UNIT 4: CURVES IN SPAE-I

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4.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- Understand the concept of curves in three-dimensional space and their analytical representation.

- Identify the class or function of a space curve and distinguish between smooth and non-smooth curves.
- Define and determine tangents, tangent lines, and different orders of contact between curves and surfaces.
- Define and determine tangents, tangent lines, and different orders of contact between curves and surfaces.
- Calculate the arc length of a curve in space.
- Describe fundamental planes such as osculating, normal, and rectifying planes.
- Understand and derive curvature, torsion, binormal vectors, and the Serret–Frenet formulas.
- Compute curvature and torsion for a general curve $r = f(t)$.

4.2 INTRODUCTION

In two-dimensional geometry, curves are studied using their projection on a plane. However, real-life physical phenomena often involve curves that exist in three-dimensional space—such as the path of a moving particle, the shape of a helix, or the orbit of celestial bodies. The geometry of such curves requires tools that describe their direction, bending, and twisting in space.

This unit introduces the fundamental geometric properties of space curves, focusing on tangents, normals, arc length, curvature, torsion, and associated planes.

4.3 SPACE CURVE

A curve in Euclidean space of three dimension is the locus of a point whose position vector r with respect to origin say O is a function of single parameter t.The Cartesian coordinates $(x, y, z)$of point P are called components of r and are functions of parameter t.

Therefore, we can express the equation of curve in terms of a single parameter t.

Thus, $r(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$

or simply $r(t) = \big(x(t), y(t), z(t)\big)$ represents a curve in space.

Examples of Space Curves:

1. Helix

$$r(t) = (cost, sint, t)$$

This curve spirals upward like a spring.

2. Circle in 3D

$$r(t) = (acost, asint, c)$$

3. Parabolic Curve in Space

$$r(t) = (t, t^2, t^3)$$

A space curve can be expressed as the intersection of two surfaces

$$f_1(x, y, z) = 0 \qquad (1)$$

and $f_2(xy, z) = 0$ (2)

The parametric form of the space curve is

$$x = x(t), y = y(t), z = z(t) \qquad (3)$$

The equation (3) can be transformed into equations (1) and (2) by eliminating t from (3).

For example:

The equations of a given curve $x = 1 + t^2$, $y = t^3$, $z = t^2$

Can be expressed in the form $y = (x - 1)^3$, $y^2 = z^3$.

Example 1: Find the equation of the curve whose parametric equations are $x = t, y = t^2, z = t^3$.

Solution: Let us take $xy = t.t^2 = t^3 = z$

$$\Rightarrow xy = z.$$

Also, we take $xz = t.t^3 = t^2.t^2 = y^2$.

4.4 CLASS OR FUNCTION OF A CURVE

Function of class m:

Let I denote a real interval and let be a positive integer.

Definition 5.4.1

A real valued function f is said to be of class m (or simply a $C^m -$ function), over a real interval I if it has m^{th} derivative at each point of and this derivative is continuous on.

Definition 5.4.2

A function is said to be class ∞ or C^∞-function, if it is infinitely differentiable. If f is single valued and possesses continuous derivatives of all orders at each point of I, then is said to be analytic over I, and f is called a function of class w or a C^w-function.

A real valued function

$$r = x(t)i + y(t)j + z(t)k \quad (4)$$

defined on I is said to be of class m if it has m^{th} derivative at each point I and this derivative is continuous on I ; or if each of its components x, y, z are of class m.

The equation (4) can also be written as

$$r = \big(x(t), y(t), z(t)\big) \quad (5)$$

The function r is said to be regular, if the derivative

$\frac{dr}{dt} \neq 0$ on the real interval I i.e. $\frac{dr}{dt}$ never vanishes on I.

This is equivalent to saying that each of the components $\frac{dx}{dt}, \frac{dy}{dt}$ and $\frac{dz}{dt}$ never vanishes simultaneously on I and the parameter t in terms of which r is represented is known as a regular parameter.

Definition 5.4.3

A regular vector – valued function of class m is known as a path of class m.

Definition 5.4.3

Two paths R_1 and R_2 of the same class m on the real intervals I_1 and I_2 respectively are called equivalent if there exist a strictly increasing function θ of class m which maps I_1 and I_2 and is such that $R_1 = R_2 o\theta$.

Note:

Since $x(t), y(t), z(t)$ are single -valued functions, a given value of t determines a unique point on the curve. But the converse is not generally true, that is for a given point on the curve, there may be more than one value of t.

Example 2:

Consider the circular helix whose parametric equation is given by

$$r = (acost, asint, ct), 0 \leq t \leq \pi \quad (6)$$

$$\therefore \frac{dr}{dt} = (-asint, acost, c)$$

We clearly observe that $\frac{dr}{dt} \neq 0$ for any value of t in $0 \leq t \leq \pi$.

Hence is a regular parameter. Let the change in parameter be $u = tan\frac{t}{2}$. (7)

i.e. $t = 2tan^{-1}u$

$\Rightarrow \frac{dt}{du} = \frac{1}{1+u^2}$ for $0 \leq u < \infty$

Hence u is also regular parameter defined by (7), in the interval $0 \leq u < \infty$

The equivalent representation of the given equation (6) in terms of parameter

$$r = \left(a\frac{1-u^2}{1+u^2}, b\frac{2au}{1+u^2}, 2c\tan^{-1}u\right).$$

4.5 ARC LENGTH

In the study of curves, especially space curves, one of the most important geometric quantities is the arc length. While measuring the distance between two points on a straight line is simple, determining the length of a curved path requires a deeper mathematical approach. A curve may bend smoothly, change direction continuously, or move in three dimensions, making ordinary distance formulas insufficient.

In this section, we develop the mathematical formula for arc length of curves in both the plane and space, explore examples, and learn techniques for evaluating the resulting integrals.

Now, we derive the expression for arc length of a curve between two points.

Let the equation of the curve be given by $r = r(t)$.

The positive direction along a curve is the direction in which the curve is traced as the parameter t increases, while the opposite direction is considered negative. Therefore, we measure the arc length in the positive direction.

Let P and Q be the two points on the curve between which it is required to find the arc length. Let at P and Q values of t be a and b respectively.

Now consider the following subdivision

$$\Delta = \{a = t_0 < t_1 < t_2 < \cdots < t_n = b\}$$

of the interval $[a, b]$.Corresponding to this subdivision the length L_Δ of the polygon inscribed to the arc joining the point P and Q is given by

$$L_\Delta = \sum_{i=1}^{n} P_{I-1}P_I = \sum_{i=1}^{n} |r(t_i) - r(t_{i-1})|$$

From this, we observe that as the number of subdivision points increases, the value of L_Δ also increases, since the sum of the lengths of any two sides of a triangle is always greater than the third side. Therefore, it is reasonable to define the arc length between P and Q as the supremum of L_Δ taken over all possible subdivisions of $[a, b]$.

Fig

Again $\left|\int_{i-1}^{i} \dot{r}(t)dt\right| = |r(t) - r(t_{i-1})|$

Thus, we have $L_\Delta = \sum_{i=1}^{n} \left|\int_{i-1}^{i} \dot{r}(t)dt\right|$ (1)

Again, by Schwarz inequality, we have

$$\sum_{i=1}^{n} \left|\int_{t_{i-1}}^{t_i} \dot{r}(t)\,dt\right| \leq \sum_{i=1}^{t_i} \int_{t_{i-1}}^{t_i} \dot{r}(t)\,dt = \int_a^b |\dot{r}(t)\,dt| \qquad (2)$$

Since $\int_a^b |\dot{r}(t)|\,dt$ is always finite and independent of Δ,so supremum of L_Δ is always finite.

Now, we shall show that the supremum of L_Δ is actually $\int_a^b |\dot{r}(t)|\,dt$.

Let $s = s(t)$ denote the arc length from a to any point t,then the arc length from T to t is given by

$$s(t) - s(T) \leq \int_{t_0}^{t} |\dot{r}(t)|\,dt \qquad (3)$$

Again, by the definition of arc length, we have

$$|r(t) - r(T)| \leq s(t) - s(T) \qquad (4)$$

From (3) and (4), we have

$$\left|\frac{r(t)-r(T)}{t-T}\right| \leq \frac{s(t)-s(T)}{t-T} \leq \frac{1}{t-T}\int_T^t |\dot{r}(T)|dt \qquad (5)$$

If $t \to T$,(5) reduces to

$$|\dot{r}(T)| \leq \dot{s}(T) \leq |\dot{r}(T)|$$

Hence $\dot{s}(t) = |\dot{r}(T)|$ (6)

Result (6) is true for all values of T in the range of t,we have

$$s = s(t) = \int_a^t |\dot{r}(t)|dt$$

Cartesian Equivalent:

Let $r = xi + yj + zk$

So that $\dot{r} = \dot{x}i + \dot{y}j + z\dot{k}$

Here $|\dot{r}| = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$

$$\therefore\ s = \int_a^t \sqrt{(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)dt}$$

Note 1:Since $\frac{ds}{dt} = \dot{s}$ will never vanish, we can use s as a new parameter. Changing the parameterfrom t to in the functions $s = s(t)$,let $t = \theta(s)$, , the parametric equation of curve with parameter $sr = r\{\theta(t)\}$.

Note 2:We shall use dashes to denote differentiation with respect to arc length s and dots to denote differentiation with respect to any other parameter t.

Thus, we have

$$\frac{dr}{ds} = r', \frac{d^2r}{ds^2} = r'' \text{ etc.}$$

and $$\frac{dr}{ds} = \dot{r}, \frac{d^2r}{ds^2} = \ddot{r} \text{ etc.}$$

Example 1:Find the length of circular helix.

$r(t) = acosti + asintj + ctk, -\infty < t < \infty$ from $(a,0,0)$ to $(a,0,2\pi)$.

Also obtain its equation in terms of parameter s.

Solution:Clearly the limits of t are from $ct = 0$ to $ct = 2\pi c$ i.e. from $t = 0$ to $t = 2\pi$.

The equation of circular helix is

$$r(t) = acosti + asintj + ctk$$

$$\therefore \dot{r} = \frac{dr}{dt} = -asinti + acostj + ck,$$

$$|\dot{r}(t)| = (a^2sint^2t + a^2sint^2t + c^2)^{\frac{1}{2}} = (a^2 + c^2)^{\frac{1}{2}}$$

Therefore, the length of the circular helix from $(a,0,0)$ to $(a,0,2\pi)$ is

$$= \int_0^{2\pi} |\dot{r}(t)|\, dt = \int_0^{2\pi} \sqrt{a^2 + c^2}\, dt$$

$$= (a^2 + c^2)[t]_0^{2\pi}$$

$$= 2\pi\sqrt{a^2 + c^2}$$

Again suppose S denotes the arc length from the point where $t = 0$ to any point t,we have

$$s = \int_0^t |\dot{r}(t)| dt$$

$$= \int_0^t \sqrt{a^2 + c^2}\, dt$$

$$= \sqrt{a^2 + c^2}[t]_0^t$$

$$= t\sqrt{a^2 + c^2}$$

$$\therefore t = \frac{S}{\sqrt{a^2 + c^2}}$$

Hence the given equation of the circular helix in terms of parameter S transforms to

$$r(t) = acos\left\{\frac{S}{\sqrt{a^2 + c^2}}\right\}i + acos\left\{\frac{S}{\sqrt{a^2 + c^2}}\right\}j + \frac{cS}{\sqrt{a^2 + c^2}}k$$

Example 2:Find the length of one complete turn of the circular helix

$$r(t) = acosti + asintj + ctk, -\infty < t < \infty$$

Solution: The range of parameter t corresponding to one complete turn of the helix is

$$t_0 \leq t \leq t_0 + 2\pi.$$

Therefore, the limit of t are from $t = t_0$ to $t = t_0 + 2\pi$.

The equation of circular helix is

$$r(t) = acosti + asintj + ctk$$

$$\therefore \dot{r} = \frac{dr}{dt} = -asinti + acostj + ck,$$

$$|\dot{r}(t)| = (a^2 sint^2 t + a^2 sint^2 t + c^2)^{\frac{1}{2}} = (a^2 + c^2)^{\frac{1}{2}}$$

$\therefore$ required arc length $= \int_{t_0}^{t_0+2\pi} |\dot{r}(t)| dt$

$$= \int_{t_0}^{t_0+2\pi} (a^2 + c^2)^{\frac{1}{2}} dt$$

$$= \sqrt{a^2 + c^2}[t]_{t_0}^{t_0+2\pi}$$

$$= 2\pi\sqrt{a^2 + c^2}$$

W CHECK YOUR PROGRESS

Q1) Find the arc length of the curve

$x = 3cosh2t, y = 3sinh2t, z = 6t$ from $t = 0$ to $t = \pi$.

Q2) Find the arc length as a function of θ along the epicycloid

$$x = (a + b)cos\theta - bcos\left(\frac{a + b}{b}\right)\theta$$

$$y = (a + b)sin\theta - bsin\left(\frac{a+b}{b}\right)\theta, z = 0.$$

Q3) Show that the length of the curve $x = 2a\left(sin^{-1}t + t\sqrt{(1-t)^2}\right), y = 2at^2, z = 4at$ between the point where $t = t_1$ and $t = t_2$ is $4a\sqrt{2}(t_2 - t_1)$

4.6 OSCULATING PLANE

Definition: The osculating plane at a point P of a curve of class ≥ 2 is the limiting position of the plane which contains the tangent line at P and a neighbouring point Q on the curve as $Q \longrightarrow P$.

Equation of the Osculating plane:

Let $r = r(s)$ be the given curve C of class ≥ 2 with respect to parameter s,the arc length. Let the arc length be measured from some point say A such that the arc $AP = s$,arc $AQ = s + \delta s$ so that $PQ = \delta s$.

The position vector of the$P$ can be taken as $r(s)$.The position vector of the point Q can be taken as $r(s + \delta s)$. Let R be the position vector of the current point Ton the plane containing the tangent line at P and the point Q.

The unit tangent vector at P is $\hat{t} = r'(s)$.

$\overrightarrow{PT} = R - r(s)$,$\hat{t} = r'(s)$ and $\overrightarrow{PQ} = r(s + \delta s) - r(s)$ line in the plane TPQ.

Hence their scalar triple product must be zero

i.e.,$[R - r(s)].r'(s) \times [r(s + \delta s) - r(s)] = 0$ (1)

Equation (1) is equation of the plane TPQ.Now expanding $r(s + \delta s)$ in power of (δs) by Taylor's theorem, we have

$$r(s + \delta S) = r(s) + \delta s r'(s) + \frac{(\delta s)^2}{2!} r''(s) + O(\delta s^3) \quad (2)$$

Putting the value of $r(s + \delta S)$ from (2) in(1), we get

$$[R - r(s)].r'(s) \times \left[\delta s r'(s) + \frac{(\delta s)^2}{2!} r''(s) + O(\delta s^3)\right] = 0$$

$$[R - r(s)].r'(s) \times [r''(s) + O(\delta s^3)] = 0 [\because\ r'(s) \times r'(s) = 0]$$

Hence the limiting position of the plane as$Q \longrightarrow P$ i.e., $\delta s \longrightarrow 0$ is

$[R - r(s)].r'(s) \times r''(s) = 0$ (3)

Provided the vectors $r'(s)$ and $r''(s)$ are linearly independent.

Equation (3) can be put as

$[R - r(s), r'(s), r''(s)] = 0$ (4)

This is the equation of the osculating plane at point P.

Remark:

From equation (3), we observe that both $\mathbf{r}'(s)$(tangent vector) and $\mathbf{r}''(s)$(principal normal direction) lie **in** the osculating plane. Therefore, the vector

$\mathbf{r}'(s) \times \mathbf{r}''(s)$

is normal to the osculating plane at P.

Alternative definition of the osculating plane:

The osculating plane at a point P of a curve is the limiting position of the plane

which contains the tangent at P and is parallel to the tangent at a consecutive point Q as $Q \longrightarrow P$.

We now find the equation of the osculating plane at P using this definition. The tangent vectors at $P(t)$ and a consecutive point $Q(t+\delta t)$ are $\dot{r}(t)$ and $\dot{r}(t+\delta t)$. Hence the plane through the tangent at $P(t)$ and parallel to the tangent at $Q(t+\delta t)$ is perpendicular to the vector

$$\dot{r}(t) \times \dot{r}(t+\delta t) = \dot{r}(t) \times [\dot{r}(t+\delta t) - \dot{r}(t)] \qquad [\because\ \dot{r} \times \dot{r} = 0]$$

and hence perpendicular to the vector $\dot{r}(t) \times \left[\frac{\dot{r}(t+\delta t)-\dot{r}(t)}{\delta t}\right]$

Therefore, as $\delta t \longrightarrow 0$, the osculating plane at P is perpendicular to the vector $\dot{r}(t) \times \dot{r}(t)$. If R is the position vector of a current point on the plane, the vector $R - r$ lies in this plane.

It follows that $(R-r).\dot{r} \times \ddot{r} = 0$

or $[R-r, \dot{r}, \ddot{r}] = 0$

which is the equation of the osculating plane.

The equation of the osculating plane in terms of general parameter

Let the equation of the curve be given in terms of general parameter t. Then

$$r' = \frac{dr}{ds} = \frac{\frac{dr}{dt}}{\frac{ds}{dt}} = \dot{r}/\dot{s} \quad \text{and} \quad r'' = \frac{\ddot{r}}{\dot{s}^2} - \frac{\dot{r}\ddot{s}}{\dot{s}^3}$$

Substituting in (2), we get

$$(R-r).\frac{\dot{r}}{\dot{s}} \times \left(\frac{\ddot{r}}{\dot{s}^2} - \frac{\dot{r}\ddot{s}}{\dot{s}^3}\right) = 0$$

or $(R-r)..\dot{r} \times \ddot{r} = 0 [\because\ \dot{r} \times \dot{r} = 0]$

Or $[R-r, \dot{r}, \ddot{r}] = 0$ (5)

which is therefore the equation of the osculating plane in terms of t.

Cartesian form:

Let (X, Y, Z) be the coordinate of a current point T on the osculating plane at P whose coordinates are (x, y, z). Then

Then $R = Xi + Yj + Zk$ and $r = xi + yj + zk$

So that $R - r = (X-x)i + (Y-y)j + (Z-z)k$

Also, $\dot{r} = \dot{x}i + \dot{y}j + \dot{z}k$

And $\ddot{r} = \ddot{x}i + \ddot{y}j + \ddot{z}k$

Hence the equation (5) is equivalent to

$$\begin{vmatrix} X-x & Y-y & Z-z \\ \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \end{vmatrix} = 0.$$

Osculating plane at a point of inflexion:

A point P where $r'' = 0$ is called a point of inflexion and the tangent line at P is called inflexional tangent.

We now examine the question of finding the equation of osculating plane at a point of inflexion. It will be shown that when a curve is analytic, there exists a definite osculating plane at a point of inflexion$P$ provided the curve is not a straight line.

Since r' is a vector of constant magnitude, it is perpendicular to its derivative r'' so that

$$r'.r'' = 0$$

Differentiating this, we get

$$r'.r''' + r''.r'' = 0 \qquad (6)$$

Since P is a point of inflexion $r'' = 0$.

Hence (6) reduces to

$$r'.r''' = 0.$$

This shows that r' is linearly independent of r''' except when $r''' = 0$.

Continuing this argument, we shall arrive at the result

$$r'.r^{(k)} = 0$$

where $r^{(k)}(k \geq 2)$ is the first non-zero derivative of r at P.We then have

$$r(s) - r(s_0) = \frac{h^k}{k!} r^{(k)}(s_0) + O\left(h^{k+1}\right) \text{ where } h = s - s_0$$

Hence the equation of the osculating plane at P is

$$\left[R - r(s_0), r'(s_0) r^{(k)}(s_0)\right] = 0.$$

Example 1: Find the equation of the osculating plane at any point 't' on the helix

$$x = a\cos t, y = a\sin t, z = ct.$$

Solution: We have

$$\dot{x} = -a\sin t, \dot{y} = a\cos t, \dot{z} = c$$

and $$\ddot{x} = -a\cos t, \ddot{y} = -a\sin t, \ddot{z} = 0$$

Hence the equation of the osculating plane is

$$\begin{vmatrix} x-acost & y-asint & z-ct \\ -asint & acost & c \\ -acost & -asint & 0 \end{vmatrix} = 0$$

Expanding the determinant, we get

$$(z-ct)[(-asint)(-asint)-(acost)(-acost)]-c[(x-acost)(-asint)-(y-asint)(-acost)] = 0$$

or $c(xsint - ycost - at) + az = 0$.

Example 2: Find the equation of the osculating plane at a general point on the cubic given by $r = (t, t^2, t^3)$ and show that the osculating planes at any three points of the curve meet at a point lying in the plane determined by these three points.

Solution:

We have $\dot{r} = (1, 2t, 3t^2)$ and $\ddot{r} = (0, 2, 6t)$

Hence the equation of the osculating plane is

$$\begin{vmatrix} X-t & Y-t^2 & Z-t^3 \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = 0.$$

Expanding the determinant, we get

$$(X-t)(2t.6t - 3t^2.2) - 1[(Y-t^2).6t - (z-t^3).2] = 0$$

or $$3t^2X - 3tY + Z - t^3 = 0.$$

If t_1, t_2, t_3 are three distinct values of the parameter, the osculating planes at these points are linearly independent and the three planes meet a point, say (X_0, Y_0, Z_0). The parameters t_1, t_2, t_3 must therefore satisfy the condition

$$t^3 - 3t^2X_0 + 3tY_0 - Z_0 = 0 \qquad (1)$$

Let the equation of the plane passing through these three points be

$$aX + bY + cZ + d = 0$$

Then the parameters must also satisfy the condition

$$at + bt^2 + ct^3 + d = 0$$

or $ct^3 + bt^2 + at + d = 0$ (2)

Since the equation (2) has three distinct roots, we have $c \neq 0$.

Then comparing the coefficients in (1) and (2), we get

$$\frac{c}{1} = \frac{b}{-3X_0} = \frac{a}{3Y_0} = \frac{d}{-Z_0}$$

i.e., $a = 3cY_0, b = -3cX_0, d = -cZ_0$.

Hence the equation of the plane is

$$3cY_0X - 3cX_0Y - cZ - cZ_0 = 0$$

or $$3Y_0X - 3X_0Y + Z - Z_0 = 0$$

The plane clearly passes through the point (X_0, Y_0, Z_0)

4.7 NORMAL LINES AND NORMAL PLANE

In the study of space curves, understanding how a curve behaves at a particular point is essential for visualizing its geometry. Two important geometric concepts that help describe this local behaviour are the normal line and the normal plane**.**

Definition 1:

The normal line to a curve at a point $P$is defined as the line passing through P that is perpendicular to the tangent at that point. In three-dimensional space, since infinitely many directions are perpendicular to a given tangent line, a space curve naturally admits infinitely many normal lines at the point P.

Definition 2:

The normal plane at a point Pon a curve is the plane that passes through P and is perpendicular to the tangent line at that point.

In other words, the normal plane at P consists of all the normal lines to the curve at P; it is the complete set (locus) of those normal directions.

Equation of the normal plane:

Let r be the position vector of a point P on a space curve. If R is the position vector of a current point on the normal plane at P, then the vector $R - r$ lies in this plane. Since the vector $\dot{r}$ is perpendicular to this plane, we have

$$(R - r).\dot{r} = 0 \qquad (1)$$

which is the required equation of the normal plane at P.

Again, the equation (1) can be put in the form

$$(R - r).t = 0 \qquad (2)$$

Equation of the normal plane in Cartesian form:

Let $R = Xi + Yj + Zk; r = xi + yj + zk$

$\therefore \dot{r} = \dot{x}i + \dot{y}j + \dot{z}k$

Putting these values in (1), we get

$$[(X-x)i+(Y-y)j+(Z-z)k].[\dot{x}i+\dot{y}j+\dot{z}k]=0$$

or $$(X-x)\dot{x}+(Y-y)\dot{y}+(Z-z)\dot{z}=0$$

Of all the normals, two normals are of special significance. These are known as Principal normal and Binormal which are defined below:

Principal Normal:

Definition: The normal lying in the osculating plane at a point P on the space curve is called the principal normal at a point P.

Thus, the principal normal is the intersection of the normal plane and the osculating plane. Also, the plane containing the tangent and the principal normal is the osculating plane.

We shall denote the unit vector along the principal normal by n.

Binormal:

Definition: The normal perpendicular to the principal normal at point P is called binormal at point P.

Thus, the binormal is the line perpendicular to the osculating plane. We shall denote the unit vector along the binormal by b.The sense of b is chosen in such a manner that the triad t, n, b form a right-handed system of axes,i.e., $b = t \times n$.

Fig.

Directions of principal normal and Binormal:

Since the binormal vector is defined to be perpendicular to the osculating plane, it must point in the same direction as the vector $\dot{r} \times \ddot{r}$. Thus, the binormal is parallel to

$$\dot{r} \times \ddot{r}.$$

Next, the principal normal is perpendicular to both the tangent and the binormal. Therefore, it must be parallel to the vector

$$(\dot{r} \times \ddot{r}) \times \dot{r}.$$

Using the vector triple product identity, this vector equals

$$(\dot{r}^2)\,\ddot{r} - (\dot{r}\cdot\ddot{r})\,\dot{r}.$$

If the parameter is the arc length s, then the unit tangent vector is

$$t = r'(s),$$

and so

$$r'^2 = 1.$$

Differentiating this gives

$$r'\cdot r'' = 0,$$

which shows that r'' is perpendicular to the tangent vector r'.

From the equation of the osculating plane, we know that r'' lies **in** this plane. Since the binormal is perpendicular to the osculating plane, r'' is also perpendicular to the binormal. Hence, r'' must be parallel to the principal normal.

Finally, because the binormal is perpendicular to both the tangent and the principal normal, its direction must be parallel to

$$r'\times r''.$$

4.8 RECTIFYING PLANE

Definition: The plane containing the tangent and binormal at P is called rectifying plane at P.i.e., it is the plane passing through P and perpendicular to principal normal at P.

The equation of the rectifying plane at $P(r)$ is

$$(R-r).n = 0$$

At each point of a space curve, we associate a set of three mutually perpendicular unit vectors $\mathrm{t}, \mathrm{n}, \mathrm{b}$. Together, these vectors form an orthonormal triad, often called a moving trihedral, at a point P. They satisfy the orthogonality relations

$$\mathrm{t}\cdot\mathrm{n} = 0, \mathrm{n}\cdot\mathrm{b} = 0, \mathrm{b}\cdot\mathrm{t} = 0,$$

and the cyclic cross product relations

$$\mathrm{n}\times\mathrm{b} = \mathrm{t}, \mathrm{b}\times\mathrm{t} = \mathrm{n}, \mathrm{t}\times\mathrm{n} = \mathrm{b}.$$

The vectors $\mathrm{t}, \mathrm{n}, \mathrm{b}$ together constitute the fundamental unit vectors associated with the curve.

4.9 FUNDAMENTAL PLANES

The three planes associated with each point of a space curve namely, the osculating plane, the normal plane, and the rectifying plane are called the **fundamental planes**. These planes are mutually perpendicular and are defined by the moving triad of unit vectors $\mathbf{t}, \mathbf{n}, \mathbf{b}$ at a given point on the curve.

Their equations are as follows:

Osculating Plane:

This plane contains the tangent vector $\mathbf{t}$ and the normal vector $\mathbf{n}$, and is perpendicular to the binormal vector $\mathbf{b}$.

Its equation is: $(\mathbf{R} - \mathbf{r}) \cdot \mathbf{b} = 0$.

Normal Plane:

This plane contains the normal vector $\mathbf{n}$ and the binormal vector $\mathbf{b}$, and is perpendicular to the tangent vector $\mathbf{t}$.

Its equation is: $(\mathbf{R} - \mathbf{r}) \cdot \mathbf{t} = 0$.

Rectifying Plane:

This plane contains the binormal vector $\mathbf{b}$ and the tangent vector $\mathbf{t}$, and is perpendicular to the normal vector $\mathbf{n}$.

Its equation is: $(\mathbf{R} - \mathbf{r}) \cdot \mathbf{n} = 0$.

Fig

4.10 EQUATION OF THE PRINCIPAL NORMAL AND BINORMAL

Let $\mathbf{r}$ be the position vector of a point P on a given curve C, at which we want to determine the equations of the principal normal and the binormal.

Let $\mathbf{R}$ be the position vector of a variable point Q on the principal normal. Then,

$$\overrightarrow{OP} = \mathbf{r}, \overrightarrow{OQ} = \mathbf{R}, \text{and} \overrightarrow{PQ} = \lambda \mathbf{n},$$

where $\mathbf{n}$ is the unit vector along the principal normal and λ is a scalar.

By the triangle law of vectors,

$$\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ},$$

Which gives

$$\mathbf{R} = \mathbf{r} + \lambda\mathbf{n}.$$

This is the required equation of the principal normal.

Fig :

Similarly, if $\mathbf{R}$ is the position vector of a variable point on the binormal, then the equation of the binormal is

$$\mathbf{R} = \mathbf{r} + \mu\mathbf{b},$$

where $\mathbf{b}$ is the unit binormal vector and μ is a scalar.

4.11CURVATURE

Definition: The curvature at a point P of a given curve is the arc rate of rotation of tangent (i.e., change in the direction of tangent) at P. Its magnitude is denoted by κ (Kappa).

We now derive a formula for the curvature of a curve at a point P.

Take a nearby point Q on the curve such that the arc length $PQ = \delta s$. Let the tangent at Q make a small angle $\delta\theta$ with the tangent at P.

The unit tangent vector changes from point to point. Let its value at P be t, and at Q be $t + \delta t$.

If

$$\overrightarrow{QM} = t, \overrightarrow{QN} = t + \delta t,$$

then

$$\overrightarrow{MN} = \delta t, \angle MQN = \delta\theta, |\overrightarrow{QM}| = |\overrightarrow{QN}| = 1.$$

Fig.

From isosceles triangle QMN, we have

$$MN = 2QM\sin\left(\frac{1}{2}\delta\theta\right) = 2\sin\left(\frac{1}{2}\delta\theta\right)$$

$$\Rightarrow |\delta t| = 2\sin\left(\frac{1}{2}\delta\theta\right)$$

$$\Rightarrow \left|\frac{\delta t}{\delta \theta}\right| = \frac{2sin\left(\frac{1}{2}\delta\theta\right)}{\frac{1}{2}\delta\theta}$$

$$\Rightarrow \left|\frac{dt}{d\theta}\right| = 1 \text{ as } \delta\theta \to 0 \quad (1)$$

$\therefore$ Curvature at $P = \kappa = \lim_{\delta s \to 0}\frac{\delta\theta}{\delta s} = \frac{d\theta}{ds}$, along the direction of the tangent.

$$= \frac{d\theta}{|dt|}\frac{|dt|}{ds}$$

$$= \left|\frac{d\theta}{dt}\right|\left|\frac{dt}{ds}\right|$$

$$= \left|\frac{dt}{ds}\right| \qquad \text{[Using (1)]}$$

$$= \left|\frac{dr'}{ds}\right|$$

$$= |r''|$$

which implies that the curvature is the scalar measure of the arc rate of turning of the unit vector t.The reciprocal of κ,i.e., $\frac{1}{\kappa}$ is called radius of curvature and is denoted by ρ.

The reciprocal

$$\rho = \frac{1}{\kappa}$$

is called the radius of curvature.

Theorem: A necessary and sufficient condition for the curve to be straight line is that the curvature $\kappa = 0$ at all points of the curve.

Proof: Necessary: Let the curve be a straight line. Its vector equation can be written as

$r = sa + b$ where a and b are constant vectors.

Hence,$t = r' = a$ and $t' = r'' = 0$

$$\therefore \kappa = |r''| = 0$$

Thus, if the curve is a straight line, then its curvature is zero. Therefore,

$$\kappa = 0$$

is a **necessary** condition for straightness.

Sufficient:

Now assume that the curvature is zero at every point of the curve. Then

$$r'' = 0. \qquad (1)$$

Integrating (1), we get $r' = a$ (2)

Integrating (2), we get $r = as + b$ (3)

where a and b are arbitrary constant vectors. The equation (3) represents a straight line for all values a and b.

Hence the curve is a straight line.

4.12 TORSION

Definition: Torsion at point P of a given curve is defined as the arc rate of the change in the direction of binormal at P.Its magnitude is denoted by τ (Tau).

We now derive the expression for the **torsion** τ of a curve at a point P.

Let Q be a point very close to P on the curve, with arc length

$$PQ = \delta s.$$

Let b and $\mathrm{b} + \delta\mathrm{b}$ denote the **unit binormal vectors** at P and Q respectively. Let $\delta\theta$ be the small angle between these two binormals.

If

$$\overrightarrow{QR} = \mathbf{b}, \overrightarrow{QS} = \mathbf{b} + \delta\mathbf{b},$$

then

$$\overrightarrow{RS} = \delta\mathbf{b}$$

Fig

Now from the isosceles triangle QRS,we have

$$\overrightarrow{RS} = 2\overrightarrow{QR}\, sin\frac{1}{2}\delta\theta$$

$$\Longrightarrow |\overrightarrow{RS}| = 2|\overrightarrow{QR}| sin\frac{1}{2}\delta\theta$$

$$\Longrightarrow |\delta b| = 2.1.\, sin\frac{1}{2}\delta\theta$$

$$\Rightarrow \left|\frac{\delta b}{\delta \theta}\right| = 2\frac{\sin\frac{1}{2}\delta\theta}{\delta\theta}$$

$$\Rightarrow \frac{db}{d\theta} = \lim_{\delta\theta\to 0}\frac{\sin\frac{1}{2}\delta\theta}{\frac{1}{2}\delta\theta} = 1$$

Thus, by definition, torsion at P

$$\tau = \lim_{\delta s\to 0}\frac{\delta\theta}{\delta s} = \frac{d\theta}{ds}$$

$$= \left|\frac{d\theta}{db}\right|\left|\frac{db}{ds}\right|$$

$$= \left|\frac{db}{ds}\right|$$

$$= |b'|$$

which shows that torsion is the **scalar rate of change of the unit binormal vector** with respect to arc length.

The reciprocal of torsion is called the radius of torsion, denoted by σ:

$$\sigma = \frac{1}{\tau}.$$

Deduction: We have $t.b = 0$

Now, differentiating

$$t.b' + t'.b = 0$$

$$\Rightarrow t.b' + \kappa n.b = 0 \qquad [\because\ t' = \kappa n]$$

$$\Rightarrow t.b' = 0 \qquad [\because\ n.b = 0]$$

i.e., b' is perpendicular to t.

Further, $b.b = 1 \Rightarrow 2b.b' = 0$

i.e., b' is perpendicular to b.

$\therefore b'$ is normal to the plane containing t and b i.e., to the rectifying plane.

Thus b' is collinear with n.

$$\therefore b' = \pm\tau n$$

Since b has the opposite direction to n,so negative sign is taken i.e.,

$$b' = -\tau n \text{ or } \frac{db}{ds} = -\tau n.$$

Theorem: A necessary and sufficient condition that a given curve is plane is that $\tau = 0$ at all points.

Proof: **Necessity: If the curve is a plane curve, then $\tau = 0$**

Let the curve be a plane curve, then the tangent and normal at all points of the curve lie in the plane of the curve, i.e., the plane of the curve is the osculating plane at all points of the curve. This implies that the unit vector b along the binormal is constant.

$$\frac{db}{ds} = 0$$

or $\tau = 0.$

Hence the condition is necessary.

Sufficiency: If $\tau = 0$everywhere, then the curve is a plane curve

Let $\tau = 0$ at all points of the curve.

This implies that $\frac{db}{ds} = 0$ i.e., b is a constant vector.

Again $\frac{d}{ds}(r.b) = \frac{dr}{ds}.b + r.\frac{db}{ds} = t.b + r.b'$

As t and b are orthogonal, we have $t.b = 0$.Also $b' = 0.$

Therefore $\frac{d}{ds}(r.b) = 0$

or $r.b =$ constant.

Again b is constant vector of magnitude unity,$r.b$ is the projection of the position vector r on b and is same at all points of the curve by the condition $r.b =$ constant.

This implies that the curve must lie in a plane.

4.13 SCREW CURVATURE

Definition: The arc rate of the rotation of the principal normal $\left(\text{i. e.}, \frac{dn}{ds}\right)$ is called the screw curvature vector and its magnitude is given by $\sqrt{(\kappa^2 + \tau^2)}$.

4.14S ERRET - FRENET FORMULAE

The following set of three relations involving space derivatives of fundamental unit

vectors t, n, b are known as Serret -Frenet Formulae.

1. $\frac{dt}{ds} = \kappa n$
2. $\frac{dn}{ds} = \tau b - \kappa t$
3. $\frac{db}{ds} = -\tau n$

Proof: (1) We know that $t = r'$ is the unit tangent vector to the curve at a point P.

Since t is of constant magnitude 1, it is perpendicular to the derivative t'

Also, the vector $t' = r''$ lies in the osculating plane.

It follows that t' lies along the principal normal at P and so is proportional to n.

Since $|t'| = k$,we may write

$$\frac{dt}{ds} = t' = \pm kn$$

We make the convention

$$\frac{dt}{ds} = kn \qquad (1)$$

(2) We now find $\frac{db}{ds}$,

Since b is of constant magnitude unity, it is perpendicular to its derivative $\frac{db}{ds}$.

Hence $\frac{db}{ds}$ lies in the osculating plane.

Also differentiating the relation $b.t = 0$ with respect to s,we get

$\frac{db}{ds}.t + b.\frac{dt}{ds} = 0$.

Now, $b.\frac{dt}{ds} = b.(kn) = 0$ so that $\frac{db}{ds}.t = 0$.

Thus $\frac{db}{ds}$ is perpendicular to t. But since $\frac{db}{ds}$ lies in the osculating plane, it must be parallel to n.

Since $\left|\frac{db}{ds}\right| = \tau$,we may write

$\frac{db}{ds} = \pm \tau n$.

By convention, we write

$$\frac{db}{ds} = b' = -\tau n \qquad (2)$$

(2) To find $\frac{dn}{ds}$, we differentiate the identity

$$n = b \times t$$

With respect to s and we get

$$\frac{dn}{ds} = \frac{db}{ds} \times t + b \times \frac{dt}{ds}$$

$= (-\tau n) \times t + b \times (\kappa n)$ [by (1) and (2)]

$= \tau b - kt$ [$\because n \times t = -b$ and $b \times n = -t$]

Thus $\frac{dn}{ds} = n' = \tau b - kt.$ (3)

Remark:

Serret – Frenet formulae can be represented in the form of matrix equation as below:

$$\begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & -\tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix}$$

4.15 CURVATURE AND TORISON OF ANY CURVE $\boldsymbol{r = f(t)}$

We have

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{ds}\frac{ds}{dt} = r'\dot{s} = t\dot{s} \quad (1)$$

$$\therefore |\dot{r}| = |t\dot{s}| = \dot{s} \quad (2)$$

Now, differentiating (1), we get

$$\ddot{r} = \frac{d^2r}{dt^2} = t'\dot{s}^2 + t\ddot{s}$$

or $$\ddot{r} = (\kappa n)\dot{s}^2 + t\ddot{s} \quad [\because t' = \kappa n] \quad (3)$$

Now taking the cross product of (1) and (3), we get

$$\dot{r} \times \ddot{r} = \dot{s}^3 k b$$

$$\Rightarrow |\dot{r} \times \ddot{r}| = \kappa \dot{s}^3 \quad (4)$$

Differentiating (4), we get

$$\dot{r} \times \dddot{r} + \ddot{r} \times \ddot{r} = \dot{s}^3 k b'\dot{s} + b\frac{d}{dt}(\dot{s}^3\kappa) \quad [\because b' = -\tau n] \quad (5)$$

Again, taking the scalar product of (3) and (5), we get

$$[\dot{r}, \ddot{r}, \dddot{r}] = -\dot{s}^6 k^2 \tau \quad (6)$$

Also, from (2) and (4), we have

$$\dot{s}^3\kappa|b| = |\dot{r}\times\ddot{r}|$$

Or $$|\dot{r}|^3\kappa|b| = |\dot{r}\times\ddot{r}|$$

Or $$\kappa = \frac{|\dot{r}\times\ddot{r}|}{|\dot{r}|^3} \quad [\because |b| = 1]$$

From (4) and (6), we have $\tau = \frac{[\dot{r},\ddot{r},\dddot{r}]}{|\dot{r}\times\ddot{r}|^2}$

An Important Result:

$$\kappa = |r'\times r''| \text{ and } \tau = \frac{[r',r'',r''']}{|\dot{r}\times\ddot{r}|^2}$$

Proof: We know that $r' = t$ and $r'' = \kappa n$

$$\therefore\ r'\times r'' = t\times\kappa n = \kappa b$$

or $$|r'\times r''| = \kappa|b| = \kappa$$

Again $$r' = t = 1.t + 0n + 0b \qquad (7)$$

$$r'' = \kappa n = 0t + \kappa n + 0b \qquad (8)$$

$$k''' = \kappa\frac{dn}{ds} + \frac{d\kappa}{ds}n = \kappa(\tau b - \kappa t) + \kappa' n$$

$$= -\kappa^2 t + \kappa' n + \kappa\tau b \qquad (9)$$

From (7), (8) and (9),we have

$$[r',r'',r'''] = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \kappa & 0 \\ -\kappa^2 & k' & k\tau \end{vmatrix}$$

$$= k^2\tau \qquad (10)$$

or $$\tau = \frac{[r',r'',r''']}{k^2} = \frac{[r',r'',r''']}{|r'\times r''|^2} \quad [\because\ \kappa = |r'\times r''|]$$

Theorem: The necessary and sufficient condition for the curve to be plane curve is

$$[r',r'',r'''] = 0.$$

Proof: From equation (10), we know that

$$[r', r'', r'''] = \kappa^2\tau.$$

We assume

$$[r', r'', r'''] = 0.$$

Then

$$\kappa^2 \tau = 0,$$

which implies that either $\kappa = 0$ or $\tau = 0$.

Suppose $\tau \neq 0$ at some point of the curve. By continuity, $\tau \neq 0$ in a neighbourhood of that point. Hence, in this neighbourhood we must have $\kappa = 0$.

But $\kappa = 0$ implies that the arc is a straight line. For a straight line, torsion must be zero, i.e., $\tau = 0$. This contradicts our assumption that $\tau \neq 0$.

Therefore, the only possibility is

$$\tau = 0 \text{ at all points of the curve.}$$

Hence the curve lies entirely in a plane.

Conversely, if the curve is a plane curve, then $\tau = 0$ everywhere. Substituting $\tau = 0$ into equation (10), we obtain

$$[r', r'', r'''] = 0.$$

Thus, the condition

$$[r', r'', r'''] = 0$$

is both necessary and sufficient.

4.17 DIRECTION COSINES OF THE PRINCIPAL NORMAL AND BINORMAL

From the Frenet formula,

$$t' = \kappa n,$$

so, the principal normal vector is

$$n = \frac{t'}{\kappa} = \frac{r''}{\kappa} = \frac{1}{\kappa}(x'' i + y'' j + z'' k), (\because t' = r''). \qquad (1)$$

Since n is a **unit vector**, equation (1) shows that its direction cosines are

$$\frac{x''}{\kappa}, \frac{y''}{\kappa}, \frac{z''}{\kappa}$$

Squaring equation (1), we obtain

$$1 = \frac{1}{\kappa^2}(x''^2 + y''^2 + z''^2),$$

which gives

$$\kappa = \sqrt{x''^2 + y''^2 + z''^2}. \qquad (2)$$

Again $b = t \times n = r' \times \frac{r''}{\kappa}$ [From (1), $n = \frac{r''}{\kappa}$ and $t = r'$] (3)

Also $r' \times r'' = \begin{vmatrix} i & j & k \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix} = (y'z'' - z'y'')i + (z'x'' - x'z'')j + (x'y'' - y'x'')k$

Therefore, on putting this value of $r' \times r''$ in (3), we get

$$b = \frac{1}{k}[(y'z'' - z'y'')i + (z'x'' - x'z'')j + (x'y'' - y'x'')k] \qquad (4)$$

Since κ is a unit vector, thedirection cosines of the binormal from (4) are

$$\frac{1}{k}(y'z'' - z'y''), \frac{1}{k}(z'x'' - x'z''), \frac{1}{k}(x'y'' - y'x'').$$

Examples 1: For the curve $x = a(3t - t^3), y = 3at^2, z = a(3t + t^3)$, show that

$$\kappa = \tau = \frac{1}{3a(1+t^2)^2}$$

Solution: The position vector of the curve is

$$r = (3at - at^3, 3at^2, 3at + at^3)$$

$$\therefore \dot{r} = (3a - 3at^2, 6at, 3a + 3at^2)$$

$$\ddot{r} = (-6at, 6a, 6at)$$

$$\dddot{r} = (-6a, 0, 6a)$$

$$\therefore |\dot{r}| = 3a\sqrt{\left[\{(1-t^2)^2 + 4t^2 + (1+t^2)^2\}\right]}$$

$$= 3\sqrt{2}a(1+t^2)$$

Again $\dot{r} \times \ddot{r} = (18a^2t^2 - 18a^2, -36a^2t, 18a^2 + 18a^2t^2)$

$$\therefore |\dot{r} \times \ddot{r}| = 18a^2\sqrt{[(t^2-1)^2 + 4t^2 + (1+t^2)^2]}$$

$$= 18\sqrt{2}a^2(1+t^2)$$

Therefore,

$$\kappa = \frac{|\dot{r} \times \ddot{r}|}{|\dot{r}|^3} = \frac{18\sqrt{2}a^2(1+t^2)}{54\sqrt{2}a^3(1+t^2)^3} = \frac{1}{3a(1+t^2)^2}$$

Again, $[\dot{r}, \ddot{r}, \dddot{r}] = \begin{vmatrix} 3a - 3at^2 & 6at & 3a + 3at^2 \\ -6at & 6a & 6at \\ 6a & 0 & 6a \end{vmatrix}$

$$= \begin{vmatrix} 6a & 6at & 3a+3at^2 \\ 0 & 6a & 6at \\ 0 & 0 & 6a \end{vmatrix} \text{ by } C_1 + C_3$$

$$= 6a(36a^2)$$

$$= 216a^3$$

Therefore

$$\tau = \frac{[\dot{r}, \ddot{r}, \dddot{r}]}{|\dot{r} \times \ddot{r}|^2} = \frac{216a^3}{\{18\sqrt{2}a^2(1+t^2)\}^2} = \frac{1}{3a(1+t^2)^2}$$

Hence

$$\kappa = \tau = \frac{1}{3a(1+t^2)^2}.$$

Example 2: Find the curvature and torsion for the curve $x = acost, y = asint, z = atcot\alpha$.

Solution: Position vector r in terms of parameter t is given by

$$r = (acost, asint, atcot\alpha)$$

$$\therefore \dot{r} = (-asint, acost, acot\alpha);$$

$$\ddot{r} = (-acost, -asint, 0)$$

$$\dddot{r} = (asint, -acost, 0)$$

Therefore, $\dot{r} \times \ddot{r} = (a^2 sintcot\alpha, -a^2 costcot\alpha, a^2)$

$$\therefore |\dot{r} \times \ddot{r}| = a^2 cosec\alpha$$

$$|\dot{r}| = (a^2 sin^2 t + a^2 cos^2 t + a^2 cot^2 \alpha)^{\frac{1}{2}} = acosec\alpha$$

Again $$[\dot{r}, \ddot{r}, \dddot{r}] = \begin{vmatrix} -asint & acost & acot\alpha \\ -acost & -asint & 0 \\ asint & -acost & 0 \end{vmatrix}$$

$$= a^3 cot\alpha \quad \text{[Expanding in terms of the third column]}$$

Therefore,

$$\kappa = \frac{|\dot{r} \times \ddot{r}|}{|\dot{r}|^3} = \frac{a^2 cosec\alpha}{a^3 cosec^3 \alpha} = \frac{1}{a} sin^2 \alpha$$

and $$\tau = \frac{[\dot{r}, \ddot{r}, \dddot{r}]}{|\dot{r} \times \ddot{r}|^2} = \frac{a^2 cot\alpha}{a^4 cosec^2 \alpha} = \frac{1}{a} sin\alpha cos\alpha.$$

Example 3: For the curve $x = 3t, y = 3t^2, z = 2t^3$, show that

$$\rho = -\sigma = \frac{3}{2}(1+2t^2)^2$$

Solution: We have $r = (3t, 3t^2, 3t^3)$

$$\therefore \dot{r} = (3,6t,6t^2) = 3(1,2t,2t^2)$$

$$\ddot{r} = (0,6,12t) = 6(0,1,2t)$$

$$\dddot{r} = (0,0,12) = 12(0,0,1)$$

Now $\dot{r} \times \ddot{r} = 18(2t^2, -2t, 1)$

$$\therefore |\dot{r} \times \ddot{r}| = 18\sqrt{4t^4 + 4t^2 + 1} = 18(2t^2 + 1)$$

and $|\dot{r}| = 3\sqrt{1 + 4t^2 + 4t^4} = 3(1 + 2t^2)$

Also

$$[\dot{r}, \ddot{r}, \dddot{r}] = 3 \times 6 \times 12 \begin{vmatrix} 1 & 2t & 2t^2 \\ 0 & 1 & 2t \\ 0 & 0 & 1 \end{vmatrix} = 216$$

Hence $\rho = \frac{1}{\kappa} = \frac{|\dot{r}|^3}{|\dot{r} \times \ddot{r}|} = \frac{27(1+2t^2)^3}{18(2t^2+1)} = \frac{3}{2}(1 + 2t^2)^2$

and $\sigma = \frac{1}{\tau} = \frac{|\dot{r} \times \ddot{r}|^2}{[\dot{r}, \ddot{r}, \dddot{r}]} = \frac{32(1+2t^2)^2}{216} = \frac{3}{2}(1 + 2t^2)^2$

Hence $\rho = \sigma = \frac{3}{2}(1 + 2t^2)^2$

For a left handed trihedral, the sign of σreverses.

Hence,

$$\rho = -\sigma = \frac{3}{2}(1 + 2t^2)^2$$

Example 4: Show that the Serret – Frenet formulae can be written in the form $\frac{dt}{ds} = \omega \times t, \frac{dn}{dt} = \omega \times n, \frac{db}{ds} = \omega \times b$ and determine ω.

Solution: We know that

$$\frac{dt}{ds} = \kappa n = \tau t \times t + \kappa b \times t \qquad [\because t \times t = 0 \text{ and } b \times t = n]$$

$$= (\tau t + b\kappa) \times n \qquad (1)$$

$$\frac{dn}{ds} = \tau b - \kappa t = \tau(t \times n) + \kappa(b \times n)$$

$$= (\tau t + b\kappa) \times n \qquad (2)$$

$$\frac{db}{ds} = -\tau n = \tau(t \times b) + \kappa b \times b \qquad [\because -n = t \times b \text{ and } b \times b = 0]$$

$$= (\tau t + \kappa b) \times b \qquad (3)$$

From equations (1), (2), (3), we have

$$\frac{dt}{ds} = \omega \times t, \frac{dn}{dt} = \omega \times n, \frac{db}{ds} = \omega \times b \quad \text{where } \omega = \tau t + \kappa b.$$

Example 5: Show that $r''' = \kappa' n - \kappa^2 t + \kappa\tau b$ and hence show that

$$r'''' = (\kappa'' - \kappa^3 - \kappa\tau^2)n - 3\kappa'\kappa t + (2\kappa'\tau + \tau'\kappa)b.$$

Solution: We have $r'' = \kappa n$

$$\therefore r''' = \kappa n' + \kappa' n$$

$$= \kappa(\tau b - \kappa t) + \kappa' n$$

$$= \kappa' n - \kappa^2 t + \kappa\tau b.$$

Hence, $r'''' = \kappa'' n + \kappa' n' - 2\kappa\kappa' t - \kappa^2 t' + (\kappa\tau' + \kappa'\tau)b + \kappa\tau b'$

$$= \kappa'' n + \kappa'(\tau b - \kappa t) - 2\kappa\kappa' t - \kappa^3 n + (\kappa\tau' + \kappa'\tau)b - \kappa\tau^2 n$$

$$= (\kappa'' - \kappa^3 - \kappa\tau^2)n - 2\kappa\kappa' t + (2\kappa'\tau + \tau'\kappa)b$$

Example 6: Prove that $[r', r'', r''] = \kappa^2\tau$.

Solution: We have $r' = t$ so that $r'' = t' = \kappa n$

and $r''' = \kappa' n - \kappa^2 t + \kappa\tau b$ [by Example 5]

$$\therefore [r', r'', r''] = r'.[r'' \times r''']$$

$$= t.[\kappa n \times (-\kappa^2 t + \kappa' n + \kappa\tau b)]$$

$$= t.[\kappa^2 b + \kappa^2\tau t]$$

$$= \kappa^2\tau \quad [\because t.b = 0 \text{ and } t.t = 1]$$

CHECK YOUR PROGRESS

Q4) For the curve $r = (t, t^2, t^3)$, prove that

$$\kappa^2 = \frac{4(9t^4+9t^2+1)}{(9t^4+4t^2+1)^3} \quad \text{and} \quad \tau = \frac{3}{(9t^4+4t^2+1)}$$

Q5) Find the curvature and torsion of the curve given by

(i) $r = (at - asint, a - acost, bt)$

(ii) $r = \left(a + acost, asint, 2asin\frac{1}{2}t\right)$

Q6) Show that the curvature and torsion of the helix $x = acos\theta, y = asin\theta, z = a\theta tan\alpha$ are given by

$$\kappa = \frac{cos^2\alpha}{a}, \quad \tau = \frac{cos^3\alpha}{asin\alpha}$$

Q7) Find the curvature and torsion of the curve given by

(a) $r = (t, 1 + t^{-1}, t^{-1} - t)$

(b) $r = \left(2asin^{-1}t + t\sqrt{1+t^2}, 2at^2, 4at\right)$

(c) $r = \left(e^t, -e^t, \sqrt{2}t\right)$

Type equation here.

Q8). Prove that for any curve

(i) $[t', t'', t'''] = [r'', r''', r''''] = \kappa^3(\kappa\tau' - \kappa'\tau) = \kappa^5 \frac{d}{ds}\left(\frac{\tau}{\kappa}\right)$

(ii) $[b', b'', b'''] = \tau^3(\kappa\tau' - \kappa'\tau) = \tau^5 \frac{d}{ds}\left(\frac{\kappa}{\tau}\right)$.

2 4.18 LET US SUM UP

- A curve in Euclidean space of three dimension is the locus of a point whose position vector r with respect to origin say O is a function of single parameter t.
- A real valued function f is said to be of class m (or simply a C^m – function), over a real interval I if it has m^{th} derivative at each point of I and this derivative is continuous on I .
- A function is said to be class ∞ or C^∞-function, if it is infinitely differentiable.If f is single valued and possesses continuous derivatives of all orders at each point of I, then is said to be analytic over I, and $f$is called a function of class w or a C^w-function.
- The osculating plane at a point P of a curve of class ≥ 2 is the limiting position of the plane which contains the tangent line at P and a neighbouring point Q on the curve as $Q \longrightarrow P$.
- A point P where $r'' = 0$ is called a point of inflexion and the tangent line at P is called inflexional tangent.
- The **normal line** to a curve at a point $P$is defined as the line passing through P that is **perpendicular to the tangent** at that point. In three-dimensional space, since infinitely many directions are perpendicular to a given tangent line, a space curve naturally admits **infinitely many normal lines** at the point P .
- The **normal plane** at a point Pon a curve is the plane that passes through P and is **perpendicular** to the tangent line at that point.

- The normal lying in the osculating plane at a point P on the space curve is called the principal normal at a point P.
- The normal perpendicular to the principal normal at point P is called binormal at point P.
- The plane containing the tangent and binormal at P is called rectifying plane at P.i.e., it is the plane passing through P and perpendicular to principal normal at P. The equation of the rectifying plane at $P(r)$ is $(R-r).n=0$
- The three planes associated with each point of a space curve namely, the osculating plane, the normal plane, and the rectifying plane are called the fundamental planes.
- The curvature at a point P of a given curve is the arc rate of rotation of tangent (i.e., change in the direction of tangent) at P. Its magnitude is denoted by κ (Kappa).
- Torsion at point *P* of a given curve is defined as the arc rate of the change in the direction of binormal at *P*.Its magnitude is denoted by τ (Tau).
- The arc rate of the rotation of the principal normal $\left(\text{i. e.,}\frac{dn}{ds}\right)$ is called the screw curvature vector and its magnitude is given by $\sqrt{(\kappa^2+\tau^2)}$.
- The following set of three relations involving space derivatives of fundamental unit vectors t,n,b are known as Serret -Frenet Formulae.

  (i) $\frac{dt}{ds}=\kappa n$

  (ii) $\frac{dn}{ds}=\tau b-\kappa t$

  (iii) $\frac{db}{ds}=-\tau n$sss

4.19 ANSWERS TO CHECK YOUR PROGRESS

Answer to Q1:

We have $x=3cosh2t, y=3sinh2t, z=6t$

$\therefore \dot{x}=6sinh2t, \dot{y}=6cosh2t, \dot{z}=6$

Therefore, $\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}=\sqrt{(6sinh2t)^2+(6cosh2t)^2+36}$

$=\sqrt{36(sinh^2t+cosh^2t+1)}$

$$= \sqrt{36.2\cosh^2(2t)}$$

$$= \sqrt{72\cosh^2(2t)}$$

$$= 6\sqrt{2}\cosh(2t)$$

$$\therefore s = \int_0^{\pi} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, dt = \int_0^{\pi} 6\sqrt{2}\cosh(2t)dt$$

$$= 6\sqrt{2}\left[\frac{\sinh 2t}{2}\right]_0^{\pi}$$

$$= 3\sqrt{2}\sinh 2\pi.$$

Answer to Q2: We have

$$x = (a+b)\cos\theta - b\cos\left(\frac{a+b}{b}\theta\right)$$

$$\therefore \dot{x} = -(a+b)\sin\theta + (a+b)\sin\left(\frac{a+b}{b}\theta\right)$$

$$y = (a+b)\sin\theta - b\sin\left(\frac{a+b}{b}\right)\theta$$

$$\therefore \dot{y} = (a+b)\cos\theta - (a+b)\cos\left(\frac{a+b}{b}\theta\right)$$

$$z = 0$$

$$\therefore \dot{z} = 0$$

Now,

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \left\{-(a+b)\sin\theta + (a+b)\sin\left(\frac{a+b}{b}\theta\right)\right\}^2 + \left\{(a+b)\cos\theta - (a+b)\cos\left(\frac{a+b}{b}\theta\right)\right\}^2 + 0$$

$$= (a+b)^2\left[\sin^2\left(\frac{a+b}{b}\theta\right) + \cos^2\left(\frac{a+b}{b}\theta\right) + \sin^2\theta + \cos^2\theta - 2\sin\theta\sin\left(\frac{a+b}{b}\theta\right) - 2\cos\theta\cos\left(\frac{a+b}{b}\theta\right)\right]$$

$$= (a+b)^2\left[1 + 1 - 2\cos\left(\frac{a+b}{b}\theta - \theta\right)\right]$$

$$= (a+b)^2\left[2 - 2\cos\left(\frac{a}{b}\theta\right)\right]$$

$$= 4(a+b)^2 sin^2 \frac{a}{2b}\theta$$

$$\therefore \sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2} = \sqrt{4(a+b)^2 sin^2 \frac{a}{2b}\theta} = 2(a+b)sin\frac{a}{2b}\theta$$

$$\therefore s = \int_0^\theta \sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}\, d\theta = \int_0^\theta \left\{2(a+b)sin\frac{a}{2b}\theta\right\} d\theta$$

$$= -2(a+b)\frac{2b}{a}\left[sin\frac{a}{2b}\theta\right]_0^\theta$$

$$= \frac{4(a+b)b}{a}\left[1-cos\frac{a}{2b}\theta\right]$$

Answer to Q3: Let the positive vector r of any point on the curve is given by

$$r = \left\{2a\left(sin^{-1}t + t\sqrt{(1-t)^2}\right), 2at^2, 4at\right\}$$

$$\frac{dr}{dt} = \dot{r}\left[2a\left\{\frac{1}{\sqrt{(1-t^2)}} + \sqrt{(1-t^2)} - \frac{2t^2}{2\sqrt{(1-t^2)}}\right\}, 4at, 4a\right]$$

$$= 16a^2(1-t^2) + 16a^2t^2 + 16a^2$$

$$= 32a^2$$

Therefore, the required length

$$s = \int_{t_1}^{t_2} \sqrt{(\dot{r}.\dot{r})}\ dt$$

$$\Longrightarrow s = \int_{t_1}^{t_2} \sqrt{(\dot{r}.\dot{r})}\ dt$$

$$= \int_{t_1}^{t_2} 4\sqrt{2}a\ dt$$

$$= 4\sqrt{2}a(t_2 - t_1)$$

Answer to Q4: Given that

$$r = (t, t^2, t^3)$$

$$\therefore \dot{r} = (1, 2t, 3t^2);\quad \ddot{r} = (0, 2, 6t);\ \dddot{r} = (0, 0, 6)$$

Therefore

$$\dot{r} \times \ddot{r} = (6t^2, -6t, 2)$$

$$\therefore |\dot{r} \times \ddot{r}| = \sqrt{\{(36t^4 + 36t^2 + 4)\}}$$

$$= 2\sqrt{\{(1 + 9t^2 + 9t^4)\}}$$

Again, $$[\dot{r}, \ddot{r}, \dddot{r}] = \begin{vmatrix} 1 & 2t & 3t^2 \\ 0 & 2 & 6t \\ 0 & 0 & 6 \end{vmatrix} = 12$$

Also $$|\dot{r}|^2 = 1 + 4t^2 + 9t^4$$

Now, $$\kappa^2 = \frac{|\dot{r} \times \ddot{r}|^2}{|\dot{r}|^6} = \frac{4\left((1+9t^2+9t^4)\right)}{\left((1+9t^2+9t^4)\right)^3}$$

and $$\tau = \frac{[\dot{r}, \ddot{r}, \dddot{r}]}{|\dot{r} \times \ddot{r}|^2} = \frac{12}{4\left((1+9t^2+9t^4)\right)} = \frac{3}{1+9t^2+9t^4}$$

Answer to Q5:

(i) Left as an exercise

(ii) Left as an exercise

Answer to Q6: Left as an exercise

Answer to Q7: (a) Left as an exercise

(b) Left as an exercise

(c) Left as an exercise

Answer to Q8:

(i) We know that

$$r' = t, r'' = t' = \kappa n = \kappa n + 0t + 0b \quad (1)$$

$$r''' = \kappa' n - \kappa^2 t + \kappa\tau b \quad [\text{by Example 5}] \qquad (2)$$

$$r'''' = (\kappa'' - \kappa^3 - \kappa\tau^2)n - 3\kappa'\kappa t + (2\kappa'\tau + \tau'\kappa)b. [\text{by Example 5}] \qquad (3)$$

From (1), (2), (3), we have

$$[t', t'', t'''] = [r'', r''', r'''']$$

$$= \begin{vmatrix} 0 & \kappa & 0 \\ -\kappa^2 & \kappa' & \kappa\tau \\ -3\kappa\kappa' & (\kappa'' - \kappa\tau^2 - \kappa^3) & (2\kappa'\tau + \kappa\tau') \end{vmatrix}$$

$$= -\kappa[-\kappa^2(2k'\tau + \kappa\tau') + 3k^2k'\tau]$$

$$= \kappa^3(\kappa\tau' - \kappa'\tau)$$

$$= \kappa^5 \frac{\kappa \frac{d\tau}{ds} - \tau \frac{d\kappa}{ds}}{k^2}$$

$$= \kappa^5 \frac{d}{ds}\left(\frac{\tau}{\kappa}\right)$$

(ii) We have

$$b' = -\tau n = 0t - \tau n + 0b \qquad (1)$$

$$b'' = -\tau n' = -\tau(\tau b - \kappa t) - \tau' n$$

$$= \tau\kappa t - \tau' n - \tau^2 b \qquad (2)$$

$$b''' = (\tau\kappa)t' + (\tau\kappa' + \tau'\kappa)t - \tau' n - \tau'' n' - \tau^2 b' - 2\tau\tau' b$$

$$= \tau\kappa(\kappa n) + (\tau\kappa' + \tau'\kappa)t - \tau'(\tau b - \kappa t) - \tau'' n - \tau^2(-\tau n) - 2\tau\tau' b$$

$$= (2\kappa\tau' + \kappa'\tau)t + (\kappa'\tau - \tau'' + \tau^3)n - 3\tau\tau' b \qquad (3)$$

Now, from (1), (2), (3), we have

$$[b', b'', b'''] = \begin{vmatrix} 0 & -\tau & 0 \\ \tau\kappa & -\tau' & -\tau^2 \\ 2k\tau' + k'\tau & \kappa^2\tau - \tau'' + \tau^3 & -3\tau\tau' \end{vmatrix}$$

$$= \tau[-3\tau^2\kappa\tau' + \tau^2(2\kappa\tau' + \kappa'\tau)]$$

$$= \tau^3(\tau\kappa' - \kappa\tau')$$

$$= \tau^5 \frac{\tau \frac{d\kappa}{ds} - \kappa \frac{d\tau}{ds}}{\tau^2}$$

$$= \tau^5 \frac{d}{ds}\left(\frac{\kappa}{\tau}\right)$$

4.20 FURTHER READING

- Weatherburn, C. E. 2016. *Differential Geometry of Three Dimensions*. Cambridge University Press.
- Graustein, W. C. 2012. *Differential Geometry*. Courier Corporation.
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- Sharma, J. N., and Vasistha, A. R. Year. Differential Geometry. Kedar Nath Ram Nath, Meerut.

M 4.21 MODEL QUESTIONS

Q1. For the curve $r = (2abt, a^2 logt, b^2t^2)$, show that

$$\kappa = \tau = \frac{2abt}{(a^2 + 2b^2t^2)^2}$$

Q2. For the curve $r = (\sqrt{6}at^3, a^2 logt, b^2t^2)$, show that

$$\kappa = \tau = \frac{2abt}{(a^2 + 2b^2t^2)^2}$$

Q3. For the curve $x = 4acos^3t, y = 4asin^3t, z = 3ccos2t$, prove that

$$\kappa = \frac{a}{6(a^2 + c^2)sin2t}$$

Q4. Find the radii of curvature and torsion at the point $\theta = \frac{\pi}{4}$ of the curve

$$r = (acos\theta, asin\theta, acos\theta)$$

What are the characteristics of computer?

Q5. Example 4: Find the osculating plane, curvature and torsion at any point of the curve $x = acos2u, y = asin2u, z = 2asinu$.

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