UNIT 14: GEODESICS CURVATURE

UNIT STRUCTURE

14.1. Learning Objectives.

14.2. Introduction.

14.3. Geodesic curvature.

14.4. Formula for geodesic curvature.

14.5. Geodesic parallels.

14.6. Geodesic coordinates.

14.7. Geodesic polar coordinates.

14.8. Angle between a curve on a surface and a geodesic through a pole.

14.9. Geodesic mapping.

14.10. Let us Sum Up.

14.11. Answers to Check Your Progress.

14.12. Further Reading.

14.13. Model questions.

14.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- Learn about geodesic curvature and understand the formula for geodesic curvature.
- Learn about geodesic parallels, geodesic coordinates and geodesic polar coordinates.
- Understand the angle between a curve on a surface and a geodesic through a pole.
- Learn about geodesic mapping.

14.2 INTRODUCTION

This chapter explores the fundamental aspects of **geodesics on surfaces**, beginning with geodesic curvature and its formula. It covers geodesic parallels, coordinates and polar coordinates, offering tools for surface analysis. The angle between a curve and a geodesic through a pole is examined, concluding with geodesic mappings which preserve geodesic structures under transformations.

14.3GEODESICS CURVATURE

We know that the curvature vector r'' at any point P on a curve can be expressed as a linear combination of vectors $N, r_1 \text{ and } r_2$ Therefore, we can write

$$r'' = K_n N + \lambda r_1 + \mu r_2 \qquad (1)$$

Obviously,

$$r'' \cdot N = K_n$$

The vector

$$K_g = \lambda r_1 + \mu r_2$$

which is tangential to the surface at P with components $(\lambda, \mu)$is called the geodesic curvature vector at P on the given curve.

Taking dot product of $r''$by r_1, we get

$$\begin{aligned} r'' \cdot r_1 &= K_n(N \cdot r_1) + \lambda r_1^2 + \mu r_2 \cdot r_1 \\ &= 0 + \lambda r_1^2 + \mu r_2 \cdot r_1 \qquad [\because N \cdot r_1 = 0] \\ \therefore U &= E\lambda + F\mu \left[\because U = \frac{d}{dt}\left(\frac{\partial T}{\partial u'}\right) - \frac{\partial T}{\partial u} = r'' \cdot r_1\right] \end{aligned} \qquad (2)$$

and

$$\begin{aligned} r'' \cdot r_2 &= 0 + \lambda r_1 \cdot r_2 + \mu r_2^2 \\ \therefore V &= \lambda F + G\mu \end{aligned} \qquad (3)$$

From (1) and (3), we get

$$\lambda = \frac{1}{H^2}(GU - FV)$$

$$\mu = \frac{1}{H^2}(EV - FU)$$

These are the components of geodesic curvature.

Illustrative Example

Example 1: Prove that the geodesic curvature vector of any curve is orthogonal to the curve.

Proof:Let

$$r'' = K_n N + \lambda r_1 + \mu r_2$$

$$\Rightarrow K_n = K_n N + \lambda r_1 + \mu r_2$$

$$\therefore K_n \cdot t = K_n N \cdot t + (\lambda r_1 + \mu r_2) \cdot t \qquad (1)$$

Again, $n \cdot t = 0$ and $N \cdot t = 0$

Therefore (1) is reduced to

$$0 = t \cdot (\lambda r_1 + \mu r_2)$$

$$\Rightarrow t \cdot K_g = 0 [\because K_g = \lambda r_1 + \mu r_2]$$

i.e. K_g is orthogonal to t.

Hence, the geodesic curvature vector K_g is orthogonal to the curve.

14.4 FORMULA FOR GEODESIC CURVATURE

We have the geodesic curvature vector K_g of a curve is orthogonal to the curve. Also, we know that K_g lies in the tangent plane and therefore perpendicular to the surface normal N. Thus K_g is orthogonal to both the unit vectors r' and N, and hence parallel to the unit vector $N \times r'$. Hence, we can write

$$K_g = k_g (N \times r'), \quad \text{where} \quad |K_g| = k_g$$

$$\therefore \; r'' = k_n N + k_g (N \times r')$$

$$\therefore (N \times r') \cdot r'' = k_g (N \times r') \cdot (N \times r') [\because (N \times r') \cdot N = 0]$$

$$\Rightarrow k_g = [n, r', r''] [\because (N \times r') \cdot (N \times r') = 1]$$

If we replace the parameter s by t,

$$k_g = \frac{[N, \dot{r}, \ddot{r}]}{\dot{s}^3} \left[\because r' = \frac{\dot{r}}{\dot{s}}, \quad etc\right]$$

But we know that

$$N = \frac{(r_1 \times r_2)}{H}$$

Therefore, we have

$$k_g = \frac{r_1 \times r_2}{H} \cdot \frac{\dot{r} \times \ddot{r}}{\dot{s}^3} = \frac{(r_1 \cdot \dot{r})(r_2 \cdot \ddot{r}) - (r_2 \cdot \dot{r})(r_1 \cdot \ddot{r})}{H \dot{s}^3} \qquad (1)$$

Also, we know that

$$\dot{r}\cdot r_1 = \frac{\partial T}{\partial \dot{u}}, \qquad \dot{r}\cdot r_2 = \frac{\partial T}{\partial \dot{v}}$$

$$\ddot{r}\cdot r_1 = U(t), \qquad \ddot{r}\cdot r_2 = V(t)$$

In view of this, (1) takes the form

$$k_g = \frac{1}{Hs^3}\left\{\frac{\partial T}{\partial \dot{u}}V(t) - \frac{\partial T}{\partial \dot{v}}U(t)\right\} \qquad (2)$$

Replacing parameter t by s in (2), we get

$$k_g = \frac{1}{H}\left\{\frac{\partial T}{\partial u'}V(s) - \frac{\partial T}{\partial v'}U(s)\right\}\left[\because \dot{s} = \frac{ds}{ds} = 1\right] \quad (3)$$

(2) and (3) give the geodesic curvature.

Another Form:

$$u'U(s) + v'V(s) = 0$$

$$\therefore U(s) = -\frac{v'}{u'}V(s)$$

Putting for $U(s)$ in (3), we get

$$k_g = \frac{V(s)}{Hu'}\left(u'\frac{\partial T}{\partial u'} + v'\frac{\partial T}{\partial v'}\right) = \frac{V(s)}{Hu'}\left[\because u'\frac{\partial T}{\partial u'} + v'\frac{\partial T}{\partial v'} = 2T = 1\right]$$

In terms of $U(s)$, we get

$$k_g = -\frac{U(s)}{Hv'}[\because u'U(s) + v'V(s) = 0]$$

Example 2: Prove that for any curve on a surface, the geodesic curvature vector is intrinsic.

Proof: To prove, (λ, μ) is intrinsic, we have to show that (λ, μ) can be found out from the metric of the surface.

$$r''\cdot r_1 = U = \frac{d}{ds}\left(\frac{\partial T}{\partial u'}\right) - \frac{\partial T}{\partial u}$$

$$r''\cdot r_2 = V = \frac{d}{ds}\left(\frac{\partial T}{\partial v'}\right) - \frac{\partial T}{\partial v} \qquad (1)$$

If (λ, μ) is the geodesic curvature vector at a point on the surface, then

$$r'' = K_n N + \lambda r_1 + \mu r_2 \quad (2)$$

Taking scalar product with r_1 and r_2 on the both sides of (2) respectively,

$$r''\cdot r_1 = K_n N\cdot r_1 + \lambda r_1\cdot r_1 + \mu r_2\cdot r_1$$

$$r''\cdot r_2 = K_n N\cdot r_2 + \lambda r_1\cdot r_2 + \mu r_2\cdot r_2 \quad (3)$$

Using $N \cdot r_1 = 0$ and $N \cdot r_2 = 0$ and the first fundamental metric coefficients, we have from (1)

$$r'' \cdot r_1 = U = Er + F\mu$$

$$r'' \cdot r_2 = V = \lambda F + \mu G \quad (4)$$

Solving for (λ, μ) in terms of U and V from (4), we obtain

$$\lambda = \frac{1}{H^2}(UG - HF)$$

$$\mu = \frac{1}{H^2}(EV - FU)$$

where

$$H^2 = EG - F^2$$

This shows that the vector (λ, μ) is intrinsic.

CHECK YOUR PROGRESS

Q1. Find the geodesic curvature of the curve $v = \text{constant}$ on the surface given by

$$r = \left(u\cos v,\ u\sin v,\ \frac{1}{2}au^2\right)$$

Q2. Obtain the geodesic curvature vector of a curve on a right helicoid

$$r = (u\cos v,\ u\sin v,\ av)$$

using different formula for it.

14.5 GEODESIC PARALLELS

Theorem: For any given family of geodesic on a surface, a parametric system can be chosen so that the metric takes the form

$$ds^2 = du^2 + G(u, v)\, dv^2.$$

The given geodesics are the parametric curves $v = \text{constant}$ and their orthogonal trajectories are given by $u = \text{constant}$, u being the distance measured along a geodesic from a fixed parallel.

Proof: For a given family of geodesic curves, let us take a system of parameters such that the geodesics of the family are given by

$$v = \text{constant}$$

and their trajectories are given by

$$u = \text{constant}.$$

Since $v = \text{constant}$ and $u = \text{constant}$ form an orthogonal parametric system,

$$F = 0.$$

We know that is geodesic iff

$$EE_2 + FE_1 - 2EF_1 = 0.$$

Since $E \neq 0$ and $F = 0$, the above condition reduces to $E_2 = 0$.

Thus E is independent of v and is a function of u only.

Therefore, the metric becomes

$$ds^2 = E(u)\,du^2 + G(u,v)\,dv^2 \qquad (1)$$

Now consider the orthogonal trajectories $u = u_1$ and $u = u_2$ and find the distance between them along the geodesic $v = \text{constant}$

Since, $v = c, \quad dv = 0$

$$\therefore \ (1) \Rightarrow ds^2 = E(u)du^2$$

$$\Longrightarrow ds = \sqrt{E(u)}du \qquad (2)$$

Integrating (2), we get

$$S = \int_{u_1}^{u_2} \sqrt{E(u)}du \qquad (3)$$

Since S is independent of v =constant, the distance between orthogonal trajectories is the same along any geodesic $v =$ constant.

Therefore, orthogonal trajectories are parallel.

Let the distance from some fixed point parallel to the neighbouring parallel be du.

Then $ds = du$ and $dv = 0$.

$$\therefore (1) \Rightarrow du^2 = E(u)\,du^2$$

$$\Rightarrow E(u) = 1$$

Hence, from (1)

$$ds^2 = du^2 + G(u,v)\,dv^2$$

Hence the proof.

Example 3: Define geodesic parametric. Write the geodesic form and give examples.

Solution:

The orthogonal trajectories of the given geodesic v =constant, on a surface are called geodesic parallels (family of geodesics) and u and v are called geodesic parameters.

For example: In the plane, we know that the straight lines are geodesics. Now, consider a family of straight lines enveloping a given curve C. This family of straight lines

envelops C so that C becomes the evolute and these family of straight lines are normals to the involute. Hence the geodesic parallels are the involutes of C.

CHECK YOUR PROGRESS

Q3. Prove that if a surface admits two orthogonal families of geodesic, then it is isometric with the plane.

14.6 GEODESIC COORDINATES

If the parametric curves are orthogonal and one of the families of parametric curves are geodesics, then the co-ordinates of any point on the surface are called a set of geodesic co-ordinates.

Since one of the parametric curves are assumed to be geodesic and other arbitrary, therefore the geodesic co-ordinate can be introduced in an arbitrary number of ways.

14.7GEODESIC POLAR COORDINATES

Theorem: Let O be a fixed point on the surface. Let us consider a family of geodesic at O. By the result of geodesics, we may say that there is a neighbourhood of O in which geodesics exist and we can take this to be the required family of geodesics at O.

Let us take the orthogonal trajectories of this family as geodesics. $u =$ constant, where u denotes the distance of the orthogonal trajectory from Oalong any geodesic. The distances are measured from O along the geodesics in the neighbourhood at$O$.

Let Pbe any point and u, vmay be distance and angle measured at O. This is same as the angle between the tangents to the geodesics through O. This angle $v$is like θin the polar plane.

The parameters (u, v) defined in this way are called the geodesic polar coordinates of$P$.

Since we measure $u$along a geodesic, the metric on the surface is

$$ds^2 = du^2 + G(u, v)\, dv^2 \qquad (1)$$

Using this we may write the following properties of metric approximation

$$\lim_{u \to 0} \frac{\sqrt{G}}{u} = 1$$

A small neighbourhood at $O$is a nearly plane and metric (1) should reduce to the polar form

$$dr^2 + r^2 d\theta^2$$

Since $u$and $v$correspondence to polar coordinates, the metric on the surface becomes

$$du^2 + u^2 dv^2$$

Identifying this metric (1), $G$is approximately $u^2$for points near O.

Hence, we have

$$\lim_{u \to 0} \frac{\sqrt{G}}{u} = 1$$

For the validity of the above metric approximation,$\sqrt{G}$ should have the power series expansion

$$\sqrt{G} = u + a_2 \frac{u^2}{2!} + a_3 \frac{u^3}{3!} + \cdots \ldots \ldots$$

The parametric system (u, v) discussed above is called geodesic polar coordinate system. The curves $u = \text{constant}$are called geodesic circles.

14.8 ANGLE BETWEEN A CURVE ON A SURFACE AND A GEODESIC THROUGH A POLE

In differential geometry, the angle between two curves on a surface is a fundamental concept. When one of the curves is a geodesic and the other is an arbitrary curve on the surface, the angle between them provides insight into how the surface is embedded and how curvature behaves.

A pole on a surface typically refers to a point from which geodesics radiate outward, like longitude lines from the north pole on a sphere. The angle between a geodesic emanating from a pole and a general curve on the surface can be studied using intrinsic geometry.

Let $S \subset \mathbb{R}^3$be a regular surface.

Let $p \in S$be a point, called the pole.

$\gamma(t)$be a curve on the surface $S$with

$$\gamma(0) = p$$

and $\sigma(s)$be a geodesic on the surface with

$$\sigma(0) = p.$$

We want to determine the angle between $\gamma$and σat the point P.

Let $v = \gamma'(0) \in T_p S$ be the tangent vector to the curve γ at point P and $w = \sigma'(0) \in T_p S$

be the tangent vector to the geodesic at P.

The angle $\theta \in [0, \pi]$ between the two curves at the point P is given by

$$\cos\theta = \frac{\langle v, w \rangle}{\| v \| \cdot \| w \|}$$

where $\langle \cdot , \cdot \rangle$ is the inner product induced by the surface metric and $\| \cdot \|$ is the norm induced by this inner product.

Now let (u, v) be local coordinates on the surface S around point P. Then the surface can be locally parametrized by

$$x(u, v): U \subset \mathbb{R}^2 \to \mathbb{R}^3$$

Let

$$\gamma(t) = x(u(t), v(t))$$

be the curve and

$$\sigma(s) = x(u_0(s), v_0(s))$$

be the geodesic.

The tangent vectors at P are

$$\gamma'(0) = \left.\frac{d\gamma}{dt}\right|_{t=0} = x_u \frac{du}{dt} + x_v \frac{dv}{dt}$$

$$\sigma'(0) = \left.\frac{d\sigma}{ds}\right|_{s=0} = x_u \frac{du}{ds} + x_v \frac{dv}{ds}$$

Then the angle θ between them at P is

$$\cos\theta = \frac{E\frac{du}{dt}\frac{du}{ds} + F\left(\frac{du}{dt}\frac{dv_0}{ds} + \frac{dv}{dt}\frac{du_0}{ds}\right) + G\frac{dv}{dt}\frac{dv_0}{ds}}{\sqrt{E\left(\frac{du}{dt}\right)^2 + 2F\frac{du}{dt}\frac{dv}{dt} + G\left(\frac{dv}{dt}\right)^2}\sqrt{E\left(\frac{du_0}{ds}\right)^2 + 2F\frac{du_0}{ds}\frac{dv_0}{ds} + G\left(\frac{dv_0}{ds}\right)^2}}$$

where,

$$E = \langle x_u, x_u \rangle, \qquad F = \langle x_u, x_v \rangle, \qquad G = \langle x_v, x_v \rangle$$

14.9 GEODESIC MAPPING

A surface S is said to be mapped geodesically onto a surface S^* if there is a differentiable homeomorphism of S onto S^* such that geodesics on S go over into geodesics on S^*.

Example 4: Prove that if a mapping is geodesic and conformal, it is necessarily an isometry or else a similarity mapping.

Solution: Let us consider a system of geodesic coordinates on a surface S, then we know that the metric in this case is given by

$$ds^2 = du^2 + G\,dv^2$$

From the conformality condition, the metric of S^* will be of the form

$$ds^{*2} = \lambda(u,v)[du^2 + G\,dv^2] \qquad (1)$$

Since the mapping is geodesic, therefore the geodesic curves u = constant are mapped onto the curves v = constant which are geodesics w.r.t. the metric (1), the condition for which being

$$EE_2 + FE_1 - 2EF_1 = 0 \qquad (2)$$

But from (1), we see that

$$E = \lambda(u,v) \text{ and } F = 0.$$

Therefore equation (2) is reduced to

$$\lambda(u,v)\frac{\partial}{\partial v}\lambda(u,v) = 0$$

$$\Rightarrow \frac{\partial \lambda}{\partial v} = 0 \qquad [\because \lambda(u,v) \neq 0]$$

i.e., λ is a function of u alone.

Consider now any geodesic on S inclined at an angle θ with the curve v = constant. Then from Liouville's formula it follows that

$$0 = d\theta + \frac{1}{2}\left(\frac{G_1}{H}\right) dv \qquad [\because K_g = 0 \text{ for a geodesic}] \qquad (3)$$

Since the mapping is both geodesic and conformal, the corresponding equation for S^* is

$$0 = d\theta + \frac{1}{2}\left(\frac{G_1^*}{H^*}\right) dv \qquad (4)$$

Comparing (3) and (4), we have

$$\frac{G_1}{H} = \frac{G_1^*}{H^*} \qquad (5)$$

Also, from since $F = 0$ and $E = 1$, therefore $H = \sqrt{G}$.

From (1), we observe that

$$G^* = \lambda G$$

Hence (5) assumes the form

$$\frac{G_1}{\sqrt{G}} = \frac{\lambda G}{\sqrt{\lambda G}}$$

But

$$(\lambda G)_1 = \lambda G_1 + \lambda_1 G$$

$$\therefore \ \frac{G_1}{\sqrt{G}} = \frac{G_1}{\sqrt{G}} + \sqrt{G}\frac{\lambda_1}{\lambda}$$

from which $\lambda_1 = 0$ and λ is thus a constant; showing that the mapping is a similarity mapping.

Example 5: If the parametric curves are at right angles, show that their geodesic curvatures are respectively given by

$$-\frac{1}{\sqrt{GE}}\frac{\partial}{\partial v}\sqrt{G} \text{ and } -\frac{1}{\sqrt{GE}}\frac{\partial}{\partial v}\sqrt{E}$$

Solution: The geodesic curvature of the parametric curve $u = \text{constant}$ and $v = \text{constant}$ are respectively given by

$$k_g = \frac{2GF_2 - GE_1 - FG_1}{2HG^{3/2}}, k_g = \frac{2EF_1 - EE_2 - FE_1}{2HE^{3/2}}$$

Since the parametric curves are orthogonal, we have

$$F = 0$$

$$\Rightarrow F_1 = 0 \ and \ H^2 = EG - F^2 = EG$$

Therefore, the geodesic curvature of the curve $v =$ constant is given by

$$k_g = \frac{2EF_1 - EE_2 - FE_1}{2HE^{3/2}}$$

$$= \frac{-EE_2}{2\sqrt{EG}\,E^{3/2}}$$

$$= -\frac{E_2}{2E\sqrt{G}}$$

$$= -\frac{1}{\sqrt{EG}}.\frac{E_2}{2\sqrt{E}}$$

$$= -\frac{1}{\sqrt{EG}}\frac{\partial}{\partial v}\left(\sqrt{E}\right)$$

Similarly, the geodesic curvature of the parametric curve $u =$ constant is given by

$$k_g = -\frac{1}{\sqrt{EG}}\frac{\partial}{\partial u}\sqrt{G}.$$

W CHECK YOUR PROGRESS

Q4. Prove that the components λ, μ of the geodesic curvature vector are given by the following formulae

$$\lambda = \frac{1}{H^2}\frac{U}{v'}\frac{\partial T}{\partial v'} = -\frac{1}{H^2}\frac{V}{u'}\frac{\partial T}{\partial v},$$

$$\mu = \frac{1}{H^2}\frac{V}{u'}\frac{\partial T}{\partial u'} = -\frac{1}{H^2}\frac{U}{v'}\frac{\partial T}{\partial u},$$

where s is the parameter.

Q.5. Prove that the curves of the family $\left(\frac{v^2}{u^2}\right) = constant$ are geodesics on a surface with metric

$$v^2\,du^2 - 2uv\,du\,dv + 2u^2\,dv^2 \quad (u > 0,\ v > 0).$$

2 14.10 LET US SUM UP

- The vector $K_g = \lambda r_1 + \mu r_2$ which is tangential to the surface at any point P with components (λ, μ) is called the geodesic curvature vector at P on the given curve.
- $K_g = \frac{V(s)}{H\,u'}$ and $K_g = -\frac{U(s)}{H\,v'}$ are called the formulae of geodesic curvature, where the symbols have their usual meanings.
- The geodesic form $du^2 + G(u,v)\,dv^2$ is called the geodesic form of ds^2.
- The geodesic coordinates can be introduced in an arbitrarily number of ways.
- $ds^2 = dr^2 + r^2 d\theta^2$ is the metric in geodesic polar coordinates.
- A surface S is mapped geodesically onto a surface S^* if there is a differentiable homeomorphism of S onto S^* such that geodesics on S go over into geodesics on S^*.

K 14.11 ANSWERS TO CHECK YOUR PROGRESS

Answer to Q1: Given,

$$r = (u\cos v,\ u\sin v,\ \frac{1}{2}au^2)$$

$$\therefore r_1 = (\cos v,\ \sin v,\ au)$$

$$r_2 = (-u\sin v,\ u\cos v,\ 0)$$

Hence,

$$E = r_1 \cdot r_1 = 1 + a^2u^2$$

$$F = 0$$

$$G = r_2 \cdot r_2 = u^2$$

and

$$H = u\sqrt{1 + a^2u^2}$$

$$E_1 = 2a^2u$$

$$E_2 = 0$$

$$G_1 = 2u$$

Using the above values in the formula for K_b, we have

$$K_b = \frac{1}{\sqrt{EG}}\frac{\partial}{\partial u}\sqrt{G} = \frac{1}{u\sqrt{1 + a^2u^2}}$$

Answer to Q2: We have discussed three different formula for the geodesic curvature. We obtain the geodesic curvature vector using these three different formulae.

Now, given

$$r = (u\cos v,\ u\sin v,\ av)$$

$$\therefore\ r_1 = (\cos v,\ \sin v,\ 0)$$

$$r_2 = (-u\sin v,\ u\cos v,\ a)$$

Hence,

$$E = r_1 \cdot r_1 = 1;\ \ F = 0$$

$$G = a^2 + u^2;\quad H = \sqrt{a^2 + u^2}$$

(i)

$$\lambda = \frac{1}{H}(VG - VF)$$

$$\mu = -\frac{1}{H^2}(EV - FU)$$

Let us find $U$and Vin the above formulae.

$$T = \frac{1}{2}[u'^2 + (u^2 + a^2)v'^2]$$

$$\frac{\partial T}{\partial u'} = u';\ \frac{\partial T}{\partial v'} = (u^2 + a^2)v'$$

$$\frac{\partial T}{\partial u} = uv'^2;\ \frac{\partial T}{\partial v} = 0 \qquad (1)$$

Hence,

$$U = \frac{d}{ds}\left(\frac{\partial T}{\partial u'}\right) - \frac{\partial T}{\partial u} = u'' - uv'^2 \qquad (2)$$

$$V = \frac{d}{ds}\left(\frac{\partial T}{\partial v'}\right) - \frac{\partial T}{\partial v} = (u^2 + a^2)v'' + 2uu'v' \qquad (3)$$

Hence,

$$\lambda = \frac{1}{a^2 + u^2}(u'' - uv'^2)(a^2 + u^2) = u'' - uv'^2$$

$$\mu = \frac{1}{a^2 + u^2}[(u^2 + a^2)v'' + 2uu'v']$$

(ii)

$$\lambda = u'' + \Gamma_{11}^1 u'^2 + 2\Gamma_{12}^1 u'v' + \Gamma_{22}^1 v'^2$$

$$\mu = v'' + \Gamma_{11}^2 u'^2 + 2\Gamma_{12}^2 u'v' + \Gamma_{22}^2 v'^2$$

First let us calculate the Christoffel symbols in the above formula for a curve on the right helicoid.

$$\Gamma_{111} = \frac{1}{2}E_1 = 0, \qquad \Gamma_{112} = \frac{1}{2}E_2 = 0$$

$$\Gamma_{122} = F_2 - \frac{1}{2}G_1 = -u, \qquad \Gamma_{211} = F_1 - \frac{1}{2}E_2 = 0$$

$$\Gamma_{212} = \frac{1}{2}G_1 = u, \qquad \Gamma_{222} = \frac{1}{2}G_2 = 0$$

Hence,

$$\Gamma_{11}^1 = \frac{1}{H^2}(G\Gamma_{111} - F\Gamma_{211}) = 0$$

$$\Gamma_{12}^1 = \frac{1}{H^2}(G\Gamma_{112} - F\Gamma_{212}) = 0$$

$$\Gamma_{22}^1 = \frac{1}{H^2}(G\Gamma_{122} - F\Gamma_{222}) = \frac{1}{a^2 + u^2}(a^2 + u^2)(-u) = -u$$

Therefore,

$$\lambda = u'' - uv'^2$$

To find μ, let us find other Christoffel symbols of the second kind.

$$\Gamma_{11}^2 = \frac{1}{H^2}(E\Gamma_{211} - F\Gamma_{111}) = 0$$

$$\Gamma_{12}^2 = \frac{1}{H^2}(E\Gamma_{212} - F\Gamma_{112}) = \frac{u}{a^2 + u^2}$$

$$\Gamma_{22}^2 = \frac{1}{H^2}(G\Gamma_{222} - F\Gamma_{122}) = 0$$

Therefore,

$$\mu = v'' + \frac{2uu'v'}{a^2+u^2}$$

$$\Rightarrow \mu = \frac{1}{a^2+u^2}[(u^2+a^2)v'' + 2uu'v']$$

(iii)

$$\lambda = \frac{1}{H^2}\frac{U}{v'}\frac{\partial T}{\partial v'} = -\frac{1}{H^2}\frac{U}{u'}\frac{\partial T}{\partial v'}$$

$$\mu = \frac{1}{H^2}\frac{V}{u'}\frac{\partial T}{\partial u'} = -\frac{1}{H^2}\frac{U}{v'}\frac{\partial T}{\partial u'}$$

Using (1), (2) and (3),

$$\lambda = \frac{1}{a^2+u^2}\left(\frac{u'' - uv'^2}{v'}\right).v'(a^2+u^2) = u'' - uv'^2$$

The other expansion gives the formula for λ in terms of v'' as follows,

$$\lambda = -\frac{1}{a^2+u^2}[(u^2+a^2)v'' + 2uu'v'].\frac{v'}{u'}(a^2+u^2)$$

$$\Rightarrow \lambda = \frac{v'}{u'}[(u^2+a^2)v'' + 2uu'v']$$

Further using (1), (2) and (3) in the formula for μ, we have

$$\mu = \frac{1}{a^2+u^2}[(u^2+a^2)v'' + 2uu'v']$$

and the alternate expression for μ is

$$\mu = -\frac{1}{a^2+u^2}\left(\frac{u'' - uv'^2}{v'}\right).u'$$

Answer to Q3: Let $U = \text{constant}$ be a family of geodesic.

Then the family of orthogonal trajectories is $u = \text{constant}$.

Suppose if we take $u = \text{constant}$ is a family of orthogonal trajectories.

Therefore, the surface admits the two orthogonal family of geodesic

$v = \text{constant}$.

Let the distance from some fixed parallel to the neighbouring parallel to du.

Hence

$$ds = du \text{ and } dv = 0$$

$$\Rightarrow ds = E\,du \text{ gives } du = E\,du \Rightarrow E(u) = 1$$

Similarly, measuring the distance along the geodesic $u = \text{constant}$. Let the distance along the geodesic $u = \text{constant}$.

Let the distance from some fixed parallel to the neighbouring parallel be dv.

Hence

$$ds = dv \text{ and } du = 0$$

$$ds = G(u,v)\,dv$$

$$dv = G(u,v)\,dv \Rightarrow G(u,v) = 1$$

Thus, the metric becomes

$$ds^2 = du^2 + dv^2$$

which is the metric of the plane.

Answer to Q4: We know that the components λ, μ of the geodesic curvature vector are given by

$$\lambda = \frac{1}{H^2}(GU - FV), \mu = \frac{1}{H^2}(EV - FU).$$

Also if s is parameter then

$$u'U + v'V = 0 \quad or \quad v = -\frac{u'U}{v'} \tag{1}$$

$$\Rightarrow \lambda = \frac{1}{H^2}\left(GU + \frac{Fu'U}{v'}\right) = \frac{U}{v'H^2}(v'G + u'F) \tag{2}$$

Again, we know that

$$T = \frac{1}{2}(Eu'^2 + 2Fu'v' + Gv'^2)$$

$$\therefore \frac{\partial T}{\partial v'} = Fu' + Gv' \tag{3}$$

In view of (3), equation (2) becomes

$$\lambda = \frac{1}{H^2}\frac{U}{v'}\frac{\partial T}{\partial v'} = -\frac{1}{H^2}\frac{V}{u'}\frac{\partial T}{\partial v'} [\because u'u + v'v = 0]$$

Similarly, putting $V = -\frac{u'V}{v'}$ from (1) and

$$\frac{\partial T}{\partial u'} = Eu' + Fv'$$

in the expression for μ, we get

$$\mu = -\frac{U}{v'H^2}\frac{\partial T}{\partial u'} = \frac{v}{u'H^2}\frac{\partial T}{\partial u'}\left[\because U = -\frac{v'v}{u'}\right]$$

Answer to Q5: The parametric equations of the curve $(v^2/u) = c\ (> 0)$ are

$$u = ct^3, v = ct^2$$

Then

$$u' = 3ct^2, v' = 2ct$$

$$\therefore T = \frac{1}{2}(Eu^2 + 2Fuv + Gv^2)$$

$$= \frac{1}{2}(v^2u'^2 - 2uvu'v' + 2u^2v'^2)$$

Now,

$$\frac{\partial T}{\partial u} = -vu'v' + 2uv'^2 = -6c^3t^5 + 8c^3t^5 = 2c^3t^5$$

$$\frac{\partial T}{\partial v} = vu'^2 - uu'v' = 3c^3t^6$$

$$\frac{\partial T}{\partial u'} = v^2u' - uvv' = c^3t^6$$

$$\frac{\partial T}{\partial v'} = c^2t^7$$

$$\therefore\ U = \frac{d}{dt}\left(\frac{\partial T}{\partial u'}\right) - \frac{\partial T}{\partial u} = \frac{d}{dt}(c^3t^6) - 2c^3t^5 = 4c^3t^5$$

$$V = \frac{d}{dt}\left(\frac{\partial T}{\partial v'}\right) - \frac{\partial T}{\partial v} = \frac{d}{dt}(c^3t^7) - 3c^3t^6 = 4c^3t^6$$

$$\Rightarrow V\frac{\partial T}{\partial u'} - U\frac{\partial T}{\partial v'} = 4c^3t^6 \cdot c^3t^6 - 4c^3t^5 \cdot c^3t^7 = 0$$

Hence,

$$V\frac{\partial T}{\partial u'} - U\frac{\partial T}{\partial v'} = 4c^6t^{12} - 4c^6t^{12} = 0$$

Hence, the curve is a geodesic for values of c.

14.12 FURTHER READING

[1] Greenstein, W. C. *Differential Geometry*, Courier Corporation, 2012.

[2] Willmore, T. J. *An Introduction to Differential Geometry*, Dover Publications Inc., 2012.

[3] Pressley, A. *Elementary Differential Geometry*, Springer, 2002.

14.13 MODEL QUESTIONS

Q.1. Find the geodesics on the catenoid of revolution

$$u = c\cosh(z/c).$$

Q.2. Prove that $k^2 = k_g^2 + k_n^2$ where k, k_g, k_n denote curvature, geodesic curvature and normal curvature respectively.

Q.3. Prove that on a surface with the metric $ds^2 = \lambda^2 du^2 + \mu^2 dv^2$ the geodesic curvature of the curve u = constant is

$$(\lambda\mu)^{-1}\frac{\partial \mu}{\partial u}$$

Q.4. If the parametric curves u = constant and v = constant are orthogonal, show that their geodesic curvatures are

$$-\frac{1}{\sqrt{G}}\frac{\partial}{\partial v}(\log\sqrt{E}), -\frac{1}{\sqrt{E}}\frac{\partial}{\partial u}(\log\sqrt{G}).$$

Q.5. Find the geodesic curvature of the parametric curve $v = c$.

Q.6. Prove that all straight lines on a surface are geodesics.

Q.7. Discuss the nature of geodesics on a plane.

Q.8. Define geodesic parallels, geodesic coordinates and geodesic mapping.

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