PG PHY S1-01 DSC

TWO YEAR MASTER'S DEGREE COURSE

(CBCS in Accordance with NEP 2020)

SEMESTER-I

CLASSICAL MECHANICS

PHYSICS

Hiranya Chandra Bhuyan School of Science and Technology

KRISHNA KANTA HANDIQUI STATE OPEN UNIVERSITY

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COURSE INTRODUCTION

Classical Mechanics is the first Discipline Specific Course (DSC) of the first semester of Master of Science in Physics programme. The main objective of this course is to develop a clear understanding of the fundamental principles of classical mechanics and the mathematical methods used to describe the motion of physical systems. The course introduces the basic concepts and analytical formulations of classical mechanics, beginning with introductory ideas such as coordinate systems, degrees of freedom, constraints, and generalized coordinates. It then discusses important principles like the principle of virtual work and D'Alembert's principle. The Lagrangian and Hamiltonian formulations of mechanics are introduced, providing powerful methods for analysing dynamical systems. Topics such as canonical transformations, Poisson brackets, and Liouville's theorem are also included to build the foundation of advanced analytical mechanics.

The course further focuses on the applications and extensions of analytical mechanics. It introduces the Hamilton–Jacobi theory as a powerful method for solving mechanical problems and discusses the two-body central force problem and the theory of small oscillations, which are important in many physical systems. In addition, the dynamics of rigid bodies is studied through Euler's equations and the concept of moments of inertia. The course concludes with an introduction to the special theory of relativity, including Lorentz transformations and their consequences, as well as the four-dimensional formulation of space–time using Minkowski space and four-vectors. Overall, this course helps learners develop a deeper understanding of the principles governing the motion of particles and rigid bodies. It also provides the analytical tools required to solve complex mechanical problems and prepares learners for advanced studies in theoretical physics.

There are 14 units in this course.

The **first unit** covers the basic concepts required for analytical mechanics, including coordinate systems and motion description. It introduces degrees of freedom, constraints, and generalized coordinates. These ideas form the foundation for advanced formulations.

The **second unit** covers the concept of equilibrium using analytical methods. It explains the principle of virtual work and develops D'Alembert's principle. This unit bridges Newtonian and Lagrangian mechanics.

The **third unit** covers the Lagrangian formulation of mechanics. It introduces the Lagrangian function and derives Lagrange's equations of motion. Applications to physical systems are also discussed.

The **fourth unit** covers the variational approach to mechanics. It explains Hamilton's principle and the principle of least action. The connection between Newtonian and Lagrangian mechanics is established.

The **fifth unit** covers the Hamiltonian formulation of mechanics. It derives canonical equations and applies them to various coordinate systems. The unit highlights problem-solving using

Hamilton's equations.

The **sixth unit** covers canonical transformations and their role in simplifying dynamics. It introduces Legendre transformations and generating functions. Conditions for canonical transformations are discussed.

The **seventh unit** covers Poisson and Lagrange brackets and their applications. It explains their physical significance in Hamiltonian mechanics. Liouville's theorem and phase space conservation are introduced.

The **eighth unit** covers the Hamilton–Jacobi method for solving dynamical problems. It derives the Hamilton–Jacobi equation and applies it to systems like harmonic oscillators and Kepler motion. The characteristic function is also discussed.

The **ninth unit** covers motion under central forces and reduction of the two-body problem to one-body motion. It derives equations of motion and orbit equations. Special focus is given to inverse-square law forces.

The **tenth unit** covers oscillations about equilibrium positions. It explains types of equilibrium and develops the theory of small oscillations. Normal modes and coupled oscillations are introduced.

The **eleventh unit** covers the basic concepts of rigid body motion. It discusses degrees of freedom and generalized coordinates. Body-fixed and space-fixed frames are introduced.

The **twelfth unit** covers rotational dynamics of rigid bodies. It introduces Euler's equations, Euler angles, and principal axes. Rotational kinetic energy is also discussed.

The **thirteenth unit** covers the fundamentals of special relativity. It explains the postulates, Lorentz transformations, and their consequences. Concepts like time dilation and length contraction are included.

The **fourteenth unit** covers the four-dimensional formulation of relativity. It introduces Minkowski space-time and four-vectors. Applications to relativistic motion are discussed.

While going through the units, the learner will notice some boxes containing additional information that help in understanding important or unfamiliar terms. Sections titled "ACTIVITY" provide opportunities for the learner to apply knowledge through questions related to the topic. Some useful concepts are also provided under the heading "LET US KNOW". At the end of each section, "CHECK YOUR PROGRESS" questions are included to help the learner assess understanding of the topic. Attempting these questions and comparing the responses with the "ANSWERS TO CHECK YOUR PROGRESS" given at the end of each unit helps the learner evaluate progress in the course.

UNIT 1: INTRODUCTORY IDEAS OF MECHANICS

UNIT STRUCTURE

1.1 Learning Objectives
1.2 Introduction
1.3 Coordinate Systems
1.4 Degrees of Freedom
1.5 Constraints
1.6 Generalized Coordinates
1.7 Let Us Sum Up
1.8 Answers to Check Your Progress
1.9 Further Reading
1.10 Model Questions

1.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- describe and apply different coordinate systems for representing the position and motion of particles
- to analyse and determine the degrees of freedom of various mechanical systems
- to classify and evaluate different types of constraints and their effects on the motion of a system
- to formulate and apply generalized coordinates to describe constrained mechanical systems efficiently

1.2 INTRODUCTION

In this unit, the learner will study the fundamental concepts required to describe motion in classical mechanics. The unit begins with an understanding of coordinate systems, which provide a mathematical framework for specifying the position and motion of particles and rigid bodies. The learner will then explore the concept of degrees of freedom, which determines the number of independent parameters needed to describe the configuration of a system. This will help in understanding how the motion of a system depends on its dimensionality and physical conditions. Further, the unit introduces the concept of constraints, which restrict the motion of a system and reduce its degrees of freedom. Different types of constraints and their physical significance will be studied. Finally, the learner will learn about generalized coordinates, which provide an efficient way to describe constrained systems and simplify the analysis of complex mechanical problems.

1.3 COORDINATE SYSTEMS

The fundamental concept involved in the motion of a particle (or system) is the position coordinate and how it changes with time. The position of a particle is represented by choosing a coordinate system. In classical mechanics, the motion of a particle or a system of particles is described relative to a chosen frame of reference with the help of a coordinate system. A coordinate system assigns numerical values to specify the position of a particle at any given instant of time.

To define a coordinate system, the following elements are required:

- A fixed reference point called the origin
- Specified directions of coordinate axes
- A suitable set of coordinates

The choice of coordinate system depends on the symmetry and nature of the physical problem. An appropriate selection simplifies the mathematical formulation of motion.

A. Types of Coordinate Systems

1. **Cartesian or Rectangular Coordinate System** (x, y, z):
   In the rectangular coordinate system, the position of a particle is represented by a position vector **r**, which is resolved into three mutually perpendicular components along the X, Y, and Z-axes. When the motion is restricted to a two-dimensional plane, the particle's position can be described either by rectangular coordinates (x, y) or by an alternative representation using radial distance and angular displacement *(r, θ)*.

2. **Cylindrical Coordinate System** (ρ, θ, z)**:**
   The cylindrical coordinate system is a three-dimensional extension of the two-dimensional polar coordinate system. In this system, the position of a point in space is specified by three quantities: the radial distance *ρ* from the reference axis, the angular coordinate *θ* measured in the horizontal plane, and the vertical coordinate *z* measured along the axis perpendicular to that plane.

3. **Spherical Coordinate System** (r, θ, Φ)
   In the spherical coordinate system, the position of a point in three-dimensional space is described by the radial distance *r* from the origin and two angular coordinates. The polar angle *θ* specifies the inclination of the line joining the origin to the point with respect to a chosen reference axis, while the azimuthal angle Φ defines the orientation of this line in the horizontal plane.

B. Relation between Cartesian, Cylindrical and Spherical Coordinate System

1. Cartesian and Polar Coordinate

In three-dimensional space, the position of a particle may be described using cylindrical coordinates *(ρ, θ, z)* or spherical coordinates *(r, θ, φ).* These coordinate representations are mathematically connected to the rectangular coordinates *(x, y, z)* through well-defined transformation relations, which allow the conversion of position vectors from one coordinate system to another

$$x = r\cos\theta\,, y = r\sin\theta$$
$$r = \sqrt{x^2 + y^2}, \theta = \tan^{-1}\left(\frac{y}{x}\right) \quad (1.1)$$

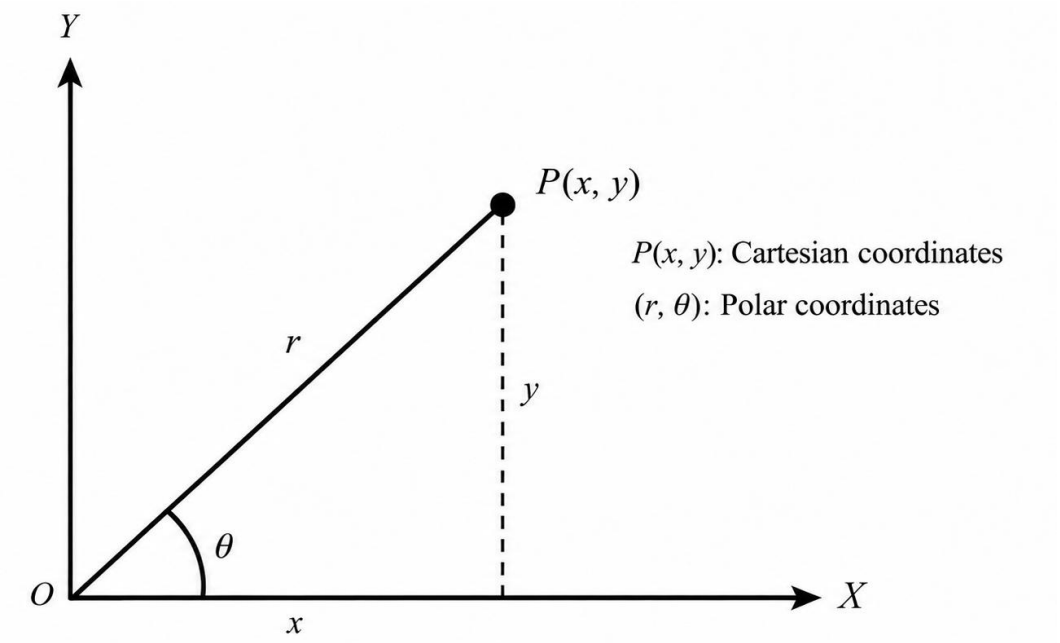


Figure. 1.1: Graphical representation of Rectangular and Polar Coordinate (Source: Classical Mechanics by J.C Upadhyay)

In three dimensions, the cylindrical coordinates (ρ, θ, z) and the spherical coordinates (r, θ, ϕ) of the position of a particle are related to the cartesian coordinates (x, y, z) as follows:

2. Cylindrical And Cartesian Coordinate

The cylindrical coordinate system (ρ, θ, z) is used to describe the position of a point in three-dimensional space. In this system, ϕ represents the angle measured in the horizontal plane, like the angle used in spherical coordinates. The coordinate z shows the height of the point above or below the reference plane and is the same as in the rectangular coordinate system. The coordinate ρ represents the distance of the point P from the z-axis**,** not from the origin. This is the main difference between cylindrical and spherical coordinate systems.

$$x = \rho\cos\theta\,, y = \rho\sin\theta\,, z = z$$
$$\rho = \sqrt{x^2 + y^2}, \theta = \tan^{-1}\left(\frac{y}{x}\right) \quad (1.2)$$

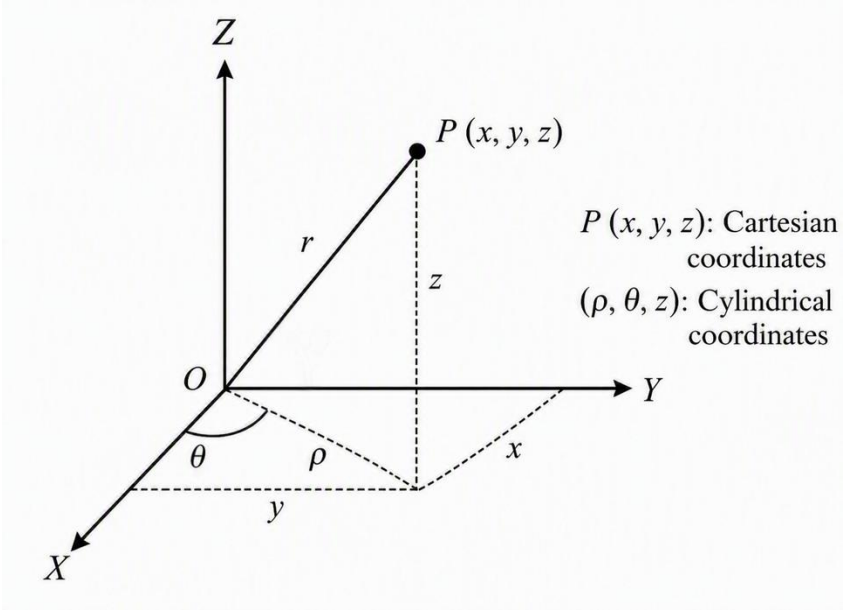


Figure1.2: Graphical representation of relation between Cartesian and Cylindrical Coordinates (Source : Classical Mechanics by J.C Upadhyay)

3. Spherical and Cartesian Coordinate

The spherical coordinate system (r, θ, ϕ) is another way of describing the position of a point in three-dimensional space. In this system, **r** represents the distance of the point from the origin, θ denotes the angle made by the line joining the point to the origin with the vertical axis, and ϕ is the angle measured in the horizontal plane. These spherical coordinates are related to the rectangular (Cartesian) coordinates (x, y, z) through mathematical relations that allow one system to be converted into the other. While Cartesian coordinates describe position using linear distances along three perpendicular axes, spherical coordinates describe the same position using distance and angular measurements.

$$x = r\sin\theta\cos\phi\,, y = r\sin\theta\sin\phi\,, z = r\cos\theta$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

Conversely, we may express $(r, \theta, \emptyset)$ in terms of (x, y, z) as

$$r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$$

$$\theta = \tan^{-1}\frac{(x^2+y^2)^{\frac{1}{2}}}{z} \qquad (1.3)$$

$$\phi = \tan^{-1}\frac{y}{x}$$

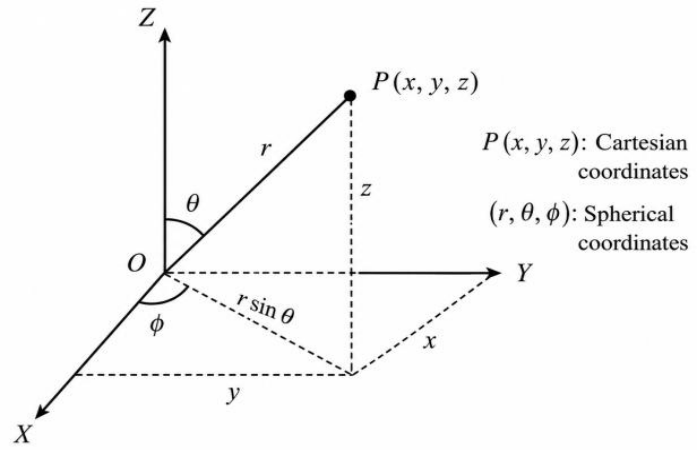


Figure 1.3: Graphical representation of relation between Cartesian and Spherical Coordinates (Source: Classical Mechanics by J.C Upadhyay)

1.4 DEGREES OF FREEDOM

In mechanics, the configuration of a system at any instant is specified by a set of independent variables. The total number of such independent variables required to completely determine the position of the system is called its degrees of freedom (DOF). In simple terms, degrees of freedom indicate the number of independent ways in which a system can move without violating any physical restriction.

For example:

- A particle moving freely in ordinary three-dimensional space requires three independent coordinates to specify its position. Therefore, it possesses three degrees of freedom**.**
- If two particles move independently in space, six independent coordinates are needed to describe the system. Hence, the system has six degrees of freedom.

Thus, for a system containing N independent particles**,** the total number of coordinates required is 3N**,** and accordingly the system has 3N degrees of freedom. When restrictions are imposed on the motion of a system, the number of independent variables decreases. Therefore, the effective degrees of freedom become less than the total number of coordinates required for a free system.

Hence, the degrees of freedom of a constrained system may also be viewed as the number of independent coordinates left after applying all the restrictions.

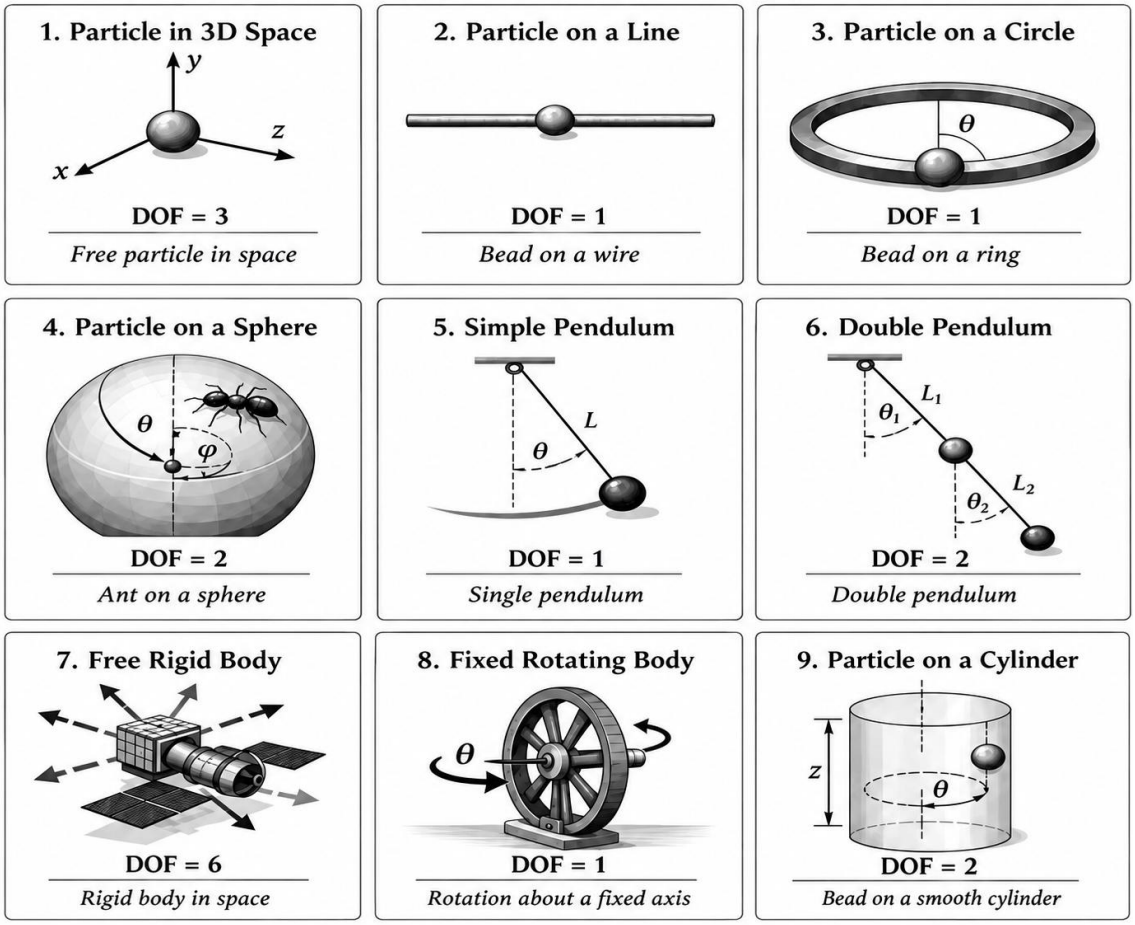


Figure 1.4: Degrees of Freedom of different physical system

1.5 CONSTRAINTS

In many mechanical systems, the motion of particles does not remain completely unrestricted. Certain physical, geometrical, or external conditions may limit the possible motion of the system. Such restrictions are known as constraints. A constraint establishes a relation among the coordinates of the system and reduces the number of independent variables needed to describe its motion.

For instance, consider a particle moving along a circular path of fixed radius. Since the distance from the centre remains unchanged, the particle cannot move freely in all directions. Its position can be specified by only one independent variable, such as the angular coordinate. Thus, constraints simplify the description of motion by reducing the available degrees of freedom.

A. Types of Constraints

Depending on their mathematical form and physical characteristics, constraints can be classified into different types, each having distinct implications for the analysis of motion.

1. Holonomic Constraints

A constraint is called **holonomic** if it can be written as an explicit mathematical relation involving the coordinates of the system and possibly time. Such constraints reduce the number of independent coordinates directly.

$$f(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N, t) = 0 \qquad (1.4)$$

If a system originally has n coordinates and is subjected to k independent holonomic constraints, then the remaining degrees of freedom become (n – k)**.**

Examples:

i. Rigid Body

In a rigid body, the separation between any two particles remains fixed throughout the motion. This fixed-distance condition provides a constraint relation among the coordinates of the particles. If r_i and r_j are the position vectors of ith and the jth particles, then the distance between them can be expressed as

$$|r_i - r_j| = C_{ij} \text{ (constant)} \qquad (1.5)$$

If (x_1, y_1, z_1) and (x_2, y_2, z_2) are the position vectors (cartesian coordinates) of the two particles, then the constraints will be expressed as

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_2 - z_1)^3 = \left(C_{ij}\right)^2 \qquad (1.6)$$

Since this relation can be written in equation form, it represents a holonomic constraint. If the relation does not involve time explicitly, it is also scleronomous.

ii. Simple Pendulum with rigid support:

For a simple pendulum, the bob remains at a constant distance from the point of suspension because the string length is fixed. If **r**denotes the position vector of the bob measured from the point of suspension, the constraint condition may be written as

$$|\vec{r}| = l = \text{constant} \qquad (1.7)$$

This constant-length condition acts as a holonomic constraint. If the pendulum swings in a vertical plane, only one independent parameter (angular displacement) is sufficient to describe the motion. Therefore, the system has one degree of freedom. In this case, the entire motion can be completely described by one generalized coordinate, namely the angular displacement θ.

2. Non-Holonomic Constraints

A constraint is termed non-holonomic when it cannot be reduced to a simple equation involving only coordinates and time. These constraints generally appear in differential form or as inequality conditions. Examples include systems where rolling conditions or velocity-dependent restrictions are present.

For example, if a particle is restricted to remain inside a spherical region of radius a, the condition is:

$$|\vec{r}| \leq a$$

In such cases, all coordinates are not independent, and the motion of the system is restricted by the nature of the constraint.

Example:

Rolling Disc

Consider a disc of radius 'a' rolling on a horizontal surface without slipping. The no-slip condition relates the translational motion of the centre to the rotational motion of the disc. The disc is assumed to roll in such a way that its plane always remains vertical. Let the coordinates (x, y) denote the position of the center of the disc. Let ϕ represent the angle through which the disc has rotated about its axis, and θdenote the angle between the axis of the disc and the X-axis.

Since the disc rolls without slipping, the linear velocity v of its center is related to the angular velocity by

$$v = a\frac{d\phi}{dt}$$

The direction of motion of the disc is always perpendicular to its axis. Therefore, the velocity components of the center along the X- and Y-axes are given by

$$v_x = \frac{dx}{dt} = v\sin\theta, v_y = \frac{dy}{dt} = -v\cos\theta.$$

Substituting the value of v, we obtain

$$\frac{dx}{dt} = a\frac{d\phi}{dt}\sin\theta, \frac{dy}{dt} = -a\frac{d\phi}{dt}\cos\theta$$

These relations may be written in differential form as

$$dx - a\sin\theta\, d\phi = 0, dy + a\cos\theta\, d\phi = 0 \qquad (1.8)$$

This relation appears in differential form and cannot be integrated into a coordinate equation. Hence, the rolling condition is an example of a non-holonomic constraint.

3. Scleronomous Constraints

A constraint is said to be scleronomous when its mathematical form does not contain time explicitly. Such constraints remain fixed throughout the motion.

Example

A particle confined to move on the surface of a fixed sphere of radius a satisfies a relation involving only spatial coordinates.

$$x^2 + y^2 + z^2 - a^2 = 0 \qquad (1.9)$$

Since the radius remains constant with time, the constraint is scleronomous. Another example is a bead moving on a fixed circular wire.

4. Rhenomomous Constraint

A constraint is called rheonomous when time appears explicitly in the constraint relation. In this case, the restriction itself changes as time progresses.

Example:
If a particle is constrained to move on the surface of a sphere whose radius varies with time $a(t)$, the constraint is given by:

$$x^2 + y^2 + z^2 - a^2(t) = 0. \qquad (1.10)$$

If a particle moves on the surface of a sphere whose radius changes with time, the constraint depends directly on time and is therefore rheonomous.
Other examples include:

- A particle moving on a wire whose shape changes with time
- A pendulum with an oscillating point of suspension

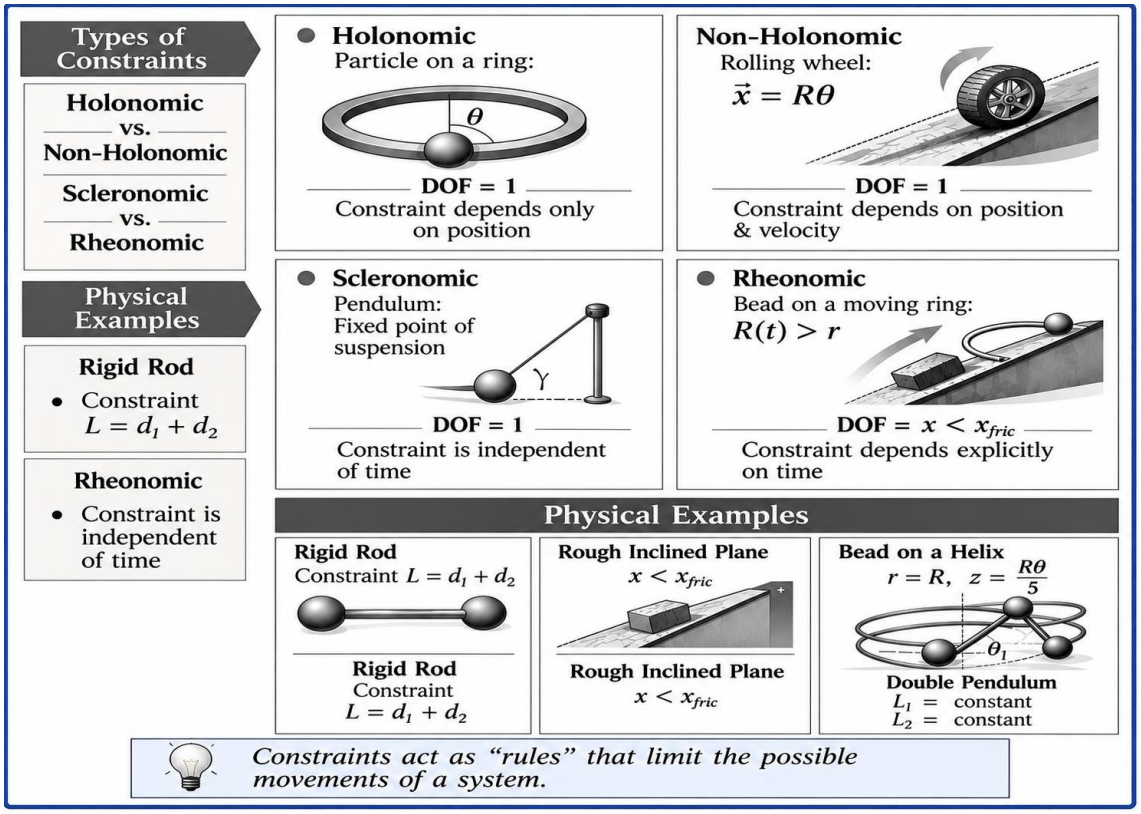


Figure 1.5 Constraints of different physical system.

5. Conservative Constraint

A constraint is termed conservative if it does not alter the total mechanical energy of the system. In such situations, the constraint force performs no network on the particle. These constraints merely guide the motion without causing energy loss.
Examples include motion on smooth surfaces and ideal pendulum motion.

6. Dissipative Constraint

A constraint becomes dissipative when the associated constraint forces remove mechanical energy from the system. This usually occurs in the presence of friction, resistance, or damping. As a result, the total mechanical energy decreases with time.

Examples include a block moving on a rough surface, a pendulum oscillating in a resisting medium such as air, or systems involving rheonomic (time-dependent) constraints, where energy dissipation generally occurs due to external driving or damping effects.

B. Difference Between Holonomic and Non-Holonomic Constraint

Holonomic Constraints	Non-holonomic Constraints
Constraints that can be expressed as equations relating the coordinates of the system.	Constraints that cannot be expressed solely as equations among the coordinates.
Each coordinate can vary independently within the limits of the constraint.	The coordinates cannot all vary independently.
The number of degrees of freedom is obtained directly by subtracting the number of constraints from the total coordinates.	The number of degrees of freedom is less than the minimum number of coordinates required to describe the system.
These constraints reduce the configuration space in a simple manner.	These constraints impose additional restrictions on the possible motions of the system.
Example: A particle constrained to move on a smooth surface or a simple pendulum of fixed length.	Example: A particle moving on a rough surface or a rolling wheel without slipping.

1.6 GENERALIZED COORDINATES

In many mechanical problems, especially those involving constraints, the ordinary Cartesian coordinates may not provide the most convenient description of motion. In such cases, it is often useful to introduce a new set of independent variables called generalized coordinates. Generalized coordinates are the minimum number of independent parameters required to specify the complete configuration of a system. These coordinates are usually denoted by:

$$q_1, q_2, q_3, \dots, q_n$$

The number of generalized coordinates is always equal to the degrees of freedom of the system. These coordinates are not restricted to Cartesian form; they may be angular variables, distances, or any suitable parameters that simplify the mathematical treatment of the problem.

For example,

- A particle constrained to move on a circle can be described by a single angular coordinate.
- A particle moving on the surface of a sphere requires two independent angular coordinates.

- A rigid body in space can be described by a suitable combination of translational and rotational coordinates.

For a system of N particles, the Cartesian coordinates of each particle may be expressed in terms of generalized coordinates and time through transformation relations.

$$\left.\begin{aligned} x_i &= x_i(q_1, q_2, q_k, \ldots\ldots q_n, t) \\ y_i &= y_i(q_1, q_2, \ldots, q_k.., q_n, t) \\ z_i &= z_i\,(q_1, q_2, \ldots., q_k, \ldots. q_n, t) \end{aligned}\right\} \quad (1.11)$$

In general, the position vector $r_i\ (x_i, y_i, z_i)$ of the i^{th} particle is

$$r_i = r_i\,(q_1, q_2, \ldots. q_k, \ldots q_n, t) \quad (1.12)$$

Equation (1.11) and (1.12) give the transformation equations. Thus, generalized coordinates provide a compact and efficient framework for studying constrained systems and form the basis of Lagrangian mechanics.

CHECK YOUR PROGRESS

Multiple Choice Questions from Q 1 to Q 6 (MCQs)

Q 1: In cylindrical coordinates *(r, ϕ, z)*, the unit vectors $\hat{r}$ and $\hat{\phi}$ are:

a) Constant in direction
b) Always perpendicular but time-independent
c) Time-dependent in general
d) Parallel to each other

Q 2: The position vector of a particle in spherical coordinates is given by:

a) $r\hat{\imath} + \theta\hat{\jmath} + \phi\hat{k}$
b) $r(\sin\theta\cos\phi\,\hat{\imath} + \sin\theta\sin\phi\,\hat{\jmath} + \cos\theta\,\hat{k})$
c) $r\hat{r} + \theta\hat{\theta} + \phi\hat{\phi}$
d) $\theta\hat{r} + r\hat{\theta}$

Q 3: Which coordinate system is most convenient for problems involving central forces?

a) Cartesian
b) Cylindrical
c) Spherical polar
d) Rectangular curvilinear

Q 4: Degrees of freedom of a system is defined as:

a) Total number of coordinates
b) Number of constraints
c) Minimum number of independent variables required to describe the system
d) Number of particles

Q 5: A system of Nfree particles in space has degrees of freedom equal to:

a) N
b) $2N$

c) $3N$
d) $N-1$

Q 6: If a system of N particles are subjected to m independent holonomic constraints, the degrees of freedom become:
a) $3N+m$
b) $3N-m$
c) $N-m$
d) $m-N$

Q 7: How many generalized coordinates are required to describe the motion of a particle on a sphere?

Q 8: State one physical example of a rheonomous constraint.

Q 9: What type of constraint is imposed by rolling without slipping?

Q10: Determine the degrees of freedom of a particle constrained to move on a circular ring and justify your answer.

⏱1.7 LET US SUM UP

- A coordinate system provides a mathematical framework to specify the position of a particle with respect to a chosen reference frame.
- Cartesian, cylindrical, and spherical coordinate systems are used depending on the symmetry of the physical problem.
- Transformation relations allow conversion of coordinates between different systems.
- Degrees of freedom represent the minimum number of independent variables required to describe a system completely.
- A system of N free particles in space has $3N$ degrees of freedom.
- The presence of constraints reduces the number of degrees of freedom.
- Constraints restrict the motion of a system and reduce the number of independent coordinates.
- Holonomic constraints can be expressed as equations involving coordinates, whereas non-holonomic constraints cannot.
- Constraints may also be classified as scleronomous, rheonomous, conservative, or dissipative based on their nature.
- Generalized coordinates are a set of independent parameters used to describe the configuration of a system.
- The number of generalized coordinates is equal to the degrees of freedom of the system.
- They provide a convenient way to analyze constrained systems and relate to Cartesian coordinates through transformation equations. Top of Form

✓ 1.8 ANSWERS TO CHECK YOUR PROGRESS

Answer to Q 1: c) Time-dependent in general

Answer to Q 2: b) $r(\sin\theta.\cos\phi\,.\hat{\imath}+\sin\theta\sin\phi\,\hat{\jmath}+\cos\theta\,\hat{k})$

Answer to Q 3: c) Spherical polar

Answer to Q 4: c) Minimum number of independent variables required to describe the system

Answer to Q 5: c) $3N$

Answer to Q 6: b) $3N - m$

Answer to Q 7: A particle on a sphere requires two generalized coordinates (e.g., θ, ϕ).

Answer to Q 8: Example: A particle constrained to move on a sphere whose radius varies with time $a(t)$.

Answer to Q 9: Rolling without slipping is a non-holonomic constraint**.**

Answer to Q 10: A particle constrained to move on a circular ring has one degree of freedom, because its motion is restricted to a fixed radius and can be described by a single coordinate (angular position).

🔍1.9 FURTHER READING

- Upadhyaya, J. C. (2010). *Classical Mechanics*. Himalayan Publishing House, Mumbai.
- Morin, D. (2008). *Introduction to Classical Mechanics*. Cambridge University Press, Cambridge.
- Gupta, A. B. (2003). *Classical Mechanics and Properties of Matter*. Books and Allied (P) Ltd., Kolkata.
- Goldstein, H., Poole, C. P., and Safko, J. L. (2002). *Classical Mechanics (3rd Edition)*. Pearson Education / Addison-Wesley, Singapore.
- Schaum's Outline Series. (1990). *Schaum's Outline of Classical Mechanics*. McGraw-Hill Education, New York.

⍰1.10 MODEL QUESTIONS

1. Define degrees of freedom of a dynamical system.
2. How many degrees of freedom does a free particle moving in 3-D space have?
3. State the formula for degrees of freedom of a system **of** N free particles.
4. What is meant by a constraint**?**
5. Write the general mathematical form of a holonomic constraint**.**
6. Identify whether the following constraint is holonomic or non-holonomic:

$$| \vec{r} | = a$$

7. Do holonomic constraints always reduce the degrees of freedom? (Yes/No)
8. Does a scleronomous constraint depend explicitly on time?
9. Name the type of constraint represented by

$$x^2 + y^2 + z^2 - a^2(t) = 0.$$

10. Are forces of constraint conservative in a smooth surface?
11. Explain how constraints affect the degrees of freedom of a system.
12. Distinguish between holonomic and non-holonomic constraints.
13. What is meant by a generalized coordinate**?**
14. Why does a simple pendulum have only one degree of freedom?
15. Give two examples of scleronomous constraints.

16. Give two examples of rheonomous constraints.
17. Why is a rolling disc without slipping a non-holonomic system?
18. State the condition under which a constraint is called conservative**.**
19. A particle is constrained to move on a sphere.
    (a) Write the constraint equation
    (b) State the type of constraint
20. How many generalized coordinates are required to describe a rigid body?
21. Explain why generalized coordinates need not be Cartesian coordinates.
22. For a system with 3 particles and 2 holonomic constraints, find the degrees of freedom.

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UNIT 2: DYNAMICAL EQUILIBRIUM

UNIT STRUCTURE

2.1 Learning Objectives

2.2 Introduction

2.3 Principle of Virtual Work

2.4 D'Alembert's Principle

2.5 Let Us Sum Up

2.6 Answers to Check Your Progress

2.7 Further Reading

2.8 Model Questions

2.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- define and explain dynamical equilibrium and its conditions
- apply Newton's laws of motion to systems in equilibrium
- draw and interpret free body diagrams for particles and rigid bodies
- analyze forces and torques to determine translational and rotational equilibrium
- solve numerical problems involving equilibrium under various force systems

2.2 INTRODUCTION

In the previous unit, we examined coordinate systems and their role in describing the position of a particle in two- and three-dimensional space. We also discussed constraints and their classifications, degrees of freedom, and the use of generalized coordinates. Building on these concepts, the present unit introduces the Principle of Virtual Work and D'Alembert's Principle**.** Although classical mechanics is fundamentally based on Newton's laws of motion, applying these laws directly becomes increasingly complex for systems subject to constraints, such as rigid bodies, pulley arrangements, and rolling motion. In many such situations, the forces of constraint are not known beforehand and are difficult to determine explicitly. To address these challenges, alternative formulations of mechanics have been developed. Among the most significant are the Principle of Virtual Work and D'Alembert's Principle. These approaches provide powerful methods for analyzing mechanical systems and serve as the basis of analytical mechanics, ultimately leading to Lagrange's equations. This unit first applies the Principle of Virtual Work to systems in equilibrium and then extends the discussion to moving systems through D'Alembert's Principle.

2.3 PRINCIPLE OF VIRTUAL WORK

Virtual Displacement

To remove the difficulties of constraints in mechanics, the concept of virtual work and virtual displacement are introduced.

A virtual displacement is defined as an infinitesimally small, imaginary displacement of a system that is consistent with the constraints of the system and occurs without the passage of time.

Important features of virtual displacement:

- It is imaginary, not an actual motion.
- It is finite
- It occurs at a fixed instant of time.
- It must satisfy all geometrical constraints of the system.
- It is denoted by δr.

Virtual displacement differs from real displacement which occurs due to motion over a finite time interval. Mathematically, it may be defined as

$$\delta \mathbf{r_i} = = \ \delta \mathbf{r_i}(q_1, q_2, \dots., q_n) \qquad (2.1)$$

Here, δr_i is an infinitesimal virtual displacement of i[th] particle of a system of N particles. It shows the displacement of position coordinates without change in time. The prefix δ is for virtual quantities and q_i are for the real quantities.

To understand the idea of virtual displacement, consider the example of a simple pendulum whose length $l(t)$ may vary with time. At any given instant t, the pendulum bob undergoes motion subject to the constraint imposed by the string.

During a small time, interval Δt, the actual (real) displacement $\Delta\vec{r}$ of the bob is not necessarily perpendicular to the force of constraint, which is the tension T in the string. This is because the length of the pendulum may change with time.

A virtual displacement $\delta\vec{r}$, on the other hand, is a hypothetical or imagined displacement that satisfies the constraint at a fixed instant of time. During the virtual displacement, the length $l(t)$ is treated as constant, and time is not allowed to change while the displacement is considered.

In contrast, the real displacement of the bob over the time interval Δt is given by

$$\Delta\vec{r}(t) = \vec{r}(t + \Delta t) - \vec{r}(t) \qquad (2.2)$$

where $\Delta\vec{r}(t)$ satisfies the constraint at time t, and $\vec{r}(t + \Delta t)$satisfies the constraint at the later time $t + \Delta t$.

Virtual Work

Virtual work is defined as the work done by a force during a small imaginary displacement of a system that is consistent with the constraints acting on the system. This imagined displacement is called a virtual displacement. It is an infinitesimally small

change in the configuration of the system that does not actually occur with time but is considered only for analysing the condition of equilibrium.

Consider a system consisting of N particles. If the system is in equilibrium, the resultant force acting on each particle must be zero. Thus, for the i^{th} particle,

$$\vec{F}_i = 0, i = 1,2, \dots, N \qquad (2.3)$$

If the particle undergoes a small virtual displacement $\delta\vec{r}_i$, the virtual work done by the force $\vec{F}_i$is given by

$$\delta W_i = \vec{F}_i \cdot \delta\vec{r}_i \qquad (2.4)$$

where $\vec{F}_i$ is the force acting on the particle and $\delta\vec{r}_i$ represents its virtual displacement. The dot product indicates that only the component of the force along the direction of displacement contributes to the work.

For a system of many particles, the total virtual work is obtained by summing the virtual work done by all the forces acting on the particles. In many constrained systems, forces of constraint such as tension in a string or the normal reaction of a surface do not perform virtual work because the virtual displacement is perpendicular to these forces. Therefore, their contribution to virtual work becomes zero. The concept of virtual work provides a convenient method for analysing mechanical systems. It forms the basis of the principle of virtual work, which states that a system is in equilibrium if the total virtual work done by all the forces acting on it for any permissible virtual displacement is zero.

Statement of Principle of Virtual Work

For a system in static equilibrium, the total force acting on the system is zero. So, the virtual work done on such a system due to an arbitrary virtual displacement must also vanish.

The total force F on the system is given by

$$\vec{F} = \vec{f} + \vec{F}^{\,a} = 0, \text{ for static equilibrium} \qquad (2.5)$$

Where,$\vec{F}$ represents the resultant force,$\vec{f}$ is the sum of the constraint forces, $\vec{F}^{\,a}$ denotes the applied force.

∴ Virtual work done by $\vec{F}$ is

$$\delta W = \vec{F} \cdot \delta\vec{r} = 0 \qquad (2.6)$$

Also, if the virtual work by the forces of constraints is also zero then

$$\vec{f}.\delta\vec{r} = 0 \qquad (2.7)$$

From eq (2.6), we observe that the virtual work done by the applied forces $\vec{F}^{a}$ also vanishes.So, the condition for static equilibrium becomes

$$\delta W_a = \vec{F}^{\,a} \cdot \delta\vec{r} = 0 \qquad (2.8)$$

For a system of n particles, we have

$$\delta W_a = \sum_i^n \vec{F}_i^{\,a} . \delta\vec{r} = 0 \qquad (2.9)$$

Equation (2.9) shows that for equilibrium of a system; the virtual work of the applied force is zero.

The Principle of Virtual Work states that 'for a system in equilibrium, the sum of the virtual work done by all applied forces during any allowable virtual displacement vanishes'

Significance of Principle of Virtual Work

The Principle of Virtual Work plays a crucial role in the study of mechanical systems. It enables us to determine the external force required to bring a moving system into a condition of equilibrium. In addition, it provides a convenient method for analysing systems with several forces and constraints without directly solving the equations of motion. The principle also simplifies the study of equilibrium in complex mechanical systems such as beams, frames, and machines. Furthermore, it forms the foundation for advanced formulations of mechanics, particularly Lagrangian mechanics, where the equations of motion of a system can be derived using generalized coordinates.

Conservative Forces and Potential Energy

If the forces acting on the system are conservative in nature**,** the work done by these forces depends only on the initial and final positions, not on the path taken.

A conservative force acting on the i^{th} particle can be expressed as the negative gradient of a scalar potential energy function V, given by

$$\vec{F}_i^{\,a} = -\nabla_i V \qquad (2.10)$$

where the potential energy V depends on the positions of the particles, $V = V(\mathrm{r}_1, \mathrm{r}_2, \ldots)$.

According to the Principle of Virtual Work, the algebraic sum of the virtual work done by all applied forces during any permissible virtual displacement must be zero. Mathematically, this condition is written as

$$\sum_i \vec{F}_i^{\,a} \cdot \delta\vec{r}_i = 0 \qquad (2.11)$$

Substituting the expression for conservative forces into this equation leads to the result

$$\delta V = 0 \qquad (2.12)$$

This implies that the system is in equilibrium when its potential energy assumes an extreme value. The condition of equilibrium can be mentioned as follows:

a) If the potential energy has a minimum value**,** the equilibrium is **stable.**

b) If the potential energy has a maximum value, the equilibrium is **unstable.**

c) When the potential energy remains unchanged with respect to the system's coordinates, the equilibrium is described as **neutral.**

Application of Principle of Virtual Work

The principle of virtual work is an important analytical tool in mechanics and is extensively used to study systems in equilibrium. One of its major advantages is that it enables us to determine equilibrium conditions without explicitly calculating constraint or reaction forces**,** which are often difficult to evaluate using Newton's laws. Since many constraint forces perform no virtual work, they automatically disappear from the analysis, thereby simplifying the problem considerably.

Applications include:

1) **Simple and Compound levers**:

   A lever is a rigid bar capable of rotating about a fixed point called the fulcrum. When forces act at different points on the lever, equilibrium can be analyzed using virtual displacement. If the lever undergoes a small virtual rotation, each applied force produces virtual work proportional to its moment arm.

   According to the principle of virtual work, the algebraic sum of the virtual works of all forces must be zero. This leads directly to the familiar law of the lever, which states that the clockwise and anticlockwise moments of the fulcrum balance each other.

   In compound levers, where several rigid bars are connected, the method becomes especially useful because it avoids calculating internal forces at the joints.

2) Pulley systems

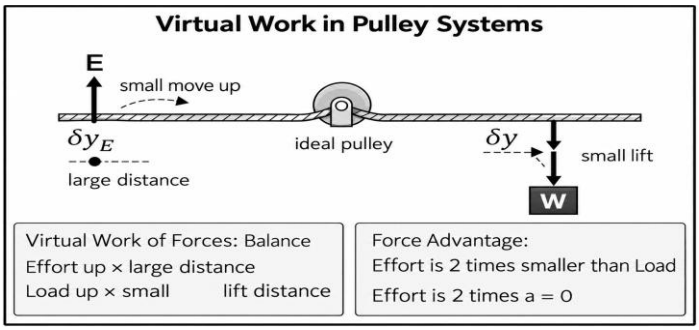


Figure 2.1: Illustration of the principle of virtual work in a pulley system where the effort moves through a larger distance while the load moves through a smaller distance, maintaining equilibrium.

Pulley arrangements are commonly used to lift loads using smaller applied forces. These systems often involve multiple strings and tensions, making direct force analysis complicated. Using the principle of virtual work, the equilibrium condition is obtained by considering a small virtual displacement of the load and relating it to the corresponding displacement of the applied force. Since the tension in an ideal string does no net virtual work internally, the relation

between effort and load can be derived easily without determining individual tensions

3) **Inclined Planes**

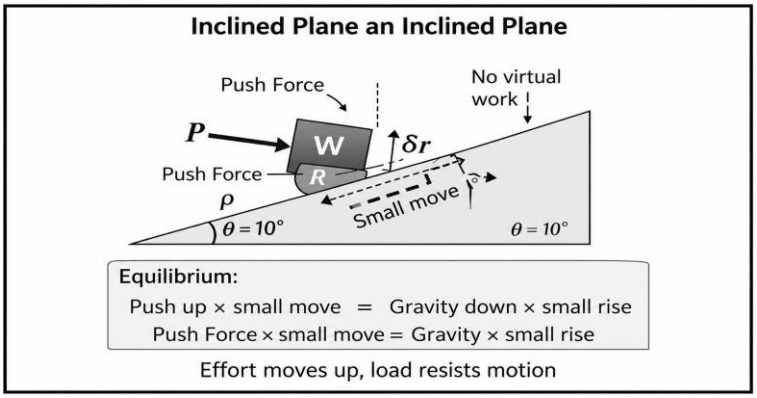


Figure 2.2: Application of the principle of virtual work on an inclined plane showing balance between the applied push force and the component of gravitational force during a small virtual displacement.

When a body rests on or moves along an inclined plane, reaction forces act perpendicular to the surface. Determining these forces directly may be unnecessary when studying equilibrium. By applying a small virtual displacement along the direction of allowed motion, the normal reaction performs no virtual work because it acts perpendicular to the displacement. Consequently, only the component of gravitational force along the plane contributes to virtual work. This simplifies the determination of equilibrium conditions and required applied forces.

4) Equilibrium of rigid bodies

The principle of virtual work is particularly powerful in analyzing rigid bodies subjected to several forces and torques. For a rigid body undergoing a small virtual displacement or rotation, equilibrium requires that the total virtual work done by all external forces and moments must vanish. This approach is widely used instructural analysis, beam and frame problems, mechanical linkages etc. Instead of resolving forces and torques separately, the equilibrium condition is obtained directly through the virtual work equation.

Limitation of Principle of Virtual Work

- The Principle of Virtual Work is directly applicable only to systems in equilibrium**.**
- It does not account for acceleration and therefore cannot be applied as such to systems in motion.
- In dynamical systems, inertial effects play an important role, which are ignored in this principle.
- Hence, the Principle of Virtual Work alone is insufficient for the analysis of moving systems.

- To extend the concept to systems undergoing motion, an additional principle that includes the effect of inertia is required.

2.4 D'ALEMBERT'S PRINCIPLE

We saw that the principle of virtual work only deals with the statics of a system of particles. However, to deal with the general motion of a system, a new principle was developed by D'Alembert known as D'Alembert's Principle.

In a system of n particles, the total force $\vec{F}_i$on the ith particle can be divided into

1) The force of constraint, $(\vec{f}_i)$
2) The applied force $(\vec{F}_i^a)$

From Newton's 2nd law of motion, we have

$$\vec{F}_i = \vec{F}_i^a + \vec{f}_i = \dot{\vec{p}}_i \qquad (2.11)$$

Where $\dot{\vec{p}}_i = \frac{d\vec{p}_i}{dt}$; is the linear momentum of the i^{th}particle of the system.

Equation (2.10) can be written as:

$$\vec{F}_i - \dot{\vec{F}}_i = 0 \qquad \text{i} = 1,2,\dots,\text{N} \qquad (2.12)$$

The above equations mean that for a system in equilibrium, the force acting on a particle is equal to the actual force plus a reverse effective force $\dot{\vec{p}}_i$**.**

The workdone in a virtual displacement of the system is given by

$$\sum_{i=1}^{n} \vec{F}_i \cdot \delta\vec{r}_i + \sum_{i=1}^{n} \vec{f}_i \cdot \delta\vec{r}_i = \sum_{i=1}^{n} \dot{\vec{p}}_i \cdot \delta\vec{r}_i \qquad (2.13)$$

Restricting oversleeves to the systems for which the virtual work by all the constraints forces is zero, i.e.

$$\sum_{i=1}^{n} \vec{f}_i \cdot \delta\vec{r}_i = 0 \qquad (2.14)$$

Now, from equation (2.13), we have

$$\sum_{i=1}^{n} \vec{F}_i \cdot \delta\vec{r}_i = \sum_{i=1}^{n} \dot{\vec{p}}i \cdot \delta\vec{r}_i$$

$$\sum_{i=1}^{n} (\vec{F}_i - \overrightarrow{p_i}) \cdot \delta\vec{r}_i = 0 \qquad (2.15)$$

This is known as D'Alembert's Principle. The total virtual work of applied forces and inertial forces vanishes for a dynamical system.

Features of D'Alembert's Principle

D'Alembert's Principle expresses the dynamics of a system through **a** single equation of motion, in contrast to Newtonian mechanics, which requires 3N separate equations for a system of N particles.

- In this formulation, forces of constraint do not explicitly appear**.** Only the applied (external) forces need to be specified. The inclusion of the inertial force acting opposite to acceleration effectively transforms a problem of dynamics into one resembling static equilibrium.
- The principle is applicable to a wide class of systems, **including** rheonomic and scleronomic systems, provided they are holonomic or homogeneous non-holonomic. Such systems are commonly referred to as D'Alembert systems.
- Explicit knowledge of the constraint equations is generally unnecessary, except for determining the virtual displacements. The principle provides a unified framework for mechanical analysis, as many formulations of mechanics can be regarded as alternative mathematical expressions of D'Alembert's Principle. An exception is Lagrange's equations of the first kind, which are more general and applicable to all holonomic and non-holonomic systems.
- D'Alembert's Principle is conceptually simpler than variational principles since it does not involve time integration. However, a limitation arises because the virtual work of the inertial forces cannot be expressed as a single scalar function, making the method less convenient when using curvilinear coordinates**.**

Application Of D'Alembert's Principle

MOTION OF A SINGLE PARTICLE

Consider a particle of mass $m$moving under the action of an external force $\vec{F}$. According to Newton's second law,

$$\vec{F} = m\vec{a} \qquad (2.16)$$

D'Alembert's principle introduces an inertial force equal to $-m\vec{a}$, allowing the equation to be rewritten as

$$\vec{F} - m\vec{a} = 0 \qquad (2.17)$$

This expression resembles the condition of equilibrium. Hence, the motion of the particle can be analyzed using equilibrium methods.

Multiplying by a virtual displacement $\delta\vec{r}$,

$$\left(\vec{F} - m\vec{a}\right) \cdot \delta\vec{r} = 0 \qquad (2.18)$$

Thus, the equation of motion arises naturally without directly solving Newton's force equations.

Numerical

Two heavy particles of weights W_1 and W_2 are connected by a light inextensible string and hang over a fixed smooth circular cylinder of radius R, the axis of which is horizontal. Find the condition of equilibrium by applying the principle of virtual work.

Solution: Let the system consist of two particles of weights $W_1$and W_2, constrained to move along a circular path of radius R.

Let the angular positions of the particles be $\theta$and ϕ, respectively.

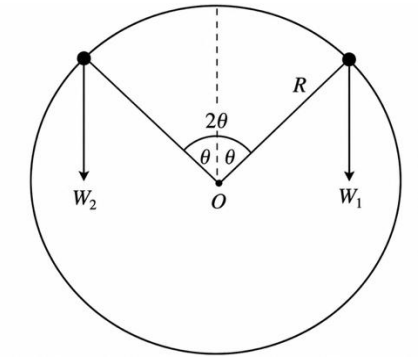


Figure 2.3: Two particles of weight $W_1$and W_2 constrained to move on circular path.

According to the principle of virtual work

$$\sum_{i=1}^{N} F_i.\delta r_i = 0 \qquad (2.19)$$

Since the particles are constrained to move on the same circular arc,

$$\delta r_1 = R\, d\theta, \delta r_2 = R\, d\phi \qquad (2.20)$$

The total virtual work done by the forces acting on the system is given by

$$W_1 \sin\theta\, \delta r_1 + W_2 \sin\phi\, \delta r_2 = 0 \qquad (2.21)$$

Substituting for the virtual displacements, we have

$$W_1 \sin\theta\, \delta\theta + W_2 \sin\phi\, \delta\phi = 0 \qquad (2.22)$$

Due to the geometrical constraint of the system, the sum of the angular displacements remains constant:

$$\theta + \phi = \text{constant} \qquad (2.23)$$

Hence, $$\delta\theta + \delta\phi = 0 \Rightarrow \delta\phi = -\delta\theta \qquad (2.24)$$

Substituting this relation into the virtual work equation, we obtain

$$(W_1 sin\,\theta - W_2 sin\,\phi)\delta\theta = 0$$

Since the system is in equilibrium ($\delta\theta \neq 0$), the required condition is

$$W_1 sin\,\theta = W_2 sin\,\phi \qquad (2.25)$$

Or

$$\frac{W_1}{W_2} = \frac{sin\phi}{sin\theta}$$

CHECK YOUR PROGRESS

Multiple Choice Questions (MCQs)

Q1.The configuration space of a system of N free particles have dimension

a) N

b) $2N$

c) $3N$

d) $6N$

Q2. A system represented by a point in a six-dimensional configuration space has

a) two degrees of freedom

b) three degrees of freedom

c) four degrees of freedom

d) six degrees of freedom

Q3. A virtual displacement is characterized by

a) finite change occurring over time

b) imaginary displacement at a fixed instant

c) displacement along the actual path of motion

d) displacement requiring force balance

Q4. Which of the following best distinguishes virtual displacement from real displacement?

a) Virtual displacement depends on time

b) Real displacement is imaginary

c) Virtual displacement satisfies constraints at a fixed time

d) Real displacement ignores constraints

Q5. The symbol used to represent virtual displacement is

a) $d\vec{\mathrm{r}}$

b) $\Delta\vec{\mathrm{r}}$

c) $\delta\vec{\mathrm{r}}$

d) $\vec{\mathrm{r}}$

Q6. Name the scalar function associated with conservative forces.

Q7. What type of equilibrium corresponds to constant potential energy?

Q8. What principle states that virtual work vanishes for equilibrium systems?

Q9. State one condition that a virtual displacement must satisfy.

Q10.How many degrees of freedom does a system of two free particles possess?

Q11.Why are generalized coordinates useful in constrained systems?

Q12.Distinguish between real displacement and virtual displacement with respect to time.

Q13. Why does the Principle of Virtual Work help eliminate the explicit calculation of constraint forces?

Q14. Why is the Principle of Virtual Work primarily applicable to systems in equilibrium?

Q15. Under what condition do constraint forces perform zero virtual work?

Q16. Why can conservative forces be expressed as the negative gradient of potential energy?

Q17.Can a system be in equilibrium if its potential energy is maximum? Explain.

Q18.What is meant by the *reverse inertial force* in D'Alembert's Principle?

2.5 LET US SUM UP

- The idea of configuration space was explained as a multidimensional space representing the complete state of a system based on its degrees of freedom.
- The concepts of virtual displacement and virtual work were defined, emphasizing that virtual displacement is an imagined displacement occurring at a fixed instant and consistent with constraints.
- The Principle of Virtual Work was formulated, stating that the total virtual work done by applied forces vanishes for a system in equilibrium.
- Virtual work principle helps eliminate the explicit calculation of constraint forces, simplifying equilibrium analysis.
- The role of conservative forces was related to the extreme values of potential energy**.**
- The conditions for stable, unstable, and neutral equilibrium were identified based on the nature of the potential energy extremum.
- The **limitations** of the Principle of Virtual Work were highlighted, noting its applicability only to systems in equilibrium.
- To analyze systems in motion, D'Alembert's Principle was introduced by incorporating inertial forces into the virtual work framework.
- The advantages of D'Alembert's Principle include the elimination of constraint forces and the reduction of dynamical problems to static-like equilibrium conditions.

- Applications of D'Alembert's Principle were illustrated through systems such as the spherical pendulum**,** Atwood's machine**,** and the simple pendulum.
- This unit provides a foundation for advanced topics in analytical mechanics**,** leading toward Lagrange's equations**.**

✓2.6 ANSWERS TO CHECK IN PROGRESS

Answer to Q1: $3N$
Answer to Q2: d) six degrees of freedom
Answer to Q3: b) imaginary displacement at a fixed instant
Answer to Q4: Virtual displacement satisfies constraints at a fixed time
Answer to Q5: c) $\delta\vec{r}$
Answer to Q6: Potential energy function $\boldsymbol{V}$
Answer to Q7: Neutral equilibrium
Answer to Q8: Principle of Virtual Work
Answer to Q9: It must satisfy the geometrical constraints of the system at a fixed instant of time.
Answer to Q10: Six
Answer to Q11: They reduce the number of variables by incorporating constraints and describe the system using independent coordinates only**.**
Answer to Q12: A real displacement occurs over a finite time interval during actual motion, whereas a virtual displacement is an infinitesimal imagined displacement considered at a fixed instant of time.
Answer to Q13: The Principle of Virtual Work eliminates constraint forces because ideal constraint forces do no virtual work for allowed virtual displacements.
Answer to Q14: The Principle of Virtual Work applies only to equilibrium systems because the total virtual work done by all applied forces must vanish when the system is in static balance.
Answer to Q15: Constraint forces perform zero virtual work when the virtual displacement is consistent with the constraints of the system.
Answer to Q16: Conservative forces can be expressed as the negative gradient of potential energy because the work done by such forces depends only on the initial and final positions.
Answer to Q17: Yes, a system can be in equilibrium at maximum potential energy, but such equilibrium is unstable since small disturbances increase displacement from equilibrium.
Answer to Q18: The reverses inertial force is an imaginary force equal to $(-m\mathbf{a})$ introduced to convert a dynamic system into an equivalent static system.

🔍2.7 FURTHER READING

- Upadhyaya, J. C. (2010). *Classical Mechanics*. Himalayan Publishing House, Mumbai.
- Schaum's Outline Series**.** (1990). *Schaum's Outline of Classical Mechanics*. McGraw-Hill Education, New York.

- Morin, D. (2008). *Introduction to Classical Mechanics*. Cambridge University Press, Cambridge.
- Gupta, A. B**.** (2003). *Classical Mechanics and Properties of Matter*. Books and Allied (P) Ltd., Kolkata.
- Goldstein, H., Poole, C. P., and Safko**,** J. L**.** (2002). *Classical Mechanics (3rd Edition)*. Pearson Education / Addison-Wesley, Singapore.

⍰2.8 MODEL QUESTIONS

1. Using the Principle of Virtual Work, derive the condition of equilibrium for a system of particles subject to constraints. Illustrate the principle with a suitable mechanical system such as a lever, pulley system, or inclined plane.
2. Explain how D'Alembert's Principle is formulated from Newton's second law. Derive the corresponding virtual-work equation and discuss how the principle facilitates the analysis of dynamical systems.
3. Discuss the D' Alembert's Principle.
4. An incline making an angle θ with the horizontal is given a horizontal acceleration 'a' in the verticle plane of the incline to prevent sliding of a frictionless block on the incline. Find the value of θ.
5. Two particles of masses are located on a frictionless double incline and connected by an inextensible massless string passing over a smooth peg. Use the principle of virtual work to show that for equilibrium, we must have

$$\frac{\sin\alpha_1}{\sin\alpha_2} = \frac{m_1}{m_2}$$

Where, α_1 and α_2 are the angles of the incline. Apply D'Alembert's principle to describe the motion.

*** *** ***

UNIT 3: LAGRANGIAN DYNAMICS I

UNIT STRUCTURE

3.1 Learning Objectives
3.2 Introduction
3.3 Lagrange's Equation of Motion
3.4 Applications of Lagrange's Equation
3.5 Let Us Sum Up
3.6 Answers to Check your Progress
3.7 Further Reading
3.8 Model Questions

3.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- recall the ideas of generalized coordinates, constraints, and the Lagrangian
- explain the derivation and physical significance of Lagrange's equations of motion
- distinguish between Newtonian and Lagrangian formulations of mechanics
- apply Lagrange's equations to systems with single and multiple degrees of freedom
- analyze mechanical systems with constraints using the Lagrangian approach
- solve numerical problems involving applications of Lagrange's equations of motion.

3.2 INTRODUCTION

In the previous unit, the role of virtual displacement and virtual work was examined as an effective method for analyzing mechanical equilibrium and constrained motion. The principle of virtual work provided a way to study equilibrium conditions without explicitly evaluating the forces of constraint. This framework naturally led to D'Alembert's principle, where dynamical problems are transformed into equivalent equilibrium problems by introducing inertial effects. These ideas provide the conceptual basis for the development of Lagrange's formulation of dynamics.

Using these foundational ideas, the present unit develops Lagrange's equations of motion as a generalized analytical framework for describing dynamical systems. The unit presents the derivation of Lagrange's equations from D'Alembert's Principle, with emphasis on the role of generalized coordinates and generalized forces. This unit emphasizes the advantages of the Lagrangian approach in analyzing constrained systems and complex mechanical configurations. The theory is supplemented with a variety of applications and solved numerical problems, enabling the learner to develop both conceptual understanding and problem-solving skills. By the end of this unit, learners will be able to apply Lagrange's equations effectively to a wide range of mechanical systems and appreciate their significance in analytical mechanics.

3.3 LAGRANGE'S EQUATION OF MOTION

Lagrange's equations provide a generalized mathematical description of mechanical motion by expressing the dynamics of a system in terms of independent generalized coordinates. For a system with generalized coordinate q_i, the Lagrange equation is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \qquad (3.1)$$

where $L = T - V$is the Lagrangian of the system, $T$is the kinetic energy, $V$is the potential energy, $q_i$is the generalized coordinate, and $\dot{q}_i$is the generalized velocity.

A) Derivation of Lagrange's Equation from D'Alembert's Principle

The Lagrange equation introduced in equation (3.1) can be derived from the D'Alembert's Principle. D'Alembert's principle reformulates a dynamical problem into an equivalent equilibrium-like form by introducing inertial terms together with the applied forces, so that the total virtual work vanishes for all admissible virtual displacements. This formulation avoids the direct computation of constraint forces and makes it possible to describe the motion through a set of independent generalized coordinates. By expressing the virtual displacements in terms of these coordinates, the dynamical equations can be systematically transformed into the Lagrangian form.

Consider, a system of N particles. Using generalized coordinates, the transformation equation for the position vectors of the particles is

$$\vec{r}_i = \vec{r}_i(q_1, q_2, \dots, q_k, \dots, q_n, t) \qquad (3.2)$$

Where$t$ is the time and $q_k(k = 1,2, \dots, n)$ are the generalized coordinates.

Differentiating eq (3.2) with respect to t, we obtain the velocity of the i^{th}particle,

$$\frac{d\vec{r}_i}{dt} = \frac{\partial \vec{r}_i}{\partial q_1}\frac{dq_1}{dt} + \frac{\partial \vec{r}_i}{\partial q_2}\frac{dq_2}{dt} + \cdots + \frac{\partial \vec{r}_i}{\partial q_k}\frac{dq_k}{dt} + \cdots + \frac{\partial \vec{r}_i}{\partial q_n}\frac{dq_n}{dt} + \frac{\partial \vec{r}_i}{\partial t}$$

$$or, \qquad \vec{v}_i = \dot{\vec{r}} = \sum_{k=1}^{n} \frac{\partial \vec{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial \vec{r}_i}{\partial t} \qquad (3.3)$$

where, $\dot{q}_k$ are the generalized velocities.

The virtual displacement is given by

$$\delta\vec{r}_i = \frac{\partial \vec{r}_i}{\partial q_1}\delta q_1 + \frac{\partial \vec{r}_i}{\partial q_2}\delta q_2 + \cdots + \frac{\partial \vec{r}_i}{\partial q_k}\delta q_k + \cdots + \frac{\partial \vec{r}_i}{\partial q_n}\delta q_n$$

Or $$\delta\vec{r}_i = \sum_{k=1}^{n} \frac{\partial \vec{r}_i}{\partial q_k}\delta q_k \qquad (3.4)$$

since, by definition, virtual displacement does not depend on time.

According to D'Alembert's Principle

$$\sum_{i=1}^{N}(\vec{F}_i - \dot{\vec{P}}_i)\cdot\delta\vec{r}_i = 0$$

Now $$\sum_{i=1}^{N}\vec{F}_i\cdot\delta\vec{r}_i = \sum_{i=1}^{N}\vec{F}_i\cdot\sum_{k=1}^{n}\frac{\partial\vec{r}_i}{\partial q_k}\delta q_k$$

$$= \sum_{k=1}^{n}\sum_{i=1}^{N}\left[\vec{F}_i\cdot\frac{\partial\vec{r}_i}{\partial q_k}\right]\delta q_k$$

$$= \sum_{k=1}^{n} G_k\ \delta q_k \qquad (3.5)$$

Where $G_k = \sum_{i=1}^{N}\vec{F}_i\cdot\frac{\partial\vec{r}_i}{\partial q_k} = \sum_{i=1}^{N}\left[F_{xi}\frac{\partial x_i}{\partial q_k} + F_{yi}\frac{\partial y_i}{\partial q_k} + F_{zi}\frac{\partial z_i}{\partial q_k}\right]$ (3.6)

These are known as the components of generalized force associated with the generalized coordinate q_k.

Since generalized coordinates are not necessarily measured in units of length, the dimension of generalized force $G_k$may differ from that of ordinary force.

Now

$$\sum_{i=1}^{N}\dot{\vec{P}}_i\cdot\delta\vec{r}_i = \sum_{i=1}^{N}m_i\ddot{\vec{r}}_i\cdot\sum_{k=1}^{n}\frac{\partial\vec{r}_i}{\partial q_k}\delta q_k$$

$$= \sum_{k=1}^{n}\left[\sum_{i=1}^{N}m_i\ddot{\vec{r}}_i\cdot\frac{\partial\vec{r}_i}{\partial q_k}\right]\delta q_k \qquad (3.7)$$

$$\sum_{i=1}^{N}m_i\ddot{\vec{r}}_i\cdot\frac{\partial\vec{r}_i}{\partial q_k} = \sum_{i=1}^{N}\frac{d}{dt}\left(m_i\dot{\vec{r}}_i\cdot\frac{\partial\vec{r}_i}{\partial q_k}\right) - m_i\dot{\vec{r}}_i\cdot\frac{d}{dt}\left(\frac{\partial\vec{r}_i}{\partial q_k}\right) \qquad (3.8)$$

Since, in equation (3.8)

$$\frac{d}{dt}\left(\frac{\partial\vec{r}_i}{\partial q_k}\right) = \frac{\partial}{\partial q_k}\left(\frac{d\vec{r}_i}{dt}\right) = \frac{\partial\vec{v}_i}{\partial q_k} \qquad (3.9)$$

And

$$\frac{\partial\vec{r}_i}{\partial q_k} = \frac{\partial\vec{v}_i}{\partial\dot{q}_k} \qquad (3.10)$$

Thus

$$\sum_{i=1}^{N} m_i \dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k} = \sum_{i=1}^{N} \left[\frac{d}{dt}\left(m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial \dot{q}_k}\right) - m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial q_k}\right] \quad (3.11)$$

Substituting in equation (3.6), we have

$$\sum_{i=1}^{N} \dot{\vec{P}}_i \cdot \delta \vec{r}_i = \sum_{k=1}^{n} \sum_{i=1}^{N} \left[\frac{d}{dt}\left(m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial \dot{q}_k}\right) - m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial q_k}\right] \delta q_k$$

$$= \sum_{k=1}^{n} \left[\frac{d}{dt}\left\{\frac{\partial}{\partial \dot{q}_k}\left(\sum_{i=1}^{N} \frac{1}{2} m_i (\vec{v}_i \cdot \vec{v}_i)\right)\right\} - \frac{\partial}{\partial q_k}\left(\sum_{i=1}^{N} \frac{1}{2} m_i (\vec{v}_i \cdot \vec{v}_i)\right)\right] \delta q_k$$

Where,

$T = \sum_{i=1}^{N} \frac{1}{2} m_i (\vec{v}_i \cdot \vec{v}_i) = \sum_{i=1}^{N} \frac{1}{2} m_i v_i^2$ is the kinetic energy of the system**.**

Substituting $\sum_i \vec{F}_i . \delta r_i$ and $\sum_i \dot{\vec{P}}_i . \overrightarrow{\delta r_i}$; from equation (3.7), D'Alembert's principle becomes

$$\sum_{k=1}^{n} \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} - G_k\right] \delta q_k = 0 \quad (3.12)$$

Since the constraints are holonomic, the virtual displacements δq_k are independent. Therefore, the coefficient of each δq_k must vanish:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} - G_k = 0$$

Or

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = G_k \quad (3.13)$$

This is the general form of Lagrange's Equation.

B) Lagrange's Equation for Conservative System

In conservative systems, the applied forces can be represented through a potential energy function, allowing the equations of motion to be written in a simplified energy-based form.

$$\vec{F}_i = -\nabla_i V = -\left(\hat{\imath} \frac{\partial V}{\partial x_i} + \hat{\jmath} \frac{\partial V}{\partial y_i} + \hat{k} \frac{\partial V}{\partial z_i}\right) \quad (3.14)$$

Where, the gradient operator for the i^{th} particle is

$$\nabla_i = \hat{\imath} \frac{\partial}{\partial x_i} + \hat{\jmath} \frac{\partial}{\partial y_i} + \hat{k} \frac{\partial}{\partial z_i}$$

From equation (3.5), the generalized force G_k can be written as

$$G_k = -\sum_{i=1}^{N}\left[\frac{\partial V}{\partial x_i}\frac{\partial x_i}{\partial q_k} + \frac{\partial V}{\partial y_i}\frac{\partial y_i}{\partial q_k} + \frac{\partial V}{\partial z_i}\frac{\partial z_i}{\partial q_k}\right] \qquad (3.15)$$

Clearly, the right-hand side represents the partial derivative of $-V$ with respect to q_k. Hence,

$$G_k = -\frac{\partial V}{\partial q_k} \qquad (3.16)$$

Equation (3.4) assumes the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = -\frac{\partial V}{\partial q_k} \qquad (3.17)$$

Or $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial (T-V)}{\partial q_k} = 0 \qquad (3.18)$$

Since the scalar potential V depends only on the generalized coordinates q_k and not on the generalized velocities $\dot{q}_k$, the above equation can be written as

$$\frac{d}{dt}\left[\frac{\partial (T-V)}{\partial \dot{q}_k}\right] - \frac{\partial (T-V)}{\partial q_k} = 0 \qquad (3.19)$$

We now define a new function

$$L = T - V$$

Here, 'L' is known as the Lagrangian of the system

Thus, equation (3.19) becomes

$$\boxed{\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_k}\right] - \frac{\partial L}{\partial q_k} = 0}$$

are called Lagrange's equations of motion for a conservative system. For a system having (n) generalized coordinates, there are n such equations— one corresponding to each generalized coordinate q_k. These equations together completely describe the motion of the system.

To apply and solve Lagrange's equations, it is essential to first determine the Lagrangian function, defined as

$$L = T - V,$$

where T is the kinetic energy and V is the potential energy of the system. Both T and V must be expressed in terms of the chosen generalized coordinates q_k and generalized velocities $\dot{q}_k$. After constructing the Lagrangian in terms of the selected coordinates and velocities, the dynamical equations follow directly through the Lagrange formalism without separately resolving individual forces. This approach

greatly simplifies the analysis of complex mechanical systems and is particularly useful when constraints are present.

C) Kinetic Energy in Generalized Coordinates

The kinetic energy T expression can be reformulated in terms of generalized coordinates by expressing the particle velocities through coordinate transformations.

In Cartesian coordinates, the kinetic energy of a system of n particles is given by

$$T = \sum_{i=1}^{n} \frac{1}{2} m_i v_i^2 \qquad (3.20)$$

Since velocity is the time derivative of the position vector,

$$T = \sum_{i=1}^{n} \frac{1}{2} m_i \dot{\vec{r}}_i . \dot{\vec{r}}_i \qquad (3.21)$$

This represents the Newtonian form of kinetic energy.

The position vector of the i^{th}particle can be expressed in terms of generalized coordinates as

$$\vec{r}_i = \vec{r}_i(q_k, t) \qquad (3.22)$$

Therefore, its velocity becomes

$$\dot{\vec{r}}_i = \sum_{k=1}^{n} \frac{\partial \vec{r}_i}{\partial q_k} \dot{q}_k + \frac{\partial \vec{r}_i}{\partial t} \qquad (3.23)$$

Substituting equation (3.23) into equation (3.21), we obtain

$$T = \sum_{i=1}^{n} \frac{1}{2} m_i \left(\sum_{k=1}^{n} \frac{\partial \vec{r}_i}{\partial q_k} \dot{q}_k + \frac{\partial \vec{r}_i}{\partial t} \right) \cdot \left(\sum_{l=1}^{n} \frac{\partial \vec{r}_i}{\partial q_l} \dot{q}_l + \frac{\partial \vec{r}_i}{\partial t} \right) \qquad (3.24)$$

After expanding and rearranging, the kinetic energy can be written in the form

$$T = M_0 + \sum_k M_k \dot{q}_k + \frac{1}{2} \sum_{k,l} M_{kl} \dot{q}_k \dot{q}_l \qquad (3.25)$$

Where $M_{kl} = \sum_i m_i \left(\frac{\partial \vec{r}_i}{\partial q_k} \cdot \frac{\partial \vec{r}_i}{\partial q_l} \right)$ is the quadratic coefficient,

$M_k = \sum_i m_i \left(\frac{\partial \vec{r}_i}{\partial t} \cdot \frac{\partial \vec{r}_i}{\partial q_k} \right)$ is the linear coefficient, and

$M_0 = \sum_i \frac{1}{2} m_i \left(\frac{\partial \vec{r}_i}{\partial t} \cdot \frac{\partial \vec{r}_i}{\partial t} \right)$ does not affect Lagrange's and often ignored

If the constraints are time-independent, then

$$\frac{\partial \vec{r}_i}{\partial t} = 0$$

Thus $M_0 = 0, M_k = 0$

and equation (3.22) reduces to

$$T = \frac{1}{2}\sum_{k,l} M_{kl}\, \dot{q}_k \dot{q}_l \qquad (3.26)$$

This expression shows that the kinetic energy is a homogeneous quadratic function of the generalized velocities**.**

D) Cyclic Coordinates

A generalized coordinate is called cyclic when the Lagrangian remains independent of that coordinate explicitly.

A coordinate $q_k$is called cyclic if

$$\boxed{\frac{\partial L}{\partial q_k} = 0} \qquad (3.27)$$

From Lagrange's Equation, we have

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_k}\right] - \frac{\partial L}{\partial q_k} = 0 \qquad (3.28)$$

If $\frac{\partial L}{\partial q_k} = 0$, the equation (3.25) becomes

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_k}\right] = 0$$

$$\Longrightarrow \frac{d}{dt}(P_k) = 0$$

$$\Longrightarrow \dot{P}_k = 0$$

Integrating, we have $\boxed{P_k = \text{Constant}}$

This result shows that the generalized momentum corresponding to a cyclic coordinate remains constant throughout the motion. Thus, the presence of a cyclic coordinate leads to a conservation law, which forms the basis of the general theory of conservation in classical mechanics.

3.4 APPLICATIONS OF LAGRANGIAN EQUATION

1. Equation of Motion of Simple Pendulum

Let us examine a pendulum consisting of a point mass$m$ suspended by a light inextensible string of length $l$and constrained to oscillate in a vertical plane.Let θbe the angular displacement of the bob from the vertical equilibrium position.

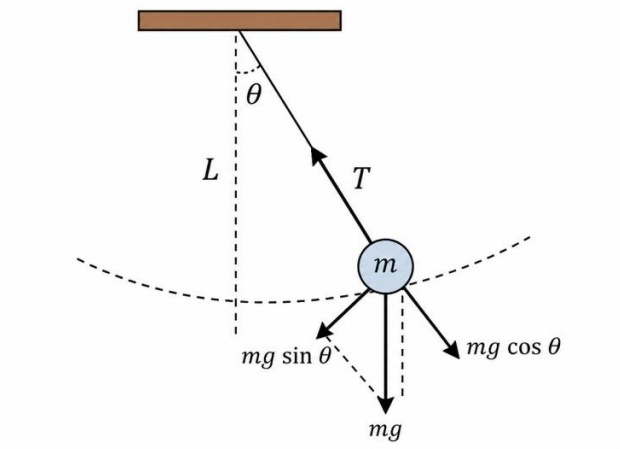


Figure 3.1: Simple pendulum illustrating angular displacement θ, string tension T, and gravitational force acting on the bob. (Source:Classical Mechanics by J.C Upadhayay)

The linear displacement of the bob along the arc is related to the angular displacement by

$$s = l\theta \tag{3.29}$$

Since$\theta = \frac{\text{arc length}}{\text{radius}} = \frac{s}{l}$

The velocity vector of the bob is $\vec{v} = \frac{d\vec{r}}{dt}$

Since motion occurs along the tangential direction,

$$\vec{v} = \frac{ds}{dt} = l\dot{\theta}$$

The kinetic energy of the pendulum is

$$T = \frac{1}{2}m\vec{v}\cdot\vec{v}$$

Substituting $\vec{v} = l\dot{\theta}$,

$$T = \frac{1}{2}ml^2\dot{\theta}^2 \tag{3.30}$$

When the bob is at the lowest position, its potential energy is taken as zero.

At angular displacement θ, the vertical rise of the bob is

$$\mathrm{h} = \mathrm{l} - \mathrm{l}\cos\theta$$

or $$h = l(1 - \cos\theta)$$

Therefore, the potential energy becomes

$$V = mgh$$
$$V = \quad mg\, l(1 - \cos\theta) \qquad (3.31)$$

Hence, the kinetic energy of the pendulum is

$$T = \frac{1}{2}mv^2 = \frac{1}{2}ml^2\dot{\theta}^2 \qquad (3.32)$$

When the bob is at the lowest position, its potential energy is taken as zero. At an angular displacement θ, the vertical rise of the bob is given by

$$h = (l - lcos\theta) = l(1 - \cos\theta)$$

Therefore, the potential energy of the system is

$$V = mgh = mgl(1 - \cos\theta) \qquad (3.33)$$

Now, The Lagrangian L is defined as

$$L = T - V$$

Substituting the expressions for kinetic and potential energies from eq. (3.29) and (3.30), we have

$$L = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1 - \cos\theta) \qquad (3.34)$$

Now, Lagrange's equation for the generalized coordinate θ is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \qquad (3.35)$$

Differentiating equation (3.31) w.r.t $\dot{\theta}$and θ, we have

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}, \quad \frac{\partial L}{\partial \theta} = -mgl\sin\theta$$

Substituting into Lagrange's equation (3.32), we have

$$\frac{d}{dt}\left[ml^2\dot{\theta}\right] + mglsin\theta$$

$$ml^2\ddot{\theta} + mgl\sin\theta = 0$$

$$\boxed{\ddot{\theta} + \frac{g}{l}\sin\theta = 0} \qquad (3.36)$$

The above relation represents the governing dynamical equation for the oscillatory motion of the pendulum.

For small oscillations, $\sin\theta \approx \theta$
Then equation (3.36) becomes

$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

This equation represents simple harmonic motion, with angular frequency and time- period given by:

$$\omega = \sqrt{\frac{g}{l}}\ ;\ T = 2\pi\sqrt{\frac{l}{g}}$$

Result:

- Equation of motion of a simple pendulum:$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$
- For small oscillations**:**

$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

- Angular frequency$\omega = \sqrt{\frac{g}{l}}$
- Time period$T = 2\pi\sqrt{\frac{l}{g}}$

2. Equation of Motion of Atwood's Machine

Atwood's arrangement consists of two masses connected by a light string passing over an ideal pulley, providing a simple constrained mechanical system with one independent coordinate. It is used to study uniformly accelerated motion and to illustrate the application of Newtonian and Lagrangian mechanics. It has one degree of freedom

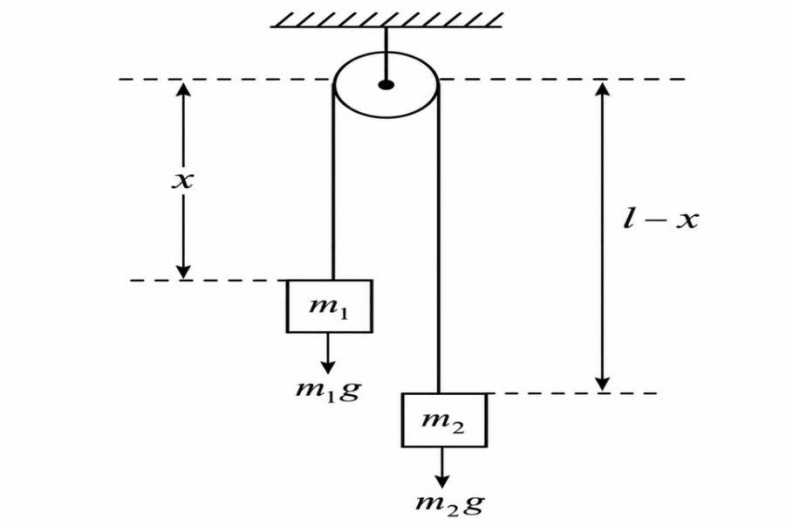


Figure 3.2: Atwood Machine: Derivation of Equation of Motion by Lagrange (Source: Classical Mechanics by J.C Upadhayay)

It is an example of a conservative system with holonomic constraint. Let, the two masses be m_1 and m_2 connected by an inextensible string of length l. Suppose x be the varying vertical distance from the pulley to the mass m_1. The mass m_2 be at a distance $(l - x)$ from the pulley.

The velocities of the two masses are given by

$$v_1 = \frac{dx}{dt} = \dot{x} \qquad \text{and} \quad v_2 = \frac{d(l-x)}{dt} = -\dot{x} \tag{3.37}$$

Therefore, Kinetic Energy is given by

$$T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{x}^2$$

$$T = \frac{1}{2}(m_1 + m_2)\dot{x}^2 \tag{3.38}$$

Thus, there is only one independent coordinate x.

Now, Potential energy of $m_1\ and\ m_2$is given by

$$V_1 = m_1gx \text{ and } V_2 = m_2g(l - x)$$

Therefore, Total potential energy of the system is given by:

$$V = V_1 + V_2 = m_1gx + m_2g(l - x) \tag{3.39}$$

Lagrangian of the system is given by

$$L = T - V \tag{3.40}$$

Substituting $T\ and\ V$ in the above equation (3.40), we have

$$L = \frac{1}{2}(m_1 + m_2)\dot{x}^2 - m_1gx - m_2g(l - x)$$

Now, Lagrange's equation of motion for generalized coordinate x is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \tag{3.41}$$

Equating the derivatives in the above equation, we have

$$\frac{\partial L}{\partial \dot{x}} = \frac{\partial}{\partial x}\left(\frac{1}{2}(m_1 + m_2)\dot{x}^2 - m_1gx - m_2g(l - x)\right) = (m_1 + m_2)\dot{x}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{d}{dt}\{(m_1 + m_2)\dot{x}\} = (m_1 + m_2)\ddot{x}$$

and $$\frac{\partial L}{\partial x} = (m_1 - m_2)g$$

Substituting in Lagrange's equation (3.41), we have

$$(m_1 + m_2)\,\ddot{x} - (m_1 - m_2)g = 0$$

Or, $$\boxed{\ddot{x} = \frac{(m_1 - m_2)g}{(m_1 + m_2)}} \tag{3.42}$$

The resulting expression describes the acceleration of the coupled masses within the Lagrangian framework.
Physical Interpretation:

- If $m_1 > m_2$, mass $m_1$moves downward with constant acceleration.
- If $m_2 > m_1$, mass $m_2$moves downward.
- The tension in the string does not appear explicitly, showing the advantage of the Lagrange method.

3. Lagrange's Equation for L-C Circuit

Consider an electrical circuit consisting of an inductor of inductance L and a capacitor of capacitance C. The capacitor initially carries a charge q, and a current i flows in the circuit. The circuit is assumed to be ideal, i.e., it contains no resistance, so energy is conserved.

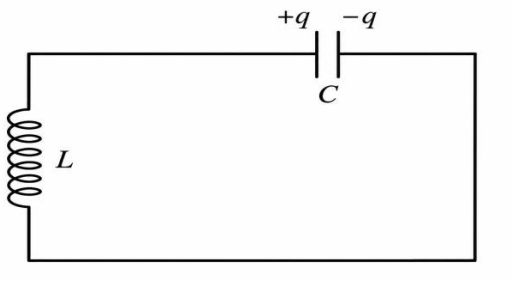


Figure 3.3: Ideal L–C circuit showing an inductor L and capacitor C with charge q oscillating in the circuit

The formalism of Lagrangian mechanics can be extended to electrical oscillatory systems by establishing an analogy between mechanical and electromagnetic energy storage. Magnetic energy stored in the inductor corresponds to kinetic energy**,** while electric energy stored in the capacitor corresponds to potential energy. The charge q plays the role of generalized displacement, and the current $i = \dot{q}$ corresponds to velocity.

Magnetic energy (analogous to kinetic energy):

$$T = \frac{1}{2}Li^2 = \frac{1}{2}L\dot{q}^2 \qquad (3.43)$$

Electric energy (analogous to potential energy):

$$V = \frac{q^2}{2C}$$

The Lagrangian is defined as

$$L = T - V$$

Substituting the expressions for $T$and V,

$$L = \frac{1}{2}L\dot{q}^2 - \frac{q^2}{2C} \qquad (3.44)$$

Taking charge q as the generalized coordinate, Lagrange's equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

Evaluating the derivates, we have

$$\left(\frac{\partial L}{\partial \dot{q}}\right) = L\dot{q}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = L\ddot{q}$$

$$\frac{\partial L}{\partial q} = -\frac{q}{C}$$

Substituting into Lagrange's equation,

$$L\ddot{q} + \frac{q}{C} = 0 \qquad (3.45)$$

Or $$\boxed{\ddot{q} + \frac{1}{LC}q = 0} \qquad (3.46)$$

This is the equation of motion of an L–C circuit. The resulting differential equation has the mathematical form of simple harmonic motion. Hence, the charge oscillates sinusoidally with angular frequency

$$\boxed{\omega = \frac{1}{\sqrt{LC}}}$$

and frequency $$\boxed{\nu = \frac{1}{2\pi\sqrt{LC}}} \qquad (3.47)$$

4. Equation of Motion of Compound Pendulum

Let point O be the fixed point of suspension taken as the origin. Let Gdenote the centre of gravity in the equilibrium position and G'its position when the body is displaced through an angle θ.

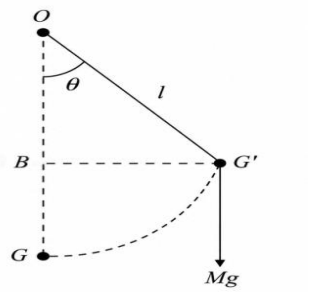


Figure 3.4: Physical pendulum showing angular displacement θ, centre of mass, length l, and gravitational force acting on the bob during oscillation. (Source: Classical Mechanics by J C Upadhayay)

When the rigid body rotates about the fixed axis through θ, its kinetic energy is

$$T = \frac{1}{2} I\dot{\theta}^2$$

where I is the moment of inertia of the body about the axis of rotation.

If the center of gravity is at a distance l from the point of suspension, then in the displaced position the potential energy is

$$V = -mgl\cos\theta$$

The Lagrangian is defined as $L = T - V$

Substituting the expressions for T and V,

$$L = \frac{1}{2} I\dot{\theta}^2 + mgl\cos\theta$$

Applying Lagrange's equation,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \qquad (3.48)$$

We have

$$\frac{\partial L}{\partial \theta} = I\dot{\theta} \ , \quad \frac{\partial L}{\partial \theta} = -mgl sin\theta$$

Therefore, substituting in equation (3.45), we have

$$I\ddot{\theta} + mgl\sin\theta = 0$$

For small angular displacements ($\sin\theta \approx \theta$),

$$I\ddot{\theta} + mgl\theta = 0$$

The time of small oscillations is given by

$$T = 2\pi\sqrt{\frac{I}{mgl}} \qquad (3.49)$$

If k is the radius of gyration about an axis parallel to the axis through the centre of gravity, then

$$I = m(k^2 + l^2)$$

Hence, $\text{Time Period, T} = 2\pi\sqrt{\frac{I}{mgl}} = 2\pi\sqrt{\frac{m(k^2+l^2)}{mgl}}$

Or $$T = 2\pi\sqrt{\frac{k^2+l^2}{l}} \qquad (3.50)$$

where k is the radius of gyration of the body about a parallel axis through the centre of gravity.

5. Numerical Analysis

Ex:1 Derive equations of motion for a particle moving under central force. What is the form of the equations, when the particle is moving under an attractive inverse square law force ($F = -\frac{k}{r^2}$)

Solution: When a particle moves under the action of a central force, the force is always directed along the line joining the particle to a fixed point (the centre) and depends only on the distance from that point. Such a force is conservative, and the motion is confined to a plane.

Let the plane polar coordinates $(r,\ \theta)$ describe the motion of a particle of mass m.

Now, the kinetic energy of the particle is given by

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$

Since the force is central and conservative, the potential energy depends only on the radial distance. The potential energy of the particle is given by

$$V = V(r)$$

Therefore, Lagrangian of the system is given by:

$$L = T - V$$

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r)$$

Lagrange's equations for the generalized coordinates r and θ are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

Evaluating the required derivatives,

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r}, \frac{\partial L}{\partial r} = mr\dot{\theta}^2 - \frac{dV}{dr}$$

Substituting into Lagrange's equation for r,

$$m\ddot{r} - mr\dot{\theta}^2 + \frac{\partial \mathrm{V}}{\partial \mathrm{r}} = 0$$

Since the Lagrangian does not explicitly depend on θ,

$$\frac{\partial L}{\partial \theta} = 0$$

And $$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

Hence, $$\frac{d}{dt}(mr^2\dot{\theta}) = 0$$

This shows that $mr^2\dot{\theta}$ is a constant of motion, representing conservation of angular momentum**.**

For an attractive inverse-square law force,

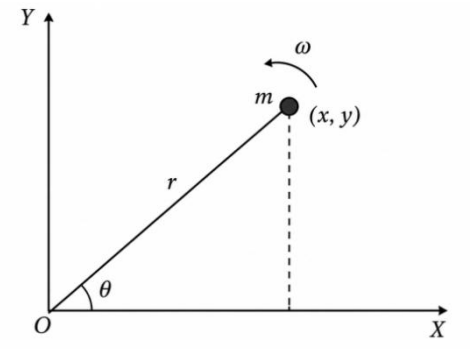
$$F = -\frac{dV}{dr} = -\frac{k}{r^2}$$

so that $$\frac{dV}{dr} = \frac{k}{r^2}$$

Substituting into the radial equation of motion,

$$\boxed{m\ddot{r} - mr\dot{\theta}^2 + \frac{k}{r^2} = 0}$$

Ex 2: A Bead slides on a smooth rod which is rotating about on end in a vertical plane with uniform angular velocity ω. Show that the equation of motion is $mr = mr\omega^2 \ddot{} - mgsin\omega t$.



Solution: The rod rotates in a vertical plane about the fixed point O. Let $r$be the distance of the bead from the pivot O. Let θ be the angular position of the rod.

Since the rod rotates uniformly,

$$\dot{\theta} = \omega \text{ and } \quad \theta = \omega t$$

The motion is described using plane polar coordinates (r, θ).
In polar coordinates, the kinetic energy of the bead is

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$

Since $\dot{\theta} = \omega$, $$T = \frac{1}{2}m(\dot{r}^2 + r^2\omega^2)$$

The vertical height of the bead is $y = r\sin\theta$

Hence, the potential energy is $V = mgr\sin\theta$

Since $\theta = \omega t$, $V = mgr\sin(\omega t)$

The Lagrangian is given by $L = T - V$

Therefore, $L = \frac{1}{2}m(\dot{r}^2 + r^2\omega^2) - mgr\sin(\omega t)$

For the generalized coordinate r,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

Evaluating derivatives, $\frac{\partial L}{\partial \dot{r}} = m\dot{r}$, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = m\ddot{r}$,

Therefore, $\frac{\partial L}{\partial r} = mr\omega^2 - mg\sin(\omega t)$

Substituting, we have

$$m\ddot{r} - mr\omega^2 + mg\sin(\omega t) = 0$$

Thus, the final equation is

$$\boxed{\ddot{r} = r\omega^2 - g\sin(\omega t)}$$

CHECK YOUR PROGRESS

Q1. State Lagrange's equations of motion.
Q2. What is meant by a generalized coordinate?
Q3. Define generalized momentum.
Q4. What is a cyclic (ignorable) coordinate?
Q5. What condition must a coordinate satisfy to be cyclic?
Q6. How is kinetic energy expressed in generalized coordinates?
Q7. What is the physical significance of generalized forces?
Q8. How many Lagrange's equations required for a system with n degrees of freedom?
Q9. What happens to Lagrange's equations when the Lagrangian is independent of time?
Q10.What form does Lagrange's equations take for conservative systems?
Q11.In the simple pendulum problem, which coordinate is taken as the generalized coordinate?
Q12.In Atwood's machine, why is only one generalized coordinate sufficient?
Q13.What expression is used for the kinetic energy of a compound pendulum?
Q14.In an L–C circuit, what quantity is treated as the generalized coordinate?
Q15.In an L–C circuit, what physical quantity corresponds to velocity in mechanics?
Q16.Why does tension not appear in the Lagrangian formulation of a simple pendulum?
Q17.In an L–C circuit, what type of motion does the charge undergo?
Q18.What is radius of gyration?
Q19.In a compound pendulum, how is the moment of inertia expressed in terms of the radius of gyration?
Q20.In Atwood's machine, what happens to constraint forces in Lagrange's method?

Q21. In a simple pendulum, which term provides the restoring torque.

⏱3.5 LET US SUM UP

- Lagrangian mechanics provides an energy-based formulation of motion using suitable generalized coordinates, particularly useful for constrained systems.
- Lagrange's equations are derived from D'Alember's principle and provide one equation for each degree of freedom.
- Generalized coordinates and generalized forces provide a convenient description of systems without explicitly resolving constraint forces.
- For a conservative system, the Lagrangian is obtained from the kinetic and potential energies, and the equations of motion follow directly from it.
- Kinetic energy in generalized coordinates is generally quadratic in generalized velocities; for time-independent constraints, it becomes a homogenous quadratic form.
- A cyclic coordinate does not appear explicitly in the Lagrangian, and its corresponding generalized momentum is conserved.
- The simple pendulum illustrates the use of an angular generalized coordinate and the small angle approximation leading to simple harmonic motion.
- Atwood's machine demonstrates the treatment of a constrained system using a single generalized coordinate, without explicitly determining the string tension.
- The L-C establishes the mechanical-electrical analogy, with change as generalized coordinate and current as generalized velocity, leading to oscillatory motion.
- The compound pendulum demonstrates the role of moment of inertia and radius of gyration in rotational motion.
- Central-force motion conveniently treated in polar coordinates, where the cyclic angular coordinate leads to conservation of angular momentum.
- The rotating rod problem illustrates the application of Lagrange's equation to a system with a time-dependent constraint.

✓3.6 ANSWERS TO CHECK YOUR PROGRESS

Answer to Q1: Lagrange's equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0.$$

Answer to Q2: A generalized coordinate is an independent variable that uniquely describes the configuration of a system.

Answer to Q3: Generalized momentum is defined as $p_k = \frac{\partial L}{\partial \dot{q}_k}$.

Answer to Q4: A cyclic coordinate is a generalized coordinate that does not appear explicitly in the Lagrangian.

Answer to Q5: A coordinate is cyclic if $\frac{\partial L}{\partial q_k} = 0$.

Answer to Q6: Kinetic energy is expressed as a quadratic function of generalized velocities.

Answer to Q7: Generalized forces represent the work done by actual forces during generalized displacements.

Answer toQ8: For a mechanical system having $\boldsymbol{n}$ independent degrees of freedom, $\boldsymbol{n}$ Lagrange equations are required to completely describe its motion.

Answer to Q9: The total energy of the system remains conserved.

Answer toQ10: For conservative systems, Lagrange's equation is written as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0,$$

where $L = T - V$.

Answer toQ11: The angular displacement θ is taken as the generalized coordinate.

Answer toQ12: Because the motion of the two masses is related by a constraint due to the inextensible string.

Answer toQ13: $T = \frac{1}{2} I\dot{\theta}^2$, where $I$is the moment of inertia about the axis of rotation.

Answer toQ14: The charge q on the capacitor is treated as the generalized coordinate.

Answer to Q15: The current $i = \dot{q}$ corresponds to velocity.

Answer toQ16: Because constraint forces do no virtual work in Lagrange's formulation.

Answer to Q17: The charge undergoes simple harmonic oscillations.

Answer to Q18: The radius of gyration is the distance from the axis at which the entire mass can be assumed concentrated without changing the moment of inertia.

Answer to Q19: The moment of inertia is given by $I = m(k^2 + l^2)$.

Answer to Q20: Constraint forces are automatically eliminated from the equations of motion.

Answer to Q21: The term $mgl\sin\theta$ provides the restoring torque.

3.7 FURTHER READING

- Upadhyaya, J. C. (2010). *Classical Mechanics*. Himalayan Publishing House, Mumbai.
- 2.Morin, D. (2008). *Introduction to Classical Mechanics*. Cambridge University Press, Cambridge.
- Goldstein, H., Poole, C. P., & Safko, J. L. (2002). *Classical Mechanics* (3rd ed.). Pearson Education / Addison-Wesley, New Delhi.
- Gupta, A. B. (2003). *Classical Mechanics and Properties of Matter*. Books and Allied Pvt. Ltd., Kolkata.

- Spiegel, M. R. (1967). *Schaum's Outline of Classical Mechanics*. McGraw-Hill Education, New York.

⍰3.8 MODEL QUESTIONS

1. If the generalized coordinate is an angle θ, what is the dimension of the corresponding generalized force?
2. A particle constrained to move along the inner surface of a fixed hemispherical bowl. What is the degree of freedom of the particle.
3. What is the Lagrangian for a charged particle in an electromagnetic field?
4. Two-point masses m are connected by a rod of length $2a$, the center of which moves on a circle of radius r. Write down kinetic energy in generalized coordinates.
5. Obtain the Lagrangian of a particle moving in force free field in spherical and cylindrical coordinate system.
6. Write down Lagrange's equation of motion for a particle of mass m falling freely under gravity near the surface of earth.
7. Write down the generalized coordinates for a simple pendulum and explain why cartesian coordinates are not suitable here.
8. Discuss the superiority of Lagrangian approach over the Newtonian Approach.

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UNIT 4: LAGRANGIAN DYNAMICS II

UNIT STRUCTURE

4.1 Learning Objectives
4.2 Introduction
4.3 Variational Principle
4.4 Derivation of Lagrange's Equation of Motion
4.5 Newton's Equation of motion from Lagrangian Equation
4.6 Let Us Sum Up
4.7 Answers to check in progress
4.8 Further Reading
4.9 Model Questions

4.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- define the action integral and Hamilton's variational principle
- explain the concept of stationary action
- derive Lagrange's equation from the variational principle
- obtain Newton's equation of motion from Lagrange's formulation
- evaluate the advantages of Lagrangian mechanics over Newtonian mechanics.

4.2 INTRODUCTION

In the previous unit, the formulation of mechanics based on generalized coordinates and the Lagrangian function was introduced as an alternative framework to Newtonian mechanics. In the present unit, we move to a deeper and more fundamental level of understanding by introducing the action integral and the variational principle, particularly Hamilton's Principle of Stationary Action. Instead of starting directly from forces, this approach begins with a scalar quantity called *action*, defined as the time integral of the Lagrangian. The actual motion of a mechanical system is determined by the condition that the action assumes a stationary value, meaning its first-order variation vanishes for the physical path followed by the system. This unit will also highlight why the Lagrangian formulation is often considered superior to the Newtonian approach in many situations. This unit will also develop the underlying variational framework of classical mechanics and demonstrates how the equations of motion emerge from a more fundamental extremum principle. The variational principle not only simplifies the treatment of complex systems but also forms the basis for advanced topics such as field theory, quantum mechanics, and modern theoretical physics. Thus, this unit not only derives Lagrange's equations from a fundamental extremum principle but also provides insight into why the Lagrangian formulation represents a powerful and unifying framework in classical mechanics.

4.3 VARIATIONAL PRINCIPLE

1. Motivation of Variational Principle

The Newtonian framework provides a direct relation between force and acceleration and works efficiently for simple mechanical systems. However, for systems involving many interacting particles or constraints, the formulation often becomes algebraically complicated. However, when dealing with complex mechanical systems, the Newtonian formulation often becomes mathematically cumbersome. These limitations motivate the introduction of variational techniques, where the motion of a system is determined through an extremum condition on a suitable scalar quantity, particularly the Lagrangian formulation. The Newtonian formulation becomes complicated in the following situations:

(i) Systems with Many Particles

For a system consisting of several interacting particles, separate vector equations must be written for each particle. This leads to many coupled differential equations. The mathematical analysis becomes lengthy and often difficult to manage.

(ii) Presence of Constraints

Many mechanical systems involve constraints such as rigid connections, inextensible strings, rolling conditions, or motion restricted to surfaces. In Newtonian mechanics, the corresponding constraint forces must be determined explicitly. The calculation of these unknown forces increases computational complexity.

(iii) Unknown Reaction Forces

Forces such as tension, normal reaction, or hinge forces are generally not known beforehand. These must first be calculated before deriving the final equations of motion. This adds additional steps and makes the solution process indirect.

(iv) Variational Methods Simplify the Analysis

Variational methods, especially the Lagrangian approach, overcome these difficulties within the framework of classical mechanics.

(a) Use of Scalar Quantities

Instead of working with vector forces, the Lagrangian formulation uses scalar quantities—kinetic energy and potential energy. This eliminates the need to resolve forces into components and simplifies the mathematical treatment.

(b) Automatic Elimination of Constraint Forces

For ideal constraints, the forces of constraint do no virtual work. Consequently, these forces do not appear explicitly in Lagrange's equations. This greatly simplifies the analysis of constrained systems.

(c) Use of Generalized Coordinates

The motion of the system can be described using generalized coordinates chosen according to the geometry or symmetry of the problem. This allows the formulation to adapt naturally to the system under consideration.

(d) Reduction in the Number of Equations

In the Newtonian method, three equations are required for each particle. In contrast, the Lagrangian method requires only as many equations as the number of degrees of freedom of the system. This significantly reduces computational effort.

2. Configuration Space and System Path

For a mechanical system with n degrees of freedom, the configuration of the system at any instant can be described by a set of generalized coordinates

$$q_1, q_2, \dots, q_n$$

The complete set of generalized coordinates may be viewed as defining an abstract space called configuration space, where each point corresponds to a particular arrangement of the system. During motion, the representative point changes continuously, generating a trajectory in configuration space between the initial and final states. Although many different paths may connect these two points, the system follows a unique path determined by the dynamical laws governing its motion.

This geometrical interpretation offers a useful framework for understanding more advanced formulations of mechanics, where the evolution of a system is represented in terms of coordinates and momenta. It leads to the development of Hamiltonian dynamics, where the motion of the system is described not only by the generalized coordinates but also by the corresponding generalized momenta. Together, these quantities define the state of the system in phase space. The evolution of the system in this space is governed by Hamilton's equations, which will be derived in the following sections. Thus, the concepts of configuration space and system trajectory form the foundation for understanding Hamiltonian mechanics.

3. Concept of Variation and Stationary Action

In variational mechanics, the motion of a system between two fixed instants of time is represented by a path in configuration space. To study how the action changes, we consider a small imaginary modification of this path known as a variation. In variational analysis, one considers neighbouring hypothetical paths that differ infinitesimally from the actual trajectory while sharing the same end conditions. If the actual path of the system is described by the generalized coordinate $q_k(t)$, then a slightly varied path may be written as

$$q_k(t) + \delta q_k(t)$$

where $\delta q_k(t)$ represents a small change in the coordinate. This variation is arbitrary except at the endpoints. Since the initial and final positions are fixed, the variation must satisfy the conditions

$$\delta q_k(t_1) = 0, \delta q_k(t_2) = 0$$

Thus, the varied path differs from the actual path only between the two endpoints.

Since velocity is the time derivative of the coordinate, we have

$$\dot{q}_k = \frac{dq_k}{dt}$$

The variation of the velocity is therefore obtained by differentiating the variation of the coordinate with respect to time:

$$\delta\dot{q}_k = \frac{d}{dt}(\delta q_k)$$

This relation is important because it allows the variation of the action integral to be expressed in terms of the variations of the generalized coordinates. It plays a key role in the derivation of Lagrange's equation from the principle of stationary action.

4. Action Integral

The key quantity in variational dynamics is the action; a scalar functional associated with the path followed by the system between two specified instants. Mathematically, the action is obtained by integrating the Lagrangian over the interval of motion.

$$S = \int_{t_1}^{t_2} L(q_k, \dot{q}_k, t)\, dt \qquad (4.1)$$

where L is the Lagrangian of the system.

For most mechanical systems, the Lagrangian is defined as the difference between the kinetic energy and the potential energy of the system, that is

$$L = T - V$$

Thus, the action represents the integral of $T - V$ over time along a particular path followed by the system in configuration space. The value of the action depends on the path taken by the system between the initial time $t_1$and the final time t_2. In variational mechanics, the actual motion of the system is determined by the principle of stationary action, which states that the true path followed by the system makes the action integral stationary compared with neighbouring paths.

5. Hamiltonian Principle

Hamilton's principle provides a fundamental variational statement of mechanical motion and serves as the basis for both Lagrangian and Hamiltonian formulations. It provides a variational formulation of the laws of motion and forms the foundation of Lagrangian and Hamiltonian dynamics.

Thus, the Hamilton's Principle can be stated as 'The actual evolution of a system between two fixed configurations occurs along a path for which the action remains stationary under infinitesimal variations. The action is defined as the time integral of the Lagrangian of the system'.

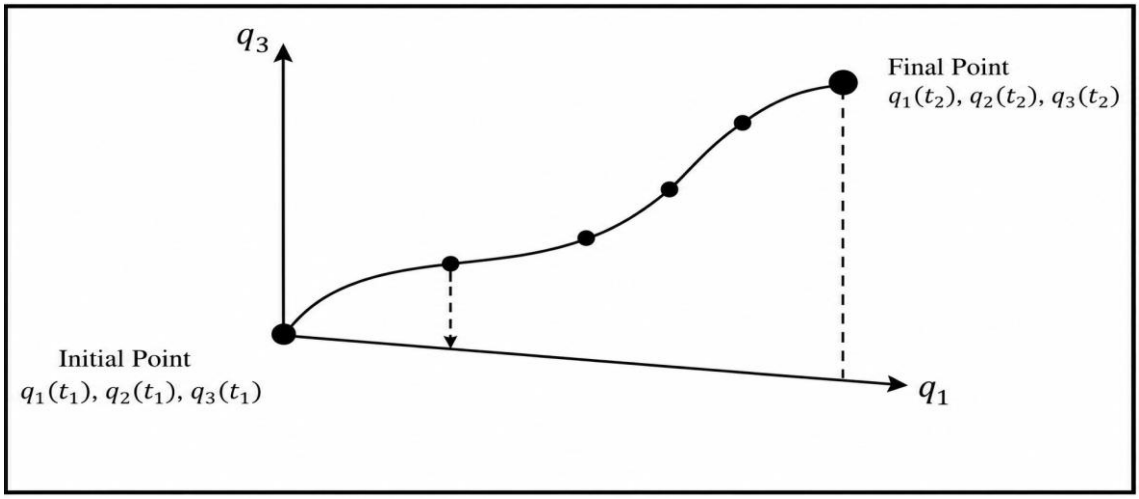


Figure 4.1: Path of the system points in configuration space

Hamilton's principle states that among all the possible paths that connect the initial configuration of the system at time $t_1$to the final configuration at time t_2, the actual path followed by the system is the one for which the action integral has a stationary value.

The condition for stationary action is therefore written as

$$\delta S = \delta \int_{t_1}^{t_2} L\, dt = 0 \qquad (4.2)$$

The term stationary does not necessarily mean that the action is minimum or maximum. It means that if the path of motion is changed by a very small amount, the corresponding change in the action is zero to first order. This principle forms the basis of variational mechanics. By applying the condition $\delta S = 0$ to the action integral, one obtains theEuler–Lagrange equations, which describe the equations of motion of the system,

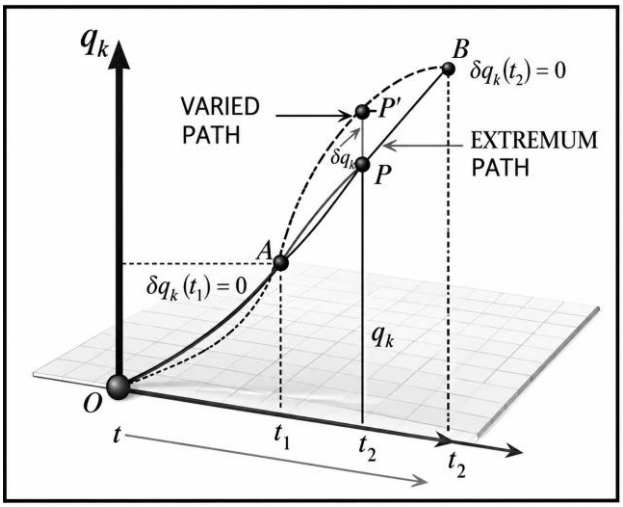


Figure 4.2: Variation of paths in configuration space showing the actual (extremum) path between fixed endpoints t_1 and t_2; nearby varied paths satisfy $\delta q_k(t_1) = \delta q_k(t_2) = 0$ in accordance with Hamilton's principle.

4.4. DERIVATION OF LAGRANGE'S EQUATION OF MOTION

The variational formulation provides a systematic procedure for deriving the equations governing the motion of a mechanical system. By applying variational calculus to the action integral, we can obtain Lagrange's equations, which describe the dynamics of systems in terms of generalized coordinates. In the following section, we use Hamilton's principle to derive Lagrange's equations in a clear and step-by-step manner.

The Lagrangian L of a system is a function of the generalized coordinates q_k, the generalized velocities $\dot{q}_k$, and possibly time t. Thus,

$$L = L(q_1, q_2, \dots, q_n, \dot{q}_1, \dot{q}_2, \dots, \dot{q}_n, t) \qquad (4.3)$$

For simplicity, consider the case where the Lagrangian does not depend explicitly on time.

The action integral of the system is defined as

$$S = \int_{t_1}^{t_2} L(\, q_k, \dot{q}_k, t)\, dt \qquad (4.4)$$

According to Hamilton's Principle, the actual motion of the system between the fixed times t_1 and t_2 is such that the action is stationary. Therefore,

$$\delta S = \delta \int_{t_1}^{t_2} L\, dt = 0$$

Now introduce a small variation in the path of motion

$$q_k \rightarrow q_k + \delta q_k \qquad (4.5)$$

The corresponding variation in the Lagrangian is

$$\delta L = \sum_{k=1}^{n} \frac{\partial L}{\partial q_k}\, \delta q_k + \sum_{k=1}^{n} \frac{\partial L}{\partial \dot{q}_k}\, \delta \dot{q}_k$$

Integrating between t_1 and t_2,

$$\delta S = \int_{t_1}^{t_2} \sum\nolimits_{k=1}^{n} \frac{\partial L}{\partial q_k} \delta q_k\, dt + \int_{t_1}^{t_2} \sum\nolimits_{k=1}^{n} \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k\, dt \qquad (4.6)$$

Since $\delta \dot{q}_k = \frac{d}{dt}(\delta q_k)$

The second term can be integrated by parts:

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k\, dt = \left[\frac{\partial L}{\partial \dot{q}_k} \delta q_k\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) \delta q_k\, dt$$

At the endpoints the coordinates are fixed, therefore

$$\delta q_k(t_1) = \delta q_k(t_2) = 0$$

Hence the boundary term vanishes.

Substituting into the expression for δS, we obtain

$$\delta S = \int_{t_1}^{t_2} \sum_{k=1}^{n} \left[\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\right] \delta q_k \, dt \qquad (4.7)$$

For a holonomic system, the variations $\delta q_k$are independent. Therefore, the coefficient of each δq_kmust vanish. Thus,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \qquad (4.8)$$

where $k = 1,2, \ldots, n$.

This (4.8) equation is known as the Euler–Lagrange equation of motion. It forms the fundamental equation of Lagrangian mechanics.

Lagrange's Equation with velocity dependent potential

The Lagrange's equation of motion derived earlier was obtained for conservative systems where the generalized forces are derived from a scalar potential function $V(q_k)$. In such systems the Lagrangian is written as

$$L = T - V$$

and the equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0$$

However, in many physical situations the generalized forces cannot be derived from a potential function depending only on coordinates. Instead, the forces may depend on both the generalized coordinates and the generalized velocities. In such cases the forces can be expressed in terms of a generalized potential function.

$$U = U(q_k, \dot{q}_k, t)$$

which may depend on coordinates, velocities and time.

If the generalized forces are derived from such a function U, then the generalized force Q_k associated with the coordinate q_k is given by

$$Q_k = -\frac{\partial U}{\partial q_k} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_k}\right) \qquad (4.9)$$

Substituting this expression of $Q_k$into the general form of Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = Q_k$$

we obtain

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = -\frac{\partial U}{\partial q_k} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_k}\right) \qquad (4.10)$$

Rearranging the terms,

$$\frac{d}{dt}\left(\frac{\partial (T-U)}{\partial \dot{q}_k}\right) - \frac{\partial (T-U)}{\partial q_k} = 0 \qquad (4.11)$$

Thus, the Lagrangian of the system can again be written as

$$L = T - U$$

and the Lagrange's equation retains the same form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \qquad (4.12)$$

Therefore, even when the potential depends on velocities, the equation of motion can still be written in the standard Lagrangian form by defining the Lagrangian as

$$L = T - U.$$

Thus, Lagrange's equations remain valid even when the potential depends on velocities, provided the generalized forces can be expressed through a generalized potential function $U(q_k, \dot{q}_k, t)$.

4.5 NEWTON'S EQUATION OF MOTION FROM LAGRANGIAN EQUATION

The general form of Lagrange's equation of motion for a system is expressed as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = G_k \qquad (4.13)$$

where T is the kinetic energy of the system, $q_k$represents the generalized coordinates, and $G_k$denotes the corresponding generalized forces.

If Cartesian coordinates are chosen as generalized coordinates, then

$$q_1 = x, q_2 = y, q_3 = z$$

The generalized force components in this case are given by

$$G_1 = F_x, G_2 = F_y, G_3 = F_z$$

Applying Lagrange's equation (4.6) to the coordinate x, we obtain

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} = F_x \qquad (4.14)$$

Since the kinetic energy does not explicitly depend on x,

$$\frac{\partial T}{\partial x} = 0 \quad \text{and} \quad \frac{\partial T}{\partial \dot{x}} = m\dot{x}$$

Substituting these results into equation (4.7),

$$\frac{d}{dt}(m\dot{x}) = F_x$$

Or $$F_x = \frac{dp_x}{dt}$$

Where, $p_x = m\dot{x}$ is the x-component of linear momentum.

Similarly, for the y- and z-coordinates,

$$F_y = \frac{dp_y}{dt}, F_z = \frac{dp_z}{dt}$$

Combining these components, the vector form becomes

$$\boxed{\mathrm{F} = \frac{d\mathrm{p}}{dt}}$$

which is the Newton's 2nd law. This result establishes that Newton's formulation appears as a special case of the Lagrangian formalism when Cartesian coordinates are employed.

Advantages of Lagrangian mechanics over Newtonian approach

The Newtonian framework directly relates force to acceleration, but its application becomes increasingly difficult in systems involving constraints or many interacting bodies. While this approach is conceptually direct, it often becomes complicated when dealing with systems involving multiple particles or constraints. The necessity of resolving forces into components along coordinate axes increases the mathematical complexity of the problem.

Another major difficulty in the Newtonian method is the treatment of constraint forces**.** These forces must either be calculated explicitly or eliminated through careful analysis. If the constraint forces are unknown, solving the equations becomes difficult. Even when they are known, the use of Cartesian (rectangular) or other standard coordinate systems may make the analysis lengthy or, in some cases, practically unmanageable.

The Lagrangian formulation avoids many of these complications by focusing on energy functions and generalized coordinates. In this formulation:

- The formulation relies on energy functions rather than direct vector-force analysis, which makes the mathematical structure more compact.

- The method uses generalized coordinates, which can be chosen according to the symmetry and constraints of the system.
- Constraint forces are often automatically accounted for, without the need to determine them explicitly.
- The mathematical complexity of solving problems is significantly reduced, especially for systems with complicated geometries or interconnected motions.
- Sometimes, choosing an appropriate generalized coordinate—such as an angular displacement or another suitable variable—instead of rectangular coordinates greatly simplifies the problem.

An additional advantage is that Lagrange's equations retain the same form under any transformation of generalized coordinates**.** This invariance makes the formulation more flexible and powerful compared to the Newtonian approach. Thus, Lagrangian mechanics provides a more elegant, systematic, and often simpler method for analyzing mechanical systems, particularly those involving constraints or multiple degrees of freedom.

CHECK YOUR PROGRESS

Q1. The action integral is defined as

A) $\int F\, dt$

B) $\int T\, dt$

C) $\int (T - V)\, dt$

D) $\int V\, dt$

Q2. Hamilton's principle states that the actual path followed by a system makes the action

A) Maximum

B) Minimum

C) Zero

D) Stationary

Q3. The variation of generalized velocity is given by

A) $\delta \dot{q}_k = \delta q_k$

B) $\delta \dot{q}_k = \frac{d}{dt}(\delta q_k)$

C) $\delta \dot{q}_k = \dot{q}_k$

D) $\delta \dot{q}_k = 0$

Q4.In Lagrangian mechanics, the Lagrangian L is defined as

A) $T + V$

B) $T - V$

C) $V - T$

D) $F - ma$

Q5. The Euler–Lagrange equation is

A) $F = ma$

B) $\frac{d}{dt}\left(\frac{\partial L}{\partial q_k}\right) - \frac{\partial L}{\partial \dot{q}_k} = 0$

C) $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0$

D) $\frac{\partial L}{\partial q_k} = 0$

Q6. If Cartesian coordinates are chosen as generalized coordinates, Lagrange's equation reduces to

A) Law of conservation of energy

B) Newton's second law

C) Law of gravitation

D) Law of inertia

Q7. In Lagrangian formulation, the number of equations equals the number of

A) Particles

B) Forces

C) Degrees of freedom

D) Constraints

Q8. What is the expression for the Lagrangian of a conservative system in terms of kinetic and potential energy?

Q9. How is Hamilton's principle expressed mathematically in terms of the action integral?

Q10.In which space do generalized coordinates describe the configuration of a system?

Q11.Which fundamental principle leads to the Euler–Lagrange equation of motion?

Q12.What is a coordinate called when the Lagrangian does not depend explicitly on it?

4.6 LET US SUM UP

- In Lagrangian mechanics, the motion of a system is described using generalized coordinates and the Lagrangian function, defined as the difference between kinetic energy and potential energy.
- Variational methods provide a powerful alternative to Newtonian mechanics, especially for systems with many particles, constraints, or complicated geometries.
- The configuration of a system with n degrees of freedom is represented by a point in configuration space described by the generalized coordinates $q_1, q_2, \ldots, q_n$.

- As time evolves, the system traces a path in configuration space between two fixed instants t_1 and t_2.
- A variation represents a small imaginary change in the path of motion while keeping the initial and final points fixed.
- The central quantity in variational mechanics is the action integral, defined as the time integral of the Lagrangian along the path of motion.
- Hamilton's principle states that the actual motion of a mechanical system between two fixed times occurs along a path for which the action integral remains stationary.
- The condition of stationary action leads directly to the Euler–Lagrange equations of motion.
- By applying Hamilton's principle and performing a variation of the action integral, the Lagrange equations can be derived systematically.
- When Cartesian coordinates are chosen as generalized coordinates, Lagrange's equations reduce to Newton's second law of motion.
- The Lagrangian formulation simplifies the analysis of constrained systems because the forces of constraint do not appear explicitly in the equations of motion.
- Lagrangian mechanics uses scalar quantities such as kinetic and potential energy instead of vector forces.
- The method also allows the use of generalized coordinates that can be chosen according to the symmetry and geometry of the system.
- Due to these advantages, Lagrangian mechanics provides a more elegant and systematic framework for studying complex mechanical systems and forms the foundation for advanced areas of physics such as quantum mechanics and field theory.

✓4.7ANSWERS TO CHECK YOUR PROGRESS

Answer to Q 1: C) $\int (T - V)\, dt$
Answer to Q 2: D) Stationary
Answer to Q 3: B) $\delta \dot{q}_k = \frac{d}{dt}(\delta q_k)$

Answer to Q 4: B) $T - V$

Answer to Q 5: C) $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0$

Answer to Q 6: B) Newton's second law
Answer to Q 7: C) Degrees of freedom.
Answer to Q 8: The Lagrangian of a conservative system is expressed as

$$L = T - V,$$

where $T$is the kinetic energy and $V$is the potential energy.
Answer to Q 9: Hamilton's principle is mathematically expressed as

$$\delta \int_{t_1}^{t_2} L\, dt = 0.$$

Answer to Q 10: Generalized coordinates describe the configuration of a system in configuration space**.**

Answer to Q 11: The Euler–Lagrange equation of motion is obtained by applying the principle of stationary action (Hamilton's principle).

Answer to Q 12: A coordinate that does not appear explicitly in the Lagrangian is called a cyclic (or ignorable) coordinate**.**

4.8 FURTHER READING

- Rao, K. Sankara. (2009). Classical Mechanics. New Delhi: PHI Learning Private Limited.
- Upadhyaya, J.C. (2010). Classical Mechanics, 2nd Edition. New Delhi: Himalaya Publishing House.
- Goldstein, Herbert. (2011). Classical Mechanics, 3rd Edition. New Delhi: Pearson Education India.
- Gupta, S.L. (1970). Classical Mechanics. New Delhi: Meenakshi Prakashan
- Takwala, R.G. and P.S. Puranik. (1980). New Delhi: Tata McGraw Hill Publishing.
- Goldstein, H., Safko, J. and Poole, (2001). Classical Mechanics, 3rd Edition,
- Rana, N.C. and Joag, P.S. (2001). Classical Mechanics, Tata McGraw-Hill,

4.9 MODEL QUESTIONS

1. What is meant by variation of a path in variational mechanics? Why must the variations vanish at the end points?
2. Define the action integral. What is its significance in Hamilton's principle?
3. State Hamilton's principle of stationary action and explain its physical meaning.
4. Explain why the action is said to be stationary rather than minimum.
5. What is meant by generalized coordinates? Why are they useful in the Lagrangian formulation?
6. A particle of mass $m$moves along a straight line under the influence of a potential
$$V(x) = \frac{1}{2}kx^2$$
   i. Write the Lagrangian of the system.
   ii. Derive the equation of motion using Lagrange's equation.
7. A particle of mass m moves in one dimension with Lagrangian
$$L = \frac{1}{2}m\dot{x}^2 - V(x)$$
   Use Hamilton's principle to derive the Euler–Lagrange equation of motion.
8. A particle of mass m moves in a potential field. $V(x) = mgx$
   i. Write the Lagrangian.
   ii. Obtain the equation of motion using Lagrange's equation.
9. Consider a particle of mass m moving in two dimensions with coordinates x and y.

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - V(x, y)$$

Derive the Euler–Lagrange equations for the coordinates $x$and y.

10. Show that if the Lagrangian does not depend explicitly on a coordinate q_k, then the corresponding generalized momentum is conserved.

*** *** ***

UNIT 5: HAMILTONIAN DYNAMICS

UNIT STRUCTURE

5.1 Learning Objectives

5.2 Introduction

5.3 Hamiltonian and Canonical Equations

5.4 Hamiltonian Equations in Different Coordinate Systems

5.4.1 Cartesian Coordinate

5.4.2 Polar Coordinate

5.43 Cylindrical Coordinate

5.4.4 Spherical Coordinate

5.5 Applications of Hamiltonian Equations

5.6 Let Us Sum Up

5.7 Answers to Check in Progress

5.8 Further Reading

5.9 Model Questions

5.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- apply Hamilton's canonical equations to determine the equations of motion for simple mechanical systems
- apply the Hamiltonian formulation to systems expressed in Cartesian coordinates
- apply Hamiltonian equations in polar–cylindrical coordinate systems to solve dynamical problems
- apply Hamilton's equations in spherical coordinate systems for systems with central forces
- analyse the various application of Hamiltonian equations

5.2 INTRODUCTION

In the previous unit, the variational formulation of mechanics was developed through the principle of stationary action, which provided a systematic method for deriving the equations of motion in terms of generalized coordinates. The Lagrangian approach established an energy-based description of mechanical systems and demonstrated its equivalence to Newtonian mechanics under suitable coordinate choices. These ideas form the basis for the transition to the Hamiltonian formulation.

In this unit, the formulation of mechanics is further developed through the Hamiltonian approach, where the description of motion is expressed in terms of generalized coordinates and their corresponding conjugate momenta $p_1, p_2, \ldots, p_n$, rather than generalized velocities $\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n$ as used in the Lagrangian framework. The unit begins with the derivation of Hamilton's canonical equations and examines their mathematical structure and physical interpretation. The formalism is then applied to different coordinate systems, including cartesian, polar, cylindrical, and spherical coordinates, to illustrate its general applicability. The unit concludes with selected applications that demonstrate the effectiveness of Hamiltonian mechanics in solving a variety of dynamical problems.

5.3 HAMILTONIAN AND CANONICAL EQUATIONS

1. Generalized Momentum and Cyclic Coordinates

Let us consider, a single particle, moving with velocity $\dot{x}$ along X-axis. The Kinetic energy of the particle is given by

$$T = \frac{1}{2} m\dot{x}^2 \qquad (5.1)$$

The momentum can be defined as

$$p = \frac{\partial T}{\partial \dot{x}} = m\dot{x}$$

If V is not a function of velocity $\dot{x}$i.e.$V = V(x)$ and $\frac{\partial V}{\partial \dot{x}} = 0$

Then, the momentum can be written as

$$p = \frac{\partial T}{\partial \dot{x}} = \frac{\partial (T-V)}{\partial \dot{x}} = \frac{\partial L}{\partial \dot{x}} \qquad (5.2)$$

Now, for a system described by the generalized coordinates $q_k's$ and generalized velocities $\dot{q}_k s$, the generalized momentum corresponding to generalized coordinate $q_k$can be defined as

$$p_k = \frac{\partial L}{\partial \dot{q}_k} \qquad (5.3)$$

The quantity defined in this way is called the conjugate (or canonical) momentum associated with the chosen generalized coordinate. Now, from Lagrange's equation for conservative system, we have

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_k}\right] - \frac{\partial L}{\partial q_k} = 0 \qquad (5.4)$$

Substituting $p_k = \frac{\partial L}{\partial \dot{q}_k}$ in equation (5.4), we have

$$\frac{dp_k}{dt} - \frac{\partial L}{\partial q_k} = 0$$

or, $$\dot{p}_k = \frac{\partial L}{\partial q_k} \quad (\because \frac{dp_k}{dt} = \dot{p}_k) \qquad (5.5)$$

Suppose in a particular formulation of the Lagrangian for a system, if a generalized coordinate does not appear explicitly, then we have

$$\frac{\partial L}{\partial q_k} = 0 \qquad (5.5a)$$

Therefore, equation (5.4) becomes

$$\dot{p}_k = \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_k}\right] = 0 \qquad (5.6)$$

Integrating the above equation (5.6), we have

$$\int \dot{p}_k = \int \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_k}\right] = \text{constant}$$

$$\Longrightarrow \qquad p_k = \frac{\partial L}{\partial \dot{q}_k} = \text{constant} \qquad (5.7)$$

Thus, equation (5.7) shows that

If a generalized coordinate $\mathrm{q_k}$ does not appear explicitly in the Lagrangian function, the corresponding generalized momentum $\mathrm{p_k}$ remains unchanged throughout the motion and therefore represents a conserved quantity. Such a coordinate $\mathrm{q_k}$ is referred to as a cyclic or ignorable coordinate**.**

Let us illustrate with an example.

Consider the motion of a particle in a central force field. In polar coordinate, the Lagrangian L is given by

$$L = T - V = \frac{1}{2}mv^2 - V(r) \qquad (5.8)$$

Since, $v^2 = \left[\dot{r}^2 + r^2\dot{\theta}^2\right]$,therefore from equation (5.8), we have

$$L = \frac{1}{2}m\left[\dot{r}^2 + r^2\dot{\theta}^2\right] - V(r) \qquad (5.9)$$

Thus, equation (5.9) shows that the Lagrangian L does not contain the coordinate θ. Hence, θ is the cyclic or ignorable coordinate.

And the generalized momentum corresponding to θ is given by,

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}}$$

Here, p_θ is known as the angular momentum and it is constant of motion in time. Thus, angular momentum of the system is conserved in central force problem. It should be noted that the generalized momentum p_k may not have the dimension of linear momentum as the generalized coordinate q_k may not have the dimension of length.

First Integral

We know, the generalized momentum P_k conjugate to cyclic coordinate q_k is constant of motion. Such a conserved quantity obtained directly from the dynamical equations is referred to as a first integral of motion.

Thus, first integral can be defined by a relation given by

$f\ (q_1, q_2 \ldots\ldots, \dot{q}_1, \dot{q}_2, \ldots. t)$ = Constant

This is first order differential equation. First integrals provide immediate physical insight into the behaviour of a system by identifying conserved quantities.

2. Conservation Theorems

Conservation laws in mechanics are closely connected with the symmetry properties of a system. Whenever the physical description remains invariant under a specific transformation, a corresponding physical quantity remains conserved during the motion. When a system remains unchanged under translation in space, the corresponding linear momentum is conserved; similarly, invariance under rotation leads to the conservation of angular momentum. Thus, conservation laws can be naturally classified based on the type of symmetry present, leading to two important cases: conservation of linear momentum and conservation of angular momentum.

A.Conservation of Linear Momentum

The Lagrange's equation of motion for a generalized coordinate q_k is given by

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_k}\right] - \frac{\partial L}{\partial q_k} = 0 \qquad (5.10)$$

$$\Rightarrow \qquad \frac{d}{dt}\left[\frac{\partial (T-V)}{\partial \dot{q}_k}\right] - \frac{\partial (T-V)}{\partial q_k} = 0 \qquad (5.11)$$

Since $L = T - V$

Let us consider a conservative system where V is not a function of velocities and T is not a function of position. Therefore,

$$\frac{\partial V}{\partial \dot{q}_k} = \frac{\partial T}{\partial q_k} = 0$$

Now, equation (5.11) can be written as

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_k}\right] + \frac{\partial V}{\partial q_k} = 0$$

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_k}\right] = -\frac{\partial V}{\partial q_k} \qquad (5.12)$$

If $\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_k}\right] = \dot{p}_k$ and $-\frac{\partial V}{\partial q_k} = G_k$

Then equation (5.12) can be written as

$$G_k = \dot{p}_k \qquad (5.13)$$

These components $G_k, \dot{p}_k$ are part of the total force and the total linear momentum of the system in the direction of translation dq_k.

For example, if a system is given a translation along X-axis, then $dq_k = dx$ and $G_k = F_k$ and $p_k = P_x$.

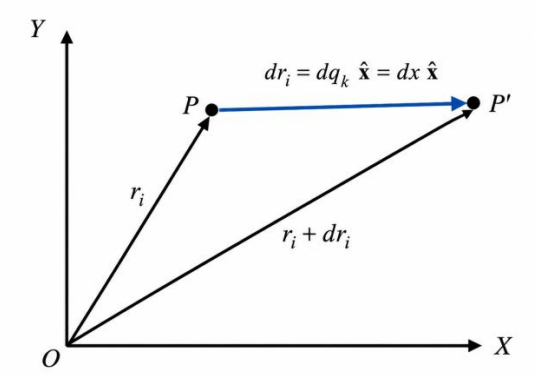


Figure 5.1: Illustration of the variation in a particle's position vector resulting from the translational displacement of the reference frame. (Source: Classical Mechanics by J C Upadhyay)

In the above fig: $dr_i = dq_k\hat{x} = dx.\hat{x}$

where, $\hat{x}$ is a unit vector along X-axis.

This gives $\frac{\partial r_i}{\partial q_k} = \frac{\partial r_i}{\partial x} = \hat{x}$

As we studied in Unit 2, the component of generalized force is given by

$$G_k = \sum_{i=1}^{N} F_i.\frac{\partial r_i}{\partial q_k} = \sum_i F_i.\hat{x} = \hat{x}.\sum_i F_i = \hat{x}.F = F_x$$

Again, we know $p_k = \frac{\partial T}{\partial \dot{q}_k}$

But, again, $T = \sum_i \frac{1}{2} m_i \dot{r}_i^{\,2}$

Therefore, $p_k = \sum_i m_i \dot{r}_i . \frac{\partial r_i}{\partial q_k}$

Since, $\dot{r}_i^{\,2} = \dot{r}_i.\dot{r}_i$ and $\frac{\partial \dot{r}_i}{\partial \dot{q}_k} = \frac{\partial r_i}{\partial q_k}$

We have, $p_k = \sum_i m_i \dot{r}_i.\hat{x} = \hat{x}.\sum_i m_i \dot{r}_i = \hat{x}.P = P_x$

Therefore, $P_k = P_x$

Thus equation (5.13) represents the equation of motion for the direction of translation.

Let us assume that the translation coordinate q_k is cyclic that is it will not appear in $L = T - V$. Thus,

$$\therefore \frac{\partial L}{\partial q_k} = -\frac{\partial V}{\partial q_k} = 0$$

Thus, from equation (5.13), we have

$$G_k = \dot{p}_k = 0. \quad or.\ p_k = \text{constant}$$

For $X-$ direction, $F_x = \dot{p}_x = 0$ or p_x = constant

This result establishes that if no external force acts along a particular direction, the corresponding component of linear momentum remains constant throughout the motion.is called conservation theorem of linear momentum**.** Therefore, if a particular component of the external (applied) force acting on a system is zero, the corresponding component of the system's linear momentum remains constant.

B. Conservation of Angular Momentum

Consider a conservative mechanical system described by generalized coordinates. Let $q_k$be a rotational coordinate representing rotation about a fixed axis. In this case, the generalized force $G_k$corresponds to the component of the total applied torque about the axis of rotation, while the generalized momentum $p_k$represents the component of the total angular momentum about the same axis.

Now, the generalized force is defined as

$$G_k = \sum_{i=1}^{N} \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k}. \qquad (5.15)$$

If the coordinate q_k represents a small rotation $d\theta$ about the Z-axis, the position vector of each particle undergoes an infinitesimal change while its magnitude remains unchanged.

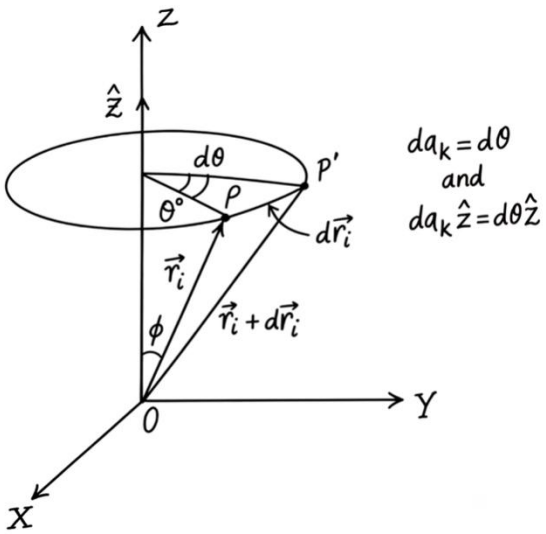


Figure 5.2: Variation in the position vector due to rotational motion of the system (Source: Classical Mechanics by J C Upadhyay)

From the rotational geometry (illustrated in Fig 5.2) the infinitesimal displacement of the particle is given by

$$d\vec{r}_i = d\theta(\hat{z} \times \vec{r}_i),$$

where $\hat{z}$ is the unit vector along the axis of rotation. Hence,

$$\frac{\partial \vec{r_i}}{\partial q_k} = \hat{z} \times \vec{r_i}.$$

Substituting into the expression for generalized force (5.15) gives

$$G_k = \sum_i \vec{F_i} \cdot (\hat{z} \times \vec{r_i}).$$

Using the scalar triple product relation, we have

$$G_k = \hat{z} \cdot \sum_i (\vec{r_i} \times \vec{F_i}).$$

Since $\sum_i (\vec{r_i} \times \vec{F_i})$ equals the total torque $\vec{\tau}$,

$$G_k = \sum \hat{z}.r_i \times F_i = \hat{z}.\sum_i r_i \times F_i$$

$$G_k = \hat{z}.\sum_i \tau_i = \hat{z}.\tau = \tau_z \qquad (5.16)$$

where $\sum_i \tau_i = \tau$ is the total applied torque and $\tau_i = \hat{z}.\tau$ is the component of τ along Z-axis. Thus, the generalized force equals the component of torque along the axis of rotation. Now, the generalized momentum associated with q_k is given by

$$p_k = \frac{\partial T}{\partial \dot{q_k}} = \sum m_i v_i \frac{\partial r_i}{\partial q_k} \qquad (5.17)$$

Substituting the rotational relation,

$$p_k = \sum_i m_i \vec{v_i} \cdot (\hat{z} \times \vec{r_i}).$$

Rearranging, we have,

$$p_k = \hat{z} \cdot \sum_i (\vec{r_i} \times m_i \vec{v_i})$$

$$p_k = \hat{z} \cdot \sum_i (\vec{r_i} \times p_i)$$

$$p_k = \hat{z} \cdot \sum_i \vec{J} \qquad (5.18)$$

The quantity inside the summation represents the angular momentum of the system $(\sum_i \vec{J} = J)$ Therefore,

$$p_k = \hat{z} \cdot \vec{J} = J_z,$$

where $\vec{J}$ is the total angular momentum and J_z is its component along the axis of rotation.

Now, from Lagrange's equation,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0$$

Or $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = -\frac{\partial V}{\partial q_k}$$

Or $$\frac{d}{dt}(p_k) = G_k$$

which gives $$\frac{dJ_z}{dt} = \tau_z \qquad (5.19)$$

This equation represents rotational motion about the chosen axis.

Statement of Conservation of Angular Momentum

If the rotational coordinate q_k is cyclic (i.e., it does not appear explicitly in the Lagrangian or potential energy), then

$$\frac{\partial V}{\partial q_k} = 0 \Rightarrow G_k = 0.$$

Hence, $$G_k = \dot{p}_k = 0$$

Or, $$\tau_z = 0 \Rightarrow \frac{dJ_z}{dt} = 0.$$

Therefore, $$J_z = \text{constant.} \qquad (5.20)$$

This relation is known as the theorem of conservation of angular momentum which states that in the absence of an external torque about a specified axis, the corresponding component of angular momentum remains conserved.

3. Importance of Translational and Rotational Cyclic Coordinates

A generalized coordinate is termed cyclic when it is absent explicitly from the Lagrangian, and this mathematical property directly leads to an associated conservation law. The absence of such a coordinate has important physical implications. If a coordinate associated with translation along a particular direction does not appear in the Lagrangian, it means that the Lagrangian remains unchanged under translation in that direction. In other words, the system possesses translational symmetry. Because of this symmetry, the corresponding linear momentum is conserved.

Similarly, if a coordinate representing rotation about a given axis does not appear explicitly in the Lagrangian, the system is invariant under rotation about that axis. This indicates the presence of rotational symmetry, and therefore the corresponding angular momentum is conserved**.**

Thus, conservation laws arise naturally from symmetry properties of the system:

- Translational symmetry → Conservation of linear momentum
- Rotational symmetry → Conservation of angular momentum

For example, in a central force problem, the Lagrangian does not depend on the angular coordinate θ. This independence reflects rotational symmetry of the system.

Since the Lagrangian remains unchanged when the system is rotated through an angle $\delta\theta$, the angular momentum about the axis is conserved.
In general, if

$$\frac{\partial L}{\partial \theta} = 0,$$

then the angular momentum conjugate to θ remains constant.

4. Hamiltonian Function H and Conservation of Energy

Consider, a general Lagrangian L of a system given by

$$L = L(q_1, q_{2,} \ldots\ldots, q_k, \ldots q_n, \dot{q}_1, \dot{q}_{2,} \ldots\ldots, \dot{q}_k, \ldots\ldots, \dot{q}_n, t)$$

Let us denote it by

$$L = L(q_k, \dot{q}_k, t) \tag{5.21}$$

Differentiating w.r.t time, we have

$$\frac{dL}{dt} = \sum_k \frac{\partial L}{\partial q_k}\frac{dq_k}{dt} + \sum_k \frac{\partial L}{\partial \dot{q}_k}\frac{d\dot{q}_k}{dt} + \frac{\partial L}{\partial t} \tag{5.22}$$

From the Lagrangian equations, we have

$$\frac{\partial L}{\partial q_k} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) \tag{5.23}$$

Substituting $\frac{\partial L}{\partial q_k}$ in (5.22), we have

$$\frac{dL}{dt} = \sum_k \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\frac{dq_k}{dt} + \sum_k \frac{\partial L}{\partial \dot{q}_k}\frac{d\dot{q}_k}{dt} + \frac{\partial L}{\partial t}$$

$$\Rightarrow \frac{dL}{dt} = \sum_k \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\dot{q}_k + \frac{\partial L}{\partial \dot{q}_k}\frac{d\dot{q}_k}{dt} + \frac{\partial L}{\partial t}$$

$$\Rightarrow \frac{d}{dt}\left[\sum_k \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L\right] = -\frac{\partial L}{\partial t} \tag{5.24}$$

Here, in equation (5.24), the quantity within the bracket is also called energy function denoted by h.

$$h(q_1, q_2, \ldots\ldots, q_n, \dot{q}_1, \dot{q}_2 \ldots. \dot{q}_n t) = \sum_k \dot{q}_k \left(\frac{\partial L}{\partial \dot{q}_k}\right)$$

Thus, from eq (5.24), we have

$$\frac{dh}{dt} = -\frac{\partial L}{\partial t} \tag{5.25}$$

If the Lagrange is not an explicit function of time (t), then eq.(5.25) becomes

$$\frac{dh}{dt} = 0 \tag{5.26}$$

This shows that the energy function is the constant of motion. This is one of the first integrals of motion also known as Jacobi's integral.

But we know that $\frac{\partial L}{\partial \dot{q}_k} = p_k$

Hence, from eq (5.24), we have

$$\frac{d}{dt}\sum_k(p_k\dot{q}_k - L) = -\frac{\partial L}{\partial t} \qquad (5.27)$$

The expression obtained from the Legendre transformation of the Lagrangian defines the Hamiltonian function H of the system given by

$$H = \sum_k(p_k\dot{q}_k - L) \qquad (5.28)$$

Thus, the Hamiltonian function H is a function generalized p_k, generlızed $\dot{\text{c}}$oordınate q_k and time t as

$$H = H(p_{k,}q_k, t) \qquad (5.29)$$

From, equation (5.24) and (5.28), it is seen that the energy function h and Hamiltonian function H has the same meaning. Only their symbols are different as H is a function of $(p_{k,}q_k, t)$ while h is a function of $(q_k, \dot{q}_k, t)$.

Now, if 't' does not appear explicitly in Lagrangian function L, then

$$\frac{\partial L}{\partial t} = 0$$

Thus, from equation (5.27) and (5.28), we have

$$\frac{d}{dt}\sum_k(p_k\dot{q}_k - L) = 0$$

$$\frac{dH}{dt} = 0 \text{ or } H = \text{constant} \qquad (5.30)$$

Which implies the Hamiltonian H is constant in time. This is known as the conservation of Hamiltonian of the system.

Under time-independent conservative conditions, the Hamiltonian coincides with the total mechanical energy of the system.

5. Physical Significance of Conservation of Energy

For a conservative mechanical system, the potential energy Vdepends only on the generalized coordinates and does not depend on the generalized velocities $\dot{q}_k$. Moreover, the transformation between Cartesian coordinates and generalized coordinates does not contain time explicitly, i.e.,

$$r_i = r_i(q_1, q_2, \dots, q_k, \dots, q_n)$$

Hence, for a conservative system,

$$\frac{\partial V}{\partial \dot{q}_k} = 0 \qquad (5.31)$$

Since the Lagrangian is defined as

$$L = T - V,$$

Thus, the generalized momentum becomes

$$p_k = \frac{\partial L}{\partial \dot{q}_k} = \frac{\partial (T - V)}{\partial \dot{q}_k} = \frac{\partial T}{\partial \dot{q}_k}$$

Again, the Hamiltonian is defined by

$$H = \sum_k p_k \dot{q}_k - L = \sum_k \frac{\partial L}{\partial \dot{q}_k} \dot{q}_k - L \qquad (5.32)$$

If the position coordinates r_i do not depend explicitly on time, the kinetic energy T becomes a of the generalized velocities. Consequently,

$$\sum_k \frac{\partial T}{\partial \dot{q}_k} \dot{q}_k = 2T$$

In a conservative system, neither the kinetic energy T nor the potential energy Vcontains explicit time dependence. Using the above relation in the Hamiltonian expression,

$$H = 2T - L = 2T - (T - V)$$

which gives $\boxed{H = T + V = E = \text{constant.}}$ (5.33)

Therefore, for systems governed by conservative forces and time-independent constraints, the Hamiltonian represents the total energy and remains invariant during the motion.

6. Hamiltonian Canonical Equations

The Hamiltonian is a function of generalized coordinate q_k, generalized momentum p_k and time t i.e

$$H = H(q_1, q_2, \ldots\ldots, q_k \ldots q_n, p_1, p_2 \ldots\ldots, p_k, \ldots, p_n, t)$$

Writing the differential of H, we have

$$dH = \sum_k \frac{\partial H}{\partial q_k} dq_k + \sum_k \frac{\partial H}{\partial p_k} dp_k + \sum_k \frac{\partial H}{\partial t} dt \qquad (5.34)$$

But, from equation (5.28), we have $H = \sum_k (p_k \dot{q}_k - L)$

Hence, differentiating the above equation, we have

$$dH = \sum_k \dot{q}_k dp_k + \sum_k p_k d\dot{q}_k - dL \qquad (5.35)$$

We know, Lagrangian is a function of generalized coordinate q_k, generalized velocity $\dot{q}_k$ and time t.

Therefore, $L = L(q_1, q_2, \ldots\ldots, q_k \ldots q_n, \dot{q_1}, \dot{q_2} \ldots\ldots \dot{q_n}, \dot{q_k}, t)$ (5.36)

Writing the differential of L, we have

$$dL = \sum_k \frac{\partial L}{\partial q_k} dq_k + \sum_k \frac{\partial L}{\partial \dot{q}_k} d\dot{q}_k + \sum_k \frac{\partial L}{\partial t} dt$$

But from equation (5.3) and (5.5), we have

$$dL = \sum_k \dot{p_k} dq_k + \sum_k p_k d\dot{q}_k + \sum_k \frac{\partial L}{\partial t}\, dt \qquad (5.37)$$

Substituting dL from equation (5.37) in (5.35), we have

$$dH = \sum_k \dot{q}_k dp_k + \sum_k p_k d\dot{q}_k - \sum_k \dot{p_k} dq_k - \sum_k p_k d\dot{q}_k - \sum_k \frac{\partial L}{\partial t}\, dt$$

Or $$dH = \sum_k \dot{q}_k dp_k + \sum_k p_k d\dot{q}_k - \sum_k \frac{\partial L}{\partial t}\, dt \qquad (5.38)$$

Now, Comparing the coefficient of dp_k and $d\dot{q}_k$ and dt in equation (5.34) and (5.38), we have

$$\dot{q}_k = \frac{\partial H}{\partial p_k} \qquad (5.39)$$

$$-\dot{p}_k = \frac{\partial H}{\partial q_k} \qquad (5.40)$$

$$-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t} \qquad (5.41)$$

Equations (5.39), (5.40) are known as Hamiltonian equations or Hamiltonian Canonical equations of motion. These equations provide a complete first-order dynamical description of the system in terms of coordinates and conjugate momenta.

Now, from equation (5.40), if any coordinate q_k is cyclic, then

$$\frac{\partial H}{\partial q_k} = 0 = \dot{p}_k = 0$$

$$\therefore\ p_k = \text{constant}$$

Thus, for any cyclic coordinate, the corresponding conjugate momentum remains constant in motion.

Again, differentiating equating (5.34) w.r.t time, we have

$$\frac{dH}{dt} = \sum_k \frac{\partial H}{\partial q_k} \dot{q}_k + \sum_k \frac{\partial H}{\partial p_k} \dot{p}_k + \sum_k \frac{\partial H}{\partial t}$$

substituting $\dot{q}_k$ and p_k in equation (5.34), we have

$$\frac{dH}{dt} = \sum_k \frac{\partial H}{\partial q_k}\frac{\partial H}{\partial p_k} - \sum_k \frac{\partial H}{\partial p_k}\frac{\partial H}{\partial q_k} - \sum_k \frac{\partial L}{\partial t}$$

$$\therefore \frac{dH}{dt} = \frac{-\partial L}{\partial t}$$

If Lagrangian L is not explicit function of time, then

$$\frac{\partial L}{\partial t} = 0$$

Thus, from (5.41), we have

$$\frac{dH}{dt} = 0$$

$$\Longrightarrow H = \text{constant}$$

If the Lagrangian does not depend explicitly on time, then the Hamiltonian is also independent of time explicitly, and consequently the Hamiltonian remains constant during the motion.

5.4 HAMILTONIAN EQUATIONS IN DIFFERENT COORDINATE SYSTEMS

The Hamiltonian formulation preserves the same mathematical structure in different coordinate systems, making it a flexible and unified method for describing mechanical motion.

5.4.1 Cartesian Coordinate

Consider a particle of mass m moving in a potential field $V(x, y, z)$.

The kinetic energy of the particle is

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2),$$

and the Lagrangian is

$$L = T - V = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z).$$

The generalized momenta $p_k = \frac{\partial L}{\partial \dot{q}_k}$corresponding to coordinates (x, y, z) are defined as

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}, p_y = m\dot{y}, p_z = m\dot{z}.$$

Hence, the velocities can be written as

$$\dot{x} = \frac{p_x}{m}, \dot{y} = \frac{p_y}{m}, \dot{z} = \frac{p_z}{m}.$$

The Hamiltonian is defined by

$$H = \sum_k p_k \dot{q}_k - L.$$

Substituting the above relations for $k = x, y, z,$ we have

$$H = p_x\dot{x} + p_y\dot{y} + p_z\dot{z} - \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z)$$

$$H = \frac{{p_x}^2}{m} + \frac{{p_y}^2}{m} + \frac{{p_z}^2}{m} - \frac{1}{2m}\left({p_x}^2 + {p_y}^2 + {p_z}^2\right) + V(x, y, z)$$

$$H = \frac{1}{2m}\left({p_x}^2 + {p_y}^2 + {p_z}^2\right) + V(x, y, z) \qquad (5.42)$$

Hamilton's equations are

$$\dot{q_x} = \frac{\partial H}{\partial p_k}, \dot{p}_k = -\frac{\partial H}{\partial q_k},$$

The Hamiltonian's equations in cartesian coordinates are

$$\dot{x} = \frac{\partial H}{\partial P_x} \quad \text{or} \quad \dot{x} = \frac{p_x}{m} \text{ or. } \dot{p}_x = -\frac{\partial H}{\partial x} \text{ or } \dot{p}_x = -\frac{\partial V}{\partial x} \qquad (5.43)$$

Similarly, $\dot{y} = \frac{p_y}{m}$ or $\dot{p}_y = -\frac{\partial V}{\partial y}$ (5.44a)

$\dot{z} = \frac{p_z}{m}$ or $\dot{p}_z = -\frac{\partial V}{\partial z}$ (5.44b)

Thus, equation of motion can be expressed as

$$\boxed{m\ddot{x} = -\frac{\partial V}{\partial x},\ m\ddot{y} = -\frac{\partial V}{\partial y},\ m\ddot{z} = -\frac{\partial V}{\partial z}} \qquad (5.45)$$

5.4.2 Polar Coordinates

Let $(r,\ \theta)$ be the polar coordinates of a particle moving in a plane. The polar coordinate system is shown in Figure 5.3.

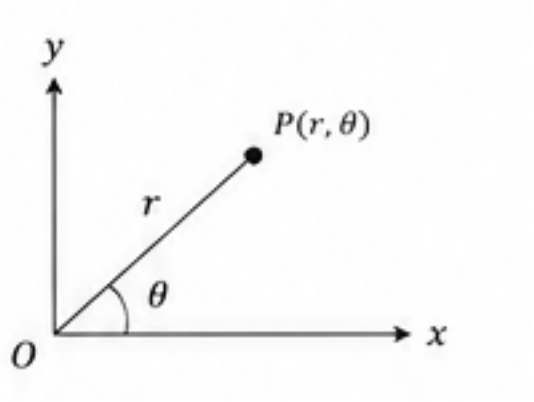


Figure 5.3: Polar coordinate system showing the radial coordinate r and angular coordinate θ describing the position of a particle in a plane. (Source: Classical Mechanics by J C Upadhyay)

The coordinate transformations are

$$x = r\cos\theta, y = r\sin\theta.$$

The kinetic energy becomes

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2),$$

and the Lagrangian is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r,\theta).$$

The generalized momenta are

$$p_r = m\dot{r}, p_\theta = mr^2\dot{\theta}.$$

Hence,

$$\dot{r} = \frac{p_r}{m}, \dot{\theta} = \frac{p_\theta}{mr^2}.$$

The Hamiltonian becomes

$$H = \frac{1}{2m}p_r^2 + \frac{p_\theta^2}{2mr^2} + V(r,\theta) \quad (5.46)$$

Hamilton's equations follow directly from partial derivatives of H

$$\dot{r} = -\frac{\partial H}{\partial p_r} = \frac{p_r}{m} \text{and } \dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{mr^3} - \frac{\partial V}{\partial r} \quad (5.47)$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} \quad \text{and} \quad \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -\frac{\partial V}{\partial \theta} \quad (5.48)$$

5.4.3 Cylindrical Coordinates

For cylindrical coordinates (r, θ, z). The cylindrical coordinate system is shown in Figure 5.4.

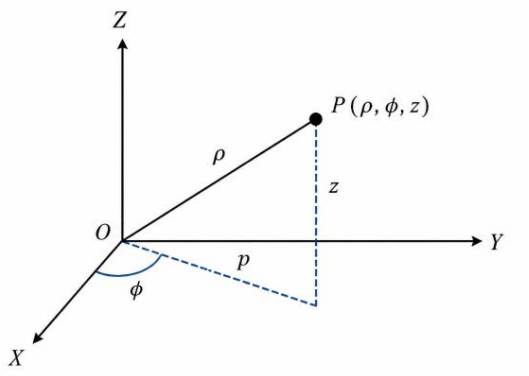


Figure 5.4: Cylindrical coordinate system showing the radial distance r, angular coordinate ϕ, and axial coordinate z..(Source: Classical Mechanics by J C Upadhyay)

$$x = r\cos\theta, y = r\sin\theta, z = z.$$

The kinetic energy is $T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2)$

Hence, Lagrangian, $L = T - V$

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x,y,z)$$

$$L = \frac{1}{2}\, m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right) - V(r,\theta,z)$$

The generalized momentum. $p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$ or $\dot{r} = \frac{p_r}{m}$

Similarly, $p_\theta = mr^2\dot{\theta}$ or $\dot{\theta} = \frac{p_\theta}{mr^2}$

$$p_z = m\dot{z} \text{ or } \dot{z} = \frac{P_z}{m}$$

Now, $H = \sum_k p_k\dot{q}_k - L.$ where $k = r,\theta,z$

Thus, $H = p_r\dot{r} + p_\theta\dot{\theta} + p_z\dot{z} - \frac{1}{2}\, m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right) - V(r,\theta,z)$

$$H = \frac{1}{2m}\left[{p_r}^2 + \frac{{p_\theta}^2}{r^2} + {p_z}^2\right] + V(r,\theta,z) \qquad (5.49)$$

The Hamiltonian equations are. $\dot{q}_k = \frac{\partial H}{\partial p_k}$ and $\dot{p}_k = -\frac{\partial H}{\partial q_k}$

Therefore, $\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m}$ and $\dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{mr^3} - \frac{\partial V}{\partial r}$ (5.50a)

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} \text{ and } \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -\frac{\partial V}{\partial \theta} \qquad (5.50b)$$

$$\dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m} \text{ and } \dot{p}_z = -\frac{\partial H}{\partial z} = -\frac{\partial V}{\partial z} \qquad (5.50c)$$

5.4.4 Spherical Coordinates

In spherical coordinates (r,θ,ϕ),

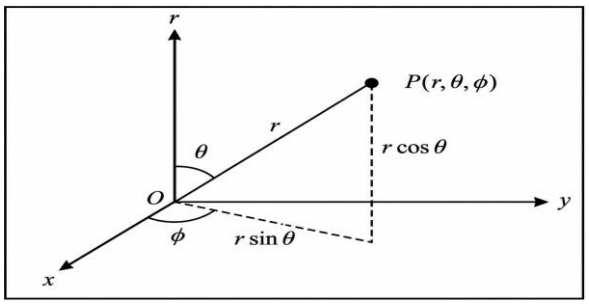


Figure 5.5: Spherical coordinate system showing the radial distance r, polar angle θ, and azimuthal angle ϕ.

$$x = r\sin\theta\cos\phi\,, y = r\sin\theta\sin\dot{\phi}\,, z = r\cos\theta$$

$$\dot{x} = \dot{r}\sin\theta\cos\phi + r\dot{\theta}\sin\theta\cos\dot{\phi} - rsin\theta\dot{\phi}\sin\phi$$

$$\dot{y} = \dot{r}\sin\theta\sin\phi + r\dot{\theta}\cos\theta\sin\phi + rsin\theta\phi\cos\phi$$

$$\dot{z} = \dot{r}cos\theta - r\dot{\theta}\ \sin\theta$$

Now, Lagrangian, $L = T - V = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z)$

Or $L = \frac{1}{2}\ m\ (\dot{r}^2 + r^2\dot{\theta}^2 + r^2 sin^2\theta\phi^2)$-$V(r, \theta, \phi)$

Therefore, $p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$ or $\dot{r} = \frac{p_r}{m}$

Similarly, $\dot{\theta} = \frac{p_\theta}{mr^2}$ and $\phi = \frac{p_\emptyset}{mr^2 sin\theta}$

Now, Hamiltonian is given by

$$H\ = \sum_{k=r,\theta\phi} p_k \dot{q}_k - L.$$

$$H = p_r\dot{r} + p_\theta\dot{\theta} + p_\emptyset\phi - \frac{1}{2}\ m\ (\dot{r}^2 + r^2\dot{\theta}^2 + r^2 sin^2\theta\phi^2)\text{-}V(r, \theta, \phi)$$

$$H = \frac{1}{2m}\left[p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\emptyset^2}{r^2 sin^2\theta}\right] + V(r, \theta, \phi) \qquad (5.51)$$

The Hamiltonian equations are. $\dot{q}_k = \frac{\partial H}{\partial p_k}$ and $\dot{p}_k = -\frac{\partial H}{\partial q_k}$

Therefore, $\dot{r} = \frac{\partial H}{\partial p_r}$ or $\dot{r} = \frac{p_r}{m}$ and $\dot{p}_r = -\frac{\partial H}{\partial r}$ or $\dot{p}_r = \frac{p_\theta^2}{mr^3} + \frac{p_\phi^2}{mr^3 \sin^2\theta} - \frac{\partial V}{\partial r}$ (5.52a)

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} \text{ or } \dot{\theta} = \frac{p_\theta}{mr^2} \quad \text{and} \quad \dot{p}_\theta = \frac{p_\phi^2 \cos\theta}{mr^2 \sin^3\theta} - \frac{\partial V}{\partial \theta} \qquad (5.52b)$$

$$\dot{\phi} = \frac{\partial H}{\partial p_\emptyset} \text{ or } \dot{\phi} = \frac{p_\emptyset}{mr^2 sin^2\theta} \text{ and } \dot{p}_\phi = -\frac{\partial V}{\partial \phi} \qquad (5.52c)$$

5.5 APPLICATIONS OF HAMILTONIAN'S EQUATIONS

A) ONE-DIMENSIONAL HARMONIC OSCILLATOR

Consider a particle of mass m moving along the x-axis under a restoring force proportional to displacement.

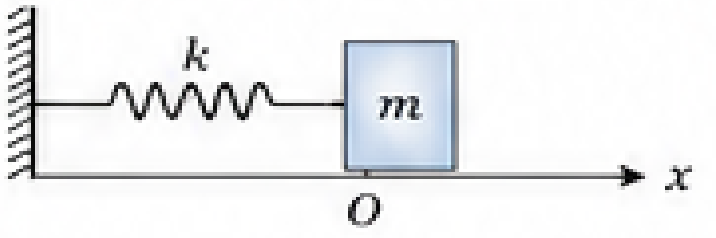


Figure 5.6: One-dimensional harmonic oscillator showing a mass attached to a spring oscillating along the x-axis under a restoring force. (Source: Classical Mechanics by J C Upadhyay)

Now, kinetic energy and potential energy is given by

$$T = \frac{1}{2}m\dot{x}^2,\ V = \frac{1}{2}kx^2$$

where k is the force constant.

We know, the Lagrangian is defined as $L = T - V$

Therefore, $L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$

Now, the momentum conjugate to coordinate x is

$$p_x = \frac{\partial L}{\partial \dot{x}} = \frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}m\dot{x}^2\right) = m\dot{x}$$

Hence, $\dot{x} = \frac{p_x}{m}$

Now, Hamiltonian is defined as

$$H = p_x\dot{x} - L$$

Substitute $\dot{x} = p_x/m$ and L in the above equation, we get

$$H = p_x\left(\frac{p_x}{m}\right) - \left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2\right) = \frac{p_x^2}{m} - \left(\frac{1}{2}\frac{p_x^2}{m} - \frac{1}{2}kx^2\right)$$

$$H = \frac{p_x^2}{2m} + \frac{1}{2}kx^2$$

Thus, Hamiltonian represents total energy of a system

Now, from Hamiltonian canonical equations, we have

$$\dot{x} = \frac{\partial H}{\partial p_x}, \dot{p}_x = -\frac{\partial H}{\partial x}$$

Therefore, $\dot{x} = \frac{\partial}{\partial p_x}\left(\frac{p_x^2}{2m} + \frac{1}{2}kx^2\right) = \frac{p_x}{m}.$

and $\dot{p}_x = -\frac{\partial}{\partial x}\left(\frac{p_x^2}{2m} + \frac{1}{2}kx^2\right) = -kx$

$$\because\ p_x = m\dot{x}$$

$$\therefore\ \dot{p}_x = m\ddot{x} = -kx$$

$$\therefore\quad m\ddot{x} + kx = 0$$

The resulting differential equation corresponds to the familiar form of simple harmonic motion.

B) TWO-DIMENSIONAL HARMONIC OSCILLATOR

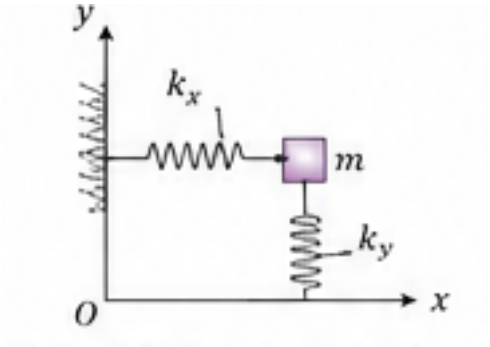


Figure 5.7: Two-dimensional harmonic oscillator showing independent oscillations of a particle along the x- and y-directions. (Source: Classical Mechanics by J C Upadhyay)

Kinetic Energy and Potential energy is given by

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2),$$

$$V = \tfrac{1}{2}k_1x^2 + \tfrac{1}{2}k_2y^2$$

Now, the Lagrangian is given by

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}k_1x^2 - \frac{1}{2}k_2y^2$$

We know the generalized momentum is given by

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}, p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y}$$

Again, we know the Hamiltonian as

$$H = p_x\dot{x} + p_y\dot{y} - L.$$

Replacing the velocities, we have

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2}k_1x^2 + \frac{1}{2}k_2y^2.$$

From the Hamiltonian Canonical Equations, we have

$$\dot{x} = \frac{p_x}{m},\ \dot{p}_x = -k_1x$$

$$\dot{y} = \frac{p_y}{m},\ \dot{p}_y = -k_2 y$$

Hence, $\boxed{m\ddot{x} + k_1 x = 0,\ \ m\ddot{y} + k_2 y = 0}$

Motion along x and y directions is independent.

C) SIMPLE PENDULUM

Consider, a simple pendulum of length l. The kinetic energy and the potential energy is given by

$$T = \frac{1}{2} m l^2 \dot{\theta}^2,$$
$$V = -mgl\cos\theta.$$

Now, the Lagrangian is given by

$$L = \frac{1}{2} m l^2 \dot{\theta}^2 - mgl\cos\theta$$

We can write the conjugate momentum as

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m l^2 \dot{\theta}.$$

Or $$\dot{\theta} = \frac{p_\theta}{m l^2}$$

Therefore, the Hamiltonian of the system can be written as

$$H = p_\theta \dot{\theta} - L.$$

Substituting the value of $p_\theta\ and\ L$ in the above equation, we have

$$H = \frac{p_\theta^2}{m l^2} - \left(\frac{1}{2} m l^2 \dot{\theta}^2 - \text{mglcos}\,\theta\right)$$

Replacing $\dot{\theta}$, $$H = \frac{p_\theta^2}{2 m l^2} - mgl\cos\theta$$

Now, from the Hamiltonian Canonical equation, we have

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{m l^2}, \qquad \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = mgl\sin\theta$$

$$\therefore \ddot{\theta} = \frac{\dot{p_\theta}}{m l^2} = -\frac{g}{l}\sin\theta = 0$$

$$\Longrightarrow \ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

Under the small-angle approximation, the nonlinear pendulum equation reduces to the simple harmonic form.

$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

D) COMPOUND PENDULUM

Consider a rigid body oscillating about a fixed horizontal axis under the action of gravity. The kinetic energy and the potential energy are given by

$$T = \frac{1}{2}I\dot{\theta}^2 \qquad \text{and} \qquad V = mglcos\theta$$

Thus, Lagrangian is given by

$$L = \frac{1}{2}I\dot{\theta}^2 - mglcos\theta$$

$$\therefore\ p_\theta = \frac{\partial L}{\partial \dot{\theta}} = I\dot{\theta}$$

$$\Longrightarrow\ \dot{\theta} = \frac{p_\theta}{I}$$

Now, the Hamiltonian is given by

$$H = \sum p_i\dot{q}_i - L = p_\theta\dot{\theta} - L$$

$$\therefore\ H = p_\theta\frac{p_\theta}{I} - \frac{1}{2}I\left(\frac{p_\theta}{I}\right)^2 - mglcos\theta$$

$$\therefore\ H = \frac{{p_\theta}^2}{2I} - mglcos\theta$$

Now, the Hamiltonian equations are

$$\dot{\theta} = \frac{\partial L}{\partial p_\theta} = \frac{p_\theta}{I}$$

$$\Rightarrow\ p_\theta = I\dot{\theta}$$

Again, $$\dot{p_\theta} = -\frac{\partial H}{\partial \dot{\theta}} = -mgl\sin\theta$$

But $$\dot{p_\theta} = I\ddot{\theta}$$

$$\therefore\ I\ddot{\theta} = -mgl\sin\theta$$

Or $$\ddot{\theta} + \frac{mgl}{I}\sin\theta = 0$$

For Small angle θ, the above equation can be written as

$$\ddot{\theta} + \frac{mgl}{I}\theta = 0$$

E) CHARGED PARTICLE MOVING IN ELECTROMAGNETIC FIELD

In an electromagnetic field, the motion of a charged particle is influenced by both electric and magnetic fields. The Lagrangian L for a charged particle is given by

$$L = T - U = T - q\phi + q(\boldsymbol{v}.\boldsymbol{A})$$

$$L = \frac{1}{2}\sum_k m{v_k}^2 + q\sum_k v_k A_k - q\phi$$

The canonical momenta for $k = x, y, z$ are

$$p_k = \frac{\partial L}{\partial \dot{q}_k} = \frac{\partial L}{\partial v_k} = mv_k + qA_k \qquad (5.53)$$

Now, $$H = \sum_k p_k \dot{q}_k - L = \sum_k mv_k^2 + qA_k v_k - L$$

$$H = mv^2 + q(v.A) - \frac{1}{2}mv^2 + q\phi - q(v.A)$$

$$H = \frac{1}{2}mv^2 + q\phi \qquad (5.54)$$

Now, $v_k = \frac{P_k}{m} - \frac{q}{m}A_k$, then H expressed as

$$H = \sum_k \frac{1}{2m}(p_k - qA_k)^2 + q\phi \qquad (5.55)$$

Or $$H = \frac{1}{2m}(p - qA)^2 + q\phi \qquad (5.56)$$

Hence, Hamilton's equation is

$$v_k = \frac{\partial H}{\partial p_k} = \frac{1}{m}(p_k - qA_k) \qquad \text{Or. } v = \frac{1}{m}(p - qA) \qquad (5.57)$$

And $$\dot{p_k} = -\frac{\partial H}{\partial q_k} \qquad \text{Or. } \dot{p} = -q\nabla\phi + q\nabla(v.A). \qquad (5.58)$$

CHECK YOUR PROGRESS

Q1. For a particle moving in a plane under a central potential, the Lagrangian is given by

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

What are the generalized momenta corresponding to the coordinates r and θ?

Q2. Write down the Hamiltonian and equation of motion for a simple pendulum.

Q3. Write down the canonical equations of motion for a conservative system.

Q4. The Lagrangian of a system is $L = ax^2 + by^2 - kxy$. Write down the Hamiltonian of the system.

Q5. What do we call a coordinate that does not appear explicitly in the Lagrangian of a system?

Q6. What is the expression for the generalized momentum p_x of a particle of mass $m$moving with velocity v_x in an electromagnetic field?

Q7. If the Lagrangian has no explicit dependence on time, which physical quantity remains conserved?

Q8. What is the dimension of the product of generalized coordinate and its conjugate momentum.

5.6 LET US SUM UP

- Hamiltonian mechanics provides an alternative and powerful formulation of classical dynamics based on generalized coordinates and conjugate momenta.
- The generalized (canonical) momentum is defined as $p_k = \frac{\partial L}{\partial \dot{q}_k}$and plays a central role in Hamiltonian formulation.
- A coordinate that does not appear explicitly in the Lagrangian is called a cyclic (ignorable) coordinate**,** and its corresponding momentum remains constant.
- The constant momentum associated with a cyclic coordinate is known as a first integral of motion**.**
- Conservation laws arise naturally from symmetry. They are translational symmetry signifying conservation of linear momentum and rotational symmetry signifying conservation of angular momentum
- The Hamiltonian function is defined as

  $$H = \sum p_k \dot{q}_k - L,$$

  and represents the energy function of the system.

- If the Lagrangian does not depend explicitly on time, the Hamiltonian remains constant, indicating conservation of energy.
- For conservative systems, the Hamiltonian is equal to the total mechanical energy$H = T + V$.
- Hamilton's canonical equation$\dot{q}_k = \frac{\partial H}{\partial p_k}, \dot{p}_k = -\frac{\partial H}{\partial q_k}$provide the equations of motion in terms of coordinates and momenta.
- Hamiltonian equations retain the same mathematical form in all coordinate systems**,** making them widely applicable.
- The formulation was applied in different coordinate systems: cartesian coordinates, polar coordinates, cylindrical coordinates, spherical coordinates
- Applications such as the harmonic oscillator, simple pendulum, compound pendulum**,** and charged particle in an electromagnetic field demonstrate the usefulness of Hamiltonian dynamics.
- Overall, Hamiltonian mechanics reveals the deep connection between symmetry, conservation laws, and energy, and offers an efficient framework for solving dynamical problems.

5.7 ANSWER TO CHECK YOUR PROGRESS

Answer Q1. $\boldsymbol{p_r = m\dot{r}}$ and $\boldsymbol{p_\theta = mr^2\dot{\theta}}$

Answer Q2. $\dot{\theta} = \frac{p_\theta}{ml^2}$, $\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = mgl\sin\theta$, $\ddot{\theta} + \frac{mgl}{I}\theta = 0$

Answer Q3. $\frac{dq_i}{dt} = \frac{\partial H}{\partial P_i}$ and $-\frac{dp_i}{dt} = \frac{\partial H}{\partial q_i}$

Answer Q4. $\boldsymbol{H = \frac{p_x^2}{4a} + \frac{p_y^2}{4b} + kxy}$

Answer Q5. Cyclic Coordinate

Answer Q6. $p_x = mv_x + qA_x$

Answer Q7. Hamiltonian

Answer Q8. Angular Momentum

5.8 FURTHER READINGS

- Upadhyaya, J.C. (2010). Classical Mechanics, 2nd Edition. New Delhi: Himalaya Publishing House.
- Rao, K. Sankara. (2009). Classical Mechanics. New Delhi: PHI Learning Private Limited.
- Goldstein, Herbert. (2011). Classical Mechanics, 3rd Edition. New Delhi: Pearson Education India.
- Takwala, R.G. and P.S. Puranik. (1980). New Delhi: Tata McGraw Hill Publishing.
- Goldstein, H., Safko, J.and Poole, Classical Mechanics,3rdEdition,2001.
- Rana, N.C. and Joag, P.S. Classical Mechanics, Tata McGraw-Hill,2001

5.9 MODEL QUESTIONS

1. A particle is moving near the surface in the earth's gravitational field. Write down the Hamiltonian and equation of motion of the particle.

2. A particle of mass m is attracted towards a given point by a force of the form $\frac{k}{r^2}$ where k is a constant. Write down the expression for the Hamiltonian of the system and derive Hamilton's equation of motion.

3. Explain the physical significance of Hamiltonian.

4. Prove that the Hamiltonian H of a conservative system is equal to the total energy of the system.

5. The Lagrangian for a problem is

$$L = \frac{1}{2m(r+rtheta.)} - V(r)$$

Identify the cyclic coordinates and the corresponding conservation law of the problem.

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UNIT 6: CANONICAL TRANSFORMATION

UNIT STRUCTURE

6.1 Learning Objectives
6.2 Introduction
6.3 Basics of Canonical Transformation
6.4 Legendre's Transformation
6.5 Generating Functions
6.6 Condition for Canonical Transformation
6.7 Let Us Sum Up
6.8 Answers to Check Your Progress
6.9 Further Reading
6.10 Model Questions

6.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- discuss the concept and significance of canonical transformations
- explain the role of Legendre transformations in analytical mechanics
- derive and apply conditions for canonical transformations
- apply canonical transformations to simplify dynamical systems.

6.2 INTRODUCTION

In the previous unit, we studied the formulation of Hamiltonian dynamics, which include Hamilton's function, canonical equations, and their applications. Building on this foundation, the present unit introduces the concept of canonical transformations, which preserve the form of Hamilton's equations under a change of variables. Canonical transformations are important because they enable the simplification of dynamical problems by transforming the system into a more convenient set of variables. They also form the basis of the Hamilton–Jacobi theory, which provides an alternative method for solving equations of motion. A systematic approach to constructing such transformations is through generating functions, which relate old and new coordinates and momenta. This unit also discusses the conditions required for a transformation to be canonical, along with illustrative examples.

6.3 BASICS OF CANONICAL TRANSFORMATION

In many dynamical problems, it is often convenient to transform the original set of canonical variables (q_k, p_k) into a new set (Q_k, P_k), such that the equations of motion take a simpler form. These new variables are related to the old ones through transformations of the form:

$$P_k = P_k(q_1, q_2, \dots, q_n;\ p_1, p_2, \dots, p_n;\ t)$$
$$Q_k = Q_k(q_1, q_2, \dots, q_n;\ p_1, p_2, \dots, p_n;\ t)$$

Such transformations are called canonical transformations if they preserve the form of Hamilton's equations.

For the new variables ($Q_k,\ P_k$), the equations of motion must retain the canonical form:

$$\dot{Q}_k = \frac{\partial H'}{\partial P_k}, \dot{P}_k = -\frac{\partial H'}{\partial Q_k} \qquad (6.1)$$

where $H'(Q, P, t)$ is the transformed Hamiltonian.

This requirement ensures that the structure of the dynamical equations remains unchanged, even though the variables describing the system have been transformed.

The new Hamiltonian H' is related to the original Lagrangian through a transformation analogous to the Legendre transformation:

$$H' = \sum_k P_k \dot{Q}_k - L'$$

where $L'$is a function expressed in terms of the new coordinates and velocities.

The function L'must satisfy Hamilton's principle:

$$\delta \int_{t_1}^{t_2} L'\ dt = 0$$

This condition guarantees that the transformed system yields the correct equations of motion.

The key idea behind canonical transformations is that they allow us to:

- Reformulate a problem in terms of more convenient variables
- Simplify the equations of motion
- Identify conserved quantities
- Reduce complex systems to integrable forms

These transformations are particularly useful in solving problems involving:

- Central force motion
- Periodic systems
- Action-angle variables

6.4 LEGENDRE TRANSFORMATION

The Legendre transformation is the mathematical method for changing the basis of one set of coordinates to another set of coordinates. Let us consider a function $f = f\ (x, y)$ of two independent variables x and y. The total differential of the function f is

$$df = \frac{\partial f}{\partial x}\ dx + \frac{\partial f}{\partial y}\ dy \qquad (6.2)$$

Let

$$u = \frac{\partial f}{\partial x} \qquad (6.3)$$

and

$$v = \frac{\partial f}{\partial y} \qquad (6.4)$$

Using equation (6.3) and (6.4) in equation (6.2) we get

$$df = udx + vdy \qquad (6.5)$$

In order to change the basis from (x, y) to (u, y), let u be an independent variable and x be a dependent variable.

Let $f' = f'(u, y)$ be a function of u and y defined by

$$f' = f - ux \qquad (6.6)$$

$$\Rightarrow df' = df - udx - xdu \qquad (6.7)$$

Using equation (6.5) in (6.7) we get

$$df' = udx + vdy - udx - xdu$$

$$\Rightarrow df' = vdy - xdu \qquad (6.8)$$

Since f' is a function of u and y, then

$$df' = \frac{\partial f'}{\partial u}\, du + \frac{\partial f'}{\partial y}\, dy \qquad (6.9)$$

Comparing equations (6.8) and (6.9), we get

$$x = -\frac{\partial f'}{\partial u} \qquad (6.10)$$

$$v = \frac{\partial f'}{\partial y} \qquad (6.11)$$

Equations (6.10) and (6.11) are called the Legendre transformation relation to change the basis from *(x, y)* to *(u, y).*

6.5 GENERATING FUNCTIONS

For canonical transformation, the Lagrangian L in $(p_k\, q_k)$ co-ordinates and $(P_k\, Q_k)$ coordinates must satisfy Hamilton's principle

$$\delta \int_{t_1}^{t_2} L dt = 0 \qquad (6.a)$$

and

$$\delta \int_{t_1}^{t_2} L' dt = 0 \qquad (6.b)$$

But $L = p_k \dot{q}_k - H$ and $L' = P_k \dot{Q}_k - H'$

Therefore, $$\delta \int_{t_2}^{t_1} [p_k \dot{q}_k - H]\, dt = 0$$

Subtracting equation (6.b) from equation (6.a) we get

$$\delta \int_{t_2}^{t_1} [(p_k \dot{q}_k - H) - (P_k \dot{Q}_k - H')]\, dt = 0$$

in the variation process the condition $\delta \int f dt = 0$ is to be satisfied in general by $f = \frac{dF}{dt}$ whereF is an arbitrary function.

Therefore, $$\delta \int_{t_2}^{t_1} \left(\frac{dF}{dt}\right) dt = 0$$

Where $\frac{dF}{dt} = L - L'$

$$\Rightarrow \frac{dF}{dt} = (p_k \dot{q}_k - H) - (P_k \dot{Q}_k - H')$$

The function F is known as the generating function and is a function of p_k, q_k, P_k, Q_k.

There are four forms of generating functions.

$F_1 = (q_k, Q_k, t)$ $F_2 = (q_k, P_k, t)$

$F_3 = (p_k, Q_k, t)$ $F_4 = (p_k, P_k, t)$

The choice of the functional form of the generating function *F* depends on the problem under consideration.

Transformation relation of different type of generating function

We will derive the transformation relation between the old set of position and momentum coordinates to the new set of position and momentum coordinates when the generating function affects the canonical transformation. Generating function has four forms

(a) Let us choose the generating function of form $F_1 = F_1(q_k, Q_k, t)$ i.e.,

$F_1 = F_1(q_1, q_2, q_3, \ldots\ldots\ldots, q_k; Q_1, Q_2, Q_3, \ldots\ldots\ldots\ldots, Q_k; t)$, then we get

$$\frac{dF_1}{dt} = \sum_k \frac{\partial F_1}{\partial q_k} \dot{q}_k + \sum_k \frac{\partial F_1}{\partial Q_k} \dot{Q}_k + \frac{\partial F_1}{\partial t} \qquad (6.12)$$

And we have

$$\frac{dF_1}{dt} = \sum_k p_k \dot{q}_k - H - \sum_k P_k \dot{Q}_k + H' \qquad (6.13)$$

Using equation (6.12) in equation (6.13) we get

$$\sum_k p_k \dot{q}_k - H - \sum_k P_k \dot{Q}_k + H' = \sum_k \frac{\partial F_1}{\partial q_k} \dot{q}_k + \sum_k \frac{\partial F_1}{\partial Q_k} \dot{Q}_k + \frac{\partial F_1}{\partial t}$$

$$\Rightarrow \sum_k \left(p_k - \frac{\partial F_1}{\partial q_k}\right) \dot{q}_k - \sum_k \left(P_k + \frac{\partial F_1}{\partial Q_k}\right) \dot{Q}_k + H' - H - \frac{\partial F_1}{\partial t} = 0$$

Thus, we get

$$p_k = \frac{\partial F_1}{\partial q_k} \tag{6.14}$$

Or

$$P_k = -\frac{\partial F_1}{\partial Q_k} \tag{6.15}$$

We also obtain

$$H' = H + \frac{\partial F_1}{\partial t} \tag{6.16}$$

(b) Let us consider the generating function of the for $F_2 = F_2\,(q_k, P_k, t)$. This form of generating function can be obtained from the generating function of the form $F_1 = F_1\,(q_k, Q_k, t)$ by changing the basis from (q_k, Q_k) to (q_k, P_k) using the Legendre transformation. So, we can write

$$F_2(q_k, P_k, t) = F_1(q_k, Q_k, t) + \sum_k P_k Q_K \tag{6.17}$$

Using equation (6.15) in equation (6.17), we get

$$\frac{\partial F_2}{\partial Q_k} = \frac{\partial F_1}{\partial Q_k} + P_k = -P_k + P_k = 0$$

So, the function F_2 is independent of the new variables Q_k.

Using equation (6.17) in equation (6.13), we get

$$\sum_k p_k \dot{q_k} - \sum_k P_k \dot{Q}_k + (H' - H) = \frac{\partial F_1}{dt} = \frac{d}{dt}\left[F_2(q_k, P_k, t) - \sum_k P_k Q_k\right] \tag{6.18}$$

$$\Rightarrow \sum_k p_k \dot{q_k} - \sum_k P_k \dot{Q}_k + (H' - H)$$

$$= \sum_k \frac{\partial F_2}{\partial q_k}\dot{q}_k + \sum_k \frac{\partial F_2}{\partial P_k}\dot{P}_k - \sum_k \dot{P}_k Q_k + \frac{\partial F_2}{\partial t} - \sum_k P_k \dot{Q}_k$$

$$\Rightarrow \sum_k p_k \dot{q_k} + (H' - H) = \sum_k \frac{\partial F_2}{\partial q_k}\dot{q}_k + \sum_k \frac{\partial F_2}{\partial P_k}\dot{P}_k - \sum_k \dot{P}_k Q_k + \frac{\partial F_2}{\partial t}$$

Comparing the coefficient of $\dot{q_k}$ and $\dot{P}_k$ on both sides of the equation, we get

$$p_k = \frac{\partial F_2}{\partial q_k} \tag{6.19}$$

$$Q_k = \frac{\partial F_2}{\partial P_k} \tag{6.20}$$

Also, we get

$$H' - H = \frac{\partial F_2}{\partial t} \Rightarrow H' = H + \frac{\partial F_2}{\partial t} \tag{6.21}$$

(c) Generating function of the form $F_3(p_k, Q_k, t)$. As in the case of F_2, $F_3(p_k, Q_k, t)$ can be obtained from $F_1(q_k, Q_k, t)$ by changing the basis of description from (q, Q) to (p, Q) by using the Legendre transformation. We get

$$F_3(p_k, Q_k, t) = F_1(q_k, Q_k, t) - \sum_k p_k q_k \tag{6.22}$$

We observe

$$\frac{\partial F_3}{\partial q_k} = \frac{\partial F_1}{\partial q_k} - p_k$$

$$\frac{\partial F_3}{\partial q_k} = p_k - p_k = 0$$

F_3 is independent of old coordinates. Using equation (6.22) in equation (6.13)

$$\sum_k p_k \dot{q}_k - \sum_k P_k \dot{Q}_k + (H' - H) = \frac{d}{dt}\left[F_3(p_k, Q_k, t) + \sum_k p_k q_k\right] \tag{6.23}$$

$$\Rightarrow \sum_k p_k \dot{q}_k - \sum_k P_k \dot{Q}_k + (H' - H) = \sum_k \frac{\partial F_3}{\partial p_k}\dot{p}_k + \sum_k \frac{\partial F_3}{\partial Q_k}\dot{Q}_k + \sum_k p_k \dot{q}_k + \frac{\partial F_3}{\partial t} + \sum_k \dot{p}_k q_k$$

$$\Rightarrow -\sum_k P_k \dot{Q}_k + (K - H) = \sum_k \frac{\partial F_3}{\partial p_k}\dot{p}_k + \sum_k \frac{\partial F_3}{\partial Q_k}\dot{Q}_k + \frac{\partial F_3}{\partial t} + \sum_k \dot{p}_k q_k$$

Equating coefficients of $\dot{Q}_k$ and $\dot{p}_k$ on both sides of the equation, we get

$$-P_k = \frac{\partial F_3}{\partial Q_k} \Rightarrow P_k = -\frac{\partial F_3}{\partial Q_k} \tag{6.24}$$

$$q_k + \frac{\partial F_3}{\partial p_k} = 0 \Rightarrow q_k = -\frac{\partial F_3}{\partial p_k} \tag{6.25}$$

Also

$$H' - H = \frac{\partial F_3}{\partial t} \Rightarrow H' = H + \frac{\partial F_3}{\partial t} \tag{6.26}$$

(d) Generating function of the form $F_4 = F_4(p_k, P_k, t)$. This form of generating function can be connected to $F_1$by using the Legendre transformation as

$$F_4(p_k, P_k, t) = F_1(q_k, Q_k, t) + \sum_k P_k Q_k - \sum_k p_k q_k \tag{6.27}$$

Using equation (6.14) and (6.15) in the above equation, we get

$$\frac{\partial F_4}{\partial Q_k} = \frac{\partial F_1}{\partial Q_k} - p_k = p_k - p_k = 0$$

$$\frac{\partial F_4}{\partial Q_k} = \frac{\partial F_1}{\partial Q_k} + P_k = -P_k + P_k = 0$$

So, F_4 does not depend on the old as well as the new coordinates.

Using equation (6.27) in equation (6.13), we get

$$\sum_k p_k \dot{q}_k - H - \sum_k P_k \dot{Q}_k + H' = \frac{d}{dt}\left[F_4(p_k, P_k, t) - \sum_k P_k Q_k + \sum_k p_k q_k\right] \quad (6.28)$$

$$\Rightarrow \sum_k p_k \dot{q}_k - H - \sum_k P_k \dot{Q}_k + H' = \sum_k \frac{\partial F_4}{\partial p_k}\dot{p}_k + \sum_k \frac{\partial F_4}{\partial P_k}\dot{P}_k - \sum_k P_k \dot{Q}_k - \sum_k \dot{P}_k Q_k + \sum_k p_k \dot{q}_k + \sum_k \dot{p}_k q_k + \frac{\partial F_4}{\partial t}$$

$$\Rightarrow H' - H = \sum_k \frac{\partial F_4}{\partial p_k}\dot{p}_k + \sum_k \frac{\partial F_4}{\partial P_k}\dot{P}_k - \sum_k \dot{P}_k Q_k + \sum_k \dot{p}_k q_k + \frac{\partial F_4}{\partial t}$$

Equating the coefficients of $\dot{p}_k$ and $\dot{P}_k$ on both sides of the above equation, we get

$$0 = \frac{\partial F_4}{\partial p_k} + q_k \Rightarrow q_k = -\frac{\partial F_4}{\partial p_k} \quad (6.29)$$

$$0 = \frac{\partial F_4}{\partial P_k} - Q_k \Rightarrow Q_k = \frac{\partial F_4}{\partial P_k} \quad (6.30)$$

Also

$$H' - H = \frac{\partial F_4}{\partial t} \Rightarrow H' = H + \frac{\partial F_4}{\partial t} \quad (6.31)$$

We can note from the result obtained that regardless of the form of generating function F, the relation between H and H' remains the same, i.e.,

$$H' = H + \frac{\partial F}{\partial t}$$

Thus, if the generating function F does not depend explicitly on time, then $\frac{\partial F}{\partial t} = 0$. Therefore, H'=H

6.6 CONDITIONS FOR CANONICAL TRANSFORMATION

Conditions for canonical transformation

i. The transformation from the old set of coordinates (q_k, p_k) to the new set of coordinates (Q_k, P_k) is said to be canonical if $\sum p_k dq_k - \sum P_k dQ_k$ is the exact differential.

Proof:

Suppose F=F (q_k, Q_k), then $\frac{\partial F}{\partial t} = 0$ and H=H'

We have, $p_k = \frac{\partial F}{\partial q_k}$ $(6.32a)$ and $P_k = -\frac{\partial F}{\partial Q_k}$ $(6.32b)$

Further, as $F = F(q, Q)$, the total differential of F is

$$dF = \sum_k \frac{\partial F}{\partial q_k} dq_k + \sum_k \frac{\partial F}{\partial Q_k} dQ_k \quad (6.33)$$

Using equations (6.32a) and (6.32b) in equation (6.33), we get

$$\Rightarrow dF = \sum_k p_k dq_k - \sum_k P_k dQ_k \qquad (6.34)$$

Since dF is an exact differential. Hence, for a given transformation to be canonical, the R.H.S of equation (6.34) must be an exact differential. That is the expression $\sum_k p_k dq_k - \sum_k P_k dQ_k$ must be an exact differential.

ii. If the transformation from (q_k, p_k) coordinates to (Q_k, P_k) coordinates are to be canonical, then the bilinear form given by $\sum_k(\delta p_k dq_k - \delta q_k dp_k)$remains invariant. That is

$$\sum_k(\delta \mathrm{p_k dq_k}\text{-}\delta \mathrm{q_k dp_k}) = \sum_k(\delta \mathrm{P_k dQ_k}\text{-}\delta \mathrm{Q_k dP_k})$$

Proof:

From Hamilton's canonical equations, we have

$$\dot{q}_k = \frac{\partial H}{\partial p_k} \qquad (6.35)$$

And

$$\dot{p}_k = -\frac{\partial H}{\partial q_k} \qquad (6.36)$$

Equations (6.35) and (6.36) can be written as

$$dq_k = \frac{\partial H}{\partial p_k}\, dt \qquad (6.37)$$

And

$$dp_k = -\frac{\partial H}{\partial q_k}\, dt \qquad (6.38)$$

From equations (6.37) and (6.38) we can write

$$\delta p_k \left(dq_k - \frac{\partial H}{\partial p_k}\, dt \right) = 0 \qquad (6.39)$$

And

$$\delta q_k \left(dp_k + \frac{\partial H}{\partial q_k}\, dt \right) = 0 \qquad (6.40)$$

Taking the sum over all the degrees of freedom, we get

$$\sum_k \left[\delta p_k \left(dq_k - \frac{\partial H}{\partial p_k}\, dt \right) \right] - \sum_k \left[\delta q_k \left(dp_k + \frac{\partial H}{\partial q_k}\, dt \right) \right] = 0$$

$$\sum_k (\delta p_k dq_k - \delta q_k dp_k) - \sum_k \left(\frac{\partial H}{\partial p_k}\, \delta p_k + \frac{\partial H}{\partial q_k}\, \delta q_k \right) dt = 0$$

$$\sum_k (\delta p_k dq_k - \delta q_k dp_k) - \delta H dt = 0 \qquad (6.41)$$

Similarly, for the transform coordinates, we get

$$\sum_k (\delta P_k dQ_k - \delta Q_k dP_k) - \delta H' dt = 0 \qquad (6.42)$$

Assuming that the generating function that affects the canonical transformation does not depend explicitly on time, then H'=H.

Then, from equations (6.41) and (6.42), we get

$$\sum_k \delta p_k dq_k - \delta q_k dp_k = \sum_k \delta P_k dQ_k - \delta Q_k dP_k$$

This shows that the bilinear form remains invariant in the canonical transformation.

(iii) Identical transformation:

Let us consider a generating function $F = F(q_k, P_k, t)$. This generating function looks like the second form.

Therefore, we have $p_k = \frac{\partial F_2}{\partial q_k} = P_k$, $Q_k = \frac{\partial F_2}{\partial P_k} = q_k$ and $\frac{\partial F_2}{\partial t} = 0$ (Since F is independent of t)

From the above, we can write

$$Q_k = q_k, \quad P_k = p_k, \quad and = H = H'$$

Therefore, the new and old variables are equal separately, and hence the generating function F generate an identical transformation.

(iv)

$$F = -\sum_k q_k P_k$$

The given generating function is of the second form of generating function, i.e. $F_2 = F_2(q_k, P_k, t)$

Therefore, we have

$$p_k = \frac{\partial F_2}{\partial q_k}, \quad Q_k = \frac{\partial F_2}{\partial P_k}, \quad and \quad H' = H + \frac{\partial F}{\partial t}$$

From the given function $F = \sum_k q_k P_k$, we get

$$\frac{\partial F}{\partial q_k} = -P_k, \frac{\partial F}{\partial P_k} = -q_k, \quad and \frac{\partial F}{\partial t} = 0$$

From the above relations, we obtain

$$Q_k = -q_k, \quad P_k = -p_k \; and \; H' = H$$

Thus, the given generating function generates a transformation leading to space inversion.

(v)

$$F = \sum_k q_k Q_k$$

The given generating function is of the first form of generating function, i.e. $F_1 = F_1(q_k, Q_k, t)$

Therefore, we have,

$$p_k = \frac{\partial F_1}{\partial q_k}, \quad P_k = -\frac{\partial F_1}{\partial Q_k}, \quad and \; H' = H + \frac{\partial F_1}{\partial t}$$

For the given function $F = \sum_k q_k Q_k$, we obtain

$$\frac{\partial F}{\partial q_k} = Q_k, \qquad \frac{\partial F}{\partial Q_k} = q_k, \quad and \ \frac{\partial F}{\partial t} = 0$$

From the above relations, we get

$$p_k = Q_k, \quad P_k = -q_k \quad and\ H' = H$$

So, the given generating function interchanges the position and momentum coordinates.

CHECK YOUR PROGESS

Q.1. Define canonical transformation.
Q.2. What is the Legendre transformation?
Q.3. Write down the different types of generating functions.
Q.4. Write down the generating function of the form $F_1(q, Q, t)$ and $F_2\ (q, p, t)$
Q.5. Define generating function in canonical transformation.
Q.6. Prove that the transformation $\mathrm{Q} = \mathrm{q}\cos\theta + \mathrm{p}\sin$ and$P = -q\sin\theta + p\cos\theta$is canonical.
Q.7. What is the concept of generating function.
Q.8. Acquire the generating function responsible for the transformation

$$Q = \ln q, P = qp$$

6.7 LET US SUM UP

- A canonical transformation is a change from one set of canonical variables (q, p) to another set (Q, P) such that the form of Hamilton's equations remains unchanged.
- These transformations are important because they help in simplifying dynamical problems, identifying conserved quantities, and converting complex systems into integrable forms.
- In a canonical transformation, the new Hamiltonian H'governs the system in the transformed variables while preserving the structure of the equations of motion.
- The Legendre transformation is a mathematical technique used to change variables and is fundamental in transforming the Lagrangian into the Hamiltonian formulation.
- A generating function (F) provides a systematic way to construct canonical transformations by establishing relations between old and new coordinates and momenta through partial derivatives.
- There are four types of generating functions based on different variable combinations: $F_1(q, Q)$, $F_2(q, P)$, $F_3(p, Q)$, and $F_4(p, P)$.
- The relation between the Hamiltonians before and after transformation remains consistent, and if the generating function does not explicitly depend on time, then $H' = H$.
- For a transformation to be canonical, the relevant expression must form an exact differential, and the bilinear form $\sum p_i dq_i = \sum P_i dQ_i$ remains invariant.
- Canonical transformations preserve the fundamental structure of phase space and are essential in advanced analytical mechanics.

- Special cases include identity transformation, space inversion, and transformations that interchange position and momentum variables.
- These transformations are widely applied in solving problems involving central forces, periodic motion, and action-angle variables.

✓6.8 ANSWERS TO CHECK YOUR PROGRESS

Answer to Q1: A canonical transformation is a transformation from one set of phase space variables $(q_k, p_k)$to another set (Q_k, P_k)such that the form of Hamilton's equations is preserved.

Answer to Q2: The Legendre transformation is a mathematical procedure used to change variables from one set to another, typically from coordinates to conjugate momenta. It is used to transform the Lagrangian into the Hamiltonian.

Answer to Q3: There are four types of generating functions:

$$F_1(q, Q, t), F_2(q, P, t), F_3(p, Q, t), F_4(p, P, t)$$

Answer to Q4: For $F_1(q, Q, t)$: $p_i = \frac{\partial F_1}{\partial q_i}, P_i = -\frac{\partial F_1}{\partial Q_i}$

For $F_2(q, P, t)$:$p_i = \frac{\partial F_2}{\partial q_i}, Q_i = \frac{\partial F_2}{\partial P_i}$

Answer to Q5: A generating function is a function that relates old and new canonical variables and generates a canonical transformation through its partial derivative.

Answer to Q6: Compute Poisson brackets:

$$\{Q, P\} = 1, \{Q, Q\} = 0, \{P, P\} = 0$$

Since canonical Poisson bracket relations are satisfied, the transformation is canonical

Answer to Q7: A generating function provides a systematic method to construct canonical transformations by relating old and new variables through partial derivatives. It simplifies the process of finding new canonical variables.

Answer to Q8: From $Q = \frac{\partial F_2}{\partial P} = l\,n\,q$

Integrating: $F_2 = P \ln q$

We have, $p = \frac{\partial F_2}{\partial q} = \frac{P}{q}$

$$\Rightarrow P = qp$$

Thus, the generating function is: $F_2 = P\ln q$

6.9 FURTHER READING

- Upadhyaya, J. C. (2026). *Classical mechanics* (3rd ed.). Himalaya Publishing House.
- Aruldhas, G. (2018). *Classical mechanics*. PHI Learning Private Limited.
- Goldstein, H., Poole, C. P., & Safko, J. L. (2011). *Classical mechanics* (3rd ed.). Pearson Education India

6.10 MODEL QUESTIONS

1. Discuss canonical transformation in detail.
2. Write down the different conditions of canonical transformation.
3. What do you understand by generating function.
4. Derive transformation equations first two generating functions.
5. Show that the transformations $Q = \log\left(1 + \sqrt{q}\cos p\right)$ and $P = 2\left(1 + \sqrt{q}\cos p\right)\sqrt{q}\sin p$ is canonical if q and p are canonically conjugate to each other.
6. Derive the bilinear invariant condition for the canonical transformation.

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UNIT 7: BRACKETS AND LIOUVILLE'S THEOREM

UNIT STRUCTURE

7.1 Learning Objectives
7.2 Introduction
7.3 Poisson's Brackets
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7.5 Relation Between Lagrange and Poisson Brackets
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7.1 LEARNING OBJECTIVES

After going through this unit, you will be able to

- explain Lagrange brackets and Poisson brackets
- recognize equations of motion in terms of Poisson bracket
- derive the relation between Lagrange and Poisson bracket
- describe phase space and Liouville's theorem.

7.2 INTRODUCTION

In the previous unit, we learned about Hamiltonian dynamics, Hamiltonian and canonical equations, Hamilton's equations in different coordinate systems, and their applications. In this unit, we will study the Poisson bracket, and how to write the equation of motion in terms of Poisson bracket. We will also learn about different kind of bracket called Lagrange bracket and its relationship with the Poisson bracket. We will also study about the phase space Liouville theorem.

7.3 POISSON'S BRACKETS

A Poisson bracket is a special type of relation between a pair of dynamical variables of any holonomic system, which is found to remain invariant under any canonical transformation.

Let us consider two functions F and G depending on the position coordinatesq_k, momentum co-ordinates p_k, and time t, then the Poisson bracket of the function$F$ and G is defined as

$$[\mathrm{F,G}]_{\mathrm{q_k,p_k}} = \left(\frac{\partial \mathrm{F}}{\partial \mathrm{q_k}} . \frac{\partial \mathrm{G}}{\partial \mathrm{p_k}} - \frac{\partial \mathrm{F}}{\partial \mathrm{p_k}} . \frac{\partial \mathrm{G}}{\partial \mathrm{q_k}}\right) \qquad (7.1)$$

Equation of motion and Poisson bracket

Let us consider a function F which is a function q_k, p_k and time t i.e. $F = F(q_k, p_k, t)$. The total time derivative can be written as

$$\frac{\mathrm{dF}}{\mathrm{dt}} = \frac{\partial \mathrm{F}}{\partial \mathrm{t}} + \left(\frac{\partial \mathrm{F}}{\partial \mathrm{q_k}}.\dot{\mathrm{q}}_\mathrm{k} - \frac{\partial \mathrm{F}}{\partial \mathrm{p_k}}.\dot{\mathrm{p}}_\mathrm{k}\right) \qquad (7.2)$$

From Hamilton's equation,

$$\dot{\mathrm{q}}_\mathrm{k} = \frac{\partial \mathrm{H}}{\partial \mathrm{p_k}} \text{ and } -\dot{\mathrm{p}}_\mathrm{k} = \frac{\partial \mathrm{H}}{\partial \mathrm{q_k}} \qquad (7.3)$$

Using equation (7.3) in equation (7.2), we can write

$$\frac{dF}{dt} = \sum_k \left(\frac{\partial F}{\partial q_k}\frac{\partial H}{\partial p_k} - \frac{\partial F}{\partial p_k}\frac{\partial H}{\partial q_k}\right) + \frac{\partial F}{\partial t}$$

$$\Rightarrow \frac{dF}{dt} = [F, H] + \frac{\partial F}{\partial t} \qquad (7.4)$$

If F is the constant of motion, then

$$\frac{dF}{dt} = 0$$

Thus, equation (7.3) reduces to

$$[F, H] + \frac{\partial F}{\partial t} = 0 \qquad (7.5)$$

If F does not depend explicitly on time, then $\frac{\partial F}{\partial t} = 0$

Therefore, equation (7.4) reduces to

$$[F, H] = 0 \qquad (7.6)$$

We can thus conclude that if a function F is a constant of motion and if it does not depend explicitly on time, then its Poisson bracket with the Hamiltonian vanishes, or conversely, a function whose Poisson bracket with the Hamiltonian vanishes is a constant of motion. The result $[F, H] = 0$ holds regardless of whether the Hamiltonian, H itself is a constant of motion itself.

Some special case

(a) $\dot{q}_k = [q_k, H]$ choosing $F = q_k$

(b) $\dot{p}_k = [p_k, H]$ choosing $F = p_k$

(c) $\dot{H} = \frac{\partial H}{\partial t}$ choosing $F = H$

The above three equations are referred to as equations of motion in Poisson bracket form.

Fundamental Poisson brackets and properties of the Poisson bracket

The Poisson has an antisymmetric property, given by

$$[F, G] = -[G, F]$$

We have,

$$[F, G] = \left[\frac{\partial F}{\partial q_k}\frac{\partial G}{\partial p_k} - \frac{\partial F}{\partial p_k}\frac{\partial G}{\partial q_k}\right] \qquad (7.7)$$

$$\Rightarrow [F, G] = -\left[\frac{\partial F}{\partial p_k}\frac{\partial G}{\partial q_k} - \frac{\partial G}{\partial p_k}\frac{\partial F}{\partial q_k}\right] = -[G, F]$$

Therefore, the Poisson bracket does not follow the commutative law of algebra.

A few of the cases where the Poisson bracket is applied

i. Substituting G with q_l in equation (7.7), we get

$$[F, q_l] = \sum_k \left[\frac{\partial F}{\partial q_k}\frac{\partial q_l}{\partial p_k} - \frac{\partial F}{\partial p_k}\frac{\partial q_l}{\partial q_k}\right] = -\sum_k \frac{\partial F}{\partial p_k}\delta_{lk}$$

Where δ_{lk} is a Kronecker delta function

$$\Rightarrow [F, q_l] = -\frac{\partial F}{\partial p_l}$$

$$\left[Since\ \frac{\partial q_l}{\partial p_k} = 0;\ \delta_{lk} = 0\ if\ l \neq k\ or\ \delta_{lk} = 1\ if\ l = k\right]$$

ii. Substituting F with q_k and G with q_l in equation (7.7), we get

$$[q_k, q_l] = -\frac{\partial q_k}{\partial p_l} = 0$$

Similarly, substituting F with p_k and G with q_l in equation (7.7), we get

$$[p_k, q_l] = -\frac{\partial p_k}{\partial p_l} = -\delta_{kl}$$

iii. Substituting G with p_l, from equation (7.7) we get

$$[F, p_l] = \sum_k \left[\frac{\partial F}{\partial q_k}\frac{\partial p_l}{\partial p_k} - \frac{\partial F}{\partial P_k}\frac{\partial p_l}{\partial q_k}\right]$$

$$[F, p_l] = \frac{\partial F}{\partial q_l}$$

Again if $F = p_k$ and $G = p_l$ then

$$[p_k, p_l] = \frac{\partial p_k}{\partial q_l} = 0$$

The above equation can be summarized as

$$[q_k, q_l] = [p_k, p_l] = 0 \qquad (7.8)$$

$$[q_k, p_l] = \delta_{kl} \qquad (7.9)$$

Equations (7.8) and (7.9) are known as the fundamental Poisson bracket.
Some identities from the definition of the Poisson bracket

i. $[F, F] = 0$
ii. $[F, C] = 0,\quad C\ is\ constant$
iii. $[CF, G] = C[F, G]$
iv. $[F_1 + F_2, G] = [F_1, G] + [F_2, G]$
v. $[F, G_1G_2] = G_1[F, G_2] + [F, G_1]G_2$
vi. $\frac{\partial}{\partial t}[F, G] = \left[\frac{\partial F}{\partial t}, G\right] + \left[F, \frac{\partial G}{\partial t}\right]$
vii. $\left[F, [G, K]\right] + \left[G, [K, F]\right] + \left[K, [F, G]\right] = 0$

Jacobi's identity

It is one of the important properties of the Poisson bracket. It states that the sum of the cyclic permutation of the double Poisson bracket of three functions is zero. i.e.,
[F, [G, K]] + [G, [K, F]] + [K, [F, G]] = 0.
Where F, G, and K are three arbitrary functions of p_k, q_k, and time.

Proof: Let us consider $\left[F, [G, K]\right] + \left[G, [K, F]\right] = \left[F, [G, K]\right] - \left[G, [F, K]\right]$(since [K, F] = − [F, K])

$$= \left[F, \sum_k \left(\frac{\partial G}{\partial q_k}\frac{\partial K}{\partial p_k} - \frac{\partial G}{\partial p_k}\frac{\partial K}{\partial q_k}\right)\right] - \left[G, \sum_k \left(\frac{\partial F}{\partial q_k}\frac{\partial K}{\partial p_k} - \frac{\partial F}{\partial p_k}\frac{\partial K}{\partial q_k}\right)\right]$$

$$= \left[F, \sum_k \left(\frac{\partial G}{\partial q_k}\frac{\partial K}{\partial p_k}\right)\right] - \left[F, \sum_k \left(\frac{\partial G}{\partial p_k}\frac{\partial K}{\partial q_k}\right)\right] - \left[G, \sum_k \left(\frac{\partial F}{\partial q_k}\frac{\partial K}{\partial p_k}\right)\right] + \left[G, \sum_k \left(\frac{\partial F}{\partial p_k}\frac{\partial K}{\partial q_k}\right)\right]$$

Using the property [F, GK] = [F, G] K + $[F, K]G$,

$$[F,[G,K]]-[G,[F,K]]$$
$$=\left[F,\sum_k\frac{\partial G}{\partial q_k}\right]\sum_k\frac{\partial K}{\partial p_k}+\left[F,\sum_k\frac{\partial K}{\partial p_k}\right]\sum_k\frac{\partial G}{\partial q_k}-\left[F,\sum_k\frac{\partial G}{\partial p_k}\right]\sum_k\frac{\partial K}{\partial q_k}$$
$$-\left[F,\sum_k\frac{\partial K}{\partial q_k}\right]\sum_k\frac{\partial G}{\partial p_k}-\left[G,\sum_k\frac{\partial F}{\partial q_k}\right]\sum_k\frac{\partial k}{\partial p_k}-\left[G,\sum_k\frac{\partial k}{\partial p_k}\right]\sum_k\frac{\partial F}{\partial q_k}$$
$$+\left[G,\sum_k\frac{\partial F}{\partial p_k}\right]\sum_k\frac{\partial K}{\partial q_k}+\left[G,\sum_k\frac{\partial K}{\partial q_k}\right]\sum_k\frac{\partial F}{\partial p_k}$$

$$\Rightarrow [F,[G,k]]-[G,[F,K]]$$
$$=\sum_k\left\{-\frac{\partial K}{\partial q_k}\left(\left[\frac{\partial F}{\partial p_k},G\right]+\left[F,\frac{\partial G}{\partial p_k}\right]\right)+\frac{\partial K}{\partial p_k}\left(\left[\frac{\partial F}{\partial q_k},G\right]+\left[F,\frac{\partial G}{\partial q_k}\right]\right)\right\}$$
$$+\sum_k\left\{\frac{\partial G}{\partial q_k}\left(\left[F,\frac{\partial K}{\partial p_k}\right]-\frac{\partial G}{\partial p_k}\left[F,\frac{\partial K}{\partial q_k}\right]\right)-\frac{\partial F}{\partial q_k}\left(\left[G,\frac{\partial K}{\partial P_k},\right]+\frac{\partial F}{\partial p_k}\left[G,\frac{\partial K}{\partial q_k}\right]\right)\right\}$$

Now, using $\frac{\partial}{\partial x}[F,G]=\left[\frac{\partial F}{\partial x},G\right]+\left[F,\frac{\partial G}{\partial x}\right]$, we get,

$$[F,[G,K]]-[G,[F,K]]=\sum_k\left[-\frac{\partial K}{\partial q_k}\frac{\partial}{\partial p_k}[F,G]+\frac{\partial K}{\partial p_k}\frac{\partial}{\partial q_k}[F,G]\right]+0$$
$$=-[K,[F,G]]$$

Therefore, $[F,[G,K]]+[G,[K,F]]+[K,[F,G]]=0$.This is the Jacobi identity.

Poisson bracket and canonical transformation

Let us consider two dynamical variables F and G, which are functions of generalized coordinates q's and generalized momentum p's, then the Poisson bracket of F and G is given by

$$[F,G]_{q,p}=\sum_k\left(\frac{\partial F}{\partial q_k}\frac{\partial G}{\partial p_k}-\frac{\partial F}{\partial p_k}\frac{\partial G}{\partial q_k}\right) \qquad (7.10)$$

Let us consider the canonical transformation of the variable q's and p's to Q's and P's, respectively, then the Poisson bracket of F and G in terms of transformed variables is

$$[F,G]_{Q,P}=\sum_k\left(\frac{\partial F}{\partial Q_k}\frac{\partial G}{\partial P_k}-\frac{\partial F}{\partial P_k}\frac{\partial G}{\partial Q_k}\right) \qquad (7.11)$$

Equation (7.11) can be written as

$$[F,G]_{Q,P}=\sum_{k,j}\left[\frac{\partial F}{\partial Q_k}\left(\frac{\partial G}{\partial q_j}\frac{\partial q_j}{\partial P_k}+\frac{\partial G}{\partial P_j}\frac{\partial P_j}{\partial P_k}\right)-\frac{\partial F}{\partial P_k}\left(\frac{\partial G}{\partial q_j}\frac{\partial q_j}{\partial Q_k}+\frac{\partial G}{\partial p_j}\frac{\partial p_j}{\partial Q_k}\right)\right] \qquad (7.12)$$

On rearranging, we get

$$[F,G]_{Q,P}=\sum_k\left\{\frac{\partial G}{\partial q_j}[F,q_j]_{Q,P}+\frac{\partial G}{\partial p_j}[F,p_j]_{Q,P}\right\} \qquad (7.13)$$

Further,

$$[F,q_j]_{Q,P}=-[q_j,F]_{Q,P}=-\sum_i\left(\frac{\partial q_j}{\partial Q_i}\frac{\partial F}{\partial P_i}-\frac{\partial q_j}{\partial P_i}\frac{\partial F}{\partial Q_i}\right)$$
$$=-\sum_{k,i}\left\{\frac{\partial q_j}{\partial Q_i}\left(\frac{\partial F}{\partial q_k}\frac{\partial q_k}{\partial P_i}+\frac{\partial F}{\partial p_k}\frac{\partial p_k}{\partial P_i}\right)-\frac{\partial q_j}{\partial P_i}\left(\frac{\partial F}{\partial q_k}\frac{\partial q_k}{\partial Q_i}+\frac{\partial F}{\partial p_k}\frac{\partial p_k}{\partial Q_i}\right)\right\}$$

$$= -\sum_k \left\{ \frac{\partial F}{\partial q_k} \sum_i \left(\frac{\partial q_j}{\partial Q_i}\frac{\partial q_k}{\partial P_i} - \frac{\partial q_j}{\partial P_i}\frac{\partial q_k}{\partial Q_i} \right) + \frac{\partial F}{\partial p_k} \sum_i \left(\frac{\partial q_j}{\partial Q_i}\frac{\partial p_k}{\partial P_i} - \frac{\partial q_j}{\partial P_i}\frac{\partial p_k}{\partial Q_i} \right) \right\}$$

$$\Rightarrow [F, q_j]_{Q,P} = -\sum_k \left\{ \frac{\partial F}{\partial q_k}[q_j, q_k]_{Q,P} + \frac{\partial F}{\partial p_k}[q_j, p_k]_{Q,P} \right\}$$

We have, $[q_j, q_k] = 0$, $[q_j, p_k] = 0 \; if \; j \neq k$, $[q_j, p_k] = 1 \; if \; j = k$

$$\therefore [F, q_j]_{Q,P} = -\sum_k \frac{\partial F}{\partial p_k}\delta_{jk} = -\frac{\partial F}{\partial p_j} \qquad (7.14)$$

Similarly,

$$[F, p_j]_{Q,P} = \frac{\partial F}{\partial q_j} \qquad (7.15)$$

Using equations (7.14) and (7.15) in equation (7.12), we get

$$[F, G]_{Q,P} = \sum_k \left\{ -\frac{\partial G}{\partial q_j}\frac{\partial F}{\partial p_j} + \frac{\partial G}{\partial p_j}\frac{\partial F}{\partial q_j} \right\}$$

$$\Rightarrow [F, G]_{Q,P} = \sum_k \left(\frac{\partial F}{\partial q_j}\frac{\partial G}{\partial p_j} - \frac{\partial F}{\partial p_j}\frac{\partial G}{\partial q_j} \right)$$

$$\Rightarrow [F, G]_{Q,P} \Rightarrow [F, G]_{q,p}$$

Hence, Poisson brackets are invariant under canonical transformation.

7.4 LAGRANGE'S BRACKETS

The Lagrange bracket of two dynamical variables $F(q_k, p_k)$ and $G\ (q_k, p_k)$ is defined as

$$\{\mathrm{F, G}\} = \sum_{\mathbf{k}} \left[\frac{\partial \mathrm{q_k}}{\partial \mathrm{F}} \cdot \frac{\partial \mathrm{p_k}}{\partial \mathrm{G}} - \frac{\partial \mathrm{p_k}}{\partial \mathrm{F}} \cdot \frac{\partial \mathrm{q_k}}{\partial \mathrm{G}} \right]$$

Properties of Lagrange bracket

i. Lagrange brackets are invariant under canonical transformation, i.e.

$$\{u, v\}_{q,p} = \{u, v\}_{Q,P}$$

Where (q, p) is the old set of generalized co-ordinates and *(Q, P)* is the new set of generalized coordinates.

Proof: The Lagrange bracket of u and v is given by

$$\{u, v\}_{q,p} = \sum_i \left(\frac{\partial q_i}{\partial u}\frac{\partial p_i}{\partial v} - \frac{\partial q_i}{\partial v}\frac{\partial p_i}{\partial u} \right) \qquad (7.16)$$

Now the new and old sets of generalized coordinates are related by a canonical transformation as

$$\left.\begin{aligned} q_i &= q_i(Q_j, P_j) \\ p_i &= p_i(Q_j, P_j) \\ Q_j &= Q_j(q_i, p_i) \\ P_j &= P_j(q_i, p_i) \end{aligned}\right\} \qquad (7.17)$$

Partially differentiating $q_i = q_i(Q_j, P_j)$ with respect to v, we get,

$$\frac{\partial q_i}{\partial v} = \sum_j \left(\frac{\partial q_i}{\partial Q_j}\frac{\partial Q_j}{\partial V} + \frac{\partial q_i}{\partial P_j}\frac{\partial P_j}{\partial v} \right) \qquad (7.18)$$

Again, partially differentiating $p_i = p_i(Q_j, P_j)$ with respect to v, we get

$$\frac{\partial p_i}{\partial v} = \sum_j \left(\frac{\partial p_i}{\partial Q_j} \frac{\partial Q_j}{\partial v} + \frac{\partial p_i}{\partial P_j} \frac{\partial P_j}{\partial v} \right) \tag{7.19}$$

We have the first form of generating function

$$p_i = \frac{\partial F_1}{\partial q_i} \qquad and \qquad -P_i = \frac{\partial F_1}{\partial Q_i} \tag{7.20}$$

$$\therefore \frac{\partial p_i}{\partial Q_j} = \frac{\partial}{\partial Q_j}\left(\frac{\partial F_1}{\partial q_i}\right)$$

$$\Rightarrow \frac{\partial p_i}{\partial Q_j} = \frac{\partial}{\partial q_i}\left(\frac{\partial F_1}{\partial Q_j}\right)$$

$$\Rightarrow \frac{\partial p_i}{\partial Q_j} = -\frac{\partial P_j}{\partial q_i} \tag{7.21}$$

Again, from the second form of generating function, we have

$$p_i = \frac{\partial F_2}{\partial q_i} \qquad and \qquad Q_i = \frac{\partial F_2}{\partial P_i} \tag{7.22}$$

$$\frac{\partial p_i}{\partial P_j} = \frac{\partial}{\partial P_j}\left(\frac{\partial F_2}{\partial q_i}\right) = \frac{\partial}{\partial q_i}\left(\frac{\partial F_2}{\partial P_j}\right)$$

$$\Rightarrow \frac{\partial p_i}{\partial P_j} = \frac{\partial Q_j}{\partial q_i} \tag{7.23}$$

We have from third generating function

$$q_i = -\frac{\partial F_3}{\partial p_i} \qquad and \qquad P_i = -\frac{\partial F_3}{\partial Q_i} \tag{7.24}$$

$$\frac{\partial q_i}{\partial Q_j} = -\frac{\partial}{\partial Q_j}\left(\frac{\partial F_3}{\partial p_i}\right) = -\frac{\partial}{\partial p_i}\left(\frac{\partial F_3}{\partial Q_j}\right)$$

$$\Rightarrow \frac{\partial q_i}{\partial Q_j} = \frac{\partial P_j}{\partial p_i} \tag{7.25}$$

Also, we have from the fourth generating function

$$q_i = -\frac{\partial F_4}{\partial p_i} \qquad and \qquad Q_i = \frac{\partial F_4}{\partial P_i} \tag{7.26}$$

$$\frac{\partial q_i}{\partial P_j} = -\frac{\partial}{\partial P_j}\left(\frac{\partial F_4}{\partial p_i}\right) = -\frac{\partial}{\partial p_i}\left(\frac{\partial F_4}{\partial P_j}\right)$$

$$\Rightarrow \frac{\partial q_i}{\partial P_j} = -\frac{\partial Q_j}{\partial p_i} \tag{7.27}$$

Using equations (7.25) and (7.27) in equation (7.18), we get

$$\frac{\partial q_i}{\partial v} = \sum_j \left(\frac{\partial P_j}{\partial p_i} \frac{\partial Q_j}{\partial v} - \frac{\partial Q_j}{\partial p_i} \frac{\partial P_j}{\partial v} \right) \tag{7.28}$$

Similarly, using equations (7.23) and (7.21) in equation (7.19) we get

$$\frac{\partial p_i}{\partial v} = \sum_j \left(\frac{\partial Q_j}{\partial q_i} \frac{\partial P_j}{\partial v} - \frac{\partial P_j}{\partial q_i} \frac{\partial Q_j}{\partial v} \right) \tag{7.29}$$

Using equations (7.28) and (7.29) in equation (7.16), we get

$$\{u,v\}_{q,p} = \sum_{i,j}\left[\frac{\partial q_i}{\partial u}\frac{\partial Q_j}{\partial q_i}\frac{\partial P_j}{\partial v} - \frac{\partial q_i}{\partial u}\frac{\partial P_j}{\partial q_i}\frac{\partial Q_j}{\partial v} - \frac{\partial P_j}{\partial p_i}\frac{\partial Q_j}{\partial v}\frac{\partial p_i}{\partial u} + \frac{\partial Q_j}{\partial p_i}\frac{\partial P_j}{\partial v}\frac{\partial p_i}{\partial u}\right]$$

$$= \sum_j\left[\frac{\partial P_j}{\partial v}\left\{\sum_i\left(\frac{\partial q_i}{\partial u}\frac{\partial Q_j}{\partial q_i} + \frac{\partial Q_j}{\partial p_i}\frac{\partial p_i}{\partial u}\right)\right\} - \frac{\partial Q_j}{\partial v}\left\{\sum_i\left(\frac{\partial q_i}{\partial u}\frac{\partial P_j}{\partial q_i} + \frac{\partial P_j}{\partial p_i}\frac{\partial p_i}{\partial u}\right)\right\}\right] \qquad (7.30)$$

Again from $Q_j = Q_j(q_i, p_i)$ and $P_j = P_j(q_i, p_i)$, we get

$$\frac{\partial Q_j}{\partial u} = \sum_i\left(\frac{\partial q_i}{\partial u}\frac{\partial Q_j}{\partial q_i} + \frac{\partial Q_j}{\partial p_i}\frac{\partial p_i}{\partial u}\right) \; and \quad \frac{\partial P_j}{\partial u} = \sum_i\left(\frac{\partial q_i}{\partial u}\frac{\partial P_j}{\partial q_i} + \frac{\partial P_j}{\partial p_i}\frac{\partial p_i}{\partial u}\right) \qquad (7.31)$$

Using equation (7.31) in (7.30) we get

$$\{u,v\}_{q,p} = \sum_j\left(\frac{\partial Q_j}{\partial u}\frac{\partial P_j}{\partial v} - \frac{\partial Q_j}{\partial v}\frac{\partial P_j}{\partial u}\right)$$

$$\Rightarrow \{u,v\}_{q,p} = \{u,v\}_{Q,P}$$

ii. Lagrange bracket does not obey the commutative law i.e. $\{F,G\} = -\{G,F\}$
Proof:

$$\{F,G\} = \sum_k\left(\frac{\partial q_k}{\partial F}\frac{\partial p_k}{\partial G} - \frac{\partial p_k}{\partial F}\frac{\partial q_k}{\partial G}\right)$$

$$= -\sum_k\left(\frac{\partial p_k}{\partial F}\frac{\partial q_k}{\partial G} - \frac{\partial q_k}{\partial F}\frac{\partial p_k}{\partial G}\right)$$

$$\{F,G\} = -\{G,F\}$$

iii. $\{q_i,q_j\} = 0 \quad \{p_i,p_j\} = 0 \quad \{q_i,p_j\} = \delta_{ij}$

$$\{q_i,q_j\} = \sum_k\left(\frac{\partial q_k}{\partial q_i}\frac{\partial p_k}{\partial q_j} - \frac{\partial q_k}{\partial q_j}\frac{\partial p_k}{\partial q_i}\right)$$

Since q's and p's are independent, we have

$$\frac{\partial p_k}{\partial q_j} = 0 \qquad and \quad \frac{\partial p_k}{\partial q_i} = 0$$

Then

$$\{q_i,q_j\} = 0$$

Similarly, we can also show

$$\{p_i,p_j\} = 0$$

Again,

$$\{q_i,p_j\} = \sum_k\left(\frac{\partial q_k}{\partial q_i}\frac{\partial p_k}{\partial p_j} - \frac{\partial q_k}{\partial p_j}\frac{\partial p_k}{\partial q_i}\right)$$

Since q's and p's are independent, then

$$\frac{\partial q_k}{\partial p_j} = 0$$

$$\therefore \{q_i,p_j\} = \sum_k\frac{\partial q_k}{\partial q_i}\frac{\partial p_k}{\partial p_j}$$

Also, we have

$$\frac{\partial q_k}{\partial q_i} = \delta_{ki} \qquad and \qquad \frac{\partial p_k}{\partial p_j} = \delta_{kj}$$

Hence

$$\{q_i, p_j\} = \sum_k \frac{\partial q_k}{\partial q_i}\frac{\partial p_k}{\partial p_j} = \sum_k \delta_{ki}\delta_{kj}$$

$$\{q_i, p_j\} = \delta_{ij}$$

7.5 RELATION BETWEEN LAGRANGE AND POISSON BRACKETS

Let u_l, where l= 1, 2, 3, ……., 2n, be the 2n independent function and u_l is a function of 2n independent coordinates $q_1, q_2, q_3, \ldots\ldots, q_n; p_1, p_2, p_1, \ldots\ldots, p_n$, then

$$\sum_{l=1}^{2n} \{u_l, u_i\}\,[u_l, u_j] = \delta_{ij} \qquad (7.32)$$

Where, $[u_l, u_j]$ is the Poisson bracket and $\{u_l, u_i\}$ is the Lagrange bracket.
Let the Lagrange bracket of dynamical variables of the function u_l, and u_j is

$$\sum_{k=1}^{n}\left(\frac{\partial q_k}{\partial u_l}\frac{\partial p_k}{\partial u_i} - \frac{\partial p_k}{\partial u_l}\frac{\partial q_k}{\partial u_i}\right) \qquad (7.33)$$

And the Poisson bracket is given by

$$\sum_{m=1}^{n}\left(\frac{\partial u_l}{\partial q_m}\frac{\partial u_j}{\partial p_m} - \frac{\partial u_l}{\partial p_m}\frac{\partial u_j}{\partial q_m}\right) \qquad (7.34)$$

Now,

$$\sum_{i=1}^{2n}\{u_l, u_i\}\,[u_l, u_j] = \sum_{l=1}^{2n}\left\{\sum_{k=1}^{n}\left(\frac{\partial q_k}{\partial u_l}\frac{\partial p_k}{\partial u_i} - \frac{\partial p_k}{\partial u_l}\frac{\partial q_k}{\partial u_i}\right)\right\}\left\{\sum_{m=1}^{n}\left(\frac{\partial u_l}{\partial q_m}\frac{\partial u_j}{\partial p_m} - \frac{\partial u_l}{\partial p_m}\frac{\partial u_j}{\partial q_m}\right)\right\}$$

$$= \sum_{l=1}^{2n}\left\{\sum_{k,m=1}^{n}\left(\frac{\partial q_k}{\partial u_l}\frac{\partial p_k}{\partial u_i}\frac{\partial u_l}{\partial q_m}\frac{\partial u_j}{\partial p_m} - \frac{\partial q_k}{\partial u_l}\frac{\partial p_k}{\partial u_i}\frac{\partial u_l}{\partial p_m}\frac{\partial u_j}{\partial q_m} - \frac{\partial p_k}{\partial u_l}\frac{\partial q_k}{\partial u_i}\frac{\partial u_l}{\partial q_m}\frac{\partial u_j}{\partial p_m} + \frac{\partial p_k}{\partial u_l}\frac{\partial q_k}{\partial u_i}\frac{\partial u_l}{\partial p_m}\frac{\partial u_j}{\partial q_m}\right)\right\}$$

$$= \sum_{k,m=1}^{n}\left(\frac{\partial p_k}{\partial u_i}\frac{\partial u_j}{\partial p_m}\right).\left(\sum_{l=1}^{2n}\frac{\partial q_k}{\partial u_l}\frac{\partial u_l}{\partial q_m}\right) - \left(\sum_{k,m=1}^{n}\frac{\partial p_k}{\partial u_i}\frac{\partial u_j}{\partial q_m}\right).\left(\sum_{l=1}^{2n}\frac{\partial q_k}{\partial u_l}\frac{\partial u_l}{\partial p_m}\right)$$
$$- \left(\sum_{k,m=1}^{n}\frac{\partial q_k}{\partial u_i}\frac{\partial u_j}{\partial p_m}\right).\left(\sum_{l=1}^{2n}\frac{\partial p_k}{\partial u_l}\frac{\partial u_l}{\partial q_m}\right) + \left(\sum_{k,m=1}^{n}\frac{\partial q_k}{\partial u_i}\frac{\partial u_j}{\partial q_m}\right).\left(\sum_{l=1}^{2n}\frac{\partial p_k}{\partial u_l}\frac{\partial u_l}{\partial p_m}\right)$$

$$= \sum_{k,m=1}^{n}\left(\frac{\partial p_k}{\partial u_i}\frac{\partial u_j}{\partial p_m}\right).\left(\frac{\partial q_k}{\partial q_m}\right) - \sum_{k,m=1}^{n}\left(\frac{\partial p_k}{\partial u_i}\frac{\partial u_j}{\partial q_m}\right).\left(\frac{\partial q_k}{\partial p_m}\right) - \sum_{k,m=1}^{n}\left(\frac{\partial q_k}{\partial u_i}\frac{\partial u_j}{\partial p_m}\right).\left(\frac{\partial p_k}{\partial q_m}\right)$$
$$+ \sum_{k,m=1}^{n}\left(\frac{\partial q_k}{\partial u_i}\frac{\partial u_j}{\partial q_m}\right).\left(\frac{\partial p_k}{\partial p_m}\right)$$

Kronecker delta function,

$\frac{\partial q_k}{\partial q_m} = \frac{\partial p_k}{\partial p_m} = \delta_{km}$ if $l = m$, then $\delta_{km} = 1$ and if $l \neq m$, then $\delta_{km} = 0$

And $\frac{\partial q_k}{\partial p_m} = \frac{\partial p_k}{\partial q_m} = 0$ (independent variables)

$$\therefore \sum_{i=1}^{2n}\{u_l, u_i\}\left[u_l, u_j\right] = \sum_{k,m=1}^{n}\left(\frac{\partial p_k}{\partial u_i}\frac{\partial u_j}{\partial p_m}\right).\delta_{km} - 0 - 0 + \sum_{k,m=1}^{n}\left(\frac{\partial q_k}{\partial u_i}\frac{\partial u_j}{\partial q_m}\right).\delta_{km}$$

$$= \sum_{k,m=1}^{n}\left(\frac{\partial p_k}{\partial u_i}\frac{\partial u_j}{\partial p_m}\right).\delta_{mk} + \sum_{k,m=1}^{n}\left(\frac{\partial q_k}{\partial u_i}\frac{\partial u_j}{\partial q_m}\right).\delta_{mk} \qquad (since\ \delta_{km} = \delta_{mk})$$

$$= \sum_{k,m=1}^{n}\left(\frac{\partial p_k}{\partial u_i}\frac{\partial u_j}{\partial p_m}\right).\left(\frac{\partial p_m}{\partial p_k}\right) + \sum_{k,m=1}^{n}\left(\frac{\partial q_k}{\partial u_i}\frac{\partial u_j}{\partial q_m}\right).\left(\frac{\partial q_m}{\partial q_k}\right)$$

$$\left(since\ \delta_{mk} = \frac{\partial p_m}{\partial p_k} = \frac{\partial q_m}{\partial q_k}\right)$$

$$\Rightarrow \sum_{i=1}^{2n}\{u_l, u_i\}\left[u_l, u_j\right] = \frac{\partial u_j}{\partial u_i} + \frac{\partial u_j}{\partial u_i} = \delta_{ji} + \delta_{ji}$$

$$\Rightarrow \sum_{i=1}^{2n}\{u_l, u_i\}\left[u_l, u_j\right] = \delta_{ji} = \delta_{ij}$$

The above relation holds good for any arbitrary transformation, even if the coordinates are not canonical.

7.6 ANGULAR MOMENTUM AND POISSON BRACKETS

By the definition of angular momentum, $\vec{J}=\vec{r}\times\vec{p}$

Here $\vec{p}$ is the linear momentum and $\vec{r}$is the radius vector.

$$\Rightarrow\vec{J} = \left(x\hat{i}+y\hat{j}+z\hat{k}\right) \times \left(p_x\hat{i}+p_y\hat{j}+p_z\hat{k}\right)$$

$$\Rightarrow\vec{J} = \left(yp_z-zp_y\right)\hat{i}+ \left(zp_x-xp_z\right)\hat{j} + \left(xp_y-yp_z\right)\hat{k} \qquad (7.35)$$

$$\therefore J_x=yp_z-zp_y$$
$$J_y=zp_x-xp_z$$
$$J_z=xp_y-yp_z$$

$$\frac{\partial J_x}{\partial p_x} = 0 \qquad \frac{\partial J_y}{\partial p_x} = z \qquad \frac{\partial J_z}{\partial p_x} = -y$$
$$\frac{\partial J_x}{\partial p_y} = -z \qquad \frac{\partial J_y}{\partial p_y} = 0 \qquad \frac{\partial J_z}{\partial p_y} = x$$
$$\frac{\partial J_x}{\partial p_z} = y \qquad \frac{\partial J_y}{\partial p_z} = -x \qquad \frac{\partial J_z}{\partial p_z} = 0$$

The Poisson bracket of the functions F and G is given by

$$[F,G]=\sum_k\left(\frac{\partial F}{\partial q_k}\frac{\partial G}{\partial p_k}-\frac{\partial F}{\partial p_k}\frac{\partial G}{\partial q_k}\right)$$

Again, we have $[p_x,p_y]=[p_y,p_z]=[p_z,p_x]=0$

Also, we have $[F,p_l]=\frac{\partial F}{\partial q_l}$,

Using the above result, we get

$$[J_x,p_y]=p_z,\qquad [J_x,p_z]=-p_y,\qquad [J_x,p_x]=0$$
$$[J_y,p_x]=-p_z,\qquad [J_y,p_y]=0,\qquad [J_y,p_z]=p_x$$
$$[J_z,p_x]=p_y,\quad [J_z,p_y]=-p_x,\quad [J_z,p_z]=0$$

Now, the Poisson bracket of the components of angular momentum *J*

$$[J_x,J_y]=\sum_k\left(\frac{\partial J_x}{\partial q_k}\frac{\partial J_y}{\partial p_k}-\frac{\partial J_x}{\partial p_k}\frac{\partial J_y}{\partial q_k}\right)$$

$$\Rightarrow [J_x,J_y]=\left(\frac{\partial J_x}{\partial x}\frac{\partial J_y}{\partial p_x}-\frac{\partial J_x}{\partial p_x}\frac{\partial J_y}{\partial x}\right)+\left(\frac{\partial J_x}{\partial y}\frac{\partial J_y}{\partial p_y}-\frac{\partial J_x}{\partial p_y}\frac{\partial J_y}{\partial y}\right)+\left(\frac{\partial J_x}{\partial z}\frac{\partial J_y}{\partial p_z}-\frac{\partial J_x}{\partial p_z}\frac{\partial J_y}{\partial z}\right)$$

$$\Rightarrow [J_x,J_y]=0-0+0-0+(-p_y)(-x)-(y)(p_x)$$

$$\Rightarrow [J_x,J_y]=xp_y-yp_x=J_z$$

Similarly, we can also prove that

$$[J_y,J_z]=J_x,\qquad [J_z,J_x]=J_y$$

7.7 LIOUVILLE'S THEOREM

The Hamiltonian is defined in terms of the position coordinates $(q_1,q_2,q_3,\ldots\ldots\ldots\ldots,q_n)$ and momentum coordinates $p_1,p_2,p_3,\ldots\ldots\ldots\ldots,p_n)$ in the Hamiltonian formulation. We can consider a space of dimension 2n in which a point with 2n coordinates provides the complete dynamical specification of a mechanical system. Such a space is called 2n dimensional phase space.

Liouville's theorem consists of two parts

i. The first part states that the density in phase space is a conserved quantity i.e.

$$\frac{d\rho}{dt}=0$$

ii. The second part states that the extension in phase space is conserved. i.e.

$$\frac{d}{dt}(\delta\Gamma)=0$$

Proof of the first part, i.e., $\frac{d\rho}{dt}=0$

Let $\rho\,(p,q,t)$ be the density of points in the p-$q$phase space. If the speed of the representative points entering the element $ABCD$ through AB is $\dot{q}$, then the number of representative points entering $ABCD$ per unit time is $\rho\dot{q}\,dp$.

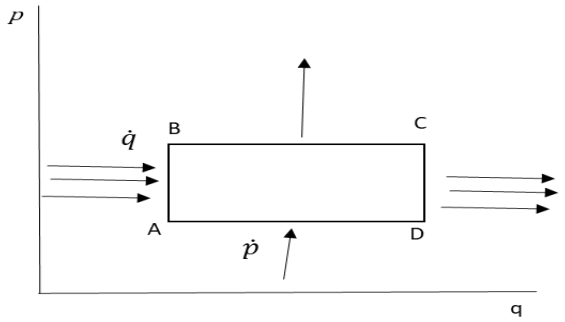


Figure 7.1: Flow of representative points through the phase space element ABCD in q-p plane.

And the number of points leaving through the sideCD is

$$\rho\dot{q}\,dp + \frac{\partial}{\partial q}(\rho\dot{q})\,dq\,dp$$

Thus, the number of points remaining in the element $ABCD$ is

$$\rho\dot{q}\,dp - \left[\rho\dot{q}\,dp + \frac{\partial}{\partial q}(\rho\dot{q})\,dq\,dp\right] = -\frac{\partial}{\partial q}(\rho\dot{q})\,dq\,dp \qquad (7.36)$$

Similarly, the number of points entering through AD and leaving through BC are

$$\rho\dot{p}\,dq$$

and

$$\rho\dot{p}\,dq + \frac{\partial}{\partial p}(\rho\dot{p})\,dp\,dq$$

respectively

Thus, the number of points which remain in the element $ABCD$ is

$$-\frac{\partial}{\partial p}(\rho\dot{p})\,dp\,dq \qquad (7.37)$$

By adding (7.36) and (7.37), we get

$$-\left[\frac{\partial}{\partial q}(\rho\dot{q}) + \frac{\partial}{\partial p}(\rho\dot{p})\right]dp\,dq \qquad (7.38)$$

Which represents the increase in the points per unit time in the element ABCD. But the rate of increase of points per unit time in the element is

$$\frac{\partial\rho}{\partial t}\,dp\,dq$$

Thus, we get

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial q}(\rho\dot{q}) + \frac{\partial}{\partial p}(\rho\dot{p}) = 0$$

$$\Rightarrow \frac{\partial\rho}{\partial t} + \dot{q}\frac{\partial\rho}{\partial q} + \dot{p}\frac{\partial\rho}{\partial p} + \rho\left(\frac{\partial\dot{q}}{\partial q} + \frac{\partial\dot{p}}{\partial p}\right) = 0 \qquad (7.39)$$

Again, we have

$$\dot{q} = \frac{\partial H}{\partial p}$$

$$\dot{p} = -\frac{\partial H}{\partial q}$$

Hence,

$$\frac{\partial \dot{q}}{\partial q} = \frac{\partial^2 H}{\partial q\, \partial p}$$
$$\frac{\partial \dot{p}}{\partial p} = -\frac{\partial^2 H}{\partial p\, \partial q}$$

If the Hamiltonian H has continuous second-order derivatives. Then,

$$\frac{\partial \dot{q}}{\partial q} = -\frac{\partial \dot{p}}{\partial p} \qquad (7.40)$$

Using equation (7.40) in equation (7.39), we get

$$\frac{\partial \rho}{\partial t} + \dot{q}\frac{\partial \rho}{\partial q} + \dot{p}\frac{\partial \rho}{\partial p} = 0 \qquad (7.41)$$

But we have $\rho = \rho(q, p, t)$. The total time derivative of ρ is

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \dot{q}\frac{\partial \rho}{\partial q} + \dot{p}\frac{\partial \rho}{\partial p} \qquad (7.42)$$

Thus, equation (7.42) becomes

$$\frac{d\rho}{dt} = 0$$

Or we can say

$$\rho = \text{constant}$$

which is the proof of Liouville's theorem.

Proof of the second part, i.e., $\frac{d}{dt}(\delta\Gamma) = 0$

We have, $\delta N = \rho\, \delta\Gamma$ (7.43)

Now, taking the total time derivative of equation (7.40) we get

$$\frac{d}{dt}(\delta N) = \frac{d\rho}{dt}\, \delta\Gamma + \rho\frac{d}{dt}(\delta\Gamma) \qquad (7.44)$$

Since the number of phase points in a given region of phase space must be fixed, then

$$\frac{d}{dt}(\delta N) = 0$$

Using the above equation in equation (7.44), we get

$$\frac{d\rho}{dt}\, \delta\Gamma + \rho\frac{d}{dt}(\delta\Gamma) = 0 \qquad (7.45)$$

We also have, $\frac{d\rho}{dt} = 0$

Then equation (7.45) can be written as

$$\rho\frac{d}{dt}(\delta\Gamma) = 0$$

Since $\rho \neq 0$, then

$$\frac{d}{dt}(\delta\Gamma) = 0$$

CHECK YOUR PROGESS

Q.1. Define Poisson bracket.
Q.2. State Jacobi identity.
Q.3. Define Lagrange bracket.

Q.4. What are the properties of Lagrange bracket.
Q.5. Write down the relation between the Lagrange and Poisson brackets.
Q.6. Define angular momentum
Q.7. State Liouville's theorem

⏱7.8 LET US SUM UP

- Poisson brackets provide a powerful mathematical tool to describe the relationship between dynamical variables in Hamiltonian mechanics and are invariant under canonical transformations.
- The Poisson bracket of two functions helps express the equations of motion in compact form and plays a crucial role in identifying constants of motion.
- A function whose Poisson bracket with the Hamiltonian is zero (and has no explicit time dependence) is a constant of motion.
- The fundamental Poisson brackets are:
  - $\{q_i, q_j\} = 0$
  - $\{p_i, p_j\} = 0$
  - $\{q_i, p_j\} = \delta_{ij}$.These are important for derivations and problem-solving.
- Important properties of Poisson brackets include Antisymmetric, Linearity, Non-commutativity and s atisfaction of Jacobi identity, which ensures consistency of the algebra.
- The Jacobi identity is a fundamental relation in Poisson bracket algebra and is frequently asked in exams.
- Lagrange brackets are defined in terms of derivatives of coordinates and momenta and are useful in studying transformations between variables.
- Key properties of Lagrange brackets: Antisymmetric, Invariant under canonical transformations and Self-bracket is zero.
- The relation between Poisson and Lagrange brackets establishes a connection between dynamical evolution and coordinate transformations and is important for theoretical understanding.
- The components of angular momentum satisfy specific Poisson bracket relations, which are useful in rotational dynamics and symmetry analysis.
- Phase space is a $2n$-dimensional space where each state of a system is represented by generalized coordinates and momenta.
- Liouville's theorem states that:
  - The density of phase space points remains constant along the motion of a system.
  - The phase space volume is conserved during time evolution.
- Liouville's theorem is fundamental in statistical mechanics and describes the conservation of phase space distribution.

✅7.9 ANSWERS TO CHECK YOUR PROGRESS

Answer to Q1: The Poisson bracket of two functions $f(q,p,t)$and $g(q,p,t)$is defined as:

$$\{f,g\} = \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right)$$

Answer to Q2: The Jacobi identity for Poisson brackets is:

$$\{f,\{g,h\}\}+\{g,\{h,f\}\}+\{h,\{f,g\}\}=0$$

Answer to Q3: The Lagrange bracket between two generalized coordinates $u$and $v$is defined as:

$$[u,v]=\sum_i\left(\frac{\partial q_i}{\partial u}\frac{\partial p_i}{\partial v}-\frac{\partial q_i}{\partial v}\frac{\partial p_i}{\partial u}\right)$$

Answer to Q4: The main properties are:

a. Antisymmetric: $[u,v]=-[v,u]$,
b. Linearity
c. Self-bracket is zero: $[u,u]=0$
d. They depend on the transformation between variables

Answer to Q5: Poisson and Lagrange brackets are reciprocal to each other:

$$\sum_k[u,x_k]\{x_k,v\}=\delta_{uv}$$

Answer to Q6: Angular momentum of a particle is defined as:

$$\vec{L}=\vec{r}\times\vec{p}$$

Answer to Q7: Liouville's theorem states that:

The density of points in phase space remains constant along the motion of a system.

7.10 FURTHER READING

- Upadhyaya, J. C. (2026). *Classical mechanics* (3rd ed.). Himalaya Publishing House.
- Aruldhas, G. (2018). *Classical mechanics*. PHI Learning Private Limited.
- Goldstein, H., Poole, C. P., & Safko, J. L. (2011). *Classical mechanics* (3rd ed.). Pearson Education India

7.11 MODEL QUESTIONS

1. Prove the invariance of the Lagrange bracket under canonical transformation.
2. Show that the Poisson bracket is invariant under canonical transformation.
3. Derive the bilinear invariant condition for the canonical transformation.
4. State and prove Jacobi's identity.
6. Prove Liouville's theorem.
7. Obtain the equation of motion in terms of the Poisson bracket.
8. Write down the relations between angular momentum and Poisson bracket.
9. Write down the properties of the Poisson bracket.
10. Derive the fundamental Poisson bracket.
11. Prove that $[J_y,J_z]=J_x$, Using the Poisson bracket.
12. Derive the Lagrange bracket formula.
13. Derive the relation between the Lagrange and the Poisson bracket.

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UNIT 8: HAMILTON JACOBI THEORY

UNIT STRUCTURE

8.1 Learning Objectives
8.2 Introduction
8.3 Hamilton Jacobi Equation
8.4 Hamilton's-Jacobi Method
8.5 Solution of Harmonic Oscillator Problem
8.6 Kepler's Problem
8.7 Hamilton Characteristic Function
8.8 Let Us Sum Up
8.9 Answers to Check Your Progress
8.10 Further Reading
8.11 Model Questions

8.1 LEARNING OBJECTIVES

After going through this unit, you will be able to

- discuss the concept of Hamilton Jacobi equation, and its physical significance
- describe about properties and characteristics of the Hamilton Jacobi equation
- construct the equation of motion for harmonic oscillator using Hamilton Jacobi method
- explain Kepler's problem for a body in central force field.

8.2 INTRODUCTION

The concepts discussed in the previous unit, particularly Poisson brackets, Lagrange brackets, and Liouville's theorem, highlight the mathematical structure and invariance properties of Hamiltonian mechanics. These tools play an important role in understanding canonical transformations and the phase space description of dynamical systems. Building on these ideas, we now introduce the Hamilton Jacobi theory, which provides another powerful formulation of classical mechanics. This approach transforms the problem of solving Hamilton's equations into solving a single partial differential equation, known as the Hamilton–Jacobi equation, and offers an alternative method for determining the motion of a mechanical system.

8.3 HAMILTON JACOBI EQUATION

A) Hamilton Jacobi Theory

In Lagrange and Hamiltonian dynamics, the basic aim is finding equation of motion to describe a physical system. The Lagrangian is a function of generalized coordinates, generalized velocity and time, whereas the Hamiltonian is a function of generalized coordinate, generalized, momenta and time. Both the system deals with energy instead

of forces. In Hamilton Jacobi theory, we choose new canonical variables, where they are assumed to be constant or cyclic.
In Hamiltonian and Lagrangian systems, a generating function is usually chosen to derive equation of motion and to interpret. But the Hamilton Jacobi theory is a method to find the generating function. Once the generating function is found its corresponding canonical transformations can be obtained and integration can be performed directly, which is the advantage of Hamilton Jacobi theory and the relation between generating function and Hamiltonian is called the Hamilton Jacobi equation.

B) The Hamilton Jacobi Equation

Under a canonical transformation from the old phase space coordinates $(q_i, p_i)$to a new set (Q_i, P_i), the equations of motion retain their canonical form, given by

$$\dot{Q}_i = \frac{\partial K}{\partial P_i}, \quad \dot{P}_i = -\frac{\partial K}{\partial Q_i}$$

where $K(Q_i, P_i, t)$is the new Hamiltonian. If the transformation is generated by a generating function F, the old Hamiltonian $H$and the new Hamiltonian $K$are related through

$$K = H + \frac{\partial F}{\partial t}.$$

A particularly important case arises when the transformation is chosen such that the new Hamiltonian vanishes, i.e., $K = 0$. In this case, the transformed system becomes trivial (all new coordinates are constants of motion), and the above relation reduces to

$$H + \frac{\partial F}{\partial t} = 0. \qquad (8.1)$$

Now, consider the generating function of type $F_2(q_i, P_i, t)$. The transformation equations then give

$$p_i = \frac{\partial F_2}{\partial q_i}, Q_i = \frac{\partial F_2}{\partial P_i}.$$

Substituting $p_i = \frac{\partial F_2}{\partial q_i}$ into equation (8.1), we obtain

$$H\left(q_i, \frac{\partial F_2}{\partial q_i}, t\right) + \frac{\partial F_2}{\partial t} = 0. \qquad (8.2)$$

Next, consider the total time derivative of the generating function F_2:

$$\frac{dF_2}{dt} = \sum_i \frac{\partial F_2}{\partial q_i}\dot{q}_i + \frac{\partial F_2}{\partial t}$$

Using $p_i = \frac{\partial F_2}{\partial q_i}$, this becomes

$$\frac{dF_2}{dt} = \sum_i p_i\,\dot{q}_i + \frac{\partial F_2}{\partial t}.$$

Substituting from equation (8.1), we get

$$\frac{dF_2}{dt} = \sum_i p_i\,\dot{q}_i - H.$$

But from classical mechanics,

$$L = \sum_i p_i\,\dot{q}_i - H,$$

where L is the Lagrangian. Hence,

$$\frac{dF_2}{dt} = L.$$

Integrating over time, we obtain

$$F_2 = \int L\, dt = S,$$

where S is the action, also known as Hamilton's principal function.
Finally, substituting $F_2 = S$ into equation (8.2), we arrive at

$$H\left(q_i, \frac{\partial S}{\partial q_i}, t\right) + \frac{\partial S}{\partial t} = 0. \qquad (8.3)$$

This equation (8.3) is a first-order partial differential equation known as the Hamilton–Jacobi equation, which provides a powerful alternative formulation of classical mechanics. It reduces the problem of motion to solving a single scalar function S, from which the complete dynamics of the system can be obtained.

C) Hamilton Characteristic Equation

The Hamilton–Jacobi equation is given by

$$H\left(q_i, \frac{\partial S}{\partial q_i}, t\right) + \frac{\partial S}{\partial t} = 0 \qquad (8.4)$$

If the Hamiltonian does not explicitly depend on time, i.e., $H = H(q_i, p_i)$, then the Hamiltonian is conserved and can be written as

$$H(q_i, p_i) = E = \text{constant}.$$

In this case, the Hamilton–Jacobi equation admits a solution in which the action S can be separated as

$$S(q_i, P_i, t) = W(q_i, P_i) - Et \qquad (8.5)$$

where $W(q_i, P_i)$is a function of coordinates and constants P_i, and is known as Hamilton's characteristic function.
Substituting this form into equation (8.4), we obtain the time-independent Hamilton–Jacobi equation:

$$H\left(q_i, \frac{\partial W}{\partial q_i}\right) = E,$$

which is an ordinary (time-independent) partial differential equation.

The function $W(q_i, P_i)$ plays the role of a generating function of type F_2, since it depends on old coordinates q_i and new momenta P_i. It generates a canonical transformation to new variables (Q_i, P_i) in which the transformed Hamiltonian becomes a constant. For a time-independent Hamiltonian, the new Hamiltonian is $K = E$ (a constant), whereas for a time-dependent Hamiltonian, one can choose the transformation such that $K = 0$. Thus, the characteristic function W(and more generally S) can be used to generate canonical transformations that simplify the equations of motion, often reducing them to trivial forms.

8.4 HAMILTON JACOBI METHOD

The Hamilton Jacobi method is an alternative formulation of classical mechanics in which the problem of motion is reduced to solving a single partial differential equation. Starting from Hamilton's equations of motion,

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \dot{p}_i = -\frac{\partial H}{\partial q_i}, \qquad (8.6)$$

we perform a canonical transformation from $(q_i,\ p_i)$ to a new set of variables (Q_i, P_i) such that the new Hamiltonian Kbecomes zero. Under this transformation, the equations of motion reduce to

$$\dot{Q}_i = 0, \dot{P}_i = 0, \qquad (8.7)$$

which shows that the new coordinates and momenta are constants of motion.

If the transformation is generated by Hamilton's principal function $S(q_i, P_i, t)$, then the old and new Hamiltonians are related by

$$K = H + \frac{\partial S}{\partial t}. \qquad (8.8)$$

For the choice $K = 0$, we obtain the Hamilton–Jacobi equation**:**

$$H\left(q_i, \frac{\partial S}{\partial q_i}, t\right) + \frac{\partial S}{\partial t} = 0. \qquad (8.9)$$

Once a complete solution S is obtained, the transformation equations are given by

$$p_i = \frac{\partial S}{\partial q_i}, Q_i = \frac{\partial S}{\partial P_i}. \qquad (8.10)$$

Time-Independent Hamiltonian

If the Hamiltonian does not explicitly depend on time, then

$$H(q_i, p_i) = E = \text{constant}. \qquad (8.11)$$

In this case, the solution of the Hamilton–Jacobi equation can be written as

$$S(q_i, P_i, t) = W(q_i, P_i) - Et, \qquad (8.12)$$

where $W(q_i, P_i)$ is called Hamilton's characteristic function**.**

Substituting this into equation (8.9), we obtain the time-independent Hamilton–Jacobi equation:

$$H\left(q_i, \frac{\partial W}{\partial q_i}\right) = E. \qquad (8.13)$$

8.5 SOLUTION OF HARMONIC OSCILLATOR PROBLEM

The Hamiltonian of a one-dimensional harmonic oscillator is given by

$$H(q, p) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2 \qquad (8.14)$$

where q is the displacement, p is the momentum, and $\omega = \sqrt{\frac{k}{m}}$ is the angular frequency.

Using the relation $p = \frac{\partial S}{\partial q}$

The Hamiltonian can be written as

$$H = \frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{1}{2}m\omega^2 q^2$$

Substituting this into the Hamilton–Jacobi equation (8.14), we obtain

$$\frac{1}{2m}\left[\left(\frac{\partial S}{\partial q}\right)^2 + m^2\omega^2 q^2\right] + \frac{\partial S}{\partial t} = 0 \qquad (8.15)$$

Assuming a separable solution of the form

$$S(q,\alpha,t) = W(q,\alpha) - \alpha t \qquad (8.16)$$

where α is a constant (total energy), Differentiating equation (8.16) w.r.t q and t and substituting into equation (8.15), we have

$$\frac{1}{2m}\left(\frac{\partial W}{\partial q}\right)^2 + \frac{1}{2}m\omega^2 q^2 = \alpha \qquad (8.17)$$

Rearranging,

$$\frac{\partial W}{\partial q} = \sqrt{2m\left(\alpha - \frac{1}{2}m\omega^2 q^2\right)}$$

Integrating, we obtain

$$W = \int \sqrt{2m\left(\alpha - \frac{1}{2}m\omega^2 q^2\right)}\, dq = m\omega \int \sqrt{\left(\frac{2\alpha}{m\omega^2} - q^2\right)}\, dq \qquad (8.18)$$

Thus, the principal function becomes

$$S = m\omega \int \sqrt{\left(\frac{2\alpha}{m\omega^2} - q^2\right)}\, dq - \alpha t \qquad (8.19)$$

Using the transformation relations

$$p = \frac{\partial S}{\partial q}, \beta = \frac{\partial S}{\partial \alpha},$$

and simplifying, we obtain

$$\omega(\beta + t) = \sin^{-1}\left(q\sqrt{\frac{m\omega^2}{2\alpha}}\right)$$

Solving for q,

$$q = \sqrt{\frac{2\alpha}{m\omega^2}}\sin[\omega(t+\beta)]. \qquad (8.20)$$

Letting $q_0 = \sqrt{\frac{2\alpha}{m\omega^2}}$

the final solution becomes

$$q = q_0 \sin[\omega(t+\beta)]. \qquad (8.21)$$

8.6 KEPLER'S PROBLEM

In this section, we apply the Hamilton–Jacobi equation to study the motion of a particle in an inverse-square central potential, known as Kepler's problem**.** The potential is given by

$$V(r) = -\frac{k}{r},$$

where $k$is a positive constant. For a conservative system, the Hamiltonian is equal to the total energy, i.e.,

$$H = E = \text{constant.}$$

The Hamilton–Jacobi equation is

$$H\left(q_i, \frac{\partial S}{\partial q_i}, t\right) + \frac{\partial S}{\partial t} = 0. \qquad (8.22)$$

For a time-independent Hamiltonian, we take

$$S(r,\phi,t) = W(r,\phi) - Et. \quad (8.23)$$

In plane polar coordinates, the Hamiltonian for a particle in a central potential becomes

$$H = \frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2} - \frac{k}{r} \quad (8.24)$$

Using the relations

$$p_r = \frac{\partial W}{\partial r}, p_\phi = \frac{\partial W}{\partial \phi}$$

and substituting (8.23) into equation (8.22), we obtain

$$\frac{1}{2m}\left(\frac{\partial W}{\partial r}\right)^2 + \frac{1}{2mr^2}\left(\frac{\partial W}{\partial \phi}\right)^2 - \frac{k}{r} = E. \quad (8.25)$$

Since the motion is separable, we write

$$W(r,\phi) = W_r(r) + W_\phi(\phi). \quad (8.26)$$

Substituting into equation (8.25), we get

$$\frac{1}{2m}\left(\frac{dW_r}{dr}\right)^2 + \frac{1}{2mr^2}\left(\frac{dW_\phi}{d\phi}\right)^2 - \frac{k}{r} = E. \quad (8.27)$$

Since ϕ is a cyclic coordinate,

$$\frac{dW_\phi}{d\phi} = L \Rightarrow W_\phi = L\phi, \quad (8.28)$$

where L is the conserved angular momentum.

Substituting into equation (8.27), we obtain

$$\frac{1}{2m}\left(\frac{dW_r}{dr}\right)^2 + \frac{L^2}{2mr^2} - \frac{k}{r} = E \quad (8.29)$$

Rearranging, $\left(\frac{dW_r}{dr}\right)^2 = 2mE + \frac{2mk}{r} - \frac{L^2}{r^2}$ (8.30)

Thus,

$$W_r = \int \sqrt{2mE + \frac{2mk}{r} - \frac{L^2}{r^2}}\, dr \quad (8.31)$$

The complete solution is

$$S = W_r(r) + L\phi - Et \quad (8.32)$$

To obtain the orbit, we use

$$\frac{\partial S}{\partial L} = \beta = \text{constant} \quad (8.33)$$

which gives $\frac{\partial W_r}{\partial L} + \phi = \beta$

After integration and simplification, the equation of the orbit is obtained as

$$r = \frac{p}{1 + e\cos\phi}, \quad (8.34)$$

Where $p = \frac{L^2}{mk}, e = \sqrt{1 + \frac{2EL^2}{mk^2}}$. (8.35)

Nature of Orbits

- For $E < 0 (e < 1)$**:** Ellipse
- For $E = 0 (e = 1)$**:** Parabola
- For $E > 0 (e > 1)$**:** Hyperbola

Thus, the Hamilton–Jacobi method successfully reduces Kepler's problem to a solvable form and yields the classical conic section orbits. This highlights the power of the method in dealing with central force problems and connecting dynamics with geometry.

8.7 HAMILTON'S CHARACTERISTIC FUNCTION

Hamilton's characteristic function arises naturally in the solution of the Hamilton–Jacobi equation when the Hamiltonian does not explicitly depend on time. In such cases, the total energy of the system remains conserved, i.e.,

$$H(q_i, p_i) = E = \text{constant.} \qquad (8.36)$$

The Hamilton–Jacobi equation is

$$H\left(q_i, \frac{\partial S}{\partial q_i}, t\right) + \frac{\partial S}{\partial t} = 0. \qquad (8.37)$$

For a time-independent Hamiltonian, the principal function Scan be separated as

$$S(q_i, P_i, t) = W(q_i, P_i) - Et, \qquad (8.38)$$

where $W(q_i, P_i)$ is called Hamilton's characteristic function.

Substituting equation (8.38) into (8.37), we obtain

$$H\left(q_i, \frac{\partial W}{\partial q_i}\right) = E. \qquad (8.39)$$

This is the time-independent Hamilton–Jacobi equation, which depends only on the coordinates and constants of motion.

The function $W(q_i, P_i)$ serves as a generating function of type F_2, since it depends on the old coordinates $q_i$and the new momenta P_i. The transformation equations are given by

$$p_i = \frac{\partial W}{\partial q_i}, Q_i = \frac{\partial W}{\partial P_i}. \qquad (8.40)$$

Thus, once the complete solution W is obtained, the motion of the system can be determined through simple differentiation. The constants P_i and Q_i correspond to conserved quantities and initial conditions of the motion.

Further, since

$$\frac{dS}{dt} = L,$$

we have

$$S = \int L\, dt, \qquad (8.41)$$

which shows that the principal function S represents the action, while W represents the reduced action or characteristic function.

Thus, Hamilton's characteristic function W plays a central role in simplifying the Hamilton Jacobi equation for time-independent systems. By determining W, the complete dynamics of the system can be obtained, making it a powerful analytical tool in classical mechanics.

Physical Significance of Hamilton's characteristic function $W(q_i, P_i)$

Hamilton's characteristic function $W(q_i, P_i)$ plays a crucial role in simplifying and interpreting the dynamics of a mechanical system:

- **Reduction to Time-Independent Form:**
  The function W transforms the Hamilton–Jacobi equation into a time-independent equation**,** thereby reducing the complexity of the problem. This

allows the motion of the system to be determined without explicitly involving time, making the analysis simpler and more tractable.

- **Generating Function for Canonical Transformations:**
  W acts as a generating function of type F_2, producing canonical transformations from the old variables (q_i, p_i) to new variables (Q_i, P_i). Under this transformation, the equations of motion become trivial, and the new variables remain constants, greatly simplifying the solution of the system.
- **Solution of Systems with Conserved Quantities:**
  For systems in which energy is conserved and variables are separable, Wenables the separation of variables, allowing the Hamilton Jacobi equation to be solved systematically. This is particularly useful in central force problems, oscillatory motion, and other symmetric systems.
- **Connection with Action and Least Principle:**
  Since W is closely related to the principal function S, which represents the **action,** it provides a direct connection to the principle of least action. Thus, it offers a deeper variational interpretation of classical mechanics.
- **Bridge to Wave Mechanics:**
  The Hamilton–Jacobi formulation establishes a strong analogy with wave mechanics**.** The function S (and hence S) resembles the phase of a wave, and this analogy forms the foundation for the transition from classical mechanics to quantum mechanics, particularly in the development of the Schrödinger equation.

CHECK YOUR PROGRESS

Q1. Solve the Hamiltonian Jacobi equation for a particle of mass m moves under the Hamiltonian H=$\frac{P^2}{2}$, construct the Hamiltonian Jacobi equation.

Q2. Obtain Hamilton's characteristic function $W$for a particle of mass $m$moving in a uniform gravitational field along the z-axis, where $V = mgz$.

Q3. Construct the Hamilton Jacobi equation for a three-dimensional Harmonic oscillator and define the generating function.

Q4. Apply Hamilton Jacobi theory for a free particle of mass m under gravity and obtain the solution.

Q5. How Hamilton's characteristic function $W(q_i, P_i)$ helps in reducing the Hamilton Jacobi equation to a time-independent form.

Q6. Show that $W(q_i, P_i)$ can act as a generating function of type F_2.

Q7. Why is the Hamilton Jacobi method useful for systems with conserved energy and separable coordinates?

Q8. Discuss the connection between Hamilton's characteristic function and wave mechanics.

8.8 LET US SUM UP

- The Hamilton–Jacobi theory provides an alternative formulation of classical mechanics, where the problem of motion is reduced to solving a single partial differential equation instead of a set of differential equations.

- The Hamilton Jacobi theory provides an alternative formulation of classical mechanics, where the problem of motion is reduced to solving a single partial differential equation instead of a set of differential equations.
- The Hamilton Jacobi equation is a first-order partial differential equation given by

$$H\left(q_i, \frac{\partial S}{\partial q_i}, t\right) + \frac{\partial S}{\partial t} = 0$$

where S is Hamilton's principal function (action).

- The principal function S plays a central role, as it contains the complete dynamical information of the system and generates canonical transformations.
- The concept of canonical transformation is fundamental in this theory, where the transformation from (q_i, p_i) to (Q_i, P_i) preserves the form of Hamilton's equations.
- For time-independent Hamiltonians, the solution can be separated as

$$S(q_i, P_i, t) = W(q_i, P_i) - Et,$$

where W is Hamilton's characteristic function**.**

- The function $W(q_i, P_i)$ reduces the Hamilton–Jacobi equation to a time-independent form, making it easier to solve.
- The characteristic function Wacts as a generating function (type F_2**)**, enabling the determination of new canonical variables and simplifying the equations of motion.
- The Hamilton Jacobi method transforms the system into one where the new variables are constants of motion, thus simplifying integration.
- For a simple harmonic oscillator, the Hamiltonian Jacobi equation is

$$\frac{1}{2m}\left[\left(\frac{\partial S}{\partial q}\right)^2 + m^2\omega^2 q^2\right] + \frac{\partial S}{\partial t} = 0$$

- Application to the simple harmonic oscillator shows that the method yields the correct sinusoidal motion:

$$q = q_0 \sin \omega(t + \beta),$$

demonstrating the effectiveness of the approach.

- Application to Kepler's problem (motion under inverse-square law) leads to the equation of conic sections:

$$r = \frac{p}{1 + e\cos\phi},$$

showing that the orbits are ellipse, parabola, or hyperbola depending on energy.

- Kepler's problem, i.e., the motion of a particle under an inverse-square central potential, can be solved using the Hamilton Jacobi method.

- For a potential $V(r) = -\frac{k}{r}$, the Hamilton–Jacobi equation separates in polar coordinates, leading to the following form:

$$\frac{1}{2m}\left(\frac{dW_r}{dr}\right)^2 + \frac{1}{2mr^2}\left(\frac{dW_\phi}{d\phi}\right)^2 - \frac{k}{r} = E$$

- The Hamilton Jacobi formulation provides a strong link between classical mechanics and quantum mechanics, where the principal function S is analogous to the phase of a wave.
- The method is particularly useful for systems with conserved quantities and separable coordinates, allowing systematic and simplified solutions.

✓ 8.9 ANSWERS TO CHECK YOUR PROGRESS

Answer to Q1: To obtain the Hamiltonian Jacobi equation for a particle moving under Hamiltonian H=$\frac{P^2}{2m}$, the following steps are adopted

Write the Standard Hamiltonian Jacobi equation

$$H\left(q_i, \frac{\partial s}{\partial q_i}, t\right) + \frac{\partial s}{\partial t} = 0$$

$$p = \frac{\partial S}{\partial q}$$

Given Hamiltonian H=$\frac{P^2}{2m}$

Therefore

$$0 = \frac{1}{2}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{\partial s}{\partial t}$$

Answer to Q2: Since the projectile moves in a uniform gravitational field along z axis, therefore the potential energy

$$v = mgz$$

and kinetic energy

$$T = \frac{P_x^2}{2m} + \frac{P_y^2}{2m} + \frac{P_z^2}{2m}$$

So, the Hamiltonian

$$H = \frac{P_x^2}{2m} + \frac{P_y^2}{2m} + \frac{P_z^2}{2m} + mgz$$

The Hamiltonian Jacobi equation

$$H\left(q_i, \frac{\partial s}{\partial q_i}, t\right) + \frac{\partial s}{\partial t} = 0$$

Where, S is principal function of Hamiltonian Jacobi equation, now employing H in the H-J equation

$$\frac{1}{2\mathrm{m}}\left[\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2\right] + mgz + \frac{\partial s}{\partial t} = 0$$

Since the projectile moves along z axis, therefore the x and y axis can be considered as cyclic, now the principal function S can be expressed as

$$S = W(q, \alpha) - Et$$

$$S = \alpha_x x + \alpha_y y + W(z) - \alpha_1 t$$

Substituting S in the above modified Hamiltonian Jacobi equation and simplifying

$$\frac{dw}{dz} = \sqrt{2m(\alpha_1 - mgz) - \alpha_x^2 - \alpha_y^2}$$

Integrating

$$W = \int \sqrt{2m(\alpha_1 - mgz) - \alpha_x^2 - \alpha_y^2}\, dz$$

Which is the characteristics function of a projectile of mass m moving in a uniform gravitational field along z axis.

Answer to Q3: To construct the Hamiltonian Jacobi equation for a three harmonic oscillator first we must start with the Hamiltonian of three-dimensional Harmonic oscillator by choosing the coordinates and spring constants.

$$H(q,p) = \frac{P_1^2}{2m} + \frac{P_2^2}{2m} + \frac{P_3^2}{2m} + \tfrac{1}{2}k_1 q_1^2 + \tfrac{1}{2}k_2 q_2^2 + \tfrac{1}{2}k_3 q_3^2$$

Where K_1, K_2 and K_3 represents spring constants.

For a conservative system total energy remains constant, i.e. *H=E*

To define principal function of the corresponding Hamiltonian

$$S = W(q, \alpha) - Et$$

$$p = \frac{\partial W}{\partial q}$$

Or

$$H = \sum_{j=1}^{3} \left[\frac{1}{2m} \left(\frac{\partial \mathrm{W}}{\partial \mathrm{q_j}} \right)^2 + \frac{1}{2} K_j q_j^2 \right] = E$$

Which is the Hamiltonian Jacobi equation for a three-dimensional harmonic oscillator. To solve this equation, the separation of variables method to be employed

$$W = w_1(q_1) + w_2(q_2) + w_3(q_3)$$

Substitution of W in the parent equation results in three equations

$$\left(\frac{\partial w_1}{\partial \mathrm{q_1}} \right)^2 + \frac{1}{2} K_1 q_1^2 = m \times \alpha_1$$

$$\left(\frac{\partial w_2}{\partial \mathrm{q_2}} \right)^2 + \frac{1}{2} K_2 q_2^2 = m \times \alpha_2$$

$$\left(\frac{\partial w_3}{\partial \mathrm{q_3}} \right)^2 + \frac{1}{2} K_3 q_3^2 = m \times \alpha_3$$

Now solving each equation and introducing new momenta we will get three solutions as similar to section 1.6 (Follow the same method of section 1.6)

$$q_1 = \sqrt{\frac{2\alpha_1}{m\omega_1^2}} \sin \omega_1(\beta_1 + t)$$

$$q_2 = \sqrt{\frac{2\alpha_2}{m\omega_2^2}} \sin \omega_2(\beta_2 + t)$$

$$q_3 = \sqrt{\frac{2\alpha_3}{m\omega_3^2}} \sin \omega_3(\beta_3 + t)$$

Answer to Q4: The Hamiltonian of a system is the total energy of the system, i.e.

$$H = T + V$$

Where T= Kinetic Energy,

$$T = \frac{P^2}{2m}$$

Where $P = mv$, linear momentum of the particle.

And V= Potential Energy

For a free particle of mass m,
$V = mgz$, g=acceleration due to gravity

Then $$H = \frac{P^2}{2m} + mgz$$

Again, $$H + \frac{\partial s}{\partial t} = 0$$

Now Hamiltonian Jacobi equation

$$\frac{1}{2\mathrm{m}}\left(\frac{\partial S}{\partial z}\right)^2 + mgz + \frac{\partial s}{\partial t} = 0$$

Where,

$$S = W(z, \alpha) - \alpha t$$

Differentiating S w.r.t to z and t

$$\frac{\partial S}{\partial z} = \frac{\partial W}{\partial z}$$

$$\frac{\partial S}{\partial t} = -\alpha$$

Substituting $\frac{\partial S}{\partial z}$ and $\frac{\partial S}{\partial t}$ in Hamiltonian Jacobi equation and integrating

$$\frac{1}{2\mathrm{m}}\left(\frac{\partial W}{\partial z}\right)^2 + mgz - \alpha = 0$$

$W = \int [2m(\alpha - mgz)^{1/2} dz + C]$, where C is integration constant.

$$S = \int [2m(\alpha - mgz)^{1/2} dz + C - \alpha t]$$

$$\frac{\partial s}{\partial \alpha} = \frac{\sqrt{2m}}{2} \int \frac{dz}{\sqrt{\alpha - mgz}} - t = \beta$$

Simplifying

$$z = \frac{\alpha}{mg} - \frac{g}{2}(\beta + t)^2$$

Again

$$P = \frac{\partial W}{\partial z} = 0$$
$$\alpha = mgz_0$$

Now at t = 0, z = z_0, $z = z_0 - \frac{1}{2} g t^2$, which is the equation of a freely falling body.

Answer to Q5. Hamilton's characteristic function $W(q_i, P_i)$ arises when the Hamiltonian does not explicitly depend on time. In such cases, the principal function can be written as

$$S(q_i, P_i, t) = W(q_i, P_i) - Et.$$

Substituting into the Hamilton–Jacobi equation removes the time dependence and gives

$$H\left(q_i, \frac{\partial W}{\partial q_i}\right) = E.$$

Thus, the problem reduces to a time-independent partial differential equation, which is easier to solve. This simplifies the analysis because time is eliminated from the equation.

Answer to Q6: The function $W(q_i, P_i)$ acts as a generating function of type F_2.
The transformation equations are

$$p_i = \frac{\partial W}{\partial q_i}, Q_i = \frac{\partial W}{\partial P_i}.$$

These relations define a canonical transformation from (q_i, p_i) to (Q_i, P_i). Under this transformation, the new Hamiltonian becomes a constant, and the equations of motion simplify, often making Q_i and P_i constants of motion.

Answer to Q7: Use in Systems with Conserved Quantities

The Hamilton–Jacobi method is particularly useful when the system has conserved energy and separable coordinates. In such cases, the solution can be written as a sum of functions of individual variables, allowing separation of variables.

For example:

- In a central force problem, the motion separates into radial and angular parts.

- In a harmonic oscillator, the equation can be solved directly to obtain sinusoidal motion.

Thus, W enables systematic solution of such systems.

Answer to Q8: The Hamilton Jacobi equation has a strong analogy with wave mechanics. The principal function $\boldsymbol{S}$ behaves like the phase of a wave, and its gradient gives momentum is

$$p_i = \frac{\partial S}{\partial q_i}$$

This idea leads to the correspondence between classical and quantum mechanics. In quantum theory, the wave function is expressed as

$$\psi \sim e^{\frac{i}{\hbar}S}$$

showing that S plays a role like the phase. This connection was fundamental in the development of the Schrödinger equation**.**

🔍 8.10 FURTHER READING

- Gupta, A. B. (2023). *Fundamentals of classical mechanics* (3rd ed.). Books & Allied (P) Ltd.
- Goldstein, H., Poole, C. P., & Safko, J. (2011). *Classical mechanics* (3rd ed.). Pearson.
- https://egyankosh.ac.in/bitstream/123456789/104617/1/Unit-9.pdf
- https://www.math.wustl.edu/~alanchang/files/alan_chang_junior_paper_hamilton_jacobi_equations.pdf
- https://www.asc.ohio-state.edu/mathur.16/821hj.pdf

⍰ 8.11 MODEL QUESTIONS

1. What is the difference between Hamiltonian and Hamiltonian Jacobi theory in classical mechanics?
2. What do you mean by Hamilton's characteristic and principal function? State their physical significance.
3. Find the expression of Hamiltonian Jacobi equation from Hamilton's equation.
4. How can Kepler's problem can be solved using Hamilton Jacobi equation.
5. Express Hamilton Jacobi equation for a particle under central potential.
6. Construct Hamilton Jacobi equation for a simple harmonic oscillator and solve.

*** *** ***

UNIT 9: TWO-BODY CENTRAL FORCE PROBLEM

UNIT STRUCTURE

9.1 Learning Objectives
9.2 Introduction
9.3 Central Force and Motion in a Plane
9.4 Reduction of Two Body Problem into Equivalent One Body Problem
9.5 Equations of Motion Under Central Force and First Integral
9.6 Numerical Problems
9.7 Differential Equation for an Orbit
9.8 Inverse Square Law of Force
9.9 Let Us Sum Up
9.10 Answer to Check Your Progress
9.11 Further Readings
9.12 Model Questions

9.1 LEARNING OBJECTIVES

After going through this unit, you will be able to

- define central force and its properties
- explain reduction of the two-body problem using reduced mass
- derive equations of motion under central force
- apply first integrals (energy and angular momentum conservation)
- analyse orbital motion under inverse square law.

9.2 INTRODUCTION

In the previous unit, we studied about the Hamilton-Jacobi equation and its method of solution, together with applications to systems such as the harmonic oscillator and Keplerian motion. We also introduced the Hamiltonian characteristic function as an important tool in analytical mechanics. Building on that foundation, this unit deals with the two-body central force problem, which is fundamental in understanding the motion of interacting particles. The discussion begins with the nature and characteristics of central forces and explains why such motion remains restricted to a plane. It then develops the reduction of a two-body system into an equivalent single-particle problem through the concepts of reduced mass and center of mass. Further, the equations of motion are obtained and the associated conservation laws, such as energy and angular momentum, are examined. The unit also introduces the orbit equation and concludes with the inverse-square law of force, which provides the basis for describing gravitational and orbital motion.

9.3 CENTRAL FORCE AND MOTION IN A PLANE

A force is said to be central when its direction always lies along the straight line joining a moving particle and a fixed reference point, while its magnitude depends solely on their instantaneous separation. Such forces act radially and are independent of the direction of motion of the particle. Central force fields are widely used in mechanics because they naturally describe gravitational and electrostatic interactions.

The mathematical form of a central force is written as

$$\vec{F} = F(r).\hat{r} \qquad (9.1)$$

Where,

r =distance of the particle from the centre,

$\hat{r}$ = radial unit vector directed along the line joining the particle to the center,

F(r) = magnitude of the force which depends only on the distance r.

F(r) can be expressed in terms of potential *V* as:

$$F(r) = -\frac{\partial V}{\partial r} \qquad (9.2)$$

Common examples of central forces include the gravitational force between two masses, the electrostatic (Coulomb) force between charges, and the spring force directed toward the equilibrium. This figure 9.1 shows that a central force always acts along the line joining a particle to a fixed centre and depends only on the distance between them.

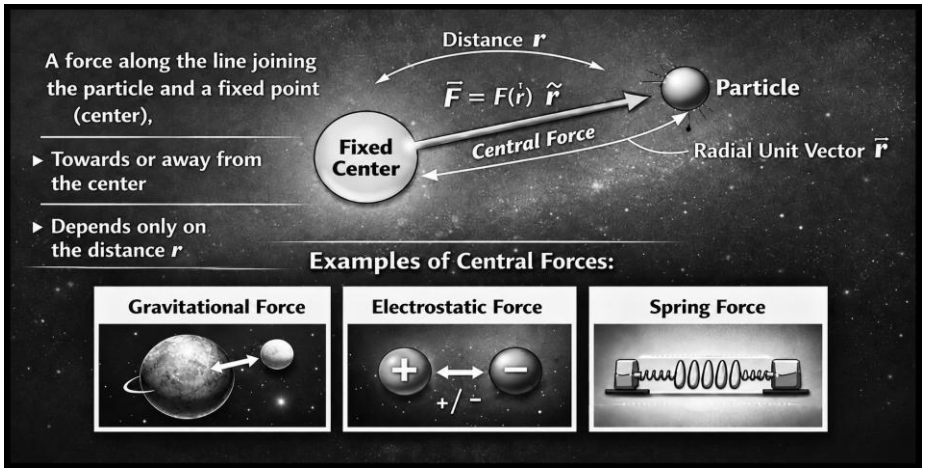


Figure 9.1: Illustration of a central force acting radially between a particle and a fixed center.

Properties of Central Force:

When a particle moves under the action of central force. The force

- Always acts along the line joining the particle to a fixed centre.
- Its magnitude depends only on the distance r from the centre.
- It may be attractive (toward the centre) or repulsive (away from the centre).
- The torque about the centre is zero i.e. $\vec{r} \times \vec{F} = 0$
- Angular momentum remains conserved.
- The motion of the particle stays confined to a single plane.
- If the force is conservative, total mechanical energy is conserved.

9.4 REDUCTION OF TWO BODY PROBLEM INTO EQUIVALENT ONE BODY PROBLEM

The motion of two mutually interacting particles can be simplified by transforming the system into two independent parts: the motion of the centre of mass and the motion of relative separation. This transformation allows the original problem to be treated as the motion of a single effective particle having reduced mass. As a result, the mathematical treatment becomes simpler while preserving the physical interaction between the bodies. Let us consider a system of two particles of masses m_1 and $m_2$located at position vectors $\vec{r}_1$ and $\vec{r}_2$ with respect to a fixed origin O. These particles interact through a central force, which depends only on the distance between them. The relative position vector is

$$\vec{r} = \vec{r}_2 - \vec{r}_1 \qquad (9.3)$$

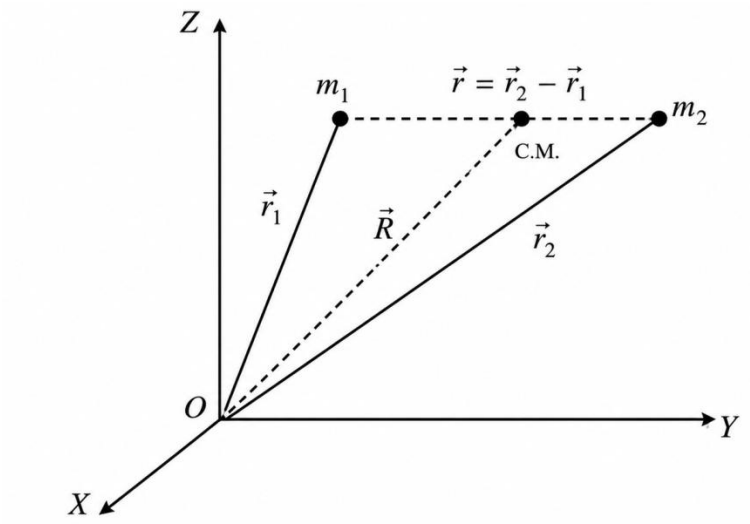


Figure 9.2: Position vectors of two particles and their centre of mass.
(Source: Goldstein, Herbert. Classical Mechanics, 3rd ed.,Pearson Education, 2002)

The total kinetic energy of the system is

$$T = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = \frac{1}{2}m_1 \dot{\vec{r}}_1^2 + \frac{1}{2}m_2 \dot{\vec{r}}_2^2 \qquad (9.4)$$

To simplify the two-body problem, we introduce the centre of mass coordinate. The centre of mass is a point where the total mass of the system may be assumed to be concentrated for describing the overall motion. It helps in separating the motion of the system into centre of mass motion and relative motion. Let distance of the centre of mass from the origin O is $\vec{R}$. The centre of mass position vector is defined as

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} \qquad (9.5)$$

Using $\vec{R}$ and$\vec{r}$, we can write

$$\vec{r}_1 = \vec{R} - \frac{m_2}{m_1 + m_2}\vec{r} \qquad (9.6)$$

$$\vec{r}_2 = \vec{R} + \frac{m_1}{m_1 + m_2}\vec{r} \qquad (9.7)$$

Differentiating with respect to time:

$$\frac{d\vec{r}_1}{dt} = \dot{\vec{r}}_1 = \dot{\vec{R}} - \frac{m_2}{m_1 + m_2}\dot{\vec{r}} \qquad (9.8)$$

$$\frac{d\vec{r}_2}{dt} = \dot{\vec{r}}_2 = \dot{\vec{R}} + \frac{m_1}{m_1 + m_2}\dot{\vec{r}} \qquad (9.9)$$

Substituting these into the kinetic energy expression and simplifying, we obtain

$$T = \frac{1}{2}(m_1 + m_2)\left(\dot{\vec{R}}\right)^2 + \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}\left(\dot{\vec{r}}\right)^2$$

$$\Longrightarrow \quad T = \frac{1}{2}(m_1 + m_2)\left(\dot{\vec{R}}\right)^2 + \frac{1}{2}\mu\left(\dot{\vec{r}}\right)^2 \qquad (9.10)$$

Where

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \qquad (9.11)$$

is called reduced mass

The potential energy depends only on the separation r, so

$$V = V(r) \qquad (9.12)$$

Hence, the Lagrangian of the system is

$$L = T - V$$

$$\Longrightarrow L = \frac{1}{2}(m_1 + m_2)\left(\dot{\vec{R}}\right)^2 + \frac{1}{2}\mu\left(\dot{\vec{r}}\right)^2 - V(r) \qquad (9.13)$$

Since no external force acts on the system, the centre of mass moves independently with constant velocity, and the central force affects only the relative separation between the particles. Hence the problem reduces to the relative motion of a particle of reduced mass μas

$$L = \frac{1}{2}\mu\left(\dot{\vec{r}}\right)^2 - V(r) \qquad (9.14)$$

Thus, the two-body problem becomes equivalent to a one-body problem of a particle of mass μ moving under the central potential V(r).

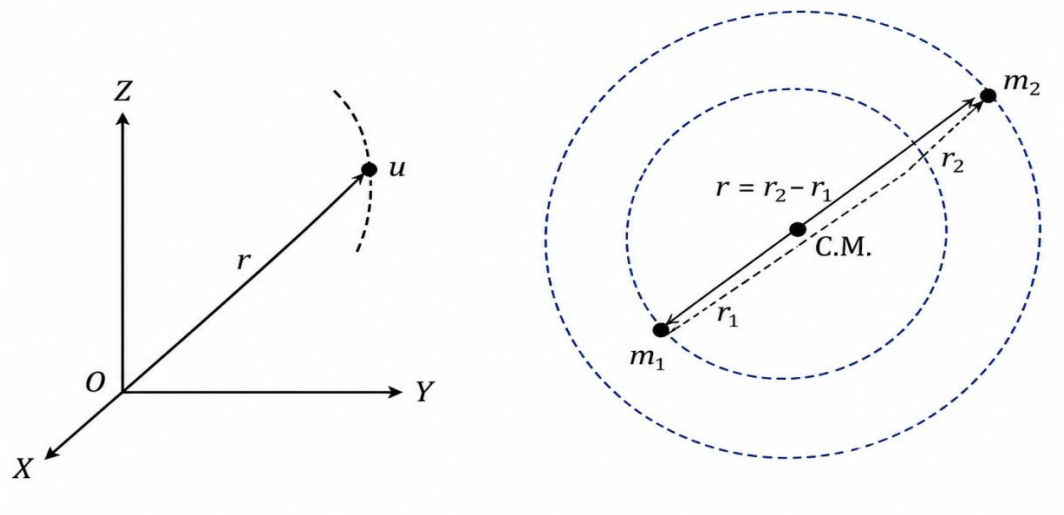


Figure 9.3: Reduction of two-body problem to one-body problem
(Source: Taylor, J. R. Classical Mechanics,
University Science Books, 2005)

9.5 EQUATIONS OF MOTION UNDER CENTRAL FORCE AND FIRST INTEGRAL

When a particle is subjected to a force directed toward a fixed point, its trajectory can be analysed using the equations of motion derived from Lagrangian mechanics. These equations provide important, conserved quantities such as angular momentum and total energy, which simplify the study of the system and help describe the orbital behaviour of the particle.

Consider a particle of mass m moving about a fixed centre O in a circular path under the action of a central force (Figure 9.4). Let the distance of the particle from the centre be r. The central is given as

$$\vec{F} = F(r).\hat{r}$$

$$\vec{F} = -\frac{\partial V}{\partial r}.\hat{r} \qquad (9.15)$$

Where *V= V(r)* is the potential energy of the system.

Initially, the particle moves from position P_0 to P_1, sweeping an angle θ at the centre. After a short interval of time, it moves further from $P_1$to a nearby position P_2, sweeping an additional small angle dθ.

From point P_1, draw a perpendicular P_1R to the radius vector OP_2. The length of this perpendicular represents the small arc displacement and is equal to rdθ.

Thus, for a very small angular displacement dθ, the linear displacement along the circular path is r dθ.

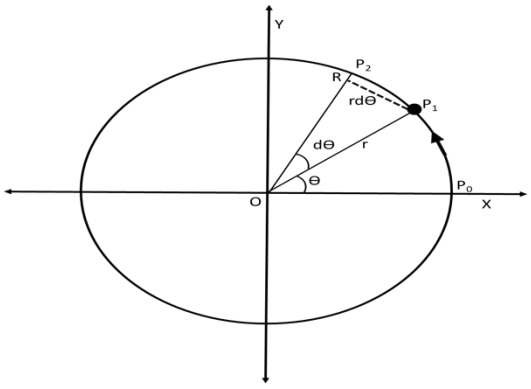


Figure 9.4: Motion of a particle in a circular path under central

The Lagrangian for the system using polar coordinates (r, θ) can be written as

$$L = T - V$$

$$\Rightarrow L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V\,(r) \qquad (9.16)$$

There are two generalised co-ordinates (r,Θ). Initially we have considered one generalised co-ordinate i.e. Θ

Now theLagrange's equation of motion of the system

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \qquad (9.17)$$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}mr^2 2\dot{\theta}\right) = 0$$

$$\Rightarrow \frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0 \qquad (9.18)$$

$$\Rightarrow mr^2\dot{\theta} = Constant\ \left(Say, \vec{J}\right) \qquad (9.19)$$

This is one of the first integral of motion. Here $mr^2\dot{\theta}$ is the magnitude of angular momentum. This value is constant throughout the motion under force and hence it is conserve.

Now we consider the next generalised coordinate r and write the Lagrange's equation of motion for the system

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \qquad (9.20)$$

$$\frac{\partial L}{\partial \dot{r}} = \frac{1}{2}.m.2\dot{r} = m\dot{r}$$

$$\frac{\partial L}{\partial r} = \frac{1}{2}m.2r.\dot{\theta}^2 - \frac{\partial V(r)}{\partial r} = mr\dot{\theta}^2 - \frac{\partial V}{\partial r}$$

Hence,

$$\frac{d}{dt}(m\dot{r}) - mr\dot{\theta}^2 + \frac{\partial V}{\partial r} = 0$$

$$\Longrightarrow m\ddot{r} - mr\dot{\theta}^2 + \frac{\partial V}{\partial r} = 0 \qquad (9.21)$$

$$\Longrightarrow m\ddot{r} = mr\dot{\theta}^2 - \frac{\partial V}{\partial r} \qquad (9.22)$$

From equation (9.19)

$$\Longrightarrow m\ddot{r} = mr\left(\frac{\vec{J}}{mr^2}\right)^2 - \frac{\partial V}{\partial r} \qquad (9.23)$$

$$\Longrightarrow m\ddot{r} = \frac{J^2}{mr^3} - \frac{\partial V}{\partial r} \qquad (9.24)$$

$$\Longrightarrow m\ddot{r} = -\frac{1}{2}\frac{\partial}{\partial r}\left(\frac{J^2}{mr^2}\right) - \frac{\partial V}{\partial r}$$

$$\Longrightarrow m\ddot{r} = -\frac{1}{2}\frac{\partial}{\partial r}\left(\frac{J^2}{mr^2} + V\right) \qquad (9.25)$$

Multiplying both sides by $\dot{r}$

$$m\ddot{r}\dot{r} = -\frac{1}{2}\frac{\partial}{\partial r}\left(\frac{J^2}{mr^2} + V\right)\dot{r}$$

$$\Longrightarrow m\ddot{r}\dot{r} = -\frac{1}{2}\frac{\partial}{\partial r}\left(\frac{J^2}{mr^2} + V\right)\dot{r}$$

$$\Longrightarrow \frac{1}{2}\frac{d}{dt}(m\dot{r}^2) = -\frac{1}{2}\frac{\partial}{\partial r}\left(\frac{J^2}{mr^2} + V\right)\frac{dr}{dt}$$

$$\Longrightarrow \frac{1}{2}\frac{d}{dt}(m\dot{r}^2) = -\frac{1}{2}\frac{d}{dt}\left(\frac{J^2}{mr^2} + V\right)$$

$$\Longrightarrow \frac{d}{dt}\left(\frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{J^2}{mr^2} + V\right) = 0$$

$$\Longrightarrow \frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{J^2}{mr^2} + V = Constant\ (Say, E) \qquad (9.26)$$

Putting the value of $\vec{J} = mr^2\dot{\theta}$ we will get

$$\frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + V = E \qquad (9.27)$$

$$\Longrightarrow Potential\ energy + Kinetic\ energy = Constant\ (E)$$

The above equation (9.27) is known as another first integral of motion

$$\vec{J} = mr^2\dot{\theta}$$

$$\Longrightarrow \dot{\theta} = \frac{\vec{J}}{mr^2}$$

$$\frac{d}{dt}\dot{r}^2 = 2\dot{r}\ddot{r}$$

$$m\frac{d}{dt}\dot{r}^2 = 2m\dot{r}\ddot{r}$$

$$\Longrightarrow m\dot{r}\ddot{r} = \frac{1}{2}$$

$$\frac{\partial}{\partial r}\left(\frac{1}{r^2}\right) = -2r^{-3}$$

$$\Longrightarrow r^{-3} = -\frac{1}{2}\frac{\partial}{\partial r}\left(\frac{1}{r^2}\right)$$

$$\Longrightarrow \frac{1}{r^3} = -\frac{1}{2}\frac{\partial}{\partial r}\left(\frac{1}{r^2}\right)$$

Multiply by $\frac{J^2}{m}$ on both sides

$$\frac{J^2}{mr^3} = -\frac{1}{2}\frac{\partial}{\partial r}\left(\frac{J^2}{mr^2}\right)$$

9.6 NUMERICAL PROBLEMS

Q1: Show that the areal velocity of particle moving under a central force is constant.

Ans:

We know that,

$$mr^2\dot{\theta} = Constant$$

Hence,

$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0 \qquad (9.28)$$

Since m≠0

$$\Rightarrow \frac{d}{dt}\left(r^2\dot{\theta}\right) = 0 \qquad (9.29)$$

Multiplying $\frac{1}{2}$ on both sides $\Rightarrow \frac{d}{dt}\left(\frac{1}{2}r^2\dot{\theta}\right) = 0$

$$\Rightarrow \frac{1}{2}r^2\dot{\theta} = Constant \qquad (9.30)$$

From the figure (9.4) it has been seen that

$$dA = \frac{1}{2}r.rd\theta = \frac{1}{2}r^2 d\theta \qquad (9.31)$$

$$\Rightarrow \frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{1}{2}r^2\dot{\theta}$$

The term $\frac{dA}{dt}$ is called areal velocity i.e. the area swept out by the radius vector per unit time.

Hence,

$$\frac{1}{2}r^2\dot{\theta} = Constant(\text{i.e. The Areal velocity}) \qquad (9.32)$$

Q2: A particle of mass m moves under an attractive central potential $(r) = -\frac{k}{r}$. Angular momentum is constant $J = mr^2\dot{\theta}$. Show that the radial motion can be written as $\frac{1}{2}m\dot{r}^2 + V_{eff}(r) = E$; where the effective potential is $V_{eff}(r) = -\frac{k}{r} + \frac{J^2}{2mr^2}$. Then find the turning points of motion of energy E<0

Solution:

From the first integral of motion

$$E = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + V(r) \qquad (9.33)$$

Using

$$\dot{\theta} = \frac{\vec{J}}{mr^2}$$

Squaring and multiplying both sides by r^2we will get,

$$r^2\dot{\theta}^2 = \frac{J^2}{m^2r^2} \qquad (9.34)$$

Substitute into energy

$$E = \frac{1}{2}m\,\dot{r}^2 + \frac{1}{2}\frac{J^2}{mr^2} - \frac{k}{r} \qquad (9.35)$$

Thus

$$E = \frac{1}{2}m\,\dot{r}^2 + V_{eff}(r) \qquad (9.36)$$

Turning point occurs when velocity is zero i.e. $\dot{r} = 0$

So,

$$E = V_{eff}(r)$$

$$E = \frac{1}{2}\frac{J^2}{mr^2} - \frac{k}{r} \qquad (9.37)$$

Multiplying by r^2 and rearranging

$$Er^2 - \frac{1}{2}\frac{J^2}{mr^2} + kr = 0 \qquad (9.38)$$

This quadratic gives turning radii

$$r = \frac{-k \pm \sqrt{k^2 + \frac{2EJ^2}{m}}}{2E} \qquad (9.39)$$

The turning points exist only if

$$k^2 + \frac{2EJ^2}{m} > 0 \qquad (9.40)$$

9.7 DIFFERENTIAL EQUATION FOR AN ORBIT

In physics, the trajectory traced by a particle moving under a central force is called its orbit. The mathematical relation that governs this trajectory is known as the orbit equation. Rather than describing the motion directly as a function of time, it is often more useful to express the motion in terms of the radial distance and angular position of the particle. This approach leads to a differential equation that characterises the geometry of the path followed by the particle. By solving this equation, one can determine the nature of the orbit, such as whether it is circular, elliptical, parabolic, or

hyperbolic. The orbit equation therefore provides an important connection between the force acting on the particle and the resulting motion, and it plays a key role in the study of planetary motion, satellite dynamics, and other orbital systems.

We know

$$m\ddot{r} - \frac{J^2}{mr^3} = -\frac{\partial V}{\partial r}$$

$$\Rightarrow m\ddot{r} - \frac{J^2}{mr^3} = f(r) \qquad (9.41)$$

Where,

$$f(r) = -\frac{\partial V}{\partial r}$$

It is convenient to introduce a new variable

$$u = \frac{1}{r} \Rightarrow r = \frac{1}{u} \qquad (9.42)$$

Now, differentiating the above equation with respect to t

$$\frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{dt}$$

$$\Rightarrow \dot{r} = -\frac{1}{u^2}\frac{du}{dt}$$

$$= -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt}$$

$$= -\frac{1}{u^2}\dot{\theta}\frac{du}{d\theta}$$

$$= -r^2\dot{\theta}\frac{du}{d\theta}$$

$$\Rightarrow \dot{r} = -\frac{\vec{J}}{m}\frac{du}{d\theta} \qquad (9.43)$$

Again we differentiate the above equation w.r.t t

$$\frac{d\dot{r}}{dt} = -\frac{\vec{J}}{m}\frac{du}{d\theta}$$

$$= -\frac{\vec{J}}{m}\frac{d}{d\theta}\left(\frac{du}{dt}\right)$$

$$= -\frac{\vec{J}}{m}\frac{d}{d\theta}\left(\frac{du}{d\theta}\cdot\frac{d\theta}{dt}\right)$$

$$= -\frac{\vec{J}}{m}\frac{d}{d\theta}\left(\frac{du}{d\theta}\dot{\theta}\right)$$

$$= -\frac{\vec{J}}{m}\dot{\theta}\left(\frac{d^2u}{d\theta^2}\right)$$

$$= -\frac{\vec{J}}{m}\frac{\vec{J}}{mr^2}\left(\frac{d^2u}{d\theta^2}\right)$$

$$= -\frac{J^2}{m^2r^2}\left(\frac{d^2u}{d\theta^2}\right)$$

$$= -\frac{J^2u^2}{m^2}\left(\frac{d^2u}{d\theta^2}\right) \qquad (9.44)$$

Now putting the values in equation (9.41) we get

$$-\frac{J^2u^2}{m}\frac{d^2u}{d\theta^2} - \frac{J^2u^3}{m} = f\left(\frac{1}{u}\right)$$

$$\Rightarrow -\frac{J^2u^2}{m}\frac{d^2u}{d\theta^2}-\frac{J^2u^3}{m}=f\left(\frac{1}{u}\right)$$
$$\Rightarrow -\frac{J^2u^2}{m}\left[\frac{d^2u}{d\theta^2}+u\right]=f\left(\frac{1}{u}\right) \qquad (9.45)$$

This is the equation of an orbit.

Exercise 1: A particle of mass m is observed to move in a spiral orbit given by the equation $= C\theta$, where C is a constant. It is moving in a central force field? If it is so find the force law.

Solution:

The differential equation of the orbit is given by

$$-\frac{J^2u^2}{m}\left[\frac{d^2u}{d\theta^2}+u\right]=f\left(\frac{1}{u}\right)$$
$$\Longrightarrow \frac{d^2u}{d\theta^2}+u=-\frac{mf\left(\frac{1}{u}\right)}{J^2u^2} \qquad (9.46)$$

Since, $u=\frac{1}{r}$ and $r=C\theta$hence $u=\frac{1}{C\theta}$

Differentiating,

$$\frac{du}{d\theta}=\frac{d}{d\theta}\left(\frac{1}{C\theta}\right)$$
$$\Longrightarrow \frac{du}{d\theta}=-\frac{1}{C\theta^2} \qquad (9.47)$$
$$\Longrightarrow \frac{d^2u}{d\theta^2}=-\frac{2}{C\theta^3} \qquad (9.48)$$
$$\Longrightarrow \frac{d^2u}{d\theta^2}=2C^2u^3$$

Substituting in equation(9.46), we get

$$2C^2u^3+u=-\frac{m}{J^2u^2}f\left(\frac{1}{u}\right)$$
$$\Longrightarrow f\left(\frac{1}{u}\right)=-\frac{J^2}{m}(u^3+2C^2u^5)$$
$$\Longrightarrow f(r)=-\frac{J^2}{m}\left(\frac{1}{r^3}+\frac{2C^2}{r^5}\right) \qquad (9.49)$$

Thus the particle is moving under a central force field and the force law is given by the above equation.

Exercise 2: A particle moving in a central force field located at r=0, describes a spiral r= $e^{-\Theta}$. Prove that the magnitude of force is inversely proportional to r^3

Solution:

Here$r=e^{-\theta}$ and therefore, $u=\frac{1}{r}=e^{\theta}$ (9.50)

Different equation of orbit is

$$\frac{d^2u}{d\theta^2}+u=-\frac{m}{J^2u^2}f\left(\frac{1}{u}\right) \qquad (9.51)$$

Substituting $u=e^{\theta}$ we get

$$e^{\theta}+e^{\theta}=-\frac{me^{-2\theta}}{J^2}f\left(\frac{1}{u}\right)$$
$$\Longrightarrow 2e^{\theta}=-\frac{me^{-2\theta}}{J^2}f\left(\frac{1}{u}\right)$$

$$\Rightarrow f\left(\frac{1}{u}\right) = -\frac{2e^{\theta}J^2}{me^{-2\theta}}$$

$$\Rightarrow f(r) = -\frac{2J^2}{mr^3} \qquad (9.52)$$

i.e.

$$f(r) \propto -\frac{1}{r^3}$$

9.8 INVERSE SQUARE LAW OF FORCE

An important class of is the inverse-square force, in which the magnitude of the force varies inversely with the square of the separation between two interacting bodies. This means that as the distance between the bodies increases, the strength of the force decreases rapidly according to the relation $F \propto \frac{1}{r^2}$. For example, if the separation is doubled, the force becomes one-fourth of its original value, and if the separation is tripled, the force reduces to one-ninth. Since the force acts along the line joining the interacting bodies, it satisfies the condition of a central force. The most important examples of inverse-square forces are the gravitational interaction between masses and the electrostatic interaction between charges. This law is fundamental in physics because it governs planetary motion, satellite trajectories, and many other natural phenomena involving long-range interactions. The inverse square law of force is given as:

$$f(r) = -\frac{K}{r^2}$$

If *V* is the potential, then

$$f(r) = -\frac{\partial V}{\partial r} = -\frac{K}{r^2}$$

And its integration gives us

$$V = -\frac{K}{r}$$

Where, the constant of integration is taken to be zero by assuming V(r)=0 at infinite separation $r = \infty$.

CHECK YOUR PROGRESS

Q1: What is a central force?
Q2: Why is the motion of a particle under a central force always confined to a plane?
Q3: State the main properties of a central force.
Q4: Why is the torque about the centre zero in central force motion?
Q5: How does conservation of angular momentum arise in a central force field?
Q6: What is meant by reduction of the two-body problem into an equivalent one-body problem?
Q7: Define reduced mass and explain its significance in two-body motion.
Q8: How is the motion of the centre of mass separated from the relative motion in a two-body system?

Q9: What is a first integral of motion? Name the first integrals obtained for central force motion.

Q10: What is the inverse square law of force and why is it important in orbital motion?

Q 11: Show that in rectangular coordinates the magnitude of the magnitude of a areal velocity is $\frac{1}{2}(x\dot{y} - y\dot{x})$

9.9 LET US SUM UP

- A central force is a force that always acts along the line joining a moving particle to a fixed centre point.
- The strength of a central force depends only on the distance between the particle and the centre.
- Central forces may be attractive, pulling the particle toward the centre, or repulsive, pushing it away.
- Examples of central forces include gravitational force between planets, electrostatic force between charges, and spring forces directed toward equilibrium.
- A key property of central force motion is that the torque about the centre is zero.
- Because torque is zero, angular momentum remains conserved throughout the motion.
- Due to conservation of angular momentum, the motion of a particle under a central force is always confined to a single plane.
- If the central force is conservative, the total mechanical energy of the system also remains constant.
- The two-body problem involves two particles interacting with each other through a central force depending only on their separation.
- This complicated two-body motion can be simplified by reducing it into an equivalent one-body problem.
- The reduction is achieved by introducing the centre of mass, which separates the overall motion of the system from the relative motion between the particles.
- The relative motion can be treated as the motion of a single effective particle with a reduced mass moving under the same central potential.
- Using Lagrangian mechanics, the equations of motion for central force systems can be derived in polar coordinates.
- Two important, conserved quantities, called first integrals of motion, arise from these equations: angular momentum conservation and energy conservation.
- Conservation of angular momentum also implies constant areal velocity, meaning equal areas are swept out in equal intervals of time.
- The path followed by a particle under a central force is called an orbit, and its shape can be studied using the differential equation of an orbit.
- Solving the orbit equation helps predict whether the trajectory will be circular, elliptical, parabolic, hyperbolic, or spiral.
- The inverse square law of force is an important central force law in which the force decreases rapidly with increasing distance.
- Gravitational and electrostatic forces follow the inverse square law, making it fundamental for understanding planetary motion and stable orbits.

- Overall, this unit explains the nature of central forces, the simplification of two-body motion, conservation laws, and the basis of orbital dynamics.

✓ 9.10 ANSWERS TO CHECK YOUR PROGRESS

Answer to Q1: A central force is a force that always acts along the line joining a moving particle to a fixed point (the centre). Its direction is purely radial (toward or away from the centre), and its magnitude depends only on the distance r between the particle and the centre, i.e., F=F(r).

Answer to Q2: The motion under a central force remains confined to a plane because the force produces no torque about the centre. Since torque is zero, angular momentum is conserved, and the particle continues to move in the same plane determined by its initial position and velocity.

Answer to Q3: The main properties of a central force are:

- It acts along the line joining the particle to the centre.
- Its magnitude depends only on the distance r.
- It may be attractive or repulsive.
- The torque about the centre is zero.
- Angular momentum is conserved.
- Motion remains planar.
- If the force is conservative, total mechanical energy is conserved.

Answer to Q4: Torque about the centre is zero because the force is always directed along the radius vector. Since the line of action passes through the centre, the moment arm is zero, hence $\vec{\tau} = \vec{r} \times \vec{F} = 0$.

Answer to Q5: Because the torque in a central force field is zero, the rate of change of angular momentum is also zero. Therefore, angular momentum remains constant throughout the motion, leading to conservation of angular momentum.

Answer to Q6: Reduction of the two-body problem means transforming the motion of two interacting particles into an equivalent problem of a single particle. This is done by introducing the reduced mass and studying the relative motion under the same central potential, which makes the problem easier to solve.

Answer to Q7: Reduced mass μ is defined as

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

It represents the effective mass of a single particle whose motion describes the relative motion of the two-body system under a central force.

Answer to Q8: In a two-body system, the motion can be separated into:

- Motion of the centre of mass, which moves with constant velocity if no external force acts.
- Relative motion of one particle with reduced mass about the other under the central potential.
- Thus, the interaction affects only the relative separation.

Answer to Q9: A first integral of motion is a conserved quantity derived from the equations of motion. In central force motion, the two important first integrals are:

- Conservation of angular momentum.
- Conservation of total mechanical energy.

Answer to Q10: The inverse square law of force states that the magnitude of the force between two bodies is inversely proportional to the square of the distance between them:

$$\vec{F} \propto \frac{1}{r^2}$$

It is important because gravitational and electrostatic forces follow this law, and it explains stable planetary and satellite orbits.

Answer to Q11: The areal velocity is defined as the rate at which the radius vector sweeps out area i.e. $\frac{dA}{dt}$

For a particle at position (x, y) the radius vector is

$$\vec{r} = x\hat{\imath} + y\hat{\jmath}$$

The velocity vector is

$$\vec{v} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath}$$

In a small time interval dt, the particle moves by

$$d\vec{r} = \vec{v}dt$$

The small triangle swept has area

$$dA = \frac{1}{2}|\vec{r} \times d\vec{r}|$$

Substituting$d\vec{r}$,

$$dA = \frac{1}{2}|\vec{r} \times \vec{v}|dt$$

Dividing both sides by dt

$$\frac{dA}{dt} = \frac{1}{2}|\vec{r} \times \vec{v}|$$

Now,

$$\vec{r} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ x & y & 0 \\ \dot{x} & \dot{y} & 0 \end{vmatrix}$$

$$= (x\dot{y} - y\dot{x})\hat{k}$$

Thus,

$$\frac{dA}{dt} = \frac{1}{2}(x\dot{y} - y\dot{x})$$

O 9.11 FURTHER READING

- Goldstein, H., Poole, C. P., & Safko, J. L. (2002). *Classical mechanics* (3rd ed.). Pearson Education.
- Rana, N. C., & Joag, P. S. (1991). *Classical mechanics*. Tata McGraw-Hill.
- Upadhyaya, J. C. (2005). *Classical mechanics*. Himalaya Publishing House.
- Marion, J. B., & Thornton, S. T. (1995). Classical dynamics of particles and systems (5th ed.). Brooks/Cole.
- Landau, L. D., & Lifshitz, E. M. (1976). Mechanics (3rd ed.). Pergamon Press.
- Lanczos, C. (1970). The variational principles of mechanics (4th ed.). University of Toronto Press.
- Hand, L. N., & Finch, J. D. (1998). Analytical mechanics. Cambridge University Press.
- Greenwood, D. T. (1988). Classical dynamics. Prentice Hall.

- Synge, J. L., & Griffith, B. A. (1970). Principles of mechanics. McGraw-Hill.
- Whittaker, E. T. (1988). A treatise on the analytical dynamics of particles and rigid bodies. Cambridge University Press.

⑦ 9.12 MODEL QUESTIONS

1. What is meant by a central force?
2. In which direction does a central force always act?
3. Give two examples of central forces.
4. What happens to the torque in a central force field?
5. Why is angular momentum conserved in central force motion?
6. Why does a particle move in a plane under a central force?
7. What is the meaning of the two-body problem in mechanics?
8. What is reduced mass?
9. Why do we reduce a two-body problem into a one-body problem?
10. What is the inverse square law of force?
11. Explain how the centre of mass helps in separating the motion of a two-body system.
12. What are the two important first integrals of motion in a central force field, and why are they significant?
13. What is meant by areal velocity, and why does it remain constant in central force motion?
14. Why is the differential equation of an orbit important in understanding orbital motion under central forces?
15. How does the concept of effective potential help in analysing radial motion in a central force problem?

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UNIT 10: SMALL OSCILLATIONS

UNIT STRUCTURE

10.1 Learning Objectives
10.2 Introduction
10.3 Potential Energy and Equilibrium
10.4 Types of Equilibrium
    10.4.1 Stable Equilibrium
    10.4.2 Unstable Equilibrium
    10.4.3 Neutral Equilibrium
10.5 General Theory of Small Oscillations
10.6 Two Coupled Oscillators
    10.6.1 Solution of Differential Equations
    10.6.2 Normal Co-ordinates
    10.6.3 Normal Modes of Frequencies
10.7 Let Us Sum Up
10.8 Answers to Check Your Progress
10.9 Further Reading
10.10 Model Questions

10.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- define potential energy and equilibrium conditions
- distinguish between stable, unstable, and neutral equilibrium
- apply Taylor expansion to analyse small oscillations
- analyse the dynamics of coupled oscillators using normal coordinates
- explain normal modes and their corresponding frequencies.

10.2 INTRODUCTION

In the previous unit, we studied the motion of particles under and analysed the two-body problem along with orbital motion. Building on that foundation, we now turn to the study of small oscillations, which arise when a system is slightly displaced from its equilibrium position. In this unit, we shall examine the relation between potential energy and equilibrium, discuss the different types of equilibrium, and develop the general theory of small oscillations using the Lagrangian method. The unit also introduces coupled oscillators, normal coordinates, and normal modes, which are essential for understanding vibrational motion in many physical systems.

10.3 POTENTIAL ENERGY AND EQUILIBRIUM

Potential energy is the energy possessed by a body due to its position or arrangement of its parts relative to each other. Its value depends on the arrangement of the system

and determines how the system responds to displacement. The concept of potential energy is particularly useful in analysing equilibrium because equilibrium positions correspond to points where the potential function becomes stationary.

Let us consider a conservative system in which the potential energy is a function of the generalised coordinates only not time. We can imagine a system of masses connected by springs, where the stored energy depends only on their positions. For such a system of consisting of *n* degrees of freedom, the potential energy is given by

$$V = V(q_{1,}q_2, \dots\dots\dots, q_n) \qquad (10.1)$$

Where,$q_{1,}q_2, \dots\dots\dots, q_n$ are the generalised coordinates of the system.

Equilibrium is a state in which a system is balanced and does not tend to change its position on its own.In terms of potential energy**,** a system is in equilibrium when its potential energy is stationary (minimum, maximum, or constant) compared to nearby positions. For example, a ball resting at the bottom of a bowl stays there because that position has the lowest potential energy.

In terms of force, a system is in equilibrium when the net force on it is zero. For example, a book resting on a table remains at rest because the downward weight is balanced by the upward support force. Analysis of equilibrium is easier through potential energy than through force, because equilibrium positions can be found by studying how the potential energy changes with position.

For a system with *n* generalized coordinates, the generalized force is given by

$$G_n = -\frac{\partial V}{\partial q_n} \qquad (10.2)$$

Where, V is the potential energy of the system and n= 1, 2, 3……

As per definition, a system is in equilibrium if the generalized force $G_n$is zero (for all *n*), i.e.

$$G_n = -\frac{\partial V}{\partial q_n} = 0 \qquad (10.3)$$

For better understanding we will consider one-dimensional potential concept where q_n=*x* as generalised coordinate for our convenience. Let us understand the equilibrium considering the one-dimensional potential curve as shown in Figure 10.1. The $\frac{\partial V(x)}{\partial x} = 0$, at all those points at which the tangent is parallel to the x-axis. Hence, A, A′, B, B′, C, C′, Dand E all are equilibrium points.

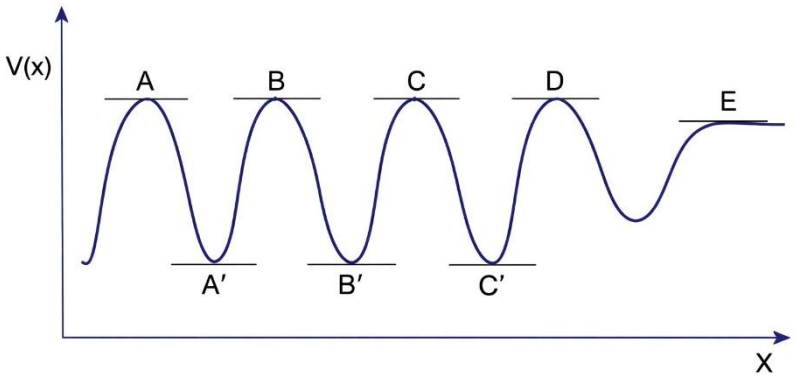


Figure 10.1: One-dimensional potential energy curve

10.4 TYPES OF EQUILIBRIUM

Equilibrium can be classified into three categories depending on how a system behaves after a small displacement from its equilibrium position: stable, unstable, and neutral equilibrium. In Figure 10.1, a particle placed at point A moves away from its original position when slightly disturbed, indicating unstable equilibrium. At point A′, there is no restoring force exists and the particle tends to return to its original position after a small displacement, representing stable equilibrium. At point E, the particle remains at its new position after being displaced, showing neutral equilibrium.

10.4.1 Stable Equilibrium

A system is said to be in stable equilibrium if, after a small displacement, it experiences a restoring force that drives it back toward the equilibrium position. In terms of potential energy, stable equilibrium corresponds to a local minimum of the potential function. The same also happens if it is pushed toward the right. Hence, these points are stable equilibrium points, and the particle is said to be in stable equilibrium. At a stable equilibrium point, aparticle experiences a negative force (we are not referring to the externally applied force). So

$$\frac{dF_x}{dx} < 0 \text{ Or} \frac{dF_x}{dx} = -ve.$$

Since $F_x = -\frac{dV}{dx}$ in one dimension, Hence

$$\frac{d}{dx}\left(-\frac{dV(x)}{dx}\right) = -ve$$

$$\Longrightarrow \frac{d}{dx}\left(-\frac{dV(x)}{dx}\right) = -ve$$

$$\Longrightarrow \frac{d^2V(x)}{dx^2} = +ve \; or \; > \; 0 \qquad (10.4)$$

This is the condition for a particle to be in a stable equilibrium position.

10.4.2 Unstable Equilibrium

A particle is said to be in unstable equilibrium when a small displacement from its equilibrium position causes it to move farther away instead of returning. In Figure 10.1, the points A, B, C, and D represent such positions. If the particle is disturbed slightly in either direction, the resulting force drives it away from the equilibrium point. Thus, the equilibrium is unstable because the system does not restore itself to the original position. At these points, the restoring tendency is absent, and the effective force acts in the direction of displacement. So,

$$\frac{dF_x}{dx} > 0 \text{ Or } \frac{dF_x}{dx} = +ve.$$

Since $F_x = -\frac{dV}{dx}$ in one dimension, Hence

$$\frac{d}{dx}\left(-\frac{dV(x)}{dx}\right) = +ve$$

$$\Rightarrow \frac{d}{dx}\left(-\frac{dV(x)}{dx}\right) = +ve$$

$$\Rightarrow \frac{d^2V(x)}{dx^2} = -ve \; or \; < \; 0 \qquad (10.5)$$

This is the condition for a particle to be in anunstable equilibrium position.

10.4.3 Neutral Equilibrium

A particle is said to be in neutral equilibrium when, after a small displacement from its equilibrium position, it remains in its new position without any tendency to return or move farther away. In Figure 10.1, point E represents such a condition. If the particle is displaced slightly to either side, no restoring or disturbing force acts on it after the displacement. Therefore, the particle stays at its new position, indicating neutral equilibrium. So,

$$\frac{dF_x}{dx} = 0$$

Since $F_x = -\frac{dV}{dx}$ in one dimension, Hence

$$\frac{d}{dx}\left(-\frac{dV(x)}{dx}\right) = 0$$

$$\Rightarrow \frac{d}{dx}\left(-\frac{dV(x)}{dx}\right) = 0$$

$$\Rightarrow \frac{d^2V(x)}{dx^2} = 0 \qquad (10.6)$$

This is the condition for a particle to be in a neutral equilibrium position.

10.5 GENERAL THEORY OF SMALL OSCILLATIONS

A system is said to undergo small oscillations when it is displaced by a very small amount from its stable equilibrium position and executes repeated motion about that position. In such cases, the restoring force acting on the system remains approximately proportional to the displacement, and the motion can be treated as. Common examples include a spring–mass system stretched slightly from equilibrium and a simple pendulum oscillating through a small angle. When the displacement becomes large, the restoring force no longer remains proportional to the displacement, and the motion becomes non-linear and more complex. The study of small oscillations is important because many physical systems, such as vibrating molecules, crystal lattice vibrations, and mechanical systems, can be analysed using this approximation. Figure 10.2 illustrates the difference between small oscillatory motion, which is nearly harmonic, and large oscillatory motion, which deviates from simple harmonic behaviour.

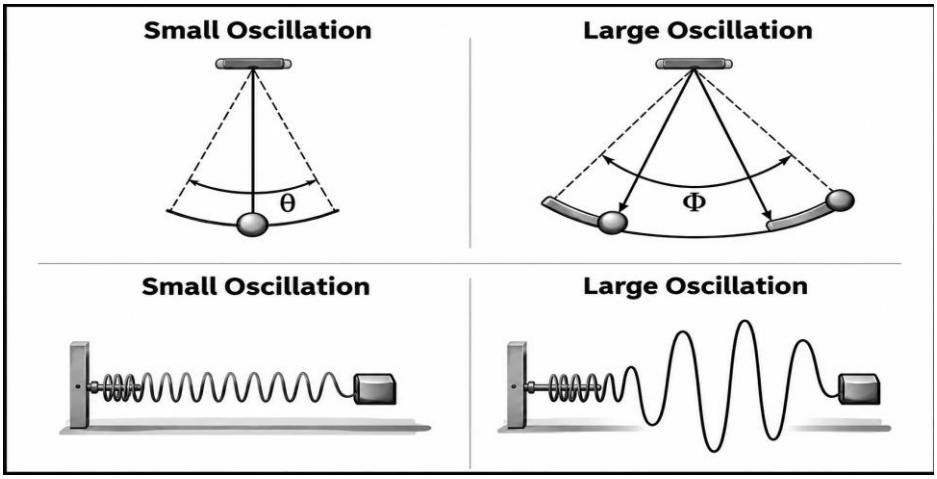


Figure10.2: Small and Large oscillations of pendulum and linear harmonic oscillator

In small oscillations, the motion is linear and simple. For small displacements, higher-order terms in the force or potential can be neglected, leaving only the linear term-this makes the motion simple. The potential energy of the system is expanded about the equilibrium position, and only the lowest-order (quadratic) term is retained. In the theory of small oscillations, we use the Taylor series expansion of the potential energy (or equivalently the force) about the stable equilibrium position. This is because, for small displacements, the system remains very close to equilibrium, and the exact form of the potential is often complicated. By expanding the potential energy about the equilibrium point and retaining only the lowest non-zero terms, the motion can be approximated as simple harmonic motion.

- The first derivative of potential energy is zero at equilibrium.
- The second derivative determines the nature of oscillation.
- Higher-order terms are neglected because the displacement is small.

The kinetic energy is usually not expanded, because it already has a simple quadratic form in velocity.

Now we will understand the theory of small oscillation of a system using the Lagrangian formulation. Consider a one-dimensional harmonic oscillator undergoing small oscillations as shown figure 10.3

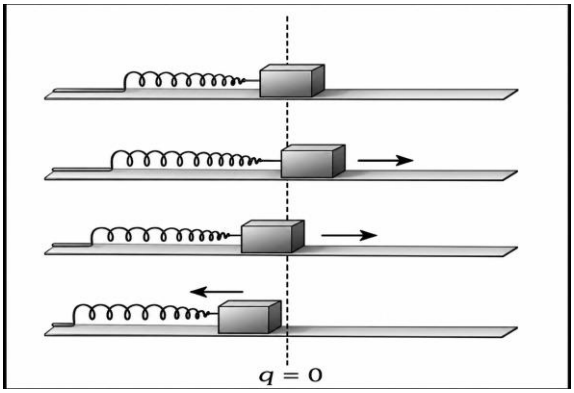


Figure10.3: 1D linear harmonic oscillator under small oscillation

For a generalized coordinate *q*, the Lagrange equation is

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\right] - \frac{\partial L}{\partial q} = 0 \qquad (10.7)$$

Where L=T-V and q is the generalised co-ordinate of the system.

Kinetic Energy:

The position of the particle with mass *m* of the system is *x* and this x is a function of generalised coordinate q i.e. $x = x(q)$
Hence the velocity of the system will be

$$\frac{dx}{dt} = \frac{d}{dt}x(q)$$

$$\Rightarrow \frac{dx}{dt} = \left(\frac{\partial x}{\partial q}\right)\left(\frac{dq}{dt}\right)$$

$$\Rightarrow \dot{x} = \left(\frac{\partial x}{\partial q}\right)\dot{q}$$

Then the kinetic energy of the system becomes

$$T = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}m\left(\frac{\partial x}{\partial q}\right)^2 \dot{q}^2$$

Here we define

$$m(q) = m\left(\frac{\partial x}{\partial q}\right)^2$$

So,

$$T = \tfrac{1}{2}\, m(q)\dot{q}^2 \qquad (10.8)$$

Here, *m(q)* is called the effective mass associated with the generalized coordinate *q*. It arises because the actual displacement x of the particle may depend on *q* in a non-linear manner. Under small oscillations, the particle executes motion very close to the stable equilibrium position q=0. Therefore, the function m(q) can be expanded in a Taylor series about q= 0:

$$m(q) = m\,(q)_0 + \left[\frac{\partial m}{\partial q}\right]_0 (q - q_0) + \cdots$$

At q=0 we will get

$$m(q) = m\,(0) + \left[\frac{\partial m}{\partial q}\right]_0 (q - 0) + \cdots$$

Now the Kinetic energy T becomes

$$T = \frac{1}{2}m\,(0)\dot{q}^2 + \frac{1}{2}\left[\frac{\partial m}{\partial q}\right]_0 q\dot{q}^2 + \cdots$$

The second term contains the product $q\dot{q}^2$, which is of higher order in small quantities, and hence it is neglected in the theory of small oscillations. So

$$T = \tfrac{1}{2}m\,(0)\dot{q}^2 \qquad (10.9)$$

Potential Energy

The potential energy expanded in a Taylor series about the stable equilibrium position q=0 as

$$V(q) = V\,(q)_0 + \left[\frac{\partial m}{\partial q}\right]_0 (q - q_0) + \frac{1}{2}\left[\frac{\partial^2 V}{\partial q^2}\right]_0 (q - q_0)^2 + \cdots$$

At q=0 we well get

$$V(q) = V\,(0) + \left[\frac{\partial V}{\partial q}\right]_0 (q - 0) + \frac{1}{2}\left[\frac{\partial^2 V}{\partial q^2}\right]_0 (q - 0)^2 + \cdots$$

At equilibrium

$$\left[\frac{\partial V}{\partial q}\right]_0 = 0$$

Hence,

$$V(q) = \frac{1}{2}\left[\frac{\partial^2 V}{\partial q^2}\right]_0 q^2 \qquad (10.10)$$

This is the lowest non-vanishing term which is retained in the equation of V (q) because it dominates the motion near equilibrium, while higher-order terms are neglected since the displacement is small; inclusion of higher-order terms would be necessary only for large oscillations.

Let, $K = \left[\frac{\partial^2 V}{\partial q^2}\right]_0$ which is called the spring constant. Therefore,

$$V(q) = \frac{1}{2}Kq^2$$

Lagrangian:

$$L = \frac{1}{2}m\,(0)\dot{q}^2 - \frac{1}{2}Kq^2 \qquad (10.11)$$

Lagrange's equation of motion

$$m(0)\ddot{q} + Kq = 0$$

$$\Rightarrow \ddot{q} + \frac{K}{m(0)}q = 0$$

$$\Rightarrow \ddot{q} + \omega^2 q = 0$$

Where, $\omega = \sqrt{\frac{K}{m(0)}}$, is called the angular frequency (or natural angular frequency) of oscillation.

The above equation of motion of the linear harmonic oscillator under small oscillations is similar to the equation of simple harmonic motion. Therefore, the solution will be simple harmonic, given by

$$q = A\cos(\omega t + \phi) \qquad (10.12)$$

Here,

q: Displacement of the particle from the equilibrium position

A: Amplitude of oscillation (maximum displacement)

ω: Angular (natural) frequency of oscillation

t: Time
ϕ: Phase constant (depends on initial conditions)
Thus, the theory of small oscillations shows that when systems such as a linear harmonic oscillator or a pendulum are slightly displaced from their stable equilibrium positions, their motion can be approximated as simple harmonic motion.

10.6 TWO COUPLED OSCILLATORS

A system of coupled oscillators consists of two or more oscillating bodies linked in such a way that the motion of each oscillator influences the others. Due to this interaction, energy can be exchanged between them during oscillation. A simple example is a system of two masses connected by springs, where an additional spring couples the two masses. If one mass is displaced and released, its oscillatory energy is gradually transferred to the other mass through the coupling. Such systems are important in understanding collective vibrations and the concept of normal modes in mechanics.

The figure 10.4 shows a system of two coupled oscillators in which two masses m_1 and m_2 are placed on a frictionless horizontal surface. The mass m_1 is attached to the left fixed wall by a spring of spring constant k, and the mass m_2 is attached to the right fixed wall by another spring of the same spring constant k. The two masses are further connected to each other by a coupling spring of spring constant k′. Due to this coupling, the motion of one mass influences the motion of the other, allowing energy to be exchanged between the two oscillators.

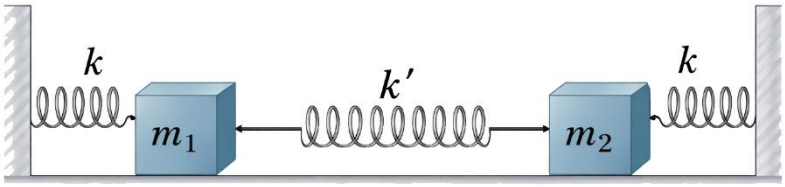


Figure 10.4: Two Coupled oscillator

10.6.1 Solution Of Differential Equations

The Kinetic energy of the oscillating system is

$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2$$

The Potential energy of the oscillating system is

$$V = \frac{1}{2}kx_1^2 + \frac{1}{2}kx_2^2 + \frac{1}{2}k'(x_1 - x_2)^2$$

Lagrangian of the system will be

$$L = T - V$$

$$L = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 - \frac{1}{2}kx_1^2 - \frac{1}{2}kx_2^2 - \frac{1}{2}k'(x_1 - x_2)^2$$

The Lagrange's equation of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0$$

$$\Rightarrow \frac{d}{dt}(m\dot{x}_1) + kx_1 + k'(x_1 - x_2) = 0$$

$$\Rightarrow m\ddot{x}_1 + kx_1 + k'(x_1 - x_2) = 0 \qquad (10.13)$$

And

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = 0$$

$$\Rightarrow \frac{d}{dt}(m\dot{x}_2) + kx_2 - k'(x_1 - x_2) = 0$$

$$\Rightarrow m\ddot{x}_2 + kx_2 - k'(x_1 - x_2) = 0 \qquad (10.14)$$

To obtain the solution for the coupled oscillators the two ordinary differential equations i.e. (10.13) and (10.14) need to be solved simultaneously. The solutions for x_1 and x_2 not necessary to be simple harmonic motion (SHM) like $x = A\cos(\omega t + \phi)$. As both equations (10.13) and (10.14) depends not only x like the equation $\ddot{x} + \omega^2 x = 0$ but also the difference$(x_1 - x_2)$. If the solutions of the equations of (10.13) and (10.14) i.e. the value of x_1 and x_2. If this is not a SHM then what it. Let's check it.

Now,

(10.13)+(10.14)⇒

$$m\ddot{x}_1 + kx_1 + k'(x_1 - x_2) + m\ddot{x}_2 + kx_2 - k'(x_1 - x_2) = 0$$

$$\Rightarrow m(\ddot{x}_1 + \ddot{x}_2) + k(x_1 + x_2) = 0$$

$$\Rightarrow (\ddot{x}_1 + \ddot{x}_2) + \frac{k}{m}(x_1 + x_2) = 0 \qquad (10.15)$$

And

(10.13)-(10.14)⇒

$$m\ddot{x}_1 + kx_1 + k'(x_1 - x_2) - m\ddot{x}_2 - kx_2 + k'(x_1 - x_2) = 0$$

$$\Rightarrow m(\ddot{x}_1 - \ddot{x}_2) + kx_1 - kx_2 + 2k'x_1 - 2k'x_2 = 0$$

$$\Rightarrow m(\ddot{x}_1 - \ddot{x}_2) + (x_1 - x_2)(k + 2k') = 0$$

$$\Rightarrow m(\ddot{x}_1 - \ddot{x}_2) + \frac{(k+2k')}{m}(x_1 - x_2) = 0 \quad (10.16)$$

The equation (10.13) corresponding to SHM of $(x_1 + x_2)$ with angular frequency $\omega_s = \sqrt{\frac{k}{m}}$ and the equation (10.14) corresponding to SHM of $(x_1 - x_2)$ with angular frequency$\omega_f = \sqrt{\frac{(k+2k')}{m}}$.

This means the individual motion of the oscillator don't execute SHM but the sum and the difference of the positions of the masses executes SHM having different frequency.

The solution of equation (10.15) will be

$$x_1 + x_2 = A_s\, Cos\,(\omega_s t + \phi_s) \quad (10.17)$$

Where A_s and ϕ_s are the constant for amplitude and phase depends on the initial conditions.

Likewise, the solution of equation (10.16) will be

$$x_1 - x_2 = A_f\, Cos\left(\omega_f t + \phi_f\right) \quad (10.18)$$

Where A_f and ϕ_f are the constant for amplitude and phase depends on the initial conditions.

Now let's figure out x_1 and x_2 separately

(10.17)+ (10.18) ⇒

$$2x_1 = A_s\, Cos\,(\omega_s t + \phi_s) + A_f\, Cos\left(\omega_f t + \phi_f\right)$$

$$\Rightarrow x_1 = \frac{A_s}{2}\, Cos\,(\omega_s t + \phi_s) + \frac{A_f}{2}\, Cos\left(\omega_f t + \phi_f\right) \quad (10.19)$$

(10.17)-(10.18) ⇒

$$2x_2 = A_s\, Cos\,(\omega_s t + \phi_s) - A_f\, Cos\left(\omega_f t + \phi_f\right)$$

$$\Rightarrow x_2 = \frac{A_s}{2}\, Cos\,(\omega_s t + \phi_s) - \frac{A_f}{2}\, Cos\left(\omega_f t + \phi_f\right) \quad (10.20)$$

The equations (10.19) and (10.20) are the solutions for first and second mass respectively.

From the above discussion, we learn that the sum and the difference of the displacements of the two masses execute simple harmonic motion (SHM). However, the individual masses do not execute SHM; instead, each mass undergoes a periodic motion. The individual displacements of the masses are linear combinations of cosine waves.

10.6.2 Normal Co-Ordinates

In a system of two coupled oscillators, certain combinations of the individual displacements can be chosen to simplify the analysis of motion. These combinations, such as (x_1+x_2) and (x_1-x_2), are known as normal coordinates. When the equations of motion are written in terms of these coordinates, the coupled equations separate into independent equations, each representing a with its own characteristic frequency. Physically, the coordinate (x_1+x_2)describes the in-phase motion, where both masses move together in the same direction, while (x_1-x_2)represents the out-of-phase motion, where the masses move in opposite directions. These normal coordinates make it easier to understand the vibrational behaviour of the coupled system.

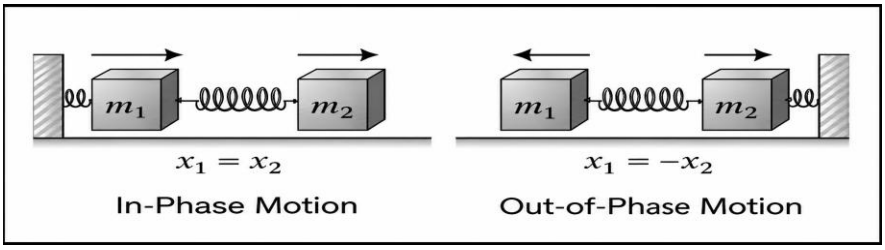


Figure 10.5: In phase and out of phase motion

10.6.3 Normal Modes Of Frequencies

A coupled oscillatory system does not vibrate at arbitrary frequencies; instead, it oscillates at certain characteristic frequencies determined by the properties of the system. These characteristic frequencies are known as normal mode frequencies, and each frequency corresponds to a particular pattern of motion called a normal mode. In the case of two coupled oscillators, the frequencies ω_s and ω_f represent the two allowed normal mode frequencies. Each of these frequencies is associated with a distinct type of oscillation, such as in-phase or out-of-phase motion. The study of normal modes helps in understanding how energy is distributed and transferred within the coupled system.

Normal Mode:

When a coupled system vibrates, it does not move randomly. Instead, it oscillates in certain definite patterns. Each such pattern is called a normal mode. If we put some initial condition in equations (10.19) and (10.20) one frequency either ω_s or ω_f came into play. This situation is known as normal mode.

If we put the initial condition $A_f = 0$ in equations (10.19) and (10.20) we get

$$x_1 = x_2 = \frac{A_s}{2}\, Cos\,(\omega_s t + \phi_s)$$

Both x_1 and x_2 executes SHM with same frequency ω_s and same amplitude A_s in phase. This is called second normal mode. First normal mode of a coupled mass-spring system showing the directions of mass displacements.

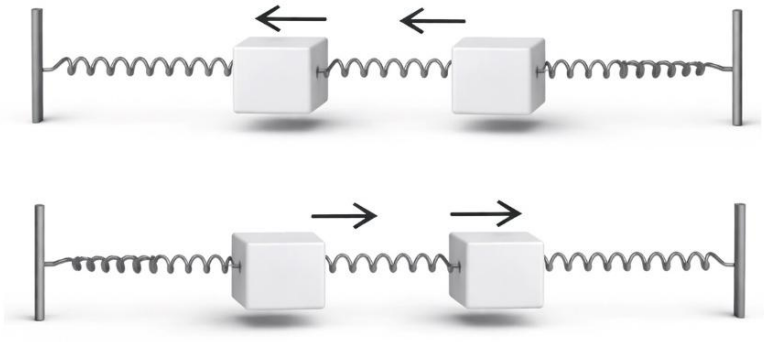


Figure 10.6: First normal mode of a coupled mass–spring system

Similarly, if we put the initial condition $A_s = 0$ in equations (10.19) and (10.20) we get

$$x_1 = -x_2 = \frac{A_f}{2}\, Cos\,(\omega_f t + \phi_f)$$

Both x_1 and x_2 are executing SHM with same frequency ω_f and same amplitude A_f in phase. This is called second normal mode. The figure 10.7 shows the second normal mode of a coupled mass–spring system, in which the two masses oscillate in opposite directions.

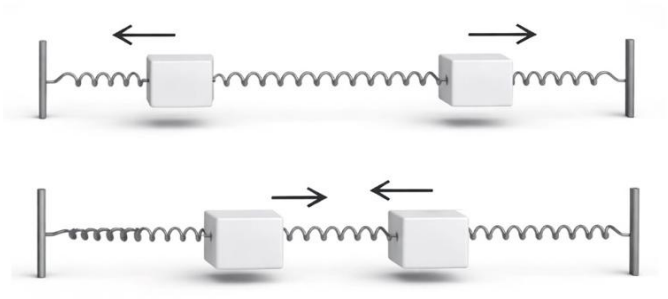


Figure 10.7: Second normal mode of a coupled mass–spring system

CHECK YOUR PROGRESS

Q1: Why is equilibrium more conveniently analysed using potential energy rather than forces?

Q2: What is the physical significance of the condition $\frac{\partial V}{\partial q_n} = 0$ for a system in equilibrium?

Q3: How does the sign of $\frac{d^2V(x)}{dx^2} = 0$ distinguish between stable and unstable equilibrium?

Q4: Why does a particle in neutral equilibrium remain at its new position after a small displacement?

Q5: Can a point where the force is zero always be an equilibrium point? Explain briefly using the potential energy curve.

Q6: Why can the motion of a system undergoing small oscillations be approximated as simple harmonic motion?

Q7: Why are higher-order terms neglected in the Taylor expansion of potential energy for small oscillations?

Q8: What physical roles do the effective mass m(0) and the spring constant K play in determining the angular frequency of small oscillations?

Q9: Why do the individual displacements x_1 and x_2 of coupled oscillators not execute simple harmonic motion, even though their sum and difference do?

Q10: What is the physical meaning of the normal coordinates (x_1+x_2) and (x_1-x_2) in a system of two coupled oscillators?

Q11: How does the coupling spring constant k′ affect the normal mode frequencies of the coupled oscillator system?

Q12: What is meant by a normal mode of vibration, and how is it obtained by choosing suitable initial conditions?

10.7 LET US SUM UP

- In a conservative system, the potential energy depends only on the configuration of the system and provides a useful method for analysing equilibrium.
- A system is in equilibrium when the generalized force becomes zero, which corresponds to a stationary value of the potential energy function.
- Depending on the response to a small disturbance, equilibrium may be classified as stable, unstable, or neutral.

- Small oscillations occur when a system is displaced slightly from a stable equilibrium position and the restoring force acts to bring it back.
- Near the equilibrium position, the potential energy can be expanded in a Taylor series, and the quadratic term mainly governs the motion.
- Under this approximation, the motion of the system becomes equivalent to .
- The frequency of small oscillations depends on the effective mass and the restoring force constant of the system.
- The approach provides a systematic method for deriving the equation of motion of oscillating systems.
- In coupled oscillator systems, the motion of one oscillator affects the other, leading to exchange of energy between them.
- By introducing normal coordinates, the coupled equations of motion can be transformed into simpler independent equations.
- The characteristic frequencies at which a coupled system oscillates are called normal mode frequencies, and the associated patterns are known as normal modes.
- The two important normal modes in a two-coupled oscillator system are **in-**phase motion and out-of-phase motion.
- The theory of small oscillations and normal modes has wide applications in molecular vibrations, crystal dynamics, and mechanical systems.

✓10.8 ANSWERS TO CHECK YOUR PROGRESS

Answer to Q No 1: Because equilibrium positions can be found directly from minima or stationary points of potential energy without resolving individual forces.

Answer to Q No 2: It indicates that the generalized force is zero, which is the condition for equilibrium.

Answer to Q No 3: If $\frac{d^2V(x)}{dx^2} > 0$, the equilibrium is stable; if, the equilibrium is $\frac{d^2V(x)}{dx^2} < 0$ unstable.

Answer to Q No 4: Because the force remains zero after a small displacement, so there is no restoring or disturbing force.

Answer to Q No 5: Yes, but stability depends on the nature of the potential energy curve near that point, not just on the force being zero.

Answer to Q No 6: Because for small displacements the restoring force is linear and proportional to displacement.

Answer to Q No 7: Because higher-order terms are very small compared to the quadratic term near equilibrium.

Answer to Q No 8: The effective mass m(0) represents inertia, and K represents the restoring force; together they determine the angular frequency $\omega = \sqrt{\frac{k}{m(0)}}$.

Answer to Q No 9: Because each mass is influenced by both its own spring and the coupling spring, making its motion a combination of two frequencies.

Answer to Q No 10: (x_1+x_2) represents in-phase motion, while (x_1-x_2) represents out-of-phase motion of the masses.

Answer to Q No 11: Increasing k′ increases the out-of-phase normal mode frequency but does not affect the in-phase frequency.

Answer to Q No 12: A normal mode is a definite pattern of oscillation at a single frequency obtained by choosing appropriate initial conditions.

10.9 FURTHER READING

- Goldstein, H., Poole, C. P., & Safko, J. L. (2002). *Classical mechanics* (3rd ed.). Pearson Education.
- Rana, N. C., & Joag, P. S. (1991). *Classical mechanics*. Tata McGraw-Hill.
- Upadhyaya, J. C. (2005). *Classical mechanics*. Himalaya Publishing House.

10.10 MODEL QUESTIONS

1. Define small oscillation.
2. State the condition for equilibrium in terms of potential energy.
3. What is meant by stable equilibrium?
4. Write the condition for neutral equilibrium in one dimension.
5. What are normal coordinates in coupled oscillators?
6. Why is equilibrium more conveniently analysed using potential energy rather than force?
7. Distinguish between stable and unstable equilibrium using potential energy curves.
8. Why are higher-order terms neglected in the theory of small oscillations?
9. What is meant by effective mass in the theory of small oscillations?
10. Explain in-phase and out-of-phase motion in a system of two coupled oscillators.
11. Explain the theory of small oscillations and show how the motion can be approximated as simple harmonic motion using Taylor expansion of potential energy.
12. Discuss the different types of equilibrium with suitable examples and necessary conditions.
13. Describe the motion of two coupled oscillators. Define normal coordinates and explain normal modes and their frequencies.

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UNIT 11: DYNAMICS OF A RIGID BODY I

UNIT STRUCTURE

11.1 Learning Objectives
11.2 Introduction
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11.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- define the concept of a rigid body and its constraints
- explain the degrees of freedom of a rigid body in 3D motion
- apply generalized coordinates to describe rigid body motion
- distinguish between space and body reference systems
- analyse coordinate transformations using direction cosines and matrices.

11.2 INTRODUCTION

In the previous unit, we studied potential energy and equilibrium, including stable unstable, and neutral equilibrium. We also discussed the general theory of small oscillations and two coupled oscillators, including normal coordinates, normal modes, and frequencies. In this unit, we will study the motion of rigid bodies. The topics include degrees of freedom, generalized coordinates, and body and space reference systems, which provide the basic framework for describing the translational and rotational motion of rigid bodies.

11.3 DEGREES OF FREEDOM OF A RIGID BODY

A rigid body is an idealized mechanical system in which the relative separation between its constituent particles remains unchanged during motion. As a result, the body preserves its shape and size while undergoing translation or rotation. This idealization simplifies the analysis of real physical bodies by neglecting very small deformations produced by external forces.

For any two particles *i* and *j* of the rigid body, the distance between them remains constant and can be expressed mathematically as

$$r_{ij} = C_{ij}$$

Where, r_{ij} is the distance between the particles and C_{ij} is a constant. These conditions restrict the independent motion of the particles and ensure that they move together as a single unit.

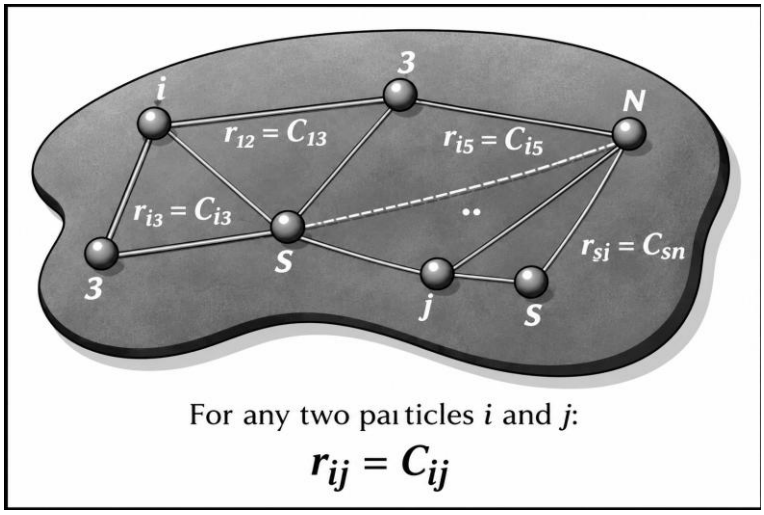


Figure 11.1: Rigid Body

The figure 11.1 shows a rigid body consisting of N particles represented by dots. The connecting lines indicate that the distance between every pair of particles remains fixed during motion. The idea of degrees of freedom (DOF) is essential in understanding the motion of mechanical systems. It refers to the smallest number of independent variables required to describe the complete configuration of a system at any instant. For a particle moving freely in three-dimensional space, three independent coordinates, such as x, y, and z, are needed; therefore, it has three degrees of freedom. If the motion is restricted to a plane, only two independent coordinates are sufficient, giving the system two degrees of freedom. Thus, the degrees of freedom of a system represent the number of independent ways in which the system can move.

For a rigid body made of N particles, each particle has three coordinates, so in total there are 3N coordinates. At first glance, one might think that the degrees of freedom can be found by subtracting the number of constraint equations from 3N i.e. DOF = 3N-number of constraints. However, this method does not work for a rigid body. Although we can write many equations saying that the distance between every pair of particles in a rigid body is constant $r_{ij} = C_{ij}$, we do not need all of them to fix the shape of the body. Once the distances between a few carefully chosen pairs of particles are fixed, the distances between the remaining pairs automatically become fixed. In other words, many distance equations are repetitions of the same restriction written in different ways. So, even though it looks like there are a very large number of constraints, they do not give new information about the motion. Because these constraints are not independent, subtracting all of them from 3N would remove too many degrees of freedom. That is why this method fails for a rigid body, and we must treat the body as a whole rather than as many separate particles.

To determine the degrees of freedom of a rigid body, consider three non-collinear particles P_1,P_2, and P_3 belonging to the body (Figure 11.2). Each particle in three-dimensional space requires three coordinates, so the total number of coordinates required to specify their positions is nine. However, since the distances between these three particles are fixed, there are three independent constraint equations corresponding to the distances r_{12}, r_{23}, and r_{13}. These constraints reduce the number of

independent coordinates from nine to six. Hence, the system of three non-collinear particles has six degrees of freedom.

Any additional particle of the rigid body has fixed distances from the three reference particles P_1,P_2, and P_3. Therefore, the position of any other particle is completely determined once the positions of these three particles are known. As a result, no additional degrees of freedom are introduced by adding more particles. Thus, the general motion of a rigid body in three-dimensional space is always described by six degrees of freedom.

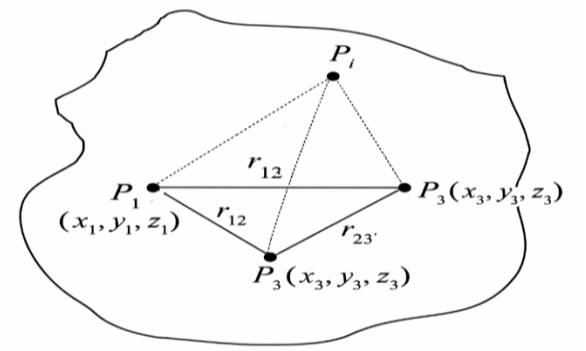


Figure 11.2: Rigid body represented by three non-collinear particles *P_1,P_2,* and *P_3* with fixed distances r_{12}, r_{23}, r_{13} the position of any other particle P_i is fully determined by these reference particles.

The six degrees of freedom of a rigid body can also be understood in an intuitive manner. One particle of the body can move freely in space and therefore requires three coordinates, contributing three degrees of freedom. A second particle, whose distance from the first particle is fixed, can move only on the surface of a sphere and hence requires two angular coordinates, contributing two degrees of freedom. A third particle, whose distances from the first two particles are fixed, can only rotate about a fixed axis and therefore requires only one coordinate, contributing one degree of freedom. The total degrees of freedom are thus 3+2+1=6.

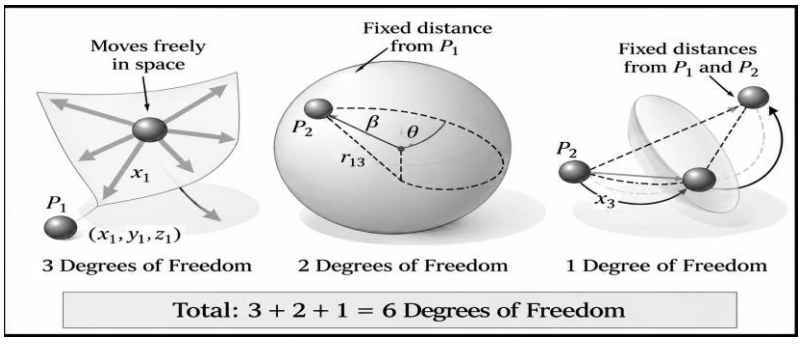


Figure 11.3: Intuitive illustration of the six degrees of freedom of a rigid body: 3 from free translation, 2 from spherical motion, and 1 from rotation about a fix axis (3+2+1 = 5)

Physically, these six degrees of freedom consist of translational and rotational motions. Three degrees of freedom correspond to the translation of the rigid body along the three mutually perpendicular axes in space. The remaining three degrees of freedom correspond to the rotation of the body about three independent axes. Any general

motion of a rigid body can therefore be considered as a combination of translation and rotation.

To describe the motion of a rigid body mathematically, six independent generalized coordinates are chosen. Usually, three of these coordinates are taken as the coordinates of a convenient reference point fixed in the body, such as the centre of mass, to describe the translational motion. The remaining three coordinates are chosen as angles, commonly known as Euler angles, which specify the orientation of the body in space and describe its rotational motion.

If, in addition to the constraints of rigidity, the body is subjected to further constraints such as being hinged, constrained to rotate about a fixed axis, or confined to move on a surface, the number of degrees of freedom is reduced accordingly. Consequently, the number of independent generalized coordinates required to describe the motion of the body also decreases.

11.4 GENERALIZED COORDINATES OF A RIGID BODY

Generalized coordinates are a set of independent variables used in classical mechanics to completely describe the configuration or position of a physical system. Unlike ordinary Cartesian coordinates, generalized coordinates are not restricted to linear distances measured along fixed axes. Any variables that uniquely specify the system and are independent of each other can be chosen as generalized coordinates. Their main purpose is to describe motion in the simplest and most convenient way, especially when the system is subjected to constraints. For example, consider a simple pendulum of fixed length suspended from a rigid support. The bob does not move freely in space; instead, its motion is constrained to a circular arc. Although its position could be described using Cartesian coordinates x and y, these are not independent because the length of the string remains constant. In this case, a single angle θ, which specifies the angular displacement of the pendulum from the vertical, is sufficient to describe the entire motion. This angle θ is a generalized coordinate, and it provides a much simpler and more natural description of the system than Cartesian coordinates.

In the study of rigid body mechanics, generalized coordinates are extremely important because a rigid body does not behave like a single particle. A rigid body is an idealized object in which the distance between any two particles remains constant throughout the motion. Because of this constraint, the motion of all particles in the body is interrelated, and it is not necessary to specify separate coordinates for each particle. Instead, a limited number of independent variables, called generalized coordinates, is sufficient to describe the configuration and motion of the entire rigid body.

For example, consider a rigid rod moving freely in space. The rod consists of a very large number of particles, and describing the position of each particle individually would require an enormous number of coordinates. However, since the distances between all particles of the rod remain fixed, the motion of the entire rod can be completely described by specifying the position of a single reference point on the rod, such as its centre of mass, along with a few angles that describe its orientation. Thus, a small set of generalized coordinates is enough to describe the motion of the whole rigid body.

A rigid body moving freely in three-dimensional space requires six independent quantities to completely describe its configuration at any instant. These quantities provide information about both the position of the rigid body in space and its orientation. Three of these quantities are needed to specify where the rigid body is located, while the remaining three are needed to specify how the rigid body is oriented. Since generalized coordinates are defined as the independent variables used to describe the configuration of a system, we choose six suitable generalized coordinates to represent these six independent quantities. Therefore, six generalized coordinates are sufficient to completely describe the motion and configuration of a rigid body in three-dimensional space.

The translational motion of a rigid body is described by specifying the position of a reference point fixed in the body. The most convenient reference point is the centre of mass. If the coordinates of the centre of mass are given by x, y, and z, these three coordinates completely determine the location of the rigid body in space. These coordinates are the translational generalized coordinates, and they describe how the rigid body moves from one position to another without considering its orientation.

However, specifying only the position of the centre of mass is not sufficient, because a rigid body can rotate while its centre of mass remains fixed. Therefore, additional generalized coordinates are required to describe the orientation of the rigid body. The orientation is determined by how the body is rotated with respect to a fixed reference frame. This rotational motion is independent of translational motion and must be described separately.

To understand how the orientation of a rigid body is described, we may imagine the rigid body as consisting of three particles. First, the position of one particle is fixed. Once this particle is fixed, the position of the second particle cannot change freely because the distance between the two particles remains constant. As a result, the second particle can move only on the surface of a sphere with the first particle at its centre. To specify the position of a point on a sphere, two angles are required. Therefore, two independent angular generalized coordinates are needed at this stage.

Next, when the positions of the first two particles are fixed, the third particle is also restricted by fixed distances. It cannot move freely in space and can only rotate about the line joining the first two particles. This motion can be described using one additional angle. Hence, one more angular generalized coordinate is required.

In this way, a total of three angles are sufficient to describe the orientation of a rigid body. These three angles serve as the generalized coordinates for rotational motion and completely specify how the rigid body is oriented in space. These rotational generalized coordinates are usually denoted by θ, ϕ, and ψ, and are commonly known as Euler angles, although other sets of angles may also be used.

Together, the three translational generalized coordinates *x, y,* and *z*, which describe the position of the centre of mass, and the three rotational generalized coordinates θ, ϕ, and ψ, which describe the orientation of the rigid body, provide a complete and consistent description of the configuration and motion of a rigid body in three-dimensional space. The table11.1 shows how each generalized coordinate helps in describing the motion of a rigid body. The figure 11.1 shows the translational and

rotational generalized coordinates used to describe the position and orientation of a rigid body in three-dimensional space.

Table 11.1: Generalized Coordinates of a Rigid Body and Their Purpose

Generalized Coordinate	Type of Motion	Purpose in Rigid Body Motion
x	Translation	Specifies the position of the rigid body along the X-direction by locating the centre of mass
y	Translation	Specifies the position of the rigid body along the Y-direction by locating the centre of mass
z	Translation	Specifies the position of the rigid body along the Z-direction by locating the centre of mass
θ	Rotation	Describes the tilt or inclination of the rigid body relative to a reference axis
ϕ	Rotation	Describes the rotation of the rigid body about a fixed reference direction
ψ	Rotation	Describes the rotation of the rigid body about its own axis

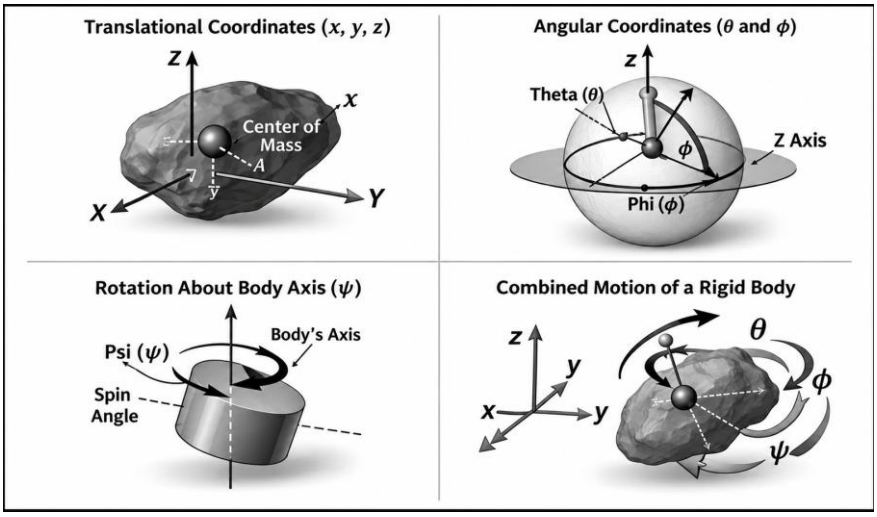


Figure 11.4: Generalized Coordinates of a Rigid Body

11.5 BODY AND SPACE REFERENCE SYSTEMS

To describe the motion of a rigid body clearly, we use two different reference systems called the space reference system and the body reference system. The space reference system is fixed in space, and its axes, usually named X, Y, and Z, do not move with the body. It acts like a permanent background or frame of reference, similar to axes drawn on the ground or in a room. The body reference system is fixed to the rigid body itself, and its axes, called X′, Y′, and Z′, move and rotate along with the body. The origin of the body reference system is generally chosen at the centre of mass of the rigid body because this makes the description of motion simpler. A rigid body has six degrees of freedom: three translational and three rotational. The translational motion is described by the position of the centre of mass relative to the space axes,

while the rotational motion is described by the changing orientation of the body axes with respect to the space axes. This orientation can be specified using three angles, commonly known as Euler angles, or equivalently by direction cosines, which represent the angles between the body axes and the space axes. Thus, by using the space reference system to describe position and the body reference system to describe orientation, we can completely describe the motion of a rigid body in a simple and systematic way. Figure 10.2 shows the body frame of reference and the space frame of reference**,** where the space axes X,Y,Z are fixed in space with origin O, and the body axes X′,Y′,Z′, are fixed in the rigid body with origin O′ (usually at the centre of mass)

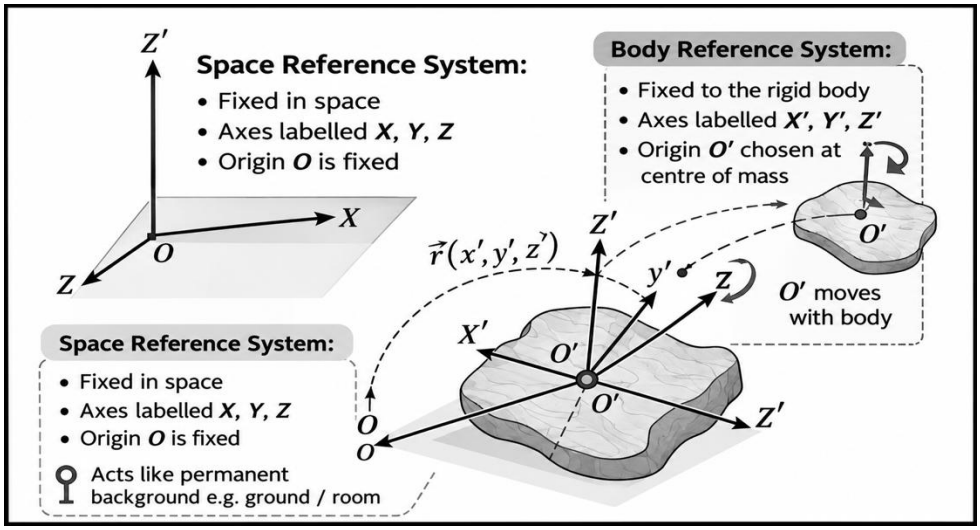


Figure 11.5: Space and Body reference frame

Transformation From One Coordinate System To Another

Consider two coordinate systems: a space (fixed) coordinate system XYZ and a body (moving) coordinate system X′Y′Z′. Let the origins of both systems coincide at a point O. A point R in space can be described with respect to both coordinate systems.(See the figure 11.5)

Let, R(x,y,z) and R(x′,y′,z′) be the coordinates of the same point R in the space frame and body frame respectively.

Let $\hat{\imath}, \hat{\jmath}, \hat{k}$ be the unit vectors along the X, Y, Z axes, and $\hat{\imath'}$, $\hat{\jmath'}$, $\hat{k'}$, be the unit vectors along the X′, Y′, Z′, axes.

The direction cosines are defined as the cosines of the angles between the body axes and the space axes.

For X' axis:

$$Cos\ \alpha_1 = \hat{\imath}'.\hat{\imath}\ ;\ Cos\ \alpha_2 = \hat{\imath}'.\hat{\jmath};\ Cos\ \alpha_3 = \hat{\imath}'.\hat{k}$$

For Y' axis:

$$Cos\ \beta_1 = \hat{\jmath}.'\hat{\imath}\ ;\ Cos\ \beta_2 = \hat{\jmath}.'\hat{\jmath};\ Cos\ \beta_3 = \hat{\jmath}.'\hat{k}$$

For Z' axis:

$$Cos\ \gamma_1 = \hat{k}.'\hat{\imath}\ ;\ Cos\ \gamma_2 = \hat{k}.'\hat{\jmath};\ Cos\ \gamma_3 = \hat{k}.'\hat{k}$$

The direction cosines are denoted as:

$$C_{11} = Cos\ \alpha_1\ ;\ C_{12} = Cos\ \alpha_2;\ C_{13} = Cos\ \alpha_3$$
$$C_{21} = Cos\ \alpha_1\ ;\ C_{22} = Cos\ \alpha_2;\ C_{23} = Cos\ \alpha_3$$
$$C_{31} = Cos\ \alpha_1\ ;\ C_{32} = Cos\ \alpha_2;\ C_{33} = Cos\ \alpha_3$$

Using the above definitions, the body-frame unit vectors can be written correctly in terms of the space-frame unit vectors as:

$$\hat{\imath}' = C_{11}\hat{\imath} + C_{12}\hat{\jmath} + C_{13}\hat{k}$$
$$\hat{\jmath}' = C_{21}\hat{\imath} + C_{22}\hat{\jmath} + C_{23}\hat{k}$$
$$\hat{k}' = C_{31}\hat{\imath} + C_{32}\hat{\jmath} + C_{33}\hat{k}$$

From these equations we obtain

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{23} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Or

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = A \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Where $A = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{23} \end{pmatrix}$

The body reference system is often represented by primed axes, while the space reference system is represented by unprimed axes.

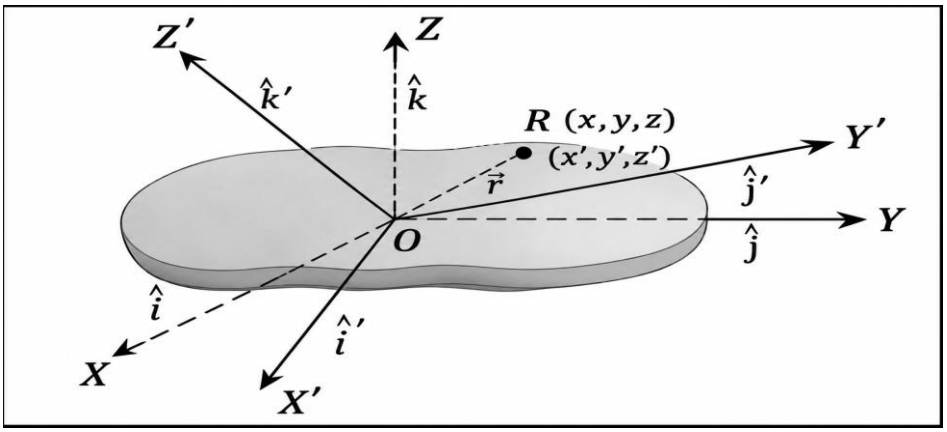


Figure11.6: Schematic representation of the unprimed (X,Y,Z) and primed (X′,Y′,Z′) coordinate systems with a common origin O, showing the position. vector r of a point R(x,y,z) within the body-fixed reference frame.

W CHECK YOUR PROGESS

Q 1: What is meant by a rigid body in dynamics?
Q 2: Why is the motion of a rigid body said to be subject to constraints?
Q 3: What are holonomic constraints in the context of rigid body motion?
Q 4: What are degrees of freedom (DOF) of a mechanical system?
Q 5: Why does the formula DOF=3N−constraints fail for a rigid body?
Q 6: How do we conclude that a rigid body in three-dimensional space has six degrees of freedom?

Q 7: What is the physical meaning of the six degrees of freedom of a rigid body?
Q 8: What are generalized coordinates in classical mechanics?
Q 9: Why are generalized coordinates preferred over Cartesian coordinates for constrained systems like a pendulum?
Q 10:How many generalized coordinates are required to describe a rigid body in three-dimensional space?
Q 11: What do the translational generalized coordinates represent for a rigid body?
Q 12:What is the difference between the space reference system and the body reference system?
Q 13:Why is the origin of the body reference system usually chosen at the centre of mass?
Q 14:How is the motion of a rigid body completely described using these two reference frames?
Q 15:What are direction cosines in rigid body dynamics?
Q 16:What is the purpose of the transformation matrix A between two coordinate systems?

2 11.6 LET US SUM UP

- A rigid body is an idealized system of particles in which the distance between any two particles remains constant during motion; therefore, the body retains its shape and size while undergoing translation or rotation.
- The fixed distances between the particles impose constraints on the motion of the rigid body, and these are called holonomic constraints because they can be expressed as coordinate equations independent of time.
- The degrees of freedom (DOF) of a system represent the minimum number of independent coordinates required to completely describe its configuration and motion.
- A free rigid body in three-dimensional space always has six degrees of freedom: three translational and three rotational. These six coordinates completely determine its motion.
- The usual formula $DOF = 3N -$ constraints cannot be directly applied to a rigid body because many of the distance constraints between particles are not independent.
- The six degrees of freedom can be understood physically as: three coordinates to specify the position of a reference point (usually the centre of mass) and three angular coordinates to specify the orientation of the body.
- Generalized coordinates are independent variables used to describe the configuration of a constrained system in the simplest and most convenient way. They are preferred over Cartesian coordinates because they naturally incorporate constraints.
- For a rigid body, the three translational generalized coordinates are usually taken as the coordinates $(x, y, z)$of the centre of mass, while the three rotational generalized coordinates are represented by the Euler angles (θ, ϕ, ψ).
- The translational coordinates describe the position of the rigid body in space, whereas the rotational coordinates describe its orientation. Together they provide a complete description of rigid body motion.

- To study rigid body motion systematically, two reference systems are used: the space reference system and the body reference system. The space frame remains fixed in space, while the body frame moves and rotates along with the rigid body.
- The origin of the body reference system is generally chosen at the centre of mass, as this simplifies the separation of translational and rotational motion.
- The orientation of the rigid body with respect to the space frame can be specified by Euler angles or equivalently by direction cosines, which represent the angles between the body axes and the space axes.
- Direction cosines are used to establish the relation between the body-fixed and space-fixed coordinate systems and are arranged in the form of a transformation matrix.
- The transformation matrix allows conversion of coordinates of a point from one reference system to another, which is essential in analysing rigid body motion in different frames of reference.
- Thus, the motion of a rigid body can be completely described as a combination of translation and rotation, making the concepts of degrees of freedom, generalized coordinates, and reference frames fundamental in rigid body dynamics.

K 11.7 ANSWERS TO CHECK YOUR PROGRESS

Answer to Q No. 1: A rigid body is an idealized system in which the distance between any two particles remains constant during motion, meaning the body does not change its shape or size while translating or rotating.

Answer to Q No. 2: Because all particles must maintain fixed distances from each other, they cannot move independently. Their motion is restricted, so the body moves as a single unit, either translating or rotating without deformation.

Answer to Q No. 3: Holonomic constraints are restrictions that can be written as equations involving only the coordinates of particles, such as $r_{ij}=C_{ij}$, and they do not explicitly depend on time.

Answer to Q No. 4: Degrees of freedom are the minimum number of independent coordinates required to completely specify the configuration and motion of a system.

Answer to Q No. 5: Because many distance constraints in a rigid body are not independent. Fixing a few key distances automatically fixes the others, so subtracting all constraints would incorrectly remove too many degrees of freedom.

Answer to Q No. 6: Three non-collinear particles require 9 coordinates, but 3 independent distance constraints reduce this to 6. Additional particles do not add new freedom, so a rigid body always has 6 DOF.

Answer to Q No. 7: A rigid body has 3 translational DOF (motion along x, y, z axes) and 3 rotational DOF (rotation about three independent axes).

Answer to Q No. 8: Generalized coordinates are independent variables chosen to uniquely describe the configuration of a system, especially in the presence of constraints, and are not limited to Cartesian coordinates.

Answer to Q No. 9: Because Cartesian coordinates may not be independent under constraints, while a single generalized coordinate (like angle θ) can describe the motion more simply and naturally.

Answer to Q No. 10: A rigid body in 3D space requires six generalized coordinates: three for translation and three for rotation.

Answer to Q No. 11: They represent the position of a fixed reference point (usually the centre of mass) using coordinates x,y,zx, y, zx,y,z, describing the location of the body in space.

Answer to Q No. 12: The space reference system (XYZ) is fixed in space, while the body reference system (X′Y′Z′) is attached to the rigid body and moves and rotates with it.

Answer to Q No. 13: Because choosing the centre of mass simplifies the description of translational motion and separates translation from rotation.

Answer to Q No. 14: Translation is described by the position of the centre of mass in the space frame, and rotation is described by the orientation of the body axes relative to the space axes.

Answer to Q No. 15: Direction cosines are the cosines of the angles between the body axes and the space axes, used to specify the orientation of the rigid body.

Answer to Q No. 16: The matrix A, formed from direction cosines, converts the coordinates of a point from the space frame (x,y,z) to the body frame (x′,y′,z′).

O 11.8 FURTHER READING

- Goldstein, H., Poole, C. P., & Safko, J. L. (2002). *Classical mechanics* (3rd ed.). Pearson Education.
- Rana, N. C., & Joag, P. S. (1991). *Classical mechanics*. Tata McGraw-Hill.
- Upadhyaya, J. C. (2005). *Classical mechanics*. Himalaya Publishing House.

M 11.9 MODEL QUESTIONS

1. Define a rigid body with one example.
2. What is meant by degrees of freedom of a rigid body?
3. Why are generalized coordinates preferred over Cartesian coordinates in constrained motion?
4. Distinguish between space reference system and body reference system.
5. What are direction cosines in rigid body dynamics?
6. Explain the concept of holonomic constraints in rigid body motion.
7. Derive why a rigid body in three-dimensional space has six degrees of freedom.
8. Discuss the role of generalized coordinates in describing rigid body motion.
9. Describe the space and body reference systems used in rigid body dynamics.
10. Explain the transformation between space and body coordinate systems using direction cosines and matrix form.

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UNIT 12: RIGID BODY DYNAMICS-II

UNIT STRUCTURE

12.1 Learning Objectives
12.2 Introduction
12.3 Euler Equations
    12.3.1 Euler Angles
    12.3.2 Components of Angular velocity
12.4 Principal Axes
12.5 Principal Moment of Inertia
12.6 Rotational Kinetic Energy of a Rigid Body
12.7 Let Us Sum Up
12.8 Answers to Check Your Progress
12.9 Further Reading
12.10 Model Questions

12.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- explain the fundamental properties of rigid bodies and analyse their rotational behaviour about specified axes
- learn the key concepts behind rigid body rotation including Eulers theorem
- obtain Eulers equation of motion of a rigid body
- visualize moment of inertia as a tensor and its role in rotational motion
- express kinetic energy of a rigid body in terms of Euler's angle.

12.2 INTRODUCTION

In the previous unit, the basic principles of rigid body motion about a fixed axis were studied. The discussion included angular displacement, angular velocity, angular acceleration, torque, angular momentum and the scalar form of moment of inertia. These concepts established the fundamental framework for understanding rotational motion under simple conditions.

In the present unit, the study is extended to the more general case of three-dimensional rotational motion, where the axis of rotation may vary with time. Such motion requires a more systematic mathematical treatment. For this purpose, Euler's equations and Euler angles are introduced to describe the orientation and rotational dynamics of rigid bodies. The concepts of principal axes, principal moments of inertia and inertia tensor are also developed to simplify the analysis of rotational systems. Finally, the rotational kinetic energy of a rigid body is expressed in terms of angular velocity components and Euler angles.

12.3 EULER EQUATIONS

A rigid body is an idealized mechanical system in which the relative distance between any two constituent particles remains constant during its motion. The dynamics of rigid bodies deals with the influence of external forces and torques on both translational and rotational motion. Leonhard Euler developed the mathematical formulation governing the rotational motion of rigid bodies subjected to external torque. Besides Euler's approach, Newtonian and Lagrangian formulations also provide important methods for studying such mechanical systems. The motion of a rigid body may take place as pure translation, pure rotation, or as a combined translational and rotational motion. To analyse the motion of a rigid body, certain fundamental physical quantities are required.

1. Angular displacement
2. Angular velocity
3. Angular acceleration
4. Angular Momentum
5. Moment of Inertia
6. Degrees of freedom

The turning effect produced by an external force on a rigid body is termed torque and is responsible for changing its rotational state. It is denoted by τ. The absence of torque leads to conservation of angular momentum. The degree of freedom is a crucial parameter to analyze its motion in different dimensions. For this the concepts of coordinate systems will be utilized to visualize the motion.

12.3.1 Euler Angles

Euler's equation of motion describes how the angular velocity of a rigid body changes with time under the influence of an external torque. For deriving Euler's equations, two coordinate systems are introduced: a body-fixed set of axes and a space-fixed set of axes. The body-fixed axes remain attached to the rigid body, whereas the space-fixed axes remain stationary in space. According to Euler's theorem, the configuration of a rigid body in space is completely specified by six independent coordinates, consisting of translational and rotational variables. Out of these, three represent rotational motion and the remaining three represent translational motion.

Euler angles are three independent angular coordinates that are used to define the orientation of a rigid body relative to a fixed reference system. The spatial orientation of a rotating rigid body in three-dimensional space can be fully represented by these three angular variables, which together specify its rotational configuration.

1. Precession Angle (ϕ) or first rotation

2. Nutation Angle (θ) or second rotation

3. Spin/Body Angle (ψ) or third rotation

The Euler angles are shown in figure12.1

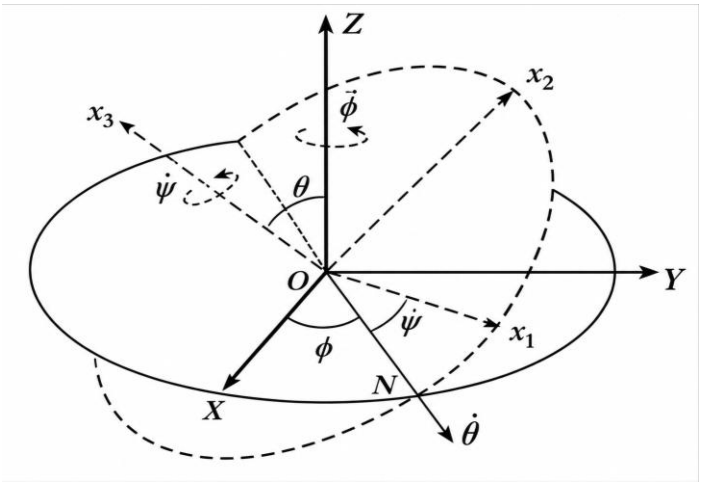


Figure 12.1 Representation of the showing the coordinates ($r,\ \theta, \phi$) and corresponding unit vectors. (Source: Classical Mechanics by Jeremy Tatum, University of Victoria')

If L is the angular momentum of a rigid body and torque $\boldsymbol{\tau}$ acts on the system, then

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

Where $\vec{L} = I\vec{\omega}$, I stand for moment of Inertia and $\vec{\omega}$ is the angular velocity. Moment of Inertia is a tensor, and it can be expressed as

$$I = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

And the angular momentum

$$\vec{L} = I\vec{\omega} = \begin{bmatrix} I_1\vec{\omega}_1 \\ I_2\vec{\omega}_2 \\ I_3\vec{\omega}_3 \end{bmatrix}$$

$\vec{\omega}_1, \vec{\omega}_2$ and $\vec{\omega}_3$ are components of angular velocity along principal axis. Now introducing the space and body set of axes and differentiating $\vec{L}$

$$\vec{\tau} = \left(\frac{d\vec{L}}{dt}\right)_{space} = \left(\frac{d\vec{L}}{dt}\right)_{body} + \vec{\omega} \times \vec{L}$$

Rearranging

$$\left(\frac{d\vec{L}}{dt}\right)_{body} = \vec{\tau} - \vec{\omega} \times \vec{L} \qquad (12.1)$$

Which is the Euler equation. Now it can be expressed in terms of three components along principal axis.

$$\left(\vec{\omega} \times \vec{L}\right)_1 = \vec{\omega}_2 L_3 - \vec{\omega}_3 L_2 = \vec{\omega}_2 I_3 \omega_3 - \vec{\omega}_3 I_2 \vec{\omega}_2 \qquad (12.2)$$

Similarly,

$$\left(\vec{\omega} \times \vec{L}\right)_2 = \vec{\omega}_3 I_1 \vec{\omega}_1 - \vec{\omega}_1 I_3 \vec{\omega}_3$$
$$\left(\vec{\omega} \times \vec{L}\right)_3 = \vec{\omega}_1 I_2 \vec{\omega}_2 - \vec{\omega}_2 I_1 \vec{\omega}_1$$

Substituting $\vec{\omega} \times \vec{L}$ in equation (1) we will get three sets of equations

$$I_1 \dot{\vec{\omega}}_1 + \vec{\omega}_2 I_3 \vec{\omega}_3 - \vec{\omega}_3 I_2 \vec{\omega}_2 = \vec{\tau}_1$$
$$I_2 \dot{\vec{\omega}}_2 + \vec{\omega}_3 I_1 \vec{\omega}_1 - \vec{\omega}_1 I_3 \vec{\omega}_3 = \vec{\tau}_2$$
$$I_3 \dot{\vec{\omega}}_3 + \vec{\omega}_1 I_2 \vec{\omega}_2 - \vec{\omega}_2 I_1 \vec{\omega}_1 = \vec{\tau}_3$$
$$\dot{\vec{\omega}}_1 = \frac{d\vec{\omega}_1}{dt}$$

This set of equations are called Euler equation of motion, and they are first order differential equation. If there is no torque working on the rigid body, then the Euler equations

$$I_1 \dot{\vec{\omega}}_1 + \vec{\omega}_2 I_3 \vec{\omega}_3 - \vec{\omega}_3 I_2 \vec{\omega}_2 = 0$$

$$I_2 \dot{\vec{\omega}}_2 + \vec{\omega}_3 I_1 \vec{\omega}_1 - \vec{\omega}_1 I_3 \vec{\omega}_3 = 0$$

$$I_3 \dot{\vec{\omega}}_3 + \vec{\omega}_1 I_2 \vec{\omega}_2 - \vec{\omega}_2 I_1 \vec{\omega}_1 = 0 \qquad (12.3)$$

The absence of torque represents free rotation of the rigid body. In absence of torque

1. Angular momentum is always conserved.
2. The kinetic energy of rotation is conserved.

12.3.2 Components of Angular Velocity

Angular velocity is the quantity that describes the rate of change of angular displacement with time and indicates how fast a rigid body changes its orientation during motion. It is denoted by $\vec{\omega}$

$$\vec{\omega} = \frac{d\theta}{dt}$$

where θ represents angular displacement. The concept of angular velocity is already familiar to us from secondary and undergraduate studies. Here we will express angular velocity in terms of Euler angles as follows in terms of body set of coordinates as well as space set of coordinates.

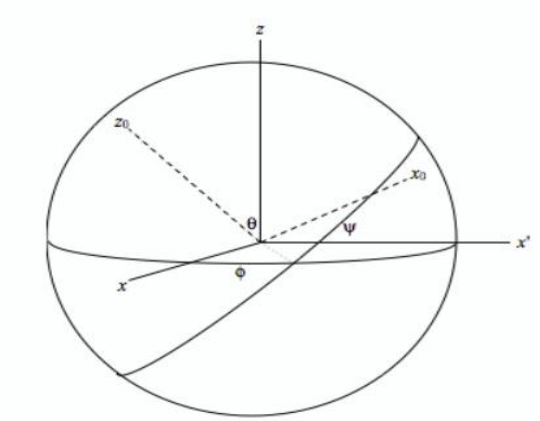


Figure 12.2: Geometrical representation of the showing the angular parameters θ and ψ on a spherical surface. (Source: Classical Mechanics by Jeremy Tatum, University of Victoria')

In the figure we have two set of coordinates x, y, z for space set of coordinates and $x_0y_0z_0$ body set of coordinates. Suppose the body is freely rotating and no torque acts on it, then the angular momentum of the body will be constant but, the angular velocity will not be constant. In the figure the Eulerian angles are indicated by $\theta, \varphi, \text{and } \psi$. Therefore, angular velocity;

Here

$\Omega_1 = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi$

$\Omega_2 = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi$

$\Omega_3 = \dot{\phi}\cos\theta + \dot{\psi}$ angular velocity in body set of coordinates.

In space set of coordinates, it is written as

$\Omega_1 = \dot{\theta}\cos\varphi + \dot{\psi}\sin\theta\sin\phi$

$\Omega_2 = \dot{\theta}\sin\varphi - \dot{\varphi}\sin\theta\cos\phi$

$\Omega_3 = \dot{\psi}\cos\theta + \dot{\phi}$

The angular velocity of rotation of rigid body about an axis is independent of choice of origin of body fixed system.

Angular Momentum

Angular momentum is an important dynamical quantity related to rotational motion and depends upon both the distribution of mass and the angular velocity of the body. It remains conserved in the absence of external torque acting on the system. For a single particle, angular momentum is defined as the vector product of its position vector $\vec{r}$ and linear momentum $\vec{p}$, and serves as the rotational counterpart of linear momentum.

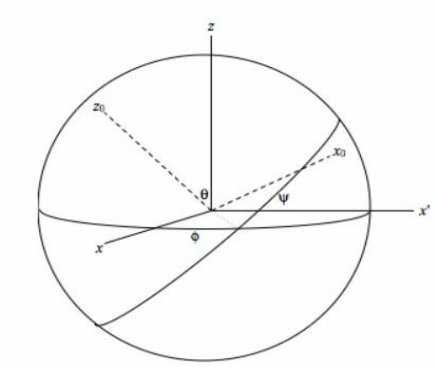


Figure 12.3: Schematic representation of a point on a sphere in illustrating the orientation angles θ, ϕ, and ψ.(Source Adapted from the book' Classical Mechanics by Jeremy Tatum, University of Victoria')

i.e. $$\overrightarrow{L} = \vec{r} \times \vec{p}$$

while for an extended object the angular momentum

i.e.

$$\vec{L} = I.\vec{\omega}$$

where *I* = Moment of Inertia and ω is angular velocity. For a system consisting of n number of particles,

$$\vec{L} = \vec{L}_1 + \vec{L}_2 + \vec{L}_3 + \cdots + \vec{L}_n$$

i.e., the sum of angular momentum of individual particles. For a rigid body consisting of n number of particles

$$L = \left(\sum_{i=1}^{n} m_i\, r_i^2\right)\vec{\omega}$$

Where, $I = \sum_{i=1}^{n} m_i\, r_i^2$ is called moment of inertia of a body.

Moment of Inertia

In rotational dynamics, moment of inertia plays a role similar to mass in translational motion by quantifying the opposition offered to angular acceleration. It depends on how the mass of a body is distributed about the axis of rotation and determines the amount of torque needed to modify its rotational motion. A greater moment of inertia implies stronger resistance to rotational acceleration, and hence a larger torque is required to produce the same angular acceleration. Two important theorems for evaluating moment of inertia are already familiar from undergraduate studies. Here we will express moment of Inertia $I=mr^2$ as a tensor.Tensor notation provides a convenient mathematical representation for physical quantities whose values depend on direction and involve components in more than one dimension.

The position vector of an object in three dimensional is represented using x, y, z axis or their components, similarly in case of tensor, especially moment of inertia we will use the notation I_{jk}, where j, k denotes element of inertia tensor.

The inertia (I) tensor is written as

$$\mathrm{I} = \begin{pmatrix} \mathrm{I_{xx}} & \mathrm{I_{xy}} & \mathrm{I_{xz}} \\ \mathrm{I_{yx}} & \mathrm{I_{yy}} & \mathrm{I_{yz}} \\ \mathrm{I_{zx}} & \mathrm{I_{zy}} & \mathrm{I_{zz}} \end{pmatrix}$$

Which is a symmetrical tensor it can be reduced to a diagonal element
Where,

$$I_{xy} = I_{xz} = I_{yx} = I_{yz} = I_{zx} = I_{zy} = 0$$

Then $I = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}$

If I_{xx}=I_1, $lyy = I_2$,$I_{zz} = I_3$

$$I = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

Symmetry plays an important role in rigid body dynamics because it simplifies the determination of principal axes and moments of inertia. For a rigid body if Two of the principal moments are equal i.e. $I_1 = I_2$, it is called symmetrical top. Three of the principal moments are different, i.e., $I_1 \neq I_2 \neq I_3$, it is called asymmetric top. If three principal moments are equal, $I_1 = I_2 = I_3$, it is called spherical top.

12.6 KINETIC ENERGY OF ROTATION OF RIGID BODY

In rotational motion, each particle of a rigid body contributes to the total kinetic energy through its individual velocity. The total rotational kinetic energy is therefore obtained by summing the kinetic energies of all constituent particles. The total rotational kinetic energy is therefore obtained by summing over all such contributions. The kinetic energy of a rigid body

$$\mathrm{T}_{rotaional} = \frac{1}{2}\sum_{i=1}^{n} m_i v_i^2 \qquad (12.4)$$

or $$\mathrm{T}_{rotaional} = \frac{1}{2} m_i v_i^2 \qquad (12.5)$$

From the relation of linear and angular velocity and substituting in equation (12.5)

$$\vec{v}_i = \vec{\omega} \times \vec{r_i}$$

$$\mathrm{T}_{rotaional} = \frac{1}{2} m_i v_i^2$$

$$= \frac{1}{2} m_i v_i . \overrightarrow{(\omega \times \vec{r_i})}$$

$$= \frac{1}{2} m_i \vec{\omega}.(\vec{v_i} \times \vec{r_i})$$
$$= \frac{1}{2} \vec{\omega}.(\vec{r_i} \times m_i \vec{v}_i)$$
$$= \tfrac{1}{2} \vec{\omega}.(\vec{r_i} \times \vec{p}_i)$$
$$= \tfrac{1}{2} \vec{\omega}.\vec{L}$$

We know that $\vec{L} = \vec{I}.\vec{\omega}$, substituting in the above equation

$$T_{rotaional} = \frac{1}{2} \vec{\omega}.\vec{L}$$
$$= \frac{1}{2} \vec{\omega}.\vec{I}.\vec{\omega}$$
$$= \frac{1}{2} I\omega^2$$

If ω_1, $\omega_2, and\ \omega_3$ are velocity components referred to principal axis and I_1, I_2 and I_3 are corresponding moment of inertia then

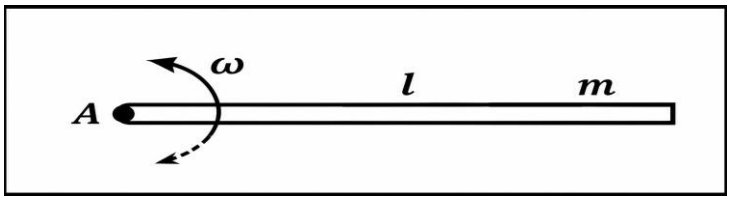
$$T_{rot} = \tfrac{1}{2} I_1 \omega_1^2 + \tfrac{1}{2} I_2 \omega_2^2 + \tfrac{1}{2} I_3 \omega_3^2 \qquad (12.6)$$

In terms of Eulerian angle, it can be expressed as

$$T_{rot} = \frac{1}{2} I_1 (\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi)^2 + \frac{1}{2} I_2 (\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi)^2 + \frac{1}{2} I_3 (\dot{\phi}\cos\theta + \dot{\psi})^2$$

WCHECK YOUR PROGRESS

Q1. What is a rigid body?
Q2. What are Eulerian Angles?
Q3. What do you mean by torque free motion of a rigid body?
Q4. Mention the conditions for a rigid body to be symmetrical top, asymmetrical top and spherical top?
Q5. Write the physical significance of moment of Inertia.
Q6. In the figure a thick string rod of mass m and length l is rotated with respect to the point A.



If angular velocity is ω, find (a) kinetic energy of rotation about A and (b) total kinetic energy of rotational and translational about center of mass.

212.8 LET US SUM UP

- A rigid body preserves the relative positions of its constituent particles during motion.
- Euler angles provide a convenient method for specifying orientation in space.
- Euler equations describe the rotational motion of rigid bodies under torque.
- In the absence of external torque, angular momentum remains conserved.

- The inertia tensor extends the scalar moment of inertia to three-dimensional motion.
- Rotational kinetic energy depends on angular velocity and mass distribution.

K12.9 ANSWERS TO CHECK YOUR PROGRESS

Answer to Q1: A rigid body is an idealized system in which the relative positions of all constituent particles remain unchanged during motion.

Answer to Q2: Euler angles are three independent angular parameters used to specify the orientation of a rigid body relative to a fixed coordinate system. The three Euler angles are**:** Precession angle**,** Nutation angle, and Spin (or Body) angle.

Answer to Q3: The torque free rotation means rotation of a rigid body without external torque, which leads to conservation of angular momentum.

$$\tau = \frac{dL}{dt}$$

Since $\tau = 0$, therefore L=const, in absence of torque, the Euler equations can be written as

$$I_1\dot{\omega}_1 + \omega_2 I_3\omega_3 - \omega_3 I_2\omega_2 = 0$$

$$I_2\dot{\omega}_2 + \omega_3 I_1\omega_1 - \omega_1 I_3\omega_3 = 0$$

$$I_3\dot{\omega}_3 + \omega_1 I_2\omega_2 - \omega_2 I_1\omega_1 = 0$$

Answer to Q4: A rigid body is said to be symmetric top if $I_1=I_2$**,** i.e. two of the principal moments are equal. Asymmetrical top if $I_1 \neq I_2 \neq I_3$, none of the principal moments are same. spherical top if $I_1 = I_2 = I_3$**,** all the principal moments are equal.

Answer to Q5: Moment of inertia quantifies the opposition offered by a body to changes in its rotational motion and depends on the mass distribution about the axis.

Answer to the Q6: The moment of Inertia of the rod about A

$$I = \frac{1}{3}ml^2$$

Therefore, kinetic energy

$$K = \frac{1}{2}I\omega^2 = \frac{1}{2}\times\frac{1}{3}ml^2\omega^2$$

$$= \frac{1}{6}ml^2\omega^2$$

The moment of Inertia of the rod

$$I_0 = \frac{1}{12}ml^2$$

Since the center of mass is moving in a circle of radius l/2
Therefore

$$v_{cm} = \frac{\omega l}{2}$$

Now kinetic energy is sum of translational and rotational energy

$$K = \frac{1}{2}MV_{\text{cm}}^2 + \frac{1}{2}I_0\omega^2$$
$$= \frac{1}{2}M\frac{l^2}{4}\omega^2 + \frac{1}{2}\frac{1}{12}Ml^2\omega^2$$
$$= Ml^2\omega^2\left(\frac{1}{8}+\frac{1}{24}\right) = \frac{1}{6}Ml^2\omega^2$$

O12.10 FURTHER READING

- Goldstein, H., Poole, C., & Safko, J. (2002). *Classical Mechanics* (3rd ed.). Pearson Education.
- Gupta, A. B. (2022). *Fundamentals of Classical Mechanics*. New Delhi: Books and Allied (P) Ltd.
- Indira Gandhi National Open University (IGNOU). (n.d.). *Classical Mechanics* [Study Material]. Retrieved from eGyanKoshRepository
- Huang, L. (2011). *A Concise Introduction to Mechanics of Rigid Bodies: Multidisciplinary Engineering*. Boca Raton, FL: CRC Press.

M12.11 MODEL QUESTIONS

1. What is an inertial system and principal axes?
2. Give a brief accounts of Eulerian angles
3. Discuss torque free motion of rigid body.
4. Write Euler theorem of rotational motion and establish Euler equation of motion and mention the physical meaning of Euler equation.
5. Derive the expression for kinetic energy of rotation of a rigid body.
6. Derive Euler equations of a rigid body of principal moments I_1, I_2 and I_3, considering no torques act on the body.
7. Define Inertia Tensor and write its properties
8. Write the importance of diagonalization of inertia tensor.

*** *** ***

UNIT 13: SPECIAL THEORY OF RELATIVITY

UNIT STRUCTURE

13.1 Learning Objectives
13.2 Introduction
13.3 Galilean Transformation
13.4 Principle of Relativity and Speed of Light
13.5 Postulates of Special Theory of Relativity
13.6 Lorentz Transformation and its Consequences
13.7 Let Us Sum Up
13.8 Answers to Check Your Progress
13.9 Further Reading
13.10 Model Questions

13.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- state the fundamental concepts of Special Theory of Relativity, including Galilean trans formation, Lorentz transformation, and the two postulates of STR
- explain the limitations of the Galilean transformation and interpret the significance of the principle of relativity and the constancy of the speed of light
- Use Lorentz transformation equations to solve numerical problems involving time dilation, length contraction, simultaneity, and relativistic velocity addition
- compare Galilean and Lorentz transformations and examine how relativistic effects arise from the transformation equations
- assess the physical implications of relativistic phenomena and justify the necessity of the Special Theory of Relativity over classical mechanics at high velocities.

13.2 INTRODUCTION

In the previous unit, we studied Euler's equations. Euler angles, components of angular velocity, principal axes and principal moments of inertia, and the rotational kinetic energy of a rigid body. In this unit, we move from rotational mechanics to the foundations of special relativity. We will study Galilean transformation, the principle of relativity and the constancy of the speed of light, the postulates of special theory of relativity, and the Lorentz transformation along with its important consequences. These concepts provide the basis for understanding how measurements of space and time are related for observers in different inertial frames

13.3 GALILEAN TRANSFORMATION

In this section, we review Newtonian Relativity, which led to the expectation that the measured speed of light should depend on the direction of its propagation; this prediction was later disproved by experiment. In the later section, we introduce the Special Theory of Relativity, developed by Albert Einstein, and examine how it resolved this difficulty by proposing that the speed of light in vacuum is the same for all observers in inertial frames, regardless of the direction of propagation or the motion of the source.

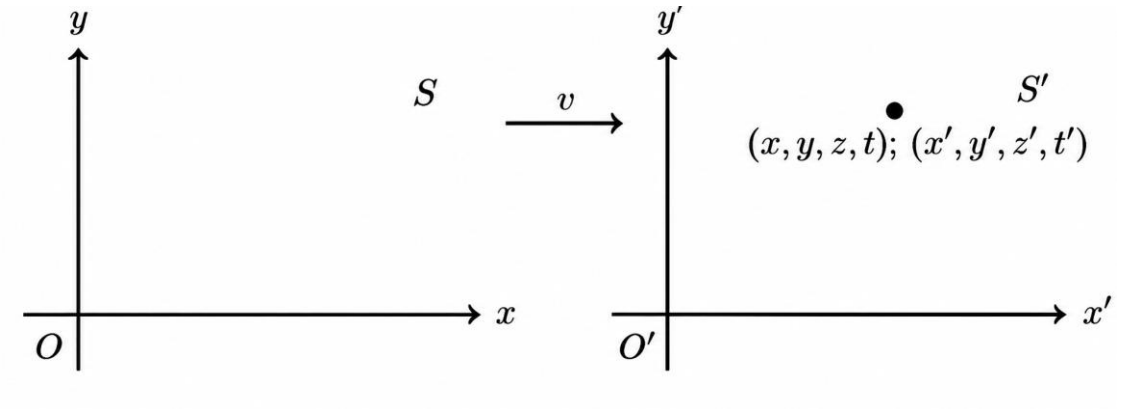


Figure 13.1: Galilean Transformation: Frame S' moves along x direction with velocity v relative to S-frame. The coordinates of the particle in the S and S' frames are (x, y, z, t) and (x', y', z', t'), respectively.

We consider two frames of reference, S and S', where the S' frame moves relative to the S frame with a uniform velocity v in the positive x or x' direction. We consider a particle of mass m that is moving under the action of force $\vec{F}$. The coordinates of the particle are given by (x, y, z, t) in S frame and (x', y', z', t') in S' frame. The coordinates t and t' give the measure of time in their respective frame and are considered the same in all reference systems. Themathematical relationship connecting the two coordinate systems is given by

$$x' = x - vt, y' = y, z' = z, t' = t \; ; \tag{13.1}$$

These relationships are known as the Galilean Transformation.

The reverse of these transformations is known as the Inverse Galilean Transformations, which are

$$x = x' + vt, y = y', z = z', t = t' \tag{13.2}$$

These equations simply reverse the relative motion between the two inertial frames. They reflect the classical assumption that time is absolute (i.e., $t'=t$) and that spatial coordinates differ only by the relative displacement v^t between the frames.

For simplicity, we only consider the motion of the frame in the x -direction. The velocities and acceleration along the x or x' direction are given by,

$$\dot{x}' = \dot{x} - v \tag{13.3}$$

$$\ddot{x}' = \ddot{x} \tag{13.4}$$

As is clear from the above velocity transformation equation (Equation (13.3)), the velocity of a particle is generally different when measured in two different reference frames. In other words, the value of velocity depends on the frame of reference from which the motion is observed. Additionally, the acceleration transformation equation (Equation (13.4)) implies that the acceleration measured of the particle in both frames is the same.

$$m\ddot{x}' = m\ddot{x} \qquad (13.4)$$

This result is consistent with Newton's law of inertia, which states that in the absence of an external force, a body at rest remains at rest and a body in motion continues to move with uniform velocity.

13.4 PRINCIPLE OF RELATIVITY AND SPEED OF LIGHT

The principle of relativity states that motion can only be described relative to a chosen frame of reference. There is no meaning to absolute velocity; a body's velocity becomes meaningful only when compared with another body or reference system. If two bodies move with constant velocity relative to each other, it is impossible to determine which one is truly at rest and which one is in motion. In contrast, acceleration has absolute significance because it can be detected by experiments performed within the system itself. According to this principle, no experiment conducted entirely inside a uniformly moving system can reveal whether that system is at rest or moving with constant velocity. For example, in a closed, non-accelerating ship moving uniformly in a straight line, all mechanical experiments will yield the same results as they would if the ship were stationary. Thus, uniform motion cannot be detected without referring to an external frame. This leads to a more general statement: the fundamental laws of physics are identical in all inertial frames of reference that move with uniform velocity relative to one another. This concept is known as the Galilean or Newtonian principle of relativity and is associated with Galilean invariance, meaning that the laws of mechanics remain unchangedunder Galilean transformations between inertial frames. Although special relativity later replaced Galilean transformations with Lorentz transformations, the core idea remains valid: the laws of physics are invariant in all inertial frames, and no inertial frame is physically preferred over another.

Following this principle, all equations, including Maxwell's equations of electrodynamics, must remain invariant in all frames. As light is an electromagnetic wave, it requires that the velocity of light must remain invariant in every frame of reference. But the transformation equation (13.1) suggests that if a source of light is placed at the origin of the frame (S), then the velocity of light as measured in another frame (S') would vary as

$$\dot{\vec{x}}' = c\vec{i} - \vec{v},$$

where c is the speed of light measured in the frame (S). According to this classical reasoning, the speed of light would depend on the direction of propagation and the relative motion between frames, implying a variation in the measured speed of light. To examine this idea, the "Michelson-Morley" experiment was conducted based on the assumption of an all-pervading ether medium. Since the Earth moves relative to the

Sun and the galaxy, motion through the ether was expected to cause measurable differences in the speed of light along different directions. However, the experiment yielded a null result: no such variation was observed. This unexpected outcome ruled out the existence of the ether and demonstrated that the speed of light remains the same regardless of the motion of the observer or the direction of propagation, posing a serious challenge to classical mechanics and motivating the development of the Special Theory of Relativity.

13.5 POSTULATES OF SPECIAL THEORY OF RELATIVITY

The postulates of the special theory of relativity are as follows:

P-1 The laws of physics should be expressed in equations having the same form in all frame of reference moving with a uniform velocity with respect to one another.

P-2 The speed of light in free space has the same value for all observers, irrespective of their state of motion.

The first postulate is the principle of relativity itself, which demands a universal frame of reference in which the laws of physics remain invariant. The second postulate follows the nature of light as an electromagnetic wave and the negative result of the "Michelson-Morley" experiment. Together, these two postulates form the foundation of Einstein's theory of special relativity.

13.6 LORENTZ TRANSFORMATION AND ITS CONSEQUENCES

Since the Galilean Transformation results in the variation of velocity measured in the S frame, which can be obtained from the S′ frame in terms of those measured in S, as

$$v'_x = v_x - v, \qquad v'_y = v_y, \qquad v'_z = v_z$$
$$\&c' = c - v$$

Thus, the Galilean transformation and the velocity transformations violate the postulates of STR; the velocity of light will depend on the frame of reference. The inability of the Galilean transformations to consistently explain the principles of the Special Theory of Relativity led to the development of a new transformation equation called the Lorentz transformations. They are developed using the following assumptions,

A1. We assume that the possible transformation equations are a linear relation

$$x' = k(x - vt'), y' = y\,, z' = z$$

where k is a proportional constant. The linearity of the equation facilitates the derivation of the Galilean transformation from this equation when k=1 and v«c. For simplification, we have considered only the x-direction.

A2. The inverse relation for the equation exists and can be written as,

$$x = k'(x' + vt'), \;\; y = y', \;\; z = z'$$

Both the transformation equations consider the time t and t' to differ with respect to the frame of reference. To determine the value of t and t' in the S and S' frames, we substitute x' in the inverse and from the transformation equation and rearrange to obtain t'.

$$x' = k(x - vt), \qquad x = k'(x' + vt')$$
$$\Rightarrow x = k'(k(x - vt) + vt') \quad \Rightarrow x = k'k(x - vt) + k'vt'$$
$$\Rightarrow k'vt' = x - kk'(x - vt) \quad \Rightarrow t' = kt + \left(\frac{1 - kk'}{k'v}\right)x$$

A3. Both k and k' can be determined using the second postulate of the STR, which states that the speed of light in all frames of reference is constant.
Both the S and S' frames of reference have identical time coordinates ($t = t'$), at ($t = 0$), indicating that the origins of the two frames coincide at the initial instant. Now, if a source of light emits a signal, the signal will propagate in both the frame of reference S and S' following the second postulate of STR as,

$$x = ct, \qquad x' = ct'$$

which signifies the distance covered by the signal in the S and S' frames. We substitute the expression for x' and t' in $x' = ct'$ to get,

$$x' = ct' \qquad \Rightarrow k(x - vt) = ckt + \left(\frac{1 - kk'}{k'v}\right)cx$$

$$\Rightarrow x = ct\left(\frac{1 + \frac{v}{c}}{1 - (1/kk' - 1)\frac{c}{v}}\right)$$

This derived equation is similar to the equation describing the distance travelled by the signal in the S frame ($x = ct$), if and only if the multiplying factor is unity, which

$$\Rightarrow \frac{1 + \frac{v}{c}}{1 - (1/kk' - 1)\frac{c}{v}} = 1 \qquad \Rightarrow 1 + \frac{v}{c} = 1 - \left(\frac{1}{kk'} - 1\right)\frac{c}{v} \qquad \Rightarrow \frac{v^2}{c^2} = 1 - \frac{1}{kk'}$$

$$\Rightarrow \frac{1}{kk'} = 1 - \frac{v^2}{c^2} \qquad \Rightarrow \sqrt{1/kk'} = \sqrt{1 - \frac{v^2}{c^2}}$$

which gives,

$$\Rightarrow \sqrt{kk'} = \sqrt{\frac{1}{1 - \frac{v^2}{c^2}}} \tag{13.6}$$

For k and k' in equation (6), we consider the argument that the above relation does not depend on the value of v and v' alone, but rather it depends on the factor v^2, thus we can choose based on this reciprocity for the choice of frames,

$$k = k' = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \tag{13.7}$$

Substituting these values in the transformation equation stated in A1, we have the following transformation equations,

$$x' = \frac{x - vt}{\sqrt{1-\frac{v^2}{c^2}}} \tag{13.8}$$

$$y' = y$$

$$z' = z$$

$$t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \tag{13.10}$$

These equations are called the Lorentz transformations.

The inverse Lorentz transformations can now be written as,

$$x = \frac{x' + vt'}{\sqrt{1-\frac{v^2}{c^2}}} \tag{13.12}$$

$$y = y'$$

$$z = z'$$

$$t = \frac{t' + \frac{vx'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \tag{13.14}$$

These transformation equations highlight the dependence of both the position and time coordinate on the frame of reference. Unlike the Galilean transformation, the Lorentz transformation accommodates the postulates of the STR theory, keeping the velocity of light constant in the frame of the observer. It is also observed that the Lorentz transformation can be reduced to the Galilean transformation when the velocity of the frame v is much less than the speed of light. This also means that the effects of STR can only be observed in frames that are moving with or close to the speed of light (c).

Consequences of Lorentz Transformations (Lt)

Due to the relative dependence of the time and position coordinates on the frame of reference, as required by the postulates of STR. The physical events can now have different physical interpretations, a few of which are stated here.

1. Relativity of Simultaneity

We have seen from *LT* that space and time are not independent, and as such, the events that seem to take place simultaneously to one observer may not be simultaneous to another observer in relative motion.

Consider two events E1 and E2 in the Figure (13.2) that take place at the same instant at two different positions, measured in the S frame as x_1 and x_2. An observer in the S' frame, which is

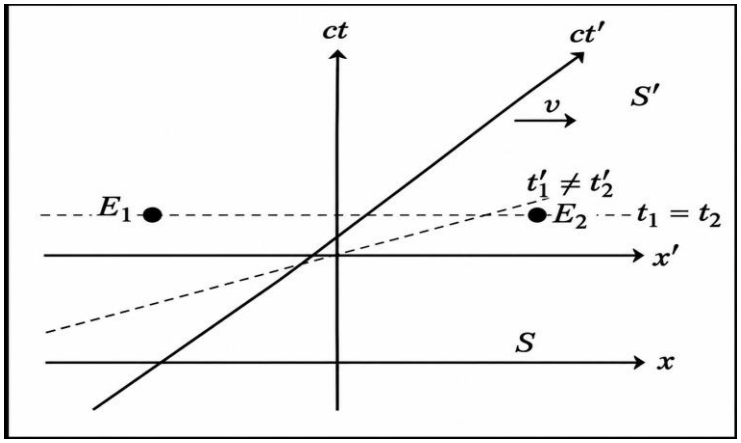


Figure 13.2: Relativity of Simultaneity

Moving relative to the S frame with a speed of *v*, will observe both the events at two different times because of this relative motion,

$$t'_1 = \frac{t_1 - \frac{vx_1}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$t'_2 = \frac{t_2 - \frac{vx_2}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

Thus, the time interval in S' frame will be

$$\Rightarrow t'_2 - t'_1 = \frac{t_2 - \frac{vx_2}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{t_1 - \frac{vx_1}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow t'_2 - t'_1 = \frac{t_2}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{\frac{vx_2}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{t_1}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{\frac{vx_1}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

If the two events were simultaneous in the S frame, then the time$t_1 = t_2$.The same simultaneity for an observer in the S' frame will be

$$\Rightarrow t'_2 - t'_1 = \frac{\frac{vx_1}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{\frac{vx_2}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\Rightarrow t'_2 - t'_1 = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \frac{v(x_1 - x_2)}{c^2}$$

Thus, an observer in S' frame will conclude that the two events were not simultaneous but rather will have a time difference of $'t'_2 - t'_1'$.

2. Length Contraction

If an object is placed along the x-direction, which dimension in the S' frame is measured as x'. We consider the relative motion of this S' frame relative to the S frame

with velocity *v*. In the S' frame, the difference between the endpoints of the object x'_1 and x'_2 will give the length of the object, which is

$$L' = x'_2 - x'_1$$

Now, an observer who is in the stationary S frame measures the same as x_1 and x_2, which gives the length in the stationary frame as $L = x_2 - x_1$. Using the Lorentz transformation,

$$x'_1 = \frac{x_1 - vt_1}{\sqrt{1 - \frac{v^2}{c^2}}}$$
$$x'_2 = \frac{x_2 - vt_2}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Here, the time coordinate t_1, $t_2$corresponds to the respective time of measurement, which should be simultaneous, i.e.,$(t_1 = t_2)$. Thus, the length is

$$L' = x'_2 - x'_1 = \frac{x_2 - vt_2}{\sqrt{1 - \frac{v^2}{c^2}}} - \frac{x_1 - vt_1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{x_2 - vt_2 - x_1 + vt_1}{\sqrt{1 - \frac{v^2}{c^2}}}$$
$$= \frac{x_2 - x_1 - v(t_2 - t_1)}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Here, $L = x_2 - x_1$ is the length of the object measured in the S frame. The second term in the numerator is zero, considering the simultaneity of the measurement process. Thus,

$$\Rightarrow L' = \frac{L}{\sqrt{1 - \frac{v^2}{c^2}}}$$
$$\Rightarrow L = L' \sqrt{1 - \frac{v^2}{c^2}}$$

That is, the length of the object in motion with respect to an observer appears shorter to the observer. This is known as Length Contraction, which was first proposed by George FitzGerald (1889) and Hendrik Lorentz (1892). As the relative velocity appears as v_2 in the above expression, the contraction is observed from both frames of reference.

3. Time dilation

Following the postulates of STR, since position and its measurement depend on the chosen frame of reference, time too must depend on the frame of reference. We consider the observed times (t) and (t') in the S and S' frames, respectively, at a given position (x). The corresponding time interval measured in the S′ frame is given as,

$$t' = t'_2 - t'_1$$

The observer in frame S measures time as,

$$t_1 = \frac{t'_1 + \frac{vx'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$
$$t_2 = \frac{t'_2 + \frac{vx'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

Therefore, in the S frame, the measured time interval will be

$$\Rightarrow t = t_2 - t_1 \quad \Rightarrow t = \frac{t'_2 + \frac{vx'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{t'_1 + \frac{vx'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\Rightarrow t = \frac{t'_2 - t'_1 + \frac{vx'}{c^2} - \frac{vx'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow t = \frac{t'_2 - t'_1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{t'}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\Rightarrow t = \frac{t'}{\sqrt{1-\frac{v^2}{c^2}}}$$

Since the factor $\sqrt{1-\frac{v^2}{c^2}} < 1$ the measured time (t') in a frame in motion will always be less than the time measured in a stationary frame. It follows that the time interval measured in the moving frame is smaller than the time interval measured in the stationary frame. This phenomenon is known as **time dilation**. The difference in time measurements observed in different frames of reference leads to the well-known 'Twin Paradox'. This effect arises because a moving clock runs slower compared to a clock at rest in a given inertial frame.

4. Addition of Velocities

Using the coordinates (x, y, z, t) and (x', y', z', t') in the respective S and S' frames, the velocity component of the 3D space can be expressed as

$$u_x = \frac{dx}{dt}, \quad u_y = \frac{dy}{dt}, \quad u_z = \frac{dz}{dt}$$
$$u'_x = \frac{dx'}{dt}, \quad u'_y = \frac{dy'}{dt}, \quad u'_z = \frac{dz'}{dt}$$

As the S' frame is moving with a relative velocity of v, using the inverse Lorentz Transformation equation

$$x = \gamma(x' + vt'), \;\; y = y', \;\; z = z', \;\; t = \gamma\left(t' + \frac{vx'}{c^2}\right)$$

where$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$.Therefore,

$$dx = \gamma(dx' + vdt'),\ \ dy = dy',\ \ dz = dz',\ \ dt = \gamma\left(dt' + \frac{vdx'}{c^2}\right)$$

Therefore, using the expression or velocity

$$\Rightarrow u_x = \frac{dx}{dt} = \frac{\gamma(dx\prime + vdt\prime)}{\gamma(dt\prime + \frac{vdx\prime}{c^2})} \Rightarrow ux = \frac{u\prime_x + v}{1 + \frac{vu\prime_x}{c^2}} \qquad (13.15)$$

Similarly, the y and z components of the velocity profile are,

$$u_y = \frac{dy}{dt} = \frac{u'_y}{\gamma(1 + \frac{vu\prime_x}{c^2})}$$
$$u_z = \frac{dz}{dt} = \frac{u'_z}{\gamma(1 + \frac{vu\prime_x}{c^2})}$$

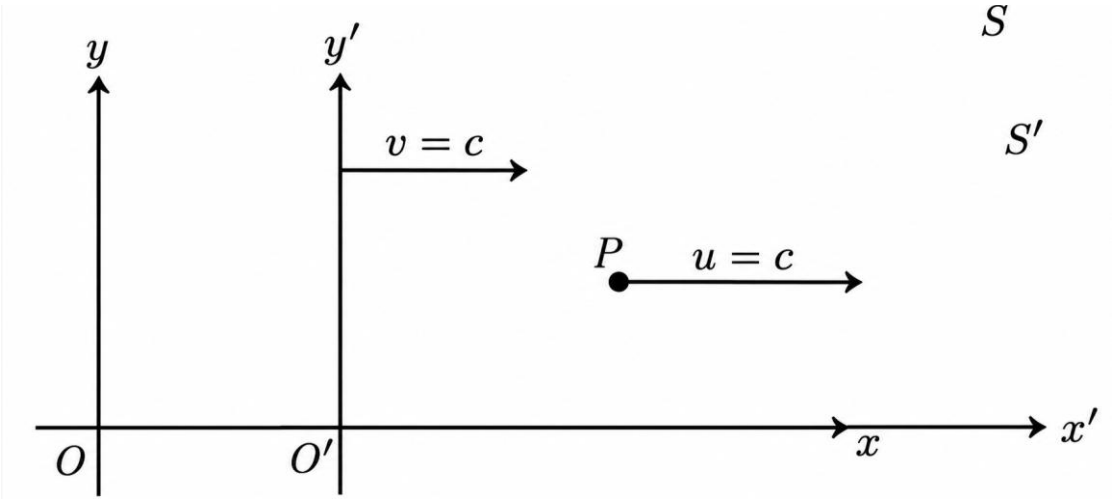


Figure 3: The two frames move relative to each other with relativistic speed, and the particle (P) moves with velocity (c) along the *x*-direction. For simplicity, the motion is analyzed only in the *x–y* plane.

This is known as the relativistic addition of velocities that conserves the second postulate of STR. Suppose a particle moves with velocity c along the x-axis in frame S', and the frame S' itself moves with velocity v along the same axis relative to frame S. According to classical (Galilean) velocity addition, one would expect the observed velocity in S to be c+v, which would exceed the speed of light. However, this contradicts the constancy of c. Using the relativistic velocity addition formula (Equation (13.15)),

$$u = \frac{c + c}{1 + \frac{cc}{c^2}} = c$$

which conserves the speed of light in a moving frame irrespective of the velocity at which the frame is moving. This result is consistent with the experimental findings of the Michel son–Morley experiment, which demonstrated that the speed of light is constant in all directions and is independent of the motion of the observer or the observer's motion in the hypothetical "ether".

W CHECK YOUR PROGRESS

Q1: Prove that "The length of the distance between two points is invariant under Galilean Transformation".
Q2: When will the measured length of the rod be maximum?
Q3: When will the measured time in the frames be maximum?

13.7 LET US SUM UP

- Galilean transformations preserve Newtonian mechanics but fail to explain the constancy of the speed of light.
- The Special Theory of Relativity is based on two postulates: the principle of relativity and the invariance of the speed of light in all inertial frames.
- Lorentz transformations replace Galilean transformations at relativistic speeds and ensure the invariance of physical laws.
- Important consequences of Lorentz transformation include time dilation, length contraction, relativity of simultaneity, and the relativistic law of velocity addition.

13.8 ANSWERS TO CHECK YOUR PROGRESS

Answer to Q1: Consider two inertial frames S and S' such that S' moves with a constant velocity v along the x-axis relative to S.

The Galilean transformations are:

$$x' = x - vt, \qquad y' = y, \qquad z' = z, \qquad t' = t.$$

Let two points have coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) in frame S at the same time t. The distance between them in frame S is

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

In frame S', their coordinates are

$$\begin{aligned} x'_1 &= x_1 - vt, \quad & x'_2 &= x_2 - vt, \\ y'_1 &= y_1, & y'_2 &= y_2, \\ z'_1 &= z_1, & z'_2 &= z_2. \end{aligned}$$

Now,

$$x'_2 - x'_1 = (x_2 - vt) - (x_1 - vt) = x_2 - x_1.$$

Similarly,

$$y'_1 - y'_1 = y_2 - y_1, \qquad z'_1 - z'_1 = z_2 - z_1.$$

Therefore, the distance in frame S' is

$$D' = \sqrt{(x'_2 - x'_1)^2 + (y'_2 - y'_1)^2 + (z'_2 - z'_1)^2}.$$

Substituting the above results,

$$D' = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Hence,

$$D' = D.$$

Thus, the distance between two points is invariant under a Galilean transformation.

Answer to Q2: In length contraction, the observed length of a moving rod is given by

$$L = L_0\sqrt{1 - \frac{v^2}{c^2}}$$

where L_0 is the proper length (measured in the rest frame of the rod), L is the length measured in the frame where the rod moves with velocity v, and c is the speed of light.

Maximum Length Measurement: The measured length L is maximum when the relative velocity $v = 0$. When $v = 0$,

$$L = L_0$$

As v increases, the factor creases. When $\sqrt{1 - \frac{v^2}{c^2}}$decreases, and hence the measured length decrease. When $v \longrightarrow c$,

$$L \to 0$$

Thus, the maximum length of the rod is its proper length, measured in the frame where the rod is at rest.

Answer to Q3: In time dilation, the measured time interval is given by

$$t = \gamma t_0 = \frac{t_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

where t_0 is the proper time (measured in the rest frame of the clock), t is the time measured in the frame where the clock moves with velocity v, and

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}.$$

Maximum Time Measurement: The measured time t becomes maximum when the relative velocity v approaches the speed of light c.
As $v \to c$,

$$\sqrt{1-\frac{v^2}{c^2}} \to 0,$$
$$\gamma \to \infty,$$
$$t \to \infty.$$

Thus, maximum time dilation occurs when the relative velocity is very close to the speed of light.

O13.9 FURTHER READING

- Upadhyaya, J. C. (2019). Classical mechanics. Himalaya Publishing House.
- Beiser, A. (2003). Concepts of modern physics (6th ed.). McGraw-Hill.
- Takwale, R. G., & Puranik, P. S. (1979). Introduction to classical mechanics. Tata McGraw-Hill.
- Prakash, S. (2023). *Relativistic Mechanics*. Pragati Prakashan.

M13.10 MODEL QUESTIONS

1. Describe the Michelson–Morley experiment with the help of a neat diagram. Explain its experimental setup, working principle, and expected outcome based on the ether hypothesis. What was the actual result of the experiment? Discuss its implications for classical physics and explain how it contributed to the development of the Special Theory of Relativity.
2. State and explain the fundamental postulates of the Special Theory of relativity and deduce the Lorentz transformations. Prove that when v « c, Lorentz transformations reduce to the Galilean ones.
3. A cube has a volume of 8000 cm^3. Find the volume as determined by an observer O' who moves at a velocity of 0.9c relative to the cube in a direction parallel to one edge. Ans: 3480cm^3
4. Pions have a half-life of 1.8×10^{-8}s. A pion beam leaves an accelerator at a speed of 0.8c. Classically, what is the expected distance over which half the pions should decay? Determine the expected distance relativistically.
5. An electron is moving with a speed of 0.85c in a direction opposite to that of a photon. Calculate the relative velocity between the electron and the photon.

*** *** ***

UNIT 14: FOUR-DIMENSIONAL FORMULATION

UNIT STRUCTURE

14.1 Learning Objectives
14.2 Introduction
14.3 Minkowski Space
14.4 Four Vectors and Examples
14.5 Let Us Sum Up
14.6 Answers to Check Your Progress
14.7 Further Reading
14.8 Model Questions

14.1 LEARNING OBJECTIVES

After going through the unit, you will be able to:

- describe the formulation and geometric structure of Minkowski space–time
- explain the concept and mathematical representation of four-vectors used to describe physical quantities in spacetime
- apply Lorentz transformations within the four-dimensional Minkowski continuum to physical problems
- identify and analyse different examples of four-vectors, such as position, velocity, and momentum four-vectors
- demonstrate and verify the invariance of the magnitude of four-vectors under Lorentz transformations.

14.2 INTRODUCTION

In the previous unit, we studied the Lorentz transformations, which reveal that space and time are not independent entities. Within the relativistic framework, the time coordinate in one inertial frame depends on both the spatial and temporal coordinates of another inertial frame. Hermann Minkowski introduced the four-dimensional formulation of spacetime in 1908 as a profound extension of Einstein's special theory of relativity. Drawing upon the Lorentz transformations and earlier work by Lorentz and Poincaré, Minkowski demonstrated that space and time are not independent quantities but are intrinsically linked, forming a unified four-dimensional continuum called spacetime. In this framework, physical events are represented as points in spacetime, and the laws of physics acquire a clear geometric interpretation. Minkowski's formulation not only clarified the invariant nature of the spacetime interval and the speed of light but also provided the mathematical language that later enabled Einstein to develop the general theory of relativity.

14.3 MINKOWSKI SPACE

In this approach, unlike in Newtonian mechanics, the spatial coordinates $x_1 = x, x_2 = y, x_3 = z$ and temporal coordinates($x_4 = t$) are treated on equal footing. Any event is therefore represented as a single point in four-dimensional spacetime. The fourth coordinate is often taken to be imaginary, written as $x_4 = ict$, so that the kinematic state of a particle can be expressed as,[1]

$$s^2 = x^2 + y^2 + z^2 + (ict)^2 = x^2 + y^2 + z^2 - ct^2 \tag{14.1}$$

which remains invariant under Lorentz transformations. Lorentz invariance ensures that the speed of light remains the same in both inertial frames S and S', where S is taken stationery and S' moves with a constant relative velocity *v*). It also requires that the spacetime interval (or four-position vector) remain unchanged under this transformation, such that,

$$s^2 = x^2 + y^2 + z^2 - (ct)^2 = x'^2 + y'^2 + z'^2 - (ct')^2 = s'^2 \tag{14.2}$$

Under the Lorentz transformation as shown in Unit 13, the space-time coordi
an event in the two inertial frames S and S' are given by $x' = \gamma(x - vt),\ y' = y,\ z' = z,$ $t' = \gamma(t - \frac{vx}{c^2})$, where $\gamma = 1/\sqrt{1 - v^2/c^2}$.

Thus,

$$s'^2 = x'^2 + y'^2 + z'^2 - (ct')^2$$

$$= \gamma^2(x - vt)^2 + y^2 + z^2 - c^2\gamma^2(t - vx/c^2)^2$$

$$= \gamma^2(1 - v^2/c^2)x^2 + y^2 + z^2 + \gamma^2(v^2 - c^2)t^2$$

$$= x^2 + y^2 + z^2 - c^2t^2 = s^2$$

The invariance of s^2 suggests that the Lorentz transformation can be considered in the four-dimensional space as a rotation with coordinate axes consisting of x,y,z, and ict.

Statement: Lorentz transformation can be regarded as an orthogonal transformation due to the rotation of axes in Minkowski space.

Proof: To prove this statement, we consider a rotation of the frame of reference by angle θ. The symbols for (x,y,z,ict) in the S frame of reference are (x_1, x_2, x_3, x_4), while for the S' frame, they are (x'_1, x'_2, x'_3, x'_4). The frame S' is moving with a constant velocity *v* along the x-axis relative to the S frame such that,

$$x_2 = x'_2 \qquad and \qquad x_3 = x'_3$$

The coordinates of the point 'P' in both systems can be written as,

$$x'_1 = x_1 cos\,\theta\ + x_4 sin\,\theta \tag{14.3}$$

(14.4)

[1] Many authors also use the metrics$s^2 = -x^2 - y^2 - z^2 + (ct)^2$, which is an equally commonly adopted convention.

$$x'_4 = -x_1 sin\,\theta + x_4 cos\,\theta$$

In matrix notation,

$$\begin{pmatrix} x'_1 \\ x'_4 \end{pmatrix} = \begin{pmatrix} \cos\theta & sin\,\theta \\ -sin\,\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_4 \end{pmatrix}$$

The inverse of this relation is,

$$\begin{pmatrix} x_1 \\ x_4 \end{pmatrix} = \begin{pmatrix} \cos\theta & -sin\,\theta \\ sin\,\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x'_1 \\ x'_4 \end{pmatrix}$$

For, $x'_1 = 0$,

$$x_1 = -x'_4 sin\,\theta$$

$$x_4 = x'_4 cos\,\theta$$

Such that

$$tan\,\theta = -\frac{x_1}{x_4} = -\frac{x}{ict} = \frac{iv}{c} \tag{14.5}$$

From this, we can write,

$$cos\,\theta = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \qquad sin\,\theta = \frac{iv/c}{\sqrt{1-\frac{v^2}{c^2}}}$$

Thus, the above transformation equations become,

$$x'_1 = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left(x_1 + i\frac{v}{c}x_4\right) = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}(x_1 - vt)$$

$$x'_2 = x_2$$

$$x'_3 = x_3$$

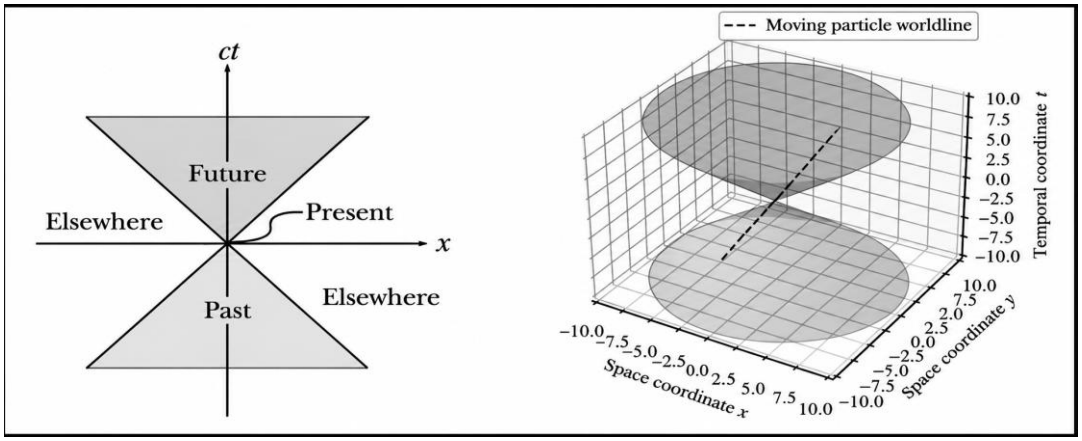
$$x'_4 = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left(-i\frac{vx}{c} + x_4\right)$$

These equations are precisely the Lorentz Transformation equations.

Minkowski Diagram

When describing events in relativity, it is convenient to represent events on a spacetime diagram. Spacetime diagrams were first used by H. Minkowski in 1908 and are often called Minkowski diagrams. Paths in Minkowski spacetime are called worldlines.

In the Figure (14.1a) one spatial coordinate x, to specify position, is used, and instead of time, t, ct is used as the other coordinate so that both coordinates will have dimensions of length.



(a) 2D spacetime diagram (x, ct) (b) 3D light cone with moving worldline

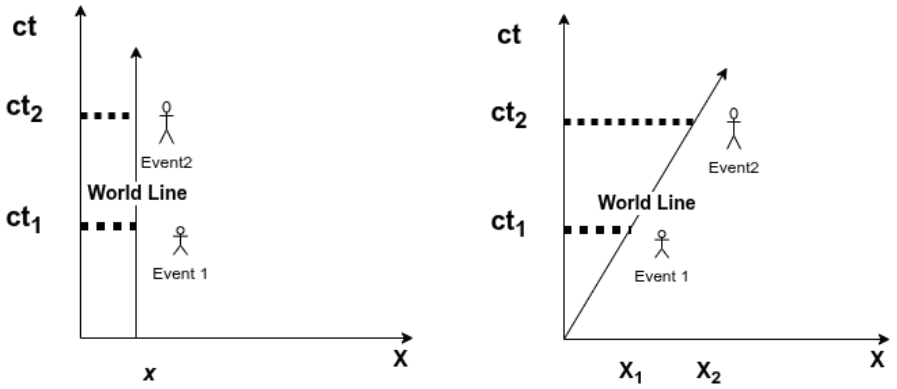
Figure 14.1: Spacetime diagrams showing light cones and worldlines

In a spacetime diagram, events are represented as points (known as the world point) plotted on a coordinate system where the horizontal axis represents position (x) and the vertical axis represents time multiplied by the speed of light (ct). This scaling ensures that both axes have units of distance, allowing for a geometric interpretation of relativity. It also ensures that the quantity

$$s^2 = x^2 + y^2 + z^2 - (ct)^2$$

remains invariant under Lorentz transformation. That is

$$s^2 = x^2 + y^2 - z^2 - (ct)^2 = s'^2 = x'^2 + y'^2 + z'^2 - (ct')^2$$



(a) (b)

Figure 14.2: (a) Space time diagram showing two events 1 and 2 in Minkowski Space where the coordinate time is changing, but the position coordinate is stationary.

(b) Spacetime diagrams showing two events 1 and 2 in Minkowski Space, where both coordinates are changing, depicting a moving object.

In Figure 14.2 (a), when a person observes a stationary object, its trajectory in the Minkowski diagram must be such that the position coordinate doesn't change, but the time coordinate does. If two events happen to be aligned vertically in a specific frame, it implies they occur at the same spatial position *($x_1 = x_2 = x$)*. In this specific scenario, the difference *($t_2 - t_1$)* represents the proper time interval Δ .

In Figure 14.2(b), the times indicated by the values on the vertical axis (such as ct_1 and ct_2) are those measured by a set of clocks synchronized within the inertial frame of the

observer represented by these axes. In such a scenario, a person witnesses an object moving at a constant speed, moving positively along the positive x-axis, and marking two events at two different positions, x_1 and x_2.

A closer look at the values of the slope of a world line reveals that,

$$\text{slope of world line } = \frac{ct_2 - ct_1}{x_2 - x_1} = \frac{c\Delta t}{\Delta x} = \frac{c}{u}$$

where,$u = \frac{\Delta x}{\Delta t}$, is the speed of the object measured by the observer.

The Spacetime Interval

The spacetime interval is a fundamental quantity related to the "distance" separating events in the diagram. Its interpretation depends on the path taken between the events:

- Inertial Frames: If the interval is measured along a straight line connecting two events in an inertial frame, it represents the "length" of that path in spacetime.
- General Paths: If the path is not straight (representing acceleration), the interval is the "arclength" of the curve connecting the events.

While the geometric analogy to "length" and "distance" may seem counterintuitive initially—as spacetime geometry differs from standard Euclidean geometry—these concepts provide the mathematical foundation for understanding how different observers perceive time and space. For two inertial frames S and S', where S' is moving with constant velocity v along the X- axis relative to S, the space-time interval between two events $E_1 (x_1,y_1,z_1,t_1)$ and $E_2(x_2,y_2,z_2,t_2)$ is given as

$$s^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - (ct_2 - ct_1)^2 = s'^2$$
$$= (x'_2 - x'_1)^2 + (y'_2 - y'_1)^2 + (z'_2 - z'_1)^2 - (ct'_2 - ct'_1)^2$$

Thus, both quantities are invariant only when one considers both the space and time interval together, such that the space-time interval remains invariant. Thus, in the Special theory of relativity, the time and space intervals are not invariant separately, which is in contrast with the Galilean approach of Newtonian mechanics. The space and time dimensions together are known as the four-dimensional space-time continuum.

- **Space-like intervals:** The above space-time intervals can be represented in a general form as

$$\Delta s^2 = (\Delta r)^2 - c^2(\Delta t)^2$$

If $(\Delta r)^2 > c^2(\Delta t)^2$, *then* $\Delta s^2 > 0$, which means that the space-time interval is positive and the space-time interval is known as space-like. In such a scenario, even if the two events occur simultaneously $(\Delta t = 0)\ implies\ (\Delta r)^2 > 0$. Thus, we cannot find a frame in which the two events occur at the same place. In the case of a space-like interval, the order of occurrence of the events also loses meaning. For one event to occur earlier than the other ($t_2 > t_1$), later than the other i.e. ($t_2 < t_1$), at the same time ($t_2 = t_1$).

$$(\Delta r)^2 > c^2(\Delta t)^2 \Rightarrow \frac{(\Delta r)}{c} > (\Delta t)$$

Thus, the time interval between two events is always less than the time taken for light (speed 'c') to cover the distance between the two events. Thus, there is no real physical process that can connect the two events under a space-like interval. Thus, the past, present, and future cannot be defined relative to these events, and the causal relation (concept of cause and effect) loses its meaning.

- **Time-like interval:** In the case of a time-like interval, $c^2(\Delta t)^2 > (\Delta r)^2$ which implies Δs^2<0 (negative).

In such a case, the space-time interval is imaginary and is said to be time-like. In such an interval, there are no events that can occur simultaneously (Δt^2*>0),* even if the two events occur at the same place (Δr^2=0). For a time-like interval, the causal relationship is definite, implying that the order of time of occurrence must be fixed, as they cannot occur simultaneously.

$$c^2(\Delta t)^2 > (\Delta r)^2 \Rightarrow (\Delta t) > \frac{(\Delta r)}{c}$$

Thus, the time interval of occurrence between two events is always greater than the time taken by the light to cover the distance between the two events, which is a physically viable process.

- **Light-like interval:** In such an interval, the information travels at the speed of light, such that the space-time interval $(\Delta s)^2 = 0\,, implying$

$$c^2(\Delta t)^2 = (\Delta r)^2 \Rightarrow (\Delta t) = \frac{(\Delta r)}{c}.$$

This means that the time interval between the occurrence of two events is equal to the time taken by light to cover the distance between the two events. Such intervals are said to be light-like or singular.

Thus, intervals are demarcated as space-like, time-like, or light-like depending on whether the value of the space-time interval $\boldsymbol{\Delta s^2}$is positive, negative or zero, respectively. In Minkowksi Diagram, such intervals can be distinguished by specific regions known as the World Regions, as depicted in Figure 14.3 (where, for convenience, we consider only the x and ct axes of the Minkowski diagram).

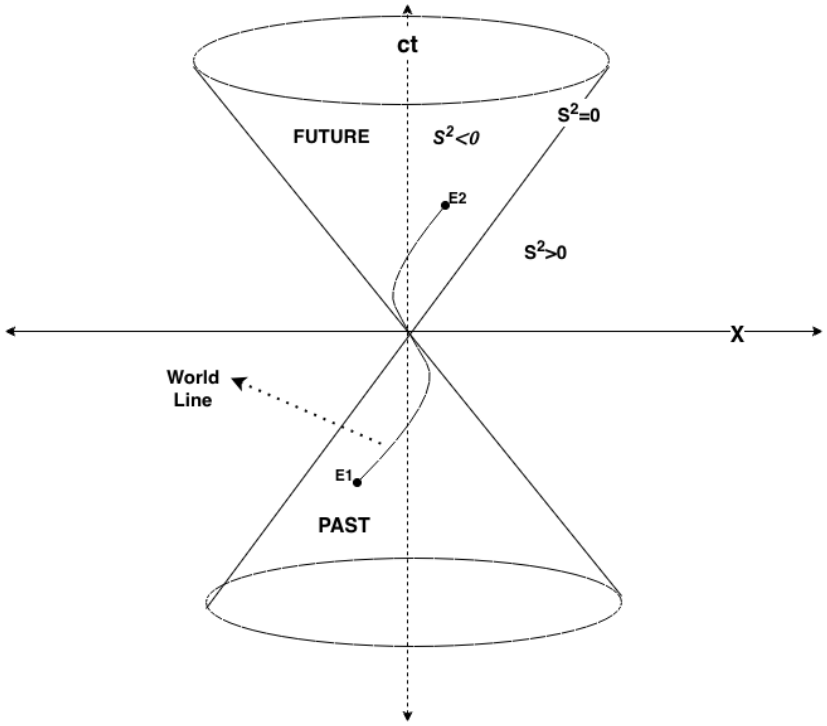


Figure 14.3: [The cone here represents the surface where the space-time interval $\Delta s^2=0$ and the interval is known as the like-like, for regions outside the cone $\Delta s^2>0$ and interval are space-like and cannot be connected by a real physical process, while regions inside the cone are $\Delta s^2<0$ and the interval is time-like. The events (E1 and E2) are bounded by a causal relationship with the lower cone depicting the absolute past and the upper cone depicting the absolute future**.** The point where the two cones are connected and through the world line is passing from the past to the future cone is the absolute present.'

14.4 FOUR VECTORS AND EXAMPLES

The four co-ordinates *x, y, z,* and *ict* define a vector in the four-dimensional Minkowski space and are called a four-vector. The four-vector remains fixed in four-space irrespective of any rotation of the coordinate system, implying that they are independent of the frame of reference. An event in four-dimensional space is represented by a world point (x_1, x_2, x_3, x_4). The Lorentz transformations from S-frame to S'-frame correspond to orthogonal transformations in the four space are represented by

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \\ x'_4 \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

Which satisfies the following condition

$$\sum_{\mu=1}^{4} x'^2_{\mu} = \sum_{\mu=1}^{4} x^2_{\mu} \qquad (14.6)$$

Mathematical Representation

A four vector is designated as a covariant tensor $A\mu$ with components A_1, A_2, A_3 and A_4,

$$A_\mu = (A_1, A_2, A_3, A_4) = (A, iA_t) \tag{14.7}$$

With A being the space component and A_4 is the time component.
They follow the same Lorentz transformation,

$$\begin{pmatrix} A'_1 \\ A'_2 \\ A'_3 \\ A'_4 \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix}$$

Which can be represented as,

$$A'_\mu = \sum_{v=1}^{4} a_{\mu v} A'_v \tag{14.8}$$

Which satisfies the following conditions

$$\sum_{\mu=1}^{4} A'^2_\mu = \sum_{\mu=1}^{4} A^2_\mu = A'_\mu A'_\mu = A_\mu A_\mu \tag{14.9}$$

• Square of the magnitude of our vector: If $A'_\mu = \left(\vec{A}, iA_t\right) = (A_1, A_2, A_3, iA_t)$,

$$A_\mu A_\mu = A_1^2 + A_2^2 + A_3^2 - A_t^2 = \sum_{\mu=1}^{4} A^2_\mu \tag{14.10}$$

• Scalar Product: For two vectors A_v and B_v,

$$A_v B_v = A_1 B_1 + A_2 B_2 + A_3 B_3 + A_4 B_4 = A_1 B_1 + A_2 B_2 + A_3 B_3 - A_t B_t \tag{14.11}$$

Examples of Four-Vectors

1. **Position**: We may represent the position vector of a world point with space and temporal coordinates as,

$$x_\mu = (x_1, x_2, x_3, x_4) = (\boldsymbol{r}, ict) \tag{14.12}$$

where r represents the spatial coordinate, and ict is the temporal coordinate.

2. **Velocity:** v_μ and its components are represented as

$$v_1 = \frac{dx_1}{d\tau} = \frac{dx_1}{dt}\frac{dt}{d\tau} = \frac{dx_1}{dt}\frac{1}{\sqrt{1 - v^2/c^2}} = \frac{v_x}{\sqrt{1 - v^2/c^2}}$$

$$v_2 = \frac{dx_2}{d\tau} = \frac{dx_2}{dt}\frac{dt}{d\tau} = \frac{dx_2}{dt}\frac{1}{\sqrt{1 - v^2/c^2}} = \frac{v_y}{\sqrt{1 - v^2/c^2}}$$

$$v_3 = \frac{dx_3}{d\tau} = \frac{dx_3}{dt}\frac{dt}{d\tau} = \frac{dx_3}{dt}\frac{1}{\sqrt{1-v^2/c^2}} = \frac{v_z}{\sqrt{1-v^2/c^2}}$$

$$v_4 = \frac{dx_4}{d\tau} = \frac{dx_4}{dt}\frac{dt}{d\tau} = \frac{d(ict)}{dt}\frac{1}{\sqrt{1-v^2/c^2}} = \frac{ic}{\sqrt{1-v^2/c^2}}$$

v_μ the four-velocity can thus be represented as,

$$v_\mu = (v_1, v_2, v_3, v_4) = \left(\frac{\boldsymbol{v}}{\sqrt{1-\frac{v^2}{c^2}}}, \frac{ic}{\sqrt{1-\frac{v^2}{c^2}}}\right) \tag{14.13}$$

where vector $\boldsymbol{v}$ is the three-dimensional velocity vector $\frac{d\boldsymbol{r}}{dt}$.

3. **Momentum:**p_μ: The component of four-momentum $\boldsymbol{p}_\mu$ is defined as

$$p_1 = m_0 v_1 = \frac{m_0 v_x}{\sqrt{1-\frac{v^2}{c^2}}} = p_x$$

$$p_2 = m_0 v_2 = \frac{m_0 v_y}{\sqrt{1-\frac{v^2}{c^2}}} = p_y$$

$$p_3 = m_0 v_3 = \frac{m_0 v_z}{\sqrt{1-\frac{v^2}{c^2}}} = p_z$$

$$p_4 = m_0 v_4 = \frac{m_0 ic}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{iE}{c}$$

where m= $\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$. Thus, the four-momentum can be expressed as,

$$\boldsymbol{p}_\mu = (p_1, p_2, p_3, p_4) = (\boldsymbol{p}, iE/c) \tag{14.14}$$

4. **Acceleration:**a_μ, similar to velocity, its components are defined as,

$$a_1 = \frac{dv_1}{d\tau} = \frac{dv_1}{dt}\frac{dt}{d\tau} = \frac{d}{dt}\left(\frac{v_x}{\sqrt{1-\frac{v^2}{c^2}}}\right).\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$= \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left(\frac{\dot{v}_x}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{v_x \boldsymbol{v}\dot{\boldsymbol{v}}}{c^2\left(1-\frac{v^2}{c^2}\right)^{3/2}}\right)$$

$$=> a_x = \frac{\dot{v}_x}{1-\frac{v^2}{c^2}} + \frac{v_x\,(\boldsymbol{v}\dot{\boldsymbol{v}})}{c^2\left(1-\frac{v^2}{c^2}\right)^2}$$

Here, we used the product rule of differentiation such that, $\frac{d(uv)}{dt} = u\frac{dv}{dt} + \frac{du}{dt}v$.
Also, as derived above, velocity and its differentiation give,

$$v = (v_x, v_y, v_z)$$

$$v^2 = v_x^2 + v_y^2 + v_z^2$$

$$v\dot{v} = v_x\dot{v_x} + v_y\dot{v_y} + v_z\dot{v_z} = \boldsymbol{v}\dot{\boldsymbol{v}}$$

Similarly, for other components a_2, we have,

$$a_2 = \frac{dv_2}{d\tau} = \frac{dv_2}{dt}\frac{dt}{d\tau} = \frac{d}{dt}\left(\frac{v_y}{\sqrt{1-\frac{v^2}{c^2}}}\right).\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$= \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left(\frac{\dot{v}_y}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{v_y\boldsymbol{v}\dot{\boldsymbol{v}}}{c^2\left(1-\frac{v^2}{c^2}\right)^{3/2}}\right)$$

$$=> a_y = \frac{\dot{v}_y}{1-\frac{v^2}{c^2}} + \frac{v_y(\boldsymbol{v}\dot{\boldsymbol{v}})}{c^2\left(1-\frac{v^2}{c^2}\right)^2}$$

And a_3, we have

$$a_3 = \frac{dv_3}{d\tau} = \frac{dv_3}{dt}\frac{dt}{d\tau} = \frac{d}{dt}\left(\frac{v_z z}{\sqrt{1-\frac{v^2}{c^2}}}\right).\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$= \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left(\frac{\dot{v}_z}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{v_z\boldsymbol{v}\dot{\boldsymbol{v}}}{c^2\left(1-\frac{v^2}{c^2}\right)^{3/2}}\right)$$

$$=> a_z = \frac{\dot{v}_z}{1-\frac{v^2}{c^2}} + \frac{v_z(\boldsymbol{v}\dot{\boldsymbol{v}})}{c^2\left(1-\frac{v^2}{c^2}\right)^2}$$

The fourth component of the acceleration vector in Minkowski space is the differentiation of the fourth velocity component

$$a_4 = \frac{dv_4}{d\tau} = \frac{dv_4}{dt}\frac{dt}{d\tau} = \frac{d}{dt}\left(\frac{ic}{\sqrt{1-\frac{v^2}{c^2}}}\right).\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = i\frac{\boldsymbol{v}\dot{\boldsymbol{v}}}{c\left(1-\frac{v^2}{c^2}\right)^2}$$

Thus, the acceleration can be expressed as,

$$a_\mu = (a_1, a_2, a_3, a_4) = \left(\frac{\boldsymbol{a}}{1-\frac{v^2}{c^2}} + \frac{\boldsymbol{v}(\boldsymbol{v}.\boldsymbol{a})}{c^2\left(1-\frac{v^2}{c^2}\right)^2}, \frac{\boldsymbol{i}(\boldsymbol{v}.\boldsymbol{a})}{c\left(1-\frac{v^2}{c^2}\right)^2} \right) \quad (14.15)$$

where a is the acceleration in three-dimensional space.

5. **Force:** F_μ isthe Minkowski force, can be defined as the rate of change of four-momentump_μ,

$$F_\mu = \frac{dp_\mu}{d\tau} = \frac{dp_\mu}{dt}\frac{dt}{d\tau}$$

the components of the four forces are

$$F_1 = \frac{dp_1}{dt}\frac{dt}{d\tau} = \frac{dp_x}{dt}\frac{dt}{d\tau} = \frac{dp_x}{dt}\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{F_x}{\sqrt{1-\frac{v^2}{c^2}}}$$

Similarly, other components can be expressed as

$$F_2 = \frac{F_y}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$F_3 = \frac{F_z}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$F_4 = \frac{dp_4}{dt}\frac{dt}{d\tau} = \frac{d}{dt}\left(\frac{iE}{c}\right).\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{i\gamma}{c}\frac{dE}{dt}$$

Here, $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$. Thus, the four vectors is defined as,

$$F_\mu = (\gamma\vec{F}, \frac{i\gamma}{c}\frac{dE}{dt}) \quad (14.16)$$

Physically, the $\vec{F}$ component represents the equations of motion, while the time component expresses conservation of energy.

CHECK YOUR PROGRESS

Q1: Prove that the scalar product of the four vectors A_v and B_v is invariant under Lorentz transformations.

Q2: Prove that the four-velocity is Lorentz Invariant?

Q3: If the wave vector associated with wave number k, can be expressed as a four-vector component $\left(k, \frac{\omega}{c}\right)$.Find its Lorentz transformation equations?

14.5 LET US SUM UP

- Minkowski spacetime unifies space and time into a single four-dimensional continuum, in which an event is specified by four coordinates (x, y, z, ict).
- Unlike Newtonian mechanics, space and time are treated on equal footing, reflecting their relativistic interdependence as revealed by the Lorentz transformations.
- Physical quantities in spacetime are represented by four-vectors, such as the four-position, four-velocity, four-momentum, and four-force.
- Lorentz transformations act as rotations in the four-dimensional spacetime continuum that preserve the Minkowski metric.
- As a consequence, the spacetime interval (or the magnitude of a four-vector) remains invariant under Lorentz transformations.
- Lorentz invariance ensures that the speed of light remains constant for all inertial observers.

14.6 ANSWERS TO CHECK YOUR PROGRESS

Answer to Q1: To prove that the scalar product $A_v B_v$ is invariant under Lorentz Transformation, we have to show that $A'_v B'_v = A_v B_v$. Thus,

$$A'_v B'_v = A'_1 B'_1 + A'_2 B'_2 + A'_3 B'_3 + A'_4 B'_4 \tag{14.17}$$

Under the Lorentz transformation and considering the S' frame to be moving relatively only along the x-axis. Using equation (8) which can be represented as the

$$\begin{pmatrix} A'_1 \\ A'_2 \\ A'_3 \\ A'_4 \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix}$$

Here, $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ and $\beta = \frac{v}{c}$. Thus, A_v components are,

$$A'_1 = \gamma(A_1 + i\beta A_4)$$

$$A'_2 = A_2$$

$$A'_3 = A_3$$

$$A'_4 = \gamma(-i\beta A_1 + A_4)$$

Similarly, for B_v

$$B'_1 = \gamma(B_1 + i\beta B_4)$$

$$B'_2 = B_2$$

$$B'_3 = B_3$$

$$B'_4 = \gamma(-i\beta B_1 + B_4)$$

Replacing the value in equation (14.17), we get

$$A'_v B'_v = \gamma^2(A_1 + i\beta A_4)(B_1 + i\beta B_4) + A_2 B_2 + A_3 B_3 + \gamma^2(-i\beta A_1 + A_1)(-i\beta B_1 + B_4)$$

$$= A_1B_1\gamma^2(1-\beta^2) + A_2B_2 + A_3B_3 + A_4B_4\gamma^2(1-\beta^2)$$

$$= A_1B_1 + A_2B_2 + A_3B_3 + A_4B_4$$

$$= A_vB_v$$

Thus, the scalar product of the four-vector is Lorentz invariant.

Answer to Q2: To prove that the four-velocity is Lorentz invariant, we take the square of the magnitude of the velocity vector,

$$v_v v_v = (\gamma \boldsymbol{v} + i\gamma c)(\gamma \boldsymbol{v} + i\gamma c)$$

$$= \gamma^2 \boldsymbol{v^2} - c^2\gamma^2$$

$$= -c^2(\gamma^2)\left(1 - \frac{v^2}{c^2}\right)$$

$$-c^2\gamma^2\gamma^{-2} = -c^2$$

which is Lorentz invariant.

Answer to Q3: The wave vector in matrix form under the Lorentz transformation gives

$$\begin{pmatrix} k'_1 \\ k'_2 \\ k'_3 \\ \frac{\omega'}{c} \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ \frac{\omega}{c} \end{pmatrix}$$

14.7 FURTHER READING

- Upadhyaya, J. C. (2019). Classical Mechanics. Himalaya Publishing House.
- Beiser, A. (2003). Concepts of Modern Physics (6th ed.). McGraw-Hill.
- Takwale, R. G., & Puranik, P. S. (1979). Introduction to Classical Mechanics. Tata McGraw-Hill.
- Prakash, S. (2023). Relativistic Mechanics. Pragati Prakashan

14.8 MODEL QUESTIONS

1. At what speed v will the Galilean and Lorentz expressions for x differ by 1%? By 15%?
2. How does the Lorentz transformation rotate the axes of a Minkowski coordinate system?
3. Considering the invariance of the four-momentum in Minkowski spacetime, derive the relation between the relativistic energy and momentum of a particle.

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