UNIT 10: BEZIER AND B-SPLINE CURVES

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10.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- Learn about the Hermite curve
- Learn about Hermite matrix
- Learn about Hermite Blending Functions
- Learn about the properties of the Bezier Curve and B-Spline Curves
- Differentiate between Bezier Curve and B-Spline Curves
- Learn about Bezier and B-Spline Surfaces

10.2 INTRODUCTION

We have already been acquainted with the idea of 3-D object representation and these 3-D objects are represented as a collection of surfaces. We have learned about quadric surfaces, spline representation, basic illumination models, and polygon rendering methods.

In this unit, we will learn about the Bezier curve and B-spline curves. Here we will also learn about the Hermite curve, Hermite matrix and Hermite blending function, Bezier and B-Spline Surfaces as well as the properties of Bezier Curve and B-Spline Curves. In the next unit, we will explore the concept of projection and the classification of projection.

10.3 HERMITE CURVE

The Hermit curve is expressed using parametric notation. The joining of multiple curves into a single curve with the help of the endpoints of each curve is termed as Hermite Curve. In the following figure 10.1, the endpoints of two different curves are shown and in figure 10.2 these two curves are joined to form the Hermite curve. Interpolation spline curve and cubic polynomial functions are required for making the Hermite equation.



Figure 10.1: curves



Figure 10.2: Hermite Curve

With the help of the following cubic polynomial, the Hermite curve is represented.

$$x(t) = a_3t^3 + a_2t^2 + a_1t + a_0$$

Here, the value of t ranges from 0 to 1. The polynomial can be specified by the position of, and gradient at, each endpoint of the curve.

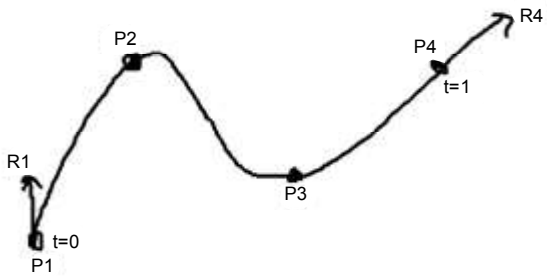


Figure 10.3: Hermite Specifications

In the above figure 10.3, p1 and p2 are the endpoints.

For making Hermite function, we need:

- **First control points, Last control points:** These two points are step position vectors.
- First derivatives of points R_1 and R_4 or P' and P'_4

$R_1 = P'_1$ (1st Derivative of P'_1)

$R_4 = P'_4$ (1st Derivative of P_4)

If we want to make a curve equation we need to make

Q(t)=[x(t) y(t) z(t)], Q(t) denotes equation in three dimension x, y, and z.

Where,

$x(t)= a_x t^3 + b_x t^2 + c_x t + d_x$

$y(t)= a_y t^3 + b_y t^2 + c_y t + d_y$

$z(t)= a_z t^3 + b_z t^2 + c_z t + d_z$

$Q(t)=\underbrace{[t^3 \quad t^2 \quad t \quad 1]}_{1x4} \quad \begin{bmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \\ d_x & d_y & d_z \end{bmatrix}$

We can also write the above equation as

Q (t)= T* C ------- (i)

Or

C=M* G

Where

M= Basic matrix: Provide Blending equation

G= Geometry Vector Matrix

By putting the value of C in equation (i), we will get

$Q(t)= T^* M_H * G_H$

M= Basic matrix: Provide Blending equation of Hermite

G= Geometry Vector Matrix of Hermite

= [t3 t2 t 1]* M_H * G_H ------- (ii)

Like equation (ii), we can make the matrix equation of the curve.

$$\begin{bmatrix} P_1 \\ P_4 \\ R_1 \\ R_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} * M_H * G_H$$

Or

$$M_H * G_H = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} P_1 \\ P_2 \\ R_1 \\ R_4 \end{bmatrix}$$

After inverse comparing both sides

$$M_H = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

And

$$G_H = \begin{bmatrix} P_1 \\ P_4 \\ R_1 \\ R_4 \end{bmatrix}$$

Now,

$$Q(t) = [t^3 \quad t^2 \quad t \quad 1] * M_H * G_H$$

$$= [t^3 \quad t^2 \quad t \quad 1] * \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} * \begin{bmatrix} P_1 \\ P_4 \\ R_1 \\ R_4 \end{bmatrix}$$

$$= \underbrace{(2t^3-3t^2+1)P_1}_{H_1} + \underbrace{(-2t^3+3t)P_4}_{H_2} + \underbrace{(t^3-2t^2+t)R_1}_{H_3} + \underbrace{(t^3-t^2)R_4}_{H_4} \quad \text{-------(iii)}$$

H_1, H_2, H_3, and H_4 are called Hermite Blending Function.

$Q(t) = H_1P_1+H_2P_4+H_3R_1+H_4R_4$

In equation (iii) we can directly put the value of t, and we can find the equation of the Hermite curve.



EXERCISE

1. Write down the Hermite blending function.

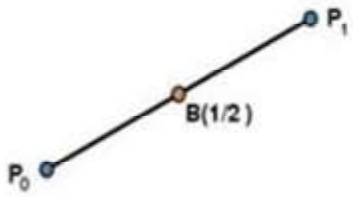
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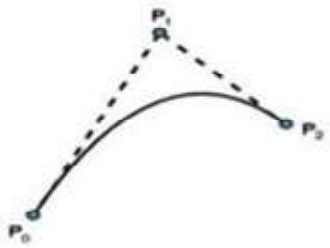
10.4 BEZIER CURVE AND B-SPLINE CURVES

10.4.1 Bezier Curveb

The bezier curve is defined by the control points. It is discovered by

Pierre Bezier. It is a spline curve. It is used in CAD, drawing applications, etc.





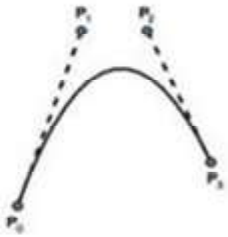


Figure 10.4: Simplest Bezier Figure 10.5: Quadratic Bezier Figure 10.6: Cubic Bezier

In the above figures P0, P1, P2, and P3 are the control points. And the curves formed from these points in the above figures are the Bezier curves.

There are three types of Bezier curves. Namely, Simplest Bezier curve, Quadratic Bezier curve and Cubic Bezier curve. The simplest Bezier curve is the straight line from the point P0 to P1, which connects two endpoints. In figure 10.4 one example of the Simplest Bezier curve is given. A quadratic Bezier curve is determined by three control points. In figure 10.5 a quadratic Bezier curve is shown. And a cubic Bezier curve is determined by four control points. Figure 10.6 shows a cubic Bezier curve. The curve order is always less than the number of control points by one. The Bezier curve always lies inside the convex hull made by the control points.

Properties of Bezier Curve:

- Bezier curve depends on the number of control points.
- Whenever we draw the Bezier curve it may only pass through the endpoints. It may not be all the control points.
- The polynomial equation of a Bezier curve is dependent on the control points. If we have n number of control points, then the degree of the polynomial function is n-1.
- Bezier curve is a global control curve, that is the change of the position of a control point can change the shape of the complete graph. Changing the curve shape locally or a portion of the graph

through the control points is not possible.

Derivation for the quadratic Bezier curve is given below:

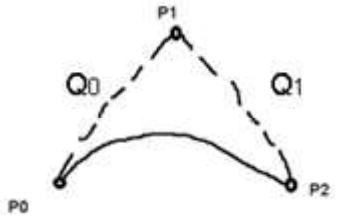


Figure 10.7: Bezier curve having three control points P1, P2 and P3

Bezier curve is a parametric curve.

Let the parameter is u and 0<=u<=1

And Q_0 and Q_1 points are the lines between P0 $\rightarrow$ P1 and P1 $\rightarrow$ P2

Q_0 = (1-u)P0+uP1 ----(i)

Q1 = (1-u)P1+uP2 ------(ii)

Suppose c(u) is the point between $Q_0 \rightarrow Q_1$

c (u)=(1-u) Q_0 + uQ_1 from equation (i) and (ii)

c (u)=(1-u) [(1-u)P0+uP1] + [u(1-u)P1+uP2]

= $(1-u)^2$ P0+u(1-u)P1+u(1-u)P1+u^2P2

= $(1-u)^2$P0+2u(1-u)P1+u^2P2 ----(iii) (for 3 control points)

Equation (iii) is the quadratic equation for the Bezier curve for three control points P0, P1, and P2.

The blending function for the Bezier curve is given below:

Suppose we have n+1 control points positions P_k (x_k y_k z_k) then the

Positional vector P(u) where 0<=u<=1 is given by

$P(u)=\sum_{k=0}^{n} P_k BEZ_{k,n}(u)$ 0<=u<=1

Blending function

$BEZ_{k,n} = c(n,k)u^k(1-u)^{n-k}$

$x(u)= \sum_{k=0}^{n} x_k BEZ_{k,n}(u)$

$y(u)= \sum_{k=0}^{n} y_k BEZ_{k,n}(u)$

$z(u)= \sum_{k=0}^{n} z_k BEZ_{k,n}(u$

10.4.2 B-SPLINE CURVES

The b-Spline curve is locally controlled that is the shape of a portion of the curve can be changed through the control points without affecting the rest of the portion of the graph. Here the degree of the polynomial does not depend on control points. It depends on the order of the polynomial. That is, it is based on the number of control points and if the order varies then the blending function degree will be different.

Properties of B-Spline Curve:

It is made of (n+1) control points and the order of the curve (k).

- It has local control over the curve
- It is used to draw open and closed curves.
- It gives us a polynomial of degree k-1.
- It has n-k+2 segments n: number of control points
- It allows us to change the number of control points without changing the degree of the polynomial.

The mathematical expression of the B-Spline curve using the Blending function is given below:

$$P(u) = \sum_{k=0}^{n} B_{ik}(u) P_k$$

$$P_k(X_k \quad Y_k \quad Z_k)$$

$$X(u) = \sum_{k=0}^{n} B_{ik}(u) X_k$$

Where 0<=u<=n-k+2 and 2<=k<=n+1

K=here k is responsible for segments (number of segments)



CHECK YOUR PROGRESS

Q 1: Define Bezier curve.

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Q 2: Write the properties of Bezier curve and B-Spline curve.

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10.5 BEZIER AND B-SPLINE SURFACES

10.5.1 BEZIER SURFACE

Bezier surface can be obtained by two sets of orthogonal Bezier curves by specifying an input mesh of control points. The bezier surface has the same properties as the Bezier curve. A cylinder/ cone can be formed from a Bézier surface. The equation of the Bezier surface is parametric and it is formed as the Cartesian product of the blending functions as given below:

$P(u,v)=\sum_{j=0}^{m}\sum_{k=0}^{n} P_{j,k}\, BEZ_{j,m}(v)\, BEZ_{k,n}(u)$

Where, $P_{j,k}$ is the location of (m+1) by (n+1) control points.

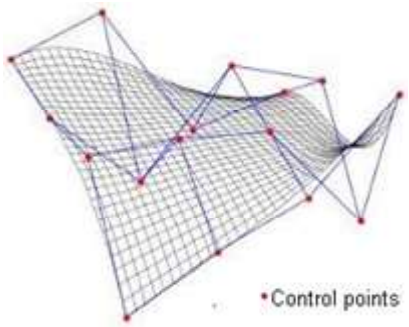


Figure 10.8: Bezier surface

10.5.2 B-SPLINE SURFACES

The b-Spline surface can be similarly formed to the Bezier Spline. The b-Spline surface has the same properties as the B-Spline curve. B-spline surface maps the unit square to a rectangular surface. The vector point function of B Spline can be obtained using the Cartesian product of the B-Spline blending function as given below:

$P(u,v)=\sum_{k1=0}^{n1}\sum_{k2=0}^{n2} P_{k1,k2}\, B_{k1,d1}(u) B_{k2,d2}(v)$

Where $P_{k1,k2}$ are the positions of the (n_1+1) by (n_2+1) control points. And d1 and d2 determine the polynomial degrees to be used.

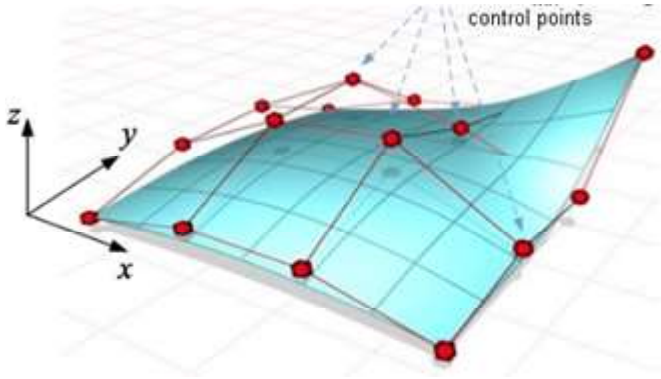


Figure 10.9: B-Spline surface



CHECK YOUR PROGRESS

Q 3: Define B-Spline surface.

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10.6 LET US SUM UP

- Hermit curve is expressed in terms of parametric notation i.e. $x(t) = a_3t^3 + a_2t^2 + a_1t + a_0$
- The result of joining different curves is called a Hermite Curve.
- Bezier curves can be generated under the control of other points.
- The Bezier surface is formed as the Cartesian product of the blending functions of two orthogonal Bezier curves.
- B-spline surface maps the unit square to a rectangular surface.



10.7 FURTHER READINGS

1) D. Hearn and P.M. Backer, Computer Graphics, Prentice Hall of India, 1986.
2) W. Newman and R.F. Sproul, Principles of Interactive Computer Graphics, McGraw Hill Publication, 1980.
3) W.K. Giloi, Interactive Computer Graphics, Prentice Hall of India, 1978.

10.8 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q No.1: Bezier curve is defined by the control points. It is discovered by Pierre Bezier. It is a spline curve. It is used in CAD and drawing applications.

c (u)=(1-u) [(1-u)P0+uP1] + [u(1-u)P1+uP2]

= $(1-u)^2$ P0+u(1-u)P1+u(1-u)P1+u^2P2

= $(1-u)^2$P0+2u(1-u)P1+u^2P2 ----(i) (for 3 control points)

Equation (i) is the quadratic equation for the Bezier curve for three control points P0, P1, and P2.

Ans. to Q No. 2: Properties of Bezier Curve:

- Bezier curve depends on the number of end control points.
- Whenever we draw the Bezier curve it may only pass through the endpoints. It may not be all the control points.

Properties of B-Spline Curve:

- It is made of (n+1) control points and the order of the curve (k).
- It has local control over the curve

Ans. to Q No. 3: B-Spline surface can be similarly formed like the Bezier Spline. B-Spline surface has same properties like B-Spline curve. B-spline surface maps the unit square to a rectangular surface.

10.9 MODEL QUESTIONS

Q 1: Write the limitation of the Hermite curve.

Q 2: How many points are needed for a piece of Hermite curve?

Q 3: What is the use of the Hermite curve?

Q 4: Write the difference between the Hermite curve and the Bezier curve.

Q 5: Elaborate on the equation of the Bezier curve.

Q 6: Explain the different types of Bezier curves.

Q 7: Explain the concept of Bezier's surface with the help of an example.

Q 8: Derive the Hermite matrix.

Q 9: Write down the properties of the Bezier curve.

Q 10: Write down the properties of the B-Spline surface.

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