UNIT 3: CHRISTOFFEL SYMBOLS COVARIANT DIFFERENTIATION

UNIT STRUCTURE

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3.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- Define and interpret Christoffel symbols in the context of curved space.
- Understand how Christoffel symbols transform under coordinate changes.
- Learn to compute the covariant derivative and understand the concept of differentiation to second-rank tensors.
- know about covariant derivative of a scalar.
- Define and compute the divergence using covariant differentiation.
- Understand what it means for a tensor to be covariantly constant.
- Learn to compute the curl of a vector in curvilinear coordinates using tensors.

3.2 INTRODUCTION

In many physical and geometrical problems, we deal with spaces that are not flat and require tools that are compatible with their curvature. Ordinary derivatives do not preserve the tensor character under general coordinate transformations. Hence, in curved spaces, we need to define new kinds of derivatives called covariant derivatives that ensure tensorial nature is retained.

This unit introduces the concept of **Christoffel symbols**, which are essential in defining covariant differentiation. The Christoffel symbols of the first and second kinds are not tensors, but they play a vital role in correcting the ordinary derivative to form the covariant derivative.

We begin by defining the Christoffel symbols and derive their transformation laws. Then we move on to define the covariant derivative of various tensor quantities, such as covariant vectors, second-rank tensors, and scalar fields. This leads us naturally to the definitions of important differential operators like **divergence**and **curl**in tensor notation.

The idea of a **covariant constant tensor**is also introduced, which refers to tensors whose covariant derivative vanishes everywhere a condition that has important implications in geometry and physics.

By the end of this unit, students will gain a clear understanding of how to differentiate tensors properly in curved spaces and how such derivatives can be used to describe physical phenomena in a coordinate independent way.

3.3 Christoffel symbols

Christoffel's Three-Index Symbols (or Brackets) and Covariant Differentiation

The Christoffel Three-Index Symbols or Christoffel Brackets

We now consider two expressions due to Christoffel involving the derivatives of the components of the fundamental tensors $g_{ij}$and g^{ij}.

The **Christoffel symbols of the first kind**and **second** kind are denoted respectively by $[ij,k]$and Γ_{ij}^{k}, and are defined as

$$[ij,k] = \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^{i}} + \frac{\partial g_{ik}}{\partial x^{j}} - \frac{\partial g_{ij}}{\partial x^{k}}\right) \tag{3.1}$$

$$\Gamma_{ij}^{k} = g^{kh}[ij,h] \tag{3.2}$$

The Christoffel symbol of the first kind can also be denoted by $\Gamma_{ij,k}$or $[k,ij]$, and the Christoffel symbol of the second kind may be represented by Γ_{ij}^{k}.

The symbols $[ij,k]$and $\Gamma_{ij}^{k}$are **nottensors**, as will be shown later. The notation follows the summation convention: the three indices in $[ij,k]$ are treated as subscripts, while the upper index k in $\Gamma_{ij}^{k}$is treated as a superscript. The repeated index h in equation is summed over.

Note: The notation

$$\partial_r A_{pq} = \frac{\partial A_{pq}}{\partial x^r}$$

may be used for partial derivatives. In this notation, the Christoffel symbol of the first kind is

$$[ij,k] = \frac{1}{2}\left(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}\right)$$

Remark: The Christoffel symbols were introduced by Christoffel and Lipschitz in 1870.

Theorem 1

Statement: The Christoffel symbols of the first and second kind defined by equations (3.1) and(3.2) are symmetric in the indices $i$and j, that is,

$[ij,k] = [ji,k] and\ \Gamma_{ij}^{k} = \Gamma_{ji}^{k}$

Proof: Recall the definition of the Christoffel symbols of the first kind

$$[ij,k] = \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right) \quad (3.3)$$

Interchange the indices $i$and $j$in (3.3)

$$[ji,k] = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right) = \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right) \quad (3.4)$$

Since the metric tensor g_{ij} is symmetric, $i.e.\ g_{ij} = g_{ji}$, it follows that

$$\frac{\partial g_{ij}}{\partial x^k} = \frac{\partial g_{ji}}{\partial x^k}$$

Therefore, from equations (3.3) and (3.4), we get

$$[ij,k] = [ji,k]. \quad (3.5)$$

Using the definition of the Christoffel symbols of the second kind

$$\Gamma_{ij}^{k} = g^{kh}[ij,h], \quad (3.6)$$

interchange $i$and $j$in (3.6) to obtain

$$\Gamma_{ji}^{k} = g^{kh}[ji,h] = g^{kh}[ij,h] = \Gamma_{ij}^{k},$$

where the symmetry of $[ij, h]$ from (3.5) is used.

Hence the Christoffel symbols of both kinds are symmetric in the lower indices i and j.

Remark: The number of independent components of the Christoffel symbols is

$$\frac{1}{2}N^2(N+1)$$

This follows because the fundamental metric tensor g_{ij} is a symmetric second-order tensor with

$$\frac{1}{2}N(N+1)$$

independent components in an N –dimensional space V_N.

Since, $\frac{\partial g_{ij}}{\partial x^k}$ involves $N \times \frac{1}{2}N(N+1) = \frac{1}{2}N^2(N+1)$ independent components, and the Christoffel symbols Γ_{ij}^k are linear combinations of these, it follows that the Christoffel symbols also have

$$\frac{1}{2}N^2(N+1)$$

independent components.

Theorem 2

Statement: The necessary and sufficient condition that all the Christoffel symbols vanish at a point is that the metric tensor components g_{ij} are constant at that point.

Proof.

Necessary condition: Assume that g_{ij} are constants, $i.e.$,

$$\frac{\partial g_{ij}}{\partial x^k} = 0, \quad \frac{\partial g_{jk}}{\partial x^i} = 0, \quad \frac{\partial g_{ki}}{\partial x^j} = 0$$

Then the Christoffel symbols of the first kind vanish

$$[ij, k] = \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right) = 0$$

Consequently, the Christoffel symbols of the second kind vanish as well, since

$$\Gamma_{ij}^k = g^{kh}[ij, h] = 0.$$

Sufficient condition: Conversely, suppose that all Christoffel symbols vanish at a point, $i.e.$,

$$\Gamma_{ij}^k = 0.$$

Then the Christoffel symbols of the first kind satisfy

$$[ij,k]=\frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i}+\frac{\partial g_{ik}}{\partial x^j}-\frac{\partial g_{ij}}{\partial x^k}\right)=0$$

Rearranging, we have

$$\frac{\partial g_{jk}}{\partial x^i}+\frac{\partial g_{ik}}{\partial x^j}=\frac{\partial g_{ij}}{\partial x^k}.$$

Since the indices i,j,k are independent variables, the only way this holds for all $i,j,k$is that

$$\frac{\partial g_{jk}}{\partial x^i}=0,\quad \frac{\partial g_{ik}}{\partial x^j}=0,\quad \frac{\partial g_{ij}}{\partial x^k}=0$$

which implies

$$g_{ij}=constant$$

Hence, the Christoffel symbols vanish at a point if and only if the metric tensor components are constant at that point.

Theorem 3

If

$$\sum_{l=1}^{N}\frac{\partial^2 y^l}{\partial x^i\partial x^j}\frac{\partial y^l}{\partial x^k}=\frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i}+\frac{\partial g_{ik}}{\partial x^j}-\frac{\partial g_{ij}}{\partial x^k}\right)$$

then the quantity on the right side is denoted by the symbol $[ij,k]$, which is the Christoffel symbol of the first kind.

Proof.

Given the metric tensor components expressed in new coordinates y^l,

$$g_{ij}=\sum_{l=1}^{N}\frac{\partial y^l}{\partial x^i}\frac{\partial y^l}{\partial x^j}$$

Differentiating $g_{jk}$with respect to x^i, we get

$$\frac{\partial g_{jk}}{\partial x^i}=\sum_{l=1}^{N}\left(\frac{\partial^2 y^l}{\partial x^i\partial x^j}\frac{\partial y^l}{\partial x^k}+\frac{\partial y^l}{\partial x^j}\frac{\partial^2 y^l}{\partial x^i\partial x^k}\right)$$

Similarly,

$$\frac{\partial g_{ik}}{\partial x^i}=\sum_{l=1}^{N}\left(\frac{\partial^2 y^l}{\partial x^j\partial x^i}\frac{\partial y^l}{\partial x^k}+\frac{\partial y^l}{\partial x^i}\frac{\partial^2 y^l}{\partial x^i\partial x^k}\right)$$

And,

$$\frac{\partial g_{ij}}{\partial x^k}=\sum_{l=1}^{N}\left(\frac{\partial^2 y^l}{\partial x^k\partial x^i}\frac{\partial y^l}{\partial x^j}+\frac{\partial y^l}{\partial x^i}\frac{\partial^2 y^l}{\partial x^k\partial x^j}\right)$$

Now, consider the combination

$$\frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i}+\frac{\partial g_{ik}}{\partial x^j}-\frac{\partial g_{ij}}{\partial x^k}\right)$$

Substituting the above expressions and rearranging terms, the symmetric second derivatives cancel appropriately, and we get

$$\frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i}+\frac{\partial g_{ik}}{\partial x^j}-\frac{\partial g_{ij}}{\partial x^k}\right)=\sum_{l=1}^{N}\frac{\partial^2 y^l}{\partial x^i \partial x^j}\frac{\partial y^l}{\partial x^k}$$

This expression is exactly the definition of the Christoffel symbol of the first kind, denoted by $[ij,k]$.

Hence,

$$[ij,k]=\frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i}+\frac{\partial g_{ik}}{\partial x^j}-\frac{\partial g_{ij}}{\partial x^k}\right)$$

which completes the proof.

Theorem 4

To establish the following results:

$(i)\quad [ij,h]=g_{kh}\Gamma_{ij}^k$

$(ii)\quad [ij,k]+[jk,i]=\dfrac{\partial g_{ik}}{\partial x^j}$

Proof:

(i) Relation between Christoffel symbols of first and second kind.

We know from the definition

$\Gamma_{ij}^k = g^{kh}[ij,h]$.

Multiplying both sides by g_{km} we get

$g_{km}\Gamma_{ij}^k = g_{km}g^{kh}[ij,h]=\delta_m^h[ij,h]=[ij,m]$

Replacing the dummy index m by h yields

$[ij,h] = g_{kh}\Gamma_{ij}^k$

This proves part (i).

(ii) Identity involving Christoffel symbols of the first kind

Recall the definition of the Christoffel symbols of the first kind,

$$[ij,k]=\frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i}+\frac{\partial g_{ik}}{\partial x^j}-\frac{\partial g_{ij}}{\partial x^k}\right)$$

Similarly,

$$[jk,i] = \frac{1}{2}\left(\frac{\partial g_{ki}}{\partial x^j} + \frac{\partial g_{ji}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^i}\right)$$

Adding these, we have

$$[ij,k] + [jk,i] = \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right) + \frac{1}{2}\left(\frac{\partial g_{ki}}{\partial x^j} + \frac{\partial g_{ji}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^i}\right)$$

Notice that the terms $\frac{\partial g_{jk}}{\partial x^i}$ cancel, so we get,

$$= \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ki}}{\partial x^j} + \frac{\partial g_{ji}}{\partial x^k}\right)$$

Using the symmetry of the metric tensor, $g_{ik} = g_{ki}$ and $g_{ij} = g_{ji}$, this simplifies to

$$= \frac{1}{2}\left(2\frac{\partial g_{ik}}{\partial x^j}\right) = \frac{\partial g_{ik}}{\partial x^j}$$

Hence,

$$[ij,k] + [jk,i] = \frac{\partial g_{ik}}{\partial x^j}$$

This proves part (ii).

Theorem 5

To show that

$$\Gamma^i_{ij} = \frac{\partial}{\partial x^j}\left(\log\sqrt{|g|}\right)$$

Where $g = \det(g_{ij})$.

Proof.

Represent the metric tensor matrix as

$$\boldsymbol{g} = \begin{pmatrix} g_{11} & g_{12} \cdots g_{1N} \\ g_{21} & g_{22} \cdots g_{2N} \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ g_{N1} g_{N2} \cdots g_{NN} \end{pmatrix}.$$

Let $g = det(g_{ij})$. Differentiating g with respect to x^j, using the formula for determinant derivative,

$$\frac{\partial g}{\partial x^j} = \sum_{i,k} cofactor(g_{ik})\frac{\partial g_{ik}}{\partial x^j} = g\, g^{ik}\frac{\partial g_{ik}}{\partial x^j}$$

Dividing both sides by g, we get

$$\frac{1}{g}\frac{\partial g}{\partial x^j} = g^{ik}\frac{\partial g_{ik}}{\partial x^j}$$

Hence,

$$\frac{\partial}{\partial x^j}(\log|g|) = g^{ik}\frac{\partial g_{ik}}{\partial x^j}$$

From the definition of Christoffel symbols of the first kind and their symmetry, it follows that

$$\Gamma^i_{ij} = \frac{1}{2}g^{ik}\left(\frac{\partial g_{kj}}{\partial x^i} + \frac{\partial g_{ij}}{\partial x^k} - \frac{\partial g_{ik}}{\partial x^j}\right) = \frac{1}{2}g^{ik}\frac{\partial g_{ik}}{\partial x^j} = \frac{1}{2}\frac{\partial}{\partial x^j}(\log|g|)$$

Taking square root,

$$\Gamma^i_{ij}$$

If the metric is Lorentzian or of indefinite signature, replace $|g|$ by $\sqrt{-g}$ accordingly.

CHECK YOUR PROGRESS 1:

Q.1 Define Christoffel's symbols of the first and second kind. Find relations between the two.

Q.2 Define the Christoffel three-index symbols

Q.3 State the relationship between the Christoffel symbols of the first and the second kind.

3.4 Tensor laws of transformation of Christoffel symbols

To prove that Christoffel symbols are not tensor quantities

Or

To determine tensor laws of transformation of Christoffel symbols

Proof: We know that g_{ij} is a covariant tensor of second order and hence law of transformation is

$$\bar{g}_{ij} = g_{uv}\frac{\partial x^u}{\partial \bar{x}^i}\frac{\partial x^v}{\partial \bar{x}^j} \tag{3.7}$$

Differentiating $(3.7) w.r.t.\, \bar{x}^k$, we get

$$\frac{\partial \bar{g}_{ij}}{\partial \bar{x}^k} = \frac{\partial g_{uv}}{\partial x^s}\frac{\partial x^s}{\partial \bar{x}^k}\frac{\partial x^u}{\partial \bar{x}^i}\frac{\partial x^v}{\partial \bar{x}^j} + g_{uv}\frac{\partial^2 x^u}{\partial \bar{x}^k \partial \bar{x}^i}\frac{\partial x^v}{\partial \bar{x}^j} + g_{uv}\frac{\partial x^u}{\partial \bar{x}^i}\frac{\partial^2 x^v}{\partial \bar{x}^k \partial \bar{x}^j} \tag{3.8}$$

[Adjusting dummy indices]

Similarly, we have

$$\frac{\partial \bar{g}_{ki}}{\partial \bar{x}^j} = \frac{\partial g_{uv}}{\partial x^s}\frac{\partial x^s}{\partial \bar{x}^j}\frac{\partial x^u}{\partial \bar{x}^k}\frac{\partial x^v}{\partial \bar{x}^i} + g_{uv}\frac{\partial^2 x^u}{\partial \bar{x}^j \partial \bar{x}^k}\frac{\partial x^v}{\partial \bar{x}^i} + g_{uv}\frac{\partial x^u}{\partial \bar{x}^k}\frac{\partial^2 x^v}{\partial \bar{x}^j \partial \bar{x}^i} \tag{3.9}$$

or

$$\frac{\partial \overline{g}_{ki}}{\partial \overline{x}^j} = \frac{\partial g_{uv}}{\partial x^s}\frac{\partial x^s}{\partial \overline{x}^j}\frac{\partial x^u}{\partial \overline{x}^k}\frac{\partial x^v}{\partial \overline{x}^i} + g_{uv}\frac{\partial^2 x^u}{\partial \overline{x}^j \partial \overline{x}^k}\frac{\partial x^v}{\partial \overline{x}^i} + g_{uv}\frac{\partial x^u}{\partial \overline{x}^k}\frac{\partial^2 x^v}{\partial \overline{x}^j \partial \overline{x}^i} \tag{3.10}$$

Applying $\frac{1}{2}[(3)+(4)-(2)]$ **and noting**

(i) $g_{uv} = g_{vu}$

(ii) $g_{uv}\frac{\partial x^u}{\partial \bar{x}^i}\frac{\partial^2 x^v}{\partial \bar{x}^j \partial \bar{x}^k} = g_{vu}\frac{\partial x^v}{\partial \bar{x}^i}\frac{\partial^2 x^u}{\partial \bar{x}^j \partial \bar{x}^k}$ and

(iii) $\frac{1}{2}\left[\frac{\partial g_{jk}}{\partial \bar{x}^i} + \frac{\partial g_{ki}}{\partial \bar{x}^j} - \frac{\partial g_{ij}}{\partial \bar{x}^k}\right] = [ij,k]$

We get,

$$\overline{[ij,k]} = [uv,w]\frac{\partial x^u}{\partial \overline{x}^i}\frac{\partial x^v}{\partial \overline{x}^j}\frac{\partial x^w}{\partial \overline{x}^k} + g_{uv}\left(\frac{\partial^2 x^u}{\partial \overline{x}^i \partial \overline{x}^j}\frac{\partial x^v}{\partial \overline{x}^k}\right) \tag{3.11}$$

Relation (3.11) is not the tensor law of transformation. Hence **Christoffel symbol of first kind** $[ij,k]$ **is not a tensor quantity.**

Now we shall prove that Christoffel symbol of second kind is not tensor quantity.

The transformation law of fundamental tensor g^{rs} is

$$\overline{g}^{kh} = g^{rs}\frac{\partial \overline{x}^k}{\partial x^r}\frac{\partial \overline{x}^h}{\partial x^s} \tag{3.12}$$

Multiplying (3.11) **by the corresponding sides of** (3.12)**, we get**

$$\overline{[ij,k]}\overline{g}^{kh} = g^{rs}[uv,w]\frac{\partial x^u}{\partial \overline{x}^i}\frac{\partial x^v}{\partial \overline{x}^j}\left(\frac{\partial x^w}{\partial \overline{x}^k}\frac{\partial \overline{x}^k}{\partial \overline{x}^r}\right)\frac{\partial \overline{x}^h}{\partial x^s} + g^{rs}g_{uv}\frac{\partial^2 x^u}{\partial \overline{x}^i \partial \overline{x}^j}\left(\frac{\partial x^v}{\partial \overline{x}^k}\frac{\partial \overline{x}^k}{\partial \overline{x}^r}\right)\frac{\partial \overline{x}^h}{\partial x^s}$$

$$\Rightarrow \overline{\Gamma}^h_{ij} = g^{rs}[uv,w]\frac{\partial x^u}{\partial \overline{x}^i}\frac{\partial x^v}{\partial \overline{x}^j}\delta^w_r\frac{\partial \overline{x}^h}{\partial x^s} + g^{rs}g_{uv}\frac{\partial^2 x^u}{\partial \overline{x}^i \partial \overline{x}^j}\delta^v_r\frac{\partial \overline{x}^h}{\partial x^s}$$

$$\Rightarrow \overline{\Gamma}^h_{ij} = g^{rs}[uv,r]\frac{\partial x^u}{\partial \overline{x}^i}\frac{\partial x^v}{\partial \overline{x}^j}\frac{\partial x^h}{\partial x^s} + g^{rs}g_{uv}\frac{\partial^2 x^u}{\partial \overline{x}^i \partial \overline{x}^j}\frac{\partial x^h}{\partial x^s}$$

$$= \Gamma^s_{uv}\frac{\partial x^u}{\partial \overline{x}^i}\frac{\partial x^v}{\partial \overline{x}^j}\frac{\partial x^h}{\partial x^s} + \delta^s_u\frac{\partial^2 x^u}{\partial \overline{x}^i \partial \overline{x}^j}\frac{\partial x^h}{\partial x^s}$$

$$= \Gamma^s_{uv}\frac{\partial x^u}{\partial \overline{x}^i}\frac{\partial x^v}{\partial \overline{x}^j}\frac{\partial x^h}{\partial x^s} + \frac{\partial^2 x^s}{\partial \overline{x}^i \partial \overline{x}^j}\frac{\partial x^h}{\partial x^s} \tag{3.13}$$

Relation (3.13) is not the tensor law of transformation. Hence **the Christoffel symbol of second kind is not a tensor quantity.**

Remark: We have proved above that Christoffel symbols are not tensor quantities. But in some very special case of linear transformations of coordinates where,

$$\frac{\partial^2 x^s}{\partial \bar{x}^i \partial \bar{x}^j} = 0$$

equation (3.11) and (3.13) reduce to give tensor law of transformations and symbols behave as tensors.

Consider a linear transformation

$$x^s = a_i^s \bar{x}^i + b^s \quad (A)$$

where a_i^s and b^s are constants, then

$$\frac{\partial^2 x^s}{\partial \bar{x}^i \partial \bar{x}^j} = 0$$

Now relations (3.11) and (3.13) reduce to

$$\overline{[ij,k]} = [uv,w]\frac{\partial x^u}{\partial \bar{x}^i}\frac{\partial x^v}{\partial \bar{x}^j}\frac{\partial x^w}{\partial \bar{x}^k} \quad (B)$$

And,

$$\bar{\Gamma}_{ij}^h = \Gamma_{uv}^s \frac{\partial x^u}{\partial \bar{x}^i}\frac{\partial x^v}{\partial \bar{x}^j}\frac{\partial \bar{x}^h}{\partial x^s} \quad (C)$$

The equations (B) and (C) give the tensor law of transformation of Christoffel symbols relative to very special case of linear transformations given by equation (A).

Theorem 7

To prove that the transformation of Christoffel's symbols form a group $i.e.$, possess the transitive property.

Proof: We consider the following transformation of coordinates

$x^i \to \bar{x}^i \to \bar{\bar{x}}^i$

Now when $x^i \to \bar{x}^i$, we have

$$\bar{\Gamma}_{ij}^h = \Gamma_{uv}^s \frac{\partial x^u}{\partial \bar{x}^i}\frac{\partial x^v}{\partial \bar{x}^j}\frac{\partial x^h}{\partial x^s} + \frac{\partial^2 x^s}{\partial \bar{x}^i \partial \bar{x}^j}\frac{\partial x^h}{\partial x^s} \quad (3.14)$$

Again, when $\bar{x}^i \to \bar{\bar{x}}^i$, we have

$$\bar{\bar{\Gamma}}_{ab}^c = \bar{\Gamma}_{ij}^h \frac{\partial \bar{x}^i}{\partial \bar{\bar{x}}^a}\frac{\partial \bar{x}^j}{\partial \bar{\bar{x}}^b}\frac{\partial \bar{\bar{x}}^c}{\partial \bar{x}^h} + \frac{\partial^2 \bar{x}^h}{\partial \bar{\bar{x}}^a \partial \bar{\bar{x}}^b}\frac{\partial \bar{\bar{x}}^c}{\partial \bar{x}^h} \quad (3.15)$$

Substituting the value from (3.14) in (3.15), we have

$$\bar{\bar{\Gamma}}_{ab}^c = \Gamma_{uv}^s \frac{\partial x^u}{\partial \bar{\bar{x}}^i}\frac{\partial x^v}{\partial \bar{\bar{x}}^j}\frac{\partial x^h}{\partial x^s}\frac{\partial \bar{x}^i}{\partial \bar{\bar{x}}^a}\frac{\partial \bar{x}^j}{\partial \bar{\bar{x}}^b}\frac{\partial \bar{\bar{x}}^c}{\partial \bar{x}^h} + \frac{\partial^2 x^s}{\partial \bar{x}^i \partial \bar{x}^j}\frac{\partial \bar{x}^h}{\partial x^s}\frac{\partial \bar{x}^i}{\partial \bar{\bar{x}}^a}\frac{\partial \bar{x}^j}{\partial \bar{\bar{x}}^b}\frac{\partial \bar{\bar{x}}^c}{\partial \bar{x}^h} + \frac{\partial^2 \bar{x}^h}{\partial \bar{\bar{x}}^a \partial \bar{\bar{x}}^b}\frac{\partial \bar{\bar{x}}^c}{\partial \bar{x}^h}$$

$$= \Gamma_{uv}^s \frac{\partial x^u}{\partial \bar{\bar{x}}^a}\frac{\partial x^v}{\partial \bar{\bar{x}}^b}\frac{\partial \bar{\bar{x}}^c}{\partial x^s} + \frac{\partial^2 x^s}{\partial \bar{x}^i \partial \bar{x}^j}\frac{\partial \bar{x}^i}{\partial \bar{\bar{x}}^a}\frac{\partial \bar{x}^j}{\partial \bar{\bar{x}}^b}\frac{\partial \bar{\bar{x}}^c}{\partial \bar{x}^s} + \frac{\partial^2 \bar{x}^h}{\partial \bar{\bar{x}}^a \partial \bar{\bar{x}}^b}\frac{\partial \bar{\bar{x}}^c}{\partial \bar{x}^h} \quad (3.16)$$

But we know that

$$\frac{\partial x^s}{\partial \bar{x}^i}\frac{\partial \bar{x}^i}{\partial \bar{\bar{x}}^a} = \frac{\partial x^s}{\partial \bar{\bar{x}}^a} \tag{3.17}$$

Differentiation of (3.17) $w.r.t.\ \bar{\bar{x}}^b$ gives

$$\frac{\partial^2 x^s}{\partial \bar{x}^i \partial \bar{x}^j}\frac{\partial \bar{x}^j}{\partial \bar{\bar{x}}^b}\frac{\partial \bar{x}^i}{\partial \bar{\bar{x}}^a} + \frac{\partial \bar{x}^s}{\partial \bar{\bar{x}}^i}\frac{\partial^2 \bar{x}^i}{\partial \bar{\bar{x}}^a \partial \bar{\bar{x}}^b} = \frac{\partial^2 \bar{x}^s}{\partial \bar{\bar{x}}^a \partial \bar{\bar{x}}^b} \tag{3.18}$$

Multiplying (3.18) by $\frac{\partial \bar{\bar{x}}^c}{\partial \bar{x}^s}$ we obtain

$$\frac{\partial^2 x^s}{\partial \bar{x}^i \partial \bar{x}^j}\frac{\partial \bar{x}^j}{\partial \bar{\bar{x}}^b}\frac{\partial \bar{x}^i}{\partial \bar{\bar{x}}^a}\frac{\partial \bar{\bar{x}}^c}{\partial \bar{x}^s} + \frac{\partial^2 \bar{x}^i}{\partial \bar{\bar{x}}^a \partial \bar{\bar{x}}^b}\frac{\partial x^s}{\partial \bar{x}^i}\frac{\partial \bar{\bar{x}}^c}{\partial \bar{x}^s} = \frac{\partial^2 \bar{x}^s}{\partial \bar{\bar{x}}^a \partial \bar{\bar{x}}^b}\frac{\partial \bar{\bar{x}}^c}{\partial \bar{x}^s}$$

Or

$$\frac{\partial^2 x^s}{\partial \bar{x}^i \partial \bar{x}^j}\frac{\partial \bar{\bar{x}}^c}{\partial \bar{x}^s}\frac{\partial \bar{x}^i}{\partial \bar{\bar{x}}^a}\frac{\partial \bar{x}^j}{\partial \bar{\bar{x}}^b} + \frac{\partial^2 \bar{x}^h}{\partial \bar{\bar{x}}^a \partial \bar{\bar{x}}^b}\frac{\partial \bar{\bar{x}}^c}{\partial \bar{x}^h} = \frac{\partial^2 \bar{x}^s}{\partial \bar{\bar{x}}^a \partial \bar{\bar{x}}^b}\frac{\partial \bar{\bar{x}}^c}{\partial \bar{x}^s} \tag{3.19}$$

[We have replaced dummy suffix i by h in second term on $L.H.S.$]

Using relation (3.19) in (3.16), we have

$$\bar{\bar{\Gamma}}^c_{ab} = \Gamma^s_{uv}\frac{\partial x^u}{\partial \bar{\bar{x}}^a}\frac{\partial x^v}{\partial \bar{\bar{x}}^b}\frac{\partial \bar{\bar{x}}^c}{\partial x^s} + \frac{\partial^2 x^s}{\partial \bar{\bar{x}}^a \partial \bar{\bar{x}}^b}\frac{\partial \bar{\bar{x}}^c}{\partial x^s} \tag{3.20}$$

Equation (3.20) is of the same form as (3.14), if we make direct transformation from x^i to $\bar{\bar{x}}^i$. This property is called that transformation of Christoffel's symbols form a group.

Example 1

Evaluate the Christoffel Symbols of both kinds for spaces, where $g_{ij} = 0$ if $i \neq j$, i.e., the coordinate system is orthogonal and the metric is diagonal.

Solution:

The Christoffel symbols of the first kind are given by:

$$[ij,k] = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right)$$

Case I : $i = j = k$

$$[ii,i] = \frac{1}{2}\left(\frac{\partial g_{ii}}{\partial x^i} + \frac{\partial g_{ii}}{\partial x^i} - \frac{\partial g_{ii}}{\partial x^i}\right) = \frac{1}{2}\frac{\partial g_{ii}}{\partial x^i}$$

Case II: $i = k \neq j$

$$[ij,i] = \frac{1}{2}\left(\frac{\partial g_{ii}}{\partial x^j} + \underbrace{\frac{\partial g_{ji}}{\partial x^i}}_{=0} - \underbrace{\frac{\partial g_{ij}}{\partial x^i}}_{=0}\right) = \frac{1}{2}\frac{\partial g_{ii}}{\partial x^j}$$

Case *III* : $i = j \neq k$

$$[ii,k] = \frac{1}{2}\left(\underbrace{\frac{\partial g_{ik}}{\partial x^i}}_{=0} + \underbrace{\frac{\partial g_{ik}}{\partial x^i}}_{=0} - \frac{\partial g_{ii}}{\partial x^k}\right) = -\frac{1}{2}\frac{\partial g_{ii}}{\partial x^k}$$

Case *IV* : $i \neq j \neq k$

$$[ij,k] = 0$$

The Christoffel symbols of the second kind are given by

$$\Gamma_{ij}^k = g^{kl}[ij,l]$$

Since the metric is diagonal, $g^{kl} = 0$ for $k \neq l$, and hence

$$\Gamma_{ij}^k = g^{kk}[ij,k]$$

Now consider the same cases

Case *I*: $i = j = k$

$$\Gamma_{ii}^i = g^{ii}[ii,i] = \frac{1}{2g_{ii}}\frac{\partial g_{ii}}{\partial x^i} = \frac{\partial}{\partial x^i}(\log\sqrt{g_{ii}})$$

Case *II*: $i = k \neq j$

$$\Gamma_{ij}^i = g^{ii}[ij,i] = \frac{1}{2g_{ii}}\frac{\partial g_{ii}}{\partial x^j} = \frac{\partial}{\partial x^j}(\log\sqrt{g_{ii}})$$

Case *III*: $i = j \neq k$

$$\Gamma_{ij}^k = g^{kk}[ii,k] = -\frac{1}{2g_{kk}}\frac{\partial g_{ii}}{\partial x^k} = -\frac{\partial}{\partial x^k}(log\sqrt{g_{ii}})$$

Case *IV*: $i \neq j \neq k$

$$\Gamma_{ij}^k = 0$$

Example 2

For the metric

$$ds^2 = -a(dx^1)^2 - b(dx^2)^2 - c(dx^3)^2 + d(dx^4)^2$$

where a, b, c, d **are functions of the coordinates** $x^i (i = 1,2,3,4)$, **evaluate the Christoffel symbols**

$$[11,1], \quad [11,2], \quad [12,3], \quad [11,4], \quad [44,4], \quad \Gamma_{12}^3, \quad \Gamma_{11}^4, \quad \Gamma_{44}^4$$

Solution:

From the given metric,

$g_{11} = -a, \quad g_{22} = -b, \quad g_{33} = -c, \quad g_{44} = d, \quad g_{ij} = 0 \ \ for\, i \neq j$

Hence, the coordinate system is orthogonal. The inverse metric components are

$$g^{11} = -\frac{1}{a}, \quad g^{22} = -\frac{1}{b}, \quad g^{33} = -\frac{1}{c}, \quad g^{44} = \frac{1}{d}$$

We use the Christoffel symbols of the first kind

$$[ij,k] = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right)$$

$$[11,1] = \frac{1}{2}\frac{\partial g_{11}}{\partial x^1} = \frac{1}{2}\frac{\partial(-a)}{\partial x^1} = -\frac{1}{2}\frac{\partial a}{\partial x^1}$$

$$\Gamma_{11}^1 = g^{11}[11,1] = -\frac{1}{a}\left(-\frac{1}{2}\frac{\partial a}{\partial x^1}\right) = \frac{1}{2a}\frac{\partial a}{\partial x^1}$$

$$[11,2] = -\frac{1}{2}\frac{\partial a}{\partial x^2}, \quad \Gamma_{11}^2 = g^{22}[11,2] = -\frac{1}{b}.\left(-\frac{1}{2}\frac{\partial a}{\partial x^2}\right) = \frac{1}{2b}\frac{\partial a}{\partial x^2}$$

$$[12,3] = 0 \Rightarrow \Gamma_{12}^3 = 0$$

$$[11,4] = -\frac{1}{2}\frac{\partial a}{\partial x^4}, \quad \Gamma_{11}^4 = g^{44}[11,4] = \frac{1}{d}\left(-\frac{1}{2}\frac{\partial a}{\partial x^4}\right) = -\frac{1}{2d}\frac{\partial a}{\partial x^4}$$

$$[44,4] = \frac{1}{2}\frac{\partial g_{44}}{\partial x^4} = \frac{1}{2}\frac{\partial d}{\partial x^4}, \quad \Gamma_{44}^4 = g^{44}[44,4] = \frac{1}{d}.\frac{1}{2}\frac{\partial d}{\partial x^4} = \frac{1}{2d}\frac{\partial d}{\partial x^4}$$

Example 3

The surface of a sphere is a two-dimensional Riemannian space. Compute the Christoffel symbols.

Solution:

The metric of the surface of a sphere in a 2 –dimensional Riemannian manifold $V_2$is given by

$$ds^2 = r^2 d\theta^2 + r^2 \sin^2\theta \, d\Phi^2$$

This gives the components of the metric tensor as

$$g_{11} = r^2, \quad g_{22} = r^2 \sin^2\theta, \quad g_{12} = g_{21} = 0$$

Hence, the coordinate system is orthogonal.

The inverse metric components are

$$g^{11} = \frac{1}{r^2}, \quad g^{22} = \frac{1}{r^2 \sin^2\theta}, \quad g^{12} = g^{21} = 0$$

We now compute the Christoffel symbols of the second kind using the formula

$$\Gamma_{ij}^k = \frac{1}{2}g^{kl}\left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l}\right)$$

Let $x^1 = \theta, \; x^2 = \Phi$. Then the non-zero Christoffel symbols are

$$\Gamma_{22}^1 = -\frac{1}{2}g^{11}\frac{\partial g_{22}}{\partial \theta} = -\frac{1}{2r^2}\cdot\frac{\partial}{\partial\theta}(r^2\sin^2\theta) = -\frac{1}{2r^2}\cdot r^2\cdot 2\sin\theta\cos\theta = -\sin\theta\cos\theta$$

$$\Gamma_{12}^2 = \Gamma_{21}^2 = \frac{1}{2}g^{22}\frac{\partial g_{22}}{\partial\theta} = \frac{1}{2}\cdot\frac{1}{r^2\sin^2\theta}\cdot\frac{\partial}{\partial\theta}(r^2\sin^2\theta) = \frac{1}{2r^2\sin^2\theta}\cdot r^2\cdot 2\sin\theta\cos\theta = \cot\theta$$

$$\Gamma^1_{22} = -\sin\theta\cos\theta, \quad \Gamma^2_{12} = \Gamma^2_{21} = \cot\theta$$

These describe the connection (i.e., how vectors change) on the surface of a sphere, a classic 2D Riemannian manifold.

CHECK YOUR PROGRESS2:

Q.1 Calculate the non-vanishing Christoffel symbols for the metric

$$ds^2 = -dx^2 - dy^2 - dz^2 + f(x,y,z)dt^2$$

Q.2 Calculate the non-vanishing Christoffel symbols of 2nd kind corresponding to the metric

$$ds^2 = (dx^1)^2 + (x^1)^2(dx^2)^2 + (x^1)^2\sin^2 x^2\,(dx^3)^2$$

3.5 Covariant Derivative of a Covariant Vector

Let A_i be a covariant vector transforming as

$$\bar{A}_i = A_p\frac{\partial x^p}{\partial \bar{x}^i} \tag{3.21}$$

Differentiating (3.21) partially with respect to $\bar{x}^j$, we get

$$\frac{\partial \bar{A}_i}{\partial \bar{x}^j} = \frac{\partial}{\partial \bar{x}^j}\left(A_p\frac{\partial x^p}{\partial \bar{x}^i}\right) = \frac{\partial A_p}{\partial \bar{x}^j}\frac{\partial x^p}{\partial \bar{x}^i} + A_p\frac{\partial^2 x^p}{\partial \bar{x}^i\partial \bar{x}^j} \tag{3.22}$$

Applying the chain rule,

$$\frac{\partial A_p}{\partial \bar{x}^j} = \frac{\partial x^m}{\partial \bar{x}^j}\frac{\partial A_p}{\partial x^m}$$

and using the formula relating second derivatives of coordinates to Christoffel symbols,

$$\frac{\partial^2 x^p}{\partial \bar{x}^i\partial \bar{x}^j} = \bar{\Gamma}^h_{ij}\frac{\partial x^p}{\partial \bar{x}^h} - \Gamma^p_{lm}\frac{\partial x^l}{\partial \bar{x}^i}\frac{\partial x^m}{\partial \bar{x}^j} \tag{3.23}$$

We get

$$\frac{\partial \bar{A}_i}{\partial \bar{x}^j} = \frac{\partial x^m}{\partial \bar{x}^j}\frac{\partial x^l}{\partial \bar{x}^i}\left(\frac{\partial A_l}{\partial x^m} - \Gamma^p_{lm}A_p\right) + A_p\bar{\Gamma}^h_{ij}\frac{\partial x^p}{\partial \bar{x}^h} \tag{3.24}$$

Defining the covariant derivative of a covariant vector as

$$A_{i,j} = \frac{\partial A_i}{\partial x^j} - \Gamma^p_{ij}A_p \tag{3.25}$$

we find that it transforms as

$$\bar{A}_{i,j} = \frac{\partial x^l}{\partial \bar{x}^i}\frac{\partial x^m}{\partial \bar{x}^j}A_{l,m}$$

which is the transformation law of a covariant tensor of rank two.

3.6 Covariant Derivative of a Covariant Tensor of Second Order

Let A_{ij} be a covariant tensor of second order. Its transformation law under a change of coordinates $x^i \to x^p$ is

$$\overline{A}_{pq} = A_{ij}\frac{\partial x^i}{\partial \overline{x}^p}\frac{\partial x^j}{\partial \overline{x}^q} \tag{3.26}$$

Differentiating both sides of (3.26) partially with respect to x^r, we get

$$\frac{\partial \overline{A}_{pq}}{\partial \overline{x}^r} = \frac{\partial A_{ij}}{\partial x^l}\frac{\partial x^l}{\partial \overline{x}^r}\frac{\partial x^i}{\partial \overline{x}^p}\frac{\partial x^j}{\partial \overline{x}^q} + A_{ij}\frac{\partial^2 x^i}{\partial \overline{x}^p \partial \overline{x}^r}\frac{\partial x^j}{\partial \overline{x}^q} + A_{ij}\frac{\partial x^i}{\partial \overline{x}^p}\frac{\partial^2 x^j}{\partial \overline{x}^q \partial \overline{x}^r} \tag{3.27}$$

We know that

$$\frac{\partial^2 x^i}{\partial \overline{x}^p \partial \overline{x}^r} = \overline{\Gamma}^l_{pr}\frac{\partial x^j}{\partial \overline{x}^l} - \Gamma^i_{uv}\frac{\partial x^u}{\partial \overline{x}^p}\frac{\partial x^v}{\partial \overline{x}^r} \tag{3.28}$$

and

$$\frac{\partial^2 x^j}{\partial \overline{x}^q \partial \overline{x}^r} = \overline{\Gamma}^l_{qr}\frac{\partial x^j}{\partial \overline{x}^l} - \Gamma^j_{uv}\frac{\partial x^u}{\partial \overline{x}^q}\frac{\partial x^v}{\partial \overline{x}^r} \tag{3.29}$$

Substituting (3.28) and (3.29) in (3.27), we get

$$\frac{\partial \overline{A}_{pq}}{\partial \overline{x}^r} = \frac{\partial A_{ij}}{\partial x^l}\frac{\partial x^l}{\partial \overline{x}^r}\frac{\partial x^i}{\partial \overline{x}^p}\frac{\partial x^j}{\partial \overline{x}^q} + A_{ij}\left[\overline{\Gamma}^l_{pr}\frac{\partial x^j}{\partial \overline{x}^l} - \Gamma^i_{uv}\frac{\partial x^u}{\partial \overline{x}^p}\frac{\partial x^v}{\partial \overline{x}^r}\right]\frac{\partial x^j}{\partial \overline{x}^q} + A_{ij}\frac{\partial x^i}{\partial \overline{x}^p}\left[\overline{\Gamma}^l_{qr}\frac{\partial x^j}{\partial \overline{x}^l} - \Gamma^j_{uv}\frac{\partial x^u}{\partial \overline{x}^q}\frac{\partial x^v}{\partial \overline{x}^r}\right] \tag{3.30}$$

Using (3.26) and replacing dummy suffix i and j by m, we get

$$\frac{\partial \overline{A}_{pq}}{\partial \overline{x}^r} = \frac{\partial A_{ij}}{\partial x^l}\frac{\partial x^l}{\partial \overline{x}^r}\frac{\partial x^i}{\partial \overline{x}^p}\frac{\partial x^j}{\partial \overline{x}^q} + \overline{A}_{lq}\overline{\Gamma}^l_{pr} - A_{mj}\Gamma^m_{uv}\frac{\partial x^u}{\partial \overline{x}^p}\frac{\partial x^v}{\partial \overline{x}^r}\frac{\partial x^j}{\partial \overline{x}^q} + \overline{A}_{lp}\overline{\Gamma}^l_{qr} - A_{mi}\Gamma^m_{uv}\frac{\partial x^u}{\partial \overline{x}^q}\frac{\partial x^v}{\partial \overline{x}^r}\frac{\partial x^i}{\partial \overline{x}^p}$$

$$\Rightarrow \frac{\partial \overline{A}_{pq}}{\partial \overline{x}^r} = \overline{A}_{lq}\overline{\Gamma}^l_{pr} + \overline{A}_{lp}\overline{\Gamma}^l_{qr} + \frac{\partial A_{ij}}{\partial x^l}\frac{\partial x^l}{\partial \overline{x}^r}\frac{\partial x^i}{\partial \overline{x}^p}\frac{\partial x^j}{\partial \overline{x}^q} - A_{mj}\Gamma^m_{uv}\frac{\partial x^u}{\partial \overline{x}^p}\frac{\partial x^v}{\partial \overline{x}^r}\frac{\partial x^j}{\partial \overline{x}^q} - A_{mi}\Gamma^m_{uv}\frac{\partial x^u}{\partial \overline{x}^q}\frac{\partial x^v}{\partial \overline{x}^r}\frac{\partial x^i}{\partial \overline{x}^p}$$

Replacing dummy suffixes u, v by i, l in $4th$ term and u, v by j, l in $5th$ term of R.H.S respectively, we get

$$\Rightarrow \frac{\partial \bar{A}_{pq}}{\partial \bar{x}^r} = \bar{A}_{lq}\bar{\Gamma}^l_{pr} + \bar{A}_{lq}\bar{\Gamma}^l_{qr} + \frac{\partial A_{ij}}{\partial x^l}\frac{\partial x^l}{\partial \bar{x}^r}\frac{\partial x^i}{\partial \bar{x}^p}\frac{\partial x^j}{\partial \bar{x}^q} - A_{mj}\Gamma^m_{il}\frac{\partial x^i}{\partial \bar{x}^p}\frac{\partial x^l}{\partial \bar{x}^r}\frac{\partial x^j}{\partial \bar{x}^q} - A_{mi}\Gamma^m_{jl}\frac{\partial x^j}{\partial \bar{x}^q}\frac{\partial x^l}{\partial \bar{x}^r}\frac{\partial x^i}{\partial \bar{x}^p}$$

$$\Rightarrow \frac{\partial \bar{A}_{pq}}{\partial \bar{x}^r} - \bar{A}_{lq}\bar{\Gamma}^l_{pr} - \bar{A}_{lq}\bar{\Gamma}^l_{qr} = \left(\frac{\partial A_{ij}}{\partial x^l} - A_{mj}\Gamma^m_{il} - A_{mi}\Gamma^m_{jl}\right)\frac{\partial x^l}{\partial \bar{x}^r}\frac{\partial x^i}{\partial \bar{x}^p}\frac{\partial x^j}{\partial \bar{x}^q} \tag{3.31}$$

Now we use the **comma notation**

$$A_{ij,l} = \frac{\partial A_{ij}}{\partial x^l} - A_{mj}\Gamma^m_{il} - A_{im}\Gamma^m_{jl} \tag{3.32}$$

Using (3.32) in (3.31), we get

$$\overline{A}_{pq,r} = A_{ij,l}\frac{\partial x^l}{\partial \overline{x}^r}\frac{\partial x^i}{\partial \overline{x}^p}\frac{\partial x^j}{\partial \overline{x}^q}$$

which is the transformation law of a covariant tensor of rank three.

Thus, the covariant derivative $A_{ij,k}$ is itself a **covariant tensor** of third order.

Note: The covariant derivative $A_{ij,k}$ consists of N^3 quantities (for an N –dimensional manifold).This completes the demonstration that the covariant derivative of a covariant tensor of second order is a covariant tensor of third order.

W CHECK YOUR PROGRESS 3:

Q.1 What do you understand by the covariant derivative of a covariant vector? Show that it is a covariant tensor of the second order.

Q.2 Find the covariant derivative of a covariant vector A_i with respect to the fundamental tensor g_{ij}.

3.7Covariant Derivative of a scalar

Let Φ be a scalar (or invariant). Its covariant derivative x^i is denoted by $\Phi_{,i}$ and is defined as the ordinary partial derivative of Φ with respect to x^i. Thus

$$\Phi_{,i} = \frac{\partial \Phi}{\partial x^i}$$

Clearly,

$$\Phi_{,i} = \frac{\partial \Phi}{\partial x^i} = grad\ \Phi = \nabla \Phi$$

Evidently $\Phi_{,i}$ is a vector which has its covariant components, ordinary derivatives of Φ.

Remark: The covariant derivative of the covariant vector $\Phi_{,ki}$ is given by

$$\Phi_{,ki} = \frac{\partial^2 \Phi}{\partial x^k \partial x^i}$$

which shows that in this case the covariant derivative is simply the second partial derivative. Since the Christoffel symbols do not appear, the covariant derivatives commute

$$\Phi_{,ki} = \Phi_{,ik}$$

3.8 Divergence of a Vector

Let A^i be a contravariant vector. The divergence of A^i is defined as the contraction of $A^i_{,i}$. Or, in other words, the contraction of covariant derivative of A^i is defined as the divergence of vector A^i. The divergence of A^i is denoted by $div\ A^i$. Thus, we may write

$$div\, A^i = A^i_{,i}$$

Theorem:

If A^i is a contravariant vector, then prove that

$$div\, A = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}\left(A^i \sqrt{g}\right)$$

Where, $g = |g_{ij}|$.

Proof: The x –covariant derivative of A^i is given by

$$A^i_{,j} = \frac{\partial A^i}{\partial x^j} + A^p \Gamma^i_{pj}$$

Setting $j = i$, we have

$$A^i_{,i} = \frac{\partial A^i}{\partial x^i} + A^p \Gamma^i_{pi}$$

$$= \frac{\partial A^i}{\partial x^i} + A^p \frac{\partial}{\partial x^p}\left(log\sqrt{g}\right) \left[Since\ \Gamma^i_{ij} = \frac{\partial}{\partial x^j}\left(log\sqrt{g}\right)\right]$$

$$= \frac{\partial A^i}{\partial x^i} + A^p \left(\frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial x^p}\right)$$

$= \frac{\partial A^i}{\partial x^i} + \frac{A^i}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial x^i}$ [Replacing dummy suffix p by i]

$$= \frac{1}{\sqrt{g}} \left[\frac{\partial A^i}{\partial x^i}\sqrt{g} + A^i \frac{\partial \sqrt{g}}{\partial x^i}\right]$$

$$= \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}(A^i \sqrt{g})$$

Hence,

$$div\, A^i = A^i_{,i} = \frac{1}{\sqrt{g}} \frac{\partial\left(A^i \sqrt{g}\right)}{\partial x^i}$$ (summed for $i = 1,2,\ldots,N$)

Theorem:

To prove that

$$div\, A_i = div\, A^i$$

where A_i and A^i are the covariant and contravariant components of the same vector *A*.

Proof: Note that the divergence of a covariant vector A_i is defined by

$$div\, A_i = g^{pq} A_{p,q}$$

$$\Rightarrow div\, A_i = \left(g^{pq} A_p\right)_{,q}$$

$= (A^q)_{,q} = A^q_{,q} = A^i_{,i}$ [Replacing dummy suffix q by i]

Thus,

$$div\, A_i = div\, A^i$$

3.9 Covariant Constants

The fundamental tensors g^{ij}, g_{ij}, and the Kronecker delta δ^i_j are treated as constants under covariant differentiation, since their covariant derivatives vanish identically. (See the following theorem for proof.) As a result, these tensors may be taken outside the sign of covariant differentiation. For example,

$$(g^{ij}A^{p\,m}_{\ j})_{;i} = g^{lj}(A^{p\,m}_{\ l})_{;i} = g^{ij}(A^{p\,m}_{\ q})_{;i}$$

Theorem: Ricci's Theorem

The covariant derivatives of the tensors g_{ij}, g^{ij}, and δ^i_j all vanish identically.

Proof.

(i) First consider the covariant derivative of the covariant metric tensor g_{ij}. By definition of covariant differentiation

$$g_{ij,m} = \frac{\partial g_{ij}}{\partial x^m} - g_{lj}\Gamma^l_{im} - g_{il}\Gamma^l_{jm}$$

But from symmetry of Christoffel symbols

$$\Gamma_{ij,m} + \Gamma_{jm,i} = \frac{\partial g_{ij}}{\partial x^m}$$

so from these two equations, we find

$$g_{ij,m} = 0.$$

Hence, the covariant derivative of vanishes identically.

(ii) Now consider the contravariant metric tensor g^{ij}. Since $g^{ij}g_{jk} = \delta^i_k$, differentiating both sides covariantly gives

$$(g^{ij}g_{jk})_{,l} = \delta^i_{k,l} = 0$$

Applying the Leibniz rule:

$$g^{ij}_{,l}g_{jk} + g^{ij}g_{jk,l} = 0$$

But we already showed that $g_{jk,l} = 0$, so this implies

$$g^{ij}_{,l} = 0$$

(iii) Finally, consider the Kronecker delta δ^i_j, which satisfies

$$\delta^i_j = g^{ik}g_{kj}$$

Differentiating covariantly

$$(\delta^i_j)_{,l} = (g^{ik}g_{kj})_{,l} = g^{ik}_{,l}g_{kj} + g^{ik}g_{kj,l} = 0 + 0 = 0$$

Thus,

$$\delta^i_{j,l} = 0$$

This completes the proof.

Hence, all three tensors g_{ij}, g^{ij}, and δ^i_j are covariant constants.

3.10 Curl of a Covariant Vector

Let A_i be a covariant vector, then we have

$$A_{i,j} = \frac{\partial A_i}{\partial x^j} - A^p\Gamma^p_{ij} \tag{3.33}$$

Interchanging suffixes i and j and noting

$$\Gamma^p_{ij} = \Gamma^p_{ji}$$

We obtain from (3.33)

$$A_{j,i} = \frac{\partial A_j}{\partial x^i} - A^p\Gamma^p_{ij} \tag{3.34}$$

Subtracting (3.34) from (3.33), we have

$$A_{i,j} - A_{j,i} = \frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i}$$

The tensor $A_{i,j} - A_{j,i}$ of rank two is defined as the curl of the vector A_i and is denoted by $curl\ A$. Thus

$$curl\ A_i = A_{i,j} - A_{j,i} \tag{3.35}$$

CHECK YOUR PROGRESS 3:

Q.1 What is the covariant derivative of a scalar function Φ?

Q.2 Write the expression for the covariant derivative of a contravariant vector A^i. and also, of a covariant vector A_i.

Q.3 Express divergence in terms of partial derivatives and the determinant of the metric tensor.

Q.4 What is the curl of a covariant vector A_i? What is the covariant erivative of the Kronecker delta δ^i_j?

2 3.11 Let Us Sum Up

- **Christoffel symbols** are defined to facilitate the computation of derivatives in curvilinear coordinates. The Christoffel symbols of the second kind are given by

  $$\Gamma_{ij}^{k} = \frac{1}{2} g^{kl} \left(\frac{\partial g_{il}}{\partial x^{j}} + \frac{\partial g_{jl}}{\partial x^{i}} - \frac{\partial g_{ij}}{\partial x^{l}} \right)$$

  Where g_{ij} are the components of the metric tensor.
- **Christoffel symbols are not tensors**, but they follow specific transformation laws under change of coordinates

  $$\overline{\Gamma}_{ij}^{k} = \frac{\partial \overline{x}^{k}}{\partial x^{r}} \frac{\partial x^{p}}{\partial \overline{x}^{i}} \frac{\partial x^{q}}{\partial \overline{x}^{j}} \Gamma_{pq}^{r} + \frac{\partial \overline{x}^{k}}{\partial x^{s}} \frac{\partial^{2} x^{s}}{\partial \overline{x}^{i} \partial \overline{x}^{j}}$$

- The **covariant derivative of a covariant vector** A_i is defined by

  $$A_{i,j} = \frac{\partial A_i}{\partial x^{j}} - \Gamma_{ij}^{k} A_k$$

- The **covariant derivative of a covariant tensor** A_{ij} of rank 2 is given by

  $$A_{ij,k} = \frac{\partial A_{ij}}{\partial x^{k}} - \Gamma_{ik}^{p} A_{pj} - \Gamma_{jk}^{p} A_{ip}$$

- The divergence of A^i denoted by $div\, A^i$ is

  $div\, A = A_{,i}^{i}$
- A tensor is said to be **covariantly constant** if its covariant derivative vanishes. For example, if $T_{ij,k} = 0$, then T_{ij} is covariantly constant.
- The curl of the vector A_i denoted by $curl\, A_i$ is

  $curl\, A_i = A_{i,j} - A_{j,i}$

K 3.12 Answers to Check Your Progress

Check Your Progress 1:

Ans to $Q.1$:

Christoffel symbols of the first kind: They are defined as

$$[ij,k] = \frac{1}{2} \left(\frac{\partial g_{jk}}{\partial x^{i}} + \frac{\partial g_{ik}}{\partial x^{j}} - \frac{\partial g_{ij}}{\partial x^{k}} \right)$$

where g_{ij} are the components of the metric tensor.

Christoffel symbols of the second kind: They are defined by raising the last index using the inverse metric tensor g^{kh}

$\Gamma_{ij}^{k} = g^{kh}[ij,h]$.

Relation between the two:

From the above definitions, the relation is

$[ij,k] = g_{kh}\Gamma_{ij}^{h}$

Thus, the Christoffel symbols of the first kind can be obtained by lowering the upper index of the Christoffel symbols of the second kind with the metric tensor, and conversely,

$\Gamma_{ij}^{k} = g^{kh}[ij,h]$.

Ans to $Q.2$:

The Christoffel symbols of the first kind are defined as

$$[ij,k] = \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^{i}} + \frac{\partial g_{ik}}{\partial x^{j}} - \frac{\partial g_{ij}}{\partial x^{k}}\right)$$

where g_{ij} are the components of the metric tensor.

The Christoffel symbols of the second kind are given by

$$\Gamma_{ij}^{k} = g^{kl}[ij,l] = \frac{1}{2}g^{kl}\left(\frac{\partial g_{jl}}{\partial x^{i}} + \frac{\partial g_{il}}{\partial x^{j}} - \frac{\partial g_{ij}}{\partial x^{l}}\right)$$

Ans to $Q.3$:

The relation connecting them is

$\Gamma_{ij}^{k} = g^{kl}[ij,l]$,

where g^{kl} is the inverse of the metric tensor g_{kl}. This means the Christoffel symbols of the second kind are obtained by raising the last index of the first kind with the inverse metric.

Check Your Progress 2:

Ans to $Q.1$:

Let us define coordinates as:

$x^{1} = x, \quad x^{2} = y, \quad x^{3} = z, \quad x^{4} = t$

Then the metric tensor g_{ij} has the components

$g_{11} = g_{22} = g_{33} = -1, \quad g_{44} = f(x,y,z), \quad g_{ij} = 0 \ for\ i \neq j$

Thus, the metric is diagonal and corresponds to an orthogonal coordinate system. The inverse metric is given by

$$g^{11} = g^{22} = g^{33} = -1, \quad g^{44} = \frac{1}{f}, \quad g^{ij} = 0 \quad for\ i \neq j$$

Christoffel Symbols of the First Kind

We use

$$[ij,k] = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^{j}} + \frac{\partial g_{jk}}{\partial x^{i}} - \frac{\partial g_{ij}}{\partial x^{k}}\right)$$

Since $g_{ij} = 0$ for $i \neq j$, and all spatial components are constants except for $g_{44} = f(x,y,z)$, the only non-vanishing first-kind Christoffel symbols are

$$[14,4] = [41,4] = \frac{1}{2}\frac{\partial g_{44}}{\partial x^1} = \frac{1}{2}\frac{\partial f}{\partial x^1}$$

$$[24,4] = [42,4] = \frac{1}{2}\frac{\partial f}{\partial x^2}$$

$$[34,4] = [43,4] = \frac{1}{2}\frac{\partial f}{\partial x^3}$$

$$[44,1] = -\frac{1}{2}\frac{\partial f}{\partial x^1}$$

$$[44,2] = -\frac{1}{2}\frac{\partial f}{\partial x^2}$$

$$[44,3] = -\frac{1}{2}\frac{\partial f}{\partial x^3}$$

Christoffel Symbols of the Second Kind

We use

$$\Gamma_{ij}^k = g^{kl}[ij,l]$$

Because the metric is diagonal, only g^{kk} contributes

$$\Gamma_{ij}^k = g^{kk}[ij,k]$$

Thus, the non-vanishing Christoffel symbols of the second kind are

$$\Gamma_{14}^4 = \Gamma_{41}^4 = g^{44}[14,4] = \frac{1}{f}\cdot\frac{1}{2}\frac{\partial f}{\partial x^1} = \frac{1}{2f}\frac{\partial f}{\partial x^1} = \frac{\partial}{\partial x^1} ln\sqrt{f}$$

$$\Gamma_{24}^4 = \Gamma_{42}^4 = \frac{1}{2f}\frac{\partial f}{\partial x^2} = \frac{\partial}{\partial x^2} ln\sqrt{f}$$

$$\Gamma_{34}^4 = \Gamma_{43}^4 = \frac{1}{2f}\frac{\partial f}{\partial x^3} = \frac{\partial}{\partial x^3} ln\sqrt{f}$$

$$\Gamma_{44}^1 = g^{11}[44,1] = (-1).\left(\frac{1}{2}\frac{\partial f}{\partial x^1}\right) = \frac{1}{2}\frac{\partial f}{\partial x^1}$$

$$\Gamma_{44}^2 = \frac{1}{2}\frac{\partial f}{\partial x^2}$$

$$\Gamma_{44}^3 = \frac{1}{2}\frac{\partial f}{\partial x^3}$$

Ans to $Q.2$:

This is the metric in spherical coordinates (without the x^4 time component). The components of the metric tensor are

$$g_{11} = 1; \quad g_{22} = (x^1)^2; \quad g_{33} = (x^1)^2 \sin^2 x^2; \quad g_{ij} = 0 \ for \ i \neq j$$

The inverse metric components

$$g^{11}=1, \quad g^{22}=\frac{1}{(x^1)^2}, \quad g^{33}=\frac{1}{(x^1)^2\sin^2 x^2}$$

Now compute non-vanishing Christoffel symbols of the second kind

$$\Gamma^2_{21}=\Gamma^2_{12}=\frac{1}{2}g^{22}\frac{\partial g_{22}}{\partial x^1}=\frac{1}{2}\cdot\frac{1}{(x^1)^2}\cdot 2x^1=\frac{1}{x^1}$$

$$\Gamma^3_{31}=\Gamma^3_{13}=\frac{1}{2}g^{33}\frac{\partial g_{33}}{\partial x^1}=\frac{1}{2}\cdot\frac{1}{(x^1)^2\sin^2 x^2}\cdot 2x^1\sin^2 x^2=\frac{1}{x^1}$$

$$\Gamma^3_{32}=\Gamma^3_{23}=\frac{1}{2}g^{33}\frac{\partial g_{33}}{\partial x^2}=\frac{1}{2}\frac{1}{(x^1)^2\sin^2 x^2}.2(x^1)^2 sin\, x^2\cos x^2=\cot x^2$$

$$\Gamma^1_{22}=-\frac{1}{2}g^{11}\frac{\partial g_{22}}{\partial x^1}=-\frac{1}{2}\cdot 2x^1=-x^1$$

$$\Gamma^1_{33}=-\frac{1}{2}g^{11}\frac{\partial g_{33}}{\partial x^1}=-\frac{1}{2}\cdot 2x^1\sin^2 x^2=-x^1\sin^2 x^2$$

$$\Gamma^2_{33}=-\frac{1}{2}g^{22}\frac{\partial g_{33}}{\partial x^2}=-\frac{1}{2}\cdot\frac{1}{(x^1)^2}\cdot 2(x^1)^2\sin x^2\cos x^2=-\sin x^2\cos x^2=-\frac{1}{2}\sin 2x^2$$

Check Your Progress 3:

Ans to $Q.1$:

Let A_i be a covariant vector field defined on a manifold with metric tensor g_{ij}. The *covariant derivative* of A_i with respect to the coordinate x^j is defined as

$$A_{i,j}=\frac{\partial A_i}{\partial x^j}-\Gamma^k_{ij}A_k$$

where Γ^k_{ij} are the Christoffel symbols of the second kind associated with the metric g_{ij}.

The covariant derivative modifies the ordinary partial derivative by subtracting terms involving the Christoffel symbols to ensure tensorial transformation properties.

To show: $A_{i,j}$ transforms as a **covariant tensor of rank two**.

Under a coordinate transformation $x^i \to \bar{x}^i$, the covariant vector transforms as

$$\overline{A}_i=\frac{\partial x^p}{\partial \overline{x}^i}A_p$$

Differentiating partially with respect to $\overline{x^j}$ and applying the chain rule and transformation properties of the Christoffel symbols, one finds

$$\overline{A}_{i,j}=\frac{\partial x^p}{\partial \overline{x}^i}\frac{\partial x^q}{\partial \overline{x}^j}A_{p,q}$$

This shows that $A_{i,j}$ transforms as a tensor with two covariant indices, $i.e.$, a covariant

tensor of second order.

Ans to $Q.2$:

Given a covariant vector A_i and the fundamental metric tensor g_{ij}, the covariant derivative of A_i is defined by

$$A_{i;j} = \frac{\partial A_i}{\partial x^j} - \Gamma_{ij}^k A_k$$

where the Christoffel symbols Γ_{ij}^k are given by

$$\Gamma_{ij}^k = \frac{1}{2} g^{km} \left(\frac{\partial g_{mj}}{\partial x^i} + \frac{\partial g_{mi}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^m} \right)$$

Thus, the covariant derivative depends explicitly on the fundamental tensor g_{ij} and its first derivatives.

The covariant derivative $A_{i,j}$ corrects the ordinary derivative by subtracting the Christoffel terms to maintain covariance under coordinate transformations.

Check Your Progress 4:

Ans to $Q.1$:

The covariant derivative of a scalar function Φ is $\Phi_i = \frac{\partial \Phi}{\partial x^i}$

Ans to $Q.2$:

The expression for the covariant derivative of a contravariant vector A^i is

$A_{,j}^i = \frac{\partial A^i}{\partial x^j} + \Gamma_{pj}^i A^p$

For the covariant derivative of a covariant vector A_i is

$$A_{i,j} = \frac{\partial A_i}{\partial x^j} - \Gamma_{ij}^p A_p$$

Ans to $Q.3$:

$$div\, A = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(A^i \sqrt{g} \right)$$

Ans to $Q.4$:

The curl of a covariant vector A_i is $curl\, A_{ij} = A_{i,j} - A_{j,i}$.

The covariant derivative of the Kronecker delta δ_j^i is $\delta_{j,k}^i = 0$

3.13 Further Reading

1. Weatherburn, C. E. Differential Geometry of Three Dimensions, Cambridge University Press,2016.

2. Graustein, W. C. Differential Geometry. Courier Corporation, 2012.
3. Wilmore T. J. An Introduction to Differential Geometry, Dover Publications Inc., 2012.
4. Pressley, A. Elementary Differential Geometry. Springer, 2002.

M 3.14 Model Questions

Q 1. Obtain tensor laws of transformation of Christoffel symbols.

Q 2. Show that transformation of Christoffel's symbols form a group.

Q 3. The laws of transformation of Christoffel's symbols possess the transitive property.

Q 4. Show that the Christoffel's symbols are not components of a tensor.

Q 5. Evaluate the Christoffel's symbols in (i) spherical coordinates (ii) cylindrical coordinates.

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