UNIT 10: CONJUGATE AND ASYMPTOTIC LINES

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10.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- learn the concept of conjugate directions.
- learn about the condition of conjugate.
- familiarize yourself with the principal property of conjugate directions.
- understand the principal directions at a point are always orthogonal conjugate.
- understand and define asymptotic lines on surfaces
- derive and analyze the differential equation of asymptotic lines

10.2 INTRODUCTION

Conjugate lines arise when we study how a surface bends, using its first and second fundamental forms. On a regular surface, a pair of tangent directions at a point is

called conjugate if the second fundamental form evaluated on them is zero, which means that, in a precise sense, these directions do not "cooperate" to produce normal curvature at that point. In a parametrized surface, this leads to the notion of conjugate coordinate curves, where each family of coordinate lines is conjugate to the other, and this special alignment simplifies the shape operator and many curvature computations. The unit develops this idea systematically: beginning with the algebraic definition via the second fundamental form, it explores geometric interpretations, relations with lines of curvature and principal directions, and typical examples such as surfaces of revolution, thereby preparing students to handle more advanced topics like orthogonal and curvature line coordinates on surfaces. In differential geometry, an asymptotic line (or asymptotic curve) on a surface is a curve along which the normal curvature is zero at every point. Intuitively, this means that although the curve lies on the surface, it does not bend away from the surface in the direction of the surface normal; all the bending happens within the tangent plane.

10.3 CONJUGATE DIRECTIONS

Conjugate direction at a given point $P(u, v)$ on a surface $\boldsymbol{r} = \boldsymbol{r}(u, v)$ may be defined as follows:

Let Q be a point on the surface adjacent to P and consider tangent planes at these two points and let PR be the line of intersection of these planes. Then as Q tends to coincide with P, the limiting directions of the lines PQ and PR are said to be conjugate directions as P.

To find an analytical expression for two directions to be conjugate

Let $\boldsymbol{r}$ be the position vector of the point P where parameter values are u, v and $\boldsymbol{r} + d\boldsymbol{r}$ that of Q a point adjacent to P in the direction (du, dv). Let the unit normals at $P(\boldsymbol{r})$ and $Q(\boldsymbol{r} + d\boldsymbol{r})$ be $\boldsymbol{N}$ and $\boldsymbol{N} + d\boldsymbol{N}$ respectively. Then

$$d\boldsymbol{N} = \frac{\partial \boldsymbol{N}}{\partial u} du + \frac{\partial \boldsymbol{N}}{\partial v} dv$$

$$\Rightarrow d\boldsymbol{N} = \boldsymbol{N}_1 du + \boldsymbol{N}_2 dv$$

Let $(\delta u, \delta v)$ be the direction of PR, the line of intersection of the tangent planes at P and Q, where $R(u + \delta u, v + \delta v)$ is a point adjacent to P in the direction $(\delta u, \delta v)$, the position vector of R be denoted by $\boldsymbol{r} + \delta \boldsymbol{r}$. Then

$$\boldsymbol{PR} = \delta \boldsymbol{r} = \boldsymbol{r}_1 \delta u + \boldsymbol{r}_2 \delta v$$

As the line PR lies in the tangent planes at P and Q, so $\delta \boldsymbol{r}$ is perpendicular to both $\boldsymbol{N}$ and $\boldsymbol{N} + d\boldsymbol{N}$, thus

$$\boldsymbol{N}.\delta \boldsymbol{r} = 0,\ (\boldsymbol{N} + d\boldsymbol{N}).\delta \boldsymbol{r} = 0 \quad \text{or} \quad d\boldsymbol{N}.\delta \boldsymbol{r} = 0$$

$$\Rightarrow (\boldsymbol{N}_1 du + \boldsymbol{N}_2 dv).(\boldsymbol{r}_1 \delta u + \boldsymbol{r}_2 \delta v) = 0$$

$$\Rightarrow \boldsymbol{N}_1.\boldsymbol{r}_1 du\delta u + \boldsymbol{N}_1.\boldsymbol{r}_2 du\delta v + \boldsymbol{N}_2.\boldsymbol{r}_1 dv\delta u + \boldsymbol{N}_2.\boldsymbol{r}_2 dv\delta v = 0$$

$$\Rightarrow L du\delta u + M(du\delta v + dv\delta u) + N dv\delta u = 0$$

[since $\boldsymbol{N}_1.\boldsymbol{r}_1 = -L, \boldsymbol{N}_1.\boldsymbol{r}_2 = \boldsymbol{N}_2.\boldsymbol{r}_1 = -M, \boldsymbol{N}_2.\boldsymbol{r}_2 = N$]

$$\Rightarrow L\frac{du}{dv}\frac{\delta u}{\delta v} + M\left(\frac{du}{dv} + \frac{\delta u}{\delta v}\right) + N = 0 \qquad (1)$$

This is the required condition that the direction (du, dv) and $(\delta u, \delta v)$ at point P on the given surface are conjugate. The expression (1) being linear in each of the ratios $\frac{du}{dv}$ and $\frac{\delta u}{\delta v}$ implies that there is a unique conjugate direction to a given direction.

Remark

The direction (l_1, m_1) and (l_2, m_2) at P are conjugate if and only if

$$Ll_1l_2 + M(l_1m_2 + l_2m_1) + Nm_1m_2 = 0$$

10.4 CONDITION OF CONJUGATE

To find the condition that the two direction given by

$$Pdu^2 + Qdudv + Rdv^2 = 0$$

be conjugate

The given quadratic may be written as

$$P\left(\frac{du}{dv}\right)^2 + Q\left(\frac{du}{dv}\right) + R = 0 \qquad (1)$$

Let the two direction given by (1) be represented by $\frac{du}{dv}$ and $\frac{\delta u}{\delta v}$, then $\frac{du}{dv}$ and $\frac{\delta u}{\delta v}$ are the roots of the equation (1), then

$$\frac{du}{dv} + \frac{\delta u}{\delta v} = -\frac{Q}{P}$$

and

$$\frac{du}{dv}\frac{\delta u}{\delta v} = \frac{R}{P}$$

Substituting these values in (1) of section **10.3**, then

$$L\left(\frac{R}{P}\right) + M\left(-\frac{Q}{P}\right) + N = 0$$

$$\Rightarrow LR + NP - MQ = 0$$

as the condition, that two directions given by the quadratic $Pdu^2 + Qdudv + Rdv^2 = 0$ are conjugate.

An Important Result

The necessary and sufficient condition that the parametric curves have conjugate direction that $M = 0$.

We know that differential equation of the parametric curves is

$$dudv = 0$$

Now comparing this equation with $Pdu^2 + Qdudv + Rdv^2 = 0$, we observe that $P = R = 0, Q = 1$. Therefore the directions of parametric curves will be conjugate if

$$LR + NP - MQ = 0$$

i.e.

$$M = 0$$

Conversely, if $M = 0$, the condition $LR + NP - MQ = 0$ is satisfied for the parametric curves $(P = 0, R = 0)$.

Principal direction at a point are always orthogonal conjugate

We know that $F = 0, M = 0$ are the necessary and sufficient conditions for parametric curves to be lines of curvature. Also the necessary and sufficient condition for parametric curves to be orthogonal is that $F = 0$ and the necessary and sufficient condition for parametric curves to have conjugate direction is that $M = 0$. Hence the direction of lines of curvature i.e. the principal directions are always orthogonal as well as conjugate.

10.5 PRINCIPAL PROPERTY OF CONJUGATE DIRECTIONS

Two directions at a point on a surface are said to be conjugate if the corresponding diameters of the Dupin indicatrix are conjugate.

Take the lines of curvatures as parameter curves, so that $F = 0$ and $M = 0$. Then the principal curvatures at P are given by

$$\kappa_a = \frac{L}{E};\ \kappa_b = \frac{N}{G}$$

Now if we take point P as origin, x –axis along the principal direction $v =$ constant at P, y –axis along the principal direction $u =$ constant at P and z –axis along the surface normal at P, then we know that Dupin's Indicatrix at P is given by the equations

$$z = h, \frac{x^2}{R_a} + \frac{y^2}{R_b} = 2h, \qquad where\ \ R_a = \frac{1}{\kappa_a}, \qquad R_b = \frac{1}{\kappa_b}$$

The directions (du, dv) and $(\delta u, \delta v)$ at P are inclined to the curve $v =$ constant at angle $\theta,\ \theta'$ such that

$$\tan\theta = \sqrt{\frac{G}{E}}\frac{dv}{du}, \tan\theta' = \sqrt{\frac{G}{E}}\frac{\delta v}{\delta u}$$

Again the direction (du, dv) and $(\delta u, \delta v)$ at P are conjugate if

$$L\frac{du}{dv}\frac{\delta u}{\delta v} + M\left(\frac{du}{dv} + \frac{\delta u}{\delta v}\right) + N = 0 \ \ or\ \ \frac{du}{dv}\frac{\delta u}{\delta v} = -N \quad [since\ M = 0]$$

$$\Rightarrow \frac{du}{dv}\frac{\delta u}{\delta v} = -\frac{L}{N} \ \ or\ \ \tan\theta'\tan\theta\frac{E}{G} = -\frac{L}{N}$$

$$\Rightarrow \tan\theta'\tan\theta = -\frac{L}{N}.\frac{G}{E} = -\frac{\kappa_a}{\kappa_b} = -\frac{R_b}{R_a}$$

We know that $\tan\theta\tan\theta' = -\frac{R_b}{R_a}$ is the necessary and sufficient condition for the direction making angles $\theta,\ \theta'$ with x –axis to be parallel to conjugate diameters of the Dupin indicatrix at point P. This prove that theorem.

10.6 CONJUGATE SYSTEMS

Consider the family of curves

$$\phi(u, v) = const. \quad (1)$$

The direction $\frac{\delta u}{\delta v}$ of a curve belonging to this family is given by

$$\phi_1\delta u + \phi_2\delta v = 0$$

i.e.

$$\frac{\delta u}{\delta v} = -\frac{\phi_2}{\phi_1}$$

If $\frac{du}{dv}$ is a direction conjugate to the direction $\frac{\delta u}{\delta v}$, then

$$L\frac{du}{dv}\frac{\delta u}{\delta v} + M\left(\frac{du}{dv} + \frac{\delta u}{\delta v}\right) + N = 0$$

$$\Rightarrow L\frac{du}{dv}\left(-\frac{\phi_2}{\phi_1}\right) + M\left(\frac{du}{dv} - \frac{\phi_2}{\phi_1}\right) + N = 0$$

$$\Rightarrow \frac{du}{dv}\left(M - L\frac{\phi_2}{\phi_1}\right) + \left(N - M\frac{\phi_2}{\phi_1}\right) = 0$$

$$\Rightarrow (M\phi_1 - L\phi_2)du + (N\phi_1 - M\phi_2)dv = 0 \qquad (2)$$

(2) is a differential equation of the first order and first degree and therefore defines a one-parameter family of curves say,

$$\psi(u, v) = const$$

The two families of curves $\phi(u, v) = const$ and $\psi(u, v) = const$ are then said to form a conjugate system. At a point of intersection of two curves, one from each family, their directions are conjugate.

CHECK YOUR PROGRESS

Q 1. State the condition for two directions (du, dv) and $(\delta u, \delta v)$ to be conjugate.

Q 2. How do conjugate directions relate to the Dupin indicatrix ?

Q 3. Prove that the parametric curves are conjugate on the surface of revolution $x = u\cos v,\ y = u\sin v,\ z = f(u)$.

10.7 ASYMPTOTIC LINES

A direction at a point on the surface is called asymptotic if it is self-conjugate. An asymptotic line is a curve whose direction at every point is self-conjugate.

10.8 DIFFERENTIAL EQUATION OF ASYMPTOTIC LINES

Let at any point (u, v) the direction of an asymptotic line be (du, dv). Then (du, dv) is a self conjugate direction.

Therefore putting $\frac{du}{dv} = \frac{\delta u}{\delta v}$ in the condition of conjugacy of two directions i.e. in

$$L\frac{du}{dv}.\frac{\delta u}{\delta v} + 2M\left(\frac{du}{dv} + \frac{\delta u}{\delta v}\right) + N = 0$$

We have

$$L\left(\frac{du}{dv}\right)^2 + 2M\frac{du}{dv} + N = 0 \qquad (1)$$

or

$$Ldu^2 + 2Mdudv + Ndv^2 = 0 \qquad (2)$$

Equation (1) or (2) is differential equation of an asymptotic line, the equation (1) being $\frac{du}{dv}$, therefore these are in general two asymptotic directions through every point on the surface. They will be real and distinct if $M^2 - LN$ is positive i.e. if the Gaussian curvature K is negative and the point is hyperbolic. They are imaginary if K is positive and the point is elliptic. They are identical if K is zero i.e., the point is parabolic, in this case the surface is a developable and the single asymptotic line through a point is the generator.

Let $\boldsymbol{r} = \boldsymbol{r}(s)$ be a curve on the surface $\boldsymbol{r} = \boldsymbol{r}(u,v)$. Along the point on the curve, the surface normal $\boldsymbol{N}$ is function of s. We have

$$\frac{d\boldsymbol{N}}{ds} = \boldsymbol{N}_1\frac{du}{ds} + \boldsymbol{N}_2\frac{dv}{ds} \quad and \quad \frac{d\boldsymbol{r}}{ds} = \boldsymbol{t} = \boldsymbol{r}_1\frac{du}{ds} + \boldsymbol{r}_2\frac{dv}{ds}$$

$$\therefore \quad \frac{d\boldsymbol{N}}{ds}.\frac{d\boldsymbol{r}}{ds} = \left(\boldsymbol{N}_1\frac{du}{ds} + \boldsymbol{N}_2\frac{dv}{ds}\right).\left(\boldsymbol{r}_1\frac{du}{ds} + \boldsymbol{r}_2\frac{dv}{ds}\right)$$

$$= L\left(\frac{du}{ds}\right)^2 + 2M\left(\frac{du}{ds}\right)\left(\frac{dv}{ds}\right) + N\left(\frac{dv}{ds}\right)^2$$

But the direction coefficients of the direction at the point $\boldsymbol{r}$ of the curve $\boldsymbol{r} = \boldsymbol{r}(s)$ are $\left(\frac{du}{ds}, \frac{dv}{ds}\right)$. Thus the necessary and sufficient condition for the curve $\boldsymbol{r} = \boldsymbol{r}(s)$ be an asymptotic line is

$$L\left(\frac{du}{ds}\right)^2 + 2M\left(\frac{du}{ds}\right)\left(\frac{dv}{ds}\right) + N\left(\frac{dv}{ds}\right)^2 = 0$$

i.e.,

$$\frac{d\boldsymbol{N}}{ds}.\frac{d\boldsymbol{r}}{ds} = 0$$

Remark

We know that normal curvature at any point is given by

$$\frac{(Ldu^2 + 2Mdudv + Ndv^2)}{Edu^2 + 2Fdudv + Gdv^2}$$

We observe that it vanishes for the asymptotic directions [using (1)]. These are thus the direction of the asymptotes of the indicatrix and hence this name.

CHECK YOUR PROGRESS

Q 4. Prove that at any point of the surface, the sum of the radii of normal curvatures in conjugate directions is constant.

Q 5. Show that the asymptotic lines of helicoids

$$x = u\cos\theta, y = u\sin\theta, z = c\theta$$

consists of the generators and the curves of intersection with coaxial right cylinders.

Q 6. Show that all straight lines on a surface are asymptotic lines.

10.9 LET US SUM UP

- The direction (l_1, m_1) and (l_2, m_2) at P are conjugate if and only if

$$Ll_1l_2 + M(l_2m_2 + l_2m_1) + n.m_1m_2 = 0$$

- The necessary and sufficient condition that the parametric curves have conjugate direction that $M = 0$.
- Principal direction at a point are always orthogonal conjugate.
- The direction which are self conjugate are called asymptotic direction and the curve whose tangents are along asymptotic direction are called asymptotic lines.
- The necessary and sufficient condition for the curve $r = r(s)$ be an asymptotic line is

$$L\left(\frac{du}{ds}\right)^2 + 2M\left(\frac{du}{ds}\right)\left(\frac{dv}{ds}\right) + N\left(\frac{dv}{ds}\right)^2 = 0$$

10.10 ANSWERS TO CHECK YOUR PROGRESS

Answer to Q 1.

$$L\frac{du}{dv}\frac{\delta u}{\delta v}+M\left(\frac{du}{dv}+\frac{\delta u}{\delta v}\right)+N=0$$

Answer to Q 2. They are conjugate diameters of the Dupin indicatrix

Answer to Q 3. The position vector $\boldsymbol{r}$ of any point P on the given surface is given by

$$\boldsymbol{r}=\{u\cos v,u\sin v,f(u)\}$$

$$\therefore\quad \boldsymbol{r}_1=(\cos v,\sin v,f_1);\quad \boldsymbol{r}_2=(-u\sin v,u\cos v,0)$$

Now

$$(\boldsymbol{r}_1\times\boldsymbol{r}_2)=(0-f_1u\cos v,-f_1u\sin v-0,u\cos^2 v+u\sin^2 v)$$

$$=(-f_1u\cos v,-f_1u\sin v,u)$$

$$\boldsymbol{r}_1.\boldsymbol{r}_2=-u\sin v\cos v+u\sin v\cos v+0=0$$

$$\boldsymbol{r}_{12}=(-\sin v,\cos v,0)$$

Again

$$F=\boldsymbol{r}_1.\boldsymbol{r}_2=0$$

and

$$HM=(\boldsymbol{r}_1\times\boldsymbol{r}_2).\boldsymbol{r}_{12}$$

$$=(-f_1u\cos v,-f_1u\sin v,u).(-\sin v,\cos v,0)$$

$$=f_1u\sin v\cos v-f_1u\sin v\cos v+0=0$$

$$\Rightarrow M=0$$

$$F=0\ \ and\ \ M=0$$

imply that the parametric curves are conjugate.

Answer to Q 4. Let the lines of curvature be taken as parametric curves, thus $F=0,M=0$. Let (du,dv) and $(\delta u,\delta v)$ be two conjugate directions through any point $P(u,v)$.

$$\therefore\quad Ldu\delta u+M(du\delta v+\delta udv)+Ndv\delta v)=0$$

$$\Rightarrow Ldu\delta u+Ndv\delta v=0\quad[\because\ M=0]$$

$$\Rightarrow\frac{\delta u}{Ndv}=\frac{\delta v}{-Ldu}=\lambda\ \ (say)\qquad(1)$$

Again the normal curvature κ_n at P in the direction (du,dv) is given by

$$\kappa_n = \frac{Ldu^2 + 2Mdudv + Ndv^2}{Edu^2 + 2Fdudv + Gdv^2} = \frac{Ldu^2 + Ndv^2}{Edu^2 + Gdv^2}$$

Suppose ρ_1 and ρ_2 are the radii of normal curvatures at P in the conjugate direction (du, dv) and $(\delta u, \delta v)$ respectively.

$$\therefore \quad \rho_1 = \frac{Edu^2 + Gdv^2}{Ldu^2 + Ndv^2} \qquad (2)$$

$$\rho_2 = \frac{E\delta u^2 + G\delta v^2}{L\delta u^2 + N\delta v^2} \qquad (3)$$

Now by substituting the values of $\delta u, \delta v$ from (1) IN (3), we get

$$\rho_2 = \frac{EN^2dv^2 + GL^2du^2}{LN^2dv^2 + NL^2du^2} = \frac{EN^2dv^2 + GL^2du^2}{LN(Ndv^2 + Ldu^2)} \qquad (4)$$

Adding (3) and (4),

$$\rho_1 + \rho_2 = \frac{Edu^2 + Gdv^2}{Ldu^2 + Ndv^2} + \frac{EN^2dv^2 + GL^2du^2}{LN(Ndv^2 + Ldu^2)}$$

$$= \frac{ELNdu^2 + GLNdv^2 + EN^2dv^2 + GL^2du^2}{LN(Ldu^2 + Ndv^2)}$$

$$= \frac{EN(Ldu^2 + Ndv^2) + GL(Ldu^2 + Ndv^2)}{LN(Ldu^2 + Ndv^2)}$$

$$= \frac{(EN + GL)(Ldu^2 + Ndv^2)}{LN(Ldu^2 + Ndv^2)}$$

$$= \frac{EN + GL}{LN}$$

$=$ independent of the conjugate directions (du, dv) and $(\delta u, \delta v) =$ constant

Answer to Q 5. The position vector $\boldsymbol{r}$ of any point on the surface is given by

$$\boldsymbol{r} = (u\cos\theta, u\sin\theta, c\theta)$$

We have shown that

$$L = 0, M = -\frac{c}{H}, V = 0, F = 0$$

Therefore the asymptotic lines namely

$$Ldu^2 + 2Mdud\theta + Nd\theta^2 = 0$$

reduces to

$$\frac{c}{H}dud\theta = 0 \quad or \quad dud\theta = 0$$

i.e.,

$$du = 0, \qquad d\theta = 0$$

giving

$$u = \text{constant}; \quad \theta = \text{constant}$$

which obviously are curves of intersection with coaxial right cylinders and the generators respectively.

Answer to Q 6. Let the equation of the surface be given by

$$\boldsymbol{r} = \boldsymbol{r}(u, v) \qquad (1)$$

and let $\boldsymbol{r} = \boldsymbol{r}(s)$ be a straight line lying on (1) we have

$$\boldsymbol{N}.\boldsymbol{t} = 0 \ \textit{where} \ \ \boldsymbol{t} = \frac{d\boldsymbol{r}}{ds}$$

Differentiating this relation w.r.t. 's', we get

$$\boldsymbol{N}.\frac{d\boldsymbol{t}}{ds} + \frac{d\boldsymbol{N}}{ds}.\boldsymbol{t} = 0$$

or

$$\boldsymbol{N}.(\kappa \boldsymbol{n}) + \frac{d\boldsymbol{N}}{ds}.\frac{d\boldsymbol{t}}{ds} = 0 \qquad (2)$$

where κ is curvature of the curve $\boldsymbol{r} = \boldsymbol{r}(s)$.

Since the curve is a straight line, so $\kappa = 0$, therefore (2) reduces to

$$\frac{d\boldsymbol{N}}{ds}.\frac{d\boldsymbol{r}}{ds} = 0$$

Hence $\boldsymbol{r} = \boldsymbol{r}(s)$ is an asymptotic line on the given surface, thus every straight line on a surface is an asymptotic line.

O 10.11 FURTHER READINGS

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2. Weatherburn, C. E. Differential Geometry of Three Dimensions, Cambridge University Press, 2016
3. Pressley, A. Elementary Differential Geometry. Springer, 2002
4. Gupta, P. P., Malik, G. S., Pundir, S. K. Differential Geometry, Pragati Prakashan, 2010

10.12 MODEL QUESTIONS

1. Given the surface

$$\boldsymbol{r}(u,v) = (u, v, u^2 - v^2)$$

(i) Verify whether the parametric curves u= constant and $v =$ constant are conjugate.

(ii) Find the condition for two arbitrary directions to be conjugate at a point.

2. Let a surface be given by

$$\boldsymbol{r}(u,v) = (u\cos v, u\sin v, v)$$

(i) Check whether the coordinate curves form a conjugate system.

(ii) Determine the differential equation of asymptotic lines.

3. Derive the condition for conjugacy of two parametric curves on a surface.

4. Define asymptotic lines and derive their differential equation.

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