UNIT 7: CONCEPT OF A SURFACE AND ENVELOPES AND DEVELOPABLE

UNIT STRUCTURE

7.1 Learning Objectives

7.2 Introduction

7.3 Class of a surface

7.4 Regular (or Ordinary) Point and Singularities on a surface

7.5 Transformation of Parameter

7.6 Geometrical Significance of Proper Transformation

7.7 Curvilinear Equations of the curve on the surface

7.8 Parametric curve

7.9 Tangent plane and Normal

7.10 Family of surfaces

7.11 Developable Surface

7.12 Let Us Sum Up

7.13 Answers to Check Your Progress

7.14 Further Reading

7.15 Model Questions

7.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- Understand and define the concept of a surface in three-dimensional space, including its analytical and geometrical representations.
- Classify different types of surfaces and interpret their properties using parametric and implicit equations.
- Explain the notion of a one-parameter family of surfaces and derive the equation of the envelope of such a family.

- Apply the conditions for the existence of an envelope and interpret its geometric significance.
- Understand the concept of developable surfaces and identify different types such as cylinders, cones, and tangent developables.
- Derive and analyze the equations of developable surfaces and examine their properties in differential geometry.
- Apply theoretical concepts of envelopes and developables to solve geometrical problems and understand their practical relevance.

7.2 INTRODUCTION

The study of surfaces constitutes an important branch of Differential Geometry, where geometrical objects are analyzed using the tools of calculus. A **surface** may be regarded as a two-dimensional locus of points in three-dimensional space, which can be represented either implicitly or parametrically. The concept of a surface helps in understanding various geometric properties such as tangent planes, normal lines, and curvature.

Another important concept discussed in this unit is that of an **envelope**. An envelope is defined as a curve or surface that is tangent to each member of a given family of curves or surfaces. It represents the limiting position of the family and provides insight into the behavior and interaction of geometrical forms.

The unit also introduces **developable surfaces**, which are special types of surfaces that can be unfolded or developed onto a plane without any distortion. Common examples include cylinders, cones, and tangent surfaces. These surfaces are of great importance in practical applications such as engineering design, architecture, and manufacturing processes.

Thus, this unit lays the foundation for understanding the analytical and geometrical aspects of surfaces, envelopes, and developables, which are essential for further study in differential geometry.

7.3 CLASS OF A SURFACE

If the equation of the surface be $x = f(u,v); y = g(u,v); z = h(u,v)$, then the

surface is said to be of class r if the functions f, g, h are single valued as well continuous and possess partial derivatives of r^{th} order.

Remark: We represent the partial differentiations with respect to the parameters u and v by the suffixes 1 and 2 respectively.

i.e.

$$\boldsymbol{r}_1 = \frac{\partial \boldsymbol{r}}{\partial u}, \boldsymbol{r}_2 = \frac{\partial \boldsymbol{r}}{\partial v}, \boldsymbol{r}_{11} = \frac{\partial^2 \boldsymbol{r}}{\partial u^2}, \boldsymbol{r}_{12} = \frac{\partial^2 \boldsymbol{r}}{\partial u \partial v}, \boldsymbol{r}_{22} = \frac{\partial^2 \boldsymbol{r}}{\partial v^2}$$

7.4 REGULAR (OR ORDINARY) POINT AND SINGULARITIES ON A SURFACE

Consider a point P on the surface whose position vector is $\boldsymbol{r}$.

Then

$$\boldsymbol{r} = (x, y, z) = [x(u, v), y(u, v), z(u, v)]$$

$$\therefore \quad \boldsymbol{r}_1 = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right); \quad \boldsymbol{r}_2 = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right)$$

Now the point P is called regular point or ordinary point if

$$\boldsymbol{r}_1 \times \boldsymbol{r}_2 \neq \boldsymbol{0}$$

i.e., if rank of the matrix

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{pmatrix}$$

is 2 (i.e. at least one of the second order minor does not vanish).

In case $\boldsymbol{r}_1 \times \boldsymbol{r}_2 = \boldsymbol{0}$ at a point P on the surface, then the point P is called singularity of the surface. We have the following two types of singularities.

1. Essential Singularities: These singularities are due to the nature (geometrical features) of the surface and are independent of the choice of parametric representation of the surface, e.g. the vertex of the cone is an essential singularity.

2. Artificial Singularities: These singularities are due to particular choice of parametric representation e.g. the pole in the plane, referred to polar coordinates is an artificial singularity. Let $\boldsymbol{r}$ be the position vector of any point in the plane then referred to polar coordinates (r, θ), we have

$$\boldsymbol{r} = (r\cos\theta, r\sin\theta, 0)$$

If $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ represents the partial derivative w.r.t. r and θ respectively, then

$$\boldsymbol{r}_1 = (\cos\theta, \sin\theta, 0)$$

$$\boldsymbol{r}_2 = (-r\sin\theta, r\cos\theta, 0)$$

$$\therefore \quad \boldsymbol{r}_1 \times \boldsymbol{r}_2 = \begin{vmatrix} \boldsymbol{i} & j & k \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \boldsymbol{rk} = 0,$$ when $\boldsymbol{r} = \boldsymbol{0}$, i.e. at the pole.

Now consider the Cartesian system, where $\boldsymbol{r} = (x, y, 0)$

$\Rightarrow \boldsymbol{r}_1 = (1,0,0), \boldsymbol{r}_2 = (0,1,0) \Rightarrow \boldsymbol{r}_1 \times \boldsymbol{r}_2 = \boldsymbol{k} \neq \boldsymbol{0}$, at the pole.

Thus there is an artificial singularity at $\boldsymbol{r} = \boldsymbol{0}$.

7.5 TRANSFORMATION OF PARAMETER

We can transform the set of parameters u and v into another set of parameters U and V by the transformation of the form

$$U = \phi(u, v), \qquad V = \psi(u, v) \qquad (1)$$

Here we shall allow only those transformations which transforms regular points into regular points. The condition for regular points in parameters u and v is $\boldsymbol{r}_1 \times \boldsymbol{r}_2 = \boldsymbol{0}$.

Again

$$\boldsymbol{r}_1 = \frac{\partial \boldsymbol{r}}{\partial u} = \frac{\partial \boldsymbol{r}}{\partial U}\frac{\partial U}{\partial u} + \frac{\partial \boldsymbol{r}}{\partial V}\frac{\partial V}{\partial u}$$

$$= \frac{\partial \boldsymbol{r}}{\partial U}\frac{\partial \phi}{\partial u} + \frac{\partial \boldsymbol{r}}{\partial V}\frac{\partial \psi}{\partial u}$$

$$\boldsymbol{r}_2 = \frac{\partial \boldsymbol{r}}{\partial v} = \frac{\partial \boldsymbol{r}}{\partial U}\frac{\partial U}{\partial v} + \frac{\partial \boldsymbol{r}}{\partial V}\frac{\partial V}{\partial v}$$

$$= \frac{\partial \boldsymbol{r}}{\partial U}\frac{\partial \phi}{\partial v} + \frac{\partial \boldsymbol{r}}{\partial V}\frac{\partial \psi}{\partial v}$$

Now

$$\boldsymbol{r}_1 \times \boldsymbol{r}_2 = \left(\frac{\partial \boldsymbol{r}}{\partial U} \times \frac{\partial \boldsymbol{r}}{\partial V}\right)\frac{\partial \phi}{\partial u}\frac{\partial \psi}{\partial v} + \left(\frac{\partial \boldsymbol{r}}{\partial V} \times \frac{\partial \boldsymbol{r}}{\partial U}\right)\frac{\partial \psi}{\partial u}\frac{\partial \phi}{\partial v}$$

$$= \frac{\partial \boldsymbol{r}}{\partial U} \times \frac{\partial \boldsymbol{r}}{\partial V}\left(\frac{\partial \phi}{\partial u}\frac{\partial \psi}{\partial v} - \frac{\partial \psi}{\partial u}\frac{\partial \phi}{\partial v}\right)$$

$$= \frac{\partial \boldsymbol{r}}{\partial U} \times \frac{\partial \boldsymbol{r}}{\partial V}\frac{\partial(\phi, \psi)}{\partial(u, v)} \qquad (2)$$

In case $\boldsymbol{r}_1 \times \boldsymbol{r}_2 \neq \boldsymbol{0}$, then from (2), we have

$$\frac{\partial(\phi,\psi)}{\partial(u,v)} \neq 0 \qquad (3)$$

in some domain D.

The transformation (1) under condition (3) is called proper transformation.

7.6 GEOMETRICAL SIGNIFICANCE OF PROPER TRANSFORMATION

Let

$$\boldsymbol{r} = \boldsymbol{r}(u,v) \qquad (1)$$

be the equation of surface and

$$U = \phi(u,v), \qquad V = \psi(u,v) \qquad (2)$$

be the proper transformation of parameters u and v to U and V, then

$$\frac{\partial(\phi,\psi)}{\partial(u,v)} \neq 0 \qquad (3)$$

Let the domain of u, v and U, V be respectively D and D'. Condition (3) is necessary and sufficient condition that the transformation (2) can be inverted near any point on D'. We should note that transformation is locally one-one but there may not exist inverse transformation defined throughout the domain D.

7.7 CURVILINEAR EQUATIONS OF THE CURVE ON THE SURFACE

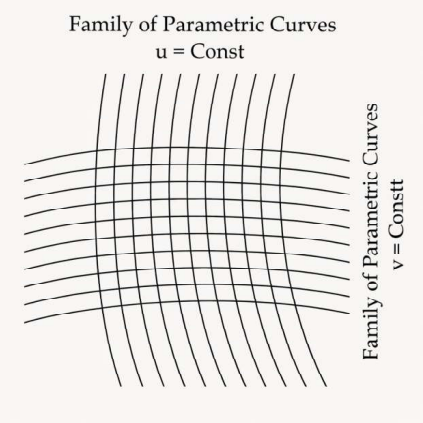
Consider a surface $\boldsymbol{r} = \boldsymbol{r}(u,v)$ and let u and v be functions of single parameter t, then $\boldsymbol{r}$ is function of single parameter t, so its locus is a curve lying on the surface $\boldsymbol{r} = \boldsymbol{r}(u,v)$.

Let $u = u(t),\ v = v(t)$; then $\boldsymbol{r} = \boldsymbol{r}\big(u(t),v(t)\big)$ represents a curve on the surface $\boldsymbol{r} = \boldsymbol{r}(u,v)$.

Again the surface $\boldsymbol{r} = \boldsymbol{r}(u,v)$ and curve $u = u(t),\ v = v(t)$ lie in the same domain and if m and n are classes of the surface and the curve respectively then the class of the curve on the surface $\boldsymbol{r} = \boldsymbol{r}\big(u(t),v(t)\big)$ is the smaller of m and n.

The equations $u = u(t),\ v = v(t)$ are called curvilinear equations of the curve.

7.8 PARAMETRIC CURVE



Let $\boldsymbol{r} = \boldsymbol{r}(u, v)$ be the equation of the surface. Now by keeping $u = c$(constant) or $v = c$(constant), we get curves of special importance and are called parametric curves. If $v = c$ and as u varies, the point $\boldsymbol{r} = \boldsymbol{r}(u, c)$ describes a parametric curve called the u –curve or the parametric curve $v = c$. Similarly if $u = c$, then as v varies, the point $\boldsymbol{r} = \boldsymbol{r}(c, v)$ describe a parametric curve called v –curve or the parametric curve $u = c$. For u –curve the tangent to the curve in the sense of u –increasing is along the vector $\boldsymbol{r}_1$ and for v –curve the tangent to the curve in the sense of v –increasing is along the vector $\boldsymbol{r}_2$.

Since $\boldsymbol{r}_1 \times \boldsymbol{r}_2 \neq \boldsymbol{0}$, thus the parametric curves of different systems cannot touch each other.

The two parametric curves through a point $P(\boldsymbol{r})$ are called orthogonal at P if $\boldsymbol{r}_1 \cdot \boldsymbol{r}_2 = 0$ and they are called orthogonal if this condition is obeyed throughout the domain for all values of u and v.

7.9 TANGENT PLANE AND NORMAL

To find the equation of the tangent plane to a surface at a given point $P(\boldsymbol{r})$.

Let $\boldsymbol{r} = \boldsymbol{r}(u, v)$ be the equation of the surface in terms of parameter u and v.

$$\therefore \frac{d\boldsymbol{r}}{ds} = \frac{\partial \boldsymbol{r}}{\partial u}\frac{du}{ds} + \frac{\partial \boldsymbol{r}}{\partial v}\frac{dv}{ds} \quad or \quad \boldsymbol{t} = \boldsymbol{r}_1\frac{du}{ds} + \boldsymbol{r}_2\frac{dv}{ds}$$

Now for the curve $u = c_1$(constant), $r_1 = 0$, so that unit vector along the tangent is parallel to r_2. Similarly the unit vector along the tangent to the curve $v = c_2$(constant) is parallel to r_1.

The tangent plane at P contains r_1 and r_2 at P, then the equation of the tangent plane is $R = r + c_1 r_1 + c_2 r_2$, where R is the position vector of a current point on the plane.

The equation of the tangent plane can be expressed as

$$(R - r) \cdot (r_1 \times r_2) = 0$$

Since $r_1 \times r_2$ gives the direction of the normal to the tangent plane, so it is perpendicular to any line $R - r$ in the plane.

Normal Line: Normal to the tangent at P is the line passing through $P(r)$ and parallel to the vector $r_1 \times r_2$. Hence the equation of the normal line at P to the surface is given by $R = r + \lambda(r_1 \times r_2)$.

The normal to the surface at P is the same as the normal to tangent plane at P and therefore the unit surface normal N is given by

$$N = \frac{r_1 \times r_2}{|r_1 \times r_2|} = \frac{r_1 \times r_2}{H}$$

where $H = |r_1 \times r_2| \neq 0$

The direction of N is fixed by convention that if N is the unit surface normal then $r_1 \times r_2$ and N form a right handed system.

- **TO FIND THE CARTESIAN EQUATION OF THE TANGENT PLANE TO THE SURFACE** $F(x, y, z) = 0$

  We know that

  $$grad\ F = \frac{\partial F}{\partial x} i + \frac{\partial F}{\partial y} j + \frac{\partial F}{\partial z} k$$

  is normal to the tangent plane at $P(x, y, z)$, thus it is perpendicular to any line in the plane. Let $R = Xi + Yj + Zk$ be any point on the plane, then $(R - r) \cdot grad\ F = 0$

  $$\Rightarrow [(X - x)i + (Y - y)j + (Z - z)k] \cdot \left[\frac{\partial F}{\partial x} i + \frac{\partial F}{\partial y} j + \frac{\partial F}{\partial z} k\right] = 0$$

  $$\Rightarrow (X - x)\frac{\partial F}{\partial x} + (Y - y)\frac{\partial F}{\partial y} + (Z - z)\frac{\partial F}{\partial z} = 0 \qquad (1)$$

  which is equation of the tangent plane at P.

  Again x, y, z are functions of u and v, thus

$$\frac{\partial F}{\partial u} \equiv \frac{\partial F}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial u} = 0 \qquad (2)$$

and

$$\frac{\partial F}{\partial v} \equiv \frac{\partial F}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial v} = 0 \qquad (3)$$

On solving (2) and (3), we get

$$\frac{\frac{\partial F}{\partial x}}{\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial y}{\partial v}} = \frac{\frac{\partial F}{\partial y}}{\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial x}{\partial u}\frac{\partial z}{\partial v}} = \frac{\frac{\partial F}{\partial z}}{\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}} = \boldsymbol{k}$$

Therefore equation of the tangent plane can be put in the form

$$\begin{vmatrix} X-x & Y-y & Z-z \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} = 0$$

❖ **TO FIND THE CARTESIAN EQUATION OF THE NORMAL TO THE SURFACE** $F(x,y,z) = 0$ **AT** $P(x,y,z)$

Let $\boldsymbol{R}$ be the position vector of any point $Q(X,Y,Z)$ on the normal and $\boldsymbol{r}$ be the position vector of $P(x,y,z)$.

$$\therefore \quad \boldsymbol{R} - \boldsymbol{r} = (X-x)\boldsymbol{i} + (Y-y)\boldsymbol{j} + (Z-z)\boldsymbol{k}$$

is vector along the normal.

Again

$$grad\ F = \frac{\partial F}{\partial x}\boldsymbol{i} + \frac{\partial F}{\partial y}\boldsymbol{j} + \frac{\partial F}{\partial z}\boldsymbol{k}$$

is normal to the tangent plane at P, so it is parallel to the normal vector PQ.

Therefore $\boldsymbol{R} - \boldsymbol{r} = \lambda\ grad\ F$, where λ is scale parameter.

or

$$(X-x)\boldsymbol{i} + (Y-y)\boldsymbol{j} + (Z-z)\boldsymbol{k} = \lambda\left[\frac{\partial F}{\partial x}\boldsymbol{i} + \frac{\partial F}{\partial y}\boldsymbol{j} + \frac{\partial F}{\partial z}\boldsymbol{k}\right]$$

or

$$\frac{X-x}{\frac{\partial F}{\partial x}} = \frac{Y-y}{\frac{\partial F}{\partial y}} = \frac{Z-z}{\frac{\partial F}{\partial z}} = \lambda$$

CHECK YOUR PROGRESS

Q 1. Find the equation of the tangent plane and normal to the surface $z = x^2 + y^2$ at the point $(1, -1, 2)$.

Q 2. Find the equation of the tangent plane and normal to the surface $xyz = 4$ at the point $(1, 2, 2)$.

Q 3. Prove that at points common to the surface $a(yz + zx + xy) = xyz$ and a sphere whose centre is the origin, the tangent plane to the surface makes intercepts on the axes whose sum is constant.

Q 4. Prove that any tangent plane to the surface $a(x^2 + y^2) + xyz = 0$ meets it again in a conic whose projection on the plane of xy is a rectangular hyperbola.

Q 5. Show that the sum of the squares of the intercepts on the coordinate axes made by the tangent plane to the surface $x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}} = a^{\frac{2}{3}}$ is constant.

Q 6. Prove that the tangent plane to the surface $xyz = a^2$ and the coordinate plane, bound a tetrahedron of constant volume.

Q 7. The normal at a point P of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ meets the coordinate planes in G_1, G_2, G_3; prove that the ratios $PG_1 : PG_2 : PG_3$ are constant.

7.10 FAMILY OF SURFACES

An equation of the form

$$F(x, y, z, a) = 0 \qquad (1)$$

where a is a constant represents a family of surface. If 'a' can take all real values i.e. if a is a parameter, then equation represents one parameter family of surface having 'a' as parameter. By assigning different values to 'a', we shall get different surfaces of this family of surfaces.

Similarly the equation of the form $F(x, y, z, a, b) = 0$, where a and b are parameters represents the family of surface having a and b as two parameters.

Characteristic Curve of Family of Surfaces

The curve of intersection of two surfaces of the family with parameter values a and $a + \delta a$ is given by the equation

$$F(x, y, z, a) = 0; \;\; F(x, y, z, a + \delta a) = 0$$

or by the equation $F(x, y, z, a) = 0$

and

$$\frac{F(x,y,z,a+\delta a)-F(x,y,z,a)}{\delta a}=0$$

The limiting position of the curve as $\delta a \to 0$, becomes

$$F(x,y,z,a)=0, \frac{\partial F}{\partial a}=0 \qquad (2)$$

This limiting position is called the characteristic of the family of surfaces given by (1).

Envelope

As the parameter a varies, the equation (2) will give different curves, so equation (2) represents a family of characteristics. The locus of all these characteristics is called the envelope of the system of surfaces.

The Equation of the Envelope

The equation of the envelope is given by eliminating a from the equations

$$F(a)=0 \quad and \quad \frac{\partial F}{\partial a}=0$$

❖ TO FIND THE ENVELOPE OF SYSTEM OF SURFACES WHOSE EQUATIONS INVOLVE TWO PARAMETERS

Let the surface be given by the equation $F(x,y,z,a,b)=0$, where a and b are parameters. Let a consecutive surface of the system be

$$F(x,y,z,a+\delta a,b+\delta b)=0$$

or

$$F(x,y,z,a,b)+\delta a\frac{\partial F}{\partial a}+\delta b\frac{\partial F}{\partial b}+\cdots=0$$

Thus, when $\delta a, \delta b$ are infinitely small, we have at a point of ultimate intersection

$$F=0, \qquad \delta a\frac{\partial F}{\partial a}+\delta b\frac{\partial F}{\partial b}=0 \qquad (1)$$

Again δa and δb being independent, we have from (1)

$$F=0, \qquad \frac{\partial F}{\partial a}=0, \qquad \frac{\partial F}{\partial b}=0 \qquad (2)$$

Thus the curve of intersection of $F=0$ with any surface consecutive to it passes

through the point satisfying the equations (2) i.e.

$$F = 0, \qquad \frac{\partial F}{\partial a} = 0, \qquad \frac{\partial F}{\partial b} = 0$$

By eliminating a and b from these equations, we get the equation of the envelope.

❖ **TO SHOW THAT THE ENVELOPE TOUCHES EACH SURFACE OF THE SYSTEM AT THE CORRESPONDING CHARACTERISTIC POINTS**

Let us consider the surface $F(x,y,z,a,b) = 0$. The normal at point (x,y,z) to a surface of the family $F = 0$ is parallel to the vector

$$\left\{\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right\} \qquad (1)$$

Again the characteristic points are given by

$$F = 0, \qquad \frac{\partial F}{\partial a} = 0, \qquad \frac{\partial F}{\partial b} = 0$$

and the equation of envelope can be regarded as

$$F(x,y,z,a,b) = 0$$

where a and b are functions of x, y, z given by

$$\frac{\partial F}{\partial a} = 0, \qquad \frac{\partial F}{\partial b} = 0 \qquad (2)$$

The normal at point (x,y,z) to the envelope is parallel to the vector

$$\left(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial a}\cdot\frac{\partial a}{\partial x} + \frac{\partial F}{\partial b}\cdot\frac{\partial b}{\partial x}, \frac{\partial F}{\partial y} + \frac{\partial F}{\partial a}\cdot\frac{\partial a}{\partial y} + \frac{\partial F}{\partial b}\cdot\frac{\partial b}{\partial y}, \frac{\partial F}{\partial z} + \frac{\partial F}{\partial a}\cdot\frac{\partial a}{\partial z} + \frac{\partial F}{\partial b}\cdot\frac{\partial b}{\partial z}\right) \qquad (3)$$

Since $\frac{\partial F}{\partial a} = 0, \frac{\partial F}{\partial b} = 0$, thus normal at point (x,y,z) to the envelope is parallel to the vector

$$\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)$$

i.e. the envelope has the normal line and therefore the envelope and the surface of the family have the same tangent plane at a characteristic point which proves the theorem.

CHECK YOUR PROGRESS

Q 8. Find the envelope of the family of planes $3a^2x - 3ay + z = a^3$.

Q 9. Find the envelope of the sphere $(x - a\cos\theta)^2 + (y - a\sin\theta)^2 + z^2 = b^2$.

Q 10. Prove that the envelope of the normal planes drawn through the generators of the cone $ax^2 + by^2 + cz^2 = 0$ is given by

$$a^{\frac{1}{3}}(b-c)^{\frac{2}{3}}x^{\frac{2}{3}} + b^{\frac{1}{3}}(d-a)^{\frac{2}{3}}y^{\frac{2}{3}} + c^{\frac{1}{3}}(a-b)^{\frac{2}{3}}z^{\frac{2}{3}} = 0.$$

Q 11. Find the envelope of $lx + my + nz = p$, when $a^2l^2 + b^2m^2 + c^2n^2 = p^2$.

Q 12. A tangent plane to ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ meets the axes in A, B, C. Show that the envelope of the sphere $OABC$ is

$$(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} + (cz)^{\frac{2}{3}} = (x^2 + y^2 + z^2)^{\frac{2}{3}}$$

Q 13. A plane makes intercepts on the axes, so that $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{k^2}$. Show that it envelopes a conicoid which has the axes as equal conjugate diameters.

7.11 DEVELOPABLE SURFACE

Definition: The envelope of a single parameter family of planes is called a developable surface or simply a developable.

The word developable is used due to the fact that the surface may be developable into a plane without stretching or tearing.

If $\boldsymbol{N}$ represents unit normal vector to the plane, the equation of one-parameter family of plane is

$$\boldsymbol{r}\cdot\boldsymbol{N}(u) = p(u) \;\; or \;\; f(u) = 0, \qquad where \; f(u) \equiv \boldsymbol{r}\cdot\boldsymbol{N}(u) - p(u)$$

where u is parameter and p is length of the perpendicular from the origin to the plane.

❖ THE NECESSARY AND SUFFICIENT CONDITION THAT SURFACE $z = f(x, y)$ SHOULD REPRESENT A DEVELOPABLE SURFACE

The equation of the tangent plane at (x, y, z) is

$$(X - x)\frac{\partial f}{\partial x} + (Y - y)\frac{\partial f}{\partial y} + (Z - z) = 0$$

or

$$p(X - x) + q(Y - y) + (Z - z) = 0$$

In case the surface is a developable surface, there exists a relation between p

and q, which may be written as $p = \phi(q)$ etc.

On differentiation, we get

$$\frac{\partial p}{\partial x} = \phi'(q)\frac{\partial q}{\partial x}, \qquad \frac{\partial p}{\partial y} = \phi'(q)\frac{\partial q}{\partial y}$$

On eliminating $\phi'(q)$, we get

$$\frac{\partial p}{\partial x}\frac{\partial q}{\partial y} = \frac{\partial p}{\partial y}\frac{\partial q}{\partial x} \quad or \quad rt = s^2$$

where

$$r = \frac{\partial p}{\partial x} = \frac{\partial^2 f}{\partial x^2}, \qquad s = \frac{\partial p}{\partial y} = \frac{\partial q}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}, \qquad t = \frac{\partial q}{\partial y} = \frac{\partial^2 f}{\partial y^2}$$

Thus $rt = s^2$ is the required condition that surface $z = f(x, y)$ may represent a developable surface.

Sufficient Condition

$$rt - s^2 = 0 \Rightarrow \frac{\partial p}{\partial x}\frac{\partial q}{\partial y} - \frac{\partial p}{\partial y}\frac{\partial q}{\partial x} = 0$$

$$\Rightarrow \begin{vmatrix} \frac{\partial p}{\partial x} & \frac{\partial p}{\partial y} \\ \frac{\partial q}{\partial x} & \frac{\partial q}{\partial y} \end{vmatrix} = 0 \Rightarrow \frac{\partial(p, q)}{\partial(x, y)} = 0$$

Jacobian of the function p and q vanishes and hence functions p and q must depend on single parameter, so shall do the tangent plane, therefore the surface is developable.

CHECK YOUR PROGRESS

Q 14. By considering the value of $rt = s^2$, determine if the surface $xyz = a^3$ is developable.

Q 15. Prove that the surface $xy = (z - c)^2$ is developable.

7.12 LET US SUM UP

- A **surface** is a two-dimensional geometric entity in three-dimensional space, usually represented parametrically, and its study forms a fundamental part of differential geometry.

- The classification of surfaces involves understanding **regular points and singularities**, where regular points satisfy rank conditions, while singularities may be essential or artificial depending on their origin.
- **Transformations of parameters** help in representing the same surface in different coordinate systems, and proper transformations preserve the regularity and geometric nature of the surface.
- Curves lying on a surface can be expressed through **curvilinear and parametric equations**, and important geometric elements such as **tangent planes and normals** describe local behaviour of the surface.
- A **family of surfaces** depending on one or more parameters leads to the concept of a **characteristic curve** and its limiting position, known as the **envelope**, which touches each member of the family.
- The **equation of an envelope** is obtained by eliminating parameters using the given family equation along with its partial derivative(s), ensuring tangency conditions.
- A **developable surface** is a special surface that can be unfolded onto a plane without distortion, characterized by zero Gaussian curvature and obtained as the envelope of a one-parameter family of planes.

7.13 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q 1. Equation of the surface is

$$F(x,y,z) = z - x^2 - y^2 = 0$$

$$\therefore \quad \frac{\partial F}{\partial x} = -2x = -2 \quad at \quad (1,-1,2)$$

$$\frac{\partial F}{\partial y} = -2y = -1 \quad at \quad (1,-1,2)$$

$$\frac{\partial F}{\partial z} = 1$$

Therefore equation of the tangent plane at the point $(1,-1,2)$ is

$$(x-1)(-2) + (y+1)(2) + (z-2)(1) = 0$$

$$\Rightarrow -2x + 2 + 2y + 2 + z - 2 = 0$$

$$\Rightarrow 2x - 2y - z = 2$$

Equation of the normal is

$$\frac{x-1}{-2}=\frac{y+1}{2}=\frac{z-2}{1}$$

Ans to Q 2. Equation of the surface is

$$F(x,y,z)=xyz-4=0$$

$$\therefore \quad \frac{\partial F}{\partial x}=yz=4 \quad at \quad (1,2,2)$$

$$\frac{\partial F}{\partial y}=xz=2 \quad at \quad (1,2,2)$$

$$\frac{\partial F}{\partial z}=xy=2 \quad at \quad (1,2,2)$$

Therefore equation of the tangent plane at $(1,2,2)$ is

$$(x-1)(4)+(y-2)(2)+(z-2)(2)=0$$

$$\Rightarrow 4x+2y+2z=12$$

$$\Rightarrow 2x+y+z=6$$

Equation of the normal at $(1,2,2)$ is

$$\frac{x-1}{4}=\frac{y-2}{2}=\frac{z-2}{2}$$

or

$$\frac{x-1}{2}=\frac{y-2}{1}=\frac{z-2}{1}$$

Ans to Q 3. Equation of the given surface is

$$F(x,y,z)=a(yz+zx+xy)-xyz=0 \qquad (1)$$

$$\therefore \quad \frac{\partial F}{\partial x}=a(z+y)-yz$$

$$\frac{\partial F}{\partial y}=a(x+z)-xz$$

$$\frac{\partial F}{\partial z}=a(x+y)-xy$$

Equation of the sphere with centre as origin and radius r is

$$x^2+y^2+z^2=r^2 \qquad (2)$$

At a common point $P(x,y,z)$ to the given surface (1) and sphere (2), the equation of the tangent plane is

$$(X-x)\{a(y+z)-yz\}+(Y-y)\{a(z+x)-zx\}+(Z-z)\{a(x+y)-xy\}=0$$

$$\Rightarrow X\{a(y+z)-yz\}+Y\{a(z+x)-zx\}+Z\{a(x+y)-xy\}=2a(yz+zx+xy)-3xyz$$

$$\Rightarrow \frac{X}{x}\{a(xy+xz)-xyz\}+\frac{Y}{y}\{a(yz+xy)-xyz\}+\frac{Z}{z}\{a(xz+yz)-xyz\}=2xyz-3xyz$$

$$\Rightarrow \frac{X}{x}(-ayz)+\frac{Y}{y}(-azx)+\frac{Z}{z}(-axy)=-xyz$$

$$\Rightarrow \frac{X}{\left(\frac{x^2}{a}\right)}+\frac{Y}{\left(\frac{y^2}{a}\right)}+\frac{Z}{\left(\frac{z^2}{a}\right)}=1$$

Intercepts made by this tangent plane with the axes are $\frac{x^2}{a},\frac{y^2}{a},\frac{z^2}{a}$.

$\therefore$ Sum of the intercepts $=\frac{x^2+y^2+z^2}{a}=\frac{r^2}{a}=$ constant.

Ans to Q 4. Equation of the given surface is

$$F(x,y,z)\equiv a(x^2+y^2)+xyz=0 \qquad (1)$$

$$\therefore \frac{\partial F}{\partial x}=2ax+yz, \qquad \frac{\partial F}{\partial y}=2ay+xy \quad and \quad \frac{\partial F}{\partial z}=xy$$

Equation of the tangent plane to (1) is

$$(X-x)(2ax+yz)+(Y-y)(2ay+xz)+(Z-z)(xy)=0 \qquad (2)$$

Taking (X,Y,Z) as current coordinates, the equation of the surface (1) is

$$a(X^2+Y^2)+XYZ=0 \qquad (3)$$

We get the projection of the curve of intersection of (2) and (3) on the plane $Z=0$ by eliminating Z between (2) and (3).

Therefore the equation of the projection is

$$(X-x)(2ax+yz)+(Y-y)(2ay+xz)-\left\{\frac{a(X^2+Y^2)}{XY}+z\right\}xy=0$$

$$\Rightarrow XY[X(2ax+yz)+Y(2ay+xz)-(2ax^2+2ay^2+2xyz)]=\{a(X^2+Y^2)+XYz\}xy$$

$$\Rightarrow XY\left[\frac{X}{x}(2ax^2+xyz)+\frac{Y}{y}(2ay^3+xyz)\right]a(X^2+Y^2)xy+xyzXY$$

$$\Rightarrow XY\left[\frac{X}{x}(ax^2-ay^3)+\frac{Y}{y}(ay^2-ax^2)\right]=a(X^2+Y^2)xy-a(x^2+y^2)XY$$

$$\Rightarrow XY(x^2-y^2)\left(\frac{X}{x}-\frac{Y}{y}\right)=Xx^2y\left(\frac{X}{x}-\frac{Y}{y}\right)-Yxy^2\left(\frac{X}{x}-\frac{Y}{y}\right)$$

$$\Rightarrow XY(x^2-y^2)=Xx^2y-Yxy^2$$

Which is a rectangular hyperbola.

Ans to Q 5. Equation of the given surface is

$$F(x,y,z)\equiv x^{\frac{2}{3}}+y^{\frac{2}{3}}+z^{\frac{2}{3}}-a^{\frac{2}{3}}=0 \qquad (1)$$

$$\therefore \quad \frac{\partial F}{\partial x} = \frac{2}{3}x^{-\frac{1}{3}}, \qquad \frac{\partial F}{\partial y} = \frac{2}{3}y^{-\frac{1}{3}}, \qquad \frac{\partial F}{\partial z} = \frac{2}{3}z^{-\frac{1}{3}}$$

Equation of the tangent plane at the point (x, y, z) is

$$(X-x)\left(\frac{2}{3}x^{-\frac{1}{3}}\right) + (Y-y)\left(\frac{2}{3}y^{-\frac{1}{3}}\right) + (Z-z)\left(\frac{2}{3}z^{-\frac{1}{3}}\right) = 0$$

$$\Rightarrow Xx^{-\frac{1}{3}} + Yy^{-\frac{1}{3}} + Zz^{-\frac{1}{3}} = x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}} = a^{\frac{2}{3}}$$

To get the intercept made by this tangent plane on the x –axis, put

$$Y = Z = 0$$

$$\therefore \quad Xx^{-\frac{1}{3}} + 0y^{-\frac{1}{3}} + 0z^{-\frac{1}{3}} = a^{\frac{2}{3}}$$

or

$$X = a^{\frac{2}{3}}x^{\frac{1}{3}}$$

Similarly the intercepts on y and z –axes are $a^{\frac{2}{3}}y^{\frac{1}{3}}$ and $a^{\frac{2}{3}}z^{\frac{1}{3}}$ respectively.

Now the sum of the squares of the intercepts

$$= \left(a^{\frac{2}{3}}\right)^2\left(x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}}\right) = a^{\frac{4}{3}}a^{\frac{2}{3}} = a^2 = \text{constant}$$

Ans to Q 6. Equation of the surface is

$$F(x,y,z) \equiv xyz - a^2 = 0$$

$$\therefore \quad \frac{\partial F}{\partial x} = yz, \qquad \frac{\partial F}{\partial y} = zx, \qquad \frac{\partial F}{\partial z} = xy$$

Equation of the tangent plane at point (x, y, z) on the surface is

$$(X-x)yz + (Y-y)zx + (Z-z)xy = 0$$

$$\Rightarrow \frac{X}{x} + \frac{Y}{y} + \frac{Z}{z} = 3$$

Its intercepts on the axes are $3x, 3y, 3z$.

Therefore the volume of the tetrahedron bounded by it and the coordinate planes is

$$= \frac{1}{6}(3x)(3y)(3z) = \frac{9}{2}xyz = \frac{9}{3}a^2 = \text{constant}$$

Ans to Q 7. We have

$$F(x,y,z) \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$$

$$\therefore \quad \frac{\partial F}{\partial x} = \frac{2x}{a^2}, \frac{\partial F}{\partial y} = \frac{2y}{b^2}, \frac{\partial F}{\partial z} = \frac{2z}{c^2}$$

Hence the equation of the normal at (x, y, z) is

$$\frac{X-x}{\left(\frac{x}{a^2}\right)}=\frac{Y-y}{\left(\frac{y}{b^2}\right)}=\frac{Z-z}{\left(\frac{z}{c^2}\right)}$$

Put $X=0$ to get the point where it meets the plane YOZ, thus we get

$$Y=y-\frac{a^2y}{b^2}, Z=z-\frac{a^2z}{c^2}$$

$\therefore$ The point is

$$\left(0, y-\frac{a^2y}{b^2}, z-\frac{a^2z}{c^2}\right)$$

Therefore

$$PG_1=\sqrt{\frac{x^2a^4}{a^2}+\frac{a^4y^2}{b^4}+\frac{a^4z^2}{c^4}}=a^2\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}$$

Similarly

$$PG_2=b^2\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}$$

and

$$PG_3=c^2\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}$$

$$\therefore \quad PG_1:PG_2:PG_3=a^2:b^2:c^2$$

Ans to Q 8. We have

$$F(x,y,z,a)\equiv 3a^2x-3ay+z-a^3=0 \qquad (1)$$

$$\therefore \quad \frac{\partial F}{\partial a}=6ax-3y-3a^2=0 \qquad (2)$$

and

$$\frac{\partial^2 F}{\partial a^2}=6x-6a=0 \qquad (3)$$

Now on eliminating a between (1) and (2), we get the equation of the envelope.

Multiply (1) by (3) and (2) by a and subtracting, we get

Now solving (2) and (4), we get

$$\frac{a^2}{2y^2-2zx}=\frac{a}{xy-z}=\frac{1}{2x^2-2y}$$

Hence on eliminating a, the envelope is

$$(xy-z)^2=4(y^2-zx)(x^2-y)$$

Ans to Q 9. The given equation can be written as

$$F(\theta) \equiv x^2 + y^2 + z^2 - 2xa\cos\theta - 2ya\sin\theta + a^2 - b^2 = 0 \qquad (1)$$

$$\therefore \quad \frac{\partial F}{\partial \theta} = 2a(x\sin\theta - y\cos\theta) = 0 \qquad (2)$$

From (2), we have $\tan\theta = \frac{y}{x}$, hence

$$\sin\theta = \frac{y}{\sqrt{x^2+y^2}}, \cos\theta = \frac{x}{\sqrt{x^2+y^2}}$$

Now putting these values of $\sin\theta$ and $\cos\theta$ in (1), we get the equation of the envelope as

$$x^2 + y^2 + z^2 - 2a\left[\frac{x^2}{\sqrt{x^2+y^2}} + \frac{y^2}{\sqrt{x^2+y^2}}\right] + (a^2 - b^2) = 0$$

$$\therefore \quad (x^2 + y^2 + z^2 + a^2 - b^2)^2 = 4a^2(x^2 + y^2)$$

Ans to Q 10. Let (u, v, w) be any point on the cone, the equation of tangent plane at this point is

$$axu + byv + czw = 0$$

The equation of the generator through this point is

$$\frac{x}{u} = \frac{y}{v} = \frac{z}{w}$$

Let $Ax + By + Cz = 0$ be the normal plane, since the generator lies in it, we have

$$Au + Bv + Cw = 0 \qquad (1)$$

Again it is perpendicular to the tangent plane, thus

$$Aau + Bbv + Ccw = 0 \qquad (2)$$

From (1) and (2), we have

$$\frac{Au}{b-c} = \frac{Bv}{c-a} = \frac{Cw}{a-b}$$

Therefore equation of the normal plane is

$$\frac{b-c}{u}x + \frac{c-a}{v}y + \frac{a-b}{w}z = 0$$

Again

$$au^2 + bv^2 + cw^2 = 0$$

Also

$$\frac{\frac{\partial F}{\partial u}}{\frac{\partial \phi}{\partial u}} = \frac{\frac{\partial F}{\partial v}}{\frac{\partial \phi}{\partial v}} = \frac{\frac{\partial F}{\partial w}}{\frac{\partial \phi}{\partial w}}$$

From this relation, we have

$$\frac{(b-c)x}{u^2.2au}=\frac{(c-a)y}{v^2.2bv}=\frac{(a-b)z}{w^2.2cw}$$

$$\Rightarrow \frac{(b-c)^{\frac{1}{3}}x^{\frac{1}{3}}}{a^{\frac{1}{3}}u}=\frac{(c-a)^{\frac{1}{3}}y^{\frac{1}{3}}}{b^{\frac{1}{3}}v}=\frac{(a-b)^{\frac{1}{3}}z^{\frac{1}{3}}}{c^{\frac{1}{3}}w}$$

$$\Rightarrow \frac{(b-c)^{\frac{2}{3}}x^{\frac{2}{3}}a^{\frac{1}{3}}}{a^{\frac{1}{3}}a^{\frac{2}{3}}u^2}=\frac{(c-a)^{\frac{2}{3}}y^{\frac{2}{3}}b^{\frac{1}{3}}}{b^{\frac{1}{3}}b^{\frac{2}{3}}v^2}=\frac{(a-b)^{\frac{2}{3}}z^{\frac{2}{3}}c^{\frac{1}{3}}}{c^{\frac{1}{3}}c^{\frac{2}{3}}w^2}$$

$$=\frac{\sum(b-c)^{\frac{2}{3}}x^{\frac{2}{3}}a^{\frac{1}{3}}}{au^2+bv^2+cw^2}$$

or

$$\sum(b-c)^{\frac{2}{3}}x^{\frac{2}{3}}a^{\frac{1}{3}}=0 \quad since \quad au^2+bv^2+cw^2=0$$

Hence the required envelope is

$$\sum(b-c)^{\frac{2}{3}}x^{\frac{2}{3}}a^{\frac{1}{3}}=0$$

Ans to Q 11. Here

$$F\equiv lx+my+nz-p=0 \qquad (1)$$

and

$$\phi\equiv a^2l^2+b^2m^2+c^2n^2-p^2=0 \qquad (2)$$

Again

$$\frac{\frac{\partial F}{\partial l}}{\frac{\partial \phi}{\partial l}}=\frac{\frac{\partial F}{\partial m}}{\frac{\partial \phi}{\partial m}}=\frac{\frac{\partial F}{\partial n}}{\frac{\partial \phi}{\partial n}}=\frac{\frac{\partial F}{\partial p}}{\frac{\partial \phi}{\partial p}}$$

i.e.

$$\frac{x}{a^2l}=\frac{y}{b^2m}=\frac{z}{c^2n}=\frac{1}{p}$$

$$\therefore \quad l=\frac{px}{a^2} \ etc. \qquad (3)$$

Now on eliminating $l.m,n,p$ from (2) and (3), the envelope is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

Ans to Q 12. Do it yourself.

Ans to Q 13. The equation of the plane is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \qquad (1)$$

Also it is given that

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{k^2} \qquad (2)$$

Now applying the relation

$$\frac{\frac{\partial F}{\partial a}}{\frac{\partial \phi}{\partial a}}=\frac{\frac{\partial F}{\partial b}}{\frac{\partial \phi}{\partial b}}=\frac{\frac{\partial F}{\partial c}}{\frac{\partial \phi}{\partial c}}$$

We have for the envelope

$$\frac{\frac{x}{a^2}}{\frac{2}{a^3}}=\frac{\frac{y}{b^2}}{\frac{2}{b^3}}=\frac{\frac{z}{c^2}}{\frac{2}{c^3}}$$

or

$$ax=by=cz=\frac{1}{\lambda}\ (say)$$

$$\therefore\ \frac{1}{a}=\lambda x\ etc. \qquad (3)$$

Now putting from (3) in (1), we have

$$x^2+y^2+z^2=\frac{1}{\lambda}$$

or

$$x^2+y^2+z^2=cz \qquad (4)$$

From (3) and (2), we have

$$x^2+y^2+z^2=\frac{c^2z^2}{k^2} \qquad (5)$$

Eliminating c from (4) and (5), we have

$$x^2+y^2+z^2=k^2$$

which is a conicoid having equal conjugate diameters.

Ans to Q 14. In this case

$$z=\frac{a^3}{xy}$$

$$\therefore\ \ p=\frac{\partial f}{\partial x}=\frac{\partial z}{\partial x}=-\frac{a^3}{x^2y}$$

$$q=\frac{\partial f}{\partial y}=\frac{\partial z}{\partial y}=-\frac{a^3}{xy^2}$$

$$r = \frac{\partial p}{\partial x} = \frac{2a^3}{x^2 y}$$

$$s = \frac{\partial p}{\partial y} = \frac{a^3}{x^2 y^2}$$

$$t = \frac{\partial q}{\partial y} = \frac{2a^3}{xy^3}$$

Now

$$rt - s^2 = \frac{4a^6}{x^4 y^4} - \frac{a^6}{x^4 y^4} = \frac{3a^4}{x^4 y^4} \neq 0$$

Hence the surface is not developable.

Ans to Q 15. Do it yourself.

O 7.14 FURTHER READINGS

1. Wilmore T. J. An Introduction to Differential Geometry, Dover Publications Inc., 2012
2. Weatherburn, C. E. Differential Geometry of Three Dimensions, Cambridge University Press, 2016
3. Pressley, A. Elementary Differential Geometry. Springer, 2002
4. Gupta, P. P., Malik, G. S., Pundir, S. K. Differential Geometry, Pragati Prakashan, 2010

7.15 MODEL QUESTIONS

1. Define a surface and explain what is meant by an ordinary point and parametric curves.
2. Find an equation of the tangent plane to the surface $2xz^2 - 3x^2y - 4x = 7$ at the point $(1, -1, 2)$.
3. Find the equation of the tangent plane to the surface $z = xy$ at the point $(2, 3, 6)$.
4. Find the unit normal vector to the surface $2xz^2 - 3xy - 4x = 7$ at the point $(1, -1, 2)$.
5. Find the envelope of the plane $lx + my + nz = 0$, when $al^2 + bm^2 + cn^2 = 0$.
6. Show that the envelope of the family of paraboloids

$$x^2 + y^2 = 4a(z - a)$$

is the circular cone $x^2 + y^2 = z^2$.

7. Find the envelope of the normal planes to the curve

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad x^2 + y^2 + z^2 = r^2$$

****