UNIT 11: ASYMPTOTIC LINES

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11.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- define asymptotic lines on a regular surface and explain their geometric interpretation in differential geometry.
- understand the condition for asymptotic lines to be orthogonal.
- familiarize yourself with the formulae for curvature and torsion of an asymptotic line.
- learn the theorem of Beltrami and Enneper.
- solve standard problems related to asymptotic lines on special surfaces such as ruled surfaces and surfaces of revolution.

11.2 INTRODUCTION

In differential geometry, asymptotic lines on a regular surface are curves whose tangent directions at every point correspond to asymptotic directions, that is, directions

along which the normal curvature vanishes. These curves exist only at points where the Gaussian curvature is non-positive, and they play a fundamental role in understanding the intrinsic and extrinsic geometry of surfaces. On a ruled surface, which is generated by a one-parameter family of straight lines (generators), one family of asymptotic lines is often formed by the generators themselves, since straight lines on such surfaces frequently satisfy the condition of zero normal curvature. The curvature and torsion of an asymptotic line can be expressed in terms of the fundamental forms of the surface, with the normal curvature being zero. A central theoretical result connecting asymptotic lines with the geometry of surfaces is the theorem of Beltrami and Enneper, which establishes important relationships between asymptotic lines and lines of curvature, particularly characterizing surfaces for which these two families of curves coincide under specific geometric conditions.

11.3 ASYMPTOTIC LINES

The directions which are self conjugate are called asymptotic directions and the curves whose tangents are along asymptotic direction are called asymptotic lines.

11.4 CONDITION FOR ASYMPTOTIC LINES TO BE ORTHOGONAL

We know that the two directions given by

$$Pdu^2 + Qdudv + Rdv^2 = 0$$

are orthogonal if and only if

$$ER - 2FQ + GP = 0 \qquad (1)$$

Again, the differential equation giving asymptotic direction is

$$Ldu^2 + 2Mdudv + Ndv^2 = 0$$

Thus the asymptotic direction are orthogonal if and only if

$$EN - 2FM + GL = 0 \qquad [using\ (1)]$$

which gives first curvature $(J) = 0$.

Also we know that first curvature is zero for the surface to be minimal and hence if this condition holds at every point of the surface, the surface is minimal surface.

Theorem 1: The necessary and sufficient condition that the parametric curves be asymptotic lines are $L = 0,\ N = 0,\ M \neq 0$.

Proof: The differential equation of the parametric curve is

$$du\,dv = 0 \qquad (1)$$

and the differential equation of the asymptotic line is

$$Ldu^2 + 2Mdudv + Ndv^2 = 0 \qquad (2)$$

In case the parametric curves are asymptotic lines, equations (1) and (2) must be identical. On comparing the coefficients of du^2, $dudv$ and dv^2, we get

$$L = 0, \qquad N = 0, \qquad M \neq 0$$

Hence the conditions are necessary.

Conversely if $L = 0, N = 0, M \neq 0$, the equation (2) reduces to equation (1) i.e., the asymptotic lines are the parametric curves.

Hence the conditions are sufficient.

Theorem 2: Along a curved asymptotic line on a surface the osculating plane coincides with the tangent plane to the surface.

Proof: Let $\boldsymbol{r} = \boldsymbol{r}(s)$ be a curved asymptotic line lying on the surface $\boldsymbol{r} = \boldsymbol{r}(u, v)$. By a curved asymptotic line we mean that the curve $\boldsymbol{r} = \boldsymbol{r}(s)$ is not a straight line.

We have

$$\boldsymbol{N} \cdot \boldsymbol{t} = 0, \qquad where\ \boldsymbol{t} = \frac{d\boldsymbol{r}}{ds}$$

Differentiating this relation with respect to s, we get

$$\frac{d\boldsymbol{N}}{ds} \cdot \boldsymbol{t} + \boldsymbol{N} \cdot \frac{d\boldsymbol{t}}{ds} = 0$$

$$\Rightarrow \frac{d\boldsymbol{N}}{ds} \cdot \frac{d\boldsymbol{r}}{ds} + \boldsymbol{N} \cdot (\kappa \boldsymbol{n}) = 0$$

$$\Rightarrow \boldsymbol{N} \cdot (\kappa \boldsymbol{n}) = 0 \quad \left[\because \boldsymbol{r} = \boldsymbol{r}(s)\ is\ an\ asymptotic\ line\ means\ that\ \frac{d\boldsymbol{N}}{ds} \cdot \frac{d\boldsymbol{r}}{ds} = 0\right]$$

$$\Rightarrow \boldsymbol{N} \cdot \boldsymbol{n} = 0 \qquad [\because \kappa \neq 0]$$

Now $\boldsymbol{N} \cdot \boldsymbol{t} = 0$ and $\boldsymbol{N} \cdot \boldsymbol{n} = 0$. Therefore $\boldsymbol{N}$ is perpendicular to both $\boldsymbol{t}$ and $\boldsymbol{n}$. Thus $\boldsymbol{n}$ is parallel to $\boldsymbol{t} \times \boldsymbol{n} = \boldsymbol{b}$. Therefore $\boldsymbol{N} = \pm \boldsymbol{b}$ i.e. at any point of a curved asymptotic line the binormal is also normal to the surface . We may take the direction of $\boldsymbol{b}$ so that

$$\boldsymbol{N} = \boldsymbol{b}$$

The equation of osculating plane to $\boldsymbol{r} = \boldsymbol{r}(s)$ at any point $\boldsymbol{r}$ on it is

$$(\boldsymbol{R} - \boldsymbol{r}) \cdot \boldsymbol{b} = 0$$

The equation of tangent plane to the surface at this point is

$$(\boldsymbol{R}-\boldsymbol{r})\cdot\boldsymbol{N}=0$$

Now $\boldsymbol{N}=\pm\boldsymbol{b}$ implies that the two planes coincide.

Hence the theorem.

Theorem 3: If the osculating plane at every point of a curve on a surface coincides with the tangent plane to the surface, then the curve must be an asymptotic line on the surface.

Proof: Let $\boldsymbol{r}=\boldsymbol{r}(s)$ be a curve lying on the surface $\boldsymbol{r}=\boldsymbol{r}(u,v)$. Suppose the osculating plane at every point of this curve coincides with the tangent plane to the surface. Then at any point of this curve, we have

$\boldsymbol{N}$ is parallel to $\boldsymbol{b}$

i.e.

$$\boldsymbol{N}=\pm\boldsymbol{b}$$

Therefore

$$\boldsymbol{N}\cdot\boldsymbol{n}=(\pm\boldsymbol{b})\cdot\boldsymbol{n}=0$$

Now

$$\boldsymbol{N}\cdot\boldsymbol{t}=0$$

Differentiating with respect to s, we get

$$\frac{d\boldsymbol{N}}{ds}\cdot\boldsymbol{t}+\boldsymbol{N}\cdot\frac{d\boldsymbol{t}}{ds}=0$$

$$\Rightarrow\frac{d\boldsymbol{N}}{ds}\cdot\frac{d\boldsymbol{r}}{ds}+\boldsymbol{N}\cdot(\kappa\boldsymbol{n})=0$$

$$\Rightarrow\frac{d\boldsymbol{N}}{ds}\cdot\frac{d\boldsymbol{r}}{ds}=0 \qquad [\because\ \boldsymbol{N}\cdot\boldsymbol{n}=0]$$

Hence the curve $\boldsymbol{r}=\boldsymbol{r}(s)$ is an asymptotic line on the surface

$$\boldsymbol{r}=\boldsymbol{r}(u,v)$$

11.5 ASYMPTOTIC LINES ON A RULED SURFACE

Before proceeding further, we would like to calculate the fundamental coefficients for a ruled surface. Equation of the ruled surface is given by

$$\boldsymbol{R}=\boldsymbol{r}(s)+v\boldsymbol{g}(s)$$

$$\therefore \quad \boldsymbol{R}_1 = \frac{\partial \boldsymbol{R}}{\partial s} = \frac{d\boldsymbol{r}}{ds} + v\frac{d\boldsymbol{g}}{ds} = \boldsymbol{r}' + v\boldsymbol{g}'$$

$$\boldsymbol{R}_2 = \frac{\partial \boldsymbol{R}}{\partial v} = \boldsymbol{g}(s) = \boldsymbol{g}$$

$$\Rightarrow E = \boldsymbol{R}_1 \cdot \boldsymbol{R}_1 = (\boldsymbol{r}' + v\boldsymbol{g}') \cdot (\boldsymbol{r}' + v\boldsymbol{g}') = \boldsymbol{r}'^2 + 2v\boldsymbol{r}' \cdot \boldsymbol{g}' + v^2\boldsymbol{g}'^2$$

$$F = \boldsymbol{R}_1 \cdot \boldsymbol{R}_2 = (\boldsymbol{r}' + v\boldsymbol{g}') \cdot \boldsymbol{g} = \boldsymbol{r}' \cdot \boldsymbol{g} + v\boldsymbol{g}' \cdot \boldsymbol{g} = \boldsymbol{r}' \cdot \boldsymbol{g}$$

$$G = \boldsymbol{R}_2 \cdot \boldsymbol{R}_2 = \boldsymbol{g} \cdot \boldsymbol{g} = 1$$

$$H^2 = EG - F^2 = (r'^2 + 2v\boldsymbol{r}' \cdot \boldsymbol{g}' + v^2\boldsymbol{g}'^2)(1) - (\boldsymbol{r}' \cdot \boldsymbol{g})^2$$

$$\boldsymbol{N} = \frac{\boldsymbol{R}_1 \times \boldsymbol{R}_2}{H} = \frac{(\boldsymbol{r}' + v\boldsymbol{g}') \times \boldsymbol{g}}{H} = \frac{\boldsymbol{r}' \times \boldsymbol{g} + v\boldsymbol{g}' \times \boldsymbol{g}}{H}$$

$$\Rightarrow H\boldsymbol{N} = \boldsymbol{r}' \times \boldsymbol{g} + v\boldsymbol{g}' \times \boldsymbol{g}$$

Further

$$\boldsymbol{R}_{11} = \boldsymbol{r}'' + v\boldsymbol{g}'', \qquad \boldsymbol{R}_{12} = g', \qquad \boldsymbol{R}_{22} = 0$$

$$\Rightarrow HL = [\boldsymbol{R}_{11}, \boldsymbol{R}_1, \boldsymbol{R}_2] = [\boldsymbol{r}'' + v\boldsymbol{g}'', \boldsymbol{r}' + v\boldsymbol{g}', \boldsymbol{g}]$$

$$= [\boldsymbol{r}'', \boldsymbol{r}', \boldsymbol{g}] + [\boldsymbol{g}'', \boldsymbol{r}', \boldsymbol{g}]v + [\boldsymbol{r}'', \boldsymbol{g}', \boldsymbol{g}]v + [\boldsymbol{g}'', \boldsymbol{g}', \boldsymbol{g}]v^2$$

$$HM = [\boldsymbol{R}_{11}, \boldsymbol{R}_1, \boldsymbol{R}_2] = [\boldsymbol{g}', \boldsymbol{r}' + v\boldsymbol{g}', \boldsymbol{g}]$$

$$= [\boldsymbol{g}', \boldsymbol{r}', \boldsymbol{g}] + [\boldsymbol{g}', \boldsymbol{g}', \boldsymbol{g}]v = [\boldsymbol{r}', \boldsymbol{g}, \boldsymbol{g}']$$

$$HN = [\boldsymbol{R}_{22}, \boldsymbol{R}_1, \boldsymbol{R}_2] = 0$$

$$\mathrm{K} = \frac{LN - M^2}{H^2} = -\frac{M^2}{H^2}$$

$$\Rightarrow \mathrm{K} \leq 0$$

i.e. for a ruled surface, K is never positive.

Now differential equation of asymptotic lines on a surface is

$$Ldu^2 + 2Mdudv + Ndv^2 = 0$$

But for a ruled surface $N = 0$, so asymptotic lines on the ruled surface are given by

$$Ldu^2 + 2Mdudv = 0 \Rightarrow du(Ldu + 2Mdv) = 0$$

Hence, one family of asymptotic lines on a ruled surface is given by $du = 0$ i.e. $u =$ constant. But the parametric curves $u =$ constant are the generators of the ruled surface. Thus the generators of a ruled surface constitute one family of asymptotic lines.

Other family of asymptotic lines is given by

$$(Ldu + 2Mdv) = 0 \Rightarrow \frac{dv}{du} = -\frac{L}{2M}$$

$$\Rightarrow \frac{dv}{du} = \frac{\{[\boldsymbol{r}'', \boldsymbol{r}', \boldsymbol{g}] + [\boldsymbol{g}'', \boldsymbol{r}', \boldsymbol{g}]v + [\boldsymbol{r}'', \boldsymbol{g}', \boldsymbol{g}]v + [\boldsymbol{g}'', \boldsymbol{g}', \boldsymbol{g}]v^2\}}{2[\boldsymbol{r}', \boldsymbol{g}, \boldsymbol{g}']}$$

which is of the form

$$\frac{dv}{du} = A + Bv + Cv^2$$

where A, B, C are functions of s alone. The most general solution of this equation is of the form $v = \frac{dP+Q}{dR+S}$, where P, Q, R, S are functions of s alone and d is any arbitrary constant. The solution result in v is called the family of curved asymptotic lines, d being different for each member of the family.

Example 1: Show that the cross ratio of the four points, in which a generator is cut by four given curved asymptotic lines, is the same for all generators.

Solution: Let us consider the intersections of four asymptotic lines d_1, d_2, d_3, d_4 with a given generator $u = u_0$. Let v_1, v_2, v_3, v_4 be the values of the parameter v at the four points, then we have

$$v_1 = \frac{d_1P + Q}{d_1R + S}, v_2 = \frac{d_2P + Q}{d_2R + Q}, v_3 = \frac{d_3P + Q}{d_3R + S}, v_4 = \frac{d_4P + Q}{d_4R + S}$$

$$\Rightarrow v_1 - v_2 = \frac{(d_1 - d_2)(PS - QR)}{(d_1R + S)(d_2R + S)}$$

$$\Rightarrow \frac{(v_1 - v_2)(v_3 - v_4)}{(v_1 - v_3)(v_2 - v_4)} = \frac{(d_1 - d_2)(d_3 - d_4)}{(d_1 - d_3)(d_3 - d_4)} = independent\ of\ u_0$$

Also v is the distance measured along the generators from the directrix which means that $v_1 - v_2$ becomes the distance between the points v_1 and v_2. Thus the cross ratio of the four points, in which a generator is cut by four given curved asymptotic lines, is the same for all generators.

CHECK YOUR PROGRESS

Q 1. Prove that the directions of lines of curvature bisect the asymptotic directions.

Q 2. Prove that the normal curvature in a direction perpendicular to an asymptotic line is twice the mean normal curvature (first curvature).

Q 3. Show that the asymptotic lines are orthogonal if and only if the surface is minimal.

Q 4. Show that the parametric curves are conjugate on the following surfaces:

(i) a surface of revolution

$$x = u\cos v, \qquad y = u\sin v, \qquad z = f(u)$$

(ii) the surface generated by the tangents to a curve, on which

$$\boldsymbol{R} = \boldsymbol{r} + u\boldsymbol{t}$$

(iii) the surface $z = f(x) + F(y)$, where x, y are parameters.

Q 5. Show that the parametric curves are asymptotic lines on the surface

$$x = u\cos v, \qquad y = u\sin v, \qquad z = cv$$

Q 6. Find the asymptotic lines on the catenoid of revolution $u = c\cosh\frac{z}{c}$.

11.6 FORMULAE FOR CURVATURE AND TORSION OF AN ASYMPTOTIC LINE

We know that the unit binormal $\boldsymbol{b}$ to an asymptotic line is the unit surface normal $\boldsymbol{N}$, i.e.

$$\boldsymbol{b} = \boldsymbol{N} \qquad (1)$$

Differentiating (1) w.r.t. to arc length s of the asymptotic line, we get

$$\boldsymbol{b}' = \boldsymbol{N}' \quad or \quad -\tau\boldsymbol{n} = \boldsymbol{N}' \qquad (2)$$

Taking dot product of both sides of (2) with $\boldsymbol{n}$, we have

$$-\tau\boldsymbol{n}\cdot\boldsymbol{n} = \boldsymbol{N}'\cdot\boldsymbol{n}$$

$$\Rightarrow \tau = -\boldsymbol{N}'\cdot\boldsymbol{n}$$

But

$$\boldsymbol{n}\times\boldsymbol{b} = \boldsymbol{t} \quad or \quad \boldsymbol{n} = \boldsymbol{N}\times\boldsymbol{t}$$

$$\therefore \quad \tau = -\boldsymbol{N}'\cdot\boldsymbol{N}\times\boldsymbol{t} = [\boldsymbol{N}, \boldsymbol{N}', \boldsymbol{t}]$$

Again we know that $\boldsymbol{t}' = \kappa\boldsymbol{n}$, so that taking dot product of both sides with $\boldsymbol{n}$, we get

$$\kappa = \boldsymbol{t}'\cdot\boldsymbol{n} = \boldsymbol{t}'\cdot\boldsymbol{N}\times\boldsymbol{t} = [\boldsymbol{N}, \boldsymbol{t}, \boldsymbol{t}']$$

$$\kappa = [\boldsymbol{N}, \boldsymbol{r}', \boldsymbol{r}'']$$

11.7 THEOREM OF BELTRAMI AND ENNEPER

Statement: At a point on a surface where the Gaussian curvature is negative and equal to K, the torsion of the asymptotic lines is $\pm\sqrt{(-\mathrm{K})}$.

Proof: The torsion τ of an asymptotic line is given by $\tau = [N, N', r']$.

Now

$$N' = \frac{dN}{ds} = \frac{\partial N}{\partial u}\frac{du}{ds} + \frac{\partial N}{\partial v}\frac{dv}{ds} = N_1 u' + N_2 v'$$

$$r' = r_1 u' + r_2 v'$$

$$\therefore \quad N' \times r' = (N_1 u' + N_2 v') \times (r_1 u' + r_2 v')$$

$$= (N_1 \times r_1)u'^2 + \{(N_1 \times r_2) + (N_2 \times r_1)\}u'v' + (N_2 \times r_2)v'^2$$

$$\therefore \quad \tau = N \cdot (N' \times r') = [N, N_1, r_1]u'^2 + \{[N, N_1, r_2] + [N, N_2, r_1]\}u'v' + [N, N_2, r_2]v'^2$$

$$= \frac{EM - FL}{H}u'^2 + \frac{EN - GL}{H}u'v' + \frac{FN - GM}{H}v'^2$$

Now we suppose that the asymptotic lines are taken as parametric curves.

Then

$$L = 0 = N, M \neq 0$$

$$\therefore \quad \tau = \frac{M}{H}(Eu'^2 - Gv'^2)$$

For the asymptotic line $u =$ constant, we have $u' = 0$.

$\therefore$ For the asymptotic line $u =$ constant, $\tau = -\frac{M}{H}Gv'^2$

Again from first fundamental form, we have

$$1 = Eu'^2 + 2DFu'v' + Gv'^2$$

From this for the curve $u =$ constant, we have

$$u' = 0 \quad and \quad Gv'^2 = 1$$

Therefore for $u =$ constant

$$\tau = -\frac{M}{H} \qquad (1)$$

Again for the asymptotic line $v =$ constant, we have

$$v' = 0 \quad and \quad Eu'^2 = 1$$

Therefore for $v =$ constant, we have

$$\tau = \frac{M}{H} \qquad (2)$$

Again Gaussian curvature

$$\mathrm{K} = \frac{LN - M^2}{H^2} = -\frac{M^2}{H^2} \quad (\because \ L = 0, N = 0)$$

or

$$\frac{M}{H} = \pm\sqrt{(-\mathrm{K})}$$

Thus from (1) and (2), we have

$$\tau = \pm\sqrt{(-K)}$$

i.e.

$$\tau^2 = K$$

CHECK YOUR PROGRESS

Q 7. Prove that on the surface $z = f(x, y)$, the asymptotic lines are

$$rdx^2 + 2sdxdy + tdy^2 = 0$$

and their torsions are

$$\pm\frac{\sqrt{s^2 - rt}}{(1 + p^2 + q^2)}$$

Q 8. Prove that on the surface of revolution $x = u\cos v, y = u\sin v, z = f(u)$, the asymptotic lines are

$$f_{11}du^2 + uf_1dv^2 = 0$$

and their torsions are

$$\frac{\pm\sqrt{(-u^3 f_1 f_{11})}}{u^2(1 + f_1^2)}$$

Q 9. Find the asymptotic lines and their torsions on the surface generated by the binormals to a twisted curve.

11.8 LET US SUM UP

- The chapter introduces asymptotic lines on a regular surface as curves whose tangents follow directions where the normal curvature vanishes, which occur only at points of non positive Gaussian curvature and are important for understanding surface geometry.
- It describes the condition under which asymptotic directions (given by the second fundamental form) are orthogonal, showing that orthogonality is characterized by the relation $EN - 2FM + GL = 0$, which implies vanishing first curvature and characterizes minimal surfaces when it holds everywhere.
- The unit establishes the necessary and sufficient conditions for parametric

curves to be asymptotic lines, namely $L = 0, N = 0$ and $M \neq 0$, by comparing the differential equations of parametric curves and asymptotic lines.

- It proves that along a curved asymptotic line the osculating plane of the curve coincides with the tangent plane of the surface, by showing that the surface normal vector and the binormal of the curve are parallel at every point of the line.
- Throughout, the chapter connects these concepts with special surfaces (such as ruled surfaces) and prepares the learner to compute curvature and torsion of asymptotic lines and to understand the role of the Beltrami–Enneper theorem relating asymptotic lines to lines of curvature.

K 11.9 ANSWERS TO CHECK YOUR PROGRESS

Answer to Q 1. Let us take parametric curves as asymptotic lines. Then $L = 0, N = 0, M \neq 0$. The differential equation of the lines of curvature is

$$(EM - FL)du^2 + (EN - GL)dudv + (FN - GM)dv^2 = 0$$

If $L = 0, N = 0, M \neq 0$, it becomes

$$Edu^2 - Gdv^2 = 0$$

Now we know that the directions given by $Edu^2 - Gdv^2 = 0$ bisect the angles between the parametric curves which are asymptotic lines in this case. Hence the directions of lines of curvature bisect the asymptotic directions.

Answer to Q 2. Let κ_a, κ_b be the principal curvatures at the point P.

Let (du, dv) be the direction of an asymptotic line through P which makes an angle ψ with the principal direction through P in which normal curvature is κ_n.

The normal curvature κ_n in the direction (du, dv) is given by

$$\kappa_n = \frac{Ldu^2 + 2Mdudv + Ndv^2}{Edu^2 + 2Fdudv + Gdv^2}$$

$= 0$ [$\because$ (du, dv) is an asymptotic direction]

$\therefore$ By Euler's theorem, we have

$$0 = \kappa_a \cos^2\psi + \kappa_b \sin^2\psi \qquad (1)$$

Let κ_1 be the normal curvature at P in a direction perpendicular to (du, dv).

Then by Euler's theorem, we have

$$\kappa_1 = \kappa_a \cos^2\left(\frac{\pi}{2}+\psi\right) + \kappa_b \sin^2\left(\frac{\pi}{2}+\psi\right)$$

$$= \kappa_a \sin^2\psi + \kappa_b \cos^2\psi \qquad (2)$$

Adding (1) and (2), we get

$$\kappa_1 = \kappa_a + \kappa_b = 2\left(\frac{\kappa_a + \kappa_b}{2}\right)$$

$$= 2 \text{ (mean normal curvature at } P\text{)}$$

Answer to Q 3. The asymptotic lines are given by

$$Ldu^2 + 2Mdudv + Ndv^2 = 0 \qquad (1)$$

The two directions given by (1) will be orthogonal if and only if

$$EN - 2FM + GL = 0$$

Also the surface is minimal if and only if

$$EN - 2FM + GL = 0$$

Hence the asymptotic lines are orthogonal if and only if the surface is minimal.

Answer to Q 4. (i) The equation of the surface is

$$\boldsymbol{r} = (u\cos v, u\sin v, f(u))$$

We have

$$\boldsymbol{r}_1 = (\cos v, \sin v, f_1)$$

where $f_1 = \frac{df}{du}$

$$\boldsymbol{r}_2 = (-u\sin v, u\cos v, 0)$$

$$\therefore \quad \boldsymbol{r}_1 \times \boldsymbol{r}_2 = (-uf_1\cos v, -uf_1\sin v, u)$$

Also

$$\boldsymbol{r}_{12} = (-\sin v, \cos v, 0)$$

Now

$$HM = \boldsymbol{r}_{12} \cdot (\boldsymbol{r}_1 \times \boldsymbol{r}_2) = 0$$

$$\therefore \quad M = 0$$

We know that $M = 0$ is a necessary and sufficient condition for parametric curves to be conjugate.

Hence on the given surface the parametric curves are conjugate.

(ii) Let $\boldsymbol{r} = \boldsymbol{r}(s)$ be a space curve. The equation of the surface generated by tangents to this curve can be taken as

$$\boldsymbol{R} = \boldsymbol{r} + u\boldsymbol{t}$$

In this equation u and s are the parameters. Note that $\boldsymbol{r}$ and $\boldsymbol{t}$ are both functions of s.

We have

$$\boldsymbol{R}_1 = \frac{\partial \boldsymbol{R}}{\partial u} = \boldsymbol{t}, \boldsymbol{R}_2 = \frac{\partial \boldsymbol{R}}{\partial s} = \frac{d\boldsymbol{r}}{ds} + u\frac{d\boldsymbol{t}}{ds} = \boldsymbol{t} + u\kappa\boldsymbol{n}$$

$$\therefore \quad \boldsymbol{R}_1 \times \boldsymbol{R}_2 = \boldsymbol{t} \times (\boldsymbol{t} + u\kappa\boldsymbol{n}) = u\kappa\boldsymbol{b}$$

Also

$$\boldsymbol{R}_{12} = \frac{d\boldsymbol{t}}{ds} = \kappa\boldsymbol{n}$$

Now

$$HM = \boldsymbol{R}_{12} \cdot (\boldsymbol{R}_1 \times \boldsymbol{R}_2) = (\kappa\boldsymbol{n}) \cdot (u\kappa\boldsymbol{b}) = 0$$

$$\therefore \quad M = 0$$

Hence on the given surface the parametric curves are conjugate.

(iii) The equation of surface is

$\boldsymbol{r} = \big(x, y, f(x) + F(y)\big)$, x and y are parameters.

We have

$$\boldsymbol{r}_1 = \frac{\partial \boldsymbol{r}}{\partial x} = \big(1, 0, f'(x)\big)$$

$$\boldsymbol{r}_2 = \frac{\partial \boldsymbol{r}}{\partial y} = \big(0, 1, F'(y)\big)$$

$$\therefore \quad \boldsymbol{r}_1 \times \boldsymbol{r}_2 = (-f'(x), -F'(y), 1)$$

Also

$$\boldsymbol{r}_{12} = (0, 0, 0)$$

Now

$$MH = \boldsymbol{r}_{12} \cdot (\boldsymbol{r}_1 \times \boldsymbol{r}_2) = 0$$

$$\therefore \quad M = 0, \qquad since\ H \neq 0$$

Hence on the given surface the parametric curves are conjugate.

Answer to Q 5. The equation of the given surface is

$$\boldsymbol{r} = (u\cos v, u\sin v, cv)$$

It can be easily calculated that for this surface

$$L = 0, M = -\frac{c}{H}, N = 0$$

Also we know that the necessary and sufficient conditions for parametric curvesto be asymptotic lines are $L = 0, N = 0, M \neq 0$.

Hence on the given surface the parametric curves are asymptotic lines.

Answer to Q 6. The equation of the catenoid of revolution is

$$\sqrt{x^2+y^2} = c\cosh\frac{z}{c} \text{ because } u = \sqrt{x^2+y^2}$$

Let the parametric equations be

$$x = u\cos v, y = u\sin v, z = c\cosh^{-1}\frac{u}{c}$$

In vector form, the equation is

$$\boldsymbol{r} = \left(u\cos v, u\sin v, c\cosh^{-1}\frac{u}{c}\right)$$

We have

$$r_1 = \left(\cos v, \sin v, \frac{c}{\sqrt{u^2-c^2}}\right)$$

$$\boldsymbol{r}_2 = (-u\sin v, u\cos v, 0)$$

$$\therefore \quad \boldsymbol{r}_1 \times \boldsymbol{r}_2 = \left(-\frac{uc\cos v}{\sqrt{u^2-c^2}}, \frac{-uc\sin v}{\sqrt{u^2-c^2}}, u\right)$$

Now

$$\boldsymbol{r}_{11} = \left(0, 0, -\frac{cu}{(u^2-c^2)^{\frac{3}{2}}}\right)$$

$$\boldsymbol{r}_{12} = (-\sin v, \cos v, 0), \boldsymbol{r}_{22} = (-u\cos v, -u\sin v, 0)$$

$$\therefore LH = \boldsymbol{r}_{11}\cdot(\boldsymbol{r}_1 \times \boldsymbol{r}_2) = -\frac{cu^2}{(u^2-c^2)^{\frac{3}{2}}}$$

$$MH = \boldsymbol{r}_{12}\cdot(\boldsymbol{r}_1 \times \boldsymbol{r}_2) = 0$$

$$NH = \boldsymbol{r}_{22}\cdot(\boldsymbol{r}_1 \times \boldsymbol{r}_2) = \frac{cu^2}{\sqrt{u^2-c^2}}$$

The asymptotic lines are given by

$$Ldu^2 + 2Mdudv + Ndv^2 = 0$$

$$\Rightarrow -\frac{cu^2}{(u^2-c^2)^{\frac{3}{2}}}du^2 + \frac{cu^2}{\sqrt{u^2-c^2}}dv^2 = 0$$

$$\Rightarrow \frac{du^2}{u^2-c^2} - dv^2 = 0$$

$$\Rightarrow \frac{dz^2}{c^2} - dv^2 = 0$$

$$\Rightarrow z \pm cv = \text{constant}$$

Answer to Q 7. The equation of the given surface is

$$\boldsymbol{r} = (x, y, z)$$

where $z = f(x, y)$.

We can take x and y as parameters. It can be easily found that for this surface

$$L = \frac{r}{H}, M = \frac{s}{H}, N = \frac{t}{H}, H^2 = 1 + p^2 + q^2$$

Now the asymptotic lines are given by

$$Ldx^2 + 2Mdxdy + Ndy^2 = 0$$

$$\Rightarrow \frac{r}{H}dx^2 + 2\frac{s}{H}\,dxdy + \frac{t}{H}\,dy^2 = 0$$

$$\Rightarrow rdx^2 + 2sdxdy + tdy^2 = 0$$

The Gaussian curvature

$$\mathrm{K} = \frac{LN - M^2}{H^2} = \frac{rt - s^2}{H^4} = \frac{rt - s^2}{(1 + p^2 + q^2)^2}$$

By the theorem of Beltrami and Enneper, the torsions of the asymptotic lines are

$\pm\sqrt{\mathrm{K}}$

i.e.

$$\pm\frac{\sqrt{s^2 - rt}}{(1 + p^2 + q^2)}$$

Answer to Q 8. The equation of the given surface is

$$\boldsymbol{r} = (u\cos v, u\sin v, f(u))$$

It can be easily found that for this surface

$$L = \frac{uf_{11}}{H}, M = 0, N = \frac{u^2 f_1}{H}, H^2 = u^2(1 + f_1^2)$$

Now the asymptotic lines are given by

$$Ldu^2 + 2Mdudv + Ndv^2 = 0$$

$$\Rightarrow uf_{11}du^2 + u^2 f_1 dv^2 = 0$$

$$\Rightarrow f_{11}du^2 + uf_1 dv^2 = 0$$

The Gaussian curvature

$$\mathrm{K} = \frac{LN - M^2}{H^2} = \frac{u^3 f_1 f_{11}}{H^4} = \frac{u^3 f_1 f_{11}}{u^4(1 + f_1^2)^2}$$

By the theorem of Beltrami and Enneper, the torsions of the asymptotic lines are

$\pm\sqrt{-\mathrm{K}}$

i.e.

$$\frac{\pm\sqrt{-u^3 f_1 f_{11}}}{u^2(1+f_1^2)}$$

Answer to Q 9. The equation of the surface generated by the binormals to the twisted curve $\boldsymbol{r} = \boldsymbol{r}(s)$ is

$$\boldsymbol{R} = \boldsymbol{r} + u\boldsymbol{b} \qquad (1)$$

where $\boldsymbol{r}$ and $\boldsymbol{b}$ are functions of s.

We have

$$\boldsymbol{R}_1 = \frac{\partial \boldsymbol{R}}{\partial u} = \boldsymbol{b}, \qquad \boldsymbol{R}_2 = \frac{\partial \boldsymbol{R}}{\partial s} = \frac{d\boldsymbol{r}}{ds} + u\frac{d\boldsymbol{b}}{ds} = \boldsymbol{t} - u\tau\boldsymbol{n}$$

$$\therefore \quad \boldsymbol{R}_1 \times \boldsymbol{R}_2 = \boldsymbol{b} \times (\boldsymbol{t} - u\tau\boldsymbol{n}) = \boldsymbol{n} + u\tau\boldsymbol{t}$$

Also

$$\boldsymbol{R}_{11} = 0, \boldsymbol{R}_{12} = \boldsymbol{b}' = -\tau\boldsymbol{n}, \boldsymbol{R}_{22} = \boldsymbol{t}' - u\tau'\boldsymbol{n} - u\tau\boldsymbol{n}'$$

$$= \kappa\boldsymbol{n} - u\tau'\boldsymbol{n} - u\tau(\tau\boldsymbol{b} - \kappa\boldsymbol{t})$$

$$= (\kappa - u\tau')\boldsymbol{n} - u\tau^2\boldsymbol{b} + u\kappa\tau\boldsymbol{t}$$

Now

$$H^2 = (\boldsymbol{R}_1 \times \boldsymbol{R}_2)^2 = 1 + u^2\tau^2$$

Also

$$LH = \boldsymbol{R}_{11} \cdot (\boldsymbol{R}_1 \times \boldsymbol{R}_2) = 0$$

$$MH = \boldsymbol{R}_{12} \cdot (\boldsymbol{R}_1 \times \boldsymbol{R}_2) = -\tau$$

$$NH = \boldsymbol{R}_{22} \cdot (\boldsymbol{R}_1 \times \boldsymbol{R}_2) = (\kappa - u\tau') + u^2\kappa\tau^2$$

The asymptotic lines are given by

$$Ldu^2 + 2Mduds + Nds^2 = 0$$

$$\Rightarrow -\frac{2\tau}{H}duds + \frac{(\kappa - u\tau' + u^2\kappa\tau^2)}{H}ds^2 = 0$$

$$\Rightarrow ds[-2\tau du + (\kappa - u\tau' + u^2\kappa\tau^2)ds] = 0$$

We see that the curves $s =$ constant, i.e. the generators are the asymptotic lines.

The Gaussian curvature

$$\mathrm{K} = \frac{LN - M^2}{H^2} = -\frac{\tau^2}{H^4} = -\frac{\tau^2}{(1+u^2\tau^2)^2}$$

The torsions of the asymptotic lines are

$$\pm\sqrt{-\mathrm{K}}$$

i.e.

$$\pm\frac{\tau}{(1+\tau^2u^2)}$$

11.10 FURTHER READINGS

1. Wilmore T. J. An Introduction to Differential Geometry, Dover Publications Inc., 2012
2. Weatherburn, C. E. Differential Geometry of Three Dimensions, Cambridge University Press, 2016
3. Pressley, A. Elementary Differential Geometry. Springer, 2002
4. Gupta, P. P., Malik, G. S., Pundir, S. K. Differential Geometry, Pragati Prakashan, 2010

11.11 MODEL QUESTIONS

1. Show that the projection on the xy-plane of an asymptotic line of the cylindroids $z(x^2+y^2)=2mxy$ is a lemniscate.
2. Show that the asymptotic lines of the helicoids $x=u\cos v, y=u\sin v, z=cv$ consist of the generators and the curves of intersection with coaxial right circular cylinders.
3. Find the asymptotic lines of the conoid

   $$x=u\cos v, y=u\sin v, z=f(v)$$

4. Show that on the surface

   $$x=3u(1+v^2)-u^3, y=3v(1+u^2)-v^3, z=3(u^2-v^2)$$

   the asymptotic lines are

   $$u\pm v=\text{constant}$$

5. Prove that the projections on the xy-plane of the asymptotic lines of the conoid:

   $$x=u\cos\theta, y=u\sin\theta, z=ae^{m\theta}$$

   are equiangular spirals.

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