UNIT 2: RIEMANNIAN METRIC

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2.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- Understand that metric is a second-rank covariant symmetric tensor used to define distances and angles.
- Learn to compute the length of a vector using the metric tensor.
- Define and identify unit tangent vectors in a coordinate system.
- Understand the concept of associated vectors through metric tensors.
- Learn to compute scalar products using tensor notation.

- Determine vector projections in various coordinate systems.
- Analyze how gradients transform under coordinate changes.
- Calculate the angle between vectors using the inner product.
- Explore how angles between coordinate curves are determined using metric properties.
- Understand the concept and mathematical representation of hypersurfaces.
- Learn to evaluate angles between hypersurfaces using tensors.

2.2 INTRODUCTION

In this unit, we investigate a range of geometric and analytic concepts that emerge naturally in the study of vector spaces and differentiable manifolds. Central to our discussion is the **metric tensor**, a second-rank covariant symmetric tensor that defines the inner product structure of the space. The metric tensor enables us to compute fundamental geometric quantities such as lengths, angles, and distances in both Euclidean and Riemannian settings.

The unit begins by establishing how the metric tensor is used to determine the **magnitude of a vector**. This in turn leads to the definition of a **unit tangent vector**, which characterizes the direction of a curve with normalized length. We then explore the concept of **associate vectors**, which connects contravariant and covariant representations of vectors through the metric, highlighting their transformation properties and geometric interpretations.

Building on the metric structure, we define the **scalar product**(or inner product) of two vectors and apply it to determine the **projection of a vector along a given direction**. These operations are foundational in expressing geometric relationships between vectors and play a crucial role in physical applications.

The unit also introduces the **gradient of a scalar function**, a covariant vector field that points in the direction of maximum increase of the function. This concept is vital in understanding field theories and differential operators in curved spaces.

A key geometric notion discussed in this unit is the **angle between two vectors**, which is derived from their inner product and magnitudes. This idea is extended to define the **angle between coordinate curves**, using their tangent vectors and the underlying

metric. The expressions derived for these angles help analyze the geometry of the coordinate system itself.

In the concluding sections, we define a **hypersurface**as a subspace formed by constraining one coordinate via a scalar function, and then study how to compute the **angle between two hypersurfaces**. This involves evaluating the inner product of their gradient vectors, again using the metric tensor.

Altogether, this unit provides the essential tools for understanding the geometry of vector fields, coordinate systems, and hypersurfaces within the framework of tensor calculus. These concepts form the basis for more advanced topics in Riemannian geometry, general relativity, and modern differential geometry.

2.3Metric g_{ij} is a second rank covariant symmetric tensor

In three-dimensional Euclidean space, the distance *ds* between two infinitesimally close points $(x, y, z)$and $(x + dx, y + dy, z + dz)$ is given by

$$ds^2 = dx^2 + dy^2 + dz^2.$$

where $ds$is called the **line element or linear element**of the curve, and the coordinate axes are rectangular Cartesian.

In a general oblique curvilinear coordinate system (u, v, w), the distance between two nearby points $(u, v, w)$and $(u + du, v + dv, w + dw)$ is given by

$$ds^2 = a\,du^2 + b\,dv^2 + c\,dw^2 + 2f\,dv\,dw + 2g\,dw\,du + 2h\,du\,dv,$$

where the coefficients $a, b, c, f, g, h$are functions of the coordinates u, v, w.

This idea of distance was extended by Riemann to an *N*-dimensional space. The distance dsbetween two adjacent points with coordinates x^i and $x^i + dx^i$is given by the quadratic differential form

$$ds^2 = g_{ij}dx^i dx^j, (i, j = 1,2, \ldots, N), m \qquad (2.1)$$

in accordance with the summation convention. The coefficients $g_{ij}$are functions of x^isuch that the determinant of the matrix $[g_{ij}]$ is non-zero, *i. e.*, $det(g_{ij}) \neq 0$.

The quadratic form

$$g_{ij}dx^i dx^j$$

is called the **Riemannian metric**or simply the **metric**or **line element**. The tensor $g_{ij}$is called the **metric tensor**or the **fundamental tensor**.

Riemannian Space:

An N –dimensional space characterized by a Riemannian metric is called a **Riemannian space**and is denoted by V_N.

Riemannian Geometry:

Riemannian geometryis the branch of geometry that is based on the Riemannian metric.

Remark 1. The line element $ds$is an invariant, since the distance between two contiguous points is independent of the choice of coordinate system.

Remark 2. In the form

$$ds^2 = dx^2 + dy^2 + dz^2,$$

the components of the fundamental tensor $g_{ij}$are

$$g_{11} = g_{22} = g_{33} = 1,\ g_{ij} = 0 \quad for\ i \neq j,$$

$i.e.$, the metric is a positive definite quadratic form for all real values of dx, dy, dz, and $ds = 0$if and only if $dx = dy = dz = 0$.

In the theory of relativity (in four-dimensional spacetime), the line element is given by

$$ds^2 = c^2(dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2.$$

Where

$$x^0 = t,\ \ x^1 = x,\ \ x^2 = y,\ \ x^3 = z.$$

In this case, the metric is positive definite along curves where $x^1, x^2, x^3$are constant. Along curves where $x^0$is constant, the metric becomes negative, implying that dsbecomes imaginary. To ensure that ds is always real, the metric must be positive definite. We may write equation (2.1) as

$$ds^2 = \varepsilon\, g_{ij}dx^i dx^j \qquad (2.2)$$

where the factor $\varepsilon$is called the *indicator*, taking values $+1$or -1so that the right-hand side is always non-negative, ensuring $ds$is always real and non-negative.

Theorem 1:

Statement: The fundamental tensor g_{ij} is a covariant symmetric tensor of rank two.

Proof.

The line element is given by

$$ds^2 = g_{ij}dx^i dx^j \qquad (2.3)$$

Let us consider a coordinate transformation from x^i to $\bar{x}^i$, where $i = 1,2,\ldots,N$. Since the line element ds is an invariant under coordinate transformations, we have:

$$ds^2 = g_{ij}dx^i dx^j = \bar{g}_{ij}d\bar{x}^i d\bar{x}^j \qquad (2.4)$$

We now prove the following three claims

(i) **dx^i is a contravariant vector**

Under a coordinate transformation $x^i \to \bar{x}^i$, we have

$$d\bar{x}^i = \frac{\partial \bar{x}^i}{\partial x^r}\, dx^r$$

which confirms that dx^i transforms like a contravariant vector. Similarly, dx^j also transforms contravariantly.

(ii) **g_{ij} is a covariant tensor of rank two**

Substitute the transformation

$$dx^i = \frac{\partial x^i}{\partial \bar{x}^p}\, d\bar{x}^p, \quad dx^j = \frac{\partial x^j}{\partial \bar{x}^q}\, d\bar{x}^q$$

into equation (2.3)

$$\begin{aligned} ds^2 &= g_{ij}\, dx^i\, dx^j \\ &= g_{ij} \left(\frac{\partial x^i}{\partial \bar{x}^p}\, d\bar{x}^p\right) \left(\frac{\partial x^j}{\partial \bar{x}^q}\, d\bar{x}^q\right) \\ &= \left(g_{ij}\, \frac{\partial x^i}{\partial \bar{x}^p}\, \frac{\partial x^j}{\partial \bar{x}^q}\right) d\bar{x}^p\, d\bar{x}^q \end{aligned}$$

Comparing this with equation (2.4), we conclude

$$\bar{g}_{pq} = g_{ij}\, \frac{\partial x^i}{\partial \bar{x}^p}\, \frac{\partial x^j}{\partial \bar{x}^q}$$

which confirms that g_{ij} transforms as a covariant tensor of rank two.

(iii) **g_{ij} is symmetric**

We express the metric tensor as

$$g_{ij} = \frac{1}{2}(g_{ij} + g_{ji}) + \frac{1}{2}(g_{ij} - g_{ji}) = A_{ij} + B_{ij}$$

Where,

$A_{ij} = \frac{1}{2}(g_{ij} + g_{ji})$ is symmetric and $B_{ij} = \frac{1}{2}(g_{ij} - g_{ji})$ is anti-symmetric

We now show that the anti-symmetric part B_{ij} contributes nothing to the line element.

Consider

$$g_{ij}dx^i dx^j = (A_{ij} + B_{ij})dx^i dx^j ,$$

Interchanging the dummy indices in the term involving B_{ij}, we obtain

$$B_{ij}dx^i dx^j = B_{ji}dx^j dx^i = -B_{ij}dx^i dx^j \Rightarrow 2B_{ij}dx^i dx^j = 0 \Rightarrow B_{ij}dx^i dx^j = 0$$

Therefore,

$(g_{ij} - A_{ij})dx^i dx^j = 0 \Rightarrow g_{ij} = A_{ij}$ since $dx^i dx^j$ are arbitrary

Hence, g_{ij} is symmetric because A_{ij} is symmetric.

Thus, we have shown that g_{ij} is a covariant symmetric tensor of second order. This is known as the **fundamental covariant tensor.**

Alternative Proof of (ii): Since ds^2 is invariant, it is a tensor of rank zero. From equation (2.3), and by the quotient theorem, it follows that $g_{ij}dx^i$ is a tensor for arbitrary contravariantvector dx^j. Applying the quotient theorem again, it follows that g_{ij} itself is a tensor for arbitrary contravariant vector dx^i.

Therefore, by two successive applications of the quotient theorem, g_{ij} is a covariant tensor of order two.

Theorem 2:

Statement: To show that $g_{ij}dx^i dx^j$ is an invariant.

Proof: Consider a coordinate transformation $x^i \to \bar{x}^i$. Since g_{ij} is a covariant tensor of rank 2, it transforms as

$$\bar{g}_{pq} = g_{ij}\frac{\partial x^i}{\partial \bar{x}^p}\frac{\partial x^j}{\partial \bar{x}^q}$$

Multiplying both sides by $d\bar{x}^p\, d\bar{x}^q$, we get

$$\bar{g}_{pq}d\bar{x}^p d\bar{x}^q = g_{ij}\frac{\partial x^i}{\partial \bar{x}^p}d\bar{x}^p\frac{\partial x^j}{\partial \bar{x}^q}d\bar{x}^q = g_{ij}dx^i dx^j$$

Thus, $g_{ij}dx^i dx^j$ is invariant, which proves the theorem.

Remark 1. Up to this point, we have defined the two fundamental tensors

- g_{ij}: the metric tensor or fundamental covariant tensor,
- g^{ij}: the conjugate metric tensor or fundamental contravariant tensor.

Remark 2. Note that

$$g_{ij}g^{iK} = \delta_j^k (if\ k = j);\ \ g_{ij}g^{ik} = 0 \quad (if\ k \neq j).$$

Thus, $g_{ij}g^{ik}$ is a mixed tensor of second order, since the Kronecker delta δ_j^k is a mixed tensor of order two.

Example 1. Find the metric of a Euclidean space referred to

(i) cylindrical coordinates

(ii) spherical coordinates

Solution: The metric in Euclidean 3 –space is given by

$$ds^2 = dx^2 + dy^2 + dz^2 (2.5)$$

Set

$$x^1 = x; \quad x^2 = y; \quad x^3 = z.$$

The metric (2.5) takes the form

$$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \qquad (2.6)$$

(i) The cylindrical coordinates are given by

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r = z \qquad (2.7)$$

Let us set,

$$\bar{x}^1 = r, \quad \bar{x}^2 = \theta, \quad \bar{x}^3 = z$$

Comparing metric (2.6) with the usual metric,

$$ds^2 = g_{ij}dx^i dx^j [i,j = 1,2,3].$$

We find,

$$g_{11} = 1, \quad g_{22} = 1, \quad g_{33} = 1, \quad g_{12} = g_{23} = \ldots = 0$$

$$\bar{g}_{11} = g_{\alpha\beta}\frac{\partial x^\alpha}{\partial \bar{x}^1}\frac{\partial x^\beta}{\partial \bar{x}^1} \quad (p,q = 1,2,3)$$

$$= g_{11}\left(\frac{\partial x^1}{\partial \bar{x}^1}\right)^2 + g_{22}\left(\frac{\partial x^2}{\partial \bar{x}^1}\right)^2 + g_{33}\left(\frac{\partial x^3}{\partial \bar{x}^1}\right)^2$$

$$= \left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2 = \cos^2\theta + \sin^2\theta + 0, \text{ using (1.7)}$$

i.e. $\bar{g}_{11} = 1$

Similarly,

$$\bar{g}_{22} = \left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2 = r^2\sin^2\theta + r^2\cos^2\theta = r^2$$

and,

$$\bar{g}_{33} = \left(\frac{\partial x}{\partial z}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + \left(\frac{\partial z}{\partial z}\right)^2 = 0 + 0 + 1 = 1.$$

$$\bar{g}_{12} = g_{\alpha\beta}\frac{\partial x^\alpha}{\partial \bar{x}^1}\frac{\partial x^\beta}{\partial \bar{x}^2}$$

$$= g_{11}\frac{\partial x^1}{\partial \bar{x}^1}\frac{\partial x^1}{\partial \bar{x}^2} + g_{22}\frac{\partial x^2}{\partial \bar{x}^1}\frac{\partial x^2}{\partial \bar{x}^2} + g_{33}\frac{\partial x^3}{\partial \bar{x}^1}\frac{\partial x^3}{\partial \bar{x}^2}$$

$$= \left(\frac{\partial x}{\partial r}\right)\left(\frac{\partial x}{\partial \theta}\right) + \left(\frac{\partial y}{\partial r}\right)\left(\frac{\partial y}{\partial \theta}\right) + \left(\frac{\partial z}{\partial r}\right)\left(\frac{\partial z}{\partial \theta}\right)$$

$$= \cos\theta(-r\sin\theta) + \sin\theta \cdot r\cos\theta + 0 = 0, \text{ using (2.7).}$$

Similarly, $g_{23} = 0, \quad g_{13} = 0$.

The metric in cylindrical coordinates is given by,

$$ds^2 = \bar{g}_{ij}d\bar{x}^i d\bar{x}^j \quad (i,j = 1,2,3)$$

$$= \bar{g}_{11}(d\bar{x}^1)^2 + \bar{g}_{22}(d\bar{x}^2)^2 + \bar{g}_{33}(d\bar{x}^3)^2$$

$$= dr^2 + r^2d\theta^2 + dz^2$$

(ii) The spherical co-ordinate is given by,

$x = r\sin\theta\cos\Phi, \; y = r\sin\theta\sin\Phi, \quad z = r\cos\theta$ (2.8)

Let us set,

$\bar{x}^1 = r, \quad \bar{x}^2 = \theta, \quad \bar{x}^3 = \Phi$ (2.9)

As shown in part (i),

$$g_{11} = g_{22} = g_{33} = 1, \qquad g_{12} = g_{23} = \cdots = 0.$$

Also,

$$\bar{g}_{11} = g_{\alpha\beta}\frac{\partial x^{\alpha}}{\partial \bar{x}^1}\frac{\partial x^{\beta}}{\partial \bar{x}^1}$$

$$= g_{11}\left(\frac{\partial x^1}{\partial \bar{x}^1}\right)^2 + g_{22}\left(\frac{\partial x^2}{\partial \bar{x}^1}\right)^2 + g_{33}\left(\frac{\partial x^3}{\partial \bar{x}^1}\right)^2$$

$$= \left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2$$

$$= \sin^2\theta\,(\cos^2\Phi + \sin^2\Phi) + \cos^2\theta = \sin^2\theta + \cos^2\theta = 1$$

Similarly,

$$\bar{g}_{22} = \left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2 = r^2$$

and,

$$\bar{g}_{23} = \left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 + \left(\frac{\partial z}{\partial \phi}\right)^2 = r^2\sin^2\theta$$

$$\bar{g}_{12} = g_{\alpha\beta}\frac{\partial x^{\alpha}}{\partial \bar{x}^1}\frac{\partial x^{\beta}}{\partial \bar{x}^2}$$

$$= g_{11}\frac{\partial x^1}{\partial \bar{x}^1}\frac{\partial x^1}{\partial \bar{x}^2} + g_{22}\frac{\partial x^2}{\partial \bar{x}^1}\frac{\partial x^2}{\partial \bar{x}^2} + g_{33}\frac{\partial x^3}{\partial \bar{x}^1}\frac{\partial x^3}{\partial \bar{x}^2}$$

$$= \left(\frac{\partial x}{\partial r}\right)\left(\frac{\partial x}{\partial \theta}\right) + \left(\frac{\partial y}{\partial r}\right)\left(\frac{\partial y}{\partial \theta}\right) + \left(\frac{\partial z}{\partial r}\right)\left(\frac{\partial z}{\partial \theta}\right) = 0$$

Similarly, $\bar{g}_{13} = 0, \quad \bar{g}_{23} = 0$.

The metric in spherical coordinates is given by

$$ds^2 = \bar{g}_{ij}d\bar{x}^i d\bar{x}^j \qquad (i,j = 1,2,3)$$

$$\bar{g}_{11}(d\bar{x}^1)^2 + g_{22}(d\bar{x}^2)^2 + g_{33}(d\bar{x}^3)^2 = dr^2 + r^2d\theta^2 + r^2\sin^2\theta\,d\Phi^2$$

CHECK YOUR PROGRESS 1:

Q.1 If the metric is given by

$$ds^2 = 5(dx^1)^2 + 3(dx^2)^2 + 4(dx^3)^2 - 6dx^1dx^2 + 4dx^2dx^3.$$

Evaluate $(i)g$ and $(ii)g^{ij}$.

2.4 Magnitude of a Vector

Let A^i denote the contravariant components of a vector $\boldsymbol{A}$. Then the **magnitude** of the vector is defined by

$$A^2 = g_{ij}A^iA^j \qquad (2.10)$$

This can also be written using the covariant components $A_i = g_{ij}A^j$ as

$$A^2 = (g_{ij}A^i)A^j = A_jA^j = g^{ij}A_iA_j \qquad (2.11)$$

Thus, the magnitude computed from the covariant components A_i is the same as that computed from the contravariant components A^i.

Thus, the magnitudes of a vector and its associated covariant and contravariant forms are equal. That is, the magnitude is invariant under raising or lowering of indices using the metric tensor.

Example 2. Prove that the magnitudes of two associate vectors are equal.

Sol:

Let A^i be a contravariant vector. Then its associated covariant vector is $A_i = g_{ij}A^j$. The magnitude of is defined by

$$A^2 = g_{ij}A^iA^j$$

Now consider the magnitude of the associated covariant vector A_i

$$A^2 = A_jA^j = g^{ij}A_iA_j$$

Using the identity $g^{ij}g_{jl} = \delta^i_l$, we simplify to obtain which is the same as the original magnitude.

Hence, the magnitudes of two associate vectors are equal.

2.5 Unit tangent vector

A vector whose magnitude is unity is called a **unit vector**.

For example, consider the curve parameterized by arc length s, with position vector components $x\hat{}i$. Then the derivative with respect to arc length, $\frac{dx^i}{ds}$, defines a unit contravariant vector since,

$$\left(\frac{ds}{ds}\right)^2 = g_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} = 1$$

This unit vector $\frac{dx^i}{ds}$ is known as the **unit tangent vector** to a curve in a Riemannian manifold V^N.

2.6 Associated Vectors

Definition. The associated vector of a contravariant vector A^i is obtained by contracting it with the metric tensor g_{ij}. The associated vector of is a covariant vector A_i, given by

$$A_i = g_{ij}A^j$$

which is called *lowering* the index.

Similarly, the contravariant vector A^j is the associated vector of the covariant vector A_i, obtained by contracting with the inverse metric tensor g^{ij}

$$A^j = g^{ij}A_i ,$$

Which is called *raising* the index.

2.7 Scalar product

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two vectors with contravariant components A^i and B^i, respectively. Then their **scalar product** is denoted $\boldsymbol{A} \cdot \boldsymbol{B}$, and is defined as

$$\boldsymbol{A} \cdot \boldsymbol{B} = g_{ij}A^iB^j$$

In terms of covariant components $A_i = g_{ij}A^j$, the scalar product can also be written as

$$\boldsymbol{A} \cdot \boldsymbol{B} = A_iB^i = A^iB_i$$

In particular, the scalar product of a vector with itself gives the square of its magnitude

$$\boldsymbol{A} \cdot \boldsymbol{A} = g_{ij}A^iA^j = A^2$$

Note: Equation (1.5) for the magnitude of a vector may also be written as:

$$A^2 = A^j(g_{ij}A^i) = A^jA_j$$

2.8 Projection of a vector along a direction

Let A^i be a contravariant vector and B^i be another vector along a certain direction. The **projection of vector A^i along the direction of B^i** is given by

$$P = \frac{g_{ij}A^iB^j}{\sqrt{g_{ij}B^iB^j}}$$

Here,

- g_{ij} is the covariant metric tensor.
- $g_{ij}A^iB^j$ is the inner product (or scalar product) of vectors A and B.
- $\sqrt{g_{ij}B^iB^j}$ is the magnitude of the vector .

This projection represents the component of A in the direction of B.

2.9 Gradient of a scalar function

Let $\Phi = \Phi(x^i)$ be a scalar field defined over a Riemannian manifold. Then the gradient of the scalar field Φ is defined as the covariant vector (or covariant derivative) given by

$$\frac{\partial \phi}{\partial x^i}$$

This quantity transforms as a covariant vector of rank one.

The **geometric meaning** of $\frac{\partial \Phi}{\partial x^i}$ is that it gives the rate of change of the scalar function Φ in the direction of increasing coordinate x^i.

The directional derivative of Φ along a vector A^i is given by the contraction

$$A^i \frac{\partial \phi}{\partial x^i}$$

Thus, the gradient vector points in the direction of the greatest rate of increase of the scalar function Φ.

CHECK YOUR PROGRESS 2:

Q.1 If $A_j = g_{jk}A^k$, show that $A^k = g^{jk}A_j$.

Q.2 Prove the following relations

(i) $A_{ij}B^{ij} = A^{ij}B_{ij}$

(ii) $A_{kl}B^{li} = A^i_k B^l_i$

2.10 Angle Between Two Vectors

Theorem 3

To define the angle θ between two vectors A and B at a point P of a Riemannian manifold V^N, and to show that this definition is consistent with the usual Euclidean notion (i.e., that it satisfies $\cos^2\theta \leq 1$).

Proof / Definition

Let A and B be two-unit vectors at a point in V^N, *i.e.*,

$$g_{ij}A^iA^j = 1, \qquad g_{ij}B^iB^j = 1.$$

Then the angle θ between A and B is defined by

$$cos\theta = g_{ij}A^iB^j \tag{2.12}$$

This definition is consistent with the Euclidean inner product when the metric tensor g_{ij} is the identity.

If A and B are not unit vectors, let $A = |A|$ and $B = |B|$ denote their magnitudes, *i.e.*,

$$A = \sqrt{g_{ij}A^iA^j}, \qquad B = \sqrt{g_{ij}B^iB^j}.$$

Then the unit vectors in the directions of A and B are $\frac{A^i}{A}$ and $\frac{B^i}{B}$, respectively.

The angle θ between them is then defined by:

$$\cos\theta = g_{ij}\left(\frac{A^i}{A}\right)\left(\frac{B^j}{B}\right) = \frac{g_{ij}A^iB^j}{AB} \tag{2.13}$$

which gives the general formula for the cosine of the angle between two vectors in a Riemannian manifold.

Justification of the Definition of Angle Between Vectors

Let θ be the angle between two-unit vectors A and B, so that

$$cos\theta = g_{ij}A^iB^j$$

It is required to prove that $\theta \in R$, i.e., $cos\theta \in [-1,1]$.

Let us consider the vector

$$C^i = lA^i + mB^i$$

Where l and m are real scalars. The square of the magnitude of C is

$$|C|^2 = g_{ij}(lA^i + mB^i)(lA^j + mB^j)$$
$$= l^2g_{ij}A^iA^j + 2lmg_{ij}A^iB^j + m^2g_{ij}B^iB^j$$

$= l^2 + 2lm\cos\theta + m^2$ (2.14)

Since A and B are unit vectors, so

$$g_{ij}A^iA^j = 1, \quad g_{ij}B^iB^j = 1, \quad g_{ij}A^iB^j = \cos\theta$$

Now, since V^N is a Riemannian manifold, the metric is positive definite, so

$$|C|^2 \geq 0 \quad for\ all\ real\ l\ and\ m.$$

Hence, from equation (2.14)

$$l^2 + 2lm\cos\theta + m^2 \geq 0 \quad for\ all\ real\ l\ and\ m$$

This quadratic expression in l must be non-negative for all l, which implies that its discriminant must be less than or equal to zero.

$$Discriminant = (2m\cos\theta)^2 - 4.1.m^2$$
$$= 4m^2\cos^2\theta - 4m^2 \leq 0$$
$$\Rightarrow \cos^2\theta \leq 1$$

Therefore, we have the desired result

$$\cos^2\theta \leq 1$$

Note.

If $A = A^i$ and $B = B^i$ are contravariant vectors, then their inner product with respect to a symmetric tensor g_{ij} is defined as

$$(A,B) = g_{ij}A^iB^j$$

Sometimes this is denoted by

$$g(A,B) = g_{ij}A^iB^j$$

Theorem 6: Orthogonality of Two Vectors

Statement: The necessary and sufficient condition for two vectors A and B at a point of a Riemannian manifold V^N to be **orthogonal** is

$g_{ij}A^iB^j = 0$ or equivalently $g(A,B) = 0$

Proof.

Let the magnitudes of A and B be A and B respectively. Let θ be the angle between these vectors. Then the cosine of the angle is given by

$$\cos\theta = \frac{g_{ij}A^iB^j}{AB} \quad (2.15)$$

Suppose **A** and **B** are orthogonal. Then the angle $\theta = \frac{\pi}{2}$ so $\cos\theta = 0$. Substituting into (2.15), we get

$$g_{ij}A^iB^j = AB \cdot \cos\theta = AB.0 = 0$$

Hence the scalar product is zero.

Conversely, suppose $g_{ij}A^iB^j = 0$. Then by equation (2.15), we have

$$\cos\theta = \frac{0}{AB} = 0 \Rightarrow \theta = \frac{\pi}{2}$$

Hence, the vectors are orthogonal.

Note.

The condition for orthogonality of two contravariant vectors A^i and B^i may also be written using the symmetric metric tensor as

$$g_{ij}A^iB^j = g(\boldsymbol{A},\boldsymbol{B}) = 0$$

2.11 Angle between co-ordinate curves

Coordinate Curves

In E_3 *i.e.* in three-dimensional Euclidean space the coordinate x curve is given by y =constant and z =constant. Thus, along x –coordinate curve y and z are constants and is a parameter.

Now if we take (x^1,x^2,x^3) as coordinates of a point in E_3, then along x^1 –coordinate curve x^2 =constant and x^3 =constant and x^1 is treated as parameter.

Now consider the coordinates $(x^1,x^2,\ldots,x^N)$ in Riemannian V_N. Along x^j –coordinate curve, the coordinate x^j alone is parameter and other coordinates $x^1,x^2,\ldots,x^{j-1},x^{j+1},\ldots,x^N$ are constants. Hence coordinate curve is defined by

$$x^i = C^i \; for \; every \; i; \; except \; i = j$$

where c^i's are constants.

Angle between two coordinate curves is defined as the angle between their tangents.

The x^j –coordinate curve is defined by $x^i = c^i$ for every i, except $i = j$.

Differentiating, $dx^i = 0$, for every i, except $i = j$ and $dx^j \neq 0$. Hence,

$$dx^i = (0,0,\ldots,0,dx^j,\ldots,0,0) \qquad (2.16)$$

Let A^i and B^i be the tangent vectors to the x^p –coordinate curve and x^q coordinate curve respectively.

In view of equation (2.16), we have

$$A^i = dx^i = (0,\ldots,0,A^p,0,\ldots,0) \qquad (2.17)$$

$$B^i = dx^i = (0,\ldots,0,B^q,0,\ldots,0) \qquad (2.18)$$

If θ is required angle, then

$$\cos\theta = \frac{g_{ij}A^iA^j}{\sqrt{\{g_{ij}A^iA^j\}\cdot\{g_{ij}B^iB^j\}}} = \frac{g_{pq}A^pB^q}{\sqrt{g_{pp}A^pA^p}\sqrt{g_{qq}B^qB^q}}$$ [No summation on p and q]

$$\cos\theta = \frac{g_{pq}A^pB^q}{\sqrt{\{g_{pp}g_{qq}\}A^pB^q}} = \frac{g_{pq}}{\sqrt{g_{pp}g_{qq}}} \qquad (2.19)$$

which is the required formula for θ.

Remark 1. The angle θ_{ij} between x^i –coordinate curve and x^j coordinate curve is given by

$$cos\theta_{ij} = \frac{g_{ij}}{\sqrt{g_{ii}g_{jj}}} \qquad (2.20)$$

Remark 2. Orthogonality: If the co-ordinate curves of parameter x^i and x^j are orthogonal then

$$\theta_{ij} = \frac{\pi}{2} \Rightarrow \cos\theta_{ij} = \cos\frac{\pi}{2} = 0$$

From (2.19), $g_{ij} = 0$

Therefore, the x^i –coordinate curve and x^j –coordinate curve are orthogonally if $g_{ij} = 0$.

Remark 3. The angles between the coordinate curves in E_3 are given by

$$\cos\theta_{12} = \frac{g_{12}}{\sqrt{g_{11}g_{22}}}, \quad \cos\theta_{23} = \frac{g_{23}}{\sqrt{g_{22}g_{33}}}, \quad \cos\theta_{31} = \frac{g_{31}}{\sqrt{g_{33}g_{11}}}$$

This result immediately follows from formula (2.20)

Obviously for an orthogonal coordinate system we have

$g_{12} = g_{23} = g_{31} = 0.$

2.12 Hypersurface

Let $t^1, t^2, \ldots, t^M$ be M parameters. The N equations

$$x^i = x^i(t^1, t^2, \ldots, t^M)(i = 1,2,\ldots,N); \quad M < N$$

defines M dimensional subspace V_M of V_N.

If we eliminate the M parameters $t^1, t^2, \ldots, t^M$, from these N equations. We shall get $(N - M)$ equations in $x^{i's}$ which represent dimensional curve in V_N.

In particular if $M = N - 1$, we get only one equation $[N - M = (n - (N - 1)) = 1]$ in $x^{i's}$ which represent $N - 1$ dimensional curve in V_N. This particular curve is called **hypersurface** of V_N. Thus

$$\Phi(x^i) = constant$$

represent a family of hypersurfaces of V_N where $\Phi(x^i)$ is a scalar function of coordinates x^i.

Angle between two hypersurfaces.

Let,

$$\Phi(x^i) = constant \quad (2.21)$$

and

$$\Psi(x^i) = constant \quad (2.22)$$

be two scalar functions, each defining a family of hypersurfaces in a Riemannian manifold V_N.

The angle ω between two hypersurfaces is defined as the angle between their respective normal. Now differentiating (2.21), we obtain

$$\frac{\partial \Phi}{\partial x^i} dx^i = 0 \quad (2.23)$$

This shows that the gradient vector $\frac{\partial \Phi}{\partial x^i}$ is normal to the hypersurface $\Phi = constant$, since dx^i is tangent to the hypersurface. Similarly, $\frac{\partial \Psi}{\partial x^i}$ is normal to the hypersurface $\Psi = constant$.

Hence, the angle ω between the two hypersurfaces is given by

$$\cos\omega = \frac{g^{ij}\frac{\partial \phi}{\partial x^i}\frac{\partial \psi}{\partial x^j}}{\sqrt{g^{ij}\frac{\partial \phi}{\partial x^i}\frac{\partial \phi}{\partial x^j}} \cdot \sqrt{g^{ij}\frac{\partial \psi}{\partial x^i}\frac{\partial \psi}{\partial x^j}}} \quad (2.24)$$

2.13 Angle between coordinate hypersurface

Let the hypersurfaces (2.21) and (2.22) be taken as co-ordinate hypersurfaces of parameters say x^p and x^q respectively, then we have

$$\Phi = x^p = constant \quad (2.25)$$

$$\Psi = x^q = constant \quad (2.26)$$

The angle ω between (2.25) and (2.26) is given by, [using the formula (2.24)]

$$\cos\omega = \frac{g^{ij}\frac{\partial x^p}{\partial x^i}\frac{\partial x^q}{\partial x^j}}{\sqrt{\left(g^{ij}\frac{\partial x^p}{\partial x^i}\frac{\partial x^p}{\partial x^j}\right)\left(g^{ij}\frac{\partial x^q}{\partial x^i}\frac{\partial x^q}{\partial x^j}\right)}}$$

$$= g^{ij}\delta^p_i\delta^q_j \Big/ \sqrt{\left(g^{ij}\delta^p_i\delta^p_j\right)\left(g^{ij}\delta^q_i\delta^q_j\right)} \quad (2.27)$$

$$= g^{pq}/\sqrt{(g^{pp})(g^{qq})}$$

Remark 1. The angle ω_{ij} between the co-ordinate hypersurfaces of parameters x^i and x^j is given by [use formula (2.27)].

$$cos\omega_{ij} = \frac{g^{ij}}{\sqrt{g^{ii}g^{jj}}} \qquad (2.28)$$

Remark 2. Orthogonality. If the co-ordinate hypersurfaces of parameters x^i and x^j are orthogonal then

$$\omega_{ij} = \pi/2 \Rightarrow \cos\omega_{ij} = \cos\frac{\pi}{2} = 0$$

We have from (2.21)

$$g_{ij} = 0 \qquad (2.29)$$

Therefore, the coordinate hypersurfaces of parameter x^i and x^j are orthogonal if $g^{ij} = 0$.

Remark 3. We know that metric of the space is given by

$$ds^2 = g_{ij}dx^i dx^j (i,j = 1,2,\ldots,N). \qquad (2.30)$$

If coordinate hypersurfaces are orthogonal then

$$g_{ij} = 0 \ \ if\ i \neq j;\ and \quad g_{ij} \neq 0 \quad if\ i = j.$$

Hence the metric of the space reduces to

$$ds^2 = \sum_{r=1}^{N} g_{rr}(dx^r)(dx^r) = \sum_{r=1}^{N} g_{rr}(dx^r)^2$$

W CHECK YOUR PROGRESS 3:

Q.1 Define the angle between two vectors in Riemannian space.

Q.2 Write the formula for the angle between two vectors A^i and B^i.

Q.3 What condition must two coordinate curves satisfy to be orthogonal at a point?

Q.4 When do two vectors make a right angle?

2 2.14 Let Us Sum Up

- The metric tensor g_{ij} is a second rank covariant symmetric tensor, which defines the geometry of the space and enables computation of lengths and angles.
- The magnitude (or length) of a contravariant vector A^i is given by $g_{ij}A^iA^j$.
- A unit tangent vector has magnitude 1 and is obtained by dividing the vector by its magnitude.
- For every contravariant vector A^i, the associated covariant vector is $A_i = g_{ij}A^j$.

Similarly, for a covariant vector B_i, the associated contravariant vector is $B^i = g^{ij}B_j$

- The scalar (inner) product of two vectors $A^i$and $B^i$is defined as $g_{ij}A^iB^j$and is invariant under coordinate transformations.
- The projection of a vector A^iin the direction of a vector $B^i$is given by $\frac{g_{ij}A^iB^j}{\sqrt{g_{ij}B^iB^j}}$.
- The gradient of a scalar function $\Phi$is a covariant vector given by $\frac{\partial \Phi}{\partial x^i}$ and its associated contravariant vector is $g^{ij}\frac{\partial \Phi}{\partial x^j}$.
- The angle θbetween two vectors $A^i$and $B^i$is defined by

$$\cos\theta = \frac{g_{ij}A^iB^j}{\sqrt{g_{ij}A^iA^j}\sqrt{g_{ij}B^iB^j}}$$

- The angle between two coordinate curves $x^i = const.$ and $x^j = const$ is given by

$$\cos\theta = \frac{g_{ij}}{\sqrt{g_{ii}g_{jj}}}$$

- A hypersurface in an n –dimensional space is an $(n-1)$ –dimensional surface defined by a scalar function $\Phi(x^1, x^2, \ldots, x^n) = constant$.
- The angle between two coordinate hypersurfaces $x^i = const.$ and $x^j = const$ is also given by

$$\cos\theta = \frac{g^{ij}}{\sqrt{g^{ii}g^{jj}}}$$

K 2.15 Answer to Check Your Progress

Check your progress 1:

Ans. to $Q.1$

(i) Comparing the given metric, with the usual metric

$ds^2 = g_{ij}dx^idx^j, \quad (i,j = 1,2,3)$

We have

$g_{11} = 5,\ g_{22} = 3,\ g_{33} = 4,\ g_{12} = g_{21} = -3,\ g_{23} = g_{32} = 2,\ g_{13} = g_{31} = 0$

Hence

$$g = \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix} = \begin{vmatrix} 5 & -3 & 0 \\ -3 & 3 & 2 \\ 0 & 2 & 4 \end{vmatrix} = 4$$

(ii) Now suppose $\xi(i,j)$ denotes the cofactor of g_{ij} in g.

Therefore, $\xi(1,1) = \text{cofactor of } g_{11} = \begin{vmatrix} 3 & 2 \\ 2 & 4 \end{vmatrix} = 8$

Similarly,

$\xi(2,2) = 20, \quad \xi(3,3) = 6$

$\xi(1,2) = \xi(2,1) = 12, \xi(1,3) = \xi(3,1) = -6, \xi(2,3) = \xi(3,2) = -10$

Since,

$$g^{ij} = \frac{\xi(i,j)}{g}$$

We obtain

$$g^{11} = 2, \ g^{22} = 5, \ g^{33} = \frac{3}{2}, \ g^{12} = g^{21} = 3, \ g^{13} = g^{31} = -\frac{3}{2},$$

$$g^{23} = g^{32} = -\frac{5}{2}$$

Hence,

$$g^{ij} = \begin{bmatrix} 2 & 3 & -\frac{3}{2} \\ 3 & 5 & -\frac{5}{2} \\ -\frac{3}{2} & -\frac{5}{2} & \frac{3}{2} \end{bmatrix}$$

Check your progress 2:

Ans. to $Q.1$

Given

$A_j = g_{jk}A^k$ (1)

Multiplying (1) by g^{iq}, we get

$$g^{iq}A_j = g^{iq}g_{jk}A^k$$

$$= \delta_k^q A^k = A^q$$

Therefore,

$$A^k = g^{ik}A_j$$

Ans. to $Q.2$

(i) We have

$A_{kl} = g_{ki}g_{lj}A^{ij}$ (1)

$$B^{kl} = g^{ki}g^{lj}B^{ij} \qquad (2)$$

Multiplying (1) and (2), we get

$$A_{kl}B^{kl} = g_{ki}g_{lj}A^{ij}g^{ki}g^{lj}B^{ij} = g^{ki}g_{ki}g^{lj}g_{lj}A^{ij}B_{ij} \Rightarrow A_{ij}B^{ij} = A^{ij}B_{ij}$$

(ii) We have

$$g^{ki}B_i^l = B^{kl} \Rightarrow g_k^i g^{ki}B_i^l = B^{il}$$

And

$$g_i^l = A_k^i = A_{kl} \Rightarrow A_{kl} = g_i^l g_{li}A_k^i$$

Therefore,

$$A_{ki}B^{li} = g_i^l g_{li}A_k^i g_k^i g^{ki}B_i^l = g_{ii}g^{ii}A_k^i B^i = A_k^i B_i^l$$

Check your progress 3:

Ans. to $Q.1$ and $Q.2$

If A^i and B^i are two vectors, then the angle θ between them is given by

$$\cos\theta = \frac{g_{ij}A^iB^j}{\sqrt{g_{ij}A^iA^j}\sqrt{g_{ij}B^iB^j}}$$

Ans. to $Q.3$

Two coordinate curves x^i =const. and x^j =const. are orthogonal if the metric coefficient $g_{ij} = 0$ for $i \neq j$.

Ans. to $Q.4$ Two vectors A^i and B^i make a right-angle if $g_{ij}A^iB^j = 0$.

O 2.16 Further Reading

1. Weatherburn, C. E. Differential Geometry of Three Dimensions, Cambridge University Press,2016.
2. Graustein, W. C. Differential Geometry. Courier Corporation, 2012.
3. Wilmore T. J. An Introduction to Differential Geometry, Dover Publications Inc., 2012.
4. Pressley, A. Elementary Differential Geometry. Springer, 2002.

M 2.17 Model Questions

$Q.1$ Prove that g_{ij} is a covariant tensor of order two.

$Q.2$ Prove that the magnitudes of two associate vectors are equal.

$Q.3$ Determine the metric tensor in (i) cylindrical and (ii) spherical coordinates.

$Q.4$ Determine the conjugate metric tensor in cylindrical and spherical coordinates.

Q.5 Prove that the cosine of the angle θ between two vectors A^i and B^i is given by

$$\cos\theta = \frac{g_{ij}A^iB^j}{\sqrt{g_{ij}A^iA^j}\sqrt{g_{ij}B^iB^j}}$$

Q.6 Prove that the cosine of the angle between the hypersurfaces

$x^i = constant, x^j = constant$ is given by

$$\frac{g^{ij}}{\sqrt{g^{ii}g^{jj}}}$$

Deduce that if the coordinate hypersurfaces are orthogonal, the metric of the space is given by

$$ds^2 = \sum_{r=1}^{N} g_{rr}(dx^r)^2$$

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