UNIT 8 : CANONICAL FORMS, MONGE'S METHOD AND METHOD OF SEPARATION OF VARIABLES

UNIT STRUCTURE

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8.1 LEARNING OBJECTIVES

After going through this unit, you will able to-

- parabolic, Hyperbolic and Elliptic partial differential equations with examples
- reduction of a partial differential equation to canonical form with examples
- monge's method of solving partial differential equation with examples
- principle of superposition of solution of P.D.E.'s
- method of separation of variables
- solutions of boundary value problems using the method of separation of variables.

8.2 INTRODUCTION

Partial differential equation carries utomost importance in many theoritical physics and engineering problem Reduction of a partial differential equation to canonical form eases the process of solving the problem. In this unit, we will discuss about various types of canonical forms and process to slove them. Monge's method is another useful technique of solving second order partial differential differential equation. Method of separation of variables is a very useful technique to solve bondary value problem.

8.3 CANONICAL FORM

If the number of independent variables is two or three, a transformation can always be found to reduce the given partial differential equation to a canonical form (also called normal form)

In general, when the number of independent variables is greater then three, it is not always possible to find such a transformation except in certain special cases. The idea of reducing the P.D.E. to a canonical form is that the transformed equation assumes a simple form, so that the subsequent analysis of solving the equation is made easy.

Let us rewrite the PDE of second order as $Au_{xx}+Bu_{xy}+Cu_{yy}+Du_x+Eu_y+Fu = G$ ..... (1)

When the coefficients A, B, C, D, E, F and G may be functions of x and y. However for the sake of simplicity, we assume them to be constants

Accordingly, the equation (1) is

elliptic if $B^2 - 4AC < 0$

parabolic if $B^2 - 4AC = 0$

hyparbolic if $B^2 - 4AC > 0$

For example, consider the one dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}$$

$$\Rightarrow u_{xx} - u_{yy} = 0 \qquad \text{..... (2)}$$

Comparing (2) with (1), we get

A = 1, B = 0, C = –1, D = 0, E = 0, F = 0, G = 0

Now $B^2 - 4AC = 0^2 - 4.1(-1) = 4 > 0$

So, one dimensional wave equation is hyperbolic similarly, consider

the one dimensional diffusion equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y}$$

$\Rightarrow u_{xx} - u_y = 0$ ..... (3)

Comparing (3) with (1), we get

$A = 1, B = 0, C = 0, D = 0, E = -1, F = 0, G = 0$

Now $B^2 - 4AC = 0^2 - 4.1.0 = 0$

So, one dimensional diffusion equation is parabolic

Now, consider the tqo dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$\Rightarrow u_{xx} + u_{yy} = 0$ ..... (4)

Comparing (4) with (1), we get

$A = 1, B = 0, C = 1, D = 0, E = 0, F = 0, G = 0$

Now $B^2 - 4AC = 0^2 - 4.1.1 = -4 < 0$

So, two dimensional Laplace equation is elliptic

Example 1

Classify the following partial differential equation

$$2\frac{\partial^2 3}{\partial x^2} + 4\frac{\partial^2 3}{\partial x\, \partial y} + 3\frac{\partial^2 3}{\partial y^2} = 2$$

Solution : The given equation can be written as

$2Z_{xx} + 4Z_{xy} + 3Z_{yy} + DZ_x + EZ_y + Fu = G$, we get

$A = 2, B = 4, C = 3, D = 0, E = 0, F = 0, G = 2$

Now $B^2 - 4AC = 4^2 - 4 \times 2 \times 3 = 16 - 24 = -8 < 0$

$\therefore$ The given equation is elliptic

Example 2

Classify the following partial differential equation

$xyr - (x^2 - y^2)s - xyt + py - qx = 2(x^2 - y^2)$

Solution : Given PDE is

$xyr - (x^2 - y^2)s - xyt + py - qx = 2(x^2 - y^2)$

$$\Rightarrow xy\frac{\partial^2 u}{\partial x^2}-(x^2-y^2)\frac{\partial^2 u}{\partial x\,\partial y}-x\,y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial x}y-\frac{\partial u}{\partial y}x=2(x^2-y^2)$$

$$\Rightarrow xyu_{xx}-(x^2-y^2)u_{xy}-xyu_{yy}+u_xy-u_yx=2(x^2-y^2) \quad \text{..... (1)}$$

Comparing (1) with

$Au_{xx}+Bu_{yy}+Cu_{yy}+Du_x+Eu_y+Fu=G$, we get

$A=xy,\ B=-(x^2-y^2),\ C=-xy,\ D=y,\ E=-x,\ F=0,\ G=2(x^2-y^2)$

Now $B^2-4AC=\{-(x^2-y^2)\}-4xy(-xy)$

$=(x^2-y^2)^2+4x^2y^2$

$=(x^2+y^2)^2>0$

$\therefore$ The given equation is hyperbolic



CHECK YOUR PROGRESS -1

Classify the following partial differential equations–

Q 1: $\frac{\partial^2 u}{\partial x^2}+4\frac{\partial^2 u}{\partial x\,\partial y}+4\frac{\partial^2 4}{\partial y^2}=2$

Q 2: $x^2(y-1)r-x(y^2-1)s+y(y-1)t+xyp-q=0$

8.4 REDUCTION TO CANONICAL FORM

Let the PDE of second order be

$AU_{xx}+BU_{xy}+CU_{yy}+DU_x+EU_y+FU=G$ ..... (1)

Now consider the most general transformation of the independent variables x and y of (1) to new variables ξ and η where

$\xi=\xi(x,y),\ \eta=\eta(x,y)$ ..... (2)

Such that the functions ξ , η are continuously differentiable and their Jacobian non zero.

(ie, $J=\frac{\partial(\xi,\eta)}{\partial(x,y)}\neq 0$) in the domain Ω

Where (1) holds. Using the chain rule of partial derivatives, we get

$$\Rightarrow U_x=U_\xi\xi_x+U_\eta\eta_x$$

$$Uy=U_\xi\xi_y+U_\eta\eta_x$$

$$U_{xx}=U_{\xi\xi}\xi^2{}_x+2U_{\xi\eta}\xi_x\eta_x+U_{\eta\eta}\eta_x^2+U_\xi\xi_{xx}+U_\eta\eta_{xx}$$

$$U_{yy}=U_{\xi\xi}\xi_y{}^2{}_x+2U_{\xi\eta}\xi_y\eta_y+U_{\eta\eta}\eta_y^2+U_\xi\xi_{yy}+U_\eta\eta_{yy}$$

$$U_{xy} = U_{\xi\xi}\xi_x\xi_y + U_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + U_{\eta\eta}\eta_y\eta_x + U_\xi\xi_{yy} + U_\eta\eta_{yy}$$

Substituting these expressions into PDE (1), we get

$$\bar{A}\,U_{\xi\xi} + \bar{B}\,U_{\xi\eta} + \bar{C}\,U_{\eta\eta} + \bar{D}\,U_\xi + \bar{E}\,U_\eta + \bar{F}\,U = \bar{G} \qquad \text{..... (3)}$$

Where

$$\bar{A} = A\xi_x^{\,2} + B\xi_x\xi_y + C\xi_y^{\,2}$$

$$\bar{B} = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y$$

$$\bar{C} = A\eta_x^{\,2} + B\eta_x\eta_y + C\eta_y^{\,2}$$

$$\bar{D} = A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y$$

$$\bar{E} = A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta$$

$$\bar{F} = F$$

$$\bar{G} = G$$

It may be noted that the transformed equation (3) has the same form as that of the origenal equation (1) under the transformation (2).

Since the classification of equation (1) depends on the coefficients A, B and C, we now rewrite the equation (1) in the form.

$$AU_{xx} + BU_{xy} + CY_{yy} = H(x, y, u, u_x, u_y) \qquad \text{..... (5)}$$

It can be shown easily that under the transformation (2), (5) takes one of the following forms–

(i) $U_{\xi\xi} - U_{\eta\eta} = \varphi(\xi, \eta, U, U_\xi, U_\eta)$

or $U_{\xi\eta} = \varphi(\xi, \eta, U, U_\xi, U_\eta)$ in the hyperbolic case

(ii) $U_{\xi\xi} + U_{\eta\eta} = \varphi(\xi, \eta, U, U_\xi, U_\eta)$ in the elliptic case

(iii) $U_{\xi\xi} = \varphi(\xi, \eta, U, U_\xi, U_\eta)$

or $U_{\eta\eta} = \varphi(\xi, \eta, U, U_\xi, U_\eta)$ in the parabolic case

Now using equation (4), it can be verified that

$$\bar{B}^2 - 4\bar{A}\,\bar{C} = (\xi_x\eta_y - \xi_y\eta_x)\,(B^2 - 4AC)$$

$$= \frac{\partial(\xi_y, \eta)}{\partial(x,\ y)}(B^2 - 4AC)$$

$$= J(B^2 - 4AC)$$

and therefore, we conclude that this transformation of the independent variables does not alter the type of the original equation.

8.5 CANONICAL FORM OF HYPERBOLIC EQUATION

In this case the discriminant $\overline{B}^2 - 4\overline{A}\,\overline{C} > 0$.

For hyperbolic case, we set $\overline{A} = 0$ and $\overline{C} = 0$ in equation (4) of article 11.4, which will give us the co-ordinates ξ and η that reduce the given PDE to canonical form in which the coefficients $U_{\xi\xi}$, $U_{\eta\eta}$ are zero.

$$\therefore\ \overline{A} = A\xi_x{}^2 + B\xi_x\xi_y + C\xi_y{}^2 = 0$$

$$\overline{C} = An_x{}^2 + B\eta_x\eta_y + C\eta_y{}^2 = 0$$

which can be written as

$$A\left(\frac{\xi_x}{\xi_y}\right)^2 + B\left(\frac{\xi_x}{\xi_y}\right) + C = 0$$

and $A\left(\frac{\eta_x}{\eta_y}\right)^2 + B\left(\frac{\eta_x}{\eta_y}\right) + C = 0$

$$\therefore\ \frac{\xi_x}{\xi_y} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

and $\dfrac{\eta_x}{\eta_y} = \dfrac{-B \pm \sqrt{B^2 - 4AC}}{2A}$

$\therefore$ To make distinction, we take

$$\left.\begin{aligned} \frac{\xi_x}{\xi_y} &= \frac{-B - \sqrt{B^2 - 4AC}}{2A} = -\frac{B + \sqrt{B^2 - 4AC}}{2A} \\ \text{and}\quad \frac{\eta_x}{\eta_y} &= \frac{-B + \sqrt{B^2 - 4AC}}{2A} = -\frac{B + \sqrt{B^2 - 4AC}}{2A} \end{aligned}\right\} \quad \text{..... (i)}$$

Thus, if $B^2 - 4AC > 0$, then at any point (x, y), there exists two solutions $\xi(x, y) = c_1$ and $\eta(x, y) = c_2$ which reduces the equation (1) in article (1.4 to canonical forms.

These solutions (curves) are called characteristic curves. Though there are two solutions for each quadratic, we have considered only one solution for each, otherwise we will end up with same co-ordinates.

Now $\xi(x, y) = c_1$ can be reduced to

$$d\xi = \xi_x dx + \xi_y dy = dc_1 = 0$$

$$\Rightarrow \xi_x dx + \xi_y dy = 0$$

$$\Rightarrow \frac{\xi_x}{\xi_y} = -\frac{dy}{dx}$$

Then (i) is replaced by

$$-\frac{dy}{dx} = -\frac{B+\sqrt{B^2-4AC}}{2A}$$

$$\Rightarrow \frac{dy}{dx} = \frac{B+\sqrt{B^2-4AC}}{2A} \qquad \rightarrow (ii)$$

Similarly,

$$-\frac{\eta_x}{\eta_y} = -\frac{B-\sqrt{B^2-4AC}}{2A}$$

$$\Rightarrow \frac{dy}{dx} = \frac{B-\sqrt{B^2-4AC}}{2A} \qquad \rightarrow (iii)$$

Integrating (ii) and (iii), we get the required characteristic curves

$\xi(x, y) = c_1$ and $\eta(x, y) = c_2$

Example 3

Reduce the following PDE to canonical form and solve

$$3\frac{\partial^2 u}{\partial x^2} + 10\frac{\partial^2 u}{\partial x \partial y} + 3\frac{\partial^2 u}{\partial y^2} = 0$$

Solution : Given PDE is

$$3\frac{\partial^2 u}{\partial x^2} + 10\frac{\partial^2 u}{\partial x \partial y} + 3\frac{\partial^2 u}{\partial y^2} = 0$$

$\Rightarrow 3u_{xx}+10u_{xy}+3u_{yy} = 0$ ..... (1)

Comparing (1) with

$AU_{xx}+BU_{xy}+CU_{yy}+DU_x+EU_y+FU = G$, we get

$A = 3, B = 10, C = 3, D = 0, E = 0, F = 0, G = 0$

Now $B^2-4AC = 10^2 - 4.3.3 = 64 > 0$

$\therefore$ Equation (1) is hyperbolic

Now the characteristic equations are

$$\frac{dy}{dx} = \frac{B+\sqrt{B^2-4AC}}{2A}$$

$$= \frac{10+\sqrt{64}}{6}$$

$$= \frac{10+8}{6}$$

= 3

$\therefore$ dy = 3dx

Integrating, $y - 3x = c_1$ ..... (2)

and $\frac{dy}{dx} = \frac{B - \sqrt{B^2 - 4AC}}{2A}$

$= \frac{10 - \sqrt{64}}{6}$

$= \frac{10 - 8}{6}$

$= \frac{1}{3}$

$\Rightarrow 3dy - dx = 0$

Integrating, $3y - x = c_2$ ..... (3)

Therefore $\xi = y - 3x$

$\eta = 3y - x$

$\therefore \xi_x = -3$

$\xi_{xx} = 0$

$\xi_y = 1$

$\xi_{yy} = 0$

$\xi_{xy} = 0$

$\eta_x = -1$

$\eta_{xx} = 0$

$\eta_y = 3$

$\eta_{yy} = 0$

$\eta_{xy} = 0$

Here, the new coefficients are

$\overline{A} = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 3(-3)^2 + 10(-3)(1) + 3(1)^2 = 0$

$\overline{B} = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y$

= 2.3.(–3) (–1) + 10((–3).3+1(–1)) + 2.3.1.3

= 18 – 100 + 18

= – 64

$\overline{C} = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 = 3.(-1)^2 + 10.(-1)3 + 3.3^2 = 0$

$\bar{D} = A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y$

= 3.0 + 10.0 + 3.0 + 0.(–3) + 0.1

= 0

$\bar{E} = A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y$

= 3.0 + 10.0 + 3.0 + 0.(–1) + 0.3

= 0

$\bar{F} = F = 0$

$\bar{G} = G = 0$

Putting these values in the equation

$$\bar{A}\,U_{\xi\xi} + \bar{B}\,U_{\xi\eta} + \bar{C}\,U_{\eta\eta} + \bar{D}\,U_{\xi} + \bar{E}\,U_{\eta} + \bar{F}\,U = \bar{G}$$

we get

$$-64\,U_{\xi_n} = 0$$

$$\Rightarrow U_{\xi_n} = 0 \qquad \text{..... (4)}$$

The equation (4) is the required conomical form.

Now (4) $\Rightarrow U_{\xi_n} = 0$

$$\Rightarrow \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \eta}\right) = 0$$

Integrating, we get

$\frac{\partial u}{\partial \eta} = f(\eta)$ when f is an arbitrary function

Again integrating

$$U = \int f(\eta)\, d\eta + \text{constant}$$

$$\Rightarrow U = G(\eta) + F(\xi)$$

$\Rightarrow U(x,y) = G(3y–x)+F(y–3x)$

where G and F are arbitrary functions

Example 4

Reduce the following PDE to canconical form and solve

U_{xx}–2sinx U_{xy}–$\cos^2 x$ U_{yy}–cosx U_y = 0

Solution : Given PDE is

U_{xx}–2sinx U_{xy}–$\cos^2 x$ U_{yy}–cosx U_y = 0 ..... (1)

Comparing (1) with

$AU_{xx}+BU_{xy}+CU_{yy}+DU_x+EU_y+FU = G$, we get

$A = 1, B = -2\sin x, C = -\cos^2 x, D = 0,$

$E = -\cos x, F = 0, G = 0$

Now $B^2 - 4AC = (-2\sin x)^2 - 4.1.(-\cos^2 x)$

$= 4\sin^2 x + 4\cos^2 x$

$= 4(\sin^2 x + \cos^2 x)$

$= 4$

$\therefore$ Equation (1) is hyperbolic

Now, the characteristic equations are

$$\frac{dy}{dx} = \frac{B + \sqrt{B^2 - 4AC}}{2A}$$

$$= \frac{-2\sin x + \sqrt{4}}{2}$$

$$= 1 - \sin x$$

$\Rightarrow dy = (1 - \sin x)dx$

Integrating

$$y = x + \cos x + c_1$$

$$\Rightarrow y - x - \cos x = c_1$$

and $$\frac{dy}{dx} = \frac{B - \sqrt{B^2 - 4AC}}{2A}$$

$$= \frac{-2\sin x - \sqrt{4}}{2}$$

$$= -1 - \sin x$$

$\Rightarrow dy = (-1 - \sin x)dx$

Integrating

$$y = -x + \cos x + c_2$$

$$\Rightarrow y + x - \cos x = c_2$$

Therefore $\xi = y - x - \cos x$ ..... (2)

$\eta = y + x - \cos x$ ..... (3)

$$\therefore \xi_x = -1 + \sin x$$

$$\xi_{xx} = \cos x$$

$$\xi_y = 1$$

$$\xi_{yy} = 0$$

$$\xi_{xy} = 0$$

$\eta_x = 1 + \sin x$

$\eta_{xx} = \cos x$

$\eta_y = 1$

$\eta_{yy} = 0$

$\eta_{xy} = 0$

The new coefficients are

$\therefore \ \overline{A} = A\xi_x^{\,2} + B\xi_x\xi_y + C\xi_y^{\,2}$

$= 1(-1 + \sin x)^2 - 2\sin x(-1 + \sin x).1 - \cos^2 x.1^2$

$= 1 + \sin^2 x - 2\sin x + 2\sin x - 2\sin^2 x - \cos^2 x$

$= 1 - (\sin^2 x + \cos^2 x)$

$= 1 - 1$

$= 0$

$\overline{B} = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y$

$= 2.1((-1) + \sin x)(1+\sin x) - 2\sin x((-1 + \sin x).1 + 1.(1+\sin x)) - 2\cos^2 x.1.1$

$= 2(\sin^2 x - 1) - 2\sin x\, 2.\sin x - 2\cos^2 x$

$= -2\cos^2 x - 4\sin^2 x - 2\cos^2 x$

$= -4(\cos^2 x + \sin^2 x)$

$= -4$

$\overline{C} = A\eta_x^{\,2} + B\eta_x\eta_y + C\eta_y^{\,2}$

$= 1.(1 + \sin x)^2 - 2\sin x.(1 + \sin x).1 - \cos^2 x.1^2$

$= 1 + \sin^2 x + 2\sin x - 2\sin x - 2\sin^2 x - \cos^2 x$

$= 1 - (\sin^2 x + \cos^2 x)$

$= 1 - 1$

$= 0$

$\overline{D} = A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y$

$= 1.\cos x - 2\sin x.0 - \cos^2 x.0 + 0.(-1+\sin x) - \cos x.1$

$= \cos x - \cos x$

$= 0$

$\overline{E} = A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y$

$= 1.\cos x - 2\sin x.0 - \cos^2 x.0 + 0.(-1+\sin x) - \cos x.1$

$= \cos x - \cos x$

$= 0$

$$\overline{F} = F = 0$$

$$\overline{G} = G = 0$$

Putting these values in the equation

$$\overline{A}\,U_{\xi\xi} + \overline{B}\,U_{\xi\eta} + \overline{C}\,U_{\eta\eta} + \overline{D}\,U_{\xi} + \overline{E}\,U_{\eta} + \overline{F}\,U = \overline{G}$$

We get

$$-4U_{\xi\eta} = 0$$

$$\Rightarrow U_{\xi\eta} = 0 \qquad \text{..... (4)}$$

(4) is the required canonical form

Now (4) $\Rightarrow U_{\xi\eta} = 0$

$$\Rightarrow \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \eta}\right) = 0$$

Integrating, we get

$$\frac{\partial u}{\partial \eta} = f(\eta)$$

Again integrating

$$U = \int f(\eta)\, d\eta + f(\xi)$$

$\Rightarrow U = G(n) + F(\xi)$

$\Rightarrow U(x, y) = G(y+x - \cos x) + F(y - x - \cos x)$

where G and F are arbitrary functions

8.6 CANONICAL FORM OF PARABOLIC EQUATION

For the parabolic equation, the discriminant $\overline{B}^2 = 4\overline{A}\,\overline{C} = 0$ which can be true if $\overline{B} = 0$ and $\overline{A}$ or $\overline{C}$ is equal to zero.

Suppose we first set $\overline{A} = 0$

Then, we obtain

$$\overline{A} = A\xi_x{}^2 + B\xi_x\xi_y + C\xi_y{}^2 = 0$$

which gives

$$A\left(\frac{\xi_x}{\xi_y}\right)^2 + B\left(\frac{\xi_x}{\xi_y}\right) + C = 0$$

As before

$$\frac{\xi_x}{\xi_y} = \frac{-B + \sqrt{B^2 - 4AC}}{2A} = \frac{-B}{2A} \qquad \text{(as } B^2\text{–}4AC = 0$$

$$\Rightarrow -\frac{dy}{dx} = -\frac{B}{2A}$$

$$\Rightarrow \frac{dy}{dx} = \frac{B}{2A}$$

Integrating, we get $\xi(x, y) = C_1$

Now $\bar{B} = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y$

$$= 2A\xi_x\eta_x + 2\sqrt{AC}(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y$$

$$\left(\because B^2 - 4AC = 0 \Rightarrow B = 2\sqrt{AC}\right)$$

$$\bar{B} = 2\left(\sqrt{A}\,\xi_x + \sqrt{C}\,\xi_y\right)\left(\sqrt{A}\,\eta_x + \sqrt{C}\,\eta_y\right) \qquad \rightarrow (1)$$

We know

$$\frac{\xi_x}{\xi_y} = \frac{-B}{2A} = \frac{-2\sqrt{AC}}{2A} = -\frac{\sqrt{C}}{\sqrt{A}}$$

$$\Rightarrow \sqrt{A}\,\xi_x + \sqrt{C}\,\xi_y = 0$$

Using this in (i), we get

$$\bar{B} = 0$$

Similarly, if we set $\bar{C} = 0$, then

$$\frac{\eta_x}{\eta_y} = -\frac{B}{2A}$$

and then $\eta(x, y) = C_1$

and $\bar{B} = 0$

Next, to determine the second characteristic curve, so that this is not parallel to the first one. In other words, we choose the second one such that the Jacobian of transformation is non-zero one. In this way, we find the second characteristic curve.

Example 5

Reduce the following PDE to canonical form and solve

$$y^2 u_{xx} - 2xyu_{xy} + x^2 u_{yy} = \frac{y^2}{x}u_x + \frac{x^2}{y}u_y$$

Solution : Given PDE is

$$y^2u_{xx} - 2xyu_{xy} + x^2u_{yy} = \frac{y^2}{x}u_x + \frac{x^2}{y}u_y$$

$$\Rightarrow y^2u_{xx} - 2xyu_{xy} + x^2u_{yy} - \frac{y^2}{x}u_x - \frac{x^2}{y}u_y = 0 \quad \text{..... (1)}$$

Comparing (1) with

$Au_{xx}+Bu_{xy}+Cu_{yy}+Du_x+Eu_y+Fu = G$, we get

$A = y^2$, $B = -2xy$, $C = x^2$, $D = -\frac{y^2}{x}$, $E = -\frac{x^2}{y}$, $F = 0$, $G = 0$

Here $B^2 - 4AC = (-2xy)^2 - 4.y^2.x^2$

$= 4x^2y^2 - 4x^2y^2$

$= 0$

Hence equation (1) is parabolic

Now, the characteristic curve is given by

$$\frac{du}{dx} = \frac{B}{2A}$$

$$= \frac{-2xy}{2y^2}$$

$$= -\frac{x}{y}$$

$$\Rightarrow xdx + ydy = 0$$

Integrating, we get

$x^2 + y^2 = c_1$

Therefore $\xi = x^2 + y^2$ ..... (2)

Now, we chose the second co-ordinate as $\eta = x^2$ ..... (3)

(or you may take $\eta = y^2$)

$$\therefore \xi_x = 2x$$

$$\xi_{xx} = 2$$

$$\xi_{xy} = 0$$

$$\xi_y = 2y$$

$$\xi_{yy} = 2$$

$$\eta_x = 2x$$

$$\eta_{xx} = 2$$

$$\eta_{xy} = 0$$

$$\eta_y = 0$$

$\eta_{yy} = 0$

$\therefore$ The new coefficients are

$\overline{A} = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2$

$= y^2(2x)^2 - 2xy(2x)(2y) + x^2(2y)^2$

$= 4x^2y^2 - 8x^2y^2 + 4x^2y^2$

$= 0$

$\overline{B} = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y$

$= 2.y^2.2x.2x - 2xy(2x.0 + 2y.2x) + 2x^2.2y.0$

$= 8x^2y^2 - 8x^2y^2$

$= 0$

$\overline{C} = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2$

$y^2(2) - 2xy.0 + x^2(2) - ()$

$= 2y^2 + 2x^2 - 2y^2 - 2x^2$

$= 0$

$\overline{D} = A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y$

$= y^2.2 - 2xy.0 + x^2.0 - ()$

$= 2y^2 - 2y^2$

$= 0$

$\overline{F} = F = 0$

$\overline{G} = G = 0$

Putting these values in the equation

$$\overline{A}\,u_{\xi\xi} + \overline{B}\,u_{\xi\eta} + \overline{C}\,u_{\eta\eta} + \overline{D}\,u_{\xi} + \overline{E}\,u_{\eta} + \overline{F}\,u = \overline{G}$$

We get

$$4x^2y^2u_{\eta\eta} = 0$$

$$\Rightarrow u_{\eta\eta} = 0$$

$$\Rightarrow \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \eta}\right) = 0$$

Integrating

$$\frac{\partial u}{\partial \eta} = f(\xi)$$

Again integrating

$$U = \int f(\xi)\, d\eta + g(\xi)$$

$$\Rightarrow U = \eta f(\xi) + g(\xi)$$

$$\Rightarrow U(x, y) = x^2 f(x^2 + y^2) + g(x^2 + y^2)$$

Where f and g are arbitrary functions

Example 6

Reduce the following PDE to canonical form.

$x^2 u_{xx} - 2xy\, u_{xy} + y^2 u_{yy} = e^x$

Solution : Given PDE is

$$x^2 u_{xx} - 2xy\, u_{xy} + y^2 u_{yy} = e^x \quad \text{..... (1)}$$

Comparing (1) with

$AU_{xx} + BU_{xy} + CU_{yy} + DU_x + EU_y + FU = G$, we get

$A = x^2$, $B = -2xy$, $C = y^2$, $D = 0$, $E = 0$, $F = 0$, $G = e^x$

Now $B^2 - 4AC = (-2xy)^2 - 4.\, x^2.y^2$

$= 4x^2y^2 - 4x^2y^2$

$= 0$

So, given equation is parabolic

Now the characteristic curve is given by

$$\frac{dy}{dx} = \frac{B}{2A} = \frac{-2xy}{2x^2} = -\frac{y}{x}$$

$$\Rightarrow \frac{dx}{x} + \frac{dy}{y} = 0$$

Integrating,

$$\log x + \log y = \log c_1$$

$$\Rightarrow \log xy = \log c_1$$

$$\Rightarrow xy = c_1$$

$$\therefore \xi = xy \qquad \rightarrow (2)$$

Now we select $\eta = y \quad \rightarrow (3)$ (or you can select $\eta = x$)

$$\therefore \xi_x = y$$

$$\xi_{xx} = 0$$

$$\xi_y = x$$

$$\xi_{yy} = 0$$

$$\xi_{xy} = 1$$

$$\eta_x = 0$$

$$\eta_{xx} = 0$$

$$\eta_y = 1$$

$$\eta_{yy} = 0$$

$$\eta_{xy} = 0$$

$\therefore$ The new coefficients are

$$\overline{A} = A\xi_x^{\,2} + B\xi_x\xi_y + C\xi_y^{\,2}$$

$= x^2.y^2 - 2xy.y.x + y^2.x^2$

$= 2x^2y^2 - 2x^2y^2$

$= 0$

$$\overline{B} = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y$$

$= 2x^2.y.0 - 2xy(y.1 + x.0) + 2y^2.x.1$

$= -2xy^2 + 2xy^2$

$= 0$

$$\overline{C} = A\eta_x^{\,2} + B\eta_x\eta_y + C\eta_y^{\,2}$$

$= x^2.0^2 - 2xy.0.1 + y^2.1^2$

$= y^2$

$$\overline{D} = A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y$$

$= x^2.0 - 2xy.1 + y^2.0 + 0.y + 0.x$

$= -2xy$

$$\overline{E} = A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y$$

$= x^2.0 - 2xy.0 + y^2.0 + 0.0 + 0.1$

$= 0$

$$\overline{F} = F = 0$$

$$\overline{G} = G = e^x$$

Putting these values in the equation

$$\overline{A}\,U_{\xi\xi} + \overline{B}\,U_{\xi\eta} + \overline{C}\,U_{\eta\eta} + \overline{D}\,U_{\xi} + \overline{E}\,U_{\eta} + \overline{F}\,U = \overline{G},$$

we get

$$y^2\,u_{\eta\eta} - 2xy\,u_{\xi} = e^x$$

$$\Rightarrow x^2\,u_{\eta\eta} - 2\,\xi\,u_{\xi} = e^{\xi/\eta}$$

This is the required canonical form

8.7 CANONICAL FORM OF ELLIPTIC EQUATION

For the elliptic equation, the discriminant $\overline{B}^2 - 4\overline{A}\,\overline{C} < 0$

$$\therefore \frac{dy}{dx} = \frac{B + \sqrt{B^2 - 4AC}}{2A}$$

and $$\frac{dy}{dx} = \frac{B - \sqrt{B^2 - 4AC}}{2A}$$

Then we have after integration

$\xi = f(x, y) + ig(x, y) = c_1$ |Since $B^2 - 4AC < 0$

$\eta = f(x, y) - ig(x, y) = c_2$ is imaginary

Here, we take the characteristic curve as

$$\xi(x, y) = f(x, y)$$

and $$\eta(x, y) = g(x, y)$$

and proceed as earlier.

Example 7

Reduce the following PDE to canonical form

$u_{xx} + x^2 u_{yy} = 0$

Solution : Given PDE is

$u_{xx} + x^2 u_{yy} = 0$ ..... (1)

Comparing (1) with

$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$, we get

$A = 1, B = 0, C = x^2, D = 0, E = 0, F = 0, G = 0$

Now $B^2 - 4AC = 0^2 - 4.1.x^2 = -4x^2 < 0$

$\therefore$ Equation (1) is elliptic

Therefore, the characteristic curve is given by

$$\frac{dy}{dx} = \frac{B + \sqrt{B^2 - 4AC}}{2A}$$

$$= \frac{0 + \sqrt{-4x^2}}{2.1}$$

$$= \frac{i.2x}{2}$$

$$= i\,x$$

$$dy = ix\,dx$$

Integrating

$$y = i\frac{x^2}{2} + c$$

$$\Rightarrow y - i\frac{x^2}{2} = c$$

Therefore $\xi = y$ ..... (2)

and $\eta = \frac{x^2}{2}$ ..... (3)

$\therefore \xi_x = 0$

$\xi_{xx} = 0$

$\xi_y = 1$

$\xi_{yy} = 0$

$\xi_{xy} = 0$

$\eta_x = x$

$\eta_{xx} = 1$

$\eta_y = 0$

$\eta_{yy} = 0$

$\eta_{xy} = 0$

$\therefore$ The new coefficient are

$\overline{A} = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2$

$= 1.0^2 + 0.0.1 + x^2.1^2$

$= x^2$

$\overline{B} = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y$

$= 2.1.0.x + 0(0.0 + 1.x) + 2x^2.1.0$

$= 0$

$\overline{C} = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2$

$= 1.x^2 - 0.x.0 + x^2.0^2$

$= x^2$

$\overline{D} = A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y$

$= 1.0 + 0.0 + x^2.0 + 0.0 + 0.1$

$= 0$

$\overline{E} = A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y$

$= 1.1 + 0.0 + x^2.0 + 0.0$

$= 1$

$\bar{F} = F = 0$

$\bar{G} = G = 0$

Putting these values in the equation

$\bar{A}\,U_{\xi\xi} + \bar{B}\,U_{\xi\eta} + \bar{C}\,U_{\eta\eta} + \bar{D}\,U_{\xi} + \bar{E}\,U_{\eta} + \bar{F}\,U = \bar{G}$,

we get

$$x^2\,u_{\xi\xi} + x^2\,u_{\eta\eta} + u_{\eta} = 0$$

$$\Rightarrow x^2\left(u_{\xi\xi} + u_{\eta\eta}\right) = -u_{\eta}$$

$$\Rightarrow u_{\xi\xi} + u_{\eta\eta} = -\frac{u_{\eta}}{x^2}$$

$$\Rightarrow u_{\xi\xi} + u_{\eta\eta} = -\frac{u_{\eta}}{2\eta} \qquad \left|\because \eta = \frac{x^2}{2} \quad \therefore x^2 = 2\eta\right.$$

This is the required canonical form



CHECK YOUR PROGRESS 2

Q 1: Reduce the following PDE to canonical form

(i) $\dfrac{\partial^2 u}{\partial x^2} + 2\dfrac{\partial^2 u}{\partial x\,\partial y} + \sin^2 x\dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial u}{\partial y} = 0$

(ii) $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial y^2} = 0$

Q 2: Reduce $3\dfrac{\partial^2 u}{\partial x^2} + 10\dfrac{\partial^2 u}{\partial x\,\partial y} + 3\dfrac{\partial^2 u}{\partial y^2} = 0$ into canonical form and solve.

8.8 MONGE'S METHOD

Let us consider the PDE of the form

$Rr + Ss + Tt = V$ ..... (1)

Where R, S, T and V are functions of x, y, z, p and q

$$\left(\text{where } r = \frac{\partial^2 z}{\partial x^2},\ s = \frac{\partial^2 z}{\partial x\,\partial y},\ t = \frac{\partial^2 z}{\partial y^2},\ p = \frac{\partial z}{\partial x},\ q = \frac{\partial z}{\partial y}\right)$$

We have $p = \dfrac{\partial z}{\partial x}$

$$q = \frac{\partial z}{\partial y}$$

$$\left.\begin{aligned} r &= \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial p}{\partial x} \\ t &= \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial q}{\partial y} \\ s &= \frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial q}{\partial x} \\ s &= \frac{\partial^2 z}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial p}{\partial y} \end{aligned}\right\} \quad \text{..... (2)}$$

Now $dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy$

$= rdx + sdy$ [using (2)] ..... (3)

$dq = \frac{\partial q}{\partial x}dx + \frac{\partial q}{\partial y}dy$

$= sdx + tdy$ [using (2)] ..... (4)

$\therefore$ From (3) and (4)

$$\left.\begin{aligned} r &= \frac{dp - sdy}{dx} \\ t &= \frac{dq - sdx}{dy} \end{aligned}\right\} \quad \text{..... (5)}$$

Substituting the values of r and s given by (5) in (1),

We get

$$R\left(\frac{dp - sdy}{dx}\right) + S\,s + T\left(\frac{dq - sdx}{dy}\right) = V$$

$$\Rightarrow \frac{R(dp - sdy)dy + Ss\,dx\,dy + T(dq - sdx)dx}{dx\,dy} = V$$

$\Rightarrow R(dp–sdy)dy + Ss\,dxdy + T(dq–sdx)dx = vdxdy$

$\Rightarrow (Rdpdy + Tdqdx – Vdxdy) – s\{R(dy)^2–Sdxdy + T(dx)^2\} = 0 \quad \rightarrow (6)$

Clearly, any relation between x, y, z, p and q which satisfies (6) must also satisfy the following two simultaneous equations (subsidiary equations)

$Rdpdy + Tdqdx = Vdxdy = 0$ ..... (7)

$(dy)^2 – Sdxdy + T(dx)^2 = 0$ ..... (8)

The equations (7) and (8) are called Monge's subsidiary equations

and the relation which satisfy these equations are called intermediate integrals.

Equation (8) being a quadratic, in general, it can be resolved into two equations, say

$dy - m_1 dx = 0$ ..... (9)

$dy - m_2 dx = 0$ ..... (10)

Now the following two cases arise–

Case I : When m_1 and m_2 are distinct in (9) and (10)

In this case, (7) and (9), if necessary by using wellknown result dz = pdx + qdy, will give two integrals $u_1 = a$ and $\nu_1 = b$, where a and b are arbitrary constants. These give

$u_1 = f_1(\upsilon_1)$ ..... (11)

Where f_1 is an arbitrary function. It is called intermediate integral of (1).

Next, taking (7) and (10) as before, we get another intermediate integral of (1), say

$U_2 = f_2(\upsilon_2$ ..... (12)

Where f_2 is an arbitrary function.

Thus, we have in this case two distinct intermadiate integrals (11) and (12). Solving (11) and (12), we obtain values of p and q in terms of x, y and z. Now substituting these values of p and q in well known relation

$dz = pdx + qdy$ ..... (13)

and then integrating (13), we get the required complete integral of (1).

Case II : When $m_1 = m_2$, ie, (8) is a perfect square

As before, in this case, we get only one intermediate integral which is in Lagrange's form

$Pq + Qq = R$ ..... (14)

Solving (14) with help of Lagrange's method, we get the required complete integral of (1)

Example 8

Solve one-dimansional wave equation by Monge's method –

$$\frac{\partial^2 z}{\partial x^2} = a^2 \frac{\partial^2 z}{\partial t^2}$$

Solution : Given equation is

$$\frac{\partial^2 z}{\partial x^2} = a^2 \frac{\partial^2 z}{\partial t^2}$$

$\Rightarrow r = a^2 t$

$\Rightarrow r - a^2 t = 0$ ..... (1)

Compaing (1) with Rr + Ss + Tt = V, we get

$R = 1, S = 0, T = -a^2, V = 0$

Hence, Monge's subsidiary equations are

$Rdpdy + Tdqdx - Vdxdy = 0$

$\Rightarrow dpdy - a^2 dqdx = 0$ ..... (2)

and $R(dy)^2 - a^2(dx)^2 = 0$ ..... (3)

Equation (3) can be factorized as

$(dy - adx)(dy + adx) = 0$

Hence, two system of equations to be considered as

$dpdy - a^2 dqdx = 0, \quad dy - adx = 0$ ..... (4)

$dpdy - a^2 dqdx = 0, \quad dy + adx = 0$ ..... (5)

Integrating, the second equation of (4), we get

$y - ax = c_1$ ..... (6)

Now elimingting $\frac{dy}{dx}$ between the equation of (4), we get $dp - adq = 0$

Integrating,

$p - aq = c_2$ ..... (7)

Hence the intermediate integral corresponding to (4) is

$p - aq = f_1(y - ax)$ ..... (8)

Similarly, another intermediate integral corresponding to (5) is

$p + aq = f_2(y + ax)$ ..... (9)

Where f_1 and f_2 are arbitrary functions

Now (9) – (8) $\Rightarrow 2aq = f_2(y+ax) - f_1(y-ax)$

$$\Rightarrow q = \frac{1}{2a}\{f_2(y+ax) - f_1(y-ax)\}$$

$$(9) \Rightarrow p + a \times \frac{1}{2a}\{f_2(y+ax) - f_1(y-ax)\} = f_2(y+ax)$$

$$\Rightarrow p = f_2(y+ax) - \frac{1}{2}f_2(y+ax) + \frac{1}{2}f_1(y-ax)$$

$$\Rightarrow p = \frac{1}{2}\{f_2(y+ax) + f_2(y-ax)\}$$

Substituting these values of p and q in

dz = pdx + qdy, we get

$$dz = \frac{1}{2}\{f_2(y+ax) + f_1(y-ax)\}dx + \frac{1}{2a}\{f_2(y+ax) - f_1(y-ax)\}dy$$

$$= \frac{1}{2a}\{af_2(y+ax)dx + af_1(y-ax)dx + f_2(y+ax) - f_1(y-ax)dy\}$$

$$dz = \frac{1}{2a}f_2(y+ax)(dy+adx) - \frac{1}{2a}f_1(y-ax)(dy-adx)$$

$\Rightarrow$ Integrating, we get

$Z = g_2(y+ax) + g_1(y-ax)$

Where g_1 and g_2 are arbitrary function.

Example 9

Solve $t - r\sec^4 y = 2q\tan y$

Solution : Given equation is

$t - r\sec^4 y = 2q\tan y$ ..... (1)

Company (1) with Rr+Ss+Tt = V, we get

$R = -\sec^4 y, S = 0, T = 1, V = 2q\tan y$ ..... (2)

Now Monge's subjidiary equations we

$Rdpdy + Tdqdx - Vdxdy = 0$ ..... (3)

and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$ ..... (4)

Putting values of R, S, T and V in (3) and (4), we get

$-\sec^4 ydpdy + dqdx - 2q\tan ydxdy = 0$ ..... (5)

$-\sec^4 y(dy)^2 + (dx)^2 = 0$ ..... (6)

Equation (6) may be factorized as

$(dx - \sec^2 ydy)(dx + \sec^2 ydy) = 0$, so that

$dx - \sec^2 ydy = 0$ ..... (7)

$dx + \sec^2 ydy = 0$ ..... (8)

Putting values of dx from (7) in (5), we get

$-\sec^4 y\,dpdy + dq\sec^2 ydy - 2q\tan ydy\sec^2 ydy = 0$

$\Rightarrow -dp + \cos^2 ydq - 2q\sin y\cos ydy = 0$

$\Rightarrow dp - (\cos^2 ydq - q \cdot 2\sin y\cos ydy) = 0$

$\Rightarrow dp - d(q\cos^2 y) = 0$

Integrating, we get

$p - q\cos^2 y = c_1$, when c_1 is an arbitrary constant ..... (9)

Integrating (7), we get

$x - \tan y = c_2$, c_2 is another arbitrary constant ..... (10)

$\therefore$ from (9) and (10), one integral of (1) as

$p - q\cos^2 y = f_1(x - \tan y)$ ..... (11)

Similarly, from (8) and (5), the other integral of (1) is

$p+q\cos^2 y = f_2(x+\tan y)$ ..... (12)

(11) + (11) $\Rightarrow 2p = f_1(x-\tan y) + f_2(x+\tan y)$

$$\Rightarrow p = \frac{1}{2}\,[f_1(x-\tan y) + f_2(x+\tan y)] \quad \text{..... (13)}$$

Putting this value of p in (11), we get

$$\frac{1}{2}[(f_1(x-\tan y) + f_2(x+\tan y)] - q\cos^2 y = f_2(x+\tan y)$$

$$\Rightarrow q = \frac{1}{2}\{f_2(x+\tan y) - f_1(x-\tan y)\}\sec^2 y \quad \text{..... (14)}$$

Now we have

$dz = pdx + qdy$

$$dz = \frac{1}{2}\{f_1(x-\tan y) + f_2(x+\tan y)\}dx + \frac{1}{2}\{f_2(x+\tan y) - f_1(x-\tan y)\}\sec^2 y\,dy$$

$$\Rightarrow dz = \frac{1}{2}f_1(x-\tan y)\,(dx - \sec^2 y\,dy) + \frac{1}{2}f_2(x+\tan y)\,(dx + \sec^2 y\,dy)$$

$$\Rightarrow dz = \frac{1}{2}f_1(x-\tan y)\,d(x-\tan y) + \frac{1}{2}f_2(x+\tan y)\,d(x+\tan y)$$

Integrating, we get

$Z = g_1(x-\tan y) + g_2(x+\tan y)$

Where g_1 and g_2 are arbitrary function.

Example 10

Solve $(1+q)^2 r - 2(1+p+q+pq)s + (1+p)^2 t = 0$

Solution : Given equation is

$(1+q)^2 r - 2(1+p+q+pq)s + (1+p)^2 t = 0$ ..... (1)

companing (1) with $Rr+Ss+Tt = V$, we get

$R = (1+q)^2$, $S = -2(1+p+q+pq)$, $T = (1+p)^2$, $V = 0$ ..... (2)

Now, Mong's subsidiary equation are

$Rdpdy + Tdqdx - Vdxdy = 0$ ..... (3)

and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$ ..... (4)

Using (2), (3) and (4) becomes

$(1+q)^2 dp\, dy + (1+p)^2 dq\, dx = 0$ ..... (5)

$(1+q)^2(dy)^2 + 2(1+p+q+pq)dx\, dy+(1+p)^2(dx)^2 = 0$ ..... (6)

Now $1+p+q+pq = (1+p)(1+q)$

(6) becomes

$(1+q)^2 (dy)^2 + 2(1+p)(1+q)dx\, dy+(1+p)^2 (dx)^2 = 0$

$\Rightarrow [(1+q)dy +(1+p)dx]^2 = 0$

$\Rightarrow (1+q)dy + (1+p)dx = 0$

$\Rightarrow (1+q)dy = -(1+p)dx$ ..... (7)

Now, (5) can be re-written as

$(1+q)dp\,\{(1+q)dy\} - (1+p)dq\,\{-(1+p)dx\} = 0$ ..... (8)

dividing both sides of (8) by (1+q)dy, or its equivalent

$-(1+p)dx$ [from (7)], we get

$(1+q)dp - (1+p)dq = 0$

$\Rightarrow (1+q)dp = (1+p)dq$

$$\Rightarrow \frac{dp}{1+p} - \frac{dq}{1+q} = 0$$

Integrating

$\log(1+p) - \log(1+q) = \log c_1$

$$\Rightarrow \log - \left(\frac{1+p}{1+q}\right) = \log c_1$$

$$\Rightarrow \frac{1+p}{1+q} = c_1$$ ..... (9), where c_1 is an arbitrary constant

Now from (7), we get

$dx + dy + pdx + qdy = 0$

$\Rightarrow dx + dy + dz = 0$ ($\because dz = pdx + qdy$)

Integrating we get

$x+y+z = c_2$ ..... (10), where c_2 is another arbitrary constant

$\therefore$ From (9) and (10), the intermediate integral of (1) is

$$\frac{1+p}{1+q} = F(x+y+z)$$

$\Rightarrow 1+p = (1+q)\, f(x+y+z)$

$\Rightarrow p - q\,F(x+y+z) = F(x+y+z) - 1$ ..... (11)

Which is of the form $Pp + Qq = R$

So, Lagrange's auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-F(x+y+z)} = \frac{dz}{F(x+y+z)-1} \quad \text{..... (12)}$$

Chooses 1, 1, 1 as multipliers, each fraction of (12), gives

$$\frac{dx+dy+dz}{0}$$

$\Rightarrow dx+dy+dz = 0$

Integrating

$x+y+z = c_2$ ..... (13)

Using (13) and taking the first two fraction of (12), we get

$$dx = -\frac{dy}{F(c_2)}$$

$\Rightarrow dy+F(c_2)dx = 0$

Integrating

$y+x\,F(c_2) = c_3$

$\Rightarrow y+x\,F(x+y+z) = c_3$ ..... (14)

Where c_3 is another arbitrary constant

$\therefore$ From (13) and (14), the required general solution is

$y+x\,F(x+y+z) = G(x+y+z)$

Where F and G and arbitrary functions

Example 11

Solve $y^2r - 2ys + t = p+6y$

Solution : Given equation is

$y^2r - 2ys + t = p+6y$ ..... (1)

Company (1) with $Rr + Ss + Tt = V$, we get

$R = y^2$ $s = -2y$ $T=1$, $V = p+6y$ ..... (2)

Now, Monge's subsidiary equations are

$Rdpdy + Tdqdx - Vdxdy = 0$ ..... (3)

and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$ ..... (4)

using (2),(3) and (4) becomes

$y^2dpdy + dqdx - (p+6xy)dxdy = 0$ ..... (5)

and $y^2(dy)^2+2ydxdy + (dx)^2 = 0$ ..... (6)

Now (6) $\Rightarrow y^2(dy)^2+2ydydx+(dx)^2 =0$

$\Rightarrow (ydy + dx)^2 = 0$

$\Rightarrow ydy + dx = 0$

$\Rightarrow dx = – ydy$ ..... (7)

Putting the value of dx from (7) in (5), we get

$y^2dpdy + dq(–y\,dy) –(p+6q)dy\,(–y\,dy) = 0$

$\Rightarrow ydp – dq + (p+6y)dy = 0$

$\Rightarrow (ydp + pdy) – dq + 6ydy = 0$

Integrating, we get

$yp – q + 3y^2 = c_1$ ..... (8), when c_1 is an arbitrary constant

Integrating, we get

$$\frac{y^2}{2} + x = \frac{c_2}{2}$$

$\Rightarrow y^2 + 2x = c_2$ ..... (9) , when c_2 is an arbitrary constant

$\therefore$ From (8) and (9), (5) becomes

$yp – q + 3y^2 = F(y^2+2x)$

$\Rightarrow yp–q = F(y^2+2x) – 3y^2$ ..... (10), where F is an arbitrary function

Which is of the four Pp + Qq = R

$\therefore$ Lagrage's auxiliary equation are

$$\frac{dx}{y} = \frac{dy}{-1} = \frac{dz}{F(y^2+2x)-3y^2}$$ ..... (11)

$\therefore$ From the first two function of (11),

$$\frac{dx}{y} = \frac{dy}{-1}$$

$\Rightarrow dx + ydy = 0$

$\Rightarrow 2ydy + 2dx = 0$

Integrating $y^2+2x = c_2$ ..... (12)

From the last two function of (11)

$$\frac{dy}{-1} = \frac{dz}{F(y^2+2x)-3y^2}$$

$$\Rightarrow \frac{dy}{-1} = \frac{dz}{F(c_2)-3y^2}$$

$\Rightarrow dz + \{F(c_2) - 3y^2\}dy = 0$

Integrating,

$Z + YF(c_2) - y^3 = c_3$

$\Rightarrow z+y\ F(xy^2+2x) - y^3 = c_3$ ..... (13)

$\therefore$ From (12) and (13), the required general solution is

$Z+ F\ (y^2+2x)-y^3 = G(y^2+2x)$

where F and G are arbitrary functions.



CHECK YOUR PROGRESS - 3

Solve the following P.D.E.'s by Monge's method

Q 1: $q(1+q)r - (p+q+2pq)s + p(1+p)t = 0$

Q 2: $x^2r - 2xs + t + q = 0$

Q 3: $r - t\cos^2x + p\tan x = 0$

8.9 METHOD OF SEPARATION OF VARIABLES

While applying the method of separation of variables we make a certain basic assumption at the outset. Suppose the given partial differential equation involves n independent variables $x_1, x_2, \ldots, x_n$ and one dependent variable u. Then, we first assume that the equation product solution of the from

$u(x_1, x_2, \ldots x_n = x_1(x_1)\ x_2(x_2) \ldots x_n(x_n)$

where x_i is a function of x_i only (i = 1, 2, . . . n). On substitution of (1) into the given equation, we shall obtain n ordinary differential equations one in each of the unknown functions x_i(1, 2, . . . n).

8.10 THE PRINCIPLE OF SUPERPOSITION

The general homogeneous partial differential equation of second order is

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x^2 \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G \quad \rightarrow (1)$$

Suppose that

(i) $u_1, u_2, \ldots u_n, \ldots$ is an infinite set of solutions of (1) in the region

R in xy plane

(ii) the infinite series $u_1+u_2+ \ldots +u_n+ \ldots$ converges and is differentiable term by term in R.

Then, by the principle of superposition, the function u, defined by

$u = \sum_{n=1}^{\alpha} u_n$ is a solution of (1) in IR.

(Here IR denotes the set of real numbers)

Example 12

Solve the boundary value problem

$\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}$, if $u(0, y) = 8e^{-3y}$

Solution : Given PDE is

$$\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y} \qquad \ldots\ldots\ldots\ldots\ldots (1)$$

and given boundary condition is

$$u(0, y) = 8e^{-3y} \qquad \ldots\ldots\ldots\ldots\ldots (2)$$

Suppose that (1) has solution of the form

$$U(x, y) = X(x)\, Y(y) \qquad \ldots\ldots\ldots\ldots\ldots (3)$$

Where x is a function of x alone and y is a function of y alone.

Substituting this value of y in (1),

$X'y = 4\, x\, y'$

$$\Rightarrow \frac{x'}{4x} = \frac{y'}{y} \qquad \ldots\ldots\ldots\ldots\ldots (4)$$

When the dashes denote derivatives w.r.t. the relevant variable.

Since x and y are independent variables, therefore (4) can only be true if each side is equal to the same constant, say k.

Then (4) gives

$$\frac{x'}{4x} = \frac{y'}{y} = k$$

$$\frac{x'}{4x} = k \qquad \frac{y'}{y} = k$$

$$\left.\begin{array}{l} \Rightarrow x' - 4kx = 0 \\ \text{and } y' - ky = 0 \end{array}\right\} \ldots\ldots\ldots\ldots\ldots (5)$$

These are ordinary differential equation of first order and first degree

Now $X' - 4kx = 0$

$$\Rightarrow \frac{x'}{x} = 4k$$

Integrating

$\log x = 4kx + \log A$

$\Rightarrow \log x = \log A + \log e^{4kx}$

$\Rightarrow \log x = \log(Ae^{4kx})$

$\Rightarrow x = Ae^{4kx}$ ..... (6)

Where A is are arbitrary constant

and $y' - ky = 0$

$$\Rightarrow \frac{y'}{y} = k$$

Integrating

$\log y = kxy + \log B$

$\Rightarrow \log y = \log(Be^{ky})$

$\Rightarrow y = Be^{ky}$ ..... (7)

Where B is another arbitrary constant

Substituting these values of X and Y in (3), a solution of (1) is

$u(x, y) = A.e^{4kx}.Be^{ky}$

$\Rightarrow u(x, y) = ABe^{4kx+ky}$

$\Rightarrow u(x, y) = Ce^{k(4x+y)}$ ..... (8)

where C = AB is another arbitrary constant.

Putting x = 0 in (8) and using the bundary condition (2)

$u(0, y) = Ce^{k(4x+y)}$

$\Rightarrow 8e^{-3y} = Ce^{ky}$

$\Rightarrow e = 8$ and $k = -3$

Putting the value of e and k in (8), we get

$u(x, y) = 8e^{-3(4x+y)}$

This is the required solution

Example 13

Solve the boundary value problem

$\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}$ if $u(0, y) = 8e^{-3y}+4e^{-5y}$

Solution : Given equation is

$$\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y} \qquad \text{..... (1)}$$

and given boundary condition is

$$u(0, y) = 8e^{-3y}+4e^{-5y} \qquad \text{..... (2)}$$

Now proceed as Example 12 and show that (8) is a solution of (1)

Since c and k are arbitrary constants, it follows that

$c_1e^{k_1(4x+y)}$ and $c_2e^{k_2(4x+y)}$ are solution of (1)

Hence by the principle of superposition, their sum is also a solution of (1). Thus a solution of (1) may be taken in the form,

$$u(x, y) = c_1e^{k_1(4x+y)}+c_2 2^{k_2(4x+y)}$$

Now putting x = 0 in (3) and using the boundary condition

$$u(0, y) = c_1e^{k_1(4x0+y)} + c_2e^{k_2y}$$

$$\Rightarrow 8e^{-3y}+4e^{-5y} = c_1^{k_1y} + c_2e^{k_2y}$$

$$\Rightarrow c_1 = 8, \quad k_1 = -3, \quad c_2 = 4, \quad k_2 = -5$$

Substituting these value in (3), we get

$$u(x, y) = 8e^{-3(4x+y)} + 4e^{-5(4x+y)}$$

This is the required solution



CHECK YOUR PROGRESS - 4

Using the method of separation of variables, solve the following P.D.E.

Q 1: $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$, where $u(x, 0) = 6\ e^{-3x}$

Q 2: $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$



8.11 LET US SUM UP

- A partial differential equation of the form

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=G$$

is elliptic if $B^2 - 4AC < 0$

parabolic if $B^2 - 4AC = 0$

hyperbolic if $B^2 - 4AC > 0$

- The Monge's subsidiary equations of the PDE of the form $Rr + Ss + Tt = V$ are

  $Rdpdy + Tdqdx - Vdxdy = 0$

  and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$

- Method of separation of variables is an useful technique to solve boundary value problems



8.12 FURTHER READINGS

1) Ross, Sheply L. (2004) : Differential Equations (3rd Ed), John Wiley & Sons, India.

2) Evans, L. C. (2014) : Partial Differential Equations (2nd Ed), The Orient Blackswan.

3) Soneddon, I. N. (2006) : Elements of Partial Differential Equations, Dover Publications.

4) Raisinghania, M. D. (2001) : Advanced Differential Equations S. Chand & Company Ltd. New Delhi.



8.13 ANSWERS TO CHECK YOUR PROGRESS

Check Your Progress-1

Ans to Q No 1: Left as exercise

Ans to Q No 2: Given PDE is

$x^2(y-1)r - x(y^2-1)s + y(y-1)t + xy\,p - q = 0$

$$\Rightarrow x^2(y-1)\frac{\partial^2 u}{\partial x^2}-x(y^2-1)\frac{\partial^2 u}{\partial x\,\partial y}+y(y-1)\frac{\partial^2 u}{\partial y^2}+xy\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y}=0$$

$\Rightarrow x^2(y-1)u_{xx}-x(y^2-1)u_{xy}+y(y-1)u_{yy} + xyu_x - u_y = 0$ ..... (1)

Company (1) with

$AU_{xx}+BU_{xy}+CU_{yy}+DU_x+EU_y+FU = G$, we get

$A = x^2(y-1)$, $B = -x(y^2-1)$, $C = y(y-1)$, $D = xy$, $E = -1$, $F = 0$, $G = 0$

Now $B^2 - 4AC = \{-x(y^2-1)\}^2 - 4.x^2(y-1).y\ (y-1)$

$= x^2(y^2-1)^2 - 4x^2(y-1)^2y$

$= x^2\{(y^2-1)^2 - 4.y\ (y-1)^2\}$

$= x^2\{y^4+1-2y^2- 4y(y^2-2y+1)\}$

$= x^2(y^4+1-2y^2-4y^3 + 8y^2- 4y\}$

$= x^2(y^4 - 4y^3 + 6y^2 - 4y +1)$

$= x^2(y^4-y^3-3y^3-3y^2+3y^2-3y-y+1)$

$= x^2\{y^3(y-1) - 3y^2(y-1) + 3y(y-1) -(y-1)\}$

$= x^2(y-1)\ (y^3-3y^2+3y-1)$

$= x^2(y-1)\ (y^3-y^2-2y^2+2y+y-1)$

$= x^2(y-1)\ \{y^2(y-1)-2y(y-1) + y(y-1)\}$

$= x^2(y-1)\ (y-1)\ (y^2-2y+1)$

$= x^2(y-1)^2\ (y-1)^2 > 0$

$\therefore$ The given equation is hyperbolic

Check Your Progress-2

Ans to Q No 1: (i) Left as exercise

(ii) Left as exercise

Ans to Q No 2: Left as exercise

Check Your Progress-3

Ans to Q No 1: Left as exercise

Ans to Q No 2: Left as exercise

Ans to Q No 3: Left as exercise

Check Your Progress-4

Ans to Q No 1: Left as exercise

Ans to Q No 2: Left as exercise



8.14 MODEL QUESTIONS

Q 1: Classify the following partial differential equations as parabolic, elliptic or hyperbolic

(i) $2y^2r - 2xys + x^2t = \frac{y^2p}{x} + \frac{x^2q}{y}$

(ii) $(x-y)\left(x\frac{\partial^2u}{\partial x^2} - (x+y)\frac{\partial^2u}{\partial x\partial y} + y\frac{\partial^2u}{\partial y^2}\right) = (x+y)\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)$

(iii) $x(xy-1)r - (x^2y^2-1)s + y(xy-1)t + xp+yq = 0$

Q 2: Reduce the following partial differential equations into canonical form–

(i) $r - 6s + 9t + 2p + 3q - z = 0$

(ii) $\frac{\partial^2z}{\partial x^2} + y^2\frac{\partial^2z}{\partial y^2} = y$

(iii) $r - (2\sin x)s - (\cos^2x)t - (\cos x)q = 0$

Q 3: Reduce the following partial differential equations into canonical form and solve

(i) $yr + (x+y)s + xt = 0$

(ii) $2r - 4s + 2t + 3z = 0$

(iii) $x\frac{\partial^2u}{\partial x^2} + \frac{\partial^2u}{\partial y^2} = x^2 (x > 0)$

Q 4: Solve the following partial differential equations using Monge's method.

(i) $r+(a+b)s + abt = xy$

(ii) $(r-t)xy - s(x^2-y^2) = qx - py$

(iii) $2yq + y^2t = 1$

Q 5: Using the method of separation of variables, solve the following PDEs

(i) $3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0$, $u(x, 0) = 4e^{-x}$

(ii) $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial y} + u$, $u(x, 0) = 3e^{-5x} + 2e^{-3x}$

(iii) $4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u$, $u(0, y) = 3e^{-y} - e^{-5y}$

*** ***** ***