UNIT 13: GEODESICS

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13.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- Learn about geodesics and understand canonical geodesic equations.
- Understand nature of geodesics on a surface of revolution and learn about normal property of geodesics and differential equations of geodesics by using normal property.
- Learn about torsion of geodesic and geodesic tangent.

13.2 INTRODUCTION

This chapter explores the fundamental concepts of geodesics, including their derivation through canonical equations and their behavior on surfaces of revolution. It examines the normal properties of geodesics and how it leads to differential equations

that describe their paths. Further, the chapter discusses the torsion of geodesics and introduces the concept of geodesic tangent, offering a comprehensive understanding of the geometry of curves constrained to surfaces.

13.3 GEODESICS

Definition (a): Geodesic in a surface is defined as the curve of stationary length (rather than strictly shortest distance) on a surface between any two points in its plane.

Definition (b): In E_3, geodesic is also defined as the curve whose curvature relative to the surface is everywhere zero.

Definition (c): Geodesic in a Riemannian V_n is defined as the curve of minimum (or maximum) length joining two points on it.

Definition (d): Geodesic is the curve whose first curvature relative to V_n is zero at all points.

DERIVATION OF EQUATIONS OF GEODESICS

A curve on a surface joining two points in an n-dimensional space is called the geodesic if its length is extremum.

The distance between any two points A and B lying on a surface is given by

$$ds^2 = g_{ij}(x^i)\, dx^i dx^j$$

Considering the variation

$$\partial(ds^2) = \left(\frac{\partial g_{ij}}{\partial x^k}\right)\partial x^k\, dx^i dx^j + g_{ij}\,\partial\left(dx^i\right)dx^j + g_{ij}\,dx^i\,\partial\left(dx^j\right) \qquad (i \leftrightarrow j)$$

$$\Rightarrow 2ds\;\partial(ds) = \frac{\partial g_{ij}}{\partial x^k}\partial x^k\, dx^i dx^j + 2g_{ij}\,\partial(dx^i)\,dx^j$$

$$\Rightarrow \partial(ds) = \left[\frac{1}{2}\frac{\partial g_{ij}}{\partial x^k}\partial x^k\frac{dx^i}{ds}\frac{dx^j}{ds} + g_{ij}\frac{\partial(dx^i)}{ds}\frac{dx^j}{ds}\right]ds$$

$$\Rightarrow \int_A^B d(\partial s) = \int_A^B \left[\frac{1}{2}\frac{\partial g_{ij}}{\partial x^k}\partial x^k\frac{dx^i}{ds}\frac{dx^j}{ds} + g_{ij}\frac{d}{ds}\left(\partial x^i\right)\frac{dx^j}{ds}\right]ds \qquad (1)$$

Now,

$$\int_A^B \left(g_{ij}\frac{d}{ds}(\partial x^i)\frac{dx^j}{ds}\right)ds$$

$$= \left[g_{ij} \frac{dx^j}{ds} \partial x^i \right]_A^B - \int_A^B \frac{d}{ds}\left(g_{ij} \frac{dx^j}{ds} \right) \partial x^i \, ds$$

$$= -\int_A^B \frac{d}{ds}\left(g_{ij} \frac{dx^j}{ds} \right) \partial x^i \, ds \qquad \left[\because \int \frac{d}{ds}(\partial x^i) ds = \partial x^i \ \text{ and } \left[g_{ij} \frac{dx^j}{ds} \delta x^i \right]_A^B = 0 \right]$$

From (1), we get

$$[\partial s]_A^B = \int_A^B \left[\frac{1}{2} \frac{\partial g_{ij}}{\partial x^k} \frac{dx^i}{ds} \frac{dx^j}{ds} \partial x^k - \frac{d}{ds}\left(g_{ij} \frac{dx^j}{ds} \right) \partial x^i \right] ds$$

$$\Rightarrow 0 = \int_A^B \left[\frac{1}{2} \frac{\partial g_{ij}}{\partial x^k} \frac{dx^i}{ds} \frac{dx^j}{ds} \partial x^k - \frac{d}{ds}\left(g_{kj} \frac{dx^j}{ds} \right) \partial x^k \right] ds \qquad (i \leftrightarrow k)$$

$$\Rightarrow \int_A^B \left[\frac{1}{2} \frac{\partial g_{ij}}{\partial x^k} \frac{dx^i}{ds} \frac{dx^j}{ds} - \frac{\partial g_{jk}}{\partial x^i} \frac{dx^i}{ds} \frac{dx^j}{ds} - g_{jk} \frac{d^2 x^j}{ds^2} \right] \partial x^k ds = 0, \quad \left[\because \frac{d}{ds}(g_{kj}) = \frac{\partial g_{jk}}{\partial x^i} \frac{dx^i}{ds} \right]$$

Now, for arbitrary ∂x^k *i.e.* $\partial x^k \neq 0$, we must have which is the differential equation of geodesics.

13.4 CANONICALGEODESIC EQUATIONS

If the parameter t is occurring in the geodesic differential equation of the curve is replaced by s, the arc length of the curve measured some fixed point, then the corresponding differential equation of the geodesics are called canonical geodesic equation.

Theorem 1: If the arc length s is the parameter of the curve, then the geodesic equations are

$$U = \frac{d}{dt}\left(\frac{\partial T}{\partial u'} \right) - \frac{\partial T}{\partial u} = 0$$

$$V = \frac{d}{dt}\left(\frac{\partial T}{\partial v'} \right) - \frac{\partial T}{\partial v} = 0 \qquad (1)$$

The equations (1) are called canonical geodesic equations.

Proof: Since the geodesic equations are true for any arbitrary parameter t, it is equally true for the parameter s also, the geodesic equations become

$$U = \frac{d}{ds}\left(\frac{\partial T}{\partial u'} \right) - \frac{\partial T}{\partial u} = \frac{1}{2T} \frac{dT}{ds} \frac{\partial T}{\partial u'} \qquad (2)$$

$$V = \frac{d}{ds}\left(\frac{\partial T}{\partial v'}\right) - \frac{\partial T}{\partial v} = \frac{1}{2T}\frac{dT}{ds}\frac{\partial T}{\partial v'} \qquad (3)$$

where

$$T = \frac{1}{2}[Eu'^2 + 2Fu'v' + Gv'^2]$$

Since,

$$u' = \frac{du}{ds} = l, \qquad v' = \frac{dv}{ds} = m$$

are the directional coefficients at a point on the curve, and

$$El^2 + 2Flm + Gm^2 = 1$$

$$\therefore \quad T = \frac{1}{2} \text{ so that } \frac{dT}{ds} = 0 \qquad (4)$$

Using (4) in the right-hand side of (2) and (3), we obtain the canonical geodesic equations (1).

Note: One should note that $T = \frac{1}{2}$ only along the curve and it is not equal to $\frac{1}{2}$ identically for all values of u, v, u', v'. The partial derivatives $\frac{dT}{du'}$ and $\frac{dT}{dv'}$ are calculated from T before we substitute the values of u' and v' in T.

(i) If the curves on a surface are not parametric curves, then the sufficient condition for a curve to be geodesic is either $U = 0$ or $V = 0$.

(ii) For a parametric curve $u = \text{constant}$ to be a geodesic, a sufficient condition is $U = 0$, and $v = \text{constant}$ to be a geodesic, the sufficient condition is $V = 0$.

Proof: (i) When s is used as a parameter, then

$$Uu' + Vv' = \frac{dT}{ds}$$

Since $\frac{dT}{ds} = 0$ by (4) of the previous theorem, we have

If the curves are not parametric curves, $u' \neq 0$ and $v' \neq 0$. This implies from (1) that U and V are not independent. Hence U is a scalar multiple of V and vice versa, so that either $U = 0$ or $V = 0$ is a sufficient condition for a curve to be geodesic.

(ii) For a curve to be geodesic on a surface, it should satisfy the following canonical equations of previous theorem 1.

$$U = \frac{d}{ds}\left(\frac{\partial T}{\partial u'}\right) - \frac{\partial T}{\partial u} = 0 \qquad (1)$$

$$V = \frac{d}{ds}\left(\frac{\partial T}{\partial v'}\right) - \frac{\partial T}{\partial v} = 0 \qquad (2)$$

If we take the parametric curve $u = \text{constant}$, so that

$$u' = 0, v' \neq 0$$

Using this, (1) implies that $V = 0$ and conversely. Hence the equation (3) is automatically satisfied for all s. So the condition for $u = \text{constant}$ to be a geodesic is $U = 0$.

In a similar manner, $V = 0$ is a sufficient condition for the parametric curve $v = \text{constant}$ to be a geodesic.

13.5 NATURE OF GEODESICS ON ASURFACE OF REVOLUTION

At every point of geodesic, the principal normal $(\hat{n})$ is normal to the surface $(\hat{N})$, i.e.

$$\hat{n} = \hat{N}$$

To prove the above, we shall establish the following identities:

$$\frac{\partial T}{\partial \dot{u}} = \dot{\mathrm{r}} \cdot \mathrm{r}_1, \qquad \frac{\partial T}{\partial \dot{v}} = \dot{\mathrm{r}} \cdot \mathrm{r}_2; \quad U(t) = \ddot{\mathrm{r}} \cdot \mathrm{r}_1, \qquad V(t) = \ddot{\mathrm{r}} \cdot \mathrm{r}_2$$

where

$$T = \frac{1}{2}(E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2)$$

and t is the parameter.

Again,

$$T = \frac{1}{2}(E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2) \qquad (1)$$

$$\dot{\mathrm{r}} = \mathrm{r}_1\dot{u} + \mathrm{r}_2\dot{v}$$

$$\dot{\mathrm{r}}^2 = r^2\dot{u}^2 + 2\,\mathrm{r}_1 \cdot \mathrm{r}_2\,\dot{u}\dot{v} + r_2^2\dot{v}^2 \qquad (2)$$

Combining (1) and (2), we get

$$T = \frac{1}{2}\dot{\mathrm{r}}^2$$

so that

$$\frac{\partial T}{\partial \dot{u}} = \dot{\mathrm{r}} \cdot \frac{\partial \dot{\mathrm{r}}}{\partial \dot{u}} = \dot{\mathrm{r}} \cdot \frac{\partial}{\partial \dot{u}}(\mathrm{r}_1\dot{u} + \mathrm{r}_2\dot{v})$$

$$= \dot{\mathrm{r}} \cdot \frac{\partial}{\partial \dot{u}}(\mathrm{r}_1\dot{u}) + \dot{\mathrm{r}} \cdot \frac{\partial}{\partial \dot{u}}(\mathrm{r}_2\dot{v}) = \dot{\mathrm{r}} \cdot \mathrm{r}_1$$

Similarly,

$$\frac{\partial T}{\partial \dot{v}} = \dot{r} \cdot \frac{\partial}{\partial \dot{v}}(r_1\dot{u} + r_2\dot{v})$$

$$= \dot{r} \cdot (r_{11}\dot{u} + r_{21}\dot{v}) = \dot{r} \cdot \frac{d}{dt}(r_1)$$

Substituting in the equations,

$$U(t) = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) - \frac{\partial T}{\partial u}$$

$$V(t) = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right) - \frac{\partial T}{\partial v}$$

We get

$$U(t) = \frac{d}{dt}(\dot{r} \cdot r_1) - \dot{r} \cdot \frac{d}{dt}(r_1) = \ddot{r} \cdot r_1$$

Also, we have

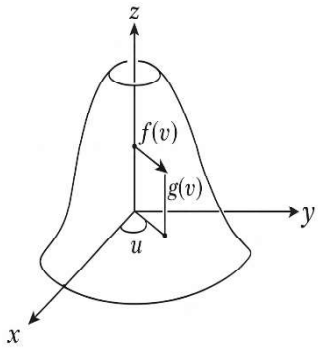
$$V(t) = \ddot{r} \cdot r_2$$

In case of s as parameter, we get

$$\ddot{r} \cdot r_1 = 0 \text{ and } \ddot{r} \cdot r_2 = 0$$

[$\because U(s) = 0$ and $V(s) = 0$ are the geodesic equations]

This shows that $\ddot{r}$ is perpendicular to both r_1 and r_2, therefore along the normal to the surface since r_1 and r_2 lie in the tangent plane to the surface. But $\ddot{r}$ is along the principal normal to the curve. Hence, we see that at every point of a geodesic, the principal normal is normal to the surface.



13.6 NORMALPROPERTY OF GEODESICS

Consider the surface of revolution parametrized by

$$X(u,v) = (f(v)\cos u,\ f(v)\sin u,\ g(v))$$

Then,

$$X_u = (-f\sin u,\ f\cos u,\ 0)$$

$$X_v = (f'\cos u,\ f'\sin u,\ g')$$

Again,

$$E = \langle X_u, X_u \rangle = f^2$$

$$F = \langle X_u, X_v \rangle = 0$$

$$G = \langle X_v, X_v \rangle = f'^2 + g'^2$$

and

$$\Gamma_{11}^1 = 0; \quad \Gamma_{11}^2 = -\frac{ff'}{(f'^2+g'^2)}; \quad \Gamma_{12}^1 = \frac{f'}{f}$$

$$\Gamma_{12}^2 = 0; \quad \Gamma_{22}^1 = 0; \quad \Gamma_{22}^2 = \frac{(f'f''+g'g'')}{(f'^2+g'^2)}$$

Now, the geodesic equations are

$$(1)u'' + u'\Gamma_{11}^1 u' + u'\Gamma_{12}^1 v' + v'\Gamma_{21}^1 u' + v'\Gamma_{22}^1 v' = 0$$

$$(2)v'' + u'\Gamma_{11}^2 u' + u'\Gamma_{12}^2 v' + v'\Gamma_{21}^2 u' + v'\Gamma_{22}^2 v' = 0$$

Inserting the actual values for the Christoffel symbols gives

$$(1)u'' + 2\left(\frac{f'}{f}\right)u'v' = 0$$

$$(2)v'' + \left(\frac{-ff'}{f^2+g^2}\right)u'^2 + \left(\frac{ff''+gg''}{f'^2+g'^2}\right)v'^2 = 0$$

Clairaut's Theorem:

Geodesics on the surface of revolution $X(u,v) = (f(v)\cos u,\ f(v)\sin u,\ g(v))$ are characterized by the equation $f\cos\alpha = \text{constant}$.

Example 1: Find geodesic curvature of the parametric curve $v = c$.

Solution: Obviously, u can be taken as parameter for the parametric curve $v = c$.

Then,

$$\dot{v} = 0, \ \dot{u} = 1 \qquad (1)$$

$$\therefore \ T = \frac{1}{2}(E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2)$$

$$\frac{\partial T}{\partial u} = \frac{1}{2}(E_1\dot{u}^2 + 2F_1\dot{u}\dot{v} + G_1\dot{v}^2)$$

$$\frac{\partial T}{\partial v} = \frac{1}{2}(E_2\dot{u}^2 + 2F_2\dot{u}\dot{v} + G_2\dot{v}^2)$$

$$\frac{\partial T}{\partial \dot{u}} = E\dot{u} + F\dot{v}, \qquad \frac{\partial T}{\partial \dot{v}} = F\dot{u} + G\dot{v} \qquad (2)$$

In view of (1), these expressions become

$$\frac{\partial T}{\partial u} = \frac{1}{2}E_1, \qquad \frac{\partial T}{\partial v} = \frac{1}{2}E_2, \qquad \frac{\partial T}{\partial \dot{u}} = E, \qquad \frac{\partial T}{\partial \dot{v}} = F.$$

Now, we know that

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$$

$$U \equiv \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) - \frac{\partial T}{\partial u},$$

$$V \equiv \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right) - \frac{\partial T}{\partial v}.$$

Putting $\dot{u} = 1,\ \dot{v} = 0$ in these expressions, we get

$$U = \frac{d}{dt}(E) - \frac{1}{2}E_1 = (E_1\dot{u} + E_2\dot{v}) - \frac{1}{2}E_1 = E_1 - \frac{1}{2}E_1 = \frac{1}{2}E_1$$

$$V = \frac{d}{dt}(F) - \frac{1}{2}E_2 = (F_1\dot{u} + F_2\dot{v}) - \frac{1}{2}E_2 = F_1 - \frac{1}{2}E_2$$

$$\dot{s}^2 = E$$

Hence, by definition of geodesic curvature

$$K_g = \frac{1}{2H\dot{s}^3}\left\{\frac{\partial T}{\partial \dot{u}}\,V(t) - \frac{\partial T}{\partial \dot{v}}\,U(t)\right\}$$

$$= \frac{2EF_1 - EE_2 - FE_1}{2HE^{3/2}}$$

Example 2: Prove that straight lines on a surface are the only asymptotic lines which are geodesic.

Solution: Let $r \equiv r(s)$ be the equation of curve on the surface which is an asymptotic line as geodesic.Then, since it is geodesic $N = n$ and asymptotic line $N = b$.

Hence, if the asymptotic lines are geodesic, then

$$b = n$$

Differentiating last relation w.r.t. the arc length, we get

$$b' = n'$$

$$\Rightarrow -\tau n = \tau b - kt$$

Equating the coefficients of t on both sides, we have

$$k \equiv 0$$

This implies that the curve must be a straight line.

Example 3: Find the differential equation of the geodesic in the space with the metric

$$ds^2 = -e^{2kt}(dx^2 + dy^2 + dz^2) + dt^2$$

Solution: In this case,

$$g_{11} = g_{22} = g_{33} = -e^{2kt}, g_{44} = 1, g_{ij} = 0 \text{ when } i \neq j$$
$$x^1 = x, x^2 = y, x^3 = z, x^4 = t$$

The Christoffel symbols are given by

$$\Gamma_{14}^1 = k, \quad \Gamma_{24}^2 = k, \quad \Gamma_{34}^3 = k$$
$$\Gamma_{11}^4 = \Gamma_{22}^4 = \Gamma_{33}^4 = ke^{2kt}$$

But the equations of geodesics are

$$\frac{d^2x^i}{ds^2} + \Gamma_{jk}^i \frac{dx^j}{ds}\frac{dx^k}{ds} = 0 \qquad (1)$$

For $i = 1$

$$\frac{d^2x^1}{ds^2} + \Gamma_{jk}^1 \frac{dx^j}{ds}\frac{dx^k}{ds} = 0$$
$$\Rightarrow \frac{d^2x}{ds^2} + \Gamma_{14}^1 \frac{dx^1}{ds}\frac{dx^4}{ds} + \Gamma_{41}^1 \frac{dx^4}{ds}\frac{dx^1}{ds} = 0$$
$$\Rightarrow \frac{d^2x}{ds^2} + 2k\frac{dx}{ds}\frac{dt}{ds} = 0 \qquad (2)$$

For $i = 2$

$$\frac{d^2x^2}{ds^2} + \Gamma_{jk}^2 \frac{dx^j}{ds}\frac{dx^k}{ds} = 0$$
$$\Rightarrow \frac{d^2y}{ds^2} + 2k\frac{dy}{ds}\frac{dt}{ds} = 0 \qquad (3)$$

For $i = 3$, similarly we get

$$\frac{d^2z}{ds^2} + 2k\frac{dz}{ds}\frac{dt}{ds} = 0 \qquad (4)$$

For $i = 4$,

$$\frac{d^2x^4}{ds^2} + \Gamma_{jk}^4 \frac{dx^j}{ds}\frac{dx^k}{ds} = 0$$
$$\Rightarrow \frac{d^2t}{ds^2} + ke^{2kt}\left[\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2\right] = 0 \qquad (5)$$

Equations (2), (3), (4) and (5) are the differential equations of geodesics.

From (2), we get

$$\frac{\frac{d}{ds}\left(\frac{dx}{ds}\right)}{\frac{dx}{ds}} = -2k\frac{dt}{ds}$$

$$\Rightarrow \frac{d\left(\frac{dx}{ds}\right)}{\frac{dx}{ds}} = -2k\,dt$$

$$\Rightarrow \log \frac{dx}{ds} = -2kt + \log A$$

$$\Rightarrow \frac{dx}{ds} = Ae^{-2kt} \qquad (6)$$

Similarly, we can write the result of (3) and (4) as

$$\frac{dy}{ds} = Be^{-2kt} \qquad (7)$$

$$\frac{dz}{ds} = Ce^{-2kt} \qquad (8)$$

Putting the values of $\frac{dx}{ds}$, $\frac{dy}{ds}$, $\frac{dz}{ds}$ in equation (5), we get

$$\frac{d^2t}{ds^2} + ke^{2kt}[(A^2 + B^2 + C^2)e^{-4kt}] = 0$$

$$\Rightarrow \frac{d^2t}{ds^2} + \alpha^2 e^{-2kt} = 0 \qquad (9)$$

Where $\alpha^2 = k(A^2 + B^2 + C^2)$

Equations (6), (7), (8) and (9) are actual differential equations of geodesic.

CHECK YOUR PROGRESS

Q.1. Discuss the nature of geodesics on a plane.

Q.2. Prove that on a surface with metric $ds^2 = \lambda^2 du^2 + \mu^2 dv^2$, the geodesic curvature of the curve $u = c$ is $(\lambda\mu)^{-1}\left(\frac{\partial \mu}{\partial u}\right)$.

Q.3. Obtain the non-vanishing 3index symbols for the metric

$$ds^2 = f(x)dx^2 + dy^2 + dz^2 + \frac{1}{f(x)}dt^2$$

and hence find the differential equations of the geodesics for this space.

13.7 DIFFERENTIAL EQUATIONS OF GEODESICS BY USING NORMAL PROPERTY

We know that

$$\vec{r} = \vec{r}(u,v)$$

$$r' = \frac{dr}{ds} = \frac{\partial r}{\partial u}\frac{du}{ds} + \frac{\partial r}{\partial v}\frac{dv}{ds}$$

or

$$r' = r_1 u' + r_2 v'$$

Differentiating w.r.t. parameter s, we get

$$r'' = r_1 u'' + r_2 v'' + (r_{11}u'^2 + 2r_{12}u'v' + r_{22}v'^2)$$

$$\therefore \quad r'' \cdot r_1 = Eu'' + Fv'' + r_1 \cdot r_{11}u'^2 + 2r_1 \cdot r_{12}u'v' + r_1 \cdot r_{22}v'^2$$

But $r'' \cdot r_1 = 0$, for a geodesic. Therefore, we have

$$Eu'' + Fv'' + r_1 \cdot r_{11}u'^2 + 2r_1 \cdot r_{12}u'v' + r_1 \cdot r_{22}v'^2 = 0 \qquad (1)$$

Similarly, by $r'' \cdot r_2 = 0$, we get

$$Fu'' + Gv'' + r_2 \cdot r_{11}u'^2 + 2r_2 \cdot r_{12}u'v' + r_2 \cdot r_{22}v'^2 = 0 \qquad (2)$$

Now,

$$r_1 \cdot r_1 = E, \qquad r_1 \cdot r_2 = F, \qquad r_2 \cdot r_2 = G$$

$$r_1 \cdot r_{11} = \frac{1}{2}\frac{\partial}{\partial u}(r_1^2) = \frac{1}{2}\frac{\partial E}{\partial u} = \frac{1}{2}E_1$$

$$r_1 \cdot r_{12} = \frac{1}{2}\frac{\partial}{\partial v}(r_1^2) = \frac{1}{2}\frac{\partial E}{\partial v} = \frac{1}{2}E_2$$

and

$$r_1 \cdot r_{22} = \frac{\partial}{\partial v}(r_1 \cdot r_2) - \frac{1}{2}\frac{\partial}{\partial u}(r_2^2) = F_2 - \frac{1}{2}G_1$$

Similarly,

$$r_2 \cdot r_{11} = F_1 - \frac{1}{2}E_2, \qquad r_2 \cdot r_{12} = \frac{1}{2}G_1, \qquad r_2 \cdot r_{22} = \frac{1}{2}G_2$$

Hence the above equations (1) and (2) may be written as

$$Eu'' + Fv'' + \frac{1}{2}E_1u'^2 + E_2u'v' + \left(F_2 - \frac{1}{2}G_1\right)v'^2 = 0 \qquad (3)$$

$$Fu'' + Gv'' + \left(F_1 - \frac{1}{2}E_2\right)u'^2 + G_1u'v' + \frac{1}{2}G_2v'^2 = 0 \qquad (4)$$

These give the general differential equations of geodesics on a surface.

Further, these equations are equivalent to the equations

$$\frac{d}{ds}(Eu' + Fv') = \frac{1}{2}(E_1u'^2 + 2F_1u'v' + G_1v'^2) \quad (3')$$

$$\frac{d}{ds}(Fu' + Gv') = \frac{1}{2}(E_2u'^2 + 2F_2u'v' + G_2v'^2) \quad (4')$$

which can be remembered very easily.

Further, the third form is obtained by solving the equations (3) and (4) for u'' and v'':

$$u'' = -(lu'^2 + 2mu'v' + nv'^2) \quad (5)$$

$$v'' = -(\lambda u'^2 + 2\mu u'v' + \nu v'^2) \quad (6)$$

where

$$l = \frac{1}{2H^2}(GE_1 - 2FF_1 + FE_2)$$

$$m = \frac{1}{2H^2}(GE_2 - FG_1)$$

$$n = \frac{1}{H^2}(2GF_2 - 2GG_1 - FE_2)$$

$$\lambda = \frac{1}{2H^2}(2EF_1 - EF_2 - FE_1)$$

$$\mu = \frac{1}{2H^2}(EG_1 - FE_1)$$

and

$$\nu = \frac{1}{2H^2}(EG_2 - 2FF_2 + FG_1)$$

But, a curve on a surface is determined if a relation between the parameters u, v can be found. Hence the above pair of differential equations can be replaced by a single equation.

Now, multiplying (5) and (6) by

$$\frac{dv}{du}\left(\frac{ds}{du}\right)^2 \text{ and } \left(\frac{ds}{du}\right)^2$$

respectively and subtracting, we obtain

$$\frac{d^2v}{du^2} = n\left(\frac{dv}{du}\right)^3 + (2m - \nu)\left(\frac{dv}{du}\right)^2 - (l - 2\mu)\frac{dv}{du} - \lambda$$

This is the single differential equation of a geodesic for a given surface.

13.8 TORSION OF GEODESICS

Let the position vector of any point P on the geodesic be r.

From the normal property of the geodesic, we have $n = N$.

Also,

$$b = t \times n = t \times N \qquad (1)$$

Differentiating w.r.t. the arc length s of the geodesic, we get

$$b' = t' \times N + t \times N'$$

$$\Rightarrow -\tau n = \kappa n \times N + t \times N' \left[\because \frac{db}{ds} = -\tau n\right]$$

$$\Rightarrow -\tau N = r' \times N' [\because n \times N = 0 \text{ and } t = r']$$

Multiplying scalarly both sides with the vector N, we get

$$-(\tau N) \cdot N = (r' \times N') \cdot N$$

$$\Rightarrow \tau = N \cdot (N' \times r') = [N, N', r'] \qquad (3)$$

The formula (3) gives the torsion of a geodesic on the surface.

Example 4: Prove that $\tau = 0$ at an umbilic.

Proof: We know that for an umbilic

$$\frac{L}{E} = \frac{M}{F} = \frac{N}{G}$$

$$EM - FL = 0, EN - GL = 0, FN - GM = 0$$

Also, we have

$$H[N, N', r'] = (EM - FL)u'^2 + (EN - GL)u'v' + (FN - GM)v'^2$$

$$\therefore H[N, N', r'] = 0 \text{ at umbilic}$$

$$\Rightarrow [N, N', r'] = 0$$

$$\Rightarrow \tau = 0 \text{ at an umbilic.}$$

13.9 GEODESIC TANGENTS

Definition: Let P be a point on the curve C. Then the geodesic tangent of curve C at point P is the geodesic which touches the curve at P. Thus, geodesic tangent at any point on a curve is the geodesic which touches the curve at the point.

Example 5: Prove that the torsion of an asymptotic line and geodesic tangent are equal.

Proof: We know that $[N, N', r']$ is the torsion of the asymptotic line at a point P.

Also, we know that $\tau = [N, N', r']$ is the torsion of a geodesic at that point P.

Since both the curves touch at P, therefore N, N' and r' are the same at point P.

Hence the result.

Note: Torsion of a geodesic in terms of principal curvatures is given by

$$\tau = (k_b - k_a)\cos\psi\sin\psi$$

where k_a and k_b are the principal curvatures.

CHECK YOUR PROGRESS

Q.4. Prove that the torsion of two geodesics at right angles are equal in magnitude but opposite in sign.

Q.5. Prove that the geodesic torsion of the direction which bisects the angles between the lines of curvature is greatest.

13.10 LET US SUM UP

- Geodesics is a curve of shortest distance between two points on the surface.
- The differential equation of a geodesic is given by

$$\frac{d^2x^\alpha}{ds^2} + \Gamma_{ij}^\alpha \frac{dx^i}{ds}\frac{dx^j}{ds} = 0$$

where the symbols have their usual meanings.

- The canonical geodesic equations are —

$$U = \frac{d}{dt}\left(\frac{\partial T}{\partial u'}\right) - \frac{\partial T}{\partial u} = 0, V = \frac{d}{dt}\left(\frac{\partial T}{\partial v'}\right) - \frac{\partial T}{\partial v} = 0.$$

- At every point of geodesic, the principal normal $(\hat{n})$ is normal to the surface $(\hat{N})$, which is the normal property of a geodesic.
- $u'' = -(l u'^2 + 2m u' v' + n v'^2), v'' = -(\lambda u'^2 + 2\mu u' v' + \nu v'^2)$

which are the differential equations of a geodesic by using normal property.

- $\tau = [N, N', r']$ gives the torsion of a geodesic on the surface, where the symbols have their usual meanings.
- Geodesic tangent at any point on a curve is the geodesic which touches the curve at that point.

13.11 ANSWERS TO CHECK YOUR PROGRESS

Answer to Q1: Let $r = r(s)$ be the equation of a curve on a plane.

Then,

$$k_g = [N, r', r''] = [N, t, kn]$$

$$= kN \cdot (t \times n) = kN \cdot b$$

Since the surface is plane, both the vectors n and t lie on the plane and therefore

$$b = t \times n = N$$

Then,

$$k_g = kN \cdot N = k$$

$$\therefore k_g = 0 \iff k = 0$$

Now, $k_g = 0$ for a geodesic and $k = 0$ for a straight line. Hence a curve on a plane is a geodesic if and only if it is a straight line. In other words, every geodesic on a plane is a straight line and vice-versa.

Answer to Q2: Comparing the given metric

$$ds^2 = \lambda^2 du^2 + \mu^2 dv^2$$

with the metric

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$$

and remembering that for the curve $u = c$, $u' = 0$, we get

$$E = \lambda^2, F = 0, G = \mu^2 \text{ and } ds^2 = \mu^2 dv^2$$

Or $$E = \lambda^2, F = 0, G = \mu^2, v'^2 = \frac{1}{\mu^2}.$$

Now, since $$H^2 = (EG - F^2)$$

Therefore,

$$H^2 = \lambda^2\mu^2 - 0$$

$$\Rightarrow H = \lambda\mu.$$

Again,

$$T = \frac{1}{2}(Eu'^2 + 2Fu'v' + Gv'^2)$$

$$= \frac{1}{2}(\lambda^2 u'^2 + \mu^2 v'^2)$$

$$\therefore \frac{\partial T}{\partial u} = \lambda \frac{\partial \lambda}{\partial u} u'^2 + \mu \frac{\partial \mu}{\partial u} v'^2$$

$$\frac{\partial T}{\partial u'} = \lambda^2 u'$$

or

$$\frac{\partial T}{\partial u} = \frac{1}{\mu} \frac{\partial \mu}{\partial u}, \frac{\partial T}{\partial u'} = 0$$

Now,

$$U \equiv \frac{d}{ds}\left(\frac{\partial T}{\partial u'}\right) - \frac{\partial T}{\partial u} = 0 - \frac{1}{\mu}\frac{\partial \mu}{\partial u} = -\frac{1}{\mu}\frac{\partial \mu}{\partial u}.$$

Therefore the geodesic curvature of the given curve $u = c$is given by

$$k_g = \frac{-U(s)}{Hv'} = \frac{\frac{1}{\mu}\frac{\partial \mu}{\partial u}}{\lambda\mu\left(\frac{1}{\mu}\right)} = \frac{1}{\lambda\mu}\frac{\partial \mu}{\partial u}$$

$$= (\lambda\mu)^{-1}\left(\frac{\partial \mu}{\partial u}\right).$$

Answer to Q3: We have

$$ds^2 = f(x)\,dx^2 + dy^2 + dz^2 + \frac{1}{f(x)}\,dt^2$$

$$\therefore\ g_{11} = f(x), g_{22} = 1, g_{33} = 1, g_{44} = \frac{1}{f(x)}$$

and $g_{ij} = 0, i \neq j$

Also, $x^1 = x, x^2 = y, x^3 = z, x^4 = t.$

∴For orthogonal spaces,

$g_{ij} = 0, i \neq j$and$g^{ii} = \frac{1}{g^{ii}}$for all i.

The non-vanishing Christoffel symbols are

$$\Gamma_{11}^1 = \frac{1}{2f}\frac{\partial f}{\partial x},$$

$$\Gamma_{44}^1 = \frac{1}{2f^3}\frac{\partial f}{\partial x},$$

$$\Gamma_{41}^4 = -\frac{1}{2f}\frac{\partial f}{\partial x}, \ldots\ldots\ldots etc$$

The equations of geodesics are

$$\frac{d^2x^i}{ds^2} + \Gamma_{jk}^i \frac{dx^j}{ds}\frac{dx^k}{ds} = 0, i = 1,2,3,4.$$

For $i = 1$:

$$\frac{d^2x^1}{ds^2} + \Gamma^1_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0$$

$$\Rightarrow \frac{d^2x}{ds^2} + \Gamma^1_{11}\frac{dx^1}{ds}\frac{dx^1}{ds} + \Gamma^1_{44}\frac{dx^4}{ds}\frac{dx^4}{ds} = 0$$

$$\Rightarrow \frac{d^2x}{ds^2} + \frac{1}{2f}\frac{\partial f}{\partial x}\left[\left(\frac{dx}{ds}\right)^2 + \frac{1}{f^2}\left(\frac{dt}{ds}\right)^2\right] = 0 \quad (1)$$

For $i = 2$:

$$\frac{d^2x^2}{ds^2} + \Gamma^2_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0$$

$$\Rightarrow \frac{d^2y}{ds^2} = 0 \quad (2)$$

For $i = 3$:

$$\frac{d^2x^3}{ds^2} + \Gamma^3_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0$$

$$\Rightarrow \frac{d^2z}{ds^2} = 0 \quad (3)$$

For $i = 4$:

$$\frac{d^2x^4}{ds^2} + \Gamma^4_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0$$

$$\Rightarrow \frac{d^2t}{ds^2} + \Gamma^4_{41}\frac{dx^4}{ds}\frac{dx^1}{ds} + \Gamma^4_{14}\frac{dx^1}{ds}\frac{dx^4}{ds} = 0$$

$$\Rightarrow \frac{d^2t}{ds^2} - \frac{1}{f}\frac{\partial f}{\partial x}\frac{dx}{ds}\frac{dt}{ds} = 0$$

$$\Rightarrow \frac{\frac{d^2t}{ds^2}}{\frac{dt}{ds}} = \frac{1}{f}\frac{\partial f}{\partial x}\frac{dx}{ds}$$

$$\Rightarrow \frac{\frac{d^2t}{ds^2}}{\frac{dt}{ds}} = \frac{d}{ds}(\log f(x))\frac{dx}{ds}$$

$$\Rightarrow \int \frac{\frac{d^2t}{ds^2}}{\frac{dt}{ds}}\,ds = \int \frac{d}{ds}(\log f(x))\frac{dx}{ds}\,ds + \log c$$

$$\Rightarrow \log \frac{dt}{ds} = \log f(x) + \log c$$

$$\Rightarrow \frac{dt}{ds} = c f(x) (4)$$

The equations (1), (2), (3) and (4) represent the required equations of geodesics.

Answer to Q4: Let τ_1 and τ_2 be the torsions of two orthogonal geodesics through a point P. Then we have

$$\tau_1 = (K_b - K_a)\cos\psi\sin\psi$$

$$\tau_2 = (K_b - K_a)\cos\left(\psi + \frac{\pi}{2}\right)\sin\left(\psi + \frac{\pi}{2}\right)$$

$$= -(K_b - K_a)\sin\psi\cos\psi = -\tau_1$$

$$\Rightarrow \tau_1 = -\tau_2$$

Answer to Q5: We know that

$$\tau = (K_b - K_a)\cos\psi\sin\psi$$

$$= \frac{1}{2}(K_b - K_a)\sin 2\psi$$

$\therefore \tau$ is maximum when $\sin 2\psi = 1$

$$\Rightarrow \psi = \frac{\pi}{4}.$$

The angle between lines of curvature is $\left(\frac{\pi}{2}\right)$; hence the result follows.

13.12 FURTHER READING

- Weatherburn, C. E. *Differential Geometry of Three Dimensions*, Cambridge University Press, 2016.
- Graustein, W. C. *Differential Geometry*, Courier Corporation, 2012.
- Pressley, A. *Elementary Differential Geometry*, Springer, 2002.

13.13 MODEL QUESTIONS

Q.1. Define geodesics. Obtain the differential equations of geodesics in a given space.

Q.2. Obtain the non-vanishing 3-index symbols for the metric

$$ds^2 = -dx^2 - dy^2 - dz^2 + f(x,y,z)\,dt^2$$

Hence obtain the equation of geodesics.

Q.3. Write the normal property of geodesics. Obtain the differential equations of geodesics by using normal property.

Q.4. Define the nature of geodesics on a sphere and a cylinder.

Q.5. Write the torsion of a geodesic. Show that the torsion of a geodesic in terms of principal curvatures is given by

$$\tau = (K_b - K_a)\cos\psi\sin\psi.$$

where K_a and K_b are the principal curvatures.

Q.6. State and prove the Clairaut's theorem of a geodesic.

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