UNIT 5: CURVES IN SPAE-II

UNIT STRUCTURE

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5.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- Determine the **curvature and torsion** of a space curve, especially when the curve is defined as the intersection of two surfaces.
- Explain and analyze special types of space curves such as **cylindrical helices** and **circular helices**, along with their key properties.
- Understand and apply the concept of **intrinsic equations** of a curve and relate them to curvature and torsion.
- State and interpret the **fundamental theorems of space curves** and understand their importance in defining a curve uniquely.
- Define and construct the **osculating circle (circle of curvature)** and explain its role in approximating a curve locally.

- Understand the concept of the **osculating sphere (sphere of curvature)** and its geometric interpretation.

5.2 INTRODUCTION

In the previous unit, we developed the foundational concepts required to study curves in three-dimensional space. We introduced the idea of a **space curve**, discussed its parametric representation, and examined important geometric elements such as the **tangent, normal,** and**binormal vectors.** We also studied **arc length**, various associated planes (osculating, normal, and rectifying planes), and important measures like **curvature**and**torsion**. Furthermore, the **Serret–Frenet formulas** provided a systematic framework to describe the local behavior of a space curve.

Building upon these fundamental ideas, the **Unit 5** extends the study to more advanced and applied aspects of space curves. While Unit 4 focused on defining and understanding the basic geometric and analytical tools, this unit emphasizes their deeper applications and interpretations.

In this unit. we compute **curvature and torsion for curves defined as the intersection of two surfaces**, an important application in geometry and physics.

Special types of curves, such as the **cylindrical helix** and **circular helix**, are studied to illustrate how curvature and torsion behave in structured geometries. The concept of **intrinsic equations** is introduced to describe curves independently of their position in space, relying only on properties like curvature and torsion.

Further, the **fundamental theorems of space curves** are discussed, which establish that a curve is uniquely determined (up to its position in space) by its curvature and torsion. This highlights the central role of these quantities in differential geometry.

The unit also introduces important geometric constructs such as the **osculating circle** and the **osculating sphere**, which provide the best local approximations of a curve at a point. These concepts help in understanding how a curve bends and twists in space.

Overall, this unit deepens the understanding of space curves by connecting geometric intuition with analytical techniques, enabling learners to analyze more complex curves and their properties in three-dimensional space.

5.3 CURVATURE AND TORSION OF A CURVE GIVEN BY THE INTERSECTION OF TWO SURFACES

Let the curve be given as the intersection of surfaces $f(r) = 0, \phi(r) = 0$.

Let $r = r(t)$ represent the equation of the curve of intersection of intersection of the two given surfaces

$f(r) = 0$ and $\phi(r) = 0$.

The unit tangent vector to the curve is orthogonal to the normals of both surfaces $f(r) = 0, \phi(r) = 0$.

Also, the vectors normal to the two surfaces are given by

$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$ and $\nabla \phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right)$ respectively.

Therefore t is parallel to $\nabla f \times \nabla \phi = h$(say).

i.e., $\nabla f \times \nabla \phi = \lambda t = \lambda r'$, (1)

i.e., $\lambda t = \lambda r' = h$.

This gives, $\lambda^2 = h^2$

Let $h = (h_1, h_2, h_3)$ and $r' = (x', y', z')$ (2)

This gives $\lambda x' = h_1, \lambda y' = h_2, \lambda z' = h_3$.

Also, $\lambda \frac{dr}{ds} = \lambda \left(\frac{\partial r}{\partial x}\frac{dx}{ds} + \frac{\partial r}{\partial y}\frac{dy}{ds} + \frac{\partial r}{\partial z}\frac{dz}{ds}\right)$

$$= \left(\lambda x' \frac{\partial r}{\partial x} + \lambda y' \frac{\partial r}{\partial y} + \lambda z' \frac{\partial r}{\partial z}\right)$$

$$= \left(h_1 \frac{\partial r}{\partial x} + h_2 \frac{\partial r}{\partial y} + h_3 \frac{\partial r}{\partial z}\right)$$

or $\lambda \frac{d}{ds} = h_1 \frac{\partial}{\partial x} + h_2 \frac{\partial}{\partial y} + h_3 \frac{\partial}{\partial z} \equiv \Delta$ (say) (3)

Therefore (1) may be written as

$\Delta r = h$ $\left[\because \lambda r' = \lambda \frac{dr}{ds} = \Delta r\right]$ (4)

Now operating on (1) by $\lambda \frac{d}{ds} \equiv \Delta$, we get

$$\lambda \frac{d}{ds}(\lambda t) = \Delta h$$

or $\lambda^2 \kappa n + \lambda \lambda' t = \Delta h$ (5)

Taking the cross product of (1) and (5), we have

$\lambda^3 \kappa b = h \times \Delta h = H$,(say) (6)

This gives $\lambda^3\kappa = |H|$. (7)

Hence, $\kappa = \frac{|H|}{\lambda^3} = \frac{|H|}{|h|^3}$ (8)

Now operating (6) with $\lambda\frac{d}{ds} \equiv \Delta$, we have

$$\lambda\frac{d}{ds}(\lambda^3\kappa b) = \Delta H$$

or $-\lambda^4\kappa\tau n + \lambda b\frac{d}{ds}(\lambda^3\kappa) = \Delta H$ (9)

Taking the scalar product of (5) and (9), we have

$$-\lambda^6\kappa^2\tau = \Delta h.\Delta H$$

or $\tau = -\frac{\Delta h.\Delta H}{\lambda^6\kappa^2} = \frac{\Delta h.\Delta H}{|H|^2}$ [by (7), $|H| = \lambda^3\kappa$] (10)

Equation (8) and (10) gives curvature and torsion of the curve of intersection of two surfaces.

Example 1: Find the curvature and torsion of the curve of intersection of the two quadratic surfaces

$ax^2 + by^2 + cz^2 = 1, a'x^2 + b'y^2 + c'z^2 = 1.$

Solution: Let $f = \frac{1}{2}(ax^2 + by^2 + cz^2 - 1)$;

$$\phi = \frac{1}{2}(a'x^2 + b'y^2 + c'z^2 - 1)$$

Therefore, $\nabla f = \left(\frac{\partial f}{\partial x'}, \frac{\partial f}{\partial y'}, \frac{\partial f}{\partial z}\right) = (ax, by, cz)$

and $\nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) = (a'x, b'y, c'z)$

Therefore, we have

$$\nabla f \times \nabla\phi = \{(bc' - b'c)gz, (ca' - ac')zx, (ab' - ba')xy\}$$

where $A = bc' - b'c, B = ca' - ac', C = ab' - ba'$

or

$$\nabla f \times \nabla\phi = xyz\left(\frac{A}{x}, \frac{B}{y}, \frac{C}{z}\right) \quad (1)$$

Since $\nabla f \times \nabla\phi$ is parallel to $t = r'$, so we have

$$\lambda t = \lambda r' = \left(\frac{A}{x}, \frac{B}{y}, \frac{C}{z}\right) \quad (2)$$

Squaring (2), so we have

$$\lambda^2 = \sum\left(\frac{A}{x}\right)^2 \quad (3)$$

Again equation (3) from section 5.3

$$\lambda\frac{d}{ds} = h_1\frac{\partial}{\partial x} + h_2\frac{\partial}{\partial y} + h_3\frac{\partial}{\partial z} \equiv \Delta$$

Gives

$$\lambda\frac{d}{ds} \equiv \left(\frac{A}{x}\frac{\partial}{\partial x} + \frac{B}{y}\frac{\partial}{\partial y} + \frac{C}{z}\frac{\partial}{\partial z}\right) \quad (4)$$

Now operating (4) on (2), we have

$$\lambda\frac{d}{ds}(\lambda t) = \left(\frac{A}{x}\frac{\partial}{\partial x} + \frac{B}{y}\frac{\partial}{\partial y} + \frac{C}{z}\frac{\partial}{\partial z}\right)\left(\frac{A}{x}, \frac{B}{y}, \frac{C}{z}\right)$$

or $\lambda^2\kappa n + \lambda\lambda' t = -\left(\frac{A^2}{x^3}, \frac{B^2}{y^3}, \frac{C^2}{z^3}\right)$ (5)

Taking cross product of (2) and (5), we get

$$\lambda^3\kappa b = \left\{\frac{BC}{y^3z^3}(Bz^2 - Cy^2), \frac{CA}{z^3x^3}(Cx^2 - Az^2), \frac{AB}{x^3y^3}(Ay^2 - Bx^2)\right\} \quad (6)$$

But

$$Bz^2 - Cy^2 = (ca' - ac')z^2 - (ab' - ba')y^2$$

$$= a'(cz^2 + by^2) - a(c'z^2 + b'z^2)$$

$$= a'(1 - ax^2) - a(1 - a'x^2) \text{ [Using the equations of given surfaces]}$$

$$= a' - a$$

Similarly, we have

$$Cx^2 - Az^2 = b' - b, Ay^2 - Bx^2 = c' - c$$

Putting these values in (6), we have

$$\lambda^3\kappa b = \frac{ABC}{x^3y^3z^3}\left\{\frac{x^3}{A}(a' - a), \frac{y^3}{B}(b' - b), \frac{z^3}{C}(c' - c)\right\} \quad (7)$$

Squaring both sides of (7), we get

$$\lambda^6\kappa^2 = \frac{A^2B^2C^2}{x^6y^6z^6}\sum\frac{x^6}{A^2}(a' - a)^2 \quad (8)$$

or $\left\{\sum\left(\frac{A}{x}\right)^2\right\}^3 k^3 = \frac{A^2B^2C^2}{x^6y^6z^6}\sum\frac{x^6}{A^2}(a' - a)^2$ [From (3) putting value of λ]

or $k^2 = \frac{A^2B^2C^2}{x^6y^6z^6}\frac{\sum\frac{x^6}{A^2}(a' - a)^2}{\left\{\sum\left(\frac{A}{x}\right)^2\right\}^3}$ (9)

Equation (7) may be put as

$$\mu b = \left\{\frac{x^3}{A}(a' - a), \frac{y^3}{B}(b' - b), \frac{z^3}{C}(c' - c)\right\} \quad (10)$$

where $\mu = \frac{\lambda^3 \kappa x^3 y^3 z^3}{ABC}$ (11)

Squaring (10), we get

$$\mu^2 = \sum \frac{x^6}{A^2(a'-a)^2} \quad (12)$$

Operating by (4) on (10), we get

$$\mu' \lambda b - \lambda \mu \tau n = 3\{x(a'-a), y(b'-b), c(c'-c)\} \quad (13)$$

Now taking dot product of (5) and (13), we get

$$\lambda^3 \mu \kappa \tau = 3 \sum \frac{A^2}{x^2}(a'-a) \quad (14)$$

Dividing (12) and (11), we get

$$\mu \lambda^3 \kappa = \frac{ABC}{x^3 y^3 z^3} \sum \frac{x^6}{A^2}(a'-a)^2 \quad (15)$$

Thus from (14) and (15), we have

$$\tau = \frac{3x^3 y^3 z^3}{ABC} \frac{\sum \frac{A^2}{x^2}(a'-a)}{\sum \frac{x^6}{A^2}(a'-a)^2} \quad (16)$$

The curvature and torsion of the intersection curve are given by equations (9) and (16).

5.4 CYLINDRICAL HELICES

Definition:A curve which lies on a cylinder and cuts the generators at a constant angle is called a cylindrical helix.

Thus, the tangent at any point on the helix makes a constant angle say α, with a fixed line, this fixed line is known as axis (generator) of the cylinder. The axis of the cylinder is also called the axis of the helix.

Characteristic property of a helix.

Theorem 5.4.1: **A necessary and sufficient condition for a curve to be a helix is that its curvature κ and torsion τ are in a constant ratio.**

Proof: **($\Rightarrow$)** Necessity

Let α be a fixed unit vector along the axis of the helix and let t be the unit tangent vector to the curve.

Since a helix meets the generators of the cylinder at a constant angle α, we have

$$t \cdot a = \cos \alpha = \text{constant.} \quad (1)$$

Fig:

Differentiating (1) with respect to arc length s,

$t' \cdot a = 0.$

Using Frenet's formula $t' = \kappa n$,

$\kappa n. a = 0$ (2)

Since $\kappa \neq 0$ for a helix, (2) implies

$n \cdot a = 0,$

i.e., the vector a is perpendicular to the principal normal n. Thus, lies in the rectifying plane, the plane spanned by t and b.

Because a makes angle α with t, it must make angle $90° - \alpha$ with b. Hence can be expressed as

$$a = \cos \alpha\, t + \sin \alpha\, b. \qquad (3)$$

Differentiating (3) with respect to s,

$$0 = \cos \alpha\, t' + \sin \alpha\, b' (\text{since } a' = 0).$$

Using Frenet's formulas $t' = \kappa n$ and $b' = -\tau n$,

$$0 = \kappa n \cos \alpha - \tau n \sin \alpha = (\kappa \cos \alpha - \tau \sin \alpha) n.$$

Since $n \neq 0$,

$$\kappa \cos \alpha - \tau \sin \alpha = 0,$$

or

$$\frac{\kappa}{\tau} = \tan \alpha = \text{constant}.$$

Thus, κ/τ is constant for a helix.

($\Leftarrow$) Sufficiency

Conversely, suppose

$$\frac{\kappa}{\tau} = \tan \alpha = \text{constant}.$$

Then

$$\kappa \cos \alpha - \tau \sin \alpha = 0,$$

or

$$(\kappa \cos \alpha - \tau \sin \alpha) n = 0.$$

Using Frenet's formulas again,

$$t' \cos \alpha + b' \sin \alpha = 0,$$

which can be written as

$$\frac{d}{ds}(t\cos\alpha + b\sin\alpha) = 0.$$

Integrating,

$$t\cos\alpha + b\sin\alpha = a,$$

where a is a constant unit vector.

Taking dot product with t,

$$t \cdot a = \cos\alpha.$$

Thus, the tangent vector makes a constant angle with the fixed direction a. Hence, the curve is a helix.

Theorem 5.4.2: Curvature κ and torsion τ of a helix C are in a constant ratio to the curvature κ_1 of the plane curve C_1 obtained by projecting on a plane perpendicular to the axis of the helix and are connected by the relations

$\kappa = \kappa_1 \sin^2\alpha$ and $\tau = \kappa_1 \sin\alpha\cos\alpha$

where α is the constant angle which the curve makes with the generators.

Proof: Let the generators be parallel to z axis and the arc length of C be measured from point A, the intersection of the curve with xy plane. The arc length s_1 of projection C_1 of C on the xy plane is measured from A upto the generators through the current point $P = (x, y, z)$.

Now if α is the constant angle at which the curve cuts the generators, then we have

$$\frac{ds_1}{ds} = s_1' = \sin\alpha \qquad (1)$$

Again, the coordinates x, y of P are functions of s_1 and $z = s\cos\alpha$. Hence the position vector r of any current point P is given by

$$r = (x, y, s\cos\alpha)$$

$$\therefore t = \frac{dr}{ds} = \left(\frac{dx}{ds_1}\frac{ds_1}{ds}, \frac{dy}{ds_1}\frac{ds_1}{ds}, \cos\alpha\right)$$

$$= \left(\frac{dx}{ds_1}\sin\alpha, \frac{dy}{ds_1}\sin\alpha, \cos\alpha\right)$$

$$\therefore t' = \left(\frac{d^2x}{ds_1^2}\frac{ds_1}{ds}\sin\alpha, \frac{d^2y}{ds_1^2}\frac{ds_1}{ds}\sin\alpha, 0\right)$$

$$= \left(\frac{d^2x}{ds_1^2}\sin^2\alpha, \frac{d^2y}{ds_1^2}\sin^2\alpha, 0\right)$$

or $\kappa n = \left(\frac{d^2x}{ds_1^2}\sin^2\alpha, \frac{d^2y}{ds_1^2}\sin^2\alpha, 0\right)$

Squaring this, we get

$$\kappa^2 = \left[\left(\frac{d^2x}{ds_1^2}\right)^2 + \left(\frac{d^2y}{ds_1^2}\right)^2\right] sin^4\alpha \qquad (2)$$

Now we shall find the curvature κ_1 of the projection C_1,let r_1 be the position vector of projection P_1 of P,then putting $\alpha = 90^0$ in the position vector of P,we have

$$r_1 = (x, y, 0) \therefore t = \frac{dr_1}{ds_1} = \left(\frac{dx}{ds_1}, \frac{dy}{ds_1}, 0\right)$$

and $t_1' = \left(\frac{d^2x}{ds_1^2}, \frac{d^2y}{ds_1^2}, 0\right)$ or $\kappa_1 n_1 = \left(\frac{d^2x}{ds_1^2}, \frac{d^2y}{ds_1^2}, 0\right)$

Squaring, we get

$$\kappa_1^2 = \left(\frac{d^2x}{ds_1^2}\right)^2 + \left(\frac{d^2y}{ds_1^2}\right)^2 \qquad (3)$$

From (2) and (3),

$$\kappa^2 = \kappa_1^2 \sin^4\alpha \Rightarrow \kappa = \kappa_1 \sin^2\alpha. \qquad (4)$$

For a helix,

$$\frac{\kappa}{\tau} = \tan\alpha.$$

Thus,

$$\tau = \kappa\cot\alpha = \kappa_1 \sin^2\alpha \cot\alpha = \kappa_1 \sin\alpha\cos\alpha. \qquad (5)$$

Note:

(a) From (4), since $\kappa$is constant for a helix, $\kappa_1$is also constant. Hence the projection $C_1$is a circle, and the helix lies on **a circular cylinder.**

(b) From (5), τ is constant as well. Therefore, the **only** space curve with both curvature and torsion constant is the **circular helix.**

5.5 CIRCULAR HELIX

Definition: A helix which lies on the surface of a circular cylinder is called helix or right circular helix.

Let the equation of the cylinder be

$$x^2 + y^2 = a^2 \qquad (1)$$

i.e., radius of the circular section is a and the z —axis is along the axis of the cylinder,x —axis is taken along the radius through the point A where the helix meets the xy —plane. Let AP be the arc of helix which makes angle α with the generator PP_1.

Let θ be the angle which the projection $AP_1(=s_1)$ of the arc $AP(=s)$ on xy –plane subtends at the origin, then $s_1 = a\theta$.

If (x, y, z) are the co-ordinates of a current point P on the helix, then we have

$$x = a\cos\theta, y = a\sin\theta$$

Also, $z = s\cos\alpha$ and $s_1 = s\sin\alpha$

$$z = s_1\cos\alpha\ \operatorname{cosec}\alpha = a\,\theta\,\cot\alpha$$

Hence, the equation of a circular helix may be put in the form

$$r = (a\cos\theta, a\sin\theta, a\,\theta\,\cot\alpha).$$

Example 5.4.1: A curve is drawn on the cylinder $x = at^2, y = 2at$, so as to cut all the generator at the same angle α, show that

$$\rho = 2a(1+t^2)^{\frac{3}{2}}\operatorname{cosec}^2\alpha \text{ and } \sigma = 2a(1+t^2)^{\frac{3}{2}}\sec\alpha\ \operatorname{cosec}\alpha.$$

Solution: Let the generators of the cylinder be parallel to the z-axis. Since the curve meets every generator at a constant angle α, the height coordinate satisfies

$$z = s\cos\ \alpha,$$

where s is the arc length of the curve.

Thus, the position vector of a general point on the curve is

$$\mathbf{r} = (at^2, 2at, s\cos\ \alpha).$$

Differentiating with respect to arc length s,

$$\mathbf{t} = \mathbf{r}' = \left(2at\,\frac{dt}{ds},\ 2a\,\frac{dt}{ds},\ \cos\ \alpha\right). \qquad (1)$$

Taking the square of its magnitude,

$$1 = (4a^2t^2 + 4a^2)\left(\frac{dt}{ds}\right)^2 + \cos^2\alpha.$$

Hence,

$$\left(\frac{dt}{ds}\right)^2 = \frac{\sin^2\alpha}{4a^2(1+t^2)}, \frac{dt}{ds} = \frac{\sin\alpha}{2a\sqrt{1+t^2}}$$

Substituting this in (1),

$$\mathbf{t} = \left(\frac{t\sin\ \alpha}{\sqrt{1+t^2}},\ \frac{\sin\alpha}{\sqrt{1+t^2}},\ \cos\ \alpha\right)$$

Differentiating $\mathbf{t}$ with respect to s,

$$t' = \left[\sin\alpha \frac{\left\{ 1\sqrt{1+t^2} - \frac{1}{2}.\frac{2t.t}{\sqrt{1+t^2}} \right\}}{(1+t^2)}, \frac{-\sin\alpha t}{(1+t^2)^{\frac{3}{2}}}, 0 \right] \frac{dt}{ds}$$

$$\mathbf{t}' = \left(\frac{\sin\alpha}{(1+t^2)^{3/2}}, -\frac{t\sin\alpha}{(1+t^2)^{3/2}}, 0 \right) \frac{dt}{ds}.$$

Using $\frac{dt}{ds} = \frac{\sin\alpha}{2a\sqrt{1+t^2}}$,

$$\kappa\mathbf{n} = \left(\frac{\sin^2\alpha}{2a(1+t^2)^2}, -\frac{t\sin^2\alpha}{2a(1+t^2)^2}, 0 \right).$$

Squaring components,

$$\kappa^2 = \frac{\sin^4\alpha(1+t^2)}{4a^2(1+t^2)^4} = \frac{\sin^4\alpha}{4a^2(1+t^2)^3}.$$

Thus,

$$\kappa = \frac{\sin^2\alpha}{2a(1+t^2)^{3/2}}, \rho = \frac{1}{\kappa} = 2a(1+t^2)^{3/2}\csc^2\alpha.$$

For a helix,

$$\frac{\kappa}{\tau} = \tan\alpha.$$

Hence,

$$\sigma = \frac{1}{\tau} = \rho\tan\alpha = 2a(1+t^2)^{3/2}\csc^2\alpha\sec\alpha.$$

Thus,

$$\sigma = 2a(1+t^2)^{3/2}\sec\alpha\csc\alpha.$$

Example 5.4.2: Show that the necessary and sufficient condition for a curve to be a helix is that its principal normals are all parallel to a fixed plane.

Solution: Assume that the given curve is a helix. Then the unit tangent vector $\mathbf{t}$ makes a constant angle α with some fixed direction represented by a constant vector $\mathbf{a}$. Thus,

$$\mathrm{t}\cdot\mathrm{a} = \cos\alpha = \text{constant.} \qquad (1)$$

Differentiating (1) with respect to arc length s,

$$\mathrm{t}'\cdot\mathrm{a} = 0.$$

Using Frenet's formula $\mathrm{t}' = \kappa\mathrm{n}$,

$$\kappa\mathrm{n}\cdot\mathrm{a} = 0.$$

Since $\kappa \neq 0$ for a helix,

$$\mathbf{n} \cdot \mathbf{a} = 0.$$

Thus, the principal normal $\mathbf{n}$ is perpendicular to the fixed vector $\mathbf{a}$, which means:

is parallel to a fixed plane (the plane whose normal is).

Hence, the condition is necessary.

Sufficiency

Conversely, suppose the principal normal $\mathbf{n}$ of the curve is parallel to a fixed plane. Let $\mathbf{a}$ be a unit vector normal to this plane. Then

$$\mathbf{n} \cdot \mathbf{a} = 0.$$

Multiplying by κ,

$$\kappa \mathbf{n} \cdot \mathbf{a} = 0,$$

and using $\mathbf{t}' = \kappa \mathbf{n}$,

$$\mathbf{t}' \cdot \mathbf{a} = 0.$$

Integrating with respect to s,

$$\mathbf{t} \cdot \mathbf{a} = \text{constant}$$

Thus, the tangent vector $\mathbf{t}$ makes a constant angle with the fixed direction $\mathbf{a}$. A curve whose tangent maintains a constant angle with a fixed direction is a helix.

Hence, the condition is sufficient.

5.6 INTRINSIC EQUATIONS

A space curve may be defined by expressing the coordinates of each point on the curve as functions of a parameter t with respect to a system of three mutually perpendicular Cartesian axes:

$$x = x(t), y = y(t), z = z(t).$$

If the same curve is described relative to a different Cartesian coordinate system, the corresponding equations will, in general, look quite different, and it is not immediately obvious that they represent the same geometric curve.

Since our interest lies in those properties of a curve that remain unchanged under rigid motions (translations and rotations), it becomes essential to have a criterion for determining when two curves are congruent—that is, identical in shape—based solely on properties that do not depend on their position in space.

For this reason, it is desirable to describe a curve intrinsically, that is, without reference to any particular coordinate axes. To achieve this, we use the intrinsic equations of the curve.

The intrinsic equations of a curve can be written in the form

$$\kappa = f(s), \tau = \phi(s),$$

where $\kappa$is the curvature, $\tau$is the torsion, and $s$is the arc length. These equations express how the curve bends and twists as functions of distance along the curve.

The advantage of using intrinsic equations is that congruent curves always have the same intrinsic equations, even though such equations do not specify the actual position of the curve in space.

5.7 FUNDAMENTAL THEOREMS FOR SPACE CURVES

Theorem 1: (Existence Theorem):

If **κ(s)** and **τ(s)** are continuous functions of a real variable $s \geq 0$, then there exists a space curve for which **κ(s)** is the curvature, **τ(s)** is the torsion, and, and **s** is the arc length measured from a suitable base point.

If $\kappa(s)$ and $\tau(s)$ are continuous functions of a real variable $s(s \geq 0)$,then there exists a space curve for which κ is the curvature,τ is the torsion, and$s$ is the arc length measured from some suitable base point.

Proof: From the existence theorem on linear differential equations, we know that the differential equations

$$\frac{d\alpha}{ds} = \kappa\alpha, \frac{d\beta}{ds} = \tau y - \kappa\alpha, \frac{d\gamma}{ds} = -\tau\beta \qquad (1)$$

admit a unique set of solutions for a given set of values of α, β, γ at $s = 0$.

Therefore, we have a unique set $\alpha_1, \beta_1, \gamma_1$ for which the values at are 1,0,0.

Similarly, there exists a unique set $\alpha_2, \beta_2, \gamma_2$ with values 0,1,0 at $s = 0$ and $\alpha_3, \beta_3, \gamma_3$ is a unique set with values 0,0,1 at $s = 0$.

Now

$$\frac{d}{ds}(\alpha_1^2 + \beta_1^2 + \gamma_1^2) = 2\left(\alpha_1 \frac{d\alpha_1}{ds} + \beta_1 \frac{d\beta_1}{ds} + \gamma_1 \frac{d\gamma_1}{ds}\right)$$

$$= 2[\alpha_1(\kappa\beta_1) + \beta_1(\tau\gamma_1 - \kappa\alpha_1) + \gamma_1(-\tau\beta_1)]$$

$$= 0 \qquad [\text{From (1)}]$$

Integrating, we get

$\alpha_1^2 + \beta_1^2 + \gamma_1^2 = C_1$(constant)

Initially at $s = 0, \alpha_1 = 1, \beta_1 = 0, \gamma_1 = 0, \therefore\ C_1 = 1$

Hence

$$\left.\begin{aligned} \alpha_1^2 + \beta_1^2 + \gamma_1^2 &= 1 \\ \alpha_2^2 + \beta_2^2 + \gamma_2^2 &= 1 \\ \alpha_3^2 + \beta_3^2 + \gamma_3^2 &= 1 \end{aligned}\right\} \qquad (2)$$

Again $\frac{d}{dr}(\alpha_1\alpha_2+\beta_1\beta_2+\gamma_1\gamma_2)$

$$=\left(\alpha_1\frac{d\alpha_2}{ds}+\beta_1\frac{d\beta_2}{ds}+\gamma_1\frac{d\gamma_1}{ds}\right)+\left(\frac{d\alpha_1}{ds}\alpha_2+\frac{d\beta_1}{ds}\beta_2+\frac{d\gamma_1}{ds}\gamma_2\right)$$

$$=\alpha_1(\kappa\beta_2)+\beta_1(\tau\gamma_2-\kappa\alpha_2)+\gamma_1(-\tau\beta_2)+\kappa\beta_1\alpha_2+(\tau\gamma_1-\kappa\alpha_1)\beta_2+(-\tau\beta_1)\gamma_2$$

$$=0$$

Thus, on integrating, we have

$$\alpha_1\alpha_2+\beta_1\beta_2+\gamma_1\gamma_2=C_2 \text{ (constant)}$$

Initially at $s=0,\alpha_1=1,\beta_1=0,\gamma_1=0,\alpha_2=0,\beta_2=1,\gamma_2=0.$

$$\therefore C_2=0$$

Hence

$$\left.\begin{aligned}\alpha_1\alpha_2+\beta_1\beta_2+\gamma_1\gamma_2=0\\ \alpha_2\alpha_3+\beta_2\beta_3+\gamma_2\gamma_3=0\\ \alpha_3\alpha_1+\beta_3\beta_1+\gamma_3\gamma_1=0\end{aligned}\right\} \quad (3)$$

Thus, we have six equations given by (2) and (3) in the elements of three sets namely $(\alpha_1,\beta_1,\gamma_1),(\alpha_2,\beta_2,\gamma_2)$ and $(\alpha_3,\beta_3,\gamma_3)$.Hence it follows that there are three mutually orthogonal unit vectors $t=(\alpha_1,\alpha_2,\alpha_3),n=(\beta_1,\beta_2,\beta_3)$ and $b=(\gamma_1,\gamma_2,\gamma_3)$ defined for each curve of s.

Now let the curve be defined by

$$r=r(s)=\int^s t(s)ds \quad (4)$$

Now $|r'|=|t|=1$ and $t'=\kappa n$ where $|n|=1$

Again $b=t\times n$

$$\because b'=t'\times n+t\times n'$$

$$=\kappa(n\times n)+t\times(-\kappa n+\tau n) \text{ putting for } t' \text{ and } n'$$

$$=\kappa(n\times n)-\kappa(t\times t)+\tau(t\times n)$$

$$=\tau b \text{ where } |b|=1$$

Thus τ is torsion of the curve, so there exists a curve given by (4) where t,n,b are unit vectors along tangent, principal normal and binormal respectively and κ and τ are its curvature and torsion respectively.

Hence, the curve defined by (4) is a space curve whose curvature and torsion are precisely the given functions κ(s) and τ(s).

Theorem 2. (Uniqueness Theorem) A curve is uniquely determined, except as to position

in space, when its curvature and torsion are given functions of its arc length s.

Proof:

Theorem 2. (Uniqueness Theorem) A curve is uniquely determined, except as to position

in space, when its curvature and torsion are given functions of its arc length s.

Proof: Suppose, if possible, that there exist two curves C and C_1 having the same curvature and same torsion for corresponding values of the arc length s. Quantities associated with will be denoted by the suffix "1".

Move the curve C_1 rigidly (without deformation) so that the points on C andcorresponding to the same value of coincide. Further rotate C_1 so that at $s = 0$, the Frenet triads also coincide:

$$(t, n, b) = (t_1, n_1, b_1) \text{at } s = 0.$$

Using Frenet–Serret formulas and differentiating the scalar products, we obtain:

$$\frac{d}{ds}(t \cdot t_1) = t \cdot t_1' + t' \cdot t_1 = t \cdot (\kappa n_1) + (\kappa n) \cdot t_1, \qquad (1)$$

$$\frac{d}{ds}(n \cdot n_1) = n \cdot n_1' + n' \cdot n_1 = n \cdot (\tau b_1 - \kappa t_1) + (\tau b - \kappa t) \cdot n_1, \qquad (2)$$

$$\frac{d}{ds}(b \cdot b_1) = b \cdot b_1' + b' \cdot b_1 = b \cdot (-\tau n_1) + (-\tau n) \cdot b_1. \qquad (3)$$

Adding (1), (2), and (3), we get:

$$\frac{d}{ds}(t \cdot t_1 + n \cdot n_1 + b \cdot b_1) = 0.$$

Integrating with respect to s:

$$t \cdot t_1 + n \cdot n_1 + b \cdot b_1 = \text{constant}.$$

At $s = 0$, the triads coincide, so

$$t = t_1, n = n_1, b = b_1,$$

which gives

$$t \cdot t_1 = n \cdot n_1 = b \cdot b_1 = 1.$$

Thus, the constant equals 3, and therefore

$$t \cdot t_1 + n \cdot n_1 + b \cdot b_1 = 3. \qquad (4)$$

Let α, β, γ be the angles between the corresponding vectors t, t_1; n, n_1; and b, b_1. Then (4) becomes:

$$\cos \alpha + \cos \beta + \cos \gamma = 3.$$

The sum of three cosines equals 3 only when each cosine equals 1, i.e.,

$$\alpha = \beta = \gamma = 0 (\text{mod } 2\pi)$$

Thus, for every corresponding point:

$$t = t_1, n = n_1, b = b_1.$$

Since $t = t_1$, we have

$$r' = r_1'.$$

Hence,

$$\frac{d}{ds}(r - r_1) = 0 \Rightarrow r - r_1 = a,$$

where a is a constant vector.

But at $s = 0$, the points coincide, so $a = 0$. Therefore,

$$r = r_1 \text{ for all } s.$$

Thus, the two curves coincide at all corresponding points.

Example: Show that the intrinsic equations of the curve given by

$x = ae^u cosu, y = ae^u sinu, z = be^u$ are

$\kappa = \frac{a\sqrt{2}}{s\sqrt{2a^2+b^2}}, \tau = \frac{b}{s\sqrt{2a^2+b^2}}$

Solution: The position vector of the curve is

$$r = (ae^u cosu, ae^u sinu, be^u)$$

Differentiating with respect to u,

$$\dot{r} = [ae^u(cosu - sinu), ae^u(sinu + cosu), be^u]$$

Hence $|\dot{r}| = \dot{s} = e^u\sqrt{[a^2(cosu - sinu)^2 + a^2(sinu + cosu)^2 + b^2]}$

$$= e^u\sqrt{2a^2 + b^2}$$

Thus $s = \int_{-\infty}^{u} |\dot{r}|\,du$

$$= \int_{-\infty}^{u} e^u\sqrt{2a^2 + b^2}\,du$$

$$= e^u\sqrt{2a^2 + b^2}$$

$= \dot{s}$ (1)

Therefore

$$\dot{s} = s.$$

$$r' = \frac{\dot{r}}{\dot{s}} = \frac{[ae^u(cosu - sinu), ae^u(sinu + cosu), be^u]}{e^u\sqrt{2a^2 + b^2}}$$

$$= \frac{[a(cosu - sinu), a(sinu + cosu), b]}{\sqrt{2a^2 + b^2}}$$

Differentiating with respect to s,

$$r'' = \kappa n = \frac{[-a(sinu+cosu), a(cosu-sinu), 0]}{\sqrt{2a^2+b^2}} \cdot \frac{1}{s} \qquad (2)$$

Now, taking modulus of both sides, we get

$$\kappa = |r''| = \frac{a\sqrt{2}}{s\sqrt{2a^2+b^2}} \left[\because \frac{1}{\dot{s}} = \frac{1}{s} \text{ from (1)} \right]$$

Again from (2)

$$sr'' = \frac{[-a(sinu + cosu), a(cosu - sinu), 0]}{\sqrt{2a^2 + b^2}}$$

Differentiate with respect to s:

$$s^2 r''' + sr'' = \frac{1}{\sqrt{2a^2 + b^2}} (-a(\cos u - \sin u), \ -a(\sin u + \cos u), \ 0).$$

Now we compute the scalar triple product

$$[r', sr'', s^2 r''' + sr'']$$

$$= \frac{1}{(2a^2 + b^2)^{\frac{3}{2}}} \begin{vmatrix} a(cosu - sinu) & a(sinu + cosu) & b \\ -a(sinu + cosu) & a(cosu - sinu) & 0 \\ -a(cosu - sinu) & -a(sinu + cosu) & 0 \end{vmatrix}$$

or $s^3[r', r'', r'''] = \frac{1}{(2a^2+b^2)^{\frac{3}{2}}} a^2 b^2 [(sinu + cosu)^2 + (cosu - sinu)^2]$

or $s^3 k^2 \tau = \frac{2a^2 b}{(2a^2+b^2)^{\frac{3}{2}}}$

or $s^3 \frac{2a^2}{(2a^2+b^2)} \frac{1}{s^2} \tau = \frac{2a^2 b}{(2a^2+b^2)^{\frac{3}{2}}}$ [on putting the value of κ]

or $\tau = \frac{b}{\sqrt{2a^2+b^2}} \frac{1}{s}$.

Hence, the intrinsic equations of the given curve are

$$\kappa = \frac{\sqrt{2}a}{s\sqrt{2a^2+b^2}} \frac{1}{s}, \tau = \frac{b}{\sqrt{2a^2+b^2}} \frac{1}{s}.$$

5.8 OSCULATING CIRCLE (OR THE CIRCLE OF CURVATURE)

Definition:

Let **P, Q, R** be three points on a smooth curve. The **circle of curvature** at point **P** is defined as the **limiting position** of the circle passing through **P, Q, R** as the points **Q** and **R** approach **P**.

Equivalently:

The osculating circle at point P on any curve is the circle which has three-point contact with the curve at P.

Radius and Centre of the Osculating Circle:

Consider the osculating plane at point **P**. Let the osculating circle be obtained as the intersection of this plane with a sphere:

$$| r - c |= a$$

or

$$(r - c)^2 = a^2$$

Where,

r= position vector of a general point on the sphere,

c= position vector of the centre of the sphere,

a= radius of the sphere.

Fig.

Let the curve be given by $r = r(s)$, where $s$is the arc length.

The points where the curve intersects the sphere satisfy:

$$F(s) = [r(s) - c]^2 - a^2 = 0$$

For three point contact, we require:

$$F(s) = 0, F'(s) = 0, F''(s) = 0$$

These give the conditions:

$$(r - c)^2 = a^2$$

$$(r - c) \cdot r' = 0$$

$$(r - c) \cdot r'' + r' \cdot r' = 0$$

Using Frenet–Serret notation:

$$r' = t, r'' = t' = \kappa n, r' \cdot r' = t^2 = 1$$

The above equations become:

$$(r - c)^2 = a^2 \qquad \text{(i)}$$

$$(r - c) \cdot t = 0 \qquad \text{(ii)}$$

$$(r - c) \cdot n = -\rho \qquad \text{(iii)}$$

where $\rho = \frac{1}{\kappa}$is the radius of curvature.

Equation (ii) shows that $r - c$ is perpendicular to the tangent vector t. Thus, $r - c$ lies in the normal plane at P.

But the osculating circle also lies in the osculating plane. Therefore, $r - c$ must lie along the principal normal n.

Hence, we set:

$$r - c = \lambda n$$

Substituting into (i) and (iii):

$$\lambda^2 = a^2, \lambda = -\rho$$

Thus,

$$a = \rho, r - c = -\rho n$$

So the centre of the osculating circle is:

$$c = r + \rho n$$

Remarks:

(i) The radius of the osculating circle is

$$a = \rho = \frac{1}{\kappa}$$

(ii) The centre of the osculating circle lies on the principal normal at a distance ρ from the point P.

(iii) The position vector of the centre is

$$c = r + \rho n$$

This shows that the osculating circle is the best circular approximation to the curve at point P.

5.9 THE OSCULATING SPHERE (OR THE SPHERE OF CURVATURE)

Definition:

Let **P, Q, R, S** be four points on a smooth space curve. The **sphere of curvature** at point **P** is defined as the **limiting position** of the sphere passing through **P, Q, R, S** as the points **Q, R, S** approach **P**.

Equivalently

The **osculating sphere** at a point **P** is the sphere that has **four point contact** with the curve at **P**. Its radius and centre are called the **radius**and**centre of spherical curvature.**

Radius and Centre of the Osculating Sphere

Let

c= position vector of the centre of the sphere, R= radius of the sphere.

The equation of the sphere is:

$$(r-c)^2 = R^2 \qquad (1)$$

where r is the position vector of a general point.

Let the curve be given by $r = r(s)$, where s is the arc length.

The points where the curve intersects the sphere satisfy:

$$F(s) = [r(s) - c]^2 - R^2 = 0$$

For four point contact, we require:

$$F(s) = 0, F'(s) = 0, F''(s) = 0, F'''(s) = 0$$

These yield the equations:

$$(r-c)^2 = R^2$$

$$(r-c)\cdot r' = 0$$

$$(r-c)\cdot r'' + r'\cdot r' = 0$$

$$(r-c)\cdot r''' + r'\cdot r'' + 2\,r'\cdot r'' = 0$$

We know:

$$r' = t, t\cdot t = 1$$

$$r'' = t' = \kappa n$$

$$r'\cdot r'' = t\cdot(\kappa n) = 0$$

$$r''' = (t')' = (\kappa n)' = \kappa n' + \kappa' n$$

But

$$n' = -\kappa t + \tau b$$

Hence:

$$r''' = \kappa(\tau b - \kappa t) + \kappa' n$$

Substituting these into the earlier equations, we obtain:

$$(r-c)^2 = R^2 \qquad (1)$$

$$(r-c)\cdot t = 0 \qquad (2)$$

$$(r-c)\cdot n = -\rho \qquad (3)$$

$$(r-c)\cdot\{\kappa(\tau b - \kappa t) + \kappa' n\} = 0 \qquad (4)$$

Using (2) and (3), equation (4) reduces to:

$$(r-c)\cdot b = -\sigma\,\rho' \qquad (5)$$

where

$\rho = \frac{1}{\kappa}$ is the radius of curvature, $\sigma = \frac{1}{\tau}$ is the radius of torsion.

From (2), $r - c$ is perpendicular to t, so it lies in the normal plane. Thus, we can write:

$$r - c = \lambda n + \mu b$$

Using (3) and (5):

$$\lambda = -\rho, \mu = -\sigma\rho'$$

Hence:

$$r - c = -\rho n - \sigma\rho' b$$

So, the centre of the osculating sphere is:

$$c = r + \rho n + \sigma\rho' b \qquad (6)$$

Substituting $r - c = \lambda n + \mu b$ into (1):

$$R^2 = \lambda^2 + \mu^2 = \rho^2 + \sigma^2(\rho')^2$$

Using curvature–torsion relations, this may also be written as:

$$R^2 = \frac{\kappa'^2 + \kappa^2\tau^2}{\kappa^2\tau^2}$$

Remark:

If the curvature $\kappa = \frac{1}{\rho}$ is constant, then $\rho' = 0$. Thus ρ is constant, and from equation **(7)** we obtain:

$$R = \rho$$

Similarly, equation **(6)** becomes:

$$c = r + \rho n$$

This shows that the centre of the osculating sphere coincides with the centre of the osculating circle when the curvature is constant.

CHECK YOUR PROGRESS

Q1) Find the equation of the osculating sphere and osculating circle at $(1,2,3)$ on the curve $r = (2t + 1{,}3t^2 + 2{,}4t^3 + 3)$

Q2) Show that the radius R of the sphere of curvature is given by

$$R^2 = \rho^4\sigma^2 r'''^2 - \sigma^2$$

Q3) If R is the radius of the spherical curvature, show that

$$R = \left|\frac{(t \times t'')}{\kappa^2\tau}\right|$$

2 5.10 LET US SUM UP

- A space curve can be studied through its curvature and torsion, which measure how the curve bends and twists in three-dimensional space.
- A cylindrical helix is a curve that lies on a cylinder and cuts its generators at a constant angle; an equivalent characterization is that the ratio $\frac{\tau}{\kappa}$ is constant.
- A circular helix is a helix on a circular cylinder; it is the unique space curve for which both curvature and torsion are constant (non zero).
- The intrinsic equations of a curve express κ and τ as functions of arc length s; they describe the shape of the curve independent of its position or orientation in space.
- The fundamental theorems of space curves state: (i) given continuous functions $\kappa(s)$ and $\tau(s)$, a space curve exists; and (ii) such a curve is unique up to rigid motions (translation and rotation).
- The osculating circle at a point provides the best circular approximation to the curve in the osculating plane; its radius is the radius of curvature (reciprocal of curvature).
- The osculating sphere has four point contact with the curve and gives the best spherical approximation; its centre and radius describe the spherical curvature of the space curve at the point.

K 5.11 ANSWERS TO CHECK YOUR PROGRESS

Answer to Q1: The equation of the curve can be put as

$$r = (2t + 1, 3t^2 + 2, 4t^3 + 3) \quad (1)$$

Evidently $t = 0$ at point $(1,2,3)$ on the curve.

Differentiating equation (1) w.r.t. 't',we get

$$\dot{r} = (2, 6t, 12t^2), \ddot{r} = (0, 6, 24t), \dddot{r} = (0, 0, 24)$$

At $t = 0$,we have

$$\dot{r} = (2, 0, 0), \ddot{r} = (0, 6, 0), \dddot{r} = (0, 0, 24)$$

Let the osculating sphere be

$$(r - c)^2 = R^2 \quad (2)$$

Where c is the position vector of the centre of osculating sphere

$c = \alpha i + \beta j + \gamma k$

Now, for a four-point contact at r,we have on differentiating equation (2), three times) w.r.t. 't',

$(r-c).\dot{r} = 0, (r-c).\ddot{r} + \dot{r}^2 = 0$ and $(r-c).\dddot{r} + 3\dot{r}.\ddot{r} = 0$

At $t = 0$,these reduce to

$[(i+2j+3k) - (\alpha i + \beta j + \gamma k)].2i = 0$ [From $(r-c).\dot{r} = 0$]

i.e., $1 - \alpha = 0$ or $\alpha = 1$

Similarly, from other two equations, we have

$(2-\beta)6 + 4 = 0$

i.e., $\beta = \frac{8}{3}$

and $(3-\gamma)24 + 0 = 0$

i.e., $\gamma = 3$

Also, osculating sphere (2) passes through $(1,2,3)$

$\therefore \left[(i+2j+3k) - \left(i + \frac{8}{3}j + 3k\right)\right]^2 = R^2$ $[\because c = i + \frac{8}{3}j + 3k]$

or $\frac{4}{9} = R^2$ or $R = \frac{2}{3}$

Hence, the equation of the osculating sphere is

$\left\{(xi + yj + zk) - \left(i + \frac{8}{3}j + 3k\right)\right\}^2 = \frac{4}{9}$ $[\because r = xi + yj + zk]$

Or $(x-1)^2 + \left(y - \frac{8}{3}\right)^2 - (z-3)^2 = \frac{4}{9}$

Or $3x^2 + 3y^2 + 3y^2 - 6x - 16y - 18z + 50 = 0$

The osculating circle is the intersection of the osculating plane and osculating sphere, and the equation of this plane is

$[R - r, \dot{r}, \ddot{r}] = 0$

i.e., at $t = 0$,it reduces to

$\{(x-1)i + (y-2)j + (z-3)k\}.12k = 0$ $[\because \dot{r} \times \ddot{r} = 12k]$

i.e., $z - 3 = 0$

Hence, the equation of the osculating circle is

$3x^2 + 3y^2 + 3y^2 - 6x - 16y - 18z + 50 = 0, z - 3 = 0.$

Answer to Q2: We have

$r'' = \kappa n = \frac{1}{\rho} n$

$$\therefore r''' = \frac{1}{\rho}(\tau b - \kappa t) - \frac{\rho'}{\rho^2} n$$

$$= -\frac{1}{\rho^2} t - \frac{\rho'}{\rho^2} n + \frac{1}{\sigma\rho} b \qquad [\because n' = \tau b - \kappa t]$$

Squaring, we get

$$r'''^2 = \frac{1}{\rho^4} + \frac{\rho'^2}{\rho^4} + \frac{1}{\sigma^2\rho^2}$$

or $\rho^2\sigma^2 r'''^2 = \sigma^2 + \rho'^2\sigma^2 + \rho^2 = \sigma^2 + R^2$

or $R^2 = \rho^2\sigma^2 r'''^2 - \sigma^2$

Answer to Q3: We have

$$t' = \kappa n = \frac{1}{\rho} n$$

$$\therefore t'' = \frac{1}{\rho}(\tau b - \kappa t) - \frac{\rho}{\rho^2} n$$

$$= -\frac{1}{\rho^2} t - \frac{\rho'}{\rho^2} n + \frac{1}{\rho\sigma} b.$$

Hence $t \times t'' = -\frac{\rho'}{\rho^2} b - \frac{1}{\rho\sigma} n$ $[\because t \times t = 0, t \times n = b, t \times b = -n]$

Then $|t \times t''| = \sqrt{\left(\frac{\rho'^2}{\rho^4} + \frac{1}{\rho\sigma}\right)}$

$$= \sqrt{\left(\frac{\rho'^2 + \rho^2}{\rho^4\sigma^2}\right)}$$

$$= \frac{R}{\rho^2\sigma}$$

or $R = |\rho^2\sigma t \times t''| = \left|\frac{(t \times t'')}{\kappa^2\tau}\right|$

O 5.12 FURTHER READINGS

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5.13 MODEL QUESTIONS

Q1. Prove that

$x'''^2 + y'''^2 + z'''^2 = \frac{1}{\rho^2\sigma^2} + \frac{1+\rho'^2}{\rho^4} = \frac{1}{\rho^4} + \frac{R^2}{\rho^4\sigma^2}$, where dashes denote differentiation with respect to s.

Q2. Show that the radius of spherical curvature R is given by

$R^2 = \rho^2 + \sigma^2\rho'^2$

Q3. Show that the osculating sphere has at least four-point contact with a curve iff either the curvature or torsion vanishes at P.

Q4. Find the radii of curvature and torsion at the point $\theta = \frac{1}{4}\pi$ of the curve given by

$x = a\cos\theta, y = a\sin\theta, z = a\cos 2\theta$

Q5. For the curve $x = 2abt, y = a^2\log t, z = b^2t^2$

Show that $\rho = \sigma = \frac{(a^2+2b^2t^2)^2}{(2abt)}$

Q6. Find the curvature and torsion of the curves given by

(i) $r = (e^t, -e^t, \sqrt{2}t)$

(ii) $r = \left[2a\sin^{-1}t + 2at\sqrt{1-t^2}, 2at^2, 4at\right]$

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