UNIT 9 : SECOND AND HIGHER ORDER P.D.E. WITH CONSTANT COEFFICIENT

UNIT STRUCTURE

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9.1 LEARNING OBJECTIVES

After going through this unit, you will able to-

- homogeneous and Non Homegeneous PDE's of Higher order
- reducible and Irreducible Homogeneous Equations
- method of solution of higher order P.D.E.' with example
- auxiliary Equation, complimentary function and particular integrals
- general method to obtain particular integral
- characteristic surfaces of second order PDE with examples
- euler cauchy equation and method to solve them with examples.

9.2 INTRODUCTION

In the previous two units, we have learned about linear and non-line or partial differential equation of first order. In this unit, we will discuss about the second and higher order lines partial differential equations. Higher order lineas P.D.E.'s can be homogeneous and non-homogeneous. In this unit, we will elaborate the methods solving them. Also, we will discuss about Partial Differential Equation with constant coefficient. This problem has great importance in many physical and technological phenomenons.

9.3 PARTIAL DIFFERENTIAL EQUATION OF SECOND ORDER

Any differential equation containing partial derivatives is called a Partial Differential Equation (P.D.E.). The independent variable (the unknown function) in any P.D.E. must be a function of at least two independent variables, otherwise partial derivatives would not arise.

In general, the solution of P.D.E. presents much more difficult problem then the solution of ordinary Differential Equatise (O.D.E.), and except for certain special types of linear P.D.E., no general method of solution is applicable.

We shall concentrate therefore on the solution of particular types of linear equation. It is both remarkable and fortunate that a large number of important equations in physies are not only linear but of second order.

We now come to one of the most important differences between the solution of P.D.E. and those of O.D.E. The general solution of a linear O.D.E. contains orbitrary constants of integration, whereas the general solution of a linear P.D.E. contains only orbitrary functions.

For example, solution of

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2} \quad \text{..... (1)}$$

is $u = f(x+y) + g(x-y)$ ..... (2)

when f and g are orbitrary functions

Which means that (2) satisfy (1) irrespective of the functional forms of f(x+y)

and g(x–y)

For example,

$u = \sin(x+y) + e^{x-y}$

and $u = (x+y)^3 + \tan(x-y)$

are both solutions of (1)

In most cases, the general solution of a P.D.E. is of little use, since it has to be made to satisfy other conditions called boundary conditions. Which arise from the physics of the problem. This is much more difficult to accomplish for P.D.E. than O.D.E. owing to great variety of choice available for the arbitray function.

For this reason, methods of solutions such as the method of separation of variables and integral transforms, which build the solution of an equation arround the boundary conditions are of great importance.

9.4 CLASSIFICATION OF EQUATIONS AND BOUNDARY CONDITIONS

Partial Differential Equation with only two independent variables are special cases of general linear homogeneous equation of second order, namely

$$a\frac{\partial^2 u}{\partial x^2} + 2h\frac{\partial^2 u}{\partial x \partial y} + b\frac{\partial^2 u}{\partial y^2} + 2f\frac{\partial u}{\partial x} + 2g\frac{\partial u}{\partial y} + eu = 0 \quad \text{..... (1)}$$

When a, h, b, f, g and e may be constants or functions of x and y.

For the moment, we note that the form of (i) resembles that of a general conic section.

$ax^2 + 2hxy + by^2 + 2yx + 2gy + e = 0$

This equation represents and ellipse, a parabola or a hyperbola when $ab–h^2 > 0$, $ab – h^2 = 0$ and $ab – h^2 < 0$ respectively.

We use a similar classification for the equation (1) and say that it is of

elliptic type when $ab–h^2 > 0$

parabolic type when $ab–h^2 = 0$

hyperbolic type when $ab–h^2 < 0$

9.5 GENERAL SOLUTIONS

When only two independent variables x and y are involved, the linear partial differential equation may be written in the symbolic form.

$F(D, D')z = f(x, y) \quad \rightarrow (1)$

When $D \equiv \frac{\partial}{\partial x}$ and $D' \equiv \frac{\partial}{\partial y}$

The operator F (D, D′) denotes a polynomial in D and D′ with coefficients, functions of x and y only when the equations (1) of the second order, we shall use the following notations –

$$p = \frac{\partial z}{\partial x}$$

$$q = \frac{\partial z}{\partial y}$$

$$r = \frac{\partial^2 z}{\partial x^2}$$

$$s = \frac{\partial^2 z}{\partial x \partial y}$$

$$t = \frac{\partial^2 z}{\partial y^2}$$

If the right hand side of (1), is zero, then

$F(D, D')z = 0$ ..... (2)

Then, we shall say that (2) is homogeneous, otherwise non-homogeneous if $f(x, y) \neq 0$.

Any function of x and y which substituted for z in a non-homogeneous equation, is called a Particular Integral (P. I.) of (1). A function satisfying the corresponding homogeneous equation (2) are called Complimentary Function. (C.F.)

9.6 REDUCIBLE AND IRREDUCIBLE HOMOGENEOUS EQUATION

The operatior F(D, D′) in a linear P.D.E. is a polynomial of D and D′ , we have what is known as a P.D.E.

Thus, we have

$F(D, D')z = 0 \quad \rightarrow (1)$

For convenience, we separate equation of class (1) into two classes of equation, which are reducible and irreducible equation. By a reducible equation, we shall mean an equation in which the operator $F(D, D')$ can be resolved into fuctors, each of which of first degree in D and D'.

For example,

$(D^2 - DD' - D + D')z = 0$

$\Rightarrow (D-D')(D-1)z = 0$

is reducible equation

An equation whose operator can't be so factored is called irreducible equation.

For example,

$(D^3-DD')z = 0$

$\Rightarrow D(D^2-D')z = 0$

is irreducible as D^2-D' can't be factorized further consider now a reducible homogeneous equation of the form

$$F(D,D')z = (\alpha_1 D - \beta_1 D' - \delta_1)(\alpha_2 D - \beta_2 D' - \delta_2)$$

$$\ldots\ldots (\alpha_m D - \beta_m D' - \delta_m)z = 0 \quad \ldots\ldots\ldots\ldots\ldots (2)$$

Where α_1's and β_1's are assumed constants, some of which may be zero.

Evidently any solution of the Lagrange equation

$$(\alpha_m D - \beta_m D' - \delta_m)z = 0$$

$$\Rightarrow p\alpha_m - \beta_m D - \delta_m z = 0$$

Will be a solution of (2)

Now the lagrange equation has the subsidiary equations

$$\frac{dx}{\alpha_m} = \frac{dy}{\beta_m} = \frac{dz}{\delta_m z}$$

From the first two ration,

$$\frac{dx}{\alpha_m} = \frac{dy}{-\beta_m}$$

$$\Rightarrow \alpha_m dy + \beta_m dx = 0$$

Integrating,

$\alpha_m y + \beta_m x$ = constant = a (say)

If $\alpha_m \neq 0$, the first and third ratio gives us

$$\frac{dx}{\alpha_m} = \frac{dz}{\delta_m z}$$

$$\Rightarrow \frac{dz}{z} = \delta_m \frac{dx}{\alpha_m}$$

Integrating,

$$\log z = \frac{\delta_m x}{\alpha_m} + \log b$$

$$\Rightarrow z = \frac{\delta_m x}{e^{\alpha_m} b}$$

$$\Rightarrow z = \frac{\delta_m x}{e^{\alpha_m}} f_m(\alpha_m y + \beta_m x) \quad \text{..... (3)}$$

Where f_m is arbitrary function of constant argument and if $\beta_m \neq 0$, we get alternatively as

$$z = \frac{-\delta_m y}{e^{\beta_m}} f_m(\alpha_m y + \beta_m x)$$

α_m and β_m both can't be zero. At least one of the above two forms (3) and (4) will serve in all cases. If $\alpha_m \neq 0$ and $\beta_m \neq 0$, either of above may be used. For definiteness, we suppose the first one to be permissible.

Now, in general

$$z = \frac{\delta_m x}{e^{\alpha_1}} f_1(\alpha_1 y + \beta_1 x) + \frac{\delta_2 x}{e^{\alpha_2}} f_2(\alpha_2 y + \beta_2 x) + \ldots + \frac{\delta_m x}{e^{\alpha_m}} f_m(\alpha_m y + \beta_m x)$$

Where $f_1, f_2, \ldots f_m$ are all arbitrary functions.

If any α_j = 0, the corresponding term will of course be replaced by a term derived above.

When the operators has repeated factors, this solution will be replaced by as follows -

Let

$$(\alpha D - \beta D' - \delta)^2 z = 0$$

$$\Rightarrow (\alpha D - \beta D' - \delta)(\alpha D - \beta D' - \delta) z = 0 \quad \text{..... (i)}$$

Let $(\alpha D - \beta D' - \delta) z = u$ ..... (ii)

Then we have from (i)

$$\left(\alpha D-\beta D'-\delta\right)u=0$$

$$\Rightarrow u=\frac{\delta x}{e^{\alpha}}f(xy+\beta x)$$

Then from (ii)

$$(\alpha D-\beta D'-\delta)z=e^{\frac{\delta_x}{\alpha}}f(\alpha\alpha+\beta x)$$

Then, the Lagrange's subsidiary equations are

$$\frac{dx}{\alpha}=\frac{dy}{-\beta}=\frac{dz}{\delta z+e^{\frac{\delta_x}{\alpha}}f(\alpha\alpha+\beta x)}$$

$\therefore$ From first and second ratios,

$$\frac{dx}{\alpha}=\frac{dy}{-\beta}$$

$$\Rightarrow \alpha dy+\beta dx=0$$

Integrating,

$\alpha y+\beta x$ = constant = b (say)

From first and third ratios,

$$\frac{dx}{\alpha}=\frac{dz}{\delta z+e^{\frac{\delta_x}{\alpha}}f(b)}$$

$$\Rightarrow\frac{dz}{dx}=\frac{\delta z+e^{\frac{\delta_x}{\alpha}}f(b)}{\alpha}$$

$$\Rightarrow\frac{dz}{dx}-\frac{\delta z}{\alpha}=\frac{e^{\frac{\delta x}{\alpha}}}{\alpha}f(b)$$

Which is a linear equation

Here IF = $e^{\int\frac{-\delta}{\alpha}dx}$

$\therefore$ its solution is

$$ze^{\frac{-\delta}{\alpha}x}=\int\frac{e^{\frac{\delta_x}{\alpha}}}{\alpha}e^{\frac{-\delta x}{\alpha}}f(b)dx+e$$

$$\Rightarrow ze^{\frac{-\delta x}{\alpha}}=\frac{1}{\alpha}f(b)dx+e$$

$$\Rightarrow z e^{\frac{-\delta x}{\alpha}} = \frac{1}{\alpha} f(b)x + e$$

$$\Rightarrow z e^{\frac{-\delta x}{\alpha}} = g(b)x + e, \text{ when } g(b) = \frac{1}{\alpha} f(b)$$

$$\Rightarrow z = e^{\frac{\delta x}{\alpha}} \left[g(b)x + e\right]$$

$$\Rightarrow z = e^{\frac{-\delta y}{\beta}} \left[f(\alpha y + \beta x) + x g(\alpha y + \beta x)\right] \qquad ((i)\, \alpha \neq 0)$$

Again, if $\beta \neq 0$

$$z = e^{\frac{-\delta y}{\beta}} \left[f(\alpha y + \beta x) + y g(\alpha y + \beta x)\right]$$

Similarly, if there are three repeated factors, then

$$z = e^{\frac{-\delta y}{\alpha}} \left[f(\alpha y + \beta x) + x g(\alpha y + \beta x) + x^2 h(\alpha y + \beta x)\right]$$

and if $\beta \neq 0$

$$z = e^{\frac{-\delta y}{3}} \left[f(\alpha y + \beta x) + y g(\alpha y + \beta x) + y^2 h(\alpha y + \beta x)\right]$$

Thus, if there are k repeated factors, then

$$z = e^{\frac{-\delta y}{\alpha}} \left[x^{k-1} f_1(\alpha y + \beta x) + x^{k-2} f_2(\alpha y + \beta x) + \ldots + f_x(\alpha y + \beta x)\right] \text{ (valid for } \alpha \neq 0)$$

and

$$z = e^{\frac{-\delta y}{\beta}} \left[y^{k-1} f_1(\alpha y + \beta x) + y^{k-2} f_2(\alpha y + \beta x) + \ldots + f_x(\alpha y + \beta x)\right] \text{ (valid for } \beta \neq 0)$$

Thus, by suitably choosing one or both of the above forms, are can obtain a solution of a reducible homogereous equation of order m containing m arbitiary functions.

Example 1

Solve $(D^2D' + DD'^2 - 2D^2 - 3DD' + 2D)z = 0$

Given

$$(D^2D' + DD'^2 - 2D^2 - 3DD' + 2D)z = 0$$

$$\Rightarrow D(DD' + D'^2 - 2D - 3D' + 2)z = 0$$

$$\Rightarrow D(DD' - 2D + D'^2 - 2D' - D' + 2)z = 0$$

$$\Rightarrow D(D(D' - 2) + D'(D' - 2) - (D' - 2)z = 0$$

$$\Rightarrow D(D' - 2)(D + D' - 1)z = 0$$

$$\Rightarrow (1.D - 0.D' - 0)(0.D - (-1)D' - 2)(1.D - (-1)D' - 1)z = 0 \quad \ldots\ldots(1)$$

Compaing (1) with

$$(\alpha_1 D - \beta_1 D' - \delta_1)(\alpha_2 D - \beta_2 D' - \delta_2)(\alpha_3 D - \beta_3 D' - \delta_3)3 = 0$$

we get

$$\alpha_1 = 1 \qquad \alpha_2 = 1 \qquad \alpha_3 = 1$$
$$\beta_1 = 1 \qquad \beta_2 = 1 \qquad \beta_3 = 1$$
$$\delta_1 = 1 \qquad \delta_2 = 1 \qquad \delta_3 = 1$$

There fore its solution is

$$z = e^{\frac{\delta_1 x}{\alpha_1}} f_1(\alpha_1 y + \beta_1 x) + e^{\frac{\delta_2 x}{\beta_2}} f_2(\alpha_2 y + \beta_2 x) + e^{\frac{\delta_3 x}{\alpha_3}} f_2(\alpha_2 y + \beta_2 x)$$

$$\Rightarrow z = e^{\frac{\delta . x}{1}} f_1(1.y + 0.x) + e^{\frac{-2y}{-1}} f_2(0.y + (-1)x) + e^{\frac{1.x}{1}} f_3(1.y + (-1)x)$$

$$\Rightarrow z = f_1(y) + e^{2y} f_2(-x) + e^{x} f_3(y - x)$$

where f_1, f_2 and f_3 we arbitrary functions

9.7 COMPLIMENTARY FUNCTION (C.F.) OF HOMOGENEOUS P.D.E

Here, we define a homogeneous equation as an equation in which the dependent variable itself is absent and all the derivatives are of the same order.

Now we are going to find the complimentary function (C.F.) of a homogeneous equation.

Let us suppose that

$z = f(y+mx)$ .....(1)

is a solution of the equation $F(D, D')z = 0$ ..... (2)

Then, from (1) $D^n z = m^n f^r (y+mx)$

and $D'^n z = f^n(y+mx)$

Then $F(D, D')z = P(m) f^n(y+mx)$

Where P(m) is a polynomial of degree n.

Now, the auxiliary equation (A.E.) is given by

$P(m) = 0$ .....(3)

Suppose, $m_1, m_2, \ldots, m_n$ are the roots of (3)

Then the solution of (2) is

$z = f(y+m_1x) + f(y+m_2x) + \ldots + f(y+m_nx)$

It is observed that the auxiliary equation (A.E.) is obtained by putting D = m and $D^{/}$= 1 in (2).

If n repeated factors occurs, then the solution is of the form.

$z = f(y+m_1x) + xf_2(y+m_2x) + x^2f_3(y+m_3x) + \ldots + x^{n-1} f_n(y+m_nx)$

Here $f_1, f_2, \ldots f_n$ are all arbitrary functions.

9.8 WORKING RULE TO FIND C.F OF LINEAR HOMOGENEOUS PDE WITH CONSTANT COEFFICIENTS

Step 1 : Put the given equation in standard form

$(A_oD^n+A_1D^{n-1}D^1+\ldots+A_nD^{/n})z = 0$ ..... (1)

Step 2 : Replacing D by m and $D^{/}$ by 1 in the coefficients of z,we obtain the auxiliay equation (A.E.) of (1) as

$A_0m^n + A_1m^{n-1}+ \ldots + A_n = 0$ ..... (2)

Step 3 : Solve (2) for m. Then two cases arise -

Cose (i) : Let $m = m_1, m_2, \ldots m_n$ (different roots), the C.Fi is

$C.F. = f_1(y+m_1x) + f_2(y+m_2x) + \ldots + f_n(y+m_nx)$

Where $f_1, f_2, \ldots, f_n$ are arbitrary functions

Cose (ii) : Let $m = m^{/}$ (repeated n times), then C.F. is

$C.F. = f_1(y+m^{/}x) + xf_2(y+m^{/}x) +x^2f_3(y+m^{/}x)+ \ldots +x^{n-1}f_n(y+m^{/}x)$

Where $f_1, f_2, \ldots, f_n$ are arbitrary functions

Example 2

Solve $r = a^2 t$

Solution : Given $r = a^2 t$ $\quad \because r = \dfrac{\partial^2 z}{\partial x^2}$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2} = a^2 \frac{\partial^2 z}{\partial y^2} \qquad t = \frac{\partial^2 z}{\partial y^2}$$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2} - a^2 \frac{\partial^2 z}{\partial y^2} = 0$$

$$\Rightarrow (D^2 - a^2 D^{/2})z = 0 \qquad \rightarrow (i)$$

Now Auxiliary equation of (1) is

$m^2 - a^2 . 1 = 0$ (Putting $D = m\ D^{/} = 1$

$\Rightarrow$ (m–a) (m+a) = 0

$\Rightarrow$ m = a, –a

$\therefore$ General solution of (1) is

$z(C.F.) = f_1(y+ax) + f_2(y–ax)$

Where f_1 and f_2 are arbitrary functions

Example 3

Solve $(D^3 – 3D^2D' + 2DD'^2)z = 0$

Given $(D^3 – 3D^2D' + 2DD'^2)z = 0$ $\rightarrow$(1)

Now, the auxiliary equation of (i) is

$m^3–3m^2+2m = 0$

$\Rightarrow m(m^2–3m+2) = 0$

$\Rightarrow m(m^2–2m–m+2) = 0$

$\Rightarrow m(m(m–2)–1(m–2) = 0$

$\Rightarrow m(m–1) (m–2) = 0$

$\Rightarrow m = 0, 1, 2$

$\therefore$ The general solution of (i) is

$z = f_1(y+0x)+f_2(y+1.x)+f_3(y+2.x)$

$\Rightarrow z = f_1(y)+f_2(y+x)+f_3(y+2x)$

Where f_1, f_2 and f_3 are arbitrary functions

Example 4

Solve $2\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial x \partial y}-2\frac{\partial^2 z}{\partial y^2}=0$

Solution : Given P.D.E. is

$$2\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial x \partial y}-2\frac{\partial^2 z}{\partial y^2}=0$$

$\Rightarrow (2D^2 – 3DD' – 2D'^2)z = 0$ ..... (1)

The A. E. of (1) is

$2m^2 – 3m – 2 = 0$

$\Rightarrow 2m^2 – 4m+m–2 = 0$

$\Rightarrow 2m(m–2)+1(m–2) = 0$

$\Rightarrow (2m+1) (m–2) = 0$

$\Rightarrow m = -\frac{1}{2}, 2$

$\therefore$ The general solution of (i) is

$z = f_1(y+(-½)x) + f_2(y+2x)$

$\Rightarrow z = f_3(2y–x)+f_2(y+2x)$

Where f_1, f_2 and f_3 are arbitrary functions.

Example 5

Solve $(D^4 – 2D^3D^{/}+2DD^{/3} – D^{1^4})3 = 0$

Solution :

Given P.D.E. is

$(D^4 – 2D^3D^{/}+2DD^{/3} – D^{1^4})3 = 0 \quad \rightarrow (1)$

The A.E. of (1) is

$m^4–2m^3+2m–1 = 0$

$\Rightarrow m^4–m^3–m^3+m^2–m^2+m+m–1 = 0$

$\Rightarrow m^3(m–1)–m^2(m–1)–m(m–1)+1(m–1) = 0$

$\Rightarrow (m–1)(m^3–m^2–m+1) = 0$

$\Rightarrow (m–1)(m^2(m–1)–1(m–1) = 0$

$\Rightarrow (m–1)(m^2–1)(m–1) = 0$

$\Rightarrow (m–1)(m+1)(m–1)(m–1) = 0$

$\Rightarrow (m+1)(m–1)^3 = 0$

$\Rightarrow m = –1, 1, 1, 1$

The general solution of (1) is

$3 = f_1(y–x) +f_2(y+x) +xf_3(y+x) +x^2f_4(y+x)$

Where f_1, f_2, f_3 and f_4 are arbitrary function.



CHECK YOUR PROGRESS-1

Q 1: Solve the following homogeneous P.D.E.'s–

(i) $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0$

(ii) $2r + 5s + 2t = 0$

(iii) $(4D^2+12DD^{/}+9D^{1^2})\, z = 0$

(iv) $r + t + 2s = 0$

(v) $(D^3–4D^2D^{/}+4DD^{1^2})\, z = 0$

(vi) $(D^3D^{1^2}+D^2D^{/3})\, z = 0$

9.9 PARTICULAR INTEGRAL (P.I.)

Suppose the given P.D.E. is

$F(D, D') z = f(x, y)$ ..... (1)

The inverse operator $\frac{1}{F(D, D')}$ of the operator $F(D, D')$ is defined by the following identity.

$$F(D, D')\left(\frac{1}{F(D, D')} f(x, y)\right) = f(x, y)$$

$\therefore$ Particular integral (P.I.) of (1) is $\frac{1}{F(D, D')} f(x, y)$

$\therefore$ General solutions of (1) is made up of the sum of C.F. (discussed in article 9.7 and 9.8) and P.I.

$\therefore$ General solution of (1) is

$z = C. F. + P. I.$

Note :

(i) D, D^2, stands for partial differentration w.r.t. x once, twice and so on

For example $Dx^2y^4 = \frac{\partial}{\partial x}(x^2y^4) = 2xy^4$

$$D^2x^2y^4 = \frac{\partial^2}{\partial x^2}(x^2y^4) = 2y^4$$

(ii) D', D'^2, stands for partial differentiation w.r.t. y once, twice and so on

For example $Dx^2y^4 = \frac{\partial}{\partial y}(x^2y^4) = 4x^2y^3$

$$D^2x^2y^4 = \frac{\partial^2}{\partial y^2}(x^2y^4) = 12x^2y^4$$

(iii) $\frac{1}{D}, \frac{1}{D^2}, \ldots$, stands for integrating partially w.r.t x once, twice and so on

For example $\frac{1}{D}x^2y^4 = \int x^2y^4 dx = \frac{x^2y^4}{3}$

$$\frac{1}{D^2}x^2y^4 = \iint x^2y^4 dx\, dx = \frac{x^2y^4}{12}$$

(iv) $\frac{1}{D}, \frac{1}{D^2}, \ldots$, stands for integrating partially w.r.t x once, twice and so on

For example $\frac{1}{D}x^2y^4 = \int x^2y^4 dy = \frac{x^2y^5}{5}$

$$\frac{1}{D'^2}x^2y^4 = \iint x^2y^4 dy\, dy = \frac{x^2y6}{30}$$

9.10 SHORT METHOD OF FINDING PARTICULAR INTEGRAL

Consider the P.D.F.

F (D, D′)z = f(x, y)

(i) If f(x, y) is of the form e^{ax+by}, then P.I. is

$$\frac{1}{F(D, D')}e^{ax+by} = \frac{e^{ax+by}}{F(a,b)}, \text{ if } F(a, b) \neq 0 \qquad \text{if } F(a, b) = 0, \text{ then}$$

$$\frac{1}{F(D, D')}e^{ax+by} = \frac{1}{\left(D - \frac{a}{b}D'\right)G(D, D')}e^{ax+by}$$

$$= \frac{1}{G(a, b)}\frac{x^r e^{ax+by}}{\lfloor r}$$

(ii) If f(x, y) is of the form sin(ax+by) (or cos(ax+by) then, P.I. is

$$\frac{1}{F(D^2, DD', DD'^2)} \sin(ax+by) \text{ (or } \cos(ax+by)),$$

$$= \frac{1}{F(-a^2, -ab, -b^2)} \sin(ax+by) \text{ (or } \cos(ax+by)), \text{ if } F(-a^2, -ab-b^2) \neq 0$$

if $F(-a^2, -ab - b^2) \neq 0$

If $F(-a^2, -ab - b^2) = 0$, then proceed as above

Example 6

Obtain a general solution of the following P.D.E.

$(D^2+2DD'+D'^2)z = e^{2x+3y}$

Given P.D.E. is

$(D^2+2DD'+D'^2)z = e^{2x+3y} \qquad \rightarrow (1)$

Here, A.E. is

$m^2+2m+1 = 0$

$\Rightarrow (m+1)^2 = 0$

$\Rightarrow m = -1, -1$

$\therefore$ C.F. = $f_1(y-x) + xf_2(y-x)$

Now $$PI = \frac{1}{D^2 + 2DD' + D'^2} e^{2x+3y}$$

$$= \frac{1}{(D+D')^2} e^{2x+3y}$$

$$= \frac{e^{2x+3y}}{(2+3)^2}$$

$$= \frac{e^{2x+3y}}{25}$$

$\therefore$ General solution of (1) is

3 = e.F. + PI

$$\therefore 3 = f_1(y-x) + xf_2(y-x) + \frac{e^{2x+3y}}{25}$$

Where f_1 and f_2 are arbitrary functions

Example 7

Solve $$\frac{\partial^2 z}{\partial x^2} - 3\frac{\partial^2 z}{\partial x \partial x} + 2\frac{\partial^2 z}{\partial y^2} = e^{2x+3y} + e^{x+y} + \sin(x-2y)$$

Solution : Given equation is

$$\frac{\partial^2 z}{\partial x^2} - 3\frac{\partial^2 z}{\partial x \partial x} + 2\frac{\partial^2 z}{\partial y^2} = e^{2x+3y} + e^{x+y} + \sin(x-2y)$$

$$\Rightarrow \left(D^2 - 3DD' + 2D'^2\right) z = e^{2x+3y} + e^{x+y} + \sin(x-2y) \quad \text{..... (1)}$$

Now, A.E. of (I) is

$m^2 - 3m + 2 = 0$

$\Rightarrow m^2 - 2m - m + 2 = 0$

$\Rightarrow m(m-2) - 1(m-2) = 0$

$\Rightarrow (m-1)(m-2) = 0$

$\Rightarrow m = 1, 2$

C.F. of (1) is $f_1(y+x) + f_2(y+2x)$, where f_1 and f_2 are arbitrary functions

Now P I for e^{2x+3y} is $$\frac{1}{D^2 - 3DD' + 2D'^2} e^{2x+3y}$$

$$= \frac{e^{2x+3y}}{2^2 - 3.2.3 + 2.3^2}$$

$$= \frac{e^{2x+3y}}{4 - 18 + 18}$$

$$= \frac{e^{2x+3y}}{4}$$

PI for e^{x+y} is $\frac{1}{D^2 - 3DD' + D'^2} e^{x+y}$

$$= \frac{1}{(D - D')(D - 2D')} e^{x+y} \qquad \because F(1, 1) = 0$$

$$= \frac{1}{(1 - 2.1)} \frac{x^2}{\lfloor 1} e^{x+y}$$

$$= -x e^{x+y}$$

P I for sin (x–2y) = $\frac{1}{D^2 - 3DD' + 2D'^2} \sin(x - 2y)$

$$= \frac{\sin(x - 2y)}{-1^2 - 3\{-1.(-2)\} + 2\left(-(-2)^2\right)}$$

$$= \frac{\sin(x - 2y)}{-1 - 6 - 8}$$

$$= \frac{\sin(x - 2y)}{-15}$$

$\therefore$ General solution of (1) is

z = C.F. + P I

$$\therefore z = f_1(y + x) + f_2(y + 2x) + \frac{1}{4} e^{2x+3y} - x e^{2x+3y} - \frac{1}{15} \sin(x - 2)$$

Example 8

Solve $(D^2 - 2DD' + D'^2)z = e^{x+2y} + x^3$

Given P.D.E. is

$$(D^2 - 2DD' + D'^2)z = e^{x+2y} + x^3$$

Here A. I. is

$m^2 - 2m + 1 = 0$

$\Rightarrow (m - 1)^2 = 0$

$\Rightarrow m = 1, 1$

$\therefore$ C.F. = $f_1(y+x) + xf_2(y+x)$, where f_1 and f_2 are arbitrary function

Now PI for $e^{x+2y} = \frac{1}{D^2 - 2DD' + 2D'^2} e^{x+2y}$

$$e^{x+2y} = \frac{e^{x+2y}}{1^2 - 2 \times 1 \times 2 + 2^2}$$

$$= \frac{e^{x+2y}}{1}$$

$$= e^{x+2y}$$

PI for x^3 is $\frac{1}{D^2 - 2DD' + 2D'^2} x^3$

$$= \frac{1}{(D - D')^2} x^3$$

$$= \frac{1}{D^2\left(1 - \frac{D'}{D}\right)^2} x^3$$

$$= \frac{1}{D^2}\left(1 - \frac{D'}{D}\right)^{-2} x^3$$

$$= \frac{1}{D^2}\left(1 + 2\left(\frac{D'}{D}\right)^2 + \ldots.\right) x^3$$

$$= \frac{1}{D^2} x^3$$

$$= \iint x^3 dx\, dx$$

$$= \int \frac{x^4}{4} dx$$

$$= \frac{x^5}{20}$$

$\therefore$ General solution of (1) is

z = C.F. + P I

$$\therefore z = f_1(y + x) + x f_2(y + x) + e^{x+2y} + \frac{x^5}{20}$$

Example 9

Solve $(D^2 - DD' + D' - 1)z = \cos(x + 2y)$

Given P.D.E. is

$(D^2 – DD' + D' – 1)z = \cos(x+2y)$

$\Rightarrow (D(D – D' + 1(D' – 1)z = \cos(x+2y)$

$\Rightarrow (D – D)\ (D – D' + 1)z = \cos(x+2y)$ ..... (1)

$\therefore$ C.F. = $e^x f_1(y) + e^{-x} f_2(y+x)$

Where f_1 and f_2 we arbitrary functions

(Refere to article 9.6 and example 1)

Now PI = $\frac{1}{D^2 - DD' + D' - 1}\cos(x + 2y)$

$= \frac{1}{-1^2 + 1.2 + D' - 1}\cos(x + 2y)$

$= \frac{1}{D'}\cos(x + 2y)$

$= \frac{1}{2}\sin(x + 2y)$

$\therefore$ General solution of (1) is

z = C.F. + P I

$\therefore z = e^x f_1(y) + e^{-x} f_2(y+x)$

Example 10

Find a general solution of the following P.D.E.

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} - 2\frac{\partial^2 z}{\partial y^2} + 2\frac{\partial z}{\partial x} + 2\frac{\partial z}{\partial y} = e^{2x+3y} + \sin(2x + y) + xy$$

Given P.D.E. is

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} - 2\frac{\partial^2 z}{\partial y^2} + 2\frac{\partial z}{\partial x} + 2\frac{\partial z}{\partial y} = e^{2x+3y} + \sin(2x + y) + xy$$

$$\Rightarrow (D^2 - DD' - 2D'^2 + 2D + 2D')z = e^{2x+3y} + \sin(2x + y) + xy$$

$$\Rightarrow (D^2 + DD' - DD' - 2D'^2 + 2D + 2D')z = e^{2x+3y} + \sin(2x + y) + xy$$

$$\Rightarrow \big(D(D + D') - 2D'(D + D') + 2(D + D')\big)z = e^{2x+3y} + \sin(2x + y) + xy$$

$$\Rightarrow (D + D')(D - 2D' + 2)z = e^{2x+3y} + \sin(2x + y) + xy$$

$$\therefore \text{C.F.} = f_1(y - x) + e^{-2x} f_2(y + 2x)$$

Where f_1 and f_2 are arbitrary functions

Now, PI for $e^{2x+3y} = \frac{1}{D^2 - DD' - 2D'^2 + 2D + 2D'} e^{2x+3y}$

$= \frac{e^{2x+3y}}{2^2 - 2.3 - 2.3^2 + 2.2 + 2.3}$

$= \frac{e^{2x+3y}}{4 - 6 - 18 + 4 + 6}$

$= \frac{1}{10} e^{2x+3y}$

$$\text{PI for } \sin(2x+3y) = \frac{1}{D^2 - DD' - 2D'^2 + 2D + 2D'} \sin(2x+y)$$

$$= \frac{1}{2^2 - \{-(2.1)\} - 2\{-(-1)^2\} + 2D + 2D'} \sin(2x+y)$$

$$= \frac{1}{2D + 2D'} \sin(2x+y)$$

$$= \frac{1}{2} \cdot \frac{1}{D + D'} \sin(2x+y)$$

$$= \frac{1}{2} \cdot \frac{D - D'}{(D+D')(D-D')} \sin(2x+y)$$

$$= \frac{1}{2} \cdot \frac{D - D'}{D^2 - D'^2} \sin(2x+y)$$

$$= \frac{1}{2} \cdot \frac{1}{-2^2 - (-1)^2} (D - D') \sin(2x+y)$$

$$= -\frac{1}{6} (D - D') \sin(2x+y)$$

$$= -\frac{1}{6} \left(D \sin(2x+y) - D' \sin(2x+y)\right)$$

$$= -\frac{1}{6} \left(2\cos(2x+y) - \cos(2x+y)\right)$$

$$= -\frac{\cos(2x+y)}{6}$$

$$\text{PI for } xy = \frac{1}{D^2 - DD' - 2D'^2 + 2D + 2D'} xy$$

$$= \frac{1}{2D\left(\frac{D^2 - DD' - 2D'^2 + 2D + 2D'}{2D}\right)} xy$$

$$= \frac{1}{2D\left\{1 - \frac{D^2 + DD' + 2D'^2 + 2D'}{2D}\right\}} xy$$

$$= \frac{1}{2D}\left\{1 - \frac{D^2 + DD' - 2D'^2 + 2D'}{2D}\right\} xy$$

$$= \frac{1}{2D}\left\{1 + \frac{D^2 + DD' - 2D'^2 + 2D'}{2D} + \left(\frac{D^2 + DD' - 2D'^2 + 2D'}{2D}\right)^2 + \ldots\right\} xy$$

$$= \frac{1}{2D}\left(1+\frac{D^2+DD'-2D'^2+2D'}{2D}\right)xy$$

$$= \frac{1}{2}\cdot\frac{1}{D}x\,y+\frac{1}{2D}\left(\frac{D^2+DD'-2D'^2+2D'}{2D}\right)xy$$

$$= \frac{1}{2}\cdot\frac{1}{D}x\,y+\frac{1}{2D}\left(\frac{1}{2D}(0+1-2x)\right)$$

$$= \frac{1}{2}\cdot x^2y+\frac{1}{2D}\left(\frac{1}{2D}\cdot 1-\frac{1}{D}\cdot x\right)$$

$$= \frac{1}{2}x^2y+\frac{1}{2D}\left(\frac{1}{2}x-\frac{x^2}{2}\right)$$

$$= \frac{1}{2}x^2y+\frac{1}{4D}x-\frac{1}{4D}x^2$$

$$= \frac{1}{2}x^2y+\frac{1}{4}\cdot\frac{x^2}{2}-\frac{1}{4}\cdot\frac{x^3}{3}$$

$$= \frac{1}{2}x^2y+\frac{1}{8}x^2-\frac{1}{12}x^3$$

∴ General solution of (1) is

z = C.F. + P I

$$\therefore\ z = f_1(y-x)+e^{-2x}f_2(y+2x)-\frac{1}{10}e^{2x+3y}-\frac{1}{6}\cos(2x+y)+\frac{1}{2}x^2y+\frac{1}{8}x^2-\frac{1}{12}x^3$$



CHECK YOUR PROGRESS-2

Solve the following P.D.E.'s -

Q 1: $(D^2+3DD'+2D'^2)z = x+y$

Q 2: $(D^2-2DD'+D'^2)z = e^{x+y}$

Q 3: $(D^3-4D^2D'+4DD'^2)z = 2\sin(3x+2y)$

Q 4: $(D^2-5DD'+4D'^2)z = \sin(4x+y)$

Q 5: $\frac{\partial^3 z}{\partial x^3}+\frac{\partial^2 z}{\partial x^2\,\partial y}-\frac{\partial^3 z}{\partial x\,\partial y^2}-\frac{\partial^3 z}{\partial y^3}=e^{y}\cos 2x$

9.11 GENERAL METHOD OF FINDING PARTICULAR INTEGRAL

Let the given equation be

$F(D, D')3 = f(x, y)$ ..... (1)

When $F(D, D')$ is homogeneous function of degree n so that

$F(D, D' = D-m_1D')\ (D-m_2D')\ \ldots\ (CD - m_nD')$

$\therefore$ PT for (1) $= \frac{1}{F(D,D')} f(x,y)$

$$= \frac{1}{(D-m_1D')\,(D-m_2D')\ldots(D-m_nD')} f(x,y) \quad \text{..... (2)}$$

In order to evaluate PI given by (2), we consider a solution of the following equation

$(D-mD')z = f(x, y)$

$\Rightarrow p - mq = f(x, y)$ ..... (3)

$\therefore$ Lagrange's auxiliary equation of (3) are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{d3}{f(x,y)} \quad \text{..... (4)}$$

Taking first two ratio of (4)

$$\frac{dx}{1} = \frac{dy}{-m}$$

$\Rightarrow dy+mdx = 0$

Integrating,

$y+mx = e$ ..... (5)

taking first and last ratios of (4),

$$\frac{dx}{1} = \frac{dz}{f(x,y)}$$

$\Rightarrow dz = f(x, y)dx$

Integrating

$$z = \int f(x,y)dx$$

$$= \int f(x, e-mx)dx$$

$$\therefore 3 = \frac{1}{D-mD'} f(x,y) = \int f(x, e-mx)dx \quad \text{..... (6)}$$

Which after integration the constant e must be replaced by y+mx since the PI. does not contain any arbitrary constant.

Hence the P.I given by (2) can be obtained by applying the operation (6) by the factors, in succession starting from the right.

Example 11

Solve the following PDE (use general method to find P.I.)

$$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} - 8\frac{\partial^2 z}{\partial y^2} = \sqrt{2x+3y}$$

Given PDE is

$$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} - 8\frac{\partial^2 z}{\partial y^2} = \sqrt{2x+3y}$$

$$\Rightarrow (D^2 - 2DD' - 8D'^2)z = \sqrt{2x+3y}$$

$$\Rightarrow (D-4D')(D+2D')z = \sqrt{2x+3y} \quad \text{..... (1)}$$

Now A. E. of (1) is

$(m-4)(m+2) = 0$

$\Rightarrow (m = 4, -2$

$\therefore$ C.F. is $f_1(y+4x) + f_2(y-2x)$

When f_1 and f_2 are arbitrary functions

$$\therefore \text{P I} = \frac{1}{(D-4D')(D+2D')}\sqrt{2x+3y}$$

$$= \frac{1}{D-4D'}\int \sqrt{2x+3y}\, dx, \text{ when } y - 2x = e$$

$$= \frac{1}{D-4D'}\int \sqrt{2x+3(e+2x)}\, dx$$

$$= \frac{1}{D-4D'}\int (8x+3e)^{1/2}\, dx$$

$$= \frac{1}{D-4D'}\frac{\frac{(8x+3e)^{1/2+1}}{\frac{1}{2}+1}}{8}$$

$$= \frac{1}{D-4D'}\frac{(8x+3e)^{3/2}}{8\times\frac{3}{2}}$$

$$= \frac{1}{12}\cdot\frac{1}{D-4D'}(8x+3e)^{3/2}$$

$$= \frac{1}{12}\cdot\frac{1}{D-4D'}(8x+3(y-2x))^{3/2}$$

$$= \frac{1}{12}\ \frac{1}{D-4D'}\ (2x+3y)^{3/2}$$

$$= \frac{1}{12}\ \int (2x+3y)^{3/2}\ dx, \qquad \text{When } y+4x = e$$

$$= \frac{1}{12}\ \int (2x+3\,(e-4x)^{3/2}\ dx$$

$$= \frac{1}{12}\ \int (3e-10x)^{3/2}\, dx$$

$$= \frac{1}{12} \times \frac{\dfrac{(3e-10x)^{3/2+1}}{\dfrac{3}{2}+1}}{-10}$$

$$= \frac{1}{12} \times \frac{(3e-10x)^{5/2}}{-10\times\dfrac{5}{2}}$$

$$= \frac{1}{300}\ (2x+3y)^{5/2} \qquad \text{(putting the value of e)}$$

$\therefore$ General solution of (1) is

$$z = f_1(y+4x) + f_2(y-2x) - \frac{1}{300}(2x+3y)^{5/2}$$

Example 12

Solve the following PDE (use general method to find PI)

$$4\frac{\partial^2 z}{dx^2} - 4\frac{\partial^2 z}{dx\,dy} + \frac{\partial^2 z}{dy^2} = 16\log(x+2y)$$

Given PDE is

$$4\frac{\partial^2 z}{dx^2} - 4\frac{\partial^2 z}{dx\,dy} + \frac{\partial^2 z}{dy^2} = 16\log(x+2y)$$

$\Rightarrow 4D^2 - 4DD' + D'^2 = 16 \log (x+2y)$

$\Rightarrow (2D - D')^2 = 16 \log (x+2y) \quad \rightarrow (1)$

Now AE of (1) is

$(2m-1)^2 = 0$

$$\Rightarrow m = \frac{1}{2}, \frac{1}{2}$$

$$\therefore\ CF = f_1\left(y+\frac{1}{2}x\right) + x\,f_2\left(y+\frac{1}{2}x\right)$$

$= g_1(2y+x) + xg_2(2y+x)$

Where g_1 and g_2 are arbitrary functions

Now $PI = \frac{1}{(2D-D')^2} 16\log(x+2y)$

$= 16\frac{1}{(2D-D')(2D-D')}\log(x+2y)$

$= 16\frac{1}{2D-D'}\int \log(x+2y)\,dx$, when 2y+x = e

$= 16 . \frac{1}{2D-D'}\int \log e.dx$

$= 16 . \frac{1}{2D-D'} x \log e$

$= 16\int x\log(2y+x)\,dx$, when 2y+x = e

$= 16\int x\log e\,dx$

$= 16\log e\int x\,dx$

$= 16\log e\times\frac{x^2}{2}$

$= 8x^2 \log(2y+x)$

General solution of (1) is

$z = g_1(2y+x) + xg_2(2y+x) + 8x^2\log(2y+x)$



CHECK YOUR PROGRESS-3

Solve the following PDE (use general method to find PI)

Q 1: $(D^2 - DD' - 2D'^2)z = (y-1)e^x$

Q 2: $r + s - 6t = y\cos x$

9.12 CHARACTERISTIC SURFACES OF SECOND ORDER EQUATIONS

We have seen that solutions obtained in above methods involve arbitrary functions of x and y. We shall now determine these under the given geometical conditions. This will lead to required sunface satisfying the given differential equation under the given geometrical conditions.

Example 13

Find a surface passing through the two lines 3 = x = 0, 3 – 1 = x–y = 0 satisfying r – 4s + 4t = 0

Given equation is

$r - 4s + 4t = 0$

$$\Rightarrow \frac{\partial^2 z}{dx^2} - 4\frac{\partial^2 z}{dx\,dy} + 4\frac{\partial^2 z}{dy^2} = 0$$

$\Rightarrow (D^2 - 4DD' + 4D'^2)z = 0$

$\Rightarrow (D - 2D')^2 z = 0$ ..... (1)

$\therefore$ its AE of (1) is

$(m-2)^2 = 0$

$\Rightarrow m = 2, 2$

$\therefore$ Solution of (1) is

$z = f_1(y+2x) + xf_2(y+2x)$ ..... (2)

When f_1 and f_2 are arbitrary functions

Since (2) passes through z = x = 0

(2) $\Rightarrow 0 = f_1(y+2.0) + 0.f_2(y+2.0)$

$\Rightarrow f_1(y) = 0$

Which gives $f_1(y+2x) = 0$

$\therefore$ (2) becomes $z = xf_2(y+2x)$ ..... (3)

Since (3) passes through z – 1 = x – y = 0

ie, z = 1 and y = x, we get

$1 = x\, f_2(x+2x)$

$\Rightarrow 1 = x\, f_2(3x)$

$$\Rightarrow f_2(3x) = \frac{1}{x} = \frac{3}{3x}$$

$$\Rightarrow f_2(y+2x) = \frac{3}{y+2x}$$

$\therefore$ From (3)

$$z = x \cdot \frac{3}{y+2x}$$

$\Rightarrow 3x = z(y+2x)$

Which is the required surface



CHECK YOUR PROGRESS-4

Q 1: Find the surface satisfying the equation $r+t-2s = 0$ and the condition that $bz = y^2$ when $r = 0$ and $az = x^2$ when $y = 0$

9.13 PDE REDUCIBLE TO EQUATIONS WITH CONSTANT COEFFICIENTS (EULER-CAUCHY EQUATIONS)

Consider the Euler-cauchy type equation $(a_0 x^n D^n + a_1 x^{n-1} D^{n-1} D' + a_2 x^{n-2} D^{n-2} D'^2 + \ldots + a_n y^n D'^n + \ldots) z = f(x, y)$ ..... (1)

When $D \equiv \frac{\partial}{\partial x}$ and $D' \equiv \frac{\partial}{\partial y}$

Now defined two variables u and ν by

$x = e^u$ and $y = e^x$

so that $u = \log x$ and $\nu = \log y$ ..... (2)

Let $D_1 = \frac{\partial}{\partial u}$ and $D_1' = \frac{\partial}{\partial \nu}$ ..... (3)

Now $D_1 = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} = \frac{1}{x} \frac{\partial z}{\partial u}$ (using (2))

$$\Rightarrow x \frac{\partial z}{\partial x} = \frac{\partial}{\partial u}$$

$$\Rightarrow x \frac{\partial}{\partial x} = \frac{\partial}{\partial u}$$

$\Rightarrow xD = D_1$ ..... (4)

Again $x \frac{\partial}{\partial x}\left(x^{n-1} \frac{\partial^{n-1} z}{\partial x^{n-1}}\right) = x^n \frac{\partial^n z}{\partial x^n} + (n-1)x^{n-1} \frac{\partial^{n-1} z}{\partial x^{n-1}}$

$$\Rightarrow x^n \frac{\partial^n z}{\partial x^n} = \left(x \frac{\partial}{\partial x} - n + 1\right) x^{n-1} \frac{\partial^{n-1} z}{\partial x^{n-1}}$$

$\Rightarrow x^n D^n = (D_1 - n+1) x^{n-1} D^{n-1}$ ..... (5)

Putting n = 2, 3, ... in (5), we have

$x^2D^2 = (D_1 - 1) xD$

$\Rightarrow x^2D^2 = D_1(D_1-1)$, using (4) ..... (6)

$x^3D^3 = (D_1-2) x^2D^2$

$\Rightarrow x^3D^3 = D_1(D_1-1)(D_1-2)$, using (6) ..... (7)

and so on

Similarly, we have

$YD' = D_1'$

$Y^2D'^2 = D_1'(D_1'-1)$ ..... (8)

$Y^3D'^3 = D_1'(D_1'-1)(D_1'-2)$

and so on

Also, we have

$xy\ DD' = D_1D_1'$ ..... (9)

and $x^m y^n D^m D'^n = D_1(D_1-1)\ldots(D_1-m+1)\ D_1'(D_1'-1)\ldots(D_1'-n+1) \rightarrow (10)$

Using the substitutions (2) and results (4), (5), (6), (7), (8), (9) and (10), the given equation (1) reduces to an equation having constant coefficients and now it can easily be solved by the methods discussed earlier in the unit. Finally, with help of (2), the solution is obtained in term of old variables x and y.

Example 14

Solve $x\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\,\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = 0$

Let $x = e^u$ $y = e^{\upsilon}$

$u = \log x$ $\upsilon = \log y$ ..... (1)

Let $D \equiv \frac{\partial}{\partial x}$

$D' \equiv \frac{\partial}{\partial y}$

$D_1 \equiv \frac{\partial}{\partial u}$

$D_1' \equiv \frac{\partial}{\partial \upsilon}$

Now the given equation can be written as

$(x^2D^2 + 2xy\ DD' + y^2D'^2)z = 0$

This reduces to

$[D_1(D_1-1) + 2D_1D_1' + D_1'(D_1'-1)]z = 0$

$\Rightarrow [(D_1+D_1')^2 - (D_1+D_1')]3 = 0$

$\Rightarrow (D_1+D_1')-(D_1+D_1'-1)z = 0$

Hence the requried solution is

$z = CF = f_1(v-u)+e^u f_2(v-u)$ When f_1 and f_2 are arbitrary functions

$\Rightarrow z = f_1(\log y - \log x) + xf_2(\log y - \log x)$ (using (1))

$\Rightarrow z = f_1\left(\log\left(\frac{y}{x}\right)\right)+x\, f_2\left(\log\left(\frac{y}{x}\right)\right)$

$\Rightarrow z = g_1\left(\frac{y}{x}\right)+x\, g_2\left(\frac{y}{x}\right)$, When g_1 and g_2 are arbitrary functions

Example 15

Solve $x^2\frac{\partial^2 z}{\partial x^2}-y^2\frac{\partial^2 z}{\partial y^2}+x\frac{\partial y}{\partial x}-y\frac{\partial z}{\partial x}=\log x$

Let $x = e^u$ $y = e^v$

$u = \log x$ $v = \log y$ ..... (1)

Let $D \equiv \frac{\partial}{\partial x}$

$D' \equiv \frac{\partial}{\partial y}$

$D_1 \equiv \frac{\partial}{\partial u}$

$D_1' \equiv \frac{\partial}{\partial v}$

Now the given equation can be written as

$(x^2 D^2 - y^2 D'^2 + xD)z = \log x$

This reduces to

$[D_1(D_1-1) - D_1'(D_1'-1)+D_1-D_1']z = u$

$\Rightarrow (D_1^2 - D_1'^2)z = u$

Here AE is

$m^2-1 = 0$

$\Rightarrow (m-1)(m+1) = 0$

$\Rightarrow m = 1, -1$

$\therefore CF = f_1(v+u)+f_2(v-u)$

$= f_1(\log y + \log x) + f_2(\log y - \log x)$ [using (1)]

When f_1 and f_2 are arbitrary functions

$$= f_1(\log xy) + f_2\left(\log \frac{y}{x}\right)$$

$$= g_1(xy) + g_2\left(\frac{y}{x}\right), \text{ When } g_1 \text{ and } g_2 \text{ are arbitrary functions}$$

$$\therefore \text{PI} = \frac{1}{D_1^2 - D_1'^2} u$$

$$= \frac{1}{D_1^2\left(1 - \frac{D_1'^2}{D_1^2}\right)} u$$

$$= \frac{1}{D_1^2}\left(1 - \frac{D_1'^2}{D_1^2}\right)^{-1} u$$

$$= \frac{1}{D_1^2}\left(1 + \frac{D_1'^2}{D_1^2} + \ldots\right) u$$

$$= \frac{1}{D_1^2} u$$

$$= \frac{1}{D_1}\left(\frac{u^2}{2}\right)$$

$$= \frac{u^3}{6}$$

$$= \frac{(\log x)^3}{6} \qquad \text{[using (1)]}$$

$\therefore$ requried solution is

z = C.F. + PI

$$z = g_1(xy) + g_2\left(\frac{y}{x}\right) + \frac{(\log x)^3}{6}$$



CHECK YOUR PROGRESS-5

Q 1: Solve : $x^2 \frac{\partial^2 z}{\partial x^2} - y^2 \frac{\partial^2 z}{\partial y^2} - y\frac{\partial z}{\partial y} + x\frac{\partial z}{\partial x} = 0$

Q 2: Solve : $x^2 \frac{\partial^2 z}{\partial x^2} - y^2 \frac{\partial^2 z}{\partial y^2} = x\,y$



9.14 LET US SUM UP

- General Solution of a partial differential equation consists of two parts - Complimentary Function (C.F.)
- Under some prescribed conditions, the solution of a partial differential equation of second order leads to an equation of surface, known as characteristic surface.
- An cuchy-Euler type equation of the form $(a_0 x^n D^n + a_1 x^{n-1} D^{n-1} D' + \ldots + a_n y^n D'^n + \ldots) z = f(x, y)$ can be redeucible to partial differential equation with constant coefficients.



9.15 FURTHER READING

1) Ross, Sheply L. (2004) : Differential Equations (3rd Ed), John Wiley & Sons, India.

2) Evans, I. C. (2014) : Partial Differential Equations (2nd Ed), The Orient Blackswan.

2) Sneddon, I.N. (2006) : Elements of Partial Differential Equations, Dover Publications.

3) Raisinghania, M. D. (2001) : Advanced Differential Equations S. Chand & Company Ltd. New Delhi.



9.16 ANSWERS TO CHECK YOUR PROGRESS

Check Your Progress-1

Ans to Q No 1: (i) Left as exercise

(ii) Given PDE is

$2r + 5s + 2t = 0$

$$\Rightarrow 2\frac{\partial^2 z}{\partial x^2}+5\frac{\partial^2 z}{\partial x \partial y}+2\frac{\partial^2 z}{\partial y^2}=0$$

$\Rightarrow (2D^2 + 5DD' + 2D'^2)z = 0$

Here A. E. is

$2m^2 + 5m + 2 = 0$

$\Rightarrow 2m^2 + 4m + m + 2 = 0$

$\Rightarrow 2m(m+2) + 1(m+2) = 0$

$\Rightarrow (2m+1)(m+2) = 0$

$\Rightarrow m = -\frac{1}{2}, -2$

$\therefore$ General solution is

$z = f_1(y - \frac{1}{2}x) + f_2(y - 2x)$

$= f_3(2y–x) + f_2(y–2x)$

Where f_1, f_2 and f_3 are arbitrary functions.

(iii) Left as exercise

(iv) Left as exercise

(v) Left as exercise

(vi) Given PDE is

$(D^3D'^2 + D^2D'^3)z = 0$

Here A. E. is

$m^3 + m^2 = 0$

$\Rightarrow m^2 (m + 1) = 0$

$\Rightarrow m = 0, 0, –1$

$\therefore$ General solution is

$z = f_1(y + 0.x) + xf_2(y+0.x) + f_3(y–x)$

$\Rightarrow z = f_1(y) + xf_2(y) + f_3(y–x)$

Where f_1, f_2 and f_3 are arbitrary functions.

Check Your Progress-2

Ans to Q No 1: Given P.D.E. is

$(D^2+3DD'+2D'^2)z = x+y$

Here $\Rightarrow (D^2 + 2DD' + DD' + 2D'^2)z = x+y$

$\Rightarrow (D(D+2D') + D'(D+2D'))\, z = x+y$

$\Rightarrow (D+D')(D+2D')z = x+y$

Here AE is

$$(m+1)(m+2) = 0$$

$$\Rightarrow m = -1, -2$$

$\therefore$ CF = $f_1(y-x) + f_2(y-2x)$, when f_1 and f_2 are arbitiary function

Now $$P.I = \frac{1}{D^2 + 3DD' + 2D'^2}(x+y)$$

$$= \frac{1}{D^2\left(1 + \frac{3DD' + 2D'^2}{D^2}\right)}(x+y)$$

$$= \frac{1}{D^2}\left(1 + \frac{3D'}{D} + \frac{2D'}{D^2}\right)^{-1}(x+y)$$

$$= \frac{1}{D^2}(x+y)$$

$$= \frac{1}{D}\,\frac{(x+y)^2}{2}$$

$$= \frac{(x+y)^3}{6}$$

$\therefore$ Solution of (1) is z = CF + PI

$$z = f_1(y-x) + f_2(y-2x) + \frac{(x+y)^3}{6}$$

Ans to Q No 2: Left as exercise

Ans to Q No 3: Left as exercise

Ans to Q No 4: Left as exercise

Ans to Q No 5: Given PDE is

$$\frac{\partial^3 z}{\partial x^3} + \frac{\partial^2 z}{\partial x^2 \partial y} - \frac{\partial^3 z}{\partial x \partial y^2} - \frac{\partial^3 z}{\partial y^3} = e^y \cos 2x$$

$$\Rightarrow (D^3 + D^2D' - DD'^2 - D'^3)z = e^y \cos 2x$$

$$\Rightarrow (D^2(D+D') - D'^2(D+D'))z = e^y \cos 2x$$

$$\Rightarrow (D+D')(D-D')(D+D')z = e^y \cos 2x$$

Here AE is

$$(m+1)(m-1)(m+1) = 0$$

$$\Rightarrow m = -1, -1, 1$$

$\therefore$ CF = $f_1(y-x) + xf_2(y-x) + f_3(y+x)$, when f_1, f_2 and f_2 are arbitiary function

Now $\text{P I} = \dfrac{1}{\text{D}^3 + \text{D}^2\text{D}' - \text{DD}'^2 - \text{D}'^3} e^{\text{y}} \cos 2\text{x}$

$= \text{R}\,e \left(\dfrac{1}{\text{D}^3 + \text{D}^2\text{D}' - \text{DD}'^2 - \text{D}'^3} e^{\text{y}} \cos 2\text{x} \right)$

($\text{R}\,e$ stands of real part for example if u = x+iy $\text{R}\,e$(u) = x)

$= \text{R}\,e \left(\dfrac{1}{\text{D}^3 + \text{D}^2\text{D}' - \text{DD}'^2 - \text{D}'^3} e^{\text{y}+2\text{i}\,\text{x}} \right)$

$= \text{R}\,e \left(\dfrac{1}{(2\text{i})^3 + (2\text{i})^2(1) - (2\text{i})(1)^2 - 1^3} \right)$

$= \text{R}\,e \left(\dfrac{e^{2\text{i}\,\text{x}+\text{y}}}{-8\text{i} - 4 - 2\text{i} - 1} \right)$

$= \text{R}\,e \left(\dfrac{e^{2\text{i}\,\text{x}+\text{y}}}{-10\text{i} - 5} \right)$

$= \dfrac{1}{5} \text{R}\,e \left(\dfrac{e^{2\text{i}\,\text{x}+\text{y}}}{1 + 2\text{i}} \right)$

$= -\dfrac{e^{\text{y}}}{5} \text{R}\,e \left(\dfrac{e^{2\text{i}\,\text{x}}}{1 + 2\text{i}} \right)$

$= -\dfrac{e^{\text{y}}}{5} \text{R}\,e \left(\dfrac{(\cos 2\text{x} - \text{i} \sin 2\text{x})(1 - 2\text{i})}{(1 + 2\text{i})(1 - 2\text{i})} \right)$

$= -\dfrac{e^{\text{y}}}{5} \text{R}\,e \left(\dfrac{\cos 2\text{x} - 2\sin 2\text{x} + \text{i}\ (\sin 2\text{x} - 2\cos \text{x})}{(1 + 2\text{i})(1 - 2\text{i})} \right)$

$= -\dfrac{e^{\text{y}}}{5} \times \dfrac{\cos 2\text{x} - 2\sin 2\text{x}}{5}$

$= -\dfrac{1}{25} e^{\text{y}}\ (\cos 2\text{x} - 2\sin 2\text{x})$

$\therefore$ Required general solution is

$z = CF + PE$

$z = f_1(y-x) + xf_2(y-x) + f_3(y+x) - \dfrac{1}{25} e^{y}\ (\cos 2x + 2\sin 2x)$

Check Your Progress-3

Ans to Q No 1: Left as exercise.

Ans to Q No 2: Given PDE is

$R + s - 6t = y \cos x$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = y\cos x$$

$\Rightarrow (D^2+DD'-6D')z = y\cos x$

$\Rightarrow (D+2D')(D+3D')z = y\cos x$

Here AE is

$(m-2)(m+3) = 0$

$\Rightarrow m = 2, -3$

$\therefore$ CF = $f_1(y+2x) + f_2(y-3x)$ when f_1 and f_2 are arbitiary function

Now $PI = \frac{1}{(D-2D')(D+3D')} y\cos x$

$= \frac{1}{D-2D'} \int y\cos x\, dx$, when $y-3x = e$

$= \frac{1}{D-2D'} \int c\,(c+3x)\cos x\, dx$

$= \frac{1}{D-2D'} \left[\int C\cos x\, dx + 3\int x\cos x\, dx\right]$

$= \frac{1}{D-2D'} \left[C\sin x + 3\left\{x\int \cos x\, dx - \int \left\{\frac{d}{dx} x \int \cos x\, dx\right\} dx\right\}\right]$

$= \frac{1}{D-2D'} \left[C\sin x + 3\left(x\sin x - \int \sin x\, dx\right)\right]$

$= \frac{1}{D-2D'} [C\sin x + 3x\sin x + 3\cos x]$

$= \frac{1}{D-2D'} [(y-3x)\sin x + 3x\sin x + 3\cos x]$

$= \frac{1}{D-2D'} [y\sin x + 3\cos x]$

$= \int y\sin x\, dx + 3\int \cos dx$, when $y+2x = e$

$= \int (e-2x)\sin x\, dx + 3\sin x$

$= e\int \sin x\, dx - 2\int x\sin x\, dx + 3\sin x$

$= -e\cos x - 2\left\{x\int \sin x\, dx - \int \left\{\frac{d}{dx} x \int \sin x\, dx\right\} dx\right\} + 3\sin x$

$= -e\cos x - 2\left\{-x\cos x - \int (-\cos x) dx\right\} + 3\sin x$

$= -e\cos x + 2x\cos x - 2\int \cos x\, dx + 3\sin x$

$= -(y+2x)\cos x + 2x\cos x - 2\sin x + 3\sin x$

$= -y\cos x - 2x\cos x + 2x\cos x + \sin x$

$= \sin x - y\cos x$

$\therefore$ Required general solution is

$z = CF + PE$

$z = f_1(y+2x) + f_2(y-3x) + \sin x - y\cos x$

Check Your Progress-4

Ans to Q No 1: Left as exercise

Check Your Progress-5

Ans to Q No 1: Left as exercise

Ans to Q No 2: Given PDE is

$$x^2\frac{\partial^2 z}{\partial x^2} - y^2\frac{\partial^2 z}{\partial y^2} = xy$$

Let $x = e^u$ $y = e^{\upsilon}$

$u = \log x$ $\upsilon = \log y$ ..... (1)

Let $D \equiv \frac{\partial}{\partial x}$

$D' \equiv \frac{\partial}{\partial y}$

$D_1 \equiv \frac{\partial}{\partial u}$

$D_1' \equiv \frac{\partial}{\partial \upsilon}$

Now given equation become

$(x^2 D^2 - y^2 D'^2 + xD)z = xy$

$\Rightarrow [D_1(D_1-1) - D_1'(D_1'-1)]z = e^u e^v$

$\Rightarrow (D_1 - D_1')(D_1 + D_1' - 1)z = e^{u+v}$

$\therefore CF = f_1(v+u) + e^u f_2(v-u)$

$= f_1(\log y + \log x) + x f_2(\log y - \log x)$ (using (1))

$= f_1(\log xy) + x f_2\left(\log\left(\frac{y}{x}\right)\right)$

$= g_1(xy) + x g_2\left(\frac{y}{x}\right)$

When f_1, f_2, g_1, g_2 are arbitrary function

Now $PI = \frac{1}{(D_1 - D_1')(D_1 + D_1' - 1)} e^{u+v}$

$= \frac{1}{D_1 - D_1'} \frac{e^{u+v}}{1+1+1}$

$= \frac{1}{D_1 - D_1'} e^{u+v}$

$= \frac{u}{\lfloor 1} e^{u+v}$

$= u e^u e^v$

= xy logx (using (1))

∴ Required general solution is

$$z = g_1(xy) + x g_2\left(\frac{y}{x}\right) + xy \log x$$



9.17 MODEL QUESTIONS

Solve the following PDE's

Q 1: $(D^2 - 3aDD' + 2a^2D'^2)z = 0$

Q 2: $25r - 40s + 16t = 0$

Q 3: $(D^3 - 3D^2D' + 3DD'^2 - D'^3)z = 0$

Q 4: $r - t = x - y$

Q 5: $(D^3 - 7DD'^2 - 6D'^3)z = \sin(x+2y) + e^{3x+2y}$

Q 6: $\frac{\partial^2 z}{\partial x^2 \partial y} - 2\frac{\partial^2 z}{\partial x \partial y^2} + \frac{\partial^2 z}{\partial y^3} = \frac{1}{x^2}$

Q 7: $(D^2+2DD'+D'^2)\, z = 2 \cos y - x \sin y$ (use general method to find PI)

Q 8: $(D^2 - D'^2 + D - D')z = e^{2x+3y}$

Q 9: $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial y} - 3 = \cos(x + 2y)$

Q 10: $x^2 \frac{\partial^2 z}{\partial x^2} - 4xy \frac{\partial^2 z}{\partial x \partial y} + 4y^2 \frac{\partial^2 z}{\partial y^2} + 6y \frac{\partial z}{\partial y} x^3 y^4$

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