Unit 1: Tensor Algebra

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1.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- Understand the distinction between contravariant and covariant vectors in different coordinate systems.
- Define tensors and explore their fundamental properties.
- Learn how gradients are represented in tensor notation.

- Understand tensor fields and their role in mathematical physics.
- Explore operations for combining tensors.
- Learn different tensor multiplication techniques, including outer products.
- Define and apply the inner product in tensor analysis.
- Understand tensor contraction and its significance.
- Learn properties of symmetric tensors, antisymmetric tensors and their applications.
- Understand how tensors follow quadratic transformation laws.
- Study reciprocal tensors and their mathematical significance.

1.2 INTRODUCTION

Tensors form the foundation of many branches of mathematics, physics, and engineering, particularly in fields such as continuum mechanics, differential geometry, and general relativity. They provide a general framework for describing multi-dimensional quantities that transform consistently under changes of coordinates.

The study of tensor algebra begins with understanding vectors both contravariant and covariant which are first-order tensors. Extending this idea, higher order tensors represent more complex quantities, such as stress, curvature, and inertia. A key feature of tensors is their ability to retain their physical meaning and form regardless of the coordinate system used, making them invaluable in formulating coordinate independent laws of nature.

This unit introduces the basic concepts and operations in tensor algebra. We begin with the definitions and transformation properties of vectors, followed by the formal definition of tensors. Concepts such as the gradient of a scalar function, tensor fields, and various algebraic operations like addition, multiplication, contraction, and inner products are developed in detail. Special classes of tensors, such as symmetric and anti-symmetric tensors, are also explored. Furthermore, essential results like the quadratic law and the concept of reciprocal tensors are presented.

By the end of this unit, learners will gain a solid understanding of the algebraic structure of tensors and their manipulation, forming a necessary base for more advanced studies in tensor calculus and its applications.

1.3 Contravariant and covariant vector

Contravariant Vectors:

Let $\{A^i\}$ for $i = 1,2,\ldots,N$ be a set of N functions defined on a coordinate system $(x^1, x^2, \ldots, x^N)$. That is, the quantities $A^1, A^2, \ldots, A^N$ are functions of the coordinates

$$A^i = A^i(x^1, x^2, \ldots, x^N)$$

We say that $\{A^i\}$ are the **components of a covariant vector** if, under a change of coordinates to a new system $(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^N)$, the components transform according to the rule

$$\bar{A}^i = \frac{\partial \bar{x}^i}{\partial x^j} A^j$$

Where summation over repeated indices is implied (Einstein summation convention).

Remarks:

- A contravariant vector is not merelya collection of functions in a given coordinate system; it is an abstract quantity. Its representation in a given coordinate systemisaset of components that obey the above transformation law.
- A contravariant vector is also called a **contravariant tensor of rank (Or order) 1**.

Properties of Contravariant vectors:

Theorem 1:

The law of transformation for a contravariant vector is transitive.

Proof: Let us consider N quantities A^iin a co-ordinate system x^i. Let these N quantites have the values $\bar{A}^i$ in another co-ordinate system say $\bar{x}^i$. Then by the transformation law of a contravariant vector, we have,

$$\bar{A}^i = \frac{\partial \bar{x}^i}{\partial x^j} A^j \qquad (1.1)$$

Now, let the N quantities $\bar{A}^i$ have the values $\bar{\bar{A}}^i$ in another co-ordinate system $\bar{\bar{x}}^i$. Then by the transformation law for a contravariant vector, we have,

$$\bar{\bar{A}}^i = \frac{\partial \bar{\bar{x}}^i}{\partial \bar{x}^j} \bar{A}^j \qquad (1.2)$$

Now, using (1.1) in (1.2), we obtain

$$\bar{\bar{A}}^i = \frac{\partial \bar{\bar{x}}^i}{\partial \bar{x}^j}\frac{\partial \bar{x}^j}{\partial x^k}A^k \Rightarrow \bar{\bar{A}}^i = \frac{\partial \bar{\bar{x}}^i}{\partial x^k}A^k$$

This shows that the transformation law of a contravariant vector is transitive.

Covariant Vectors:

Let $\{A_i\}$ for $i = 1,2,\ldots,N$ be a set of *N* functions defined on a coordinate system $(x^1,x^2,\ldots,x^N)$. That is, the quantities $A_1,A_2,\ldots,A_N$ are functions of the coordinates:

$$A_i = A_i(x^1,x^2,\ldots,x^N)$$

We say that $\{A_i\}$ are the **components of a covariant vector** if, under a change of coordinates to a new system $(\bar{x}^1,\bar{x}^2,\ldots,\bar{x}^N)$, the components transform according to the rule:

$$\bar{A}_i = \frac{\partial x^j}{\partial \bar{x}^i}A_j,$$

Where summation over repeated indices is implied (Einstein summation convention).

Remarks:

- A covariant vector is not merely a collection of *N* functions in a given coordinate system; it is an abstract quantity. Its representation in a given coordinate system is a set of components $A_i(x)$ that obey the above transformation law.
- A covariant vector is also called a covariant tensor of rank (or order) 1.
- A *superscript* (upper index) is used to indicate contravariant components, while a *subscript* (lower index) is used to indicate covariant components.
- The rank of a tensor is defined as the total number of indices per component.

1.4 Definition of Tensor

A tensor of rank (or order) (r,s) on a differentiable manifold is an abstract multilinear map that takes *r* one-forms and *s* vectors as input and returns a scalar.

In a given coordinate system $(x^1,x^2,\ldots,x^N)$, a tensor is represented by a set of components $T^{i_1\ldots i_r}_{j_1\ldots j_r}$, were

$$T^{i_1\ldots i_r}_{j_1\ldots j_s} = T\left(dx^{j_1},\ldots,dx^{j_s},\frac{\partial}{\partial x^{i_1}},\ldots,\frac{\partial}{\partial x^{i_r}}\right)$$

Under a change of coordinates $x^i \to \bar{x}^i$, the components of the tensor transform according to the law

$$\overline{T}^{a_1\ldots a_r}_{b_1\ldots b_s} = \frac{\partial \overline{x}^{a_1}}{\partial x^{i_1}}\cdots\frac{\partial \overline{x}^{a_r}}{\partial x^{i_r}}\frac{\partial x^{j_1}}{\partial \overline{x}^{b_1}}\cdots\frac{\partial x^{j_s}}{\partial \overline{x}^{b_s}}T^{i_1\ldots i_r}_{j_1\ldots j_s}.$$

Remarks:

- A tensor of rank $(1,0)$ is a *contravariant vector*.
- A tensor of rank $(0,1)$ is a *covariant vector* $(or\ 1-form)$.
- A tensor of rank $(0,0)$ is a *scalar*.
- The total number of indices $r+s$ is called the rank or order of the tensor.

1.5 Gradient

Consider a scalar function *f* of co-ordinates x^i. The gradient of f *is* a vector denoted by Δf or, is defined by

$$\Delta f = grad\ f = \frac{\partial f}{\partial x^i}$$

If $\frac{dx^i}{ds}$ are the components of a unit tangent vector then the projection of Δf in the direction of unit tangent vector $= \Delta f.\frac{dx^i}{ds} = \frac{\partial f}{\partial x^i}.\frac{dx^i}{ds} = \frac{df}{ds}$

$\frac{df}{ds}$ is called the intrinsic derivative of the function f.

Note: The gradient Δf is a covariant tensor of rank 1.

1.6 Tensor field

A tensor field is a tensor that is defined at every point of a region in an *N*-dimensional space. In other words, if a tensor *V* is defined at all points of a region in R^N (or more generally on an *N*-dimensional manifold), then *V* is said to form a tensor field.

- If the tensor is of rank zero, the field is called a scalar field.
- If the tensor is of rank one, the field is called a vector field.
- Tensors of higher rank (rank 2) give rise to higher-order tensor fields.

Thus, a tensor field is a spatial distribution of a tensor that varies smoothly (or continuously) from point in a region.

1.7 Addition and subtraction of tensors

Addition

The **sum** of two or more tensors is defined only when the tensors are of the same rank and type. That is, the tensors must have the same number of *covariant* and *contravariant* indices. In this case, their sum is a new tensor of the **same rank and type**, defined by adding the corresponding components.

If A^p_{qr} and B^p_{qr} two tensors of the same type (one contravariant and two covariant indices), their sum is the tensor C^p_{qr}given by

$$C^p_{qr} = A^p_{qr} + B^p_{qr}.$$

Subtraction

Similarly, the difference of two tensors is defined if the tensors are of the same rank and type. The result is again a tensor of the same rank and type, obtained by subtracting corresponding components.

If and are two such tensors, their difference is

$$D^p_{qr} = A^p_{qr} - B^p_{qr}$$

Remark

By *tensors of the same rank and type*, we mean that

- They have the same total rank (total number of indices),
- They have the same number of covariant indices (lower indices),
- They have the same number of contravariant indices (upper indices).

For example,A^p_{qr} andB^p_{qr} are both tensors of type $(1,2)$, and thus can be added or subtracted component-wise.

Theorem 2:

The sum or difference of two tensors of the same rank and type is again a tensor of the same rank and type.

Proof: Let

$$A(x^1, x^2, \ldots, x^N) \text{ and } B(x^1, x^2, \ldots, x^N)$$

be two tensors of the same rank and type, defined at the same point *P*, so that they satisfy the tensor transformation law. That is, under a change of coordinates $x^i \to \bar{x}^i$, their components transform as

$$\bar{A}^{m_1 m_2 \ldots m_r}_{n_1 n_2 \ldots n_s}(\bar{x}^1, \ldots, \bar{x}^N) = \frac{\partial \bar{x}^{m_1}}{\partial x^{p_1}} \cdots \frac{\partial \bar{x}^{m_r}}{\partial x^{p_r}} \frac{\partial x^{q_1}}{\partial \bar{x}^{n_1}} \cdots \frac{\partial x^{q_s}}{\partial \bar{x}^{n_s}} A^{p_1 p_2 \ldots p_r}_{q_1 q_2 \ldots q_s}(x^1, \ldots, x^N) \qquad (1.3)$$

$$\bar{B}^{m_1 m_2 \ldots m_r}_{n_1 n_2 \ldots n_s}(\bar{x}^1, \ldots, \bar{x}^N) = \frac{\partial \bar{x}^{m_1}}{\partial x^{p_1}} \cdots \frac{\partial \bar{x}^{m_r}}{\partial x^{p_r}} \frac{\partial x^{q_1}}{\partial \bar{x}^{n_1}} \cdots \frac{\partial x^{q_s}}{\partial \bar{x}^{n_s}} B^{p_1 p_2 \ldots p_r}_{q_1 q_2 \ldots q_s}(x^1, \ldots, x^N) \qquad (1.4)$$

Now consider the sum $C = A + B$. Adding equations (1.3) and (1.4), we get

$$\bar{C}^{m_1 m_2 \ldots m_r}_{n_1 n_2 \ldots n_s}(\bar{x}^1, \ldots, \bar{x}^N) = \bar{A}^{m_1 m_2 \ldots m_r}_{n_1 n_2 \ldots n_s} + \bar{B}^{m_1 m_2 \ldots m_r}_{n_1 n_2 \ldots n_s}$$

$$= \frac{\partial \bar{x}^{m_1}}{\partial x^{p_1}} \cdots \frac{\partial \bar{x}^{m_r}}{\partial x^{p_r}} \frac{\partial x^{q_1}}{\partial \bar{x}^{n_1}} \cdots \frac{\partial x^{q_s}}{\partial \bar{x}^{n_s}} (A^{p_1 p_2 \ldots p_r}_{q_1 q_2 \ldots q_s} + B^{p_1 p_2 \ldots p_r}_{q_1 q_2 \ldots q_s})$$

This shows that $C = A + B$ satisfies the same transformation law as A and B, and hence is also a tensor of the same rank and type.

Similarly, the difference $C = A - B$ also satisfies the same transformation law and is a tensor of the same type.

Corollary

Any linear combination of tensors of the same rank and type, with scalar (invariant) coefficients, is a tensor of the same rank and type.

Proof: Let *A* and *B* be two tensors of the same rank and type as in equations (1.3) and (1.4), and let *a* and *b* be scalars (invariant under coordinate transformations). Then

$$\bar{T}^{m_1 \ldots m_r}_{n_1 \ldots n_s} = a\bar{A}^{m_1 \ldots m_r}_{n_1 \ldots n_s} + b\bar{B}^{m_1 \ldots m_r}_{n_1 \ldots n_s}$$

$$= \frac{\partial \bar{x}^{m_1}}{\partial x^{p_1}} \cdots \frac{\partial \bar{x}^{m_r}}{\partial x^{p_r}} \frac{\partial x^{q_1}}{\partial \bar{x}^{n_1}} \cdots \frac{\partial x^{q_s}}{\partial \bar{x}^{n_s}} \left(aA^{p_1 \ldots p_r}_{q_1 \ldots q_s} + bB^{p_1 \ldots p_r}_{q_1 \ldots q_s}\right),$$

which is the transformation law for a tensor of the same type. Hence $aA + bB$ is a tensor of the same rank and type.

CHECK YOUR PROGRESS 1

Q1) Define a contravariant vector and write its transformation law under a coordinate change.

Q2) What is the difference between contravariant and covariant vectors?

Q3) State the transformation law for a covariant vector.

Q4) What does the rank of a tensor signify?

Q5) What are the ranks of contravariant and covariant vectors considered as tensors?

Q6) How does the gradient of a scalarfunction transform? What type of tensor is it?

Q7) When can two tensors be added or subtracted?

1.8 Multiplication of tensors

Outer (Open) Product of Tensors

Definition: Let A and B be tensors of type (r_1, s_1) and (r_2, s_2) respectively. The *outer* (or *open*) product of and is defined as the tensor of type given by the component-wise product

$$C^{\mu_1 \dots \mu_{r_1} \alpha_1 \dots \alpha_{r_2}}_{\nu_1 \dots \nu_{s_1} \beta_1 \dots \beta_{s_2}} = A^{\mu_1 \dots \mu_{r_1}}_{\nu_1 \dots \nu_{s_1}} \cdot B^{\alpha_1 \dots \alpha_{r_2}}_{\beta_1 \dots \beta_{s_2}}.$$

Example. Let $A^{\mu\nu\rho}$ be a contravariant tensor of rank 3 and B_{ijklm} a covariant tensor of rank 5. Then their outer product is

$$C^{\mu\nu\rho}_{ijklm} = A^{\mu\nu\rho}.B_{ijklm}$$

which is a mixed tensor of rank 8 (3 contravariant, 5 covariant).

Theorem 3:

Statement: The outer product of two vectors is a tensor of order two. Is the converse true?

Proof. Let A^i be a contravariant vector and B_j be a covariant vector. Define

$$C^i_j = A^i B_j$$

From the transformation laws of tensors, we have

$$\bar{A}^u = \frac{\partial \bar{x}^u}{\partial x^i} A^i, \quad \bar{B}_v = \frac{\partial x^j}{\partial \bar{x}^v} B_j$$

Multiplying these terms, we obtain

$$\bar{C}^u_v = \bar{A}^u \bar{B}_v = \left(\frac{\partial \bar{x}^u}{\partial x^i}\right)\left(\frac{\partial x^j}{\partial \bar{x}^v}\right) A^i B_j = \frac{\partial \bar{x}^u}{\partial x^i}\frac{\partial x^j}{\partial \bar{x}^v} C^i_j$$

Hence, C^i_j transforms like a tensor of type $(1,1)$, i.e., a tensor of order 2.

Converse

The converse is **not** true. Not every rank-2 tensor is the outer product of two vectors.

For instance, the metric tensor g_{ij} is a rank-2 covariant tensor that, in general, cannot be expressed as the outer product of two vectors. Also, any linear combination of tensor products or contraction operations can yield a tensor that is not simply a product of two vectors.

Theorem 4:

Statement: If A^i and B_j are components of a contravariant and covariant vector respectively, then $C^i_j = A^i B_j$ are the components of a mixed tensor of rank two.

Proof:

Given

$$C^i_j = A^i B_j$$

Under coordinate transformation:

$$\bar{A}^m = \frac{\partial \bar{x}^m}{\partial x^i} A^i \ , \ \bar{B}_n = \frac{\partial x^j}{\partial \bar{x}^n} B_j$$

Multiplying

$$\bar{C}^m_n = \bar{A}^m \bar{B}_n = \left(\frac{\partial \bar{x}^m}{\partial x^i} A^i\right)\left(\frac{\partial x^j}{\partial \bar{x}^n} B_j\right) = \frac{\partial \bar{x}^m}{\partial x^i}\frac{\partial x^j}{\partial \bar{x}^n} C^i_j$$

This is the transformation law of a tensor of type $(1,1)$. Hence, C^i_j is a mixed tensor of rank 2.

Theorem 5:

Statement: The outer product of two tensors is a tensor whose rank is the sum of the ranks of the two tensors.

Proof:

Let A^{ij}_k be a tensor of type $(2,1)$ and B^p_q be a tensor of type . Define their outer product

$$C^{ijp}_{kq} = A^{ij}_k \cdot B^p_q$$

This tensor has

- 3 contravariant indices: i, j, p
- 2 covariant indices: k, q

Thus, C_{kq}^{ijp} is a mixed tensor of type $(3,2)$ and rank 5.

Under coordinate transformation, the components transform as

$$\bar{C}_{rs}^{abc} = \frac{\partial \bar{x}^a}{\partial x^i}\frac{\partial \bar{x}^b}{\partial x^j}\frac{\partial \bar{x}^c}{\partial x^p}\frac{\partial x^k}{\partial \bar{x}^r}\frac{\partial x^q}{\partial \bar{x}^s}C_{kq}^{ijp}$$

This satisfies the transformation law of a tensor of rank 5. Hence, the outer product is a tensor, and the rank is the sum of the ranks of the original tensors.

Theorem 6:

Statement: The outer product of two tensors is a tensor whose rank (or order) is the sum of the ranks of the two tensors.

Proof. Consider two tensors A_l^{ij} and B_q^p. Let the outer product of these two tensors be a tensor C_{lq}^{ijp}, that is,

$$C_{lq}^{ijp} = A_l^{ij}.B_q^p \qquad (1.5)$$

Now we are to show that C_{lq}^{ijp} is a tensor of rank 5. We know that

$$\bar{A}_w^{uv} = A_l^{ij}.\frac{\partial \bar{x}^u}{\partial x^i}.\frac{\partial \bar{x}^v}{\partial x^j}.\frac{\partial x^l}{\partial \bar{x}^w} \qquad (1.6)$$

And

$$\overline{B_l^k} = B_q^p.\frac{\partial \bar{x}^k}{\partial x^p}.\frac{\partial x^q}{\partial \bar{x}^t} \qquad (1.7)$$

Multiplying (1.6) and (1.7) and using (1.5), we have

$$\overline{A}_w^{uv} \cdot \overline{B}_t^k = C_{lq}^{ijp} \cdot \frac{\partial \bar{x}^u}{\partial x^i} \cdot \frac{\partial \bar{x}^v}{\partial x^j} \cdot \frac{\partial x^l}{\partial \bar{x}^w} \cdot \frac{\partial \bar{x}^k}{\partial x^p} \cdot \frac{\partial x^q}{\partial \bar{x}^t}$$

In view of equation (1.5), we may write the above equation as

$$\overline{C}_{wt}^{uvk} = C_{lq}^{ijp} \cdot \frac{\partial \bar{x}^u}{\partial x^i} \cdot \frac{\partial \bar{x}^v}{\partial x^j} \cdot \frac{\partial \bar{x}^k}{\partial x^p} \cdot \frac{\partial x^l}{\partial \bar{x}^w} \cdot \frac{\partial x^q}{\partial \bar{x}^t} \qquad (1.8)$$

Relation (1.8) is the law of transformation of a mixed tensor of rank five. Hence C_{lq}^{ijp} is a mixed tensor of rank 5.

1.9Inner Product

Definition. By performing the *outer product* of two tensors followed by a *contraction* over one or more pairs of indices (one covariant and one contravariant), we obtain a new tensor called the inner product of the given tensors. This operation is also referred to as inner multiplication.

Example

Let $A_l^{ij}$and B_q^pbe two tensors. Then, the outer product is given by

$$A_l^{ij} \otimes B_q^p = A_l^{ij} B_q^p.$$

Now, by contracting the index p with the first index l of A, say $p = l$, we get

$$A_l^{ij} B_q^l = C_q^{ij}$$

Alternatively, contracting $p = l$ and $j = q$, we obtain another inner product

$$A_l^{ij} B_q^l = D^i$$

Theorem 7:

Statement: The inner product of two tensors

$$A_q^p \text{and} \quad B_l^{ij}$$

is a tensor of rank three.

Proof. Consider the contraction of the covariant index $q$in $A_q^p$with one of the covariant indices in B_l^{ij} say $q = i$. The resulting expression is

$$C_l^{pj} = A_q^p B_l^{qj}$$

We must show that this object C_l^{pj}transforms like a rank-3 tensor under a change of coordinates. Let $A_q^p$and B_l^{qj} transform under coordinate transformations as follows

$$\bar{A}_q^p = \frac{\partial \bar{x}^p}{\partial x^r}\frac{\partial x^s}{\partial \bar{x}^q} A_s^r,$$
$$\bar{B}_l^{qj} = \frac{\partial x^q}{\partial \bar{x}^m}\frac{\partial x^j}{\partial \bar{x}^n}\frac{\partial \bar{x}^t}{\partial x^l} B_t^{mn}$$

Now, from the product

$$\bar{C}_l^{pj} = \bar{A}_q^p \bar{B}_l^{qj} = \left(\frac{\partial \bar{x}^p}{\partial x^r}\frac{\partial x^s}{\partial \bar{x}^q} A_s^r\right)\left(\frac{\partial x^q}{\partial \bar{x}^m}\frac{\partial x^j}{\partial \bar{x}^n}\frac{\partial \bar{x}^t}{\partial x^l} B_t^{mn}\right) = \frac{\partial \bar{x}^p}{\partial x^r}\frac{\partial x^s}{\partial \bar{x}^q}\frac{\partial x^q}{\partial \bar{x}^m}\frac{\partial x^j}{\partial \bar{x}^n}\frac{\partial \bar{x}^t}{\partial x^l} A_s^r B_t^{mn}$$

Note that

$\frac{\partial x^s}{\partial \bar{x}^q}\frac{\partial x^q}{\partial \bar{x}^m} = \delta^s_m$, (Using summation over q)

Thus, we simplify

$$\bar{C}^{pj}_{l} = \frac{\partial \bar{x}^p}{\partial x^r}\frac{\partial x^j}{\partial \bar{x}^n}\frac{\partial \bar{x}^t}{\partial x^l}A^r_s B^{sn}_t$$

which is the transformation law for a tensor with two contravariant indices (p, j) and one covariant index (l). Hence, C^{pj}_l is a tensor of rank 3 and mixed type $(2,1)$.

Theorem 8:

The inner multiplication of tensors is commutative and associative

Proof. This result follows directly from the properties of outer multiplication and contraction, and is a consequence of Theorem 7. Since contraction is performed after outer multiplication, and both operations are bilinear and obey index arithmetic independently of order (as long as index positions are compatible), the inner multiplication is both commutative (under contraction of compatible indices) and associative.

1.10Contraction

If in a tensor we put one contravariant and one covariant indices equal (*i.e.* same), the process is called contraction of a tensor. *For example:* Consider a tensor $A^{ij}_{pqr}$of rank five. If we put $j = r$, we get $A^{ir}_{pqr}$and is a tensor (of rank 3) obtained by contracting (as explained in theorem below). Other tensors obtained by contracting are

$$A^{ij}_{iqr}, \quad A^{ij}_{pqi}, \quad A^{ij}_{pir}, \quad A^{ij}_{pjr}, \quad A^{ij}_{pqj}, \quad A^{ij}_{jqr}, \quad A^{ij}_{pij}, \quad A^{ij}_{iqj}, \quad A^{ij}_{ijr}$$

Theorem 9:

If in a mixed tensor, contravariant of rank p and covariant of rank q, we equate a covariant and contravariant index and sum with regard to that index, then the resulting set of N^{p+q-2}sums is a mixed tensor, contravariant of rank $p-1$and covariant of rank $q-1$.

Proof. There is no loss of generality by taking a mixed tensorA^{ij}_{lmn} of rank five, contravariant of rank two and covariant of rank three. [There is convenience of writing by taking this tensor].

Then we have

$$A^{ij}_{lmn} = \frac{\partial \bar{x}^i}{\partial x^k}\frac{\partial \bar{x}^j}{\partial x^t}\frac{\partial x^u}{\partial \bar{x}^l}\frac{\partial x^v}{\partial \bar{x}^m}\frac{\partial x^w}{\partial \bar{x}^n}A^{kt}_{uvw}$$

Let $j = n$, we get

$$\begin{aligned}
\bar{A}^{in}_{lmn} &= \frac{\partial \bar{x}^i}{\partial x^k}\frac{\partial \bar{x}^n}{\partial x^t}\frac{\partial x^u}{\partial \bar{x}^l}\frac{\partial x^v}{\partial \bar{x}^m}\frac{\partial x^w}{\partial \bar{x}^n}A^{kt}_{uvw} \\
&= \frac{\partial \bar{x}^i}{\partial x^k}\frac{\partial x^u}{\partial \bar{x}^l}\frac{\partial x^v}{\partial \bar{x}^m}\left(\frac{\partial \bar{x}^n}{\partial x^t}\frac{\partial x^w}{\partial \bar{x}^n}\right)A^{kt}_{uvw} \\
&= \frac{\partial \bar{x}^i}{\partial x^k}\frac{\partial x^u}{\partial \bar{x}^l}\frac{\partial x^v}{\partial \bar{x}^m}\frac{\partial x^w}{\partial x^t}A^{kt}_{uvw} \\
&= \frac{\partial \bar{x}^i}{\partial x^k}\frac{\partial x^u}{\partial \bar{x}^l}\frac{\partial x^v}{\partial \bar{x}^m}\delta^w_l A^{kt}_{uvw} \\
&= \frac{\partial \bar{x}^i}{\partial x^k}\frac{\partial x^u}{\partial \bar{x}^l}\frac{\partial x^v}{\partial \bar{x}^m}A^{kw}_{uvw} \\
&= \frac{\partial \bar{x}^i}{\partial x^k}\frac{\partial x^u}{\partial \bar{x}^l}\frac{\partial x^v}{\partial \bar{x}^m}A^{k}_{uv} \quad \text{Putting } A^{kw}_{uvw} = A^{k}_{uv}
\end{aligned}$$

This is clearly the law of transformation of a tensor of rank three.

Thus $\bar{A}^{in}_{lmn} = \bar{A}^{i}_{lm}$ is a tensor of rank 3, contravariant of rank $2-1\ i.e.\ 1$ and covariant of rank $3-1\ i.e.\ 2$.

This proves our theorem.

Remark. We never put equal two indices of the same type $(i.e.$ either both in superscript or both in subscript$)$, since in general, the summation convention is applied to two indices one of which is subscript and other a superscript.

1.11 Symmetric Tensor

Definition. A covariant tensor A_{pq} of rank two is said to be symmetric if

$$A_{pq} = A_{qp} \tag{1.9}$$

Similarly, a contravariant tensor A^{pq} of rank two is symmetric if

$$A^{pq} = A^{qp} \tag{1.10}$$

In general, a tensor is said to be symmetric in a pair of similar indices if interchanging those indices does not change the value of the tensor. For instance, A^{lmn}_{pqr} is symmetric in the second and third contravariant indices if $A^{lmn}_{pqr} = A^{lnm}_{pqr}$ or A^{lmn}_{pqr} is symmetric in the first two covariant indices if $A^{lmn}_{pqr} = A^{lmn}_{qpr}$.

Theorem 10

If a tensor is symmetric in two similar indices in a coordinate system $(x^1, x^2, \ldots, x^N)$, then it remains symmetric in any other coordinate system $(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^N)$. *That is, symmetry is preserved under tensor transformation laws.*

Proof. Suppose $A_{pq} = A_{qp}$ in one coordinate system. Then under a coordinate transformation, the tensor components transform as

$$\bar{A}_{pq} = \frac{\partial x^r}{\partial \bar{x}^p} \frac{\partial x^s}{\partial \bar{x}^q} A_{rs}$$

By symmetry of A_{rs}, we also have

$$\bar{A}_{qp} = \frac{\partial x^r}{\partial \bar{x}^q} \frac{\partial x^s}{\partial \bar{x}^p} A_{rs}$$

Now, since $A_{rs} = A_{sr}$, we can interchange dummy indices $r \leftrightarrow s$ to write:

$$\bar{A}_{qp} = \frac{\partial x^s}{\partial \bar{x}^p} \frac{\partial x^r}{\partial \bar{x}^q} A_{sr} = \bar{A}_{pq}$$

Thus, $\bar{A}_{pq} = \bar{A}_{qp}$ and symmetry is preserved in the transformed components.

Remark. Symmetry of a tensor is independent of the choice of coordinates and is therefore an intrinsic property.

Theorem 11:

A symmetric tensor A_{pq} of rank two in an N –dimensional space V_N has at most

$$\frac{1}{2}N(N+1)$$

independent components.

Proof. A general rank-2 tensor A_{pq} has N^2 components since both indices p and q independently range from 1 to N. In the symmetric case, we have

$A_{pq} = A_{qp}$,

so, components like A_{12} and A_{21} are equal.

Let us count

- Diagonal components $A_{11}, A_{22}, \ldots, A_{NN}$: there are *N* of them.
- Off-diagonal pairs (p, q) with $p \neq q$: there are $N^2 - N$ such pairs. But due to symmetry, each pair with determines a unique independent component. So,

the number of independent off- diagonal components is

$$\frac{N^2 - N}{2}$$

Therefore, total number of independent components is

$$N + \frac{N^2 - N}{2} = \frac{N(N+1)}{2}$$

Corollary. If A_{pqr}, is a rank-3 tensor symmetric in indices p and q, then the number of independent components is

$$\frac{1}{2}N(N+1) \cdot N = \frac{1}{2}N^2(N+1)$$

Here, $\frac{1}{2}N(N+1)$ choices of symmetric index pairs (p, q), and N values of r.

1.12 SkewSymmetric (or Anti-Symmetric) Tensor

Definition. A tensor A_{pq} is said to be skew-symmetric (or anti-symmetric) if

$A_{pq} = -A_{qp}$.

Similarly, a tensor A_{pq}^{lmn} is skew-symmetric in the second and third contravariant indices if

$$A_{pq}^{lmn} = -A_{pq}^{lnm}$$

Theorem 12

If a tensor is skew-symmetric with respect to a pair of similar indices in one system of coordinates, it remains skew-symmetric with respect to the same pair in every coordinate system.

Proof. Let A_{pq} be a skew-symmetric tensor in a coordinate system $(x^1, x^2, \ldots, x^N)$, so that

$$A_{pq} = -A_{qp} \qquad (1.11)$$

Let $(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^N)$ be another coordinate system. Under a change of coordinates, the tensor components transform as

$$\bar{A}_{pq} = \frac{\partial x^j}{\partial \bar{x}^p}\frac{\partial x^k}{\partial \bar{x}^q}A_{jk}$$

Using skew-symmetry of A_{jk}, we have

$$\bar{A}_{pq} = -\frac{\partial x^j}{\partial \bar{x}^p}\frac{\partial x^k}{\partial \bar{x}^q}A_{kj}$$

Now relabel $j \leftrightarrow k$, and rearrange

$$\bar{A}_{pq} = -\frac{\partial x^k}{\partial \bar{x}^q}\frac{\partial x^j}{\partial \bar{x}^p}A_{kj} = -\bar{A}_{qp}$$

Hence, $\bar{A}_{pq} = -\bar{A}_{qp}$, which proves that skew-symmetry is preserved.

Remark. Like symmetry, skew-symmetry is also an intrinsic property of tensors, $i.e.$, it is invariant under coordinate transformation.

Theorem 13:

A skew-symmetric tensor of rank two in an N –dimensional space V_N has at most

$$\frac{1}{2}N(N-1)$$

independent components.

Proof. Let A_{pq} be a skew-symmetric tensor, so

$A_{pq} = -A_{(qp)}$,

If $p = q$, we have

$$A_{pp} = -A_{pp} \Rightarrow 2A_{pp} = 0 \Rightarrow A_{pp} = 0.$$

Thus, all diagonal components vanish.

The total number of components in a rank-2 tensor in V_N is N^2. The number of components with repeated indices (diagonal terms) is N, and all of these are zero in the skew-symmetric case.

So, the number of components with distinct indices is $N^2 - N$. By skew-symmetry, for each such pair (p, q), we have $A_{pq} = -A_{qp}$, so only half of these are independent. Hence, the number of independent components is

$$\frac{1}{2}(N^2 - N) = \frac{1}{2}N(N-1)$$

Theorem 14:

If a mixed tensor A^p_q is symmetric in one coordinate system, then it is not (in general) symmetric in another coordinate system.

Proof. Suppose in a coordinate system x^i, the mixed tensor satisfies

$$A_q^p = A_p^q$$

In another coordinate system $\bar{x}^i$, the transformation rule for a mixed tensor is

$$\bar{A}_q^p = \frac{\partial \bar{x}^p}{\partial x^j}\frac{\partial x^i}{\partial \bar{x}^q}A_i^j \qquad (1.12)$$

Now, interchange $p \leftrightarrow q$, and $i \leftrightarrow j$, to get

$$\bar{A}_q^p = \frac{\partial \bar{x}^q}{\partial x^i}\frac{\partial x^j}{\partial \bar{x}^p}A_j^i \qquad (1.13)$$

In general, the right-hand sides of (1.12) and (1.13) are not equal, because

$$\frac{\partial \bar{x}^p}{\partial x^j}\frac{\partial x^i}{\partial \bar{x}^q} \neq \frac{\partial \bar{x}^q}{\partial x^i}\frac{\partial x^j}{\partial \bar{x}^p}$$

and even if $A_i^j = A_j^i$, the asymmetry in the transformation law prevents symmetry from being preserved.

Therefore, $\bar{A}_q^p \neq \bar{A}_p^q$, in general.

CHECK YOUR PROGRESS 2

Q.1 When is a tensor said to be symmetric?

Q.2 What does it mean for a tensor to be skew-symmetric?

1.13 Quotient Law

This law is used to test whether a given quantity is a tensor. Suppose a quantity X is given and we do not know whether it is a tensor. We test it by forming an inner product of X with an *arbitrary* tensor. If the result is a tensor, then X itself is a tensor. This is known as the Quotient Law.

Theorem 15: (Quotient Theorem of Tensors)

Statement: A quantity which on inner multiplication by an arbitrary vector always gives a tensor, is itself a tensor.

Proof. Let the inner product of $A(i,j,k)$ with a vector A^u be denoted by $A(u,j,k)A^u$ whether $A(i,j,k)$ is a tensor or not.

To prove the actual theorem, let ζ^u be an arbitrary vector. Let the inner product of

$A(u,j,k)$ with ζ^u be a tensor of the type A^j_k, *i. e.*,

$$A(u,j,k)\zeta^u = A^j_k \qquad (1.14)$$

Then to prove $A(i,j,k)$ is a tensor.

Now transform $\bar{A}(u,j,k)\bar{\zeta}^u$ of $A(u,j,k)\zeta^u$ is given by the law [in view of (1.14)]

$$A(u,j,k)\,\bar{\zeta}^u = A(v,q,r)\,\zeta^v \frac{\partial x^r}{\partial \bar{x}^k}\frac{\partial \bar{x}^j}{\partial x^q} \qquad (1.15)$$

Were

$$\zeta^v = \bar{\zeta}^u \frac{\partial x^v}{\partial \bar{x}^u} \qquad (1.16)$$

Using (1.16) in (1.15), we have

$$\begin{aligned}\bar{A}(u,j,k)\,\bar{\zeta}^u &= A(v,q,r)\,\bar{\zeta}^u \frac{\partial x^v}{\partial \bar{x}^u}\frac{\partial x^r}{\partial \bar{x}^k}\frac{\partial \bar{x}^j}{\partial x^q} \\ &= \bar{\zeta}^u\left(\bar{A}(u,j,k) - A(v,q,r)\frac{\partial x^v}{\partial \bar{x}^u}\frac{\partial x^r}{\partial \bar{x}^k}\frac{\partial \bar{x}^j}{\partial x^q}\right) = 0\end{aligned}$$

Since ζ^u is an arbitrary vector, hence the expression within the bracket must be zero. Therefore,

$$A(u,j,k) = A(v,q,r)\frac{\partial x^v}{\partial \bar{x}^u}\frac{\partial x^r}{\partial \bar{x}^k}\frac{\partial \bar{x}^j}{\partial x^q}$$

This is the law of transformation of a tensor of the type A^j_{ik}. Hence the quantity $A(i,j,k)$ is a tensor A^j_{ik}.

Note 1.

The quantity does not mean a single quantity but term quantity stands for a set of quantities.

Note 2. Generalised Quotient Law

A set of quantities which on inner multiplication by an arbitrary tensor gives a tensor, is itself a tensor.

1.14 Reciprocal tensor or Conjugate tensor

Definition. Consider a covariant symmetric tensor A_{pq} of rank two. Let d denote the determinant $|A_{pq}|$ with elements A_{pq}, *i. e.*, $d = |A_{pq}|$ and assume $d \neq 0$.

Now we define A^{pq} by

$$A^{pq} = \frac{cofactor\ of\ A_{pq}\ in\ the\ determinant\ of\ |A_{pq}|}{d}$$

A^{pq} is a contravariant symmetric tensor of rank two and is said to be reciprocal (or conjugate) tensor of A_{pq}

Theorem 16:

If $\xi(p,q)$ is the cofactor of A_{pq} in the determinant $d = |A_{pq}| \neq 0$ and A^{pq} is defined as $A^{pq} = \frac{\xi(p,q)}{d}$ (1.17)

Then prove that

$$A_{pq}A^{rq} = \delta_p^r$$

Proof. From properties of determinants, we have the following two results

(i) $A_{pq}\xi(p,q) = d \Rightarrow A_{pq}\frac{\xi(p,q)}{d} = 1 \Rightarrow A_{pq}A^{pq} = 1$, using (1), summation being taken over q

(ii) $A_{pq}\xi(r,q) = 0 \Rightarrow A_{pq}\frac{\xi(p,q)}{d} = 0$, as $d \neq 0$

$\Rightarrow A_{pq}A^{pq} = 0$ if $r \neq p$, using (1.17)

Thus (i) and $(ii) \Rightarrow$

$$A_{pq}A^{rq} = \begin{cases} 1 & if\ p = r \\ 0 & if\ p \neq r \end{cases}$$

$$i.e.$$

$$A_{pq}A^{rq} = \delta_p^r \qquad (1.18)$$

CHECK YOUR PROGRESS 3

$Q.1$ What is the purpose of the Quotient Law?

$Q.2$ Under what condition does the reciprocal tensor A^{pq} exist for the tensor A_{pq}?

$Q.3$ Suppose A_{pq} is a covariant symmetric tensor with

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

Find its reciprocal tensor .

2 1.15 Let Us Sum Up

- **Contravariant and Covariant Vectors:** A contravariant vector A^i transforms according to the partial derivatives of the new coordinates with respect to the old, while a covariant vector A_i transforms with the partial derivatives of the old coordinates with respect to the new.
- **Definition of Tensor:** A tensor of rank (m,n) is a mathematical object that transforms under a change of coordinates following a specific law involving m contravariant and n covariant indices.
- **Gradient:** The gradient of a scalar function Φ is a covariant vector given by $\frac{\partial \Phi}{\partial x^i}$. It points in the direction of maximum rate of increase of Φ.
- **Tensor Field:** A tensor field assigns a tensor to every point in a region of space, allowing the components to vary with position.
- **Addition and Subtraction of Tensors:** Tensors of the same type (same rank and variance) can be added or subtracted component-wise.
- **Multiplication of Tensors:** Tensors can be multiplied using the outer product, resulting in a tensor of higher rank. Tensor multiplication is distributive over addition.
- **Inner Product:** The inner product (or scalar product) of two vectors A^i and B_i is defined as $g_{ij}A^iB^j$, resulting in a scalar (rank 0 tensor).
- **Contraction:** Contraction is the process of summing over one covariant and one contravariant index of a tensor, resulting in a tensor of lower rank.
- **Symmetric Tensor:** A tensor T_{ij} is symmetric if $T_{ij} = T_{ji}$ for all i and j.
- **Antisymmetric Tensor:** A tensor T_{ij} is antisymmetric if $T_{ij} = -T_{ji}$, and thus $T_{ii} = 0$ for all i.
- **Quadratic Law:** A second-rank tensor can be defined through the quadratic form $T = a_{ij}x^ix^j$, which remains invariant under coordinate transformations when a_{ij} follows the tensor transformation law.
- **Reciprocal Tensor:** Given a covariant symmetric tensor A_{ij} of rank 2 with non-zero determinant, the reciprocal tensor A^{ij} is defined such that $A_{ik}A^{kj} = \delta_i^j$. It satisfies the same symmetry and transformation laws as A_{ij}

K 1.16 Ans to Check Your Progress

W Check Your Progress 1:

Ans to Q.1: A contravariant vector A^i is a set of components that transform under a coordinate change $x^i \to \bar{x}^i$ as

$$\bar{A}^i = \frac{\partial \bar{x}^i}{\partial x^j} A^j$$

where summation over repeated indices is implied.

Ans to Q.2: Contravariant vectors have components with upper indices A^i and transform with the partial derivative of new coordinates with respect to old

$$\overline{A}^i = \frac{\partial \overline{x}^i}{\partial x^j} A^j$$

Covariant vectors have components with lower indices A_i and transform with the partial derivative of old coordinates with respect to new

$$\overline{A}_i = \frac{\partial x^j}{\partial \overline{x}^i} A_j$$

Ans to Q.3: Under coordinate change, the covariant vector components transform as

$$\overline{A}_i = \frac{\partial x^j}{\partial \overline{x}^i} A_j$$

Ans to Q.4: The rank (or order) of a tensor is the total number of indices it has. For a tensor $T^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s}$, the rank is $r + s$.

Ans. to Q.5: A contravariant vector is a tensor of rank $(1,0)$ and a covariant vector is a tensor of rank $(0,1)$.

Ans. to Q.6: The gradient of a scalar f is

$$\Delta f = \frac{\partial f}{\partial x^i}$$

It transforms as a covariant vector (tensor of rank $(0,1)$) under coordinate changes.

Ans. to Q.7: Two tensors can be added or subtracted if and only if they have the same rank and type, $i.e.$, the same number of contravariant and covariant indices.

Check Your Progress 2:

Ans. to Q.1: A tensor is symmetric in a pair of indices if interchanging those indices does not change the tensor. For example, a covariant tensor A_{pq} is symmetric if $A_{pq} = A_{qp}$.

Ans. to Q.2: A tensor is skew-symmetric (or anti-symmetric) in a pair of indices if interchanging those those indices change the sign of the tensor. For example, $A_{pq} = -A_{qp}$.

Check Your Progress 3:

Ans. to Q.1: The Quotient Law is used to determine whether a given quantity X is a tensor by testing if the inner product of X with any arbitrary tensor results in a tensor.

Ans. to Q.2: The reciprocal tensor A^{pq} exists only if the determinant $d = |A_{pq}| \neq 0$.

Ans to Q.3: First, calculate the determinant

$$d = (2)(3) - (1)(1) = 6 - 1 - 5 \neq 0$$

Next, find the cofactor matrix

$$Cofactor(A) = \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$$

Then,

$$A^{pq} = \frac{1}{d} \times cofactor(A) = \frac{1}{5}\begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix}$$

Hence, the reciprocal tensor is

$$A^{pq} = \begin{bmatrix} 0.6 & -0.2 \\ -0.2 & 0.4 \end{bmatrix}$$

1.17 Further Reading

1. Weatherburn, C. E. Differential Geometry of Three Dimensions, Cambridge University Press,2016.
2. Graustein, W. C. Differential Geometry. Courier Corporation, 2012.
3. Wilmore T. J. An Introduction to Differential Geometry, Dover Publications Inc., 2012.
4. Pressley, A. Elementary Differential Geometry. Springer, 2002.

M 1.18 Model Questions

Q.1 Explain the transformation law of a contravariant vector and prove that the transformation is transitive.

Q.2 Define a covariant vector and explain its transformation properties.

Q.3 Prove that the sum (or difference) of two tensors of the same rank and type is again a tensor of the same rank and type.

Q.4 Show that the outer product of a contravariant vector and a covariant vector is a mixed tensor of rank 2.

Q.5 Define the reciprocal (or conjugate) tensor A^{pq} corresponding to a covariant symmetric tensor A_{pq} of rank two. Explain how A^{pq} is constructed from A_{pq}.

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