UNIT 12: 3-D GEOMETRIC TRANSFORMATIONS

UNIT STRUCTURE

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12.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- Know about the different 3D Geometric transformations.
- Learn how the transformations are done in graphics
- Know about the composite transformation in computer graphics.

12.2 INTRODUCTION

In the previous unit we have discussed the projections and their classifications. We have also gone through the perspective projections as well as the hidden layer projections.

To create a realistic computer model of the world or anything in it, the model should be 3D as well. Representation of any object in 3D requires the understanding of the shape of the object. In computer graphics, animations are done on the screen using the different transformation techniques. Graphics is now used almost everywhere and it is rapidly growing. The three dimensional (3D) graphics are used extensively by the interactive software in business and education.

In this unit, we will introduce you to the transformation techniques of the pictures and images. Transformations are important part of the process of building an image. Construction of any geometrical object requires the use of geometrical transformations. If an image has two identical or similar parts, then only one of them needs to be constructed. The other part can be obtained by copying the first part and then moving, reflecting, scaling and rotating it to get the required shape, size and orientation. Operations such as moving, reflecting, scaling, rotating an image are called geometric transformations and will be discussed in this unit for three dimensions.

Three-dimensional transformation is similar to 2D transformations, with some minor differences. Here, we have 4 X 4 transformation matrices (in homogeneous coordinate system) instead of 3 X 3.

12.3 TRANSLATION

A translation refers to moving every point at a constant distance in a particular direction. In other words, repositioning an object along a straight line path from one coordinate location to another coordinate location is called translation transformation. t_x and t_y are the translation parameters or translation distances.

A point (x,y,z) is translated to a new location (x', y', z') by adding the translation parameters t_x, t_y and t_z to the original coordinates x, y and z.

$$x' = x + t_x$$
$$x' = y + t_y$$
$$z' = y + t_z$$

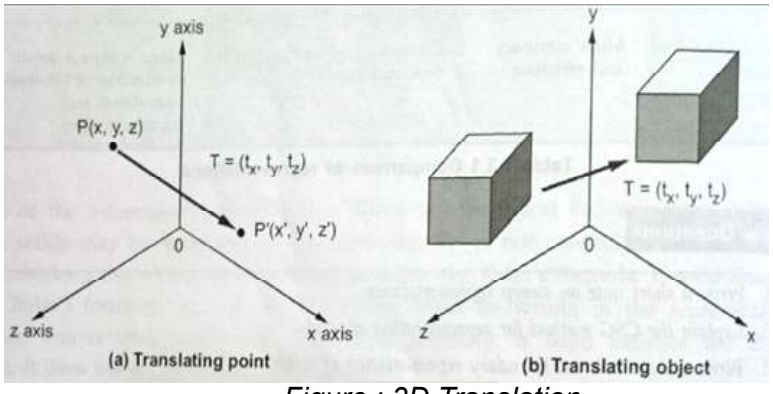


Figure : 3D Translation

Translation in 3D is a simple extension from that in 2D. In matrix form,

$$T(t_x,t_y,t_z) = \begin{pmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This can be written as,

P' = T.P

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

- Translation is a rigid body transformation which moves the objects without deformation.
- 3D translation is very similar to 2D translation.
- There are now three coordinates to translate (t_x, t_y, t_z). A 4×4 matrix is used to perform the operation.

12.4 ROTATION

Rotation is movement in circular motion. The rotation transformation describes the motion of a rigid body around a fixed point. In 2-dimension, all rotations are defined about the Z-axis, or an axis parallel to it. However, in 3D, there are three basic rotations, with respect to each of the principle axes X, Y and Z. performing rotation transformation in 3D is much more complex than that of 2D because 3D transformation rotation can be specified around any line in space.

The transformation matrix for rotation about an arbitrary axis (any axis of rotation other than the principle axis) is more complicated.

Coordinate axes rotations

The 2D z-axis rotation equations can be extended to 3D as

$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta \quad \text{.....(12.1)}$$
$$z' = z$$

where θ is the rotation angle. Thus, the 3D z-axis rotation can be expressed in homogeneous coordinates as

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

which can be rewritten as

$$P' = R_z(\theta).P$$

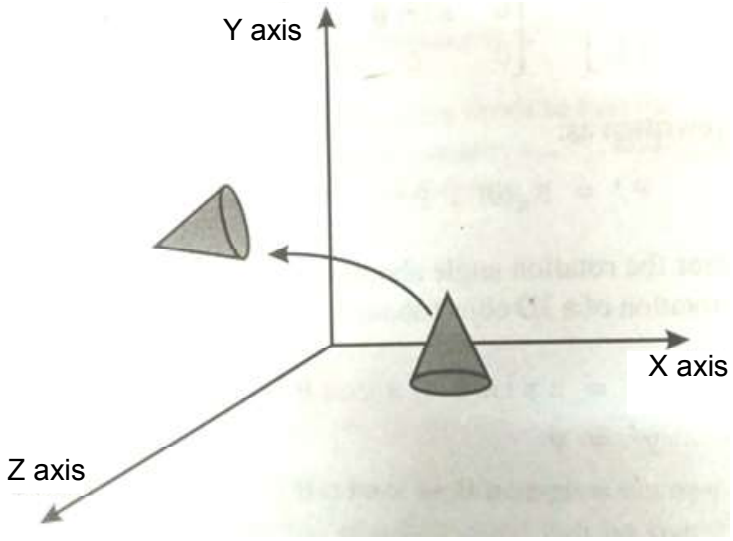


Figure : Rotation of an object about the z-axis

Rotations about the other two axes can be obtained by the following replacement.

X $\rightarrow$ y $\rightarrow$ z $\rightarrow$ x …….. (12.2)

Substituting the (12.2) in the equation 12.1, the equations for the x-axis rotation are obtained as follows:

$$y' = y\cos\theta - z\sin\theta$$
$$z' = y\sin\theta + z\cos\theta \rightarrow (12.3)$$
$$x' = x$$

which can be expressed in homogeneous coordinates as

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

This can be rewritten as,

$$P' = R_x(\theta).P$$

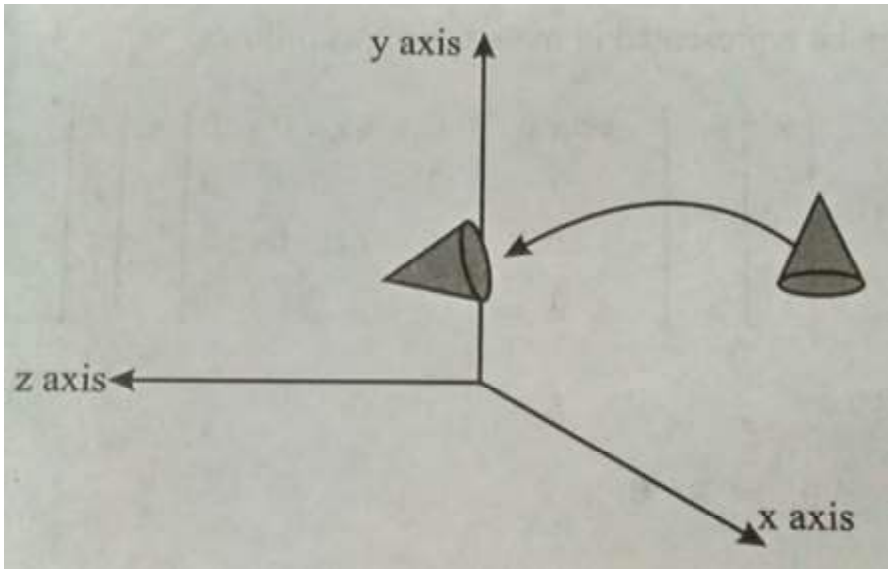


Figure : Rotation of an object about the x-axis

Substituting (12.2) in the equation 12.3, the equations for the y-axis rotation are obtained as follows:

$$\begin{aligned} z' &= z\cos\theta - x\sin\theta \\ x' &= z\sin\theta + x\cos\theta \quad \rightarrow (12.4) \\ y' &= y \end{aligned}$$

which can be expressed in homogeneous coordinates as

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

This can be rewritten as,

$$P' = R_y(\theta) \,.\, \mathrm{P}$$

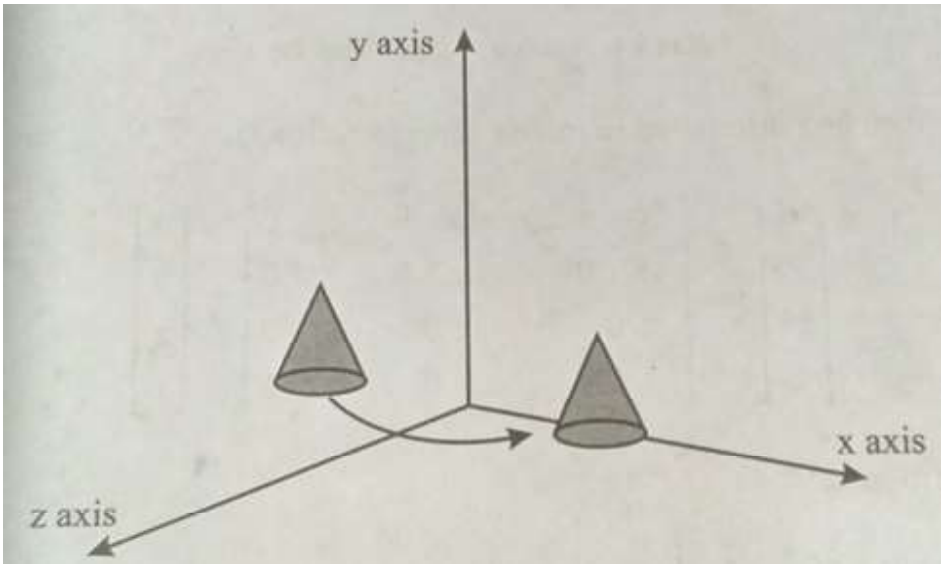


Figure : Rotation of an object about the y-axis

Rotate an object about an axis parallel to one of the coordinate axis:

For rotating an object about an axis parallel to one of the coordinate axis, the following steps needs to be performed.

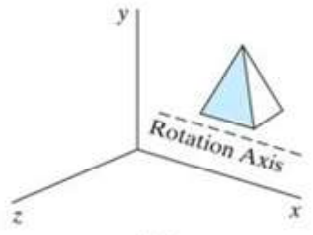
- **Translate :** object is translated in such a way that the rotation axis coincides with the parallel coordinate axis.
- **Rotate :** object is then rotated by the specified angle about that axis.
- **Translate :** the object is finally translated so that the rotation axis is moved backed to the original position.

Mathematically, it can be expressed as,

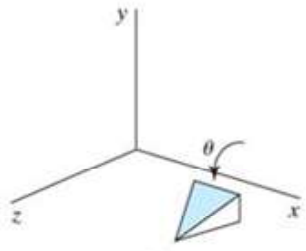
$P' = T^{-1} \cdot R(\theta).T.P$

where $R(\theta) = T^{-1}.R_x(\theta).T$ is the composite transformation.

The steps are shown in the figure below.



(a)
Original Position of Object



(c)
Rotate Object Through Angle θ

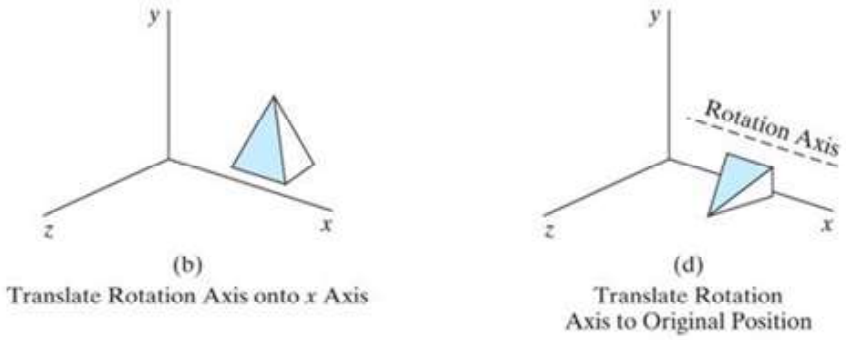
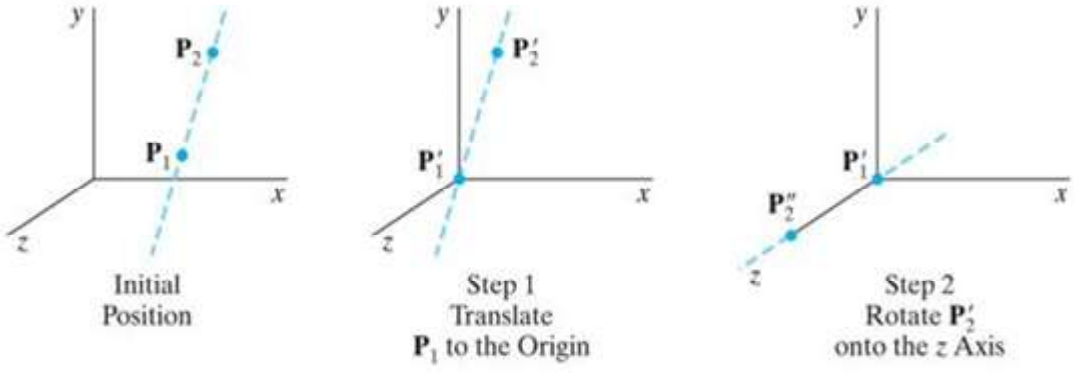


Figure : Rotating an object about an axis parallel to the x-axis

Rotate an object about an axis not parallel to any of the coordinate axis:

For rotating an object about an axis not parallel to any of the coordinate axis, the following steps needs to be performed.

- **Translate:** The object is translated such that the rotation axis passes through the coordinate origin.
- **Rotate :** Object is rotated such that its axis of rotation coincides with any of the coordinate axes
- **Rotate:** It is again rotated by the specified angle about the coordinate axis fro where the axis of rotation has passed.
- **Inverse rotation:** This brings the rotation axis to its original position.
- **Inverse translation:** This is done to bring back the rotation axis to its original position.



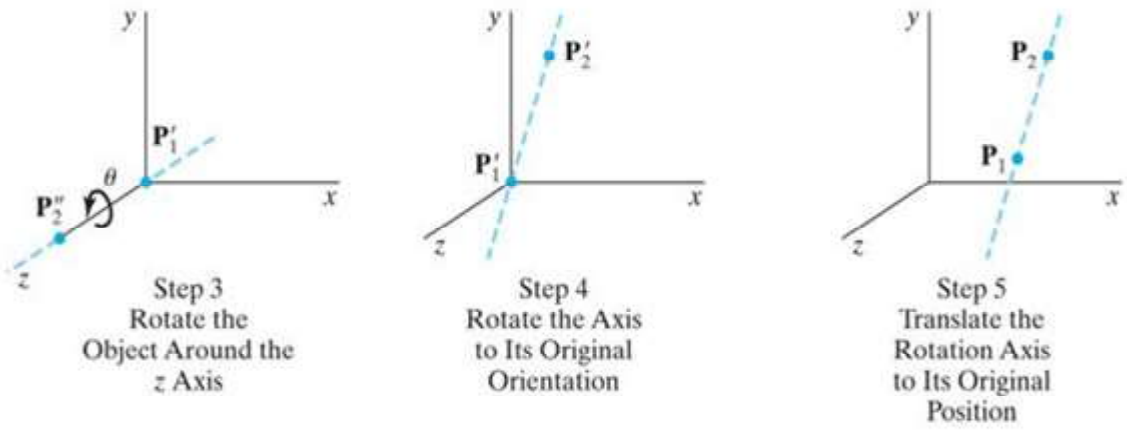


Figure : Rotating an object about an arbitrary axis

12.5 SCALING

Scaling changes the size of an object and repositions the object relative to the coordinate origin. Like 2D scaling, in 3D scaling also the size of an object is changed by compressing or expanding its dimensions. For example, the point P will be scaled from its original position P(x,y,z) to a new position $P'(x', y', z')$ relative to the coordinate origin from the scaling matrix expression.

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

where s_x, s_y and s_z are the scaling parameters that can be assigned any positive values.

$$x' = x.S_x$$
$$y' = y.S_y$$
$$z' = z.S_z$$

Equation 12.5 can also be written as

$$P' = S.P$$

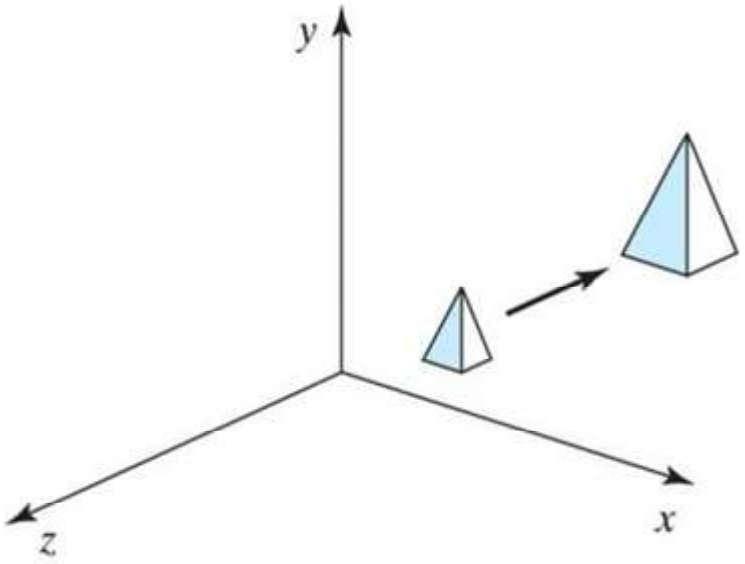


Figure : Scaling an object with the scaling parameters set to 2

If the transformation parameters are not all equal, relative dimensions in the object are changed. The original shape of an object is preserved by uniform scaling that is $s_x = s_y = s_z$. In the figure above, the scaling parameters $s_x = s_y = s_z = 2$. This has increased both the size and the distance of the object by a factor of 2.



CHECK YOUR PROGRESS

Q 1: Fill in the blanks :

a) Repositioning an object along a straight line path from one coordinate location to another coordinate location is called_____.

b) A _____ matrix is used to perform the 3D transformations.

c) In 3D, there are _____ basic rotations, with respect to each of the principle axes X, Y and Z.

d) Rotation transformation in 3D is much more complex than that of 2D because 3D transformation rotation can be specified around any _____ in space.

e) The steps needed to be performed for rotating an object about an axis parallel to one of the coordinate axis are _____, _____ and _____.

f) _____ changes the size of an object and repositions the object relative to the coordinate origin.

g) If the transformation parameters in scaling are not all equal, relative dimensions in the object are _____.

h) Operations such as moving, reflecting, scaling, rotating an image are called _____ transformations.

i) _____ is a rigid body transformation which moves the objects without deformation.

j) The transformation matrix for rotation about an _____ axis is more complicated.

12.6 REFLECTION AND SHEARING TRANSFORMATION

Reflection produces a mirror image of an object. An object is rotated a particular angle about the axis of reflection to get the reflected image.

A 3D reflection can be performed in one of the following ways:

- With respect to a selected reflection axis
- With respect to a selected reflection plane.

When reflection is performed with respect to a selected reflection axis, the object of reflection is rotated by 1800 about that axis. If the reflection is performed with respect to a selected reflection plane then also the object is rotated by 1800. However, the rotation will be performed in 4D space.

Reflection about the xy- plane

Reflection relative to the xy-plane can be obtained by the following set of instructions.

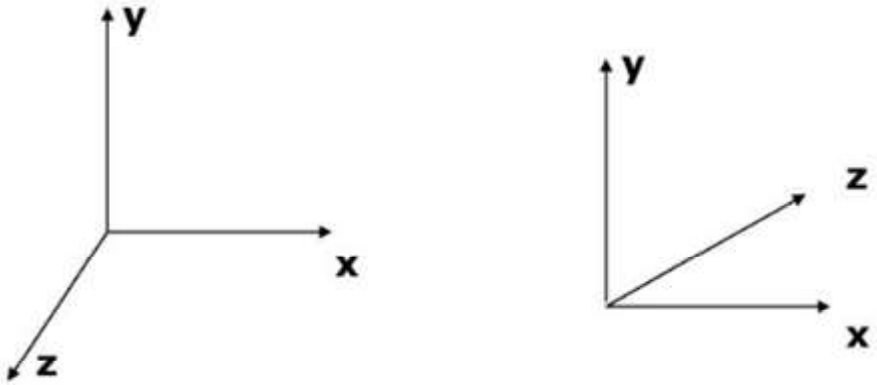
$x' = x \quad y' = y \quad z' = -z$

In homogeneous matrix form,

$$RF_{xy} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Reflections about other planes can also be obtained by symmetry.

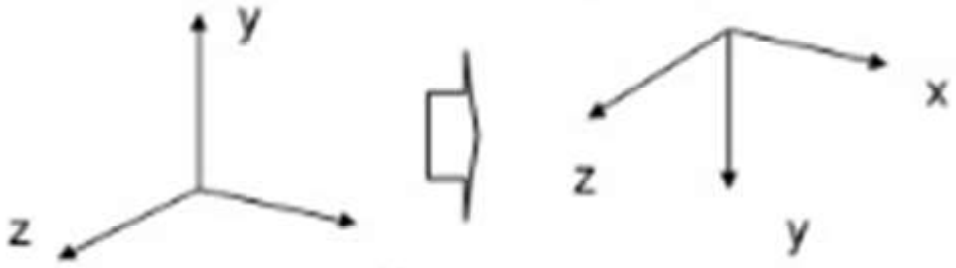
Reflection about xy-plane is a useful reflection as it converts a right-handed coordinate system into a left-handed coordinate system.



Similarly, the transformation matrix representation for the reflection of points with respect to the xz-plane (in which only the sign of the y coordinate is changed) is given as follows:

$$RF_{xy} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

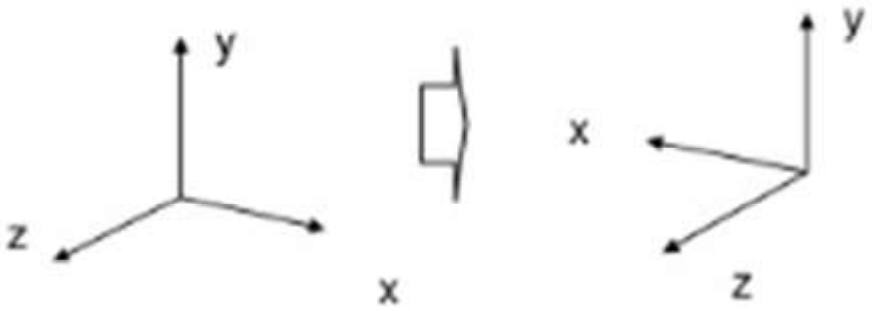
The figure below shows the reflection transformation with respect to the xz-plane.



The transformation matrix representation for the reflection in the yz-plane (where only the sign of the x-coordinate value is changed) is given as follows:

$$RF_{yz} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The figure below shows the reflection transformation with respect to the yz-plane.



3D- SHEARING TRANSFORMATION

This kind of transformation distorts the shape of the objects and they appear to look as if the internal layers are caused to slide over each other. The object is pulled horizontally by different amounts.



Original After Shear

3D shearing transformation is similar to 2D shearing transformation except that in 3D shearing one can shear along the z-axis. Shearing along the x-axis keeps the y-coordinate and the z-coordinate unchanged. Only the x-coordinate gets changed. This makes the object tilt towards the x-direction.

The transformation matrix of 3D shearing along the x-axis is as follows:

$$SH_x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ b & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Where a and b are parameters that can be assigned any real values.

Similarly, the transformation matrix of 3D shearing along the y-axis and the z-axis is as follows:

$$SH_y = \begin{pmatrix} 1 & a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & b & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$SH_z = \begin{pmatrix} 1 & 0 & a & 0 \\ 0 & 1 & b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Shearing in the y-axis and the z-axis tilts the object towards the y- and z- directions respectively.

12.7 COMPOSITE TRANSFORMATION

Complex geometric and coordinate transformations can be done from the basic transformations. Composite transformation can be setup as a sequence of transformation using matrix representation. In composite transformations, we multiply the basic matrices. As matrix multiplication is not commutative, the sequence of the matrix is very important. Wrong sequence can give wrong results. The composite transformation is carried out from left to right, where the rightmost matrix is the first transformation to be applied and the leftmost matrix is the last one to be applied. That is, each successive transformation matrix premultiplies the product of the preceding transformation matrices.

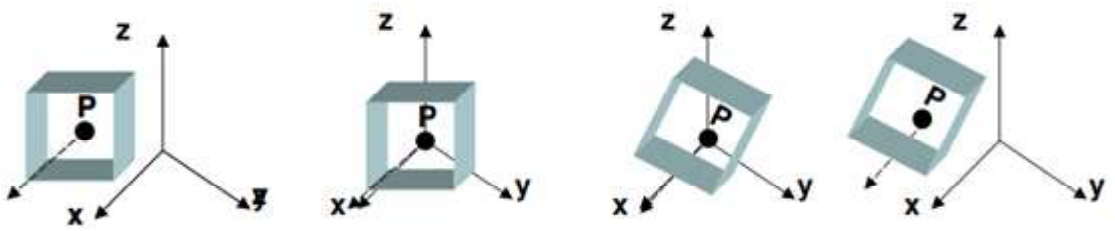
Thus, if we want to apply two transformations to a point at position P, the transformation would be calculated as

$$P' = M_2.M_1.P$$
$$= M.P$$

For example, to rotate about a fixed point, the following transformations are needed to be performed.

- Translate point to origin
- Rotate the object
- Reverse the previous translation.

The figure below shows the composite transformation.



Translation

$$T(-P) = \begin{pmatrix} 1 & 0 & 0 & -p_x \\ 0 & 1 & 0 & -p_y \\ 0 & 0 & 1 & -p_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Rotation around x

$$R(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Translation

$$T(P) = \begin{pmatrix} 1 & 0 & 0 & p_x \\ 0 & 1 & 0 & p_y \\ 0 & 0 & 1 & p_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M = T(P)R(\theta)T(-P)$$



CHECK YOUR PROGRESS

Q 2: Fill in the blanks:

a. Reflection produces a _____ image of an object.

b. If the reflection is performed with respect to a selected reflection plane then the object is rotated by_____.

c. Reflection about xy-plane is a useful reflection as it converts a right-handed coordinate system into a _____

coordinate system.

d. _____ transformation distorts the shape of the objects and they appear to look as if the internal layers are caused to slide over each other.

e. 3D reflection can be performed with respect to reflection axis and reflection _____.

f. Composite transformation can be setup as a _____ of transformation using matrix representation.

g. The composite transformation is carried out from _____ to right.



12.8 LET US SUM UP

- Repositioning an object along a straight line path from one coordinate location to another coordinate location is called translation transformation
- A 4X4 matrix is used to perform the 3D transformation operations.
- In 3D, there are three basic rotations, with respect to each of the principle axes X, Y and Z.
- Scaling changes the size of an object and repositions the object relative to the coordinate origin.
- Like 2D scaling, in 3D scaling also the size of an object is changed by compressing or expanding its dimensions.
- The original shape of an object is preserved by uniform scaling that is sx = sy = sz
- Reflection produces a mirror image of an object.
- When reflection is performed with respect to a selected reflection axis, the object of reflection is rotated by 1800 about that axis.
- If the reflection is performed with respect to a selected reflection plane then also the object is rotated by 1800
- Shear transformation distorts the shape of the objects and they appear to look as if the internal layers are caused to slide over each other.
- 3D shearing transformation is similar to 2D shearing transformation

except that in 3D shearing one can shear along the z-axis.

- Complex geometric and coordinate transformations can be done from the basic transformations.
- Composite transformation can be setup as a sequence of transformation using matrix representation.
- The composite transformation is carried out from left to right, where the rightmost matrix is the first transformation to be applied and the leftmost matrix is the last one to be applied.



12.9 FURTHER READINGS

1) Computer Graphics By Donald Hearn and M. Pauline Baker
2) Computer Graphics, Schaum's Outlines, Plastock and Kalley, McGraw-Hill © 1986
3) Computer graphics: principles and practice by James D. Foley.



12.10 ANSWER TO CHECK YOUR PROGRESS

Ans to Q No 1:

a. Translation
b. 4X4
c. Three
d. Line
e. Translate, rotate, translate
f. Scaling
g. Changed
h. Geometric
i. Translation
j. arbitrary

Ans to Q No 2:

a. mirror.
b. 1800
c. left-handed
d. Shear

e. plane

f. sequence

g. left



12.11 MODEL QUESTIONS

Q 1: Explain translation of an object in 3D space.

Q 2: How can you rotate an object about an axis parallel to one of the coordinate axis?

Q 3: Show the steps to rotate an object about an axis not parallel to any of the coordinate axis.

Q 4: Explain relection and shearing of objects in 3-dimension

Q 5: What is composite transformation?

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