UNIT 12 : FUNDAMENTAL EQUATIONS OF SURFACE THEORY AND PARALLEL SURFACES

UNIT STRUCTURE

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12.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- Understand the fundamental equations of surface theory, including the Gauss and Mainardi-Codazzi equations, and explain their role in describing the intrinsic and extrinsic geometry of surfaces.
- Explain the concept of parallel surfaces and describe how a surface can generate a family of parallel surfaces at a constant normal distance.
- Derive and use the relations between the fundamental quantities of a surface and those of its parallel surfaces.
- Solve theoretical and illustrative problems related to the fundamental equations and properties of parallel surfaces in differential geometry.

12.2 INTRODUCTION

In the study of Differential Geometry, surfaces embedded in three–dimensional space are analyzed through various mathematical tools that describe their geometric

properties. Among these tools, the fundamental equations of surface theory play a central role in establishing the relationship between the intrinsic and extrinsic geometry of a surface. These equations connect the first and second fundamental forms and provide conditions that must be satisfied for a surface to exist with given metric and curvature properties. In particular, the well-known Gauss and Mainardi-Codazzi equations express important compatibility conditions that relate the curvature of the surface to its fundamental forms.

Another important concept in surface theory is that of parallel surfaces. A parallel surface is obtained by moving each point of a given surface along its normal direction by a constant distance. Studying such surfaces helps in understanding how geometric quantities such as curvature and fundamental coefficients change when the surface is shifted in space. The analysis of parallel surfaces provides deeper insight into the geometric structure of surfaces and the relationships among their fundamental quantities.

12.3 THE FUNDAMENTAL EQUATIONS OF SURFACE THEORY

In this section, we shall obtain relation between the fundamental coefficients E, F, G and L, M, N.

The Formula of Gauss: In theory of curves in space the Frenet formulae expresses $\boldsymbol{t}', \boldsymbol{n}', \boldsymbol{b}'$ as linear combination of $\boldsymbol{t}, \boldsymbol{n}$ and $\boldsymbol{b}$. In the theory of surface the Gauss's formula giving relation between fundamental coefficients and their partial derivatives w.r.t. u and v are of great importance.

Statement: The following partial differential equation of face theory are called Gauss equations

$$\boldsymbol{r}_{11} = L\boldsymbol{N} + l\boldsymbol{r}_1 + \lambda \boldsymbol{r}_2$$

$$\boldsymbol{r}_{12} = M\boldsymbol{N} + m\boldsymbol{r}_1 + \mu \boldsymbol{r}_2$$

$$\boldsymbol{r}_{22} = N\boldsymbol{N} + n\boldsymbol{r}_1 + \nu \boldsymbol{r}_2$$

or

$$\begin{bmatrix} \boldsymbol{r}_{11} \\ \boldsymbol{r}_{12} \\ \boldsymbol{r}_{22} \end{bmatrix} = \begin{bmatrix} l & \lambda & L \\ m & \mu & M \\ n & \nu & N \end{bmatrix} \begin{bmatrix} \boldsymbol{r}_1 \\ \boldsymbol{r}_2 \\ \boldsymbol{N} \end{bmatrix}$$

where $l, m, n; \lambda, \mu, \nu$ (Christoffel symbols) are suitable functions of E, F, G and their

partial derivatives with respect to u and v.

Proof: We know that L, M, N are the resolved parts of $\boldsymbol{r}_{11}, \boldsymbol{r}_{12}, \boldsymbol{r}_{22}$ are along the normal to the surface, therefore we may write

$$\boldsymbol{r}_{11} = L\boldsymbol{N} + l\boldsymbol{r}_1 + \lambda \boldsymbol{r}_2 \qquad (1)$$

$$\boldsymbol{r}_{12} = M\boldsymbol{N} + m\boldsymbol{r}_1 + \mu \boldsymbol{r}_2 \qquad (2)$$

$$\boldsymbol{r}_{22} = N\boldsymbol{N} + n\boldsymbol{r}_1 + \nu \boldsymbol{r}_2 \qquad (3)$$

where $l, m, n; \lambda, \mu, \nu$ are to be determined as follows:

Taking dot product of both sides of (1) with $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ successively, we get

$$\left.\begin{aligned} \boldsymbol{r}_1 \cdot \boldsymbol{r}_{11} &= L\boldsymbol{r}_1 \cdot \boldsymbol{N} + l\boldsymbol{r}_1 \cdot \boldsymbol{r}_1 + \lambda \boldsymbol{r}_1 \cdot \boldsymbol{r}_2 \\ \boldsymbol{r}_2 \cdot \boldsymbol{r}_{11} &= L\boldsymbol{r}_2 \cdot \boldsymbol{N} + l\boldsymbol{r}_2 \cdot \boldsymbol{r}_1 + \lambda \boldsymbol{r}_2 \cdot \boldsymbol{r}_2 \end{aligned}\right\} \qquad (4)$$

Since $\boldsymbol{r}_1 \cdot \boldsymbol{N} = 0 = \boldsymbol{r}_2 \cdot \boldsymbol{N}$

and $\boldsymbol{r}_1 \cdot \boldsymbol{r}_1 = E,\ \boldsymbol{r}_1 \cdot \boldsymbol{r}_2 = F, \boldsymbol{r}_2 \cdot \boldsymbol{r}_2 = G,$

the equations (4) can be put as

$$\left.\begin{aligned} \boldsymbol{r}_1 \cdot \boldsymbol{r}_{11} &= lE + \lambda F \\ \boldsymbol{r}_2 \cdot \boldsymbol{r}_{11} &= lF + \lambda G \end{aligned}\right\} \qquad (5)$$

$$\boldsymbol{r}_1 \cdot \boldsymbol{r}_{11} = \frac{1}{2}\frac{\partial}{\partial u}(\boldsymbol{r}_1^2) = \frac{1}{2}\frac{\partial}{\partial u}(E) = \frac{1}{2}E_1$$

and

$$\boldsymbol{r}_{11} \cdot \boldsymbol{r}_2 = \frac{\partial}{\partial u}(\boldsymbol{r}_1 \cdot \boldsymbol{r}_2) - \boldsymbol{r}_1 \cdot \boldsymbol{r}_{12} = \frac{\partial}{\partial u}(F) - \frac{1}{2}\frac{\partial}{\partial v}(\boldsymbol{r}_1^2)$$

$$= F_1 - \frac{1}{2}\frac{\partial}{\partial v}(E) = F_1 - \frac{1}{2}E_2$$

Putting these values of $\boldsymbol{r}_1 \cdot \boldsymbol{r}_{11}$ and $\boldsymbol{r}_{11} \cdot \boldsymbol{r}_2$ in (5), we have

$$\left.\begin{aligned} \frac{1}{2}E_1 &= lE + \lambda F \\ F_1 - \frac{1}{2}E_2 &= lF + \lambda G \end{aligned}\right\} \qquad (6)$$

Solving equations (6) for l and λ, we get

$$l = \frac{1}{2H^2}(GE_1 - 2FF_1 + FE_2), \qquad \lambda = \frac{1}{2H^2}(2EF_1 - 2EE_2 - FE_1) \qquad (7)$$

To find the values of m and μ, taking dot product of both sides of (2) with the vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ successively, we get

$$\left.\begin{aligned} \boldsymbol{r}_1 \cdot \boldsymbol{r}_{12} &= mE + \mu F \\ \boldsymbol{r}_2 \cdot \boldsymbol{r}_{12} &= mF + \mu G \end{aligned}\right\} \qquad (8)$$

Again

$$\boldsymbol{r}_1 \cdot \boldsymbol{r}_{12} = \frac{1}{2}\frac{\partial}{\partial v}(\boldsymbol{r}_1^2) = \frac{1}{2}\frac{\partial}{\partial v}(E) = \frac{1}{2}E_2, \qquad \boldsymbol{r}_2 \cdot \boldsymbol{r}_{11} = F_1 - \frac{1}{2}E_2$$

and

$$\boldsymbol{r}_2 \cdot \boldsymbol{r}_{22} = \frac{1}{2}\frac{\partial}{\partial v}(\boldsymbol{r}_2^2) = \frac{1}{2}\frac{\partial}{\partial v}(G) = \frac{1}{2}G_2, \qquad \boldsymbol{r}_2 \cdot \boldsymbol{r}_{12} = \frac{1}{2}G_1$$

Hence equations (8) can be written as

$$\frac{1}{2}E_2 = mE + \mu F;\ \frac{1}{2}G_1 = mF + \mu G$$

Solving these equations for m and μ, we get

$$m = \frac{1}{2H^2}(GE_2 - FG_1);\ \mu = \frac{1}{2H^2}(EG_1 - FE_2) \qquad (9)$$

To find the values of n and v, we take dot product of both sides of (3) with $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ successively, then we get

$$\boldsymbol{r}_1 \cdot \boldsymbol{r}_{22} = nE + vF;\ \boldsymbol{r}_2 \cdot \boldsymbol{r}_{22} = nF + vG \qquad (10)$$

Now

$$\boldsymbol{r}_1 \cdot \boldsymbol{r}_{22} = \frac{\partial}{\partial v}(\boldsymbol{r}_1 \cdot \boldsymbol{r}_{22}) - \boldsymbol{r}_{12} \cdot \boldsymbol{r}_2 = \frac{\partial}{\partial v}(F) - \frac{1}{2}\frac{\partial}{\partial u}(\boldsymbol{r}_2^2)$$

$$= F_2 - \frac{1}{2}\frac{\partial}{\partial u}(G) = F_2 - \frac{1}{2}G_2$$

and

$$\boldsymbol{r}_2 \cdot \boldsymbol{r}_{22} = \frac{1}{2}\frac{\partial}{\partial v}(\boldsymbol{r}_2^2) = \frac{1}{2}\frac{\partial}{\partial v}(G) = \frac{1}{2}G_2$$

Therefore equations (10) can be written as

$$F_2 - 3G_1 = nE + vF;\ \frac{1}{2}G_2 = nF + vG$$

Solving for n and v, we get

$$\left.\begin{aligned} n &= \frac{1}{2H^2}(2GF_2 - GG_1 + FG_2) \\ v &= \frac{1}{2H^2}(EG_2 - 2FF_2 + FG) \end{aligned}\right\} \qquad (11)$$

Deduction: In case, the parametric curves are orthogonal then $F = 0$ and $H^2 = EG$, thus the values of $l, m, n; \lambda, \mu, \nu$ reduce to

$$l_1 = \frac{1}{2}\frac{E_1}{E}, m = \frac{1}{2}\frac{E_2}{E}, n = -\frac{1}{2}\frac{G_1}{E};\ \lambda = -\frac{1}{2}\frac{E_2}{G}, \mu = \frac{1}{2}\frac{G_1}{G}, \nu = \frac{1}{2}\frac{G_2}{G}$$

There in this case Gauss' formulae reduces to

$$\boldsymbol{r}_{11} = L\boldsymbol{N} + \frac{1}{2}\frac{E_1}{E}\boldsymbol{r}_1 - \frac{1}{2}\frac{E_2}{G}\boldsymbol{r}_2;\ \ \boldsymbol{r}_{12} = M\boldsymbol{N} + \frac{1}{2}\frac{E_2}{E}\boldsymbol{r}_1 + \frac{1}{2}\frac{G_1}{G}\boldsymbol{r}_2$$

$$\boldsymbol{r}_{22} = N\boldsymbol{N} + \frac{1}{2}\frac{G_1}{E}\boldsymbol{r}_1 + \frac{1}{2}\frac{G_2}{G}\boldsymbol{r}_2$$

❖ GAUSS CHARACTERISTIC EQUATION:

It states that

$$T^2 = \frac{1}{2}(2F_{12} - E_{22} - G_{11}) + (m^2E + 2m\mu F + \mu^2 G) - \{l\,nE + (lv + \lambda n)F + \lambda vG\}$$

where $T^2 = LN - M^2$

Proof: From Gauss' formulae, we have

$$\boldsymbol{r}_{11} = L\boldsymbol{N} + l\boldsymbol{r}_1 + \lambda\boldsymbol{r}_2;\ \ \boldsymbol{r}_{12} = M\boldsymbol{N} + m\boldsymbol{r}_1 + \mu\boldsymbol{r}_2;\ \ \boldsymbol{r}_{22} = N\boldsymbol{N} + n\boldsymbol{r}_1 + v\boldsymbol{r}_2$$

$$\therefore\ \ \boldsymbol{r}_{11}\cdot\boldsymbol{r}_{22} = LN + l\,nE + (lv + \lambda n)F + \lambda vG$$

and

$$\boldsymbol{r}_{12}^2 = \boldsymbol{r}_{12}\cdot\boldsymbol{r}_{12} = M^2 + m^2E + 2m\mu F + \mu^2 G$$

$$\therefore\ \ \boldsymbol{r}_{11}\cdot\boldsymbol{r}_{22} - \boldsymbol{r}_{12}^2$$

$$= (LN - M^2) + \{l\,nE + (lv + \lambda n)F + \lambda vG\} - (m^2E + 2m\mu F + \mu^2 G) \qquad (1)$$

Now we shall prove that

$$\boldsymbol{r}_{11}\cdot\boldsymbol{r}_{22} - \boldsymbol{r}_{12}^2 = \frac{1}{2}(2F_{12} - E_{22} - G_{11})$$

Since

$$E = \boldsymbol{r}_1\cdot\boldsymbol{r}_1 = \boldsymbol{r}_1^2$$

$$\therefore\ \ E_2 = 2\boldsymbol{r}_1\cdot\boldsymbol{r}_{12} \ \ and \ \ E_{22} = 2\boldsymbol{r}_{12}\cdot\boldsymbol{r}_{12} + 2\boldsymbol{r}_2\cdot\boldsymbol{r}_{122}$$

$$G = \boldsymbol{r}_2\cdot\boldsymbol{r}_2 = \boldsymbol{r}_2^2$$

$$\therefore\ \ G_1 = 2\boldsymbol{r}_2\cdot\boldsymbol{r}_{12} \ \ and \ \ G_{11} = 2\boldsymbol{r}_{12}\cdot\boldsymbol{r}_{12} + 2\boldsymbol{r}_2\cdot\boldsymbol{r}_{112}$$

$$F = \boldsymbol{r}_1\cdot\boldsymbol{r}_2$$

$$\therefore\ \ F_1 = \boldsymbol{r}_{11}\cdot\boldsymbol{r}_2 + \boldsymbol{r}_1\cdot\boldsymbol{r}_{12} \ \ and \ \ F_{12} = \boldsymbol{r}_{11}\cdot\boldsymbol{r}_{22} + \boldsymbol{r}_{12}\cdot\boldsymbol{r}_{22} + \boldsymbol{r}_{12}\cdot\boldsymbol{r}_{12} + \boldsymbol{r}_1\cdot\boldsymbol{r}_{122}$$

Therefore

$$2F_{12} - F_{22} - G_{11} = 2(\boldsymbol{r}_{11}\cdot\boldsymbol{r}_{22} - \boldsymbol{r}_{12}\cdot\boldsymbol{r}_{12})$$

$$\boldsymbol{r}_{11}\cdot\boldsymbol{r}_{22} - \boldsymbol{r}_{12}^2 = \frac{1}{2}(2F_{12} - E_{22} - G_{11})$$

On putting this value of $\boldsymbol{r}_{11}\cdot\boldsymbol{r}_{22} - \boldsymbol{r}_{12}^2$ in (1), we get

$$\frac{1}{2}(2F_{12} - E_{22} - G_{11}) = (LN - M^2) + \{l\,nE + (lv + \lambda n)F + \lambda vG\} - (m^2E + 2m\mu F + \mu^2 G)$$

or

$$LN - M^2 = \frac{1}{2}(2F_{12} - E_{22} - G_{11}) + (m^2E + 2m\mu F + \mu^2 G) - \{l\,nE + (lv + \lambda n)F + \lambda v G\} \qquad (2)$$

Deduction 1: Equation (2) may be put in the form

$$LN - M^2 = \frac{1}{2}H\frac{\partial}{\partial u}\left\{\frac{F}{EH}\frac{\partial E}{\partial v} - \frac{1}{H}\frac{\partial G}{\partial u}\right\} + \frac{1}{2}H\frac{\partial}{\partial u}\left\{\frac{2}{H}\frac{\partial F}{\partial u} - \frac{1}{H}\frac{\partial E}{\partial v} - \frac{F}{EH}\frac{\partial E}{\partial u}\right\}$$

$$= \frac{1}{2}H\frac{\partial}{\partial u}\left(\frac{FE_2}{EH} - \frac{G_1}{H}\right) + \frac{1}{2}H\frac{\partial}{\partial v}\left(\frac{2F_1}{H} - \frac{E_2}{H} - \frac{FE_1}{EH}\right) \qquad (3)$$

Deduction 2: Intrinsic formula for Gaussian curvature

Suppose $\boldsymbol{r} = \boldsymbol{r}(u, v)$ represents a surface. We know that at any point (u, v) on the surface, the Gaussian curvature K is given by

$$\mathrm{K} = \frac{LN - M^2}{H^2} = \frac{1}{2H}\frac{\partial}{\partial u}\left(\frac{FE_2}{EH} - \frac{G_1}{H}\right) + \frac{1}{2H}\frac{\partial}{\partial v}\left(\frac{2F_1}{H} - \frac{E_2}{H} - \frac{FE_1}{EH}\right)$$

$$= \frac{1}{H}\frac{\partial}{\partial u}\left(\frac{FE_2 - EG_1}{2HE}\right) + \frac{1}{H}\frac{\partial}{\partial v}\left(\frac{2EF_1 - FE_1 - EE_2}{2HE}\right) \qquad (4)$$

where, $H^2 = EG - F^2$.

The formula (4) gives Gaussian curvature K in terms of first fundamental magnitudes E, F, G and their partial derivatives with respect to u and v. Thus (4) is intrinsic formula for Gaussian curvature.

In case, the parametric curves are orthogonal, $F = 0$ and then $F_1 = 0$.

$$\Rightarrow LN - M^2 = \frac{1}{2}H\frac{\partial}{\partial u}\left\{-\frac{G_1}{H}\right\} + \frac{1}{2}H\frac{\partial}{\partial v}\left(-\frac{E_2}{H}\right)$$

But

$$H^2 = EG - F^2 = EG$$

$$\Rightarrow \mathrm{K} = \frac{LN - M^2}{H^2} = -\frac{1}{2H}\left[\frac{\partial}{\partial u}\left(\frac{G_1}{H}\right) + \frac{\partial}{\partial v}\left(\frac{E_2}{H}\right)\right]$$

$$= -\frac{1}{H}\left[\frac{1}{2}\frac{\partial}{\partial u}\left(\frac{G_1}{\sqrt{EG}}\right) + \frac{1}{2}\frac{\partial}{\partial v}\left(\frac{E_2}{\sqrt{EG}}\right)\right]$$

$$= -\frac{1}{\sqrt{EG}}\left[\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{1}{\sqrt{G}}\cdot\frac{\partial\sqrt{E}}{\partial v}\right)\right]$$

❖ MAINARDI-CODAZZI EQUATIONS:

Statement: The equations

$$L_2 - M_1 = mL - (l - \mu)M - \lambda N$$

and

$$M_2 - N_1 = nL - (m - v)M - \mu N$$

are called Mainardi-Codazzi equations.

Proof: Consider the identity

$$\frac{\partial}{\partial v}(\boldsymbol{r}_{11}) = \frac{\partial}{\partial u}(\boldsymbol{r}_{12}) \qquad (1)$$

Putting the values of $\boldsymbol{r}_{11}$ and $\boldsymbol{r}_{12}$ from Gauss' formula in (1), we get

$$\frac{\partial}{\partial v}(L\boldsymbol{N} + l\boldsymbol{r}_1 + \lambda \boldsymbol{r}_2) = \frac{\partial}{\partial u}(M\boldsymbol{N} + m\boldsymbol{r}_1 + \mu \boldsymbol{r}_2)$$

or

$$L_2\boldsymbol{N} + l_2\boldsymbol{r}_1 + \lambda_2\boldsymbol{r}_2 + L\boldsymbol{N}_2 + l\boldsymbol{r}_{12} + \lambda \boldsymbol{r}_{22}$$
$$= M_1\boldsymbol{N} + m\boldsymbol{r}_1 + \mu_1\boldsymbol{r}_2 + M\boldsymbol{N}_1 + m\boldsymbol{r}_{11} + \mu \boldsymbol{r}_{12} \qquad (2)$$

Now substituting the values of $\boldsymbol{r}_{11}, \boldsymbol{r}_{22}, \boldsymbol{r}_{12}$ from Gauss formulae and the values of $\boldsymbol{N}_1$ and $\boldsymbol{N}_2$ from Weingarten equations, namely

$$\boldsymbol{N}_1 = \frac{(FM - GL)}{H^2}\boldsymbol{r}_1 + \frac{(FL - EM)}{H^2}\boldsymbol{r}_2$$

and

$$\boldsymbol{N}_2 = \frac{(FN - GM)}{H^2}\boldsymbol{r}_1 + \frac{(FM - EN)}{H^2}\boldsymbol{r}_2$$

the identity (2) is expressed in terms of vectors $\boldsymbol{G}, \boldsymbol{r}_1, \boldsymbol{r}_2$ (non-coplanar vectors).

On equating the coefficients of $\boldsymbol{N}$ on the both sides, we get

$$L_2 + lM + \lambda N = M_1 + mL + \mu M$$

or

$$L_2 - M_1 = mL - (l - \mu)M - \lambda N$$

which is the first Mainardi-Codazzi equation.

Now consider the identity

$$\frac{\partial}{\partial v}(\boldsymbol{r}_{12}) = \frac{\partial}{\partial u}(\boldsymbol{r}_{22}) \qquad (3)$$

From Gauss formula, we have

$$\boldsymbol{r}_{12} = M\boldsymbol{N} + m\boldsymbol{r}_1 + \mu \boldsymbol{r}_2; \quad \boldsymbol{r}_{22} = N\boldsymbol{N} + n\boldsymbol{r}_1 + v\boldsymbol{r}_2$$

Putting these values in (3), we get

$$M_2\boldsymbol{N} + m_1\boldsymbol{r}_1 + M\boldsymbol{N}_2 + m\boldsymbol{r}_{12} + \mu \boldsymbol{r}_{22}$$
$$= N_1\boldsymbol{N} + n_1\boldsymbol{r}_1 + v\boldsymbol{r}_2 + N\boldsymbol{N}_1 + n\boldsymbol{r}_{11} + v\boldsymbol{r}_{12} \qquad (4)$$

Substituting in (4) the values of $\boldsymbol{r}_{11}, \boldsymbol{r}_{22}, \boldsymbol{r}_{12}$ from Gauss formulae and the values

of N_1 and N_2 from Weingarten equations (relations given above), we get a vector identity and then equating coefficients of N on both sides of the identity, we get

$$M_2 + mM + \mu N = N_1 + nL + vM$$

or

$$M_2 - N_1 = nL - (m - v)M - \mu N$$

which is second Mainardi-Codazzi equation.

CHECK YOUR PROGRESS

Q 1. Show that for the right helicoid $\boldsymbol{r} = (u\cos v, u\sin v, cv)$

$$l = 0, m = 0, n = -u;\ \lambda = 0, \mu = \frac{u}{(n^2 + c^2)}, v = 0$$

Q 2. For any surface prove that

$$\frac{\partial}{\partial u}(\log H) = l + \mu, \frac{\partial}{\partial v}(\log H) = m + v$$

where u and v are parameters and symbols have their usual meaning.

Q 3. Show that for a surface whose metric is given by $ds^2 = du^2 + D^2 dv^2$

$$l = 0, m = 0, n = -DD_1;\ \lambda = 0, \mu = \frac{D_1}{D}, v = \frac{D_2}{D}$$

Q 4. From the Gauss characteristic equation deduce that, when the parametric curves are orthogonal

$$\kappa = -\frac{1}{\sqrt{EG}}\left[\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}\right)\right]$$

Q 5. Show that when the lines of curvature are chosen as parametric curves, the Codazzi relations expressed in terms of E, G, L, N and their derivatives are

$$L_2 = \frac{1}{2}E_2\left(\frac{L}{E} + \frac{N}{G}\right), N_1 = \frac{1}{2}G_1\left(\frac{L}{E} + \frac{N}{G}\right)$$

Show also that the equation of Gauss may be written as

$$\frac{LN}{\sqrt{EG}} + \frac{\partial}{\partial u}\left(\frac{1}{\sqrt{G}}\frac{\partial\sqrt{G}}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}\right) = 0$$

Q 6. For a surface given by $ds^2 = \phi(du^2 + dv^2)$, prove that

$$l = \frac{\phi_1}{2\phi}, m = \frac{\phi_2}{2\phi}, n = -\frac{\phi_1}{2\phi}, \lambda = \frac{\phi_2}{2\phi}, \mu = \frac{\phi_1}{2\phi}, v = \frac{\phi_2}{2\phi}$$

and further show that Mainardi-Codazzi relations become

$$L_2 - M_1 = \frac{1}{2}(L+N)\frac{\phi_2}{\phi}$$

$$N_1 - M_2 = \frac{1}{2}(L+N)\frac{\phi_1}{\phi}$$

12.4 PARALLEL SURFACES

A surface S^* is to be parallel to another surface S if S^* is at a constant distance along the normal form S. e.g. two concentric spheres are parallel surface. Since the constant distance may be chosen arbitrarily the number of such parallel surface is infinite. Consider a point $P(\boldsymbol{r})$ on the surface S, let $\boldsymbol{N}$ be the unit normal to S at P, then position vector $\boldsymbol{r}^*$ of the corresponding point Q on S^* is given by

$$\boldsymbol{r}^* = \boldsymbol{r} - c\boldsymbol{N}$$

where c is the scalar constant may be taken positive or negative and represents distance between P and Q along $\boldsymbol{N}$.

❖ FUNDAMENTAL COEFFICIENTS FOR PARALLEL SURFACES WHEN THE LINES OF CURVATURE ON THE GIVEN SURFACES ARE TAKEN AS PARAMETRIC CURVES

Let $\boldsymbol{r}$ be the position vector of any point P on the surface S and $\boldsymbol{N}$ the unit normal to S at P.

Let $\boldsymbol{r}^*$ be the position vector of corresponding point Q on parallel surface S^*. Thus

$$\boldsymbol{r}^* = \boldsymbol{r} - c\boldsymbol{N} \qquad (1)$$

Differentiating (1) partially w.r.t. parameters u and v, we get

$$\boldsymbol{r}_1^* = \boldsymbol{r}_1 - c\boldsymbol{N}_1 \qquad (2)$$

$$\boldsymbol{r}_2^* = \boldsymbol{r}_2 - c\boldsymbol{N}_2 \qquad (3)$$

Let us assume the lines of curvatures to be parametric curves. So that we have $F = 0, M = 0$. Hence values of $\boldsymbol{N}_1$ and $\boldsymbol{N}_2$ in Weingarten equation namely

$$H^2\boldsymbol{N}_1 = (GM - GL)\boldsymbol{r}_1 + (FL - EM)\boldsymbol{r}_2$$

and

$$H^2\boldsymbol{N}_2 = (FN - MG)\boldsymbol{r}_1 + (FM - EN)\boldsymbol{r}_2$$

reduce to

$$EG - \boldsymbol{N}_1 = -GL\boldsymbol{r}_1 \quad or \quad \boldsymbol{N}_1 = -\frac{L}{E}\boldsymbol{r}_1 \quad [\because \ F = 0, M = 0, H^2 = EG - F^2 = EG]$$

and

$$EG\boldsymbol{N}_2 = -EN\boldsymbol{r}_2 \Rightarrow \boldsymbol{N}_2 = -\frac{N}{G}\boldsymbol{r}_2$$

Substituting these values of $\boldsymbol{N}_1$ and $\boldsymbol{N}_2$ in (2) and (3), we get

$$\boldsymbol{r}_1^* = \left(1 + c\frac{L}{E}\right)\boldsymbol{r}_1 \qquad (4)$$

$$\boldsymbol{r}_2^* = \left(1 + c\frac{N}{G}\right)\boldsymbol{r}_2 \qquad (5)$$

Let κ_a and κ_b be the principal curvature of the surface at P. Since the lines of curvature are parametric curves, so κ_a and κ_b are normal curvatures at P in the direction of parametric curves.

We know that normal curvature κ_n at P in the direction (du, dv) is given by

$$\kappa_n = \frac{Ldu^2 + 2Mdudv + Ndv^2}{Edu^2 + 2Fdudv + Gdv^2} = \frac{Ldu^2 + Ndv^2}{Edu^2 + Gdv^2} \qquad [\because F = 0, M = 0]$$

In case κ_a and κ_b are normal curvatures at P along the line $dv = 0$ and $du = 0$ respectively, then we have

$$\kappa_a = \frac{L}{E} \quad and \quad \kappa_b = \frac{N}{G} \qquad (6)$$

Thus (4) and (5) can be written as

$$\boldsymbol{r}_1^* = (1 + c\kappa_a)\boldsymbol{r}_1 \qquad (7)$$

$$\boldsymbol{r}_2^* = (1 + c\kappa_b)\boldsymbol{r}_2 \qquad (8)$$

Let E^*, G^*, L^*, M^*, N^* denote the fundamental coefficients of S^*, then we have

$$E^* = \boldsymbol{r}_1^{*2} = (1 + c\kappa_a)\boldsymbol{r}_1^2 = (1 + c\kappa_a)^2 E$$

$$F^* = \boldsymbol{r}_1^* \cdot \boldsymbol{r}_2^* = \boldsymbol{r}_1 \cdot \boldsymbol{r}_2(1 + c\kappa_a)(1 + c\kappa_b) = 0 \qquad [\because \boldsymbol{r}_1 \cdot \boldsymbol{r}_2 = F = 0]$$

$$G^* = \boldsymbol{r}_2^{*2} = \boldsymbol{r}_2^2(1 + c\kappa_b)^2 = G(1 + c\kappa_b)^2$$

$$H^{*2} = E^*G^* - F^* = EG(1 + c\kappa_a)^2(1 + c\kappa_b)^2$$

$$= H^2(1 + c\kappa_a)^2(1 + c\kappa_b)^2$$

Again taking cross product of (6) and (7), we get

$$H^*\boldsymbol{N}^* = \boldsymbol{r}_1^* \times \boldsymbol{r}_1^* = \boldsymbol{r}_1 \times \boldsymbol{r}_2(1 + c\kappa_a)(1 + c\kappa_b)$$

$$\Rightarrow H^*\boldsymbol{N}^* = H\{1 + c(\kappa_a + \kappa_b) + \kappa_a\kappa_b c^2\}\boldsymbol{N}$$

$$\Rightarrow H^*\boldsymbol{N}^* = H(1 + \mu c + \mathrm{K}c^2)\boldsymbol{N}$$

where κ and μ represent respectively Gaussian curvature and mean curvature of S.

We observe that expression on right hand side of (8) will vanish if $c = -\frac{1}{\kappa_a}$ or $c = -\frac{1}{\kappa_b}$, so excluding these two cases of c, from (8), we have

$$\boldsymbol{N}^* = \pm \boldsymbol{N} \qquad (9)$$

where $H^* = H(1 + 2\mu c + \mathrm{K}c^2)$.

Equation (9) shows that although S and S^* have the same normal lines, they may be directed in opposite directions.

Again

$$L^* = -\boldsymbol{r}_1^* \cdot \boldsymbol{N}_1^* = \pm(1 + c\kappa_a)\boldsymbol{r}_1 \cdot \boldsymbol{N}_1 = \pm(1 + c\kappa_a)L$$

$$M^* = -\boldsymbol{r}_2^* \cdot \boldsymbol{N}_1^* = \pm(1 + c\kappa_b)\boldsymbol{r}_2 \cdot \boldsymbol{N}_1 = 0$$

$$N^* = -\boldsymbol{r}_2^* \cdot \boldsymbol{N}_2^* = \pm(1 + c\kappa_b)\boldsymbol{r}_2 \cdot \boldsymbol{N}_2 = \pm(1 + c\kappa_b)N$$

Since $F^* = M^* = 0$, so we conclude that the curve on S^* corresponding to lines of curvature of S are also lines of curvature on S^*.

12.5 GAUSSIAN CURVATURE AND MEAN CURVATURE FOR THE PARALLEL SURFACE

We have

$$\mathrm{K}^* = \frac{L^*N^* - M^{*2}}{H^{*2}} = \frac{LN(1 + c\kappa_a)(1 + c\kappa_b) - 0}{H^2(1 + c\kappa_a)^2(1 + c\kappa_b)^2}$$

$$= \frac{\mathrm{K}}{(1 + c\kappa_a)(1 + c\kappa_b)} = \frac{\mathrm{K}}{(1 + 2\mu c + \kappa c^2)} \quad \left[\because \mathrm{K} = \frac{LN - 0}{H^2}\right] \qquad (1)$$

Again

$$2\mu^* = \kappa_a^* + \kappa_b^* = \frac{L^*}{E^*} + \frac{N^*}{G^*}$$

$$= \pm\left[\frac{L(1 + c\kappa_a)}{E(1 + c\kappa_a)^2} + \frac{N(1 + c\kappa_a)}{G(1 + c\kappa_b)^2}\right]$$

$$= \pm\left(\frac{\kappa_a}{1 + c\kappa_a} + \frac{\kappa_b}{1 + c\kappa_b}\right) \quad \left[\because \kappa_a = \frac{L}{F}; \kappa_b = \frac{N}{G}\right]$$

$$= \pm\left[\frac{(\kappa_a + \kappa_b) + 2c\kappa_a\kappa_b}{(1 + c\kappa_a)(1 + c\kappa_b)}\right] = \pm\left\{\frac{2\mu + 2c\mathrm{K}}{1 + 2\mu c + \mathrm{K}c^2}\right\}$$

or

$$\mu^* = \pm\left\{\frac{\mu + c\mathrm{K}}{1 + 2\mu c + \mathrm{K}c^2}\right\}$$

Theorem: With every surface of constant mean curvature $(2A)^{-1}$ there is a parallel surface of constant Gaussian curvature at distance A from it.

Proof: We know that the Gaussian curvature κ of the parallel surface is given by

$$K^* = \frac{K}{(1 + 2\mu c + Kc^2)}$$

where K is the Gaussian curvature and μ is mean curvature of the given surface S.

Now putting $\mu = (2A)^{-1}$ and $c = -A$ in (1), we have

$$K^* = \frac{K}{1 - \frac{2A}{2A + KA^2}} = \frac{1}{A^2} = constant$$

Theorem 2: Bonnet's Theorem on Parallel Surface

Statement: In general for every surface S of constant positive Gaussian curvature A^{-2} there are associated two surfaces of constant mean curvature $\pm(2A)^{-1}$ which are parallel to the former S and distant $\pm A$ from it.

Proof: We know that the mean curvature μ^* of the parallel surface S^* is given by

$$\mu^* = \pm\left\{\frac{\mu + cK}{1 + 2\mu c + Kc^2}\right\} \qquad (1)$$

where μ is the mean curvature and K is the Gaussian curvature of the given surface S.

Here $K = A^{-2}$ and $c = \pm A$

Putting these values in (1), we get

$$\mu^* = \pm\left\{\frac{\mu \pm \frac{A}{A^2}}{1 \pm 2\mu A + 1}\right\} = \pm\frac{\mu \pm \frac{1}{A}}{2(1 \pm \mu A)} = \pm\frac{1}{2A}$$

$$= \pm\frac{1}{2A} = constant$$

which proves the theorem.

While proving Bonnet's theorem, we have excluded the possibilities that

$$c = -\frac{1}{\kappa_a} \quad or \quad c = \frac{1}{\kappa_b}$$

But we cannot exclude these possibilities altogether e.g., if the surface S is of radius A, then

$$\kappa_a = \kappa_b = \frac{1}{A} \quad and \quad K = \kappa_a\kappa_b = \frac{1}{A^2} = A^{-2}$$

Also

$$A^2 = \frac{1}{K_a^2} \quad and \quad c = \pm A = \pm\frac{1}{\kappa_a}$$

In case we take $c = -\frac{1}{\kappa_a}$, then the parallel surface S^* of constant mean curvature is concentric sphere of radius $2A$, if we take $c = -\frac{1}{\kappa_a}$, then the parallel surface S^* degenerates into a single point, the centre of the sphere S, which a singularity. In order to prevent S^* from having singularities, the same signs of c and principal curvatures are taken. Then there will be one parallel surface of constant mean curvature and the second parallel surface will definitely have singularities at the umbilics of S. Therefore we may slightly modify the statement of Bonnet's theorem as follows:

With every surface of constant positive Gaussian curvature, there exists at leats one (non-singular) parallel surface with constant mean curvature.

CHECK YOUR PROGRESS

Q 7. Show that Gaussian curvature of a parallel surface is given by

$$K^* = \frac{K}{(1+c\kappa_a)(1+c\kappa_b)}$$

Q 8. Show that Mean curvature of a parallel surface is given by

$$\mu^* = \pm\left\{\frac{\mu + cK}{1 + 2\mu c + Kc^2}\right\}$$

Q 9. Show that if S^* is a surface parallel to a surface S, then the curves on S^* corresponding to the lines of curvature on S are lines of curvature of S^*.

Q 10. State and prove the Bonnet's theorem in the theory of parallel surfaces.

12.6 LET US SUM UP

- The **fundamental equations of surface theory**-namely the Gauss and Mainardi-Codazzi equations establish intrinsic relationships between the coefficients of the first and second fundamental forms, ensuring the compatibility of a surface in space.
- The **first fundamental form** describes the metric properties of a surface such as length, angle, and area, while the **second fundamental form** captures the curvature behaviour of the surface.

- The **Gaussian curvature** is an intrinsic invariant depending only on the first fundamental form, whereas **mean curvature** reflects the bending of the surface in space.
- A **parallel surface** is obtained by moving each point of a given surface along its normal direction by a constant distance, forming a one-parameter family of surfaces.
- The **principal curvatures of parallel surfaces** transform according to $\kappa_i^t = \frac{\kappa_i}{1-t\kappa_i}$, which leads to corresponding changes in Gaussian and mean curvatures.
- Parallel surfaces may develop **singularities (focal surfaces)** when $t = \frac{1}{\kappa_a}$ or $t = \frac{1}{\kappa_b}$, highlighting the geometric limitations of normal displacement.

K 12.7 ANSWERS TO CHECK YOUR PROGRESS

Answer to Q 1. $\boldsymbol{r} = (u\cos v, u\sin v, cv)$

$$\therefore \quad \boldsymbol{r}_1 = (\cos v, \sin v, 0), \qquad \boldsymbol{r}_2 = (-u\sin v, u\cos v, c)$$

$$\therefore \quad E = \boldsymbol{r}_1^2 = 1, F = \boldsymbol{r}_1 \cdot \boldsymbol{r}_2 = 0, G = \boldsymbol{r}_2^2 = u^2 + c^2, H^2 = EG - F^2 = u^2 + c^2$$

Therefore

$$E_1 = 0; E_2 = 0; F_1 = 0 = F_2; G_1 = 2u; G_2 = 0$$

Again

$$l = \frac{1}{2H^2}(GE_1 - 2FF_1 + FE_2) = 0$$

$$\lambda = \frac{1}{2H^2}(2EF_1 - EE_2 - FE_1) = 0$$

$$m = \frac{1}{2H^2}(GE_2 - FG_1) = 0$$

$$\mu = \frac{1}{2H^2}(EF_1 - FE_2) = \frac{u}{(u^2 + c^2)}$$

$$n = \frac{1}{2H^2}(2GF_2 - GG_1 - FG_2) = \frac{1}{2(u^2 + c^2)}[-(u^2 + c^2)2u] = -u$$

$$v = \frac{1}{2H^2}(EG_2 - 2FF_2 + EG_1) = 0$$

Answer to Q 2. We have

$$H^2 = EG - F^2$$

$$\therefore \quad \frac{\partial}{\partial u}(\log H) = \frac{\partial}{\partial u}\left(\frac{1}{2}\log H^2\right) = \frac{1}{2}\frac{1}{H^2}\frac{\partial}{\partial u}(H^2)$$

$$= \frac{1}{2H^2}\frac{\partial}{\partial u}(EG - F^2) = \frac{E_1G + EG_1 - 2FF_1}{2H^2} \qquad (1)$$

Now from (9) and (11) of fundamental equation of surface theory section, we have

$$l + \mu = \frac{GE_1 + FE_2 - 2FF_1 + EG_1 - FE_2}{2H^2} = \frac{E_1G + EG_1 - 2FF_1}{2H^2} \qquad (2)$$

$\therefore$ From (1) and (2), we have

$$\frac{\partial}{\partial u}(\log H) = 1 + \mu$$

Similarly we can prove the second result. **(Do it yourself)**

Answer to Q 3. Comparing $ds^2 = du^2 + D^2dv^2$ with

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2$$

we have in this case

$$E = 1, F = 0, C = D^2, H^2 = EG - F^2 = D^2$$

$$\therefore \quad E_1 = 0, G_1 = 2DD_1; E_2 = 0, G_2 = 2DD_2$$

$$\therefore \ l = \frac{E_1}{2E} = 0, \quad m = \frac{E_2}{2E} = 0, \quad n = -\frac{G_1}{2E} = -DD_1, \quad \lambda = \frac{E_2}{2G} = 0, \quad \mu = \frac{G_1}{2G} = \frac{D_1}{2D},$$
$$v = \frac{G_2}{2G} = \frac{D_2}{D}$$

Answer to Q 4. In case parametric curves are orthogonal, we have

$$F = 0$$

$$\therefore F_1 = 0$$

Hence we have

$$LN - M^2 = \frac{1}{2}H\frac{\partial}{\partial u}\left(-\frac{G_1}{H}\right) + \frac{1}{2}H\frac{\partial}{\partial v}\left(-\frac{E_2}{H}\right)$$

But

$$H^2 = EG - F = EG$$

$$\therefore \quad \kappa = \frac{LN - M^2}{H^2} = -\frac{1}{2H}\left[\frac{\partial}{\partial u}\left(\frac{G_1}{H}\right) + \frac{\partial}{\partial v}\left(\frac{E_2}{H}\right)\right]$$

$$= \frac{1}{H}\left[\frac{1}{2}\frac{\partial}{\partial u}\left\{\frac{G_1}{\sqrt{EG}}\right\} + \frac{1}{2}\frac{\partial}{\partial v}\left\{\frac{E_2}{\sqrt{EG}}\right\}\right]$$

$$= -\frac{1}{\sqrt{EG}}\left[\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E}}\frac{\partial}{\partial u}\sqrt{G}\right) + \frac{\partial}{\partial v}\left(\frac{1}{\sqrt{G}}\frac{\partial}{\partial v}\sqrt{E}\right)\right]$$

Answer to Q 5. The Codazzi equations are

$$L_2 - M_1 = mL - (l - \mu)M - \lambda N \qquad (1)$$

$$M_2 - N_1 = nL - (m - v)M - \mu N \qquad (2)$$

Since the lines of curvature are parametric curves, we have $F = 0, M = 0$.

Also then the Gauss coefficients are

$$l = \frac{E_2}{2E}, m = \frac{E_2}{2E}, n = -\frac{G_2}{2E}, \lambda = -\frac{E_2}{2G}, \mu = \frac{G_1}{2G}, \nu = \frac{G_2}{2G}$$

Thus the equation (1) reduces to

$$L_2 = \frac{E_2}{2E}L + \frac{E_2}{2G}N = \frac{1}{2}E_2\left(\frac{L}{E} + \frac{N}{G}\right)$$

and equation (2) reduces to

$$-N_1 = -\frac{G_1}{2E}L - \frac{G_1}{2G}N \quad or \quad N_1 = \frac{1}{2}G_1\left(\frac{L}{E} + \frac{N}{G}\right)$$

For the second part, from the above question **(Q 4.)**, we have

$$\kappa = -\frac{1}{\sqrt{LG}}\left[\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E}}\frac{\partial}{\partial u}\sqrt{G}\right) + \frac{\partial}{\partial v}\left(\frac{1}{\sqrt{G}}\frac{\partial}{\partial v}\sqrt{E}\right)\right] \qquad (3)$$

In this case

$$\kappa = \frac{LN - M^2}{EG - F^2} = \frac{LN}{EG} \qquad [\because F = 0, M = 0]$$

Putting this value of κ in (3), we get the required result.

Answer to Q 6. Comparing $ds^2 = \phi(du^2 + dv^2)$ with $ds^2 = Edu^2 + 2Fdudv + Gdv^2$

$$E = \phi, F = 0, G = \phi$$

As $F = 0$, the Gauss' coefficient reduces to

$$l = \frac{1}{2}\frac{\phi_1}{\phi}, m = \frac{1}{2}\frac{\phi_2}{\phi}, n = -\frac{1}{2}\frac{\phi_1}{\phi}, \lambda = -\frac{1}{2}\frac{\phi_2}{\phi}, \mu = \frac{1}{2}\frac{\phi_1}{\phi}, \nu = \frac{1}{2}\frac{\phi_2}{\phi}$$

Hence, the Mainnardi-Codazzi relations become

$$L_2 - M_1 = \frac{1}{2}\frac{\phi_2}{\phi}L - \left(\frac{1}{2}\frac{\phi_1}{\phi} - \frac{1}{2}\frac{\phi_1}{\phi}\right)M + \frac{1}{2}\frac{\phi_2}{\phi}N = \frac{1}{2}(L + N)\frac{\phi_2}{\phi}$$

and

$$M_2 - N_1 = -\frac{1}{2}\frac{\phi_1}{\phi}L - \left(\frac{1}{2}\frac{\phi_2}{\phi} - \frac{1}{2}\frac{\phi_2}{\phi}\right)M - \frac{1}{2}\frac{\phi_1}{\phi}N = -\frac{1}{2}(L + N)\frac{\phi_1}{\phi}$$

$$\Rightarrow N_1 - M_2 = \frac{1}{2}(L + N)\frac{\phi_1}{\phi}$$

Answer to Q 7. We have

$$\mathrm{K}^* = \frac{L^*N^* - M^{*2}}{H^{*2}} = \frac{LN(1 + c\kappa_a)(1 + c\kappa_b) - 0}{H^2(1 + c\kappa_a)^2(1 + c\kappa_b)^2}$$

$$= \frac{\mathrm{K}}{(1 + c\kappa_a)(1 + c\kappa_b)}$$

Answer to Q 8. We have

$$2\mu^* = \kappa_a^* + \kappa_b^* = \frac{L^*}{E^*} + \frac{N^*}{G^*}$$

$$= \pm\left[\frac{L(1+c\kappa_a)}{E(1+c\kappa_a)^2} + \frac{N(1+c\kappa_a)}{G(1+c\kappa_b)^2}\right]$$

$$= \pm\left(\frac{\kappa_a}{1+c\kappa_a} + \frac{\kappa_b}{1+c\kappa_b}\right) \qquad \left[\because \kappa_a = \frac{L}{F}; \kappa_b = \frac{N}{G}\right]$$

$$= \pm\left[\frac{(\kappa_a+\kappa_b)+2c\kappa_a\kappa_b}{(1+c\kappa_a)(1+c\kappa_b)}\right] = \pm\left\{\frac{2\mu+2c\mathrm{K}}{1+2\mu c+\mathrm{K}c^2}\right\}$$

or

$$\mu^* = \pm\left\{\frac{\mu+c\mathrm{K}}{1+2\mu c+\mathrm{K}c^2}\right\}$$

Answer to Q 9. Do it yourself.

Answer to Q 10. We know that the mean curvature μ^* of the parallel surface S^* is given by

$$\mu^* = \pm\left\{\frac{\mu+c\mathrm{K}}{1+2\mu c+\mathrm{K}c^2}\right\} \qquad (1)$$

where μ is the mean curvature and K is the Gaussian curvature of the given surface S.

Here $\mathrm{K} = A^{-2}$ and $c = \pm A$

Putting these values in (1), we get

$$\mu^* = \pm\left\{\frac{\mu \pm \frac{A}{A^2}}{1 \pm 2\mu A + 1}\right\} = \pm\frac{\mu \pm \frac{1}{A}}{2(1 \pm \mu A)} = \pm\frac{1}{2A}$$

$$= \pm\frac{1}{2A} = constant$$

which proves the theorem.

While proving Bonnet's theorem, we have excluded the possibilities that

$$c = -\frac{1}{\kappa_a} \quad or \quad c = \frac{1}{\kappa_b}$$

But we cannot exclude these possibilities altogether e.g., if the surface S is of radius A, then

$$\kappa_a = \kappa_b = \frac{1}{A} \quad and \quad \mathrm{K} = \kappa_a\kappa_b = \frac{1}{A^2} = A^{-2}$$

Also

$$A^2 = \frac{1}{K_a^2} \quad and \quad c = \pm A = \pm \frac{1}{\kappa_a}$$

In case we take $c = -\frac{1}{\kappa_a}$, then the parallel surface S^* of constant mean curvature is concentric sphere of radius $2A$, if we take $c = -\frac{1}{\kappa_a}$, then the parallel surface S^* degenerates into a single point, the centre of the sphere S, which a singularity. In order to prevent S^* from having singularities, the same signs of c and principal curvatures are taken. Then there will be one parallel surface of constant mean curvature and the second parallel surface will definitely have singularities at the umbilics of S. Therefore we may slightly modify the statement of Bonnet's theorem as follows:

With every surface of constant positive Gaussian curvature, there exists at leats one (non-singular) parallel surface with constant mean curvature.

O 12.8 FURTHER READINGS

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2. Weatherburn, C. E. Differential Geometry of Three Dimensions, Cambridge University Press, 2016
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4. Gupta, P. P., Malik, G. S., Pundir, S. K. Differential Geometry, Pragati Prakashan, 2010

12.9 MODEL QUESTIONS

1. Obtain the fundamental equation of surface theory.
2. Prove that the Gaussian curvature at a point is expressible in terms of the fundamental magnitudes of the first order and their derivatives of the first two orders.
3. Obtain the equation of Weingarten and use them to establish Mainardi-Codazzi equations.
4. For a surface with first fundamental form $ds^2 = Edu^2 + 2Gdudv + Gdv^2$, where $E = 1,\ F = 0,\ G = e^{2u}$. Find the Christoffel symbols and verify one of the Mainardi-Codazzi equations.

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