UNIT 6 : ORTHOGONALITY AND INDEPENDENCE IN O.D.E.'S

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6.1 LEARNING OBJECTIVES

After going through this unit, you will able to-

- orthogonal and orthonormal set of functions with examples
- orthogonal and orthonormal set of functions with respect to a weight function with examples
- gram-schmidt orthonomalization process with examples

- singular solution of an ordinary differential equation
- p-discriminant and e-discriminant
- determination of singular solution with examples.

6.2 INTRODUCTION

Orthogonality and orthonomality of a set of functions bears significant importance in linear algebra as well as in differential equations. In this unit, we have discussed throughly about orthogonal and orthonomal set of functions (with or without weight function) and Gram-schmidt process of orthonormalizations.

Singular solution is a special type of solution of ordinary differential equation. In this unit, we have throughly discussed about the methods of finding singular solution.

6.3 ORTHOGONALITY

Two functions f(x) and g(x) defined on same interval $a \le x \le b$ are said to be orthogonal on $a \le x \le b$ if

$$\int_a^b f(x)g(x)dx = 0$$

The norm ||f(x)|| of f(x) is defined by $\|f(x)\| = \left\{\int_a^b f(x)^2 dx\right\}^{1/2}$

6.4 ORTHOGONAL SET OF FUNCTIONS

Let $\{f_n(x)\}$, where n=1, 2, 3, . . . be a set of functions defined on some interval $a \le x \le b$. Then the set $\{f_n(x)\}$ is said to be and orthogonal set of functions on the interval $a \le x \le b$ if

$$\int_a^b f_m(x)f_n(x)dx = 0 \text{ whenever } m \ne n$$

6.5 ORTHONORMAL SET OF FUNCTIONS

Let $\{f_n(x)\}$, where n = 1, 2, 3, . . . be a set of function defined on some interval $a \le x \le b$. Then the set $\{f_n(x)\}$ is said to be orthonormal on $a \le x \le b$ if they are orthogonal on $a \le x \le b$ and all have norm 1. Thus, set $\{f_n(x)\}$ is orthonomal on $a \le x \le b$ if

$$\int_a^b f_m(x) f_n(x) dx = \begin{cases} 0, & \text{When } m \neq n \\ 1, & \text{When } m = n \end{cases}$$

i.e., $\int_a^b f_m(x) f_n(x) dx = \delta_{mn}$

Where dmn denotes kronecker delta which is defined as

$$\delta_{mn} = \begin{cases} 0, & \text{When } m \neq n \\ 1, & \text{When } m = n \end{cases}$$

6.6 ORTHOGONALITY WITH RESPECT TO A WEIGHT FUNCTION

Let p(x) >0. Then two functions f(x) and g(x) defined on some interval $a \le x \le b$ are said to be orthogonal on $a \le x \le b$ with respect to weight function p(x), if

$$\int_a^b p(x) f(x) g(x) dx = 0$$

Then, the norm ||f(x)|| of f(x) is defined by

$$\| f(x) \| = \left\{ \int_a^b p(x) (f(x))^2 dx \right\}^{1/2}$$

6.7 ORTHOGONAL SET OF FUNCTIONS WITH RESPECT TO A WEIGHT FUNCTION

Let $\{f_n(x)\}$, where n = 1, 2, 3, . . . be a set of functions defined on some interval $a \le x \le b$. Then the set $\{f_n(x)\}$ is said to be orthogonal on $a \le x \le b$ with respect to weight function p(x) >0, if

$$\int_a^b p(x) f_m(x) f_n(x)\,dx = 0 \text{, whenever } m \neq n$$

6.8 ORTHONOMAL SET OF FUNCTIONS WITH RESPECT TO A WEIGHT FUNCTION

Let $\{f_n(x)\}$, where n = 1, 2, 3, . . . be a set of functions defined on some interval $a \leq x \leq b$. Then the set $\{f_n(x)\}$ is said to be orthonormal with respect to a weight function p(x) > 0 if they are orthogonal with respect to weight function p(x) on $a \leq x \leq b$ and all have norm 1. Thus, the set $\{f_n(x)\}$ is orthonomal with respect to weight function p(x), if

$$\int_a^b p(x) f_m(x) f_n(x)\,dx = \delta_{mn} = \begin{cases} 0, & \text{When } m \neq n \\ 1, & \text{When } m = n \end{cases}$$

6.9 WORKING RULE FOR GETTING ORTHONORMAL SET $\{F_N(X)\}$ OF FUNCTION CORRESPONDING TO A KNOWN ORTHOGONAL SET $\{F_N(X)\}$ WHERE NONE OF THE FUNCTIONS $F_N(X)$ HAVE ZERO NORM

Divide each function $f_n(x)$, n = 1, 2, 3, . . . by its norm $\|f_n(x)\|$ and get a new function $\phi_n(x) = \dfrac{f_n(x)}{\|f_n(x)\|}$, n = 1, 2, 3, . . . Then we have

$$\|\varphi_n(x)\| = \left\{\int_a^b (\varphi_n(x))^2\,dx\right.$$

$$= \left[\int_a^b \left\{\frac{f_n(x)}{\|f_n(x)\|}\right\}^2 dx\right]^{1/2}$$

$$= \frac{1}{\|f_n(x)\|}\left\{\int_a^b (f_n(x))^2\,dx\right\}^{1/2}$$

$$= \frac{1}{\|f_n(x)\|} \cdot \|f_n(x)\|$$

= 1

This shows that norm of $\phi_n(x)$ is 1.

Hence the set $\{\varphi_n(x)\}$, i.e., $\left\{\frac{f_n(x)}{\|f_n(x)\|}\right\}$ is an orthonormal set of functions.

6.10 GRAM-SCHMIDT PROCESS OF ORTHONORMALIZATION

Let $\{f_n(x)\}$, where n = 1, 2, 3,. . . be a set of linearly independent function for each of which norm $\|f_n(x)\|$, exists and is non-zero. Then, we wish to obtain an orthonormal set $\{\varphi_n(x)\}$, where n = 1, 2, 3, . . . such that

$$\int_a^b \varphi_m(x)\varphi_n(x)dx = \begin{cases} 0, \text{ when } m \neq n \\ 1, \text{ when } m = n \end{cases} \qquad (1)$$

We select $f_1(x)$ and obtain

$$\varphi_n(x) = \frac{f_1(x)}{\|f_1(x)\|} \qquad (2)$$

Next, we choose $f_2(x)$ and let

$$F_2(x) = f_2 + c\varphi_1 \qquad (3)$$

Where c is choosen in such a way that F_2 may be orthogonal to φ_1

$$\therefore \int_a^b F_2(x)\varphi_1(x)dx = 0$$

$$\Rightarrow \int_a^b (f_2 + c\varphi_1)\varphi_1 dx = 0 \quad \text{using (3)}$$

$$\Rightarrow \int_a^b f_2\,\varphi_1\,dx + c\int_a^b \varphi_1^{\,2}dx = 0$$

$$\Rightarrow \int_a^b f_2\,\varphi_1\,dx + c\,.1 = 0 \quad \{\text{using (1)}\}$$

$$\Rightarrow \int_a^b f_2\,\varphi_1\,dx + e = 0$$

$$\Rightarrow c = -\int_a^b f_2\,\varphi_1\,dx$$

Putting this value of c in (3), we get

$$F_2 = f_2 - \varphi_1\int_a^b f_2\,\varphi_1\,dx \qquad (4)$$

Now, we take $\varphi_2(x) = \frac{F_2(x)}{\|F_2(x)\|}$ (5)

Now, we choose f_3 and let

$F_3(x) = f_3 + e_1\varphi_1 + e_2\varphi_2$ (6)

Where e_1 and e_2 are chosen in such a way that F_2 may be orthogonal to φ_1 and φ_2, ie.

$$\int_a^b (f_3 + e_1\varphi_1 + e_2\varphi_2)\,dx = 0$$

$$\Rightarrow \int_a^b f_3\,\varphi_1 dx + e_1 \int_a^b \varphi_1^2 dx + e_2 \int_a^b \varphi_2\,\varphi_1 dx = 0$$

$$\Rightarrow \int_a^b f_3\,\varphi_1 dx + e_1.1 + e_2.0 = 0 \quad (7)$$

$$\Rightarrow e_1 = -\int_a^b f_3\,\varphi_1 dx$$

and $$\int_a^b (f_3 + e_1\varphi_1 + e_2\varphi_2)\varphi_2 d_2 = 0$$

$$\Rightarrow \int_a^b f_3\,\varphi_2\,dx + e_1 \int_a^b \varphi_1\,\varphi_2\,dx + e_2 \int_a^b \varphi_2^2 dx = 0$$

$$\Rightarrow \int_a^b f_3\,\varphi_2\,dx + e_1.0 + e_2.1 = 0$$

$$\Rightarrow e_2 = -\int_a^b f_3\,\varphi_2\,dx \quad (8)$$

Using (7) and (8), (6) gives

$$F_3 = f_3 - \varphi_1 \int_a^b f_3\,\varphi_1\,dx - \varphi_2 \int_a^b f_3\,\varphi_1 dx \quad (9)$$

So, we take $\varphi_3(x) = \frac{F_3(x)}{\|F_3(x)\|}$ (10)

By continuing the above process, the n^{th} nomalized function φ_n is given by

$$\varphi_n(x) = \frac{F_n(x)}{\|F_n(x)\|} \quad (11)$$

Where $F_n = f_n - \varphi_1 \int_a^b f_n \varphi_1 dx - \varphi_2 \int_a^b f_n \varphi_2 dx - \ldots - \varphi_{n-1} \int_a^b f_n \varphi_{n-1} dx$ (12)

The above process will fail if and only if at some stage $F_n = 0$ for $1 \le r \le n$.

But $F_r = 0$

$$\Rightarrow f_r - \varphi_1 \int_a^b f_r \varphi_1 dx - \varphi_2 \int_a^b f_r \varphi_2 dx - \ldots - \varphi_{r-1} \int_a^b f_r \varphi_{r-1} dx = 0$$

$\Rightarrow f_r$ is a linear combination of $\varphi_1, \varphi_2, \ldots, \varphi_{r-1}$

$\Rightarrow f_r$ is a linear combination of $f_1, f_2, \ldots f_{r-1}$ (by our construction of new function $\varphi_1, \varphi_2, \ldots, \varphi_{r-1}$)

$\Rightarrow f_1, f_2, \ldots f_r$ is a linearly dependent set for $1 \le r \le n$.

But this is a contradiction because $\{f_n(x)\}$, where n = 1, 2, 3, . . , is linearly independent set and its every subset would also be linearly independent.

Hence $F_r \neq 0$, for $1 \le r \le n$

So, we would always get a set $\{\varphi_n(x)\}$, where n = 1, 2, 3, . . . which would be orthonormal.

Remark

Sometimes orthonormalization is required with respect to a weight function p(x)>0.

Then (1) takes the form

$$\int_a^b \rho(x)\varphi_m(x)\varphi_n(x)dx = \begin{cases} 0, & \text{when } m \neq n \\ 1, & \text{when } m = n \end{cases}$$

Also, $\int_a^b F_2 \varphi_1 dx = 0$ world takes the form $\int_a^b \rho(x) F_2(x) dx = 0$, etc. rest of the procedure is similar.

Example 1

Show that the set of functions $\left\{ \sin\left(\frac{n\pi x}{e}\right) \right\}$, n = 1, 2, 3, . . . is orthogonal on the interval and find the corresponding orthonormal set.

Given functions are

$f_n(x) = \sin\left(\frac{n\pi x}{e}\right)$, n = 1, 2, 3, . . .

For $m \neq n$, we have

$$\int_0^e f_m(x) f_n(x)\,dx = \int_0^e \sin\left(\frac{m\pi x}{e}\right)\sin\left(\frac{n\pi x}{e}\right)dx$$

$$= \frac{1}{2}\int_0^e 2\sin\left(\frac{m\pi x}{e}\right)\sin\left(\frac{n\pi x}{e}\right)dx$$

$$= \frac{1}{2}\int_0^e \left\{\cos\left(\frac{m\pi x}{e} - \frac{n\pi x}{e}\right) - \cos\left(\frac{m\pi x}{e} + \frac{n\pi x}{e}\right)\right\}dx$$

$$= \frac{1}{2}\int_0^e \left\{\cos\left(\frac{(m-n)\pi x}{e}\right) - \cos\left(\frac{(m+n)\pi x}{e}\right)\right\}dx$$

$$= \frac{1}{2}\left\{\int_0^e \cos\left(\frac{(m-n)\pi x}{e}\right)dx - \int_0^e \cos\left(\frac{(m+n)\pi x}{e}\right)dx\right\}$$

$$= \frac{1}{2}\left[\frac{e}{\pi(m-n)}\sin\left(\frac{(m-n)\pi x}{e}\right)\right]_0^e - \left[\frac{e}{\pi(m+n)}\sin\left(\frac{(m+n)\pi x}{e}\right)\right]_0^e$$

$$= \frac{1}{2}\left\{\left[\frac{e}{\pi(m-n)}\sin\left(\frac{(m-n)\pi e}{e}\right) - \frac{e}{\pi(m-n)}\sin\left(\frac{(m-n)\pi 0}{e}\right)\right]\right.$$

$$\left. - \left[\frac{e}{\pi(m+n)}\sin\left(\frac{(m+n)\pi e}{e}\right) - \frac{e}{\pi(m+n)}\sin\left(\frac{(m+n)\pi.0}{e}\right)\right]\right.$$

$$= \frac{1}{2}\{0 + 0 - (0+0)\}$$

$$= 0$$

Therefore the given set of functions is orthogonal let

Now norm of $f_n(x) = \|f_n(x)\| = \left\{\int_0^e \sin^2\left(\frac{n\pi x}{e}\right)dx\right\}^{1/2}$

$$= \left[\frac{1}{2}\int_0^e \left\{1 - \cos\left(\frac{2n\pi x}{e}\right)\right\}dx\right]^{1/2}$$

$$\left(\because \cos^2\theta = 1 - 2\sin^2\theta \;\Rightarrow \sin^2\theta = \frac{1}{2}(1 - \cos^2\theta)\right)$$

$$= \left[\frac{1}{2}\int_0^e dx - \frac{1}{2}\int_0^e \cos\left(\frac{2n\pi x}{e}\right)dx\right]^{1/2}$$

$$= \left\{\frac{1}{2}[x]_0^e - \frac{1}{2}\times\frac{e}{2n\pi}\left[\sin\left(\frac{2n\pi x}{e}\right)\right]_0^e\right\}^{1/2}$$

$$= \left\{\frac{1}{2}(e-0) - \frac{e}{4n\pi}\left[\frac{\sin 2n\pi e}{e} - \left(\frac{\sin 2n\pi 0}{e}\right)\right]\right\}^{1/2}$$

$$= \left\{\frac{1}{2}e - \frac{e}{4n\pi}(0-0)\right\}^{1/2}$$

$$= \left(\frac{e}{2}\right)^{1/2}$$

Let $\varphi_n(x) = \dfrac{f_n(x)}{\|f_n(x)\|}$

$$= \frac{\sin\left(\frac{n\pi x}{e}\right)}{\left(e/2\right)^{1/2}}$$

$$= \left(2/e\right)^{1/2}\sin\left(\frac{n\pi x}{e}\right)$$

Hence, the required orthonormal set is given by

$\{\varphi_n(x)\}$, i.e, $\left\{\left(\frac{2}{e}\right)^{1/2}\sin\left(\frac{n\pi x}{e}\right)\right\}$ where n = 1,2, 3, . . .

Example 2

Show that the set of functions {cos nx}, n = 0, 1, 2, 3, . . . is orthogonal in the interval $-\pi \le x \le \pi$ and find the corresponding orthonormal set of function.

Given functions are

$f_n(x) = \cos nx$, n = 0, 1, 2, . . .

For $m \neq n$, we have

$$\int_{-\pi}^{\pi} f_m(x) f_n(x)dx = \int_{-\pi}^{\pi} \cos mx \cos nx\, dx$$

$$= 2\int_{-0}^{\pi} \cos mx \cos nx\,dx$$

$$= 2\int_{0}^{\pi} 2\cos mx \cos nx\,dx$$

$$= \int_{0}^{\pi} \{\cos(m+n)x + \cos(m-n)x\}\,dx$$

$(\because 2\cos A \cos B = \cos(a+B)+\cos(A-B))$

$$= \int_{0}^{\pi} \cos(m+n)x\,dx + \int_{0}^{\pi} \cos(m-n)x\,dx$$

$$= \left[\frac{\sin(m+n)\pi}{m+n} - \frac{\sin(m+n)0}{m+n}\right] + \left[\frac{\sin(m-n)\pi}{m-n} - \frac{\sin(m-n)0}{m-n}\right]$$

$$= [0-0]+[0-0]$$

$= 0$

Therefore, the given set of functions is orthogonal

Now, for $n \neq 0$, norm of $f_n(x)$ is

$$\|f_n(x)\| = \left\{\int_{-\pi}^{\pi} (f_n(x))^2\,dx\right\}^{1/2}$$

$$= \left\{\int_{-\pi}^{\pi} \cos^2 nx\,dx\right\}^{1/2}$$

$$= \left\{2\int_{0}^{\pi} \cos^2 nx\,dx\right\}^{1/2}$$

$$= \left\{2\int_{0}^{\pi} \frac{1}{2}(1+\cos 2nx)dx\right\}^{1/2}$$

$$\because \cos 2\theta = 2\cos^2\theta - 1 \Rightarrow \cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$$

$$= \left\{\int_{0}^{\pi} (1+\cos 2n\pi x)dx\right\}^{1/2}$$

$$= \left\{\int_{-\pi}^{\pi} dx + \int_{-\pi}^{\pi} \cos 2nx\,dx\right\}^{1/2}$$

$$= \left\{ [x]_0^{\pi} + \left[\frac{1}{2n} \sin 2nx \right]_0^{\pi} \right\}^{1/2}$$

$$= \left\{ \pi - 0 + \frac{1}{2n} (\sin 2n\pi - \sin 2n.0) \right\}^{1/2}$$

$$= \left\{ \pi + \frac{1}{2n} (0 - 0) \right\}^{1/2}$$

$$= \pi^{1/2}$$

$$= \sqrt{\pi}$$

Again, for n = 0

$$\| f_n(x) \| = \left\{ \int_{-\pi}^{\pi} \cos^2 \theta x \, dx \right\}^{1/2}$$

$$= \left\{ \int_{-\pi}^{\pi} .1.dx \right\}^{1/2}$$

$$= \left\{ [x]_{-\pi}^{\pi} \right\}^{1/2}$$

$$= \{ \pi - (-\pi) \}^{1/2}$$

$$= (2\pi)^{1/2}$$

$$= \sqrt{2\pi}$$

Thus, $\varphi_n(x) = \dfrac{f_n(x)}{\| f_n(x) \|}$

$$= \begin{cases} \dfrac{1}{\sqrt{2\pi}}, & \text{if } n = 0 \\ \dfrac{1}{\sqrt{\pi}} \cos nx, & \text{if } n \neq 0 \end{cases}$$

Hence, the required orthonormal set is

$$\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos x, \frac{1}{\sqrt{\pi}} \cos 2x, \frac{1}{\sqrt{\pi}} \cos 3x, \text{ - - -}$$

Example 3

Given that $f_1(x) = a_0$, $f_2(x) = b_0 + b_1 x$ and $f_3(x) = e_0 + e_1 x + e_2 x^2$. Determine the constants a_0, b_0, e_0, b_1, e_1 and e_2 so that the given functions form an orthonormal set on the interval $-1 \leq x \leq 1$.

Since f_1, f_2 and f_3 form an orthonormal set on $-1 \leq x \leq 1$, There fore,

we have

(i) $\int_{-1}^{1} f_1 f_2 \, dx = 0$

(ii) $\int_{-1}^{1} f_2 f_3 \, dx = 0$

(iii) $\int_{-1}^{1} f_3 f_1 \, dx = 0$

(iv) $\| f_1 \| \neq 0$

(v) $\| f_2 \| \neq 0$

(vi) $\| f_3 \| \neq 0$

Now (I) $\Rightarrow \int_{-1}^{1} f_1 f_2 \, dx = 0$

$$\Rightarrow \int_{-1}^{1} a_0 (b_0 + b_1 x) dx = 0$$

$$\Rightarrow \int_{-1}^{1} (a_0 b_0 + a_0 b_1 x) dx = 0$$

$$\Rightarrow \left[a_0 b_0 x + \frac{a_0 b_1 x^2}{2} \right]_{-1}^{1} = 0$$

$$\Rightarrow a_0 b_0 . 1 + \frac{a_0 b_1 1^2}{2} - a_0 b_0 (-1) - \frac{a_0 b_1 (-1)^2}{2} = 0$$

$\Rightarrow 2a_0 b_0 = 0$

$\Rightarrow a_0 b_0 = 0$ (1)

(II) $\Rightarrow \int_{-1}^{1} f_2 f_3 \, dx = 0$

$$\Rightarrow \int_{-1}^{1} (b_0 + b_1 x) + (e_0 + e_1 x + e_2 x^2) dx = 0$$

$$\Rightarrow \int_{-1}^{1} \left(b_0 e_0 + b - e_1 x + b_0 e_2 x^2 + b_1 e_0 x + b_1 e_1 x^2 + b_1 e_2 x^3 \right) dx = 0$$

$$\Rightarrow \left[b_0 e_0 x + \frac{1}{2} b_0 e_1 x^2 + \frac{1}{3} b_0 e_2 x^3 + \frac{1}{2} b_1 e_0 x^2 + \frac{1}{3} b_1 e_1 x^3 + \frac{1}{4} b_1 e_2 x^4 \right]_{-1}^{1} = 0$$

$$\Rightarrow b_0e_0(1-(-1))+\frac{1}{2}b_0e_1\left(1^2-(-1)^2\right)+\frac{1}{3}b_0e_2\left(1^3-(-1)^3\right)$$

$$+\frac{1}{2}b_1e_0\left(1^2-(-1)^2\right)+\frac{1}{3}b_1e_1\left(1^3-(-1)^3\right)+\frac{1}{4}b_1e_2\left(1^4-(-1)^4\right)=0$$

$$\Rightarrow 2b_0e_0+\frac{1}{2}b_0e_1\times 0+\frac{1}{3}\times 2b_0e_2+\frac{1}{2}b_1e_0\times 0$$

$$+\frac{1}{3}b_1e_1\times 2+\frac{1}{4}b_1e_2\times 0=0$$

$$\Rightarrow 2b_0e_0+\frac{2}{3}b_0e_2+\frac{2}{3}b_1e_1=0$$

$$\Rightarrow \frac{6b_0e_0+2b_0e_2+2b_1e_1}{3}=0$$

$$\Rightarrow 2b_0(3e_0+e_2)=-2b_1e_1$$

$$\Rightarrow b_0(3e_0+e_2)+b_1e_1=0 \qquad (2)$$

(III) $$\Rightarrow \int_{-1}^{1} f_3\, f_1\, dx=0$$

$$\Rightarrow \int_{-1}^{1}\left(e_3+e_1x+e_2x^2\right)a_0\, dx=0$$

$$\Rightarrow \int_{-1}^{1}\left(a_0e_0+a_0e_1x+a_0e_2x^2\right)dx=0$$

$$\Rightarrow \left[a_0e_0+\frac{1}{2}a_0e_1x^2+\frac{1}{3}a_0e_2x^3\right]_{-1}^{1}=0$$

$$\Rightarrow a_0e_0(1-(-1))+\frac{1}{2}a_0e_1\left(1^2-(-1)^2\right)+\frac{1}{3}a_0e_2\left(1^3-(-1)^3\right)=0$$

$$\Rightarrow a_0e_0\times 2+\frac{1}{2}a_0e_1\times 0+\frac{1}{3}a_0e_2\times 2=0$$

$$\Rightarrow 2a_0e_0+\frac{2}{3}a_0e_2=0$$

$$\Rightarrow \frac{6a_0e_0+2a_0e_2}{3}=0$$

$$\Rightarrow 2a_0(3e_0+e_2)=0$$

$$\Rightarrow a_0(3e_0+e_2)=0 \qquad (3)$$

Now (IV) $\Rightarrow \|f_1\|\neq 0$

$$\Rightarrow \left\{\int_{-1}^{1}(f_1(x))^2\,dx\right\}^{1/2} \neq 0$$

$$\Rightarrow \left\{\int_{-1}^{1}a_0{}^2\,dx\right\}^{1/2} \neq 0$$

$$\Rightarrow \left\{a_0{}^2[x]_{-1}^{1}\right\}^{1/2} \neq 0$$

$$\Rightarrow \left\{a_0{}^2(1-(-1))\right\}^{1/2} \neq 0$$

$$\Rightarrow a_0 2^{1/2} \neq 0$$

$$\Rightarrow a_0 \neq 0 \qquad (4)$$

Now (1) and (4) gives

$$b_0 = 0 \qquad (5)$$

Again since $b_0 = 0$, therefor (2) gives

$$b_1 e_1 = 0 \qquad (6)$$

Now (V) $\Rightarrow \|f_2(x)\| \neq 0$

$$\Rightarrow \left\{\int_{-1}^{1}(f_2(x))^2\,dx\right\}^{1/2} \neq 0$$

$$\Rightarrow \left\{\int_{-1}^{1}(b_0+b_1x)^2\,dx\right\}^{1/2} \neq 0$$

$$\Rightarrow \left\{\int_{-1}^{1}(b_1x)^2\,dx\right\}^{1/2} \neq 0 \quad \text{(Since (5) } \Rightarrow b_0 = 0\text{)}$$

$$\Rightarrow \left\{\left[b_1^2\frac{x^3}{3}\right]_{-1}^{1}\right\}^{1/2} \neq 0$$

$$\Rightarrow \left\{\frac{b_1^2}{3}\left(1^3-(-1)^3\right)\right\}^{1/2} \neq 0$$

$$\Rightarrow b_1\left(\frac{2}{3}\right)^{1/2} \neq 0$$

$$\Rightarrow b_1 \neq 0 \qquad (7)$$

Now (7) and (6) gives

$e_1 = 0$ (8)

Now (VI) $\Rightarrow \| f_3 \| \neq 0$

$$\Rightarrow \left\{ \int_{-1}^{1} (f_3(x))^2 dx \right\}^{1/2} \neq 0$$

$$\Rightarrow \left\{ \int_{-1}^{1} \left(e_0 + e_1 x + e_2 x^2\right)^2 dx \right\}^{1/2} \neq 0$$

$$\Rightarrow \left\{ \int_{-1}^{1} \left(e_0 + e_2 x^2\right)^2 dx \right\}^{1/2} \neq 0 \qquad \text{(Since (8) } \Rightarrow e_1 = 0\text{)}$$

$$\Rightarrow \left\{ \int_{-1}^{1} \left(e_0^2 + 2e_0 e_2 x^2 + e_2^2 x^4\right) dx \right\}^{1/2} \neq 0$$

$$\Rightarrow \left\{ \left[e_0^2 x + \frac{2}{3} e_0 e_2 x^3 + \frac{1}{5} e_2^2 x^5 \right]_{-1}^{1} \right\}^{1/2} \neq 0$$

$$\Rightarrow \left\{ e_0^2 (1 - (-1)) + \frac{2}{3} e_0 e_2 \left(1^3 - (-1)^3\right) + \frac{1}{5} e_2^2 \left(1^5 - (-1)^5\right) \right\}^{1/2} \neq 0$$

$$\Rightarrow \left\{ 2e_0^2 + \frac{4}{3} e_0 e_2 + \frac{2}{5} e_2^2 \right\}^{1/2} \neq 0$$

$$\Rightarrow \left\{ e_0^2 + \frac{2}{3} e_0 e_2 + \frac{1}{5} e_2^2 \right\}^{1/2} \neq 0 \qquad (9)$$

From (3) and (5), we get

$3e_0 + e_2 = 0$

$\Rightarrow e_2 = -3e_0$ (10)

Using (10) in (9), we get

$$\left\{ e_0^2 + \frac{2}{3} e_0 (-3e_0) + \frac{1}{5} (-3e_0)^2 \right\}^{1/2} \neq 0$$

$$\Rightarrow \left\{ e_0^2 - 2e_0^2 + \frac{9}{5} e_0^2 \right\}^{1/2} \neq 0$$

$$\Rightarrow \left\{ \frac{5e_0^2 - 10e_0^2 + 9e_0^2}{5} \right\}^{1/2} \neq 0$$

$$\Rightarrow \frac{\left\{4e_0^2\right\}^{1/2}}{\sqrt{5}} \neq 0$$

$$\Rightarrow \frac{2}{\sqrt{5}} e_0 \neq 0$$

$$\Rightarrow e_0 \neq 0 \qquad (11)$$

$\therefore$ From (4), (5), (7), (8), (10) and (11), we find that $b_0 = 0$, $e_1 = 0$, $e_2 = -3e_0$. and a_0, e_0, can take any arbitrary real values.

Example 4

With help of 1, x, x^2, construct three functions φ_0, φ_1, and φ_2, which are orthogonal over $-1 \leq x \leq L$.

We take

$$\varphi_0,(x) = 1 \qquad (1)$$

Next, we chose

$$\varphi_1, = x+e \qquad (2)$$

Let φ_1 be orthogonal to φ_0

$$\therefore \int_{-1}^{1} \varphi_0 \varphi_1 dx = 0$$

$$\Rightarrow \int_{-1}^{1} 1(x+e)dx = 0$$

$$\Rightarrow \left[\frac{x^2}{2} + ex\right]_{-1}^{1} = 0$$

$$\Rightarrow \left(\frac{1^2}{2} + e.1\right) - \frac{(-1)^2}{2} + e(-1) = 0$$

$$\Rightarrow 2e = 0$$

$$\Rightarrow e = 0$$

Hence (2) gives

$$\varphi_1, = x \qquad (3)$$

Next, we take

$$\varphi_2 = x^2 + e_1\varphi_0 + e_2\varphi_1$$

$$\Rightarrow \varphi_2 = x^2 + e_1 + e_2 x \qquad (4)$$

Let φ_2 be orthogonal to both φ_0 and φ_1

$$\therefore \int_{-1}^{1} \varphi_2\, \varphi_0 \, dx = 0$$

$$\Rightarrow \int_{-1}^{1} (x^2 + e_1 + e_2 x).1 \, dx = 0$$

$$\Rightarrow \left[\frac{x^3}{3} + e_1 x + e_2 \frac{x^2}{2} \right]_{-1}^{1} = 0$$

$$\Rightarrow \left(\frac{1^3}{3} + e_1.1 + e_2.\frac{1^2}{2} \right) - \left(\frac{(-1)^3}{3} + e_1(-1) + e_2 \frac{(-1)^2}{2} \right) = 0$$

$$\Rightarrow \frac{2}{3} + 2e_1 = 0$$

$$\Rightarrow 2e_1 = -2/3$$

$$\Rightarrow e_1 = -1/3$$

and $\int_{-1}^{1} \varphi_2\, \varphi_1 \, dx = 0$

$$\Rightarrow \int_{-1}^{1} (x^2 + e_1 + e_2 x).x \, dx = 0$$

$$\Rightarrow \int_{-1}^{1} \left(x^3 + e_1 x + e_2 x^2 \right) dx = 0$$

$$\Rightarrow \left[\frac{x^3}{4} + e_1 \frac{x^2}{2} + e_2 \frac{x^3}{3} \right]_{-1}^{1} = 0$$

$$\Rightarrow \left(\frac{1^4}{4} + e_1.\frac{1^2}{2} + e_2.\frac{1^3}{3} \right) - \left(\frac{(-1)^4}{4} + e_1 \frac{(-1)^2}{2} + e_2 \frac{(-1)^2}{3} \right) = 0$$

$$\Rightarrow \frac{2}{3} e_2 = 0$$

$$\Rightarrow e_2 = 0$$

(4) becomes

$$\varphi_2(x) = x^2 - \frac{1}{3}$$

So, the required functions are

$$\varphi_1 = 1$$

$$\varphi_1 = x$$

$$\varphi_2 = x^2 - \frac{1}{3}$$



CHECK YOUR PROGRESS - 1

Q 1: Show that the functions $f_1(x) = 1$, $f_2(x) = x$ are orthogonal on the interval $(-1, 1)$ and determine the constants A and B so that the function $f_3(x) = 1+Ax+Bx^2$ is orthogonal to both f_1 and f_2 on the interval $(-1, 1)$.

Q 2: With the help of 1, x, x^2 construct three functions φ_0, φ_1 and φ_2 which are orthogonal with respect to e^{-x} over $0 \leq x < \infty$.

Q 3: Show that the functions sinx, sin2x, sin3x, . . . are orthogonal on the interval $(0, \pi)$.

6.11 SINGULAR SOLUTIONS

Sometimes a differential equation possesses a solution which does not contain any arbitrary constant and which cannot be obtained from the general solution of the given differential equation by assigning a particular value to the arbitrary constant. Such an extra solution is called a singular solution of the given differential equation.

6.12 ENVELOPE

A curve which touches each member of a one-parameter family of curves and at each point is touched by some member of the family, is called the envelope of that one parameter family of curves.

Whenever the family of curves

$\varphi(x, y, c) = 0$

represented by the differential equation

$$f(x, y, p) \text{ where } p = \frac{dy}{dx} \qquad (1)$$

possesses an envelope, the equation of the envelope is the singular solution of the differential equation (1).

6.13 C-DISCRIMINANT AND P-DISCRIMINANT

The discriminant of an equation involving a single parameter is the simplest function of the coefficients in the rational integral form, whose vanishing is the condition that the equation have two equal roots. When the equation is quadratic in parameter p or c, the corresponding p and c-discriminant relations can be obtained by using the following simple formula of theory of equations.

If $Ap^2 + Bp + c = 0$ ($Ac^2+Bc+c = 0$) be quadratic in p (or c) where A, B, C are functions of x an y, then p-discriminant (or c-discriminent) relation is given by $B^2–4Ac = 0$

However, if the equation in p(or c) is of higher degree (tan 2), then p (or c) discriminant relation is found in the following way -

Method of getting p discriminant relation

Let the given equation be

$f(x, y, p) = 0$

Then, the p discriminant relation is obtained by eliminating p between the equations.

$f(x, y, p) = 0$

and $\frac{\partial f}{\partial p} = 0$

Thus, the p discriminant relation represents the locus, for each point of which $f(x, y, p) = 0$ has equal values of p.

Method of getting e-discriminant relation.

Let $\varphi(x, y, c) = 0$ be the general solution of the given differential equation

$f(x, y, p) = 0$

Let the arbitrary constant e be regarded as a parameter. Then, the e-discriminant relation is obtained by eliminating e between the equations.

$\varphi(x, y, c) = 0$

and $\frac{\partial f}{\partial c} = 0$

Thus, the c-discriminant relation represents the locus, for each point of which $\varphi(x, y, c) = 0$ has equal values of c.

6.14 DETERMINATION OF SINGULAR SOLUTION

There are two methods of determining singular solution of a given differential equation. We my find it either from the general solution of the given differential equation or directly from the given differential equation.

First Method : Determination of singular solution from the general solution $\varphi(x, y, c) = 0$ of the given differential equation $f(x, y, p) = 0$.

We know that the envelope of the family of curves $\varphi(x, y, c) = 0$ is contained in e-discriminant relation say $\psi(x, y) = 0$

As the c-discriminant $\psi(x, y) = 0$ may respresent loci other than the envelope, it follows that only that part of locus $\psi(x, y) = 0$ is the singular solution which also satisfies the given differential equation $f(x, y, p) = 0$. Accordingly, if the equation $\psi(x, y) = 0$ fails to satisfy $f(x, y, p) = 0$, then $\psi(x, y) = 0$ should be resolved into now simpler factors. Now, try to verify if any part so obtained is or is not a solution of $f(x, y, p) = 0$. Only those parts will constitute the required envelope which satisfy $f(x, y, p) = 0$.

Second Method : Determination of singular solution directly from the given differential equation $f(x, y, p) = 0$.

Let $\phi(x, y, c) = 0$ (1)

be the general solution of

$f(x, y, p) = 0$ (2)

Let $\psi(x, y) = 0$ (3)

be the c-discriminant relation.

From the theory of equations, it is known a that the equation (3) represents the locus of points (x, y) such that at least two of the corresponding values of e, are equal, i.e., such that at least two of the curves of the family through (x, y) coincide. As the equation(2) determines the slopes of the tangents to the curves of the family through (x, y), it follows that for a point (x, y) satisfying (3) at least two of the corresponding values of p must coincide. Hence, we see that the envelope and hence the singular solution is also cotained in p-discriminant relation.

6.15 WORKING RULE FOR FINDING SINGULAR SOLUTION

Method I

Let f(x, y, c) = 0 (1)

be the given differential equation.

Differentiating (1) partially with respect to p, we get

$$\frac{\partial f}{\partial p} = 0 \qquad (2)$$

Eliminate p between (1) and (2) to get p-discriminant relation

F(x, y) = 0 (3)

If (3) satisfies (1), then (3) is the required solution. If (3) does not satisfy (1), then resolve F(x, y) into simpler factors. Now it is necessary to verify if any part (3) is or is not a solution of (1). Only those parts will constitute the singular solution which satisfy (1).

From the above discussion, if follows that if p occurs only in the first degree in the given differential equation, there will be no singular solution. Similarly, if the differential equation can be resolved into a number of factors, each of linear in p, there will be no singular solution.

Method II

Let ϕ(x, y, c) = 0 (4)

be the general solution of given differential equation (1)

Differentiating (4) partially with respect to e.

$$\frac{\partial \varphi}{\partial c} = 0 \qquad (5)$$

Eliminate p between (4) and (5) to get e-discriminant relation

ψ (x, y) = 0 (6)

If (6) satisfies (1), then (6) is the required singular solution. On the other hand, if (6) does not satisfy (1), then resolve ψ(x, y) into simpler factors. Now, it is necessary to verify if any part of (6) is or is not a solution of (1). Only those parts will be the singular solutions which satisfy (1).

Particulas Case : Singular Solution of Clairauts equation

The general solution of the clairayts equation

$y = px+f(p)$ (1)

is $y = cx + f(c)$ (2)

Differentiating (2) partially w.r.t.c., we get

$0 = x+f'(c)$ (3)

The singular solution, which is the envelope of (2), is obtained by elimination of e between (2) and (3).

Now, differentiating (1) partially wrt 'p'.

$0 = x+f'(p) = 0$ (4)

Note that the equations (1) and (4) different from the equations (2) and (3) only in having p in place of c.

Accordingly, the c discriminant relations from (2) and (3) and p discriminant relation from (1) and (4) are identical and both them gives us the required singular solution.

Therefore, it is advised to make use of this fact in doing problems. Thus, there is no need to get general solution of (2) and subsequently, e-discriminant relation while dealing with clairaut's form of differential equation. Accordingly, the singular solution of the clairaut's equation (1) should always obtained from p-discriminant relation.

Example 5

Find general and singular solutions of

$9p^2(2-y)^2 = 4(3-y)$

Given equation is

$9p^2(2-y)^2 = 4(3-y)$

$$\Rightarrow p^2 = \frac{4}{9}\,\frac{3-y}{(2-y)^2}$$

$$\Rightarrow p = \pm\frac{2}{3}\,\frac{\sqrt{3-y}}{2-y}$$

$$\Rightarrow \frac{dy}{dx} = \pm\frac{2}{3}\,\frac{\sqrt{3-y}}{2-y}$$

$$\Rightarrow dx = \pm\frac{3}{2}\,\frac{2-y}{\sqrt{3-y}}dy$$

$$\Rightarrow dx = \pm\frac{3}{2}\,\frac{3-y-1}{\sqrt{3-y}}dy$$

$$\Rightarrow dx = \pm\frac{3}{2}\left(\sqrt{3-y} - \frac{1}{\sqrt{3-y}}\right)dy$$

Integrating,

$$\int dx = \pm\frac{3}{2}\left[\int (3-y)^{1/2}\,dy - (3-y)^{-1/2}\right]dy$$

$$\Rightarrow x + c = \pm\frac{3}{2}\left[\frac{(3-y)^{1/2+1}}{1/2+1} - \frac{(3-y)^{-1/2+1}}{-1/2+1}\right]$$

$$\Rightarrow x + c = \pm\frac{3}{2}\left[\frac{2}{3}(3-y)^{3/2} - 2(3-y)^{1/2}\right]$$

$$\Rightarrow x + c = \pm\frac{3}{2}\times\frac{2}{3}(3-y)^{1/2}\left[\sqrt{3-y} - 3\right]$$

$$\Rightarrow x + c = \pm(3-y)^{1/2}(-y)$$

$$\Rightarrow x + c = \mp\, y\,(3-y)^{1/2}$$

$$\Rightarrow (x+c)^2 = \left(\mp\, y\,(3-y)^{1/2}\right)^2$$

$$\Rightarrow (x+c)^2 = y^2(3-y)$$

This is the general solution of the given differential equation.

The singular solution can be obtained by any of the two following methods–

First method

The general solution is

$(x+c)^2 = y^2\,(3-y)$

$\Rightarrow x^2 + c^2 + 2cx = y^2(3-y)$

$\Rightarrow c^2 + 2xc + \{x^2 - y^2(3-y)\} = 0$

This is a quadratic equation in the parameter e.

So, the c-discriminant relation is

$4x^2 - 4\times 1\times\{x^2 - y^2(3-y)\} = 0 \qquad [B^2 - 4AC = 0]$

$\Rightarrow 4\{x^2 - x^2 + y^2(3-y)\} = 0$

$\Rightarrow y^2(3-y) = 0$

Now $y = 0$ gives, $p = \frac{dy}{dx} = 0$

Substitution of $y = 0$ and $p = 0$ in the given differential equation does not satisfy the given differential equation ($\because 9.0^2\,(2-0)^2 \neq 4(3-0)$

Hence $y = 0$ is not a singular solution.

Again $3 - y = 0 \Rightarrow y = 3$

This gives $p = \frac{dy}{dx} = \frac{d}{dx}(3) = 0$

Substitution of y = 3 and p = 0 in the given differential equation satisfy it ($\because 9.0^2 (2-3)^2 = 4(3-3)$)

Hence, y = 3 is the required singular solution

Second Method

Given equation is

$9p^2(2-y)^2 = 4(3-y)$

$\Rightarrow 9p^2 (2-y)^2 + 0.p - 4(3-y) = 0$

This is a quadratic equation in the parameter p.

So, p-discriminant relation is given by

$0^2 - 4.9 (2-y)^2 . \{-4(3-y)\} = 0 \qquad [B^2 - 4AC = 0]$

$\Rightarrow 144 (2-y)^2 (3-y) = 0$

$\Rightarrow (2-y)^2 (3-y) = 0$

Now $2 - y = 0$ gives, $y = 2$ so $p = \frac{dy}{dx} = \frac{d}{dx}(2) = 0$

Substitution of y = 2 and p = 0 in the given equation does not satisfy it ($\because 9.0^2 (2-2)^2 \neq 4(3-2)$)

Hence y = 0 is not a singular solution.

Again $3 - y = 0$ gives $y = 3$, so $p = \frac{dy}{dx} = \frac{d}{dx}(3) = 0$

Substitution of y = 3 and p = 0 in the given differential equation satisfy it ($\because 9.0^2 (2-3)^2 = 4(3-3)$)

Hence, y = 3 is the required singular solution

Example 6

Find general and singular solution of $8ap^3 = 27y$

Given equation

$8ap^3 = 27y$

$\Rightarrow p^3 = \frac{27y}{8a}$

$\Rightarrow p^3 = \left(\frac{27y}{8a}\right)^{1/3}$

$$\Rightarrow \frac{dy}{dx} = \frac{3}{2} \times \frac{y^{1/3}}{a^{1/3}}$$

$$\Rightarrow dx = \frac{2}{3} a^{1/3} y^{-1/3} dy$$

Integrating,

$$\int dx = \frac{2}{3} a^{1/3} \int y^{-1/3} dy$$

$$\Rightarrow x + c = \frac{2a^{1/3}}{3} \cdot \frac{y^{-1/3+1}}{-1/3+1}$$

$$\Rightarrow x + c = \frac{2a^{1/3}}{3} \cdot \frac{y^{2/3}}{2/3}$$

$$\Rightarrow x + c = a^{1/3} y^{2/3}$$

$$\Rightarrow (x+c)^3 = \left(a^{1/3} y^{2/3}\right)^3$$

$$\Rightarrow (x+c)^3 = a y^2 \qquad (1)$$

(1) Is the general solution of the given differential equation. Now differentiating (1) partially wrt. 'c', we get

$3(x+c)^2 = 0$

$\Rightarrow x + c = 0$

$\Rightarrow c = -x \qquad (2)$

Hence, eliminating e between (1) & (s), we get

$(x-x)^3 = ay^2$

$\Rightarrow ay^2 = 0$

$\Rightarrow y = 0$

This is the e-discriminant relation.

Now leation y = 0, $p = \frac{dy}{dx} = \frac{d}{dx}(0) = 0$

Substituting y = 0 and p = 0 in the given differential equation, we see that these value satisfy it ($\because$ 8 a.0^3 .27.0)

Hence, y = 0 is the required singular solution

Example 7

Find general and singular solution of

$3xy = 2px^2 - 2p^2$

Given equation is

$3xy = 2px^2 - 2p^2$ (1)

$$\Rightarrow y = \frac{2px^2 - 2p^2}{3x}$$

$$\Rightarrow y = \frac{2}{3}px - \frac{2}{3}p^2x^{-1} \quad (2)$$

differentiating (2) wrt. x,

$$\frac{dy}{dx} = \frac{2}{3}p + \frac{2x}{3}\frac{dp}{dx} - \frac{2}{3}\left(p^2(-1)x^{-1-1} + x^{-1}.2p\frac{dp}{dx}\right)$$

$$\Rightarrow p = \frac{2}{3}p + \frac{2x}{3}\frac{dp}{dx} + \frac{2p^2}{3}.\frac{1}{x^2} - \frac{4p}{3x}\frac{dp}{dx}$$

$$\Rightarrow 3p = 2p + 2x\frac{dp}{dx} + \frac{2p^2}{x^2} - \frac{4p}{x}\frac{dp}{dx}$$

$$\Rightarrow p - \frac{2p^2}{x^2} - 2\frac{dp}{dx}\left(x - \frac{2p}{x}\right) = 0$$

$$\Rightarrow p\left(1 - \frac{2p}{x^2}\right) - 2x\frac{dp}{dx}\left(1 - \frac{2p}{x^2}\right) = 0$$

$$\Rightarrow \left(1 - \frac{2p}{x^2}\right)\left(p - 2x\frac{dp}{dx}\right) = 0 \quad (3)$$

Omitting the first fuction which does not contain $\frac{dp}{dx}$, we get

$$p - 2x\frac{dp}{dx} = 0$$

$$\Rightarrow p = 2x\frac{dp}{dx}$$

$$\Rightarrow \frac{2}{p}dp = \frac{1}{x}dx$$

Integrating,

$$2\int\frac{1}{p}dp = \int\frac{1}{x}dx$$

$$\Rightarrow 2\log p = \log x + \log c$$

$$\Rightarrow \log p^2 = \log(cx)$$

$\Rightarrow p^2 = cx$

$$\Rightarrow p = \pm\sqrt{cx}$$

Putting this value of p in (2),

$$y = \pm\frac{2}{3}x\sqrt{cx} - \frac{2}{3}cx \times \frac{1}{x}$$

$\Rightarrow y = \pm\frac{2}{3}x\sqrt{cx} - \frac{2}{3}c$

$\Rightarrow 3y = \pm 2.x\sqrt{cx} - 2c$

$\Rightarrow 3y + 2c = \pm 2x\sqrt{cx}$

$\Rightarrow (3y+2c)^2 = \left(\pm 2x\sqrt{cx}\right)^2$

$\Rightarrow (3y+2c)^2 = 4x^2\,cx$

$\Rightarrow (3y+2c)^2 = 4cx^3$ (4)

This is the general solution of (1)

Now (4) $\Rightarrow (3y+2c)^2 = 4cx^3$

$\Rightarrow 9y^2+4c^2 + 12cy = 4cx^3$

$\Rightarrow 4c^2+4c^2(3y-x^3) + 9y^2 = 0$ (5)

From (5), the c-discriminant relation is

$\left(4c(3y-x^3)\right)^2 - 4.4.9y^2 = 0$ ($B^2 - 4AC = 0$)

$\Rightarrow 16(3y - x^3)^2 - 16.9y^2 = 0$

$\Rightarrow (9y^2+x^6-6yx^3) - 9y^2 = 0$

$\Rightarrow x^3(x^3-6y) = 0$

Now $x^3 = 0 \Rightarrow x = 0$ and then

$\frac{1}{p} = \frac{dx}{dy} = \frac{d}{dy}(0) = 0$

But these does not satisfy

So, x = 0 is not a singular solution

Again $x^3-6y = 0$ gives $y = \frac{x^3}{6}$

and then $p = \frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^3}{6}\right) = \frac{3x^2}{6} = \frac{x^2}{2}$

This value satisfy (1)

Hence $x^3-6y = 0$ is a singular solution of (1)

Example 8

Find the general and singular solution of $y = px + \frac{a}{p}$

The given equation is

$y = px + \frac{a}{p}$ (1)

Which is in clairaut's form

So replacting p by c in (1), the solution of (1) is

$$y = cx + \frac{a}{c}$$

$\Rightarrow cy = c^2x^2 + a$

$\Rightarrow c^2y - cy + a = 0$ (2)

Now c-discrimenant relation of (2) is

$(-y)^2 - 4 \cdot x\,(a) = 0$

$\Rightarrow y^2 = 4ax$ (1)

Now $y^2 = 4ax$ given

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(4ax)$$

$$\Rightarrow 2y\frac{dy}{dx} = 4a$$

$$\Rightarrow 2yp = 4a \qquad \left(\because p = \frac{dy}{dx}\right)$$

$$\Rightarrow p = \frac{2a}{y}$$

Putting this value of p in (1), we get

$$y = \frac{2a}{y}.x + \frac{a}{\frac{2a}{y}}$$

$$\Rightarrow y = \frac{2ax}{y} + \frac{y}{2}$$

$$\Rightarrow y = \frac{y^2 + 4ax}{2y}$$

$$\Rightarrow 2y^2 = y^2 + 4ax$$

$$\Rightarrow y^2 = 4ax$$

Which is true by (3)

Hence (3) satisfies (1)

$\therefore$ Required singular solution of (1) is $y^2 = 4ax$

Example 9

Find the general solution and singular solution of the ordinary differential equation $(xp-y)^2 = p^2-1$. Also, interpret the results geometrically.

Given Equation is

$(xp-y)^2 = p^2-1$ (1)

$\Rightarrow xp - y = \pm\sqrt{p^2-1}$

$\Rightarrow y = px \mp \sqrt{p^2-1}$ (2)

Which is in clairaut's form.

Reptacing p by c, the required general solution is

$y = cx \mp \sqrt{c^2-1}$

Now (3) $\Rightarrow y = cx \mp \sqrt{c^2-1}$

$\Rightarrow (y-cx)^2 = \left(\mp\sqrt{c^2-1}\right)^2$

$\Rightarrow y^2 - c^2x^2 - 2ycx = c^2 - 1$

$\Rightarrow c^2(x^2-1) - 2cxy + y^2 + 1 = 0$ (4)

From (4) the c-discriminant relations is

$(-2xy)^2 - 4(x^2-1)(y^2+1) = 0$ $(B^2-4ac = 0)$

$\Rightarrow 4x^2y^2 - 4x^2y^2 - 4x^2 + 4y^2 + 4 = 0$

$\Rightarrow 4x^2 - 4y^2 = 4$

$\Rightarrow x^2 - y^2 = 1$ (5)

Relation (5) satisfy (1)

Hence it is a singular solution of (1)

Geometical intepretation

The general solution given by (3) represents a family of straight lines, each member of which touches the rectangular hyperbola $x^2-y^2 = 1$, which is the envelope of this family of straight lines.



CHECK YOUR PROGRESS - 2

Q 1: Find the general and singular solution of the following ordinary differential equation.

1. $xp^2 = (x-a)^2$
2. $xp^2 - (y-x)p - y = 1$
3. $y = x - 2ap + ap^2$
4. $(px - y)^2 = p^2 + m^2$
5. $y = px + p - p^2$



6.16 LET US SUM UP

- Suppose $\{f_n(x)\}$, n = 1, 2, 3,. . . be a set of function defined on some interval [a, b].

  Then $\{f_n(x)\}$ is said to be orthogonal set of function on [a, b] if

  $$\int_a^b f_m(x) f_n(x)\,dx = 0 \text{ whenever } m \neq n$$

  And $\{f_n(x)\}$ is said to be orthogonal set of function on [a, b] if

  $$\int_a^b f_m(x) f_n(x)\,dx = \begin{cases} 0, & \text{When } m \neq n \\ 1, & \text{When } m = n \end{cases}$$

- Let $\{f_n(x)\}$, n = 1, 2, 3, . . . be a set of function defined on some interval [a, b].

  Then $\{f_n(x)\}$ is said to be orthogonal on [a, b] with respect to weight function p(x) > 0 if

  $$\int_a^b p(x) f_m(x) f_n(x)\,dx = 0, \text{ whenever } m \neq n$$

  And $\{f_n(x)\}$ is said to be orthonomal on [a, b] with respect to weight function p(x) > 0, if

  $$\int_a^b p(x) f_m(x) f_n(x)\,dx = 0$$

- If the ordinary differential equation possesses a solution, then such solution is termed as the singular solution of the differential equation.
- Whenever the family of curves represented by the solution of a differential equation possesses are envelope then the equation of the envelope is the singular solution of the differential equation.
- If $Ap^2+Bp+c = 0$ be a quadratic in p where A, B, c are function of x and y, then p-discriminent relation is given by $B^2 - 4Ac = 0$

  Similarly, if $Ac^2+Bc+c = 0$ be a quadratic in c where A, B, C are functions of x and y, then e-discriminent is also given by $B^2-4Ac = 0$.



6.17 FURTHER READING

1) Ross, Sheply L. (2004) : Differential Equations (3rd Ed), John Wiley & Sons, India.

2) Reid, W.T. (1971) : Ordinary Differential Equations, John Wiley & Sons, New York.

3) Raisinghania, M. D. : Ordinary and Partial Differential Equations, S. Chand & Co.

4) Raisinghania, M. D. : Advanced Differential Equations S. Chand & Co.



6.18 ANSWERS TO CHECK YOUR PROGRESS

Check your progress-1

Ans to Q No 1: Given function are

$f_1(x) = 1$

$f_2(x) = x$

$f_3(x) = 1+Ax+Bx^2$

Now $\int_{-1}^{1} f_1(x)f_2(x)dx = \int_{-1}^{1} 1.x.dx = \left[\frac{x^2}{2}\right]_{-1}^{1} = \frac{1^2}{2} - \frac{(-1)^2}{2} = 0$

This shows that $f_1(x)$ and $f_2(x)$ are orthogonal on the interval (–1, 1)

If $f_3(x)$ is orthogonal to both $f_1(x)$ and $f_2(x)$ on the interval (–1, 1), then by definition.

$$\int_{-1}^{1} f_1(x)f_3(x)dx = 0$$

$$\Rightarrow \int_{-1}^{1} 1.(1+Ax+Bx^2)dx = 0$$

$$\Rightarrow \left[x + A\frac{x^2}{2} + B\frac{x^3}{3}\right]_{-1}^{1} = 0$$

$$\Rightarrow 1+A.\frac{1^2}{2}+B.\frac{(-1)^3}{3}-\left(1+A.\frac{(-1)^2}{2}+B.\frac{(-1)^3}{3}\right)=0$$

$$\Rightarrow 2+\frac{2B}{3}=0$$

$$\Rightarrow 6 + 2B = 0$$

$$\Rightarrow B = -3$$

and $\int_{-1}^{1} f_2(x)f_3(x)dx = 0$

$$\Rightarrow \int_{-1}^{1} x(1+Ax+Bx^2)dx = 0$$

$$\Rightarrow \int_{-1}^{1} x+Ax^2+Bx^3)dx = 0$$

$$\Rightarrow \left[\frac{x^2}{2}+A\frac{x^3}{3}+B\frac{x^4}{4}\right]_{-1}^{1} = 0$$

$$\Rightarrow \left(\frac{1^2}{2}+A\frac{1^3}{3}+B.\frac{1^4}{4}\right)-\left(\frac{(-1)^2}{2}+A.\frac{(-1)^3}{3}+B.\frac{(-1)^4}{4}\right)=0$$

$$\Rightarrow \frac{2A}{3}=0$$

$$\Rightarrow A = 0$$

$$\therefore A = 0$$

$$B = -3$$

Ans to Q No 2: Let us take $\varphi_0(x) = 1$ (1)

Next, let us chose $\varphi_1 = x + c\,\varphi_0(x) = x+c$ (2)

Let φ_1 be orthogonal to φ_0 with respect to e^{-x}. Then we have

$$\int_0^{\infty} e^{-x}\varphi_0(x)\varphi_1 dx = 0$$

$$\Rightarrow \int_0^{\infty} e^{-x}(x+c)dx = 0$$

$$\Rightarrow \int_0^{\infty} xe^{-x}.x\,dx+e\int_0^{\infty} c^{-x}\,dx = 0 \quad (3)$$

We know that $\int_0^\infty e^{-x}x^n\,dx = \underline{|n}$ (4)

Using (4), (3) gives

$\underline{|1}+c\,x\,\underline{|0} = 0$

$\Rightarrow 1+c = 0$

$\Rightarrow c = -1$

Hence (2) gives

$\varphi_1 = x-1$ (5)

Finally, let $\varphi_2 = x^2 + c_1\varphi_0 + c_2\varphi_1 = x^2 + c_1 + c_2(x-1)$ (6)

Let φ_2 be orthogonal to both φ_0 and φ_1 with respect to e^{-x}.

Then we have

$$\int_0^\infty e^{-x}\varphi_2(x)\varphi_0(x)dx = 0$$

$$\Rightarrow \int_0^\infty e^{-x}\left(x^2 + c_1 + c_2(x-1)\right).1\,dx = 0$$

$$\Rightarrow \int_0^\infty e^{-x}.x^2\,dx + c_1\int_0^\infty e^{-x}\,dx + c_2\int_0^\infty x\,e^{-x}dx - c_2\int_0^\infty e^{-x}\,dx = 0$$

$$\Rightarrow \underline{|2} + c_1\,\underline{|0} + c_2\,\underline{|1} - c_2\,\underline{|0} = 0$$

$$\Rightarrow 2 + c_1 + c_2 - c_2 = 0$$

$$\Rightarrow c_1 = -2$$

and $\int_0^\infty e^{-x}\varphi_2(x)\varphi_1(x)dx = 0$

$$\Rightarrow \int_0^\infty e^{-x}\left(x^2 + c_1 + c_2(x-1)\right)(x-1)dx = 0$$

$$\Rightarrow \int_0^\infty e^{-x}\left(x^2 - 2 + c_2(x-1)\right)(x-1)dx = 0$$

$$\Rightarrow \int_0^\infty e^{-x}\left(x^3 - x^2 - 2x + 2 + c_2(x^2 - 2x + 1)\right)dx = 0$$

$$\Rightarrow \int_0^\infty e^{-x}.x^3dx + (c_2 - 1)\int_0^\infty e^{-x}x^2dx + \int_0^\infty (-2c_2 - 2)\int_0^\infty e^{-x}x\,dx + (2 + c_2)\int_0^\infty e^{-x}.dx = 0$$

$\Rightarrow |\underline{3}+(c_2-1)\ |\underline{2}+(c_0-2c_1-2)\ |\underline{1}+(2+c_1)\ |\underline{0}= 0$

$\Rightarrow 6 + 2c_2 - 2 - 2c_2 - 2 + 2 + c_2 = 0$

$\Rightarrow c_2 = -4$

Putting the values of c_1 and c_2 in (6), we get

$\varphi_2 = x^2 - 2 - 4(x-1)$

$\Rightarrow \varphi_2 = x^2 - 4x + 2$

Hence, the required function are

$\varphi_0 = 1$

$\varphi_1(x) = x - 1$

$\varphi_2(x) = x^2 - 4x + 2$

Ans to Q No 3: Left as exercise.

Check your progress-2

Ans to Q No 1: Left as exercise

Ans to Q No 2: Given equation is

$xp^2 - (y-x)\,p - y = 1 \qquad \rightarrow (1)$

$\Rightarrow xp^2 - (y-x)\,p - (y+1) = 0$

The p-discriminant relation is

$(y-x)^2 - 4x(-(y+1)) = 0 \qquad (B^2 - 4Ae = 0)$

$\Rightarrow (y-x)^2+4x(y+1) = 0$

$\Rightarrow y^2 + x^2 - 2yx + 4xy + 4x = 0$

$\Rightarrow x^2 + 2xy + y^2 = -4x$

$\Rightarrow (x+y)^2 = -4x \qquad \rightarrow (2)$

Differentiating (2) wrt. x

$$2(x+y)\frac{d}{dx}(x+y) = -4$$

$$\Rightarrow (x+y)\left(1+\frac{dy}{dx}\right) = -2$$

$\Rightarrow (x+y)\ (1+p) = -2$

$\Rightarrow p(x+y) + x+y = -2$

$\Rightarrow p(x+y) = -(x+y+2)$

$$\Rightarrow p = -\frac{x+y+2}{x+y}$$

Now substituting this value of p in (1)

$$\text{LHS}\quad x\left(-\frac{x+y+2}{x+y}\right)^2 - (y-x)\left(-\frac{x+y+2}{x+y}\right) - y$$

$$= \frac{x+y+2}{x+y}\left(\frac{x(x+y+2)}{x+y}+y-x\right)-y$$

$$= \frac{x+y+2}{x+y}\left(\frac{x^2+xy+2x+xy+y^2-x^2-xy)}{x+y}\right)-y$$

$$= \frac{x+y+2}{x+y}\cdot\frac{2x+xy+y^2}{x+y}-y$$

$$= \frac{2x^2+2xy+4x+x^2y+xy^2+2xy+xy^2+y^3+2y^2-y(x+y)^2}{(x+y)^2}$$

$$= \frac{2x^2+4xy+4x+x^2y+2xy^2+y^3+2y^2-yx^2-y^3-2xy^2}{(x+y)^2}$$

$$= \frac{2(x^2+2xy+y^2)+4x}{(x+y)^2}$$

$$= \frac{2(x+y)^2-(x+y)^2}{(x+y)^2} \qquad \text{(using (2))}$$

$$= \frac{(x+y)^2}{(x+y)^2}$$

= 1 = RHS

$\therefore$ Equation (1) is sutisfied

Hence $(x+y)^2 = -4x$ is the required singular solution of (1).

Ans to Q No 3: Left as exercise

Ans to Q No 4: Given equation is

$(px - y)^2 = p^2+m^2$ (1)

$$\Rightarrow (px-y)^2 = \pm\sqrt{p^2+m^2}$$

$$\Rightarrow y = px \mp \sqrt{p^2+m^2}$$

Which is in clairaut's form,

So, replacing p by orbitray constant c, the general solution of (1) is

$$(y = cx \pm \sqrt{c^2+m^2}) \qquad (2)$$

$$\Rightarrow y-cx = \mp\sqrt{c^2+m^2}$$

$$\Rightarrow (y-cx)^2 = \left(\mp\sqrt{c^2+m^2}\right)^2$$

$\Rightarrow y^2+c^2x^2 - 2cxy = c^2+m^2$

$\Rightarrow c^2(x^2-1) - 2cxy + (y^2-m^2) = 0$

$\therefore$ The e-discriminant relation is

$(-2xy)^2 - 4(x^2-1)(y^2-m^2) = 0 \qquad (B^2 - 4AC = 0)$

$\Rightarrow 4x^2y^2 - 4x^2y^2 + 4x^2m^2 + 4y^2 - 4m^2 = 0$

$\Rightarrow 4y^2 + 4m^2x^2 = 4m^2$

$\Rightarrow y^2 + m^2x^2 = m^2 \qquad (3)$

The relation (3) satisfies (1)

Hence singular solution of (1) is $y^2 + m^2x^2 = m^2$

Ans to Q No 5: Left as exercise.



6.19 MODEL QUESTIONS

Q 1: Show that the function sinx, sin2x, sin3x, . . . are orthogonal on the interval $(0, \pi)$

Q 2: Show that the function 1, $\cos\pi x$, $\sin\pi x$, $\cos2\pi x$, $\sin2\pi x$, . . . form are orthogonal set on $2 \le x \le 2$ and find the corresponding orthonormal set.

Find the general and singular solutions of the differential equation (3) to (8).

Q 3: $4xp^2 = (3x-a)^2$

Q 4: $p^2(1-x^2) = 1-y^2$

Q 5: $xp^2 - yp - y = 0$

Q 6: $y = xp + \frac{1}{2p}$

Q 7: $y = px - p^2$

Q 8: $p^3+px-y = 0$.

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