UNIT 6: INVOLUTE AND EVOLUTE

UNIT STRUCTURE

6.1 Learning Objectives

6.2 Introduction

6.3 Involute of a given space curve

6.4 Evolute of a given curve

6.5 Curvature and Torsion of the Evolute

6.6 Let Us Sum Up

6.7 Answers to Check Your Progress

6.8 Further Reading

6.9 Model Questions

6.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- understand the **concepts of involute and evolute of a curve**, including their geometric interpretation and relationship.
- derive and analyze the **parametric equations of involutes and evolutes** for a given space curve.
- study the **properties of evolutes**, particularly their connection with the center of curvature and envelope of normal lines.
- examine the **relationship between curvature and evolute**, and how the evolute represents the locus of centers of curvature.
- apply the theory to **solve problems involving specific curves** (such as circles, helices, and plane curves) and interpret the results geometrically.

6.2 INTRODUCTION

In differential geometry, the concepts of **involute** and **evolute** arise in the study of the geometric behaviour of curves and their curvature. These curves are closely related

and provide deep insight into how a given curve bends in space.

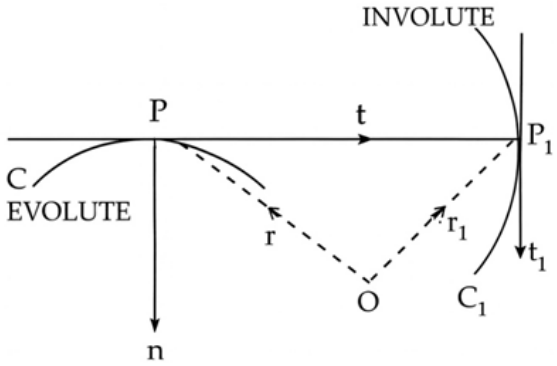
An **evolute** of a curve is defined as the *locus of the centres of curvature* of the original curve. In other words, as one moves along a curve, each point has a corresponding centre of curvature (the centre of the osculating circle), and the collection of all such centres forms the evolute. Thus, the evolute captures how the curvature of the original curve changes from point to point.

On the other hand, an **involute** of a curve can be visualized as the path traced by the end of a taut string as it is unwound from the given curve. This construction provides a natural geometric interpretation and shows that the involute is always orthogonal to the tangents of the original curve. Importantly, the involute and evolute are inverse constructions of each other: the original curve is the involute of its evolute.

These concepts are fundamental in understanding curvature, envelopes of normals, and geometric constructions of curves. They also have practical applications in engineering, particularly in the design of gear teeth and mechanical systems where smooth transmission of motion is required.

Definition: If there be one-one correspondence between points of two curves C and C_1 such that the tangent at any point C is a normal to the corresponding point of C_1, then C_1 is called an involute of C and C is called an evolute of C_1.

6.3 INVOLUTE OF A GIVEN SPACE CURVE



Let $\boldsymbol{r} = \boldsymbol{r}(s)$ be the given curve C and let C_1 be the involute of C . The quantities belonging to C_1 will be distinguished by using the suffix unity. Then the position vector $\boldsymbol{r}$ of any point P_1 on C_1 is given by

$$\boldsymbol{r}_1 = \boldsymbol{r} + \lambda \boldsymbol{t} \qquad (1)$$

where λ is to be determined.

Differentiating (1) w.r.t. s_1, we get

$$\boldsymbol{t}_1 = (\boldsymbol{t} + \lambda' \boldsymbol{t} + \lambda \kappa \boldsymbol{n}) \frac{ds}{ds_1} \qquad (2)$$

By definition $\boldsymbol{t}$ is perpendicular to $\boldsymbol{t}_1$, so taking dot product of both sides of (2) with $\boldsymbol{t}$ and using $\boldsymbol{t} \cdot \boldsymbol{t}_1 = 0$, we get

$$(1 + \lambda') \frac{ds}{ds_1} = 0$$

or

$$1 + \lambda' = 0$$

which on integration gives $s + \lambda = c$ or $\lambda = c - s$, where c is constant of integration.

$$\therefore \quad \boldsymbol{r}_1 = \boldsymbol{r} + (c - s)\boldsymbol{t} \qquad (3)$$

which is the required equation of involute of C_1 of C.

Again putting the value of λ in (2), the unit tangent vector $\boldsymbol{t}_1$ is given by

$$\boldsymbol{t}_1 = (c - s)\kappa \frac{ds}{ds_1} \boldsymbol{n} \qquad [\because \lambda' = -1] \qquad (4)$$

From (4), we observe that $\boldsymbol{t}_1$ is parallel to $\boldsymbol{n}$. Taking the positive direction along the involute such that

$$\boldsymbol{t}_1 = \boldsymbol{n}$$

Thus from (4)

$$\frac{ds_1}{ds} = \kappa(c - s)$$

Curvature κ_1 and τ_1 Torsion of the Involute

Differentiating $\boldsymbol{t}_1 = \boldsymbol{n}$ w.r.t. s_1, we get

$$\kappa_1 \boldsymbol{n}_1 = (\tau \boldsymbol{b} - \kappa \boldsymbol{t}) \frac{ds}{ds_1}$$

$$= \frac{\tau \boldsymbol{b} - \kappa \boldsymbol{t}}{\kappa(c - s)}$$

Therefore, on squaring both sides, we get

$$\kappa_1^2 = \frac{\tau^2 + \kappa^2}{\kappa^2 (c - s)^2} \quad or \quad \kappa_1 = \frac{(\tau^2 + \kappa^2)^{\frac{1}{2}}}{\kappa(c - s)}$$

Obviously the unit principal normal to involute is

$$\boldsymbol{n}_1 = \frac{(\tau\boldsymbol{b} - \kappa\boldsymbol{t})}{\kappa\kappa_1(c-s)} = \frac{(\tau\boldsymbol{b} - \kappa\boldsymbol{t})}{(\tau^2 - \kappa^2)^{\frac{1}{2}}} \qquad (5)$$

$$\boldsymbol{b}_1 = \boldsymbol{t}_1 \times \boldsymbol{n}_1 = \boldsymbol{n} \times \boldsymbol{n}_1 = \frac{(\kappa\boldsymbol{b} + \tau\boldsymbol{t})}{\kappa\kappa_1(c-s)} = \frac{\kappa\boldsymbol{b} + \tau\boldsymbol{t}}{(\tau^2 + \kappa^2)^{\frac{1}{2}}} \qquad (6)$$

Now differentiating (6) w.r.t. s, we get

$$-\tau_1\boldsymbol{n}_1\frac{ds_1}{ds} = \frac{(\kappa^2+\tau^2)(\kappa'\boldsymbol{b} + \tau'\boldsymbol{t} + \kappa\boldsymbol{b}' + \tau\boldsymbol{t}') - (\kappa\boldsymbol{b} + \tau\boldsymbol{t})(\kappa\kappa' + \tau\tau')}{(\kappa^2+\tau^2)^{\frac{3}{2}}}$$

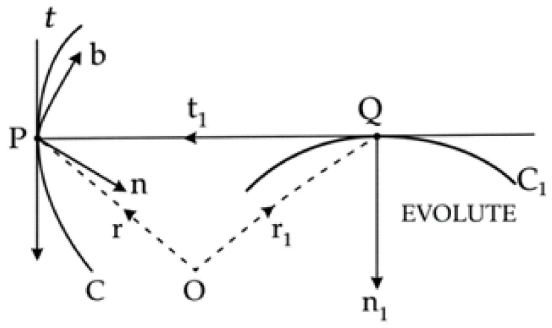
Squaring both sides and putting $\frac{ds_1}{ds} = \kappa(c-s)$, we get

$$\tau_1^2\kappa^2(c-s)^2 = \frac{(\kappa\tau' - \kappa'\tau)^2(\kappa^2+\tau^2)}{(\kappa^2+\tau^2)^3}$$

or

$$\tau_1 = \frac{(\kappa\tau' - \kappa'\tau)}{\kappa(c-s)(\kappa^2+\tau^2)}$$

6.4 EVOLUTE OF A GIVEN CURVE



In this case we will find a curve C_1 such that C is involute of C_1 and consequently C_1 will be evolute of C.

Let $\boldsymbol{r} = \boldsymbol{r}(s)$ be equation of curve C. We shall use the suffix unit for quantities belonging to curve C_1. Let $\boldsymbol{r}_1$ be the position vector of any point Q on C_1 and that of corresponding point P on C be $\boldsymbol{r}$. Since the tangents to curve C_1 are normals to curve C, the vector PQ must lie in the normal plane to the curve C at P.

$$\Rightarrow \boldsymbol{r}_1 - \boldsymbol{r} = \lambda\boldsymbol{n} + \mu\boldsymbol{b} \Rightarrow \boldsymbol{r}_1 = \boldsymbol{r} + \lambda\boldsymbol{n} + \mu\boldsymbol{b} \qquad (1) \quad [\because PQ = \boldsymbol{r}_1 - \boldsymbol{r} = \lambda\boldsymbol{n} + \mu\boldsymbol{b}]$$

where the values of λ and μ are to be determined.

Now differentiating (1) w.r.t. s_1, we get

$$\frac{d\boldsymbol{r}_1}{ds_1} = [\boldsymbol{r}' + \lambda'\boldsymbol{n} + \lambda\boldsymbol{n}' + \mu'\boldsymbol{b} + \mu\boldsymbol{b}']\frac{ds}{ds_1}$$

$$\Rightarrow \boldsymbol{t}_1 = [\boldsymbol{t} + \lambda'\boldsymbol{n} + \lambda(\tau\boldsymbol{b} - \kappa\boldsymbol{t}) + \mu'\boldsymbol{b} + \mu(-\tau\boldsymbol{n})]\frac{ds}{ds_1}$$

$$= [(1 - \kappa\lambda)\boldsymbol{t} + (\lambda' - \mu\tau)\boldsymbol{n} + (\mu' + \lambda\tau)\boldsymbol{b}]\frac{ds}{ds_1} \qquad (2)$$

Since $\boldsymbol{t}_1$ lies in the normal plane at P to the curve C, so it must be parallel to $\lambda\boldsymbol{n} + \mu\boldsymbol{b}$ and $\boldsymbol{t}_1 = (\lambda\boldsymbol{n} + \mu\boldsymbol{b})P$. Therefore on comparing the coefficients of this with that of relation (2), we get

$$1 - \kappa\lambda = 0 \quad or \quad \lambda = \frac{1}{\kappa} = \rho \quad and \quad \lambda' - \mu\tau = P\lambda, \qquad \mu' + \lambda t = P\mu$$

$$\Rightarrow \frac{\lambda' - \mu\tau}{\lambda} = \frac{\mu' + \lambda\tau}{\mu}, i.e.\ \tau = \frac{\lambda'\mu - \lambda\mu'}{\lambda^2 + \mu^2} = \frac{d}{ds}tan^{-1}\left(\frac{\lambda}{\mu}\right)$$

$$\Rightarrow \tau = \frac{d}{ds}tan^{-1}\left(\frac{\lambda}{\mu}\right) \qquad (3)$$

Integrating (3), we get

$$a + \int \tau ds = tan^{-1}\left(\frac{\lambda}{\mu}\right)$$

where a is an arbitrary constant.

$$\Rightarrow \lambda = \mu\, tan\left(\int \tau ds + a\right)$$

$$\Rightarrow \mu = \lambda\, cot\left(\int \tau ds + a\right)$$

$$\Rightarrow \mu = \rho\, cot\left(\int \tau ds + a\right)$$

Therefore on putting the values of λ and μ in (1), we get

$$\boldsymbol{r}_1 = \boldsymbol{r} + \rho\boldsymbol{n} + \rho\, cot\left(\int \tau ds + a\right)\boldsymbol{b} \qquad (4)$$

which gives the required equation of evolute C_1 of C. In case we give different values of a, we shall have an infinite system of evolutes of given curve.

In case we assume

$$\int \tau ds = \phi(s) \quad and \quad c - \frac{\pi}{2} = a$$

Equation (4) may be put as

$$\boldsymbol{r}_1 = \boldsymbol{r} + \rho \boldsymbol{n} - \rho \, tan[\phi(s) + c]\boldsymbol{b}$$

6.5 CURVATURE AND TORSION OF THE EVOLUTE

From the previous section, the position vector $\boldsymbol{r}_1$ on the evolute is given by

$$\boldsymbol{r}_1 = \boldsymbol{r} + \rho \boldsymbol{n} - \rho \tan[\phi(s) + c]\boldsymbol{b}$$

or

$$\boldsymbol{r}_1 = \boldsymbol{r} + \rho \boldsymbol{n} - \rho \tan\psi \, \boldsymbol{b} \quad [\text{putting } \phi + c = \psi]$$

Differentiating above equation w.r.t. s, we get

$$\boldsymbol{t}_1 \left(\frac{ds_1}{ds}\right) = \frac{d\boldsymbol{r}_1}{ds} = (\boldsymbol{r} + \rho \boldsymbol{n} - \rho \tan\psi \, \boldsymbol{b})$$

$$= \boldsymbol{t} + \rho(\tau \boldsymbol{b} - \kappa \boldsymbol{t}) + \boldsymbol{n}\rho' - \rho \tan\psi \boldsymbol{b} + \tau \boldsymbol{n} \rho \tan\psi - \rho \sec^2\psi \, \tau \boldsymbol{b}$$

$$= [\rho' + \rho\tau \tan\psi]\boldsymbol{n} - [\rho' \tan\psi + \rho\tau \tan^2\psi]\boldsymbol{b}$$

Squaring both sides, we get

$$\left(\frac{ds_1}{ds}\right)^2 = (\rho' + \rho\tau \tan\psi)^2(1 + \tan^2\psi)$$

$$\frac{ds_1}{ds} = \frac{\kappa\tau \sin\psi - \kappa\cos\psi}{\kappa^2 \cos^2\psi} \qquad \left(\because \rho' = \frac{\kappa'}{\kappa^2}\right)$$

Thus

$$\boldsymbol{t}_1 = \cos\psi \, \boldsymbol{n} - \sin\psi \, \boldsymbol{b} \qquad (1)$$

Now differentiating equation (1) w.r.t. s, we get

$$\frac{d\boldsymbol{t}_1}{ds} = \frac{d\boldsymbol{t}_1}{ds_1} \cdot \frac{ds_1}{ds} = [-\sin\psi \, \boldsymbol{n}\tau + \cos\psi(\tau \boldsymbol{b} - \kappa \boldsymbol{t}) - \cos\psi \, \tau \boldsymbol{b} + \sin\psi \, \tau \boldsymbol{n}] \quad \left[\because \frac{d\psi}{ds} = \tau\right]$$

or

$$\frac{ds_1}{ds} \kappa_1 \boldsymbol{n}_1 = -\kappa \cos\psi \, \boldsymbol{t}$$

This equation shows that the principal normal to the evolute is parallel to $\mathbf{t}$. We may take the positive direction along the evolute such that $n_1 = -t$.

Therefore

$$\kappa_1 = \kappa cos\ \psi \frac{ds}{ds_1} = \frac{\kappa^3 cos^3\psi}{\kappa\tau sin\ \psi - \kappa' cos\ \psi}$$

To find torsion, we have

$$\boldsymbol{b}_1 = \boldsymbol{t}_1 \times \boldsymbol{n}_1 = [\cos\psi\, \boldsymbol{n} - \sin\psi\, \boldsymbol{b}\,] \times (-\boldsymbol{t})$$

or

$$\boldsymbol{b}_1 = \cos\psi \boldsymbol{b} + \sin\psi\, \boldsymbol{n} \quad (2)$$

Differentiating (2) w.r.t. s, we get

$$\frac{db_1}{ds_1}, \frac{ds_1}{ds} = [-\sin\psi\tau b - \cos\psi\tau n + \cos\psi\tau n + \sin\psi(\tau b - kt)]$$

or

$$-\tau_1 \boldsymbol{n}_1 \frac{ds_1}{ds} = -\kappa \sin\psi\, \boldsymbol{t}$$

or

$$\tau_1 = -\kappa \sin\psi\, \frac{ds}{ds_1}$$

or

$$\tau_1 = \frac{\kappa^3 \sin\psi\ \cos^2\psi}{\kappa\tau sin\ \psi - \kappa' cos\ \psi}$$

Also

$$\frac{\tau_1}{\kappa_1} = -\tan\psi \quad (\because \boldsymbol{n}_1 = -\boldsymbol{t})$$

CHECK YOUR PROGRESS

Q 1. Find the involute of a circular helix.

Q 2. Show that the distance between corresponding points of two involutes is constant.

Q 3. Prove that the involute of a circular helix are plane curves.

Q 4. The locus of the centre of curvature is an evolute only when the curve is plane.

Q 5. Find the involutes and evolutes of the circular helix

$$x = a\cos\theta, y = a\sin\theta, z = a\theta\tan\alpha.$$

6.6 LET US SUM UP

- An **involute** of a curve is obtained by tracing the path of the end of a taut string as it is unwound from the given curve, while the **evolute** is the locus of the centers of curvature of that curve.
- The involute and evolute are geometrically related: **the original curve is the evolute of all its involutes**, and conversely, involutes can be constructed from

the evolute.

- The concept of **curvature** plays a central role, as the evolute represents the centers of curvature corresponding to each point on the curve.
- The parametric equations of involutes can be derived using arc length, and they depend on the tangent direction of the original curve.
- Evolutes often exhibit **cusps (singular points)**, which correspond to points where the curvature of the original curve is extremal.
- These concepts have practical applications in areas such as **mechanical design (gear teeth profiles)**, physics, and the study of geometrical properties of curves in differential geometry.

K 6.7 ANSWERS TO CHECK YOUR PROGRESS

Answer to Q 1. The equation of circular helix is

$$\boldsymbol{r} = [a\cos u, a\sin u, bu]$$

$$\dot{\boldsymbol{r}} = [-a\sin u, a\cos u, b], \qquad \dot{s} = |\dot{\boldsymbol{r}}| = \sqrt{a^2+b^2}$$

$$\boldsymbol{r}' = \frac{\boldsymbol{r}}{\dot{s}} = \frac{1}{\sqrt{a^2+b^2}}[-a\sin u, a\cos u, b]$$

$$\therefore \quad s = \int_0^u |\dot{\boldsymbol{r}}|du = \int_0^u \sqrt{a^2+b^2}\,du$$

Again equation of an involute is

$$\boldsymbol{r}_1 = \boldsymbol{r} + (c-s)\boldsymbol{t}$$

$$= [a\cos u, a\sin u, bu] + (c-s)\frac{1}{\sqrt{a^2+b^2}}[-a\sin u, a\cos u, b]$$

$$\boldsymbol{r}_1 = \left[a\cos u - \frac{a(c-s)}{\sqrt{a^2+b^2}}\sin u,\ a\sin u + \frac{a(c-s)}{\sqrt{a^2+b^2}}\cos u,\ bu + \frac{b(c-s)}{\sqrt{a^2+b^2}}\right]$$

where

$$s = \sqrt{a^2+b^2}\,u$$

Answer to Q 2. Let $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ denote the position vectors of the corresponding points on the two involutes determined by $c = c_1$ and $c = c_2$. Then

$$\boldsymbol{r}_1 = \boldsymbol{r} + (c_1 - s)\boldsymbol{t}$$

and

$$\boldsymbol{r}_2 = \boldsymbol{r} + (c_2 - s)\boldsymbol{t}$$

Hence

$$\boldsymbol{r}_1 - \boldsymbol{r}_2 = (c_1 - c_2)\boldsymbol{t}$$

so that

$$|\boldsymbol{r}_1 - \boldsymbol{r}_2| = |c_1 - c_2| = \text{constant}$$

Answer to Q 3. For a circular helix, $\frac{\kappa}{\tau} = a$ (constant)

$$\kappa' = a\tau'$$

Again torsion of an involute of a given curve $r = r(s)$ is given by

$$\tau_1 = \frac{\kappa\tau' - \kappa'\tau}{\kappa(c-s)(\tau^2 + \kappa^2)} \qquad (1)$$

On putting the values of κ and κ' in terms of τ and τ' in (1), we have

$$\tau_1 = 0$$

i.e. the torsion for the involute is zero and hence the involute is a plane curve.

Answer to Q 4. The position vector of a current point on the evolute is given by

$$\boldsymbol{r}_1 = \boldsymbol{r} + \rho\boldsymbol{n} - \rho\tan(\phi + c)\boldsymbol{b} \qquad (1)$$

where c is an arbitrary constant and for its various values we get an infinite system of evolutes.

Also the locus of the centre of curvature is given by

$$\boldsymbol{c} = \boldsymbol{r} + \rho\boldsymbol{n} \qquad (2)$$

we observe that equations (1) and (2) will coincide, i.e. the locus of the centre $\boldsymbol{c}$ is an evolute $\boldsymbol{r}_1$, then we must have

$$\tan(\phi + c) = 0$$

i.e.

$$\phi + c = m\pi$$

or

$$\frac{d\phi}{ds} = 0$$

i.e.

$$\tau = 0 \qquad \left(\because \frac{d\phi}{ds} = \tau\right)$$

Hence the curve must be a plane.

Answer to Q 5. The equation of the circular helix can be put as

$$\boldsymbol{r} = (a\cos\theta, a\sin\theta, a\theta\tan\alpha)$$

$$\therefore \quad \boldsymbol{t} = \boldsymbol{r}' = (-a\sin\theta, a\cos\theta, a\tan\alpha)\frac{d\theta}{ds} \qquad (1)$$

Taking module of the both sides, we get

$$1 = \sqrt{a^2 \sec^2 \alpha}\frac{d\theta}{ds}$$

or

$$\frac{ds}{d\theta} = a \sec \alpha \qquad (2)$$

$$s = \int_0^{\theta} a \sec \alpha d\theta = a\theta \sec \alpha$$

We know that involute of a space curve $\boldsymbol{r} = \boldsymbol{r}(s)$ is given by

$$\boldsymbol{r}_1 = \boldsymbol{r} + (c - s)\boldsymbol{t}$$

$$\therefore r_1 = (a\cos\theta, a\sin\theta, a\theta\tan\alpha) + (c - a\theta\sec\alpha)\cos\alpha\,(-\sin\theta, \cos\theta, \tan\alpha)$$

In case $\boldsymbol{r}_1 = (x, y, z)$, then the equation of involutes are given by

$$\left.\begin{aligned} x &= a\cos\theta - \cos\alpha\sin\theta\,(c - a\theta\sec\alpha) \\ y &= a\sin\theta + \cos\alpha\cos\theta\,(c - a\theta\sec\alpha) \\ z &= a\theta\tan\alpha + \sin\alpha\,(c - a\theta\sec\alpha) \end{aligned}\right\}$$

Again equation of evolutes of space curve $\boldsymbol{r} = \boldsymbol{r}(s)$ are given by

$$\boldsymbol{r}_1 = \boldsymbol{r} + \boldsymbol{n} - \rho\tan(\phi + c)\boldsymbol{b} \qquad (4)$$

where

$$\phi = \int \tau ds \qquad (5)$$

From (1) and (2), we have

$$\boldsymbol{t} = \cos\alpha(-\sin\theta, \cos\theta, \tan\alpha)$$

$$\therefore \quad \boldsymbol{t}' = \kappa\boldsymbol{n} = \frac{\cos^2\alpha}{a}(-\cos\theta, -\sin\theta, 0) \qquad [by\ (2)]$$

which gives

$$\kappa = \frac{\cos^2\alpha}{a}$$

or

$$\rho = a\sec^2\alpha$$

and

$$\boldsymbol{n} = (-\cos\theta, -\sin\theta, 0)$$

Therefore

$$\boldsymbol{b} = \boldsymbol{t} \times \boldsymbol{n} = \cos\alpha(\sin\theta\tan\alpha, -\cos\theta\tan\alpha, 1)$$

$$\therefore \quad \boldsymbol{b}' = -\tau\boldsymbol{n} = \frac{\cos^2\alpha}{a}.\tan\alpha(\cos\theta, -\sin\theta, 0)$$

which gives

$$\tau = \frac{1}{a}\sin\alpha\cos\alpha$$

$$\therefore \phi = \int \tau ds = \int \frac{1}{a}\sin\alpha\cos\alpha ds = \frac{s}{a}\sin\alpha\cos\alpha$$

Now let $\boldsymbol{r}_1 = (x, y, z)$ be the position vector of a current point on the evolute, then by making these substitutions in (4), we get

$$x = a\cos\theta - a\cos\theta\sec^2\alpha - a\sec^2\alpha\tan(c + \theta\sin\alpha)\sin\theta\sin\alpha$$

or

$$x = a\cos\theta(1 - \sec^2\alpha) - a\tan\alpha\sec\alpha\sin\theta\tan(c + \theta\sin\alpha)$$

or

$$\left.\begin{aligned} x &= -a\cos\theta\tan^2\alpha - a\tan\alpha\sec\alpha\sin\theta\tan(c + \theta\sin\alpha) \\ y &= -a\sin\theta\tan^2\alpha + a\tan\alpha\sec\alpha\cos\theta\tan(c + \theta\sin\alpha) \\ z &= a\theta\tan\alpha - a\tan\alpha\sec\alpha\tan(\theta\sin\alpha + c) \end{aligned}\right\} \quad (6)$$

O 6.8 FURTHER READINGS

1. Wilmore T. J. An Introduction to Differential Geometry, Dover Publications Inc., 2012
2. Weatherburn, C. E. Differential Geometry of Three Dimensions, Cambridge University Press, 2016
3. Pressley, A. Elementary Differential Geometry. Springer, 2002
4. Gupta, P. P., Malik, G. S., Pundir, S. K. Differential Geometry, Pragati Prakashan, 2010

M 6.9 MODEL QUESTIONS

1. Show that the torsion of an involute of a curve is equal to

$$\frac{\rho(\sigma\rho' - \sigma\rho)}{(\rho^2 + \sigma^2)(c - s)}$$

2. Find the involutes and evolutes of the twisted cubic given by

$$x = u, \quad y = u^2, \quad z = u^3.$$

3. Show that the tangents to two different evolutes corresponding to the values a_1 and a_2 of the parameter a drawn from the same point of the given curve are inclined to each other at a constant angle $a_1 - a_2$.