UNIT 7 : FIRST ORDER LINEAR PARTIAL DIFFERENTIAL EQUATION

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7.1 LEARNING OBJECTIVES

After going through this unit, you will able to know

- origin and formation of partial differential equation.
- solution of a partial differential equation.
- lagrange's method.

7.2 INTRODUCTION

If a differential equation contains one or more partial derivatives, it is called a partial differential equation. Partial derivatives occur only when there are two or more than two independent variables.

For example

$$\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=z+xy$$

$$\left(\frac{\partial z}{\partial x}\right)^2-\frac{\partial^3 z}{\partial y^3}=3x\left(\frac{\partial z}{\partial x}\right)$$

Notations:

Whenever we consider the case of two independent variables, we

shall assume them to be *x, y* and take *z* to be the dependent variable. We shall adopt the following notations.

$$\frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q, \quad \frac{\partial^2 z}{\partial x^2} = r, \quad \frac{\partial^2 z}{\partial x \partial y} = s \text{ and } \quad \frac{\partial^2 z}{\partial y^2} = t$$

7.3

7.3 ORIGIN OF PARTIAL DIFFERENTIAL EQUATION

Partial differential equations are formed by eliminating the arbitrary constants or arbitrary functions.

7.3.1 Eliminating arbitrary constants:

Case I : When number of independent variable is more than the number of arbitrary constants. In that case usually we get the PDE of order one but may not be unique.

Case II : When number of independent variable is equal to the number of arbitrary constants, then the PDE is unique of order one.

Case III : When number of independent variable is less than the number of arbitrary constants, then usually the PDE is formed of order more than one.

Ex. 1: Form partial differential equations by eliminating the arbitrary constants of the following:

(i) $Z = ax + by + a^2 + b$

(ii) $Z = (x + a)\,(y + b)$

(i) **Solution :** Given $Z = ax + by + a^2 + b$ ..... (1)

Differentiating (i) partially with respect to *x* and *y*, we get

$$\frac{\partial z}{\partial x} = a \quad \text{..... (2)}$$

$$\frac{\partial z}{\partial y} = b \quad \text{..... (3)}$$

Using (2) & (3) in (1), we get

$$Z = \left(\frac{\partial z}{\partial x}\right) x + \left(\frac{\partial z}{\partial y}\right) y + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)$$

Which is the required PDE.

(ii) **Solution :** Given $Z = (x + a)\ (y + b)$ .....(1)

Differentiating (1) partially with respect to *x* and *y*, we get

$$\frac{\partial z}{\partial x} = y + b \text{ ..... (2)}$$

$$\frac{\partial z}{\partial y} = x + a \text{ ..... (3)}$$

Using (2) & (3) in (1), we get

$$Z = \left(\frac{\partial z}{\partial y}\right)\left(\frac{\partial z}{\partial x}\right)$$

which is the required PDE.

Ex. 2: Form a PDE by eliminating the arbitrary constants from $2z = (ax + y)^2 + b$

Solution: Given $2z = (ax + y)^2 + b$ ..... (1)

Differentiating (1) partially with respect to *x* and *y*, we get

$$2\frac{\partial z}{\partial x} = 2a(ax + y)$$

$$\Rightarrow \frac{\partial z}{\partial x} = a(ax + y) \qquad (2)$$

$$2\frac{\partial z}{\partial y} = 2(ax + y) \qquad (3)$$

$$\Rightarrow \frac{\partial z}{\partial y} = (ax + y)$$

Now, dividing (2) by (3), gives

$$\left(\frac{\partial z}{\partial x}\right) \Big/ \frac{\partial z}{\partial y} = a \qquad (4)$$

Using (4) in (3), we have

$$\frac{\partial z}{\partial y} = \left\{\frac{\left(\frac{\partial z}{\partial x}\right)}{\left(\frac{\partial z}{\partial y}\right)} x + y\right\}$$

$$\Rightarrow q = \left(\frac{px}{q} + y\right)$$

$$\left[\because \frac{\partial z}{\partial x} = p, \frac{\partial z}{\partial y} = q\right]$$

$$\Rightarrow q^2 = px + yq$$

$$\Rightarrow px + qy = q^2$$

Which is the required PDE.



CHECK YOUR PROGRESS

Q. 1 Form a PDEs by eliminating the arbitrary constants of the following :

(i) $z = a(x + y) + b$

(ii) $z = ax + a^2y^2 + b$

Q. 2 Eliminate *a* and *b* from $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$.

Q. 3 Form a PDE by eliminating *a* and *b* from $\log(az - 1) = x + ay + b$

7.3.2 By elimination arbitrary functions

Let *u* and *v* be the two functions of *x, y, z*. Which are connected by the relation.

$f(u, v) = 0$ ..... (1)

We treat *z* as dependent variable and *x* and *y* are independent variables so that

$\frac{\partial z}{\partial x} = p, \frac{\partial z}{\partial y} = q, \frac{\partial y}{\partial x} = 0$ and $\frac{\partial x}{\partial y} = 0$

Differentiating (1) w.r.t *x* and *y*, we have

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial y}{\partial z}p\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}p\right) = 0 \quad \text{..... (2)}$$

and

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}q\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}q\right) = 0 \quad \text{.....(3)}$$

Eliminating $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ from (2) and (3), we have

$$\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}p\right)+\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q\right)=\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q\right)\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p\right)$$

$$\Rightarrow\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial v}{\partial y}\frac{\partial u}{\partial z}\right)p+\left(\frac{\partial v}{\partial x}\frac{\partial y}{\partial z}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}\right)q=\frac{\partial u}{\partial x}.\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}.\frac{\partial v}{\partial x}$$

$\Rightarrow Pp+Qq=R$ ..... (4)

Where

$$P=\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial v}{\partial y}\frac{\partial u}{\partial z}=\frac{\partial(u,v)}{\partial(y,z)}$$

$$Q=\frac{\partial v}{\partial x}\frac{\partial u}{\partial z}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}=\frac{\partial(u,v)}{\partial(z,x)}$$

$$R=\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}.\frac{\partial v}{\partial x}=\frac{\partial(u,v)}{\partial(x,y)}$$

Equation (4) is the required PDE.

Ex. 3 : Form a PDE by eliminating the function f from $f\left(x+y+z,\ x^2+y^2-z^2\right)=0$

Solution: Here $f\left(x+y+z,\ x^2+y^2-z^2\right)=0$ ..... (1)

Let $u=x+y+z$, and $v=x^2+y^2-z^2$ ..... (2)

$\because$ (1) $\Rightarrow f(u,v)=0$ ..... (3)

Differentiating (3) w.r.t. *x* partially, we get

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial x}\right)+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)=0$$ ..... (4)

From (2), $\frac{\partial u}{\partial x}=1,\ \frac{\partial u}{\partial z}=1,\ \frac{\partial v}{\partial x}=2x,\ \frac{\partial v}{\partial z}=2z,\ \frac{\partial u}{\partial y}=1,\ \frac{\partial v}{\partial y}=2y$ ... (5)

$\therefore$ (4) & (5) $\Rightarrow \frac{\partial f}{\partial u}(1+p)+\frac{\partial f}{\partial v}(2x-2pz)=0$

$$\Rightarrow \left(\frac{\partial f}{\partial u}\right)\Big/\left(\frac{\partial f}{\partial v}\right)=\frac{-2(x-pz)}{(1+p)}$$ ..... (6)

Again differentiating (3) w.r.t. *y* partially, we get

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right)=0$$

Or $\Rightarrow \frac{\partial f}{\partial u}(1+q)+2\left(\frac{\partial f}{\partial v}\right)(y-zq)=0$

$$\Rightarrow \left(\frac{\partial f}{\partial u}\right) \Big/ \left(\frac{\partial f}{\partial v}\right) = \frac{-2(y-qz)}{1+q} \quad \text{..... (7)}$$

From (6) & (7) eliminating *f*, we have

$$(x-pz)/(1+p) = (y-qz)/(1+q)$$

$$\Rightarrow (1+q)\,(x-pz)\,(y-qz)$$

$$\Rightarrow (y+z)\,p-(x+z)\,q = x-y$$

Which is the required PDE of first order.

Ex.4: Form the PDE by eliminating the arbitrary function from $z=f(x+it)+g(x-it)$, where $i=\sqrt{-1}$

Solution: Given $z=f(x+it)+g(x-it)$ ..... (1)

Differentiating (1) twice partially w.r.t. *x* and *t*, we get

$$\frac{\partial z}{\partial x} = f'(x+it)+g'(x-it)$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x+it)+g''(x-it) \quad \text{..... (2)}$$

Also $\frac{\partial z}{\partial t} = f'(x+it).i - g'(x-it).i$

$$\frac{\partial^2 z}{\partial t^2} = f''(x+it).i^2 + g''(x-it).i^2$$

$$\frac{\partial^2 z}{\partial t^2} = -f''(x+it) - g''(x-it) \quad \text{..... (3)}$$

From (2) and (3),

$$\frac{\partial^2 z}{\partial x^2} = -\frac{\partial^2 z}{\partial t^2}$$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial t^2} = 0 \quad \text{..... (4)}$$

Which is the required PDE.

Ex.5: Form a PDE by eliminating the arbitrary function *f* and *g* from $z=y\,f(x)+x\,g(y)$

Solution: Given $z = y f(x) + x g(y)$ ..... (1)

Differentiating (1) partially w.r.t. *x* and *y*, we get

$$\frac{\partial z}{\partial x} = y f'(x) + g(y) \quad \text{..... (2)}$$

$$\frac{\partial z}{\partial y} = f(x) + x g'(y) \quad \text{..... (3)}$$

Again differentiating (3) partially w.r.t. *x*, we get

$$\frac{d^2 z}{dxdy} = f'(x) + g'(y) \quad \text{..... (4)}$$

Equation (2) & (3) can be arranged as

$$f'(x) = \frac{1}{y}\left[\frac{\partial z}{\partial x} - g(y)\right]$$

$$g'(y) = \frac{1}{x}\left[\frac{\partial z}{\partial y} - f(x)\right]$$

Putting the above values in (4), we have

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{1}{y}\left[\frac{\partial z}{\partial x} - g(y)\right] + \frac{1}{x}\left[\frac{\partial z}{\partial y} - f(x)\right]$$

$$\Rightarrow xy\frac{\partial^2 z}{\partial x \partial y} = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} - y f(x) - x g(y)$$

$$\Rightarrow xy\frac{\partial^2 z}{\partial x \partial y} = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} - z$$ using (1) which is the required PDE.



CHECK YOUR PROGRESS

Q4: From a PDE by eliminating the arbitrary function from $z - xy + f\left(x^2 + y^2\right)$

Q5: Eliminate the arbitrary functions and hence obtain the PDE from $z = f(xy/z)$

Q6: Eliminate the arbitrary function *f* and F from

$$y = f(x - at) + F(x + at)$$

Q7: Obtained the PDE by eliminating the function from

$$f(x^2 + y^2 + z^2,\ z^2 - 2xy) = 0$$

7.4 SOLUTION OF A PDE

Definitions :

Complete Integrals : If from the PDE $f(x, y, z, p, q) = 0$ we can find a relation F(*x, y, z, a, b*) = 0, which contains as many arbitrary constants as there are independent variables, the relation F(*x, y, z, a, b*) = 0 is known as complete integral of the given equation.

Particular Integral : Giving the particular values to the constants a and b occurring in the complete integral, we obtain a solution of the given PDE which we shall call as particular integral.

Singular Integral : The equation of the envelope of the surfaces represented by the complete integral of the given PDE is called its singular integrals. Thus if F(*x, y, z, a, b*) = 0 is the complete integral then singular integral is obtained by eliminating a and b from $f = 0, \frac{\partial f}{\partial a} = 0$ and $\frac{\partial f}{\partial b} = 0.$

General Integral : If two functions *u, v* of *x, y, z* are connected by an arbitrary function *f*(*u, v*) = 0, then eliminating f we get a PDE of the form Pp + Qq = R. The solution of the equation is called the general integral of the differential equation.

Cauchy's Problem for first order PDE:

If (a) $x_0(s)$, $y_0(s)$ and $z_0(s)$ are arbitrary functions which together with theiri first derivatives are continuous in the integral I defined by $s_1 < s < s_2$

(b) And if $f(x, y, z, p, q)$ is a continuous function of *x,y,z,p* and *q* in a certain region 'U' of the *xyzpa* space, then it is required to establish the existence of a function $\Phi(x, y)$ with the following properties:

(i) $\Phi(x, y)$ and its partial derivatives of *x* and *y* in a region R of the *xy* space.

(ii) For all values of *x* and *y* lying in R, the point

$\{x, y, \Phi(x, y), \Phi_x(x, y), \Phi_y(x, y)\}$ lies in $\frac{U}{x}$ and

$f[x, y\ \Phi(x, y), \Phi_x(x, y), \Phi_y(x, y] = 0$

(iii) For all s belonging to the interval I, the point $\{x_o(s), y_o(s)\}$ belongs to the region R and $\Phi\{x_o(s), y_o(s)\} = z_o$ stated geometrically, what we wish

to prove is that there exists a surface $z = \Phi(x, y)$ which passes through the curve C whose parametric equations are given by $x = x_o(s)$, $y = y_o(s)$, $z = z_o(s)$ and at every point of which the directions $(p, q, -1)$ of the normal is such that $f(x, y, z, p, q) = 0$.

Ex.6: Solve the Cauchy's problem for $zp + q = 1$, when the initial data curve is $x_o = s,\ y_o = s,\ z_o = s/2,\ 0 \le s \le 1$.

Solution: Given $f(x, y, z, p, q) = zp + q - 1 = 0$ ..... (1)

And initial data curve $x_o = s,\ y_o = s,\ z_o = s/2,\ 0 \le s \le 1$

From (1), we have

$$\frac{\partial f}{\partial p} = z, \quad \frac{\partial f}{\partial q} = 1$$

and $\frac{\partial f}{\partial q}.\frac{dx_o}{ds} - \frac{\partial f}{\partial p}.\frac{dy_o}{ds} = 1 \times 1 - z \times 1 = 1 - \frac{1}{2}s \neq 0$, for $0 \le s \le 1$

Now, $\frac{dx}{dt} = \frac{\partial f}{\partial p}, \frac{\partial y}{\partial t} = \frac{\partial f}{\partial q}$ and $\frac{dz}{dt} = \frac{\partial z}{\partial x}.\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$

$\Rightarrow \frac{dx}{dt} = z, \frac{dy}{dt} = 1$ ..... (3) and

$$\frac{dz}{dt} = p\left(\frac{\partial f}{\partial p}\right) + q\left(\frac{\partial f}{\partial q}\right)$$

$= pz + q$ [Using 1]

= 1 ..... (4)

integrating (3) & (4), we get

$y = t + c_1$ and $y = t + c_2$ ..... (5)

From (2) at $t = 0,\ x(s,0) = s,\ y(s,0) = s$ and $z(s,0).\frac{s}{2}$ ..... (6)

Using (5) & (6), we have

$y = t + s,\ z = t + s/2$ .....(7)

and

Then, from (3) & (7), $\frac{dx}{dt} = t + s/2$ so that

$$x = \frac{1}{2} \times t^2 + \frac{1}{2} \times st + c_3 \text{ ..... (8)}$$

Now (6) & (8) gives

$$x = \frac{1}{2}t^2 + \frac{1}{2}st + s \text{ ..... (9)}$$

Again, solving $y = t + s$ with (9) for s and t in terms of x, and y, we get

$$t = \frac{y - x}{1 - y/2} \text{ and } s = \frac{x - y^2/2}{1 - y/2}$$

Substituting these values in $z = t + s/2$, we get

$$z = \left\{2(y - x) + x - y^2/2\right\}/(2 - y)$$

Which is the required solution passing through the initial data curve.

7.5 SOLUTION TO LINEAR PDE OF FIRST ORDER

A PDE is said to be linear if the dependent variables and its partial derivatives occur only in the first degree and are not multiplied.

Lagrange's Linear Equation:

The partial differential equation of the form Pp + Qq = R, where P, Q, R are functions of *x,y,z* is the standard form of the linear partial differential equation of the order one and is called Lagrange's Linear Equation.

Langrange's Solution of the linear equation:

Theorem : The general solution of Langrange equation

Pp + Qq = R ..... (1)

is $\phi(u, v) = 0$ ..... (2)

where $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ ..... (3) are two independent solution of

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Here c_1 and c_2 are arbitrary constants and at least one of *u, v* must contain *z*. Also recall that *u* and *v* are said to be independent if *u/v* is not merely a constant.

Proof : Differentiating (2) partially w.r.t. '*x*' and '*y*', we get

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0 \text{ ..... (5)}$$

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)+\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right)=0 \quad \text{..... (6)}$$

and

Eliminating $\frac{\partial \phi}{\partial u}$ and $\frac{\partial \phi}{\partial v}$ between (5) and (6), we have

$$\begin{vmatrix} \frac{\partial u}{\partial x}+p\left(\frac{\partial u}{\partial z}\right) & \frac{\partial v}{\partial x}+p\left(\frac{\partial v}{\partial z}\right) \\ \frac{\partial u}{\partial y}+q\left(\frac{\partial u}{\partial z}\right) & \frac{\partial v}{\partial y}+q\left(\frac{\partial v}{\partial z}\right) \end{vmatrix}=0$$

$$\Rightarrow\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right)-\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)=0$$

$$\Rightarrow\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}\right)p+\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}\right)q=\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}\frac{\partial u}{\partial y}\frac{\partial u}{\partial x}$$

Thus (2) is a solution of equation (7)

Now, taking the differentials of $u\,(x,y,z) = c_1$ and $v\,(x,y,z) = c_2$, we have

$$\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz=0 \quad \text{..... (8)}$$

& $$\frac{dv}{dx}dx+\frac{dv}{dy}dy+\frac{dv}{dz}dz=0 \quad \text{..... (9)}$$

Since *u* and *v* are independent function, hence solving (8) & (9) gives

$$\frac{dx}{\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}}=\frac{dy}{\frac{\partial y}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}}=\frac{dx}{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial y}{\partial y}\frac{\partial v}{\partial x}} \quad \text{..... (10)}$$

From (4) & (10), we obtain

$$\frac{\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}}{P}=\frac{\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}}{Q}=\frac{\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}}{R}=K \quad \text{(let)}$$

$$\Rightarrow \frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}=KP,\ \frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}=KQ,\ \frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}=KR \quad \text{.....}$$

(11)

From (7) & (11), we have

K (Pp + Qq) = KR

$\Rightarrow$ Pp + Qq = R

Therefore if $u(x,y,z)=c_1$ and $v(x,y,z)=c_2$ are two independent solutions

of the system of differential equations $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$, then $\phi(uv)=0$ is a solution of Pp + Qq = R, where ϕ being an arbitrary functions.

Ex.1: Solve $\frac{y^2z}{x}p+xzq=y^2$

Solution: Given $\frac{y^2z}{x}p+xzq=y^2$ ..... (1)

The Lagrange's auxilliary equation of (1) are

$$\frac{dx}{\frac{y^2z}{x}}=\frac{dy}{xz}=\frac{dz}{y^2} \quad \text{..... (2)}$$

From the first two fractions, we get

$$\frac{dx}{\frac{y^2z}{x}}=\frac{dy}{xz}$$

$$\Rightarrow x^2dx=y^2dy$$

$$\Rightarrow 3x^2dx-3y^2dy=0$$

integrating, we have

$x^3-y^3=c_1$ ..... (3), $c_1\rightarrow$ being arbitrary constant.

Now, taking the first and last fraction, we get

$$\frac{dx}{\frac{y^2z}{x}}=\frac{dz}{y^2}$$

$$\Rightarrow xdx=zdz$$

$$\Rightarrow 2xdx-2zdz=0$$

integrating, we have

$x^2-z^2=c_2$, ... (4) $c_2\rightarrow$ being arbitrary constants.

Therefore the required general integral is given by

$\phi\left(x^3-y^3,\ x^2-y^2\right)=0$, ϕ being an arbitrary functions.

Ex.2 : Find the general integral of the PDE, $y^2p-xyq=x(z-2y)$

Soln: Given $y^2p-xyq=x(z-2y)$ ..... (1)

The Lagrange's auxiliary equations are

$$\frac{dx}{y^2}=\frac{dy}{-xy}=\frac{dz}{x(z-2y)} \quad \text{..... (2)}$$

From the first two fractions, we have

$$\frac{dx}{y^2} = \frac{dy}{-xy}$$

$$\Rightarrow xdx + ydy = 0$$

integrating,

$$x^2 + y^2 = c_1 \text{ ..... (3)}$$

From the last two fractions of (2), we have

$$\frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

$$\Rightarrow zdy - 2ydy = -ydz$$

$$\Rightarrow 2dy(zdy + ydz) = 0$$

integrating, we have

$$y^2 - yz = c_2 \text{ ..... (4)}$$

Hence the curve given by equation (3) & (4) generate the required integral surface as

$$\phi\left(x^2 + y^2,\ y^2 - yz\right) = 0$$

Ex.3: Solve $p + 3q = 5z + \tan(y - 3x)$

Solution: Given $p + 3q = 5z + \tan(y - 3x)$ ..... (1)

The Lagrange's auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{5z + \tan(y - 3x)} \text{ ..... (2)}$$

Taking the first two fractions, we have

$$\frac{dx}{1} = \frac{dy}{3}$$

$$\Rightarrow dy - 3dx = 0$$

$\Rightarrow y - 3x = c_1$ ..... (3), $c_1 \rightarrow$ being arbitrary constants.

Taking the first & last fractions of the equation (2) & using (3), we have

$$\frac{dx}{1} = \frac{dz}{5z + \tan c_1}$$

integrating,

$$x = \frac{1}{5}\log(5z + \tan c_1) + \frac{1}{5}c_2,$$

$\Rightarrow 5x - \log(5z + \tan c_1) = c_2$, $c_2 \rightarrow$ being arbitrary constants.

$$\Rightarrow 5x - \log[5z + \tan(y - 3x)] = c_2$$

The required general integral is given by

$$\phi\left(y - 3x,\ 5x - \log[5z + \tan(y - 3x)]\right) = 0$$

$\phi \rightarrow$ being arbitrary function.

Ex.4: Solve $px(z-2y^2)=(z-qy)(z-y^2-2x^3)$

Solution: Given

$$px(z-2y^2)=(z-qy)(z-y^2-2x^3)$$

$$\Rightarrow x(z-2y^2)\,p+y(z-y^2-2x^3)\,q=z(z-y^2-2x^3) \quad \text{..... (1)}$$

$\therefore$ Lagrange's auxiliary equations are

$$\frac{dx}{x(z-2y^2)}=\frac{dy}{y(z-y^2-2x^3)}=\frac{dz}{z(z-y^2-2x^3)} \quad \text{..... (2)}$$

From the last two fractions, we have

$$\frac{dy}{y}=\frac{dz}{z}$$

$$\Rightarrow \log z-\log y=\log c_1$$

$$\Rightarrow z/y=c_1 \text{ — (3)}$$

Taking first and second fractions & using (3) we have

$$\frac{dx}{x(yc_1-2y^2)}=\frac{dy}{y(yc_1-y^2-2x^3)}$$

$$\Rightarrow (c_1y-y^2-2x^3)dx+x(2y-yc_1)dy=0 \quad \text{..... (4)}$$

Comparing (4) with $Mdx+Ndy=0$,

$M=c_1y-y^2-2x^3$, $N=x(2y-yc_1)$, then

$\frac{dM}{dy}=c_1-2y$ and $\frac{dN}{dx}=2y-c_1$

$$\therefore \frac{1}{N}\left(\frac{dM}{dy}-\frac{dN}{dx}\right)=\frac{1}{x(2y-c_1)}\times 2(c_1-2y)$$

$$=\frac{-2}{x}$$

Which is a function of *x* alone.

Hence, integrating factor of (1) $=\int_e \frac{-2}{x}dx$

$$=e^{-2\log x}$$

$$=e^{x-2}$$

$$=x^{-2}$$

Multiplying (4) by integrating factor x^{-2} and solving

$$\int \{(c_1 y - y^2)x^{-2} - 2x\}dx + \int x^{-1}(2y-4)\,dy = c_2$$

(Treating *y* as constant) (Integrating terms free from *x*)

$$\Rightarrow (c_1 y - y^2) \times \left(-\frac{1}{x}\right) - x^2 = c_2$$

$$\Rightarrow (y^2 - c_1 x)/x - x^2 = c_2$$

$$\Rightarrow (y^2 - c_1 x - x^3)/x = c_2, \; c_2 \to \text{ being arbitrary constant.}$$

Therefore, the required solutions is given by

$$(y^2 - c_1 x - x^3)/x = \phi\left(z/y\right) \quad \phi \text{ being arbitrary function.}$$

Ex.5 : Solve $(mz - ny)p + (nx - lz)q = ly - mx$

Soln. Given $(mz - ny)p + (nx - lz)q = ly - mx$ ..... (1)

The Lagrange's auxiliary equations are given by

$$\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx} \qquad \text{..... (2)}$$

Taking *x, y, z* as multipliers, we have

$$\text{Each fraction} = \frac{xdx + ydy + zdx}{x(mz - ny) + y(nx - lz) + z(ly - mx)} = \frac{xdx + ydy + zdz}{0}$$

$$\therefore \; xdx + ydy + zdz = 0$$

Integrating $x^2 + y^2 + z^2 = c_1$ ..... (3)

Again, taking *l, m, n* as multipliers of (1), each fraction

$$= \frac{ldx + mdy + ndz}{l(mz - ny) + n(nx - lz) + (ly + mx)} = \frac{ldx + mdy + ndz}{0}$$

$$\therefore \; ldx + mdy + ndz = 0$$

Integrating, $lx + my + nz = c_2$ ..... (4)

The general solution is given by

$$\phi\left(x^2 + y^2 + z^2, \, lx + my + nz\right) = 0, \quad \phi \text{ being arbitrary function.}$$

Ex.6 : Solve $x\left(y^2 + z\right)p - y\left(x^2 + z\right)q = z\left(x^2 - y^2\right)$

Soln. Given $x\left(y^2 + z\right)p - y\left(x^2 + z\right)q = z\left(x^2 - y^2\right)$ ..... (1)

The Lagrange's auxiliary equations are

$$\frac{dx}{x\left(y^2 + z\right)} = \frac{dy}{-y\left(x^2 + z\right)} = \frac{dz}{z\left(x^2 - y^2\right)} \qquad \text{..... (2)}$$

Taking $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers, each fraction

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{y^2 + z - x^2 - z + x^2 - y^2} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$$\therefore \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

Integrating $\log x + \log y + \log z = \log c_1$

$\Rightarrow xyz = c_1$ ..... (3)

Again taking $x, y, -1$ as multipliers, each fraction

$$= \frac{xdx + ydy - dz}{x^2y^2 + x^2z - y^2x^2 - y^2z + zx^2 - y^2z} = \frac{xdx + ydy - dz}{0}$$

$$\therefore \; xdx + ydy - dz = 0$$

Integrating $x^2 + y^2 - 2z = c_2$ ..... (4)

$\therefore$ The general solution is given by

$\phi\left(xyz,\ x^2 + y^2 - 2z\right) = 0$

Ex.7 : Solve $(y+z)\,p + (z+x)\,q = x + y$

Soln. Given $(y+z)\,p + (z+x)\,q = x + y$ ..... (1)

The Lagrange's auxiliary equations are

$$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y} \quad \text{..... (2)}$$

Choosing, 1, –1, 0 as multipliers, each fraction

$$= \frac{dx - dy}{(y+z) - (z+x)} = \frac{d(x-y)}{-(x-y)} \quad \text{..... (3)}$$

Again, choosing 0, 1, –1 as multipliers, each fraction

$$= \frac{dy - dz}{(z+x) - (x+y)} = \frac{d(x-y)}{-(y-z)} \quad \text{..... (4)}$$

Finally, choosing 1, 1, 1 as multipliers, each fraction

$$= \frac{dx + dy + dz}{(y+z) + (z+x) + (x+y)} = \frac{d(x+y+z)}{2(x+y+z)} \quad \text{..... (5)}$$

Now, using (3), (4) & (5)

$$\frac{d(x-y)}{-(x-y)} = \frac{d(y-z)}{-(y-z)} = \frac{d(x+y+z)}{2(x+y+z)} \quad \text{..... (6)}$$

Taking the first two fractions of

$$\frac{d(x-y)}{-(x-y)} = \frac{d(y-z)}{-(y-z)}$$

Integrating $\log(x-y) = \log(y-z) + \log c_1$

$$\Rightarrow \frac{x-y}{y-z} = c_1 \quad \text{..... (7)}$$

Taking the first and last fraction of (6)

$$= \frac{d(x-y)}{-(x-y)} = \frac{d(x+y+z)}{2(x+y+z)}$$

$$\Rightarrow 2\frac{d(x-y)}{x-y} + \frac{d(x+y+z)}{(x+y+z)} = 0$$

Integrating $2\log(x-y) + \log(x+y+z) = \log c_2$

$$\Rightarrow (x-y)^2 (x+y+z) = c_2 \quad \text{..... (8)}$$

The required general solution is given by

$$\phi\left[(x-y)^2(x+y+z), \frac{x-y}{y-z}\right] = 0$$

Ex.8 : Solve $\left(x^2 - y^2 - z^2\right) p + 2xyq = 2xz$

Or $\left(y^2 + z^2 - x^2\right) p - 2xyq = -2xz$

Soln. Given $\left(y^2 + z^2 - x^2\right) p - 2xyq = -2xz$ ..... (1)

Here Lagrange's auxiliary equations are

$$\frac{dx}{y^2 + z^2 - x^2} = \frac{dy}{-2xy} = \frac{dz}{-2xz} \quad \text{..... (2)}$$

Taking the last two fractions

$$\frac{dy}{-2xy} = \frac{dz}{-2xz}$$

$$\Rightarrow \frac{1}{y}dy = \frac{1}{z}dz$$

Integrating $\log y = \log z + \log c_1$

$$\Rightarrow y/z = c_1 \quad \text{..... (3)}$$

Choosing *x, y, z* as multipliers, each fraction of (2)

$$= \frac{xdx + ydy + zdz}{xy^2 + xz^2 - x^3 - 2x^2y - 2xz^2} = \frac{xdx + ydy + zdz}{-x\left(x^2 + y^2 + z^2\right)} \quad \text{..... (4)}$$

Combining third fraction of (2) with fraction of (4), we have

$$\frac{xdx + ydy + zdz}{-x\left(x^2 + y^2 + z^2\right)} = \frac{dz}{-2xz}$$

$$\Rightarrow \frac{2xdx + 2ydy + 2dz}{x^2 + y^2 + z^2} - \frac{dz}{z} = 0$$

Integrating, $\log\left(x^2+y^2+z^2\right)-\log z=\log c_2$

$$\Rightarrow \frac{x^2+y^2+z^2}{z}=c_2 \quad \text{..... (5)}$$

$\therefore$ The general solution is given by

$$\phi\left(\frac{x^2+y^2+z^2}{z}, \frac{y}{z}\right)=0$$

Ex.9 : Solve $x(z+2a)\,p+(xz+2yz+2ay)\,q=z(z+a)$

Soln. The Lagrange's auxiliary equations are

$$\frac{dx}{x(z+2a)}=\frac{dy}{xz+2yz+2ay}=\frac{dz}{z(z+a)} \quad \text{..... (1)}$$

Each fraction of (1)

$$=\frac{dx+dy}{2(x+y)\,(z+a)}=\frac{dz}{z(z+a)}$$

Or $$\frac{d(x+y)}{(x+y)}=\frac{2dz}{z}$$

Integrating,

$$\log(x+y)=2\log z+\log c_1$$

$$\Rightarrow (x+y)\big/_{z^2}=c_1 \quad \text{..... (2)}$$

Taking the first and third fractions of (1)

$$\frac{dx}{x}=\frac{z+2a}{z(z+a)}dz$$

$$\Rightarrow \frac{dx}{x}=\left(\frac{2}{z}-\frac{1}{z+a}\right)dz$$

Integrating $\log x=2\log z-\log(z+a)+\log c_2$

$$\Rightarrow x(z+a)\big/_{z^2}=c_2 \quad \text{..... (3)}$$

Thus the general solution is given by

$$\phi\left[\frac{(x+y)}{z^2}, \frac{x(z+a)}{z^2}\right]=0$$

Ex.10 : Solve $\left(x^2-yz\right)p+\left(y^2-zx\right)q=z^2-xy$

Soln. Lagrange's auxiliary equations

$$\frac{dx}{x^2-yz}=\frac{dy}{y^2-zx}=\frac{dz}{z^2-xy} \quad \text{..... (1)}$$

Choosing 1, –1, 0 and 0, 1, – 1 as multipliers in turn each fraction of (1)

$$= \frac{dx-dy}{x^2-y^2+z(x-y)} = \frac{dy-dz}{(y-z)(y+z+x)}$$

$$\Rightarrow \frac{dx-dy}{(x-y)(x+y+z)} = \frac{dy-dz}{(y-z)(y+z+x)}$$

$$\Rightarrow \frac{d(x-y)}{x-y} - \frac{d(y-z)}{y-z} = 0$$

Integrating, $\log(x-y) - \log(y-z) = \log c_1$

$$\Rightarrow (x-y)/(y-z) = c_1 \quad \text{— (2)}$$

Choosing *x, y, z* as multipliers, each fraction of (1)

$$= \frac{xdx+ydy+zdz}{x^3+y^3+z^3-3xyz} = \frac{xdx+ydy+zdz}{(x+y+z)\left(x^2+y^2+z^2-xy-yz-zx\right)} \quad \text{— (3)}$$

Again, choosing 1, 1, 1 as multipliers, each fraction of (1)

$$= \frac{dx+dy+dz}{x^2+y^2+z^2-xy-yz-zx} \quad \text{..... (4)}$$

From (3) & (4)

$$\frac{xdx+ydy+zdz}{(x+y+z)} = dx+dy+dz$$

$$\Rightarrow 2(x+y+z)\,d(x+y+z) - (2xdx+2ydy+2zdz) = 0$$

Integrating, $(x+y+z)^2 - \left(x^2+y^2+z^2\right) = 2c_2$

$$\Rightarrow x^2+y^2+z^2+2xy+2yz+2zx-x^2-y^2-z^2 = 2c_2$$

$$\Rightarrow xy+yz+zx = c_2$$

Hence the required general solution is given by

$$\phi\left[xy+yz+zx,\ (x-y)/(y-z)\right] = 0$$



CHECK YOUR PROGRESS

Q8: Solve the following PDEs.

(i) $xp+yq=z$

(ii) $x^2p+y^2q=z^2$

(iii) $yzp+zxq=xy$

(iv) $p-q=\dfrac{z}{x+y}$

(v) $z(p-q)=z^2+(x+y)^2$

(vi) $xzp+yzq=xy$

(vii) $x^2p+y^2q=nxy$

(viii) $x\left(y^2-z^2\right)p+y\left(z^2-x^2\right)q=z\left(x^2-y^2\right)$

(ix) $z(xp-yq)=y^2-x^2$

(x) $y^2(x-y)p+x^2(y-x)q=z(x^2+y^2)$

(xi) $xp+zq+y=0$

(xii) $x^2p+y^2q=(x+y)z$

7.6 ANSWER TO CHECK YOUR PROGRESS

Ans to Q No 1: (i) $\dfrac{\partial z}{\partial x}=\dfrac{\partial z}{\partial y}$

(ii) $\dfrac{\partial z}{\partial y}=2y\left(\dfrac{\partial z}{\partial x}\right)^2$

Ans to Q No 2: $2z=x\dfrac{\partial z}{\partial x}+y\dfrac{dz}{dy}$

Ans to Q No 3: $\left(1+\dfrac{\partial z}{\partial y}\right)\dfrac{\partial z}{\partial x}=z\dfrac{\partial z}{\partial y}$

Ans to Q No 4: $py-qx=y^2-x^2$

Ans to Q No 5: $px-qy=0$

Ans to Q No 6: $\dfrac{\partial^2 z}{\partial t^2}=a^2\left(\dfrac{\partial^2 y}{\partial x^2}\right)$

Ans to Q No 7: $(p-q)Z=y-x$

Ans to Q No 8: (i) $\phi\left(x/z,\ y/z\right)=0$

(ii) $\phi\left(\dfrac{1}{x}-\dfrac{1}{y},\ \dfrac{1}{y}-\dfrac{1}{2}\right)=0$

(iii) $\phi\left(x^2-y^2,\ x^2-z^2\right)=0$

(iv) $x-(x+y)\log z=\phi(x+y)$

(v) $e^{2y}\left[z^2+(x+y)^2\right]=\phi(x+y)$

(vi) $\phi\left(xy-z^2, \frac{x}{y}\right)=0$

(vii) $\phi\left(\frac{y-x}{xy}, z-\frac{nxy}{y-x}\log\frac{y}{x}\right)=0$

(viii) $\phi\left(x^2+y^2+z^2, xyz\right)=0$

(ix) $\phi\left(x^2+y^2+z^2, xy\right)=0$

(x) $\phi\left(x^3+y^3, \frac{x-y}{z}\right)=0$

(xi) $\log|x|-\tan^{-1}\left(\frac{y}{z}\right)=\phi\left(y^2+z^2\right)$

(ix) $\phi\left[\left(\frac{xy}{z}\right), \left(\frac{x-y}{z}\right)\right]=0$



7.7 FURTHER READINGS

1) Ross, S.L. 'Differential Equations', 3rd edition, Wiley India, 2007

2) Sneddow, I.N. 'Elements of partial Differential Equations', Dover Publications, 2006

3) Reisinghania, M.D. 'Advanced Differential Equations', S. Chand & Company Ltd, New Delhi, 2001.

4) Gupta, R.K. and Sharma, J.N. 'Differential Equations', Krishna Prakashan Media (P) Ltd, 2017.



7.8 MODEL QUESTIONS

Q 1: Eliminate the arbitrary constants and form the PDE

$z=(x^2+a)(y^2+b)$

Q 2: Eliminate the arbitrary function and form the PDE

$z=x+y+f(xy)$

Q 3: Solve using Lagrange's method:

$xzp+yzq=xy$

Q 4: Solve: $x^2p+y^2q=z^2$

Q 5: Solve $(y^2+z^2)p-xyq=-3x$

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