\left(\frac{1}{2} - \frac{1}{3}\right) - x^3e^x\left(\frac{1}{4} - \frac{1}{9}\right)$$

$$= \left(\frac{1}{6}\right) \times x^3e^x \log x - \left(\frac{5}{36}\right) \times x^3e^x$$

Hence, the general solution is y = C.F+P.I.

$\therefore\ y = C_1e^x + C_2xe^x + \left(\frac{1}{6}\right) \times x^3e^x \log x - \left(\frac{5}{36}\right) \times x^3e^x$.

(iii) Do Yourself.

The required solution is

$y = e^x\left(C_1 \cos x + C_2 \sin x\right) - e^x \cos x \log\left(\sin x + \tan x\right)$

(iv) Do youself.

The required solution is

$y = C_1e^{-x} + C_2e^{-2x} + \frac{x}{2} - \left(\frac{3}{4}\right) + \left(\frac{1}{10}\right) \times \left(3\sin x - \cos x\right)$.

CHECK YOUR PROGRESS – 2

Ans to Q.No. 1:

(i) Try yourself

(ii) Given, $(D^2 + 3D + 2)y = x + \cos x$ (i)

Auxiliary Equation is

$$m^2 + 3m + 2 = 0$$

$$\Rightarrow m = -1, -2$$

$\therefore$ C.F. $= c_1 e^{-x} + c_2 e^{-2x}, c_1 c_2$ being arbitrary constants.

Corresponding to special form x of R.H.S. of (i), we choose trial solution for P.I. as $A_0 \cos x + A_3 \sin x$. Combining these, we attempt a trial for P.I. as

$$y^* = A_0 + A_1 x + A_2 \cos x + A_3 \sin x \qquad \text{(ii)}$$

where A_0, A_1, A_2 and A_n are constants to be determined. Since y* must satisfy (i), we get

$$(D^2 + 3D + 2)y^* = x + \cos x$$

$$\Rightarrow D^2 y^* + 3Dy^* + 2y^* = x + \cos x \qquad \text{(iii)}$$

from (ii)

$$Dy^* = A_1 - A_2 \sin x + A_3 \cos x \qquad \text{(iv)}$$

$$D^2 y^* = -A_2 \cos x - A_3 \sin x \qquad \text{(v)}$$

Using (iii), (iv), (v), (iii) reduces to

$$-A_2 \cos x - A_3 \sin x + 3(A_1 - A_2 \sin x + A_3 \cos x)$$

$$+ \ 2(A_0 + A_1 x + A_2 \cos x + A_3 \sin x) = x + \cos x$$

$$\Rightarrow 3A_1 + 2A_0 + 2A_1 x + (A_2 + 3A_3)\cos x$$

$$+ (A_3 - 3A_2)\sin x = x + \cos x$$

Equating the coefficient of like terms, we get

$$3A_1 + 2A_0 = 0 \qquad 2A_1 = 0 \qquad A_2 + 3A_3 = 1 \qquad A_3 - 3A_2 = 0$$

Solving, $A_0 = -\frac{3}{4}, A_1 = \frac{1}{2}, A_2 = \frac{1}{10}, A_3 = \frac{3}{10}$

Then from (ii), we get

$$y^* = -\frac{3}{4} + \frac{x}{2} + \frac{1}{10}(\cos x + 3\sin x)$$

$\therefore$ Required general solution is

Y = CF + PI

= CF + y*

$$= c_1 e^{-x} + c_2 e^{-2x} - \frac{3}{4} + \frac{x}{2} + \frac{1}{10}(\cos x + 3\sin x)$$

(iii) Try yourself.



5.8 MODEL QUESTIONS

Q 1 : Solve the following differential equations

(a) $(3x+2)^2\frac{d^2y}{dx^2}+5(3x+2)\frac{dy}{dx}-3y=x^2+x+1$

(b) $x^3\frac{d^3y}{dx^3}+3x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=\log x+x$

(c) $\left(x^3D^3-3x^2D^2+6xD-6\right)y=(\log x)^2$

(d) $(x+1)^2\frac{d^2y}{dx^2}-3(x+1)+4y=x$

(e) $\left\{(1+2x)^2D^2-6(1+2x)D+16\right\}y=8(1+2x)^2$

Q 2 : Solve the following differential equations by the method of variation of parameters :

(a) $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=2x$

(b) $\frac{d^2y}{dx^2}+y=\cos ecx\cot x$

(c) $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+3y=x+\sin x$

(d) $\left(D^2-9\right)y=e^{2x}+x$

(e) $\frac{d^2y}{dx^2}+y=\sin^2 x$

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