UNIT 8:FUNDAMENTAL FORMS AND CURVATURE OF SURFACES

UNIT STRUCTURE

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8.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- learn the concept of cyberspace, understanding its scope, significance, and impact on modern society.
- learn about the architecture of cyberspace that support the seamless functioning of digital communications and online activities.
- familiarize yourself with the essential aspects of computers, including their characteristics, fundamental components, and functionalities.
- acquire an overview of web technology and its various related technologies.

8.2 INTRODUCTION

In differential geometry, the study of surfaces deals with understanding their shape and how they bend in space. The fundamental forms are mathematical tools used to describe these properties. The first fundamental form helps measure lengths, angles, and areas on the surface, while the second fundamental form explains how the surface curves in three-dimensional space.

An important idea in this topic is principal curvatures and their corresponding lines of curvature. These show the directions in which the surface bends the most and the least. The behavior of curves on the surface can also be studied using quantities like torsion, which depends on the direction of the curve relative to these principal directions.

It can be shown that torsion is maximum when the direction lies exactly between the two principal directions. Since the lines of curvature are perpendicular to each other, this happens when the direction makes equal angles with them. This result helps us understand the relationship between curvature and direction on a surface.

8.3 FIRST FUNDAMENTAL FORM OR METRIC

Let $\boldsymbol{r} = \boldsymbol{r}(u,v)$ be the equation of a surface and let $E = \boldsymbol{r}_1^2 = \boldsymbol{r}_1 \cdot \boldsymbol{r}_1$, $F = \boldsymbol{r}_1 \cdot \boldsymbol{r}_2 = \boldsymbol{r}_2 . \boldsymbol{r}_1$, and $G = \boldsymbol{r}_2^2 = \boldsymbol{r}_2 \cdot \boldsymbol{r}_2$. The quadratic differential form

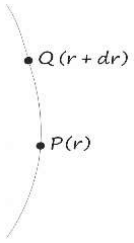
$$E du^2 + 2F\, du dv + G dv^2$$

in du, dv is called metric or first fundamental form of the surface and the quantities E, F, G are called first fundamental coefficients or first order fundamental magnitudes. Since E, F and G are functions of u, v, the quantities will generally vary from point to point on the surface. These quantities are of great importance and will appear frequently in the rest of the discussion.

4.4 GEOMETRICAL INTERPRETATION OF THE METRIC

Let $\boldsymbol{r} = \boldsymbol{r}(u,v)$ be the equation of a surface and $u = u(t)$ and $v = v(t)$ be a curve on the surface. Let P and Q be two neighbouring points on a curve on the surface with position vectors $\boldsymbol{r}$ and $\boldsymbol{r} + d\boldsymbol{r}$, corresponding to the parameter values u, v and $u + du, v + dv$ respectively. Then

$$d\boldsymbol{r} = \frac{\partial \boldsymbol{r}}{\partial u}du + \frac{\partial \boldsymbol{r}}{\partial v}dv = \boldsymbol{r_1}du + \boldsymbol{r_2}dv$$



Since the two points P and Q are adjacent points on a curve passing through them, the length ds of the element of arc joining them is equal to their actual distance $|\ d\boldsymbol{r}\ |$ apart. Thus, we have

$$ds^2 = |\ d\boldsymbol{r}\ |^2 = (\boldsymbol{r_1}du + \boldsymbol{r_2}dv)^2$$

$$= r_1^2 du^2 + 2r_1.r_2 dudv + r_2^2 dv^2$$

$$= Edu^2 + 2Fdudv + Gdv^2$$

Thus, we have the following geometrical interpretation of the first fundamental form:

If ds denotes the element of arc lying on the surface and joining the points (u, v) and $(u + du, v + dv)$ on the surface, then

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2$$

Since the first fundamental form is chiefly used for the calculation of arc lengths on the surface, therefore it is called the metric of the surface. If $u = u(t), v = v(t)$ are the equations of a curve lying on the surface, then the arc length s of the curve is related to the parameter t by

On the parametric curve $v = \text{constant}$, we have $dv = 0$ and the metric reduces to $ds^2 = Edu^2$.

Similarly on parametric curve $u = \text{constant}$, we have $ds^2 = Gdv^2$

8.5 RELATIONS BETWEEN THE COEFFICIENTS E, F, G

If ω is the angle between the vectors $\boldsymbol{r_1}$ and $\boldsymbol{r_2}$, then from the definition of cross product of two vectors, we have

$$\boldsymbol{r_1} \times \boldsymbol{r_2} = |\ \boldsymbol{r_1}\ |\ \ |\ \boldsymbol{r_2}\ | \sin\omega\, \boldsymbol{N}$$

where $\boldsymbol{N}$ is the unit normal vector to the surface at the point $\boldsymbol{r}$.

$$\therefore (\boldsymbol{r_1} \times \boldsymbol{r_2})^2 = |\ \boldsymbol{r_1}\ |^2 |\ \boldsymbol{r_2}\ |^2 \sin^2\omega (\boldsymbol{N}^2 = 1)$$

$$\Rightarrow (HN)^2 = |\ \boldsymbol{r_1}\ |^2 |\ \boldsymbol{r_2}\ |^2 (1 - \cos^2\omega)$$

$$\Rightarrow H^2 = |\ \boldsymbol{r_1}\ |^2 |\ \boldsymbol{r_2}\ |^2 - |\ \boldsymbol{r_1}\ |^2 |\ \boldsymbol{r_2}\ |^2 \cos^2\omega$$

$$\Rightarrow H^2 = \boldsymbol{r_1^2 r_2^2} - (\boldsymbol{r_1}.\boldsymbol{r_2})^2$$

Since H is a positive quantity, thus $EG - F^2$ is always a positive quantity and $H = \sqrt{EG - F^2}$. Also, at any ordinary point $\boldsymbol{r_1} \neq 0, \boldsymbol{r_2} \neq 0$, therefore, $\boldsymbol{r_1^2} = E > 0$ and $\boldsymbol{r_2^2} = G > 0$ and thus $EG - F^2 > 0$.

8.6 PROPERTIES OF THE METRIC

Property 1: The first fundamental form orthemetric is a positive definite quadratic form in du, dv.

Proof: We have

$$E\,du^2 + 2F\,du\,dv + G\,dv^2 = (\boldsymbol{r_1}\,du + \boldsymbol{r_2}\,dv)^2$$
$$= |\,\boldsymbol{r_1}\,du + \boldsymbol{r_2}\,dv\,|^2 \;[\because (\boldsymbol{a})^2 = |\boldsymbol{a}|^2]$$
$$\geq 0 \text{ for all real values of } du \text{ and } dv.$$

Again

$$E\,du^2 + 2F\,du\,dv + G\,dv^2 = 0$$
$$\Rightarrow |\,\boldsymbol{r_1}\,du + \boldsymbol{r_2}\,dv\,|^2 = 0$$
$$\Rightarrow \boldsymbol{r_1}\,du + \boldsymbol{r_2}\,dv = 0$$
$$\Rightarrow du = 0 \text{ and } dv = 0 \text{ because } \boldsymbol{r_1} \text{ and } \boldsymbol{r_2} \text{ are two non-collinear vectors.}$$

Hence $E\,du^2 + 2F\,du\,dv + G\,dv^2$is a positive definite quadratic form in du, dv.

Property 2: The metric is invariant under a transformation of parameters.

Proof: Suppose in the equation of a surface $\boldsymbol{r} = \boldsymbol{r}(u,v)$, the parameters $u, v$are transformed to the parameters $u', v'$by the relations

$$u' = \phi(u,v), \quad v' = \psi(u,v).$$

Then

$$\boldsymbol{r_1'} = \frac{\partial \boldsymbol{r}}{\partial u'} = \frac{\partial \boldsymbol{r}}{\partial u}\frac{\partial u}{\partial u'} + \frac{\partial \boldsymbol{r}}{\partial v}\frac{\partial v}{\partial u'} = \boldsymbol{r_1}\frac{\partial u}{\partial u'} + \boldsymbol{r_2}\frac{\partial v}{\partial u'}$$
$$\boldsymbol{r_2'} = \frac{\partial \boldsymbol{r}}{\partial v'} = \boldsymbol{r_1}\frac{\partial u}{\partial v'} + \boldsymbol{r_2}\frac{\partial v}{\partial v'}$$

Also

$$du = \frac{\partial u}{\partial u'}\,du' + \frac{\partial u}{\partial v'}\,dv'$$
$$dv = \frac{\partial v}{\partial u'}\,du' + \frac{\partial v}{\partial v'}\,dv'$$

Now

$$E'\,du'^2 + 2F'\,du'\,dv' + G'\,dv'^2$$
$$= \boldsymbol{r_1'^2}\,du'^2 + 2\boldsymbol{r_1'}\cdot\boldsymbol{r_2'}\,du'\,dv' + \boldsymbol{r_2'^2}\,dv'^2$$
$$= (\boldsymbol{r_1'}\,du' + \boldsymbol{r_2'}\,dv')^2$$

$$= \left[\left(\boldsymbol{r}_1\frac{\partial u}{\partial u'} + \boldsymbol{r}_2\frac{\partial v}{\partial u'}\right)du' + \left(\boldsymbol{r}_1\frac{\partial u}{\partial v'} + \boldsymbol{r}_2\frac{\partial v}{\partial v'}\right)dv'\right]^2$$

$$= \left[\boldsymbol{r}_1\left(\frac{\partial u}{\partial u'}du' + \frac{\partial u}{\partial v'}dv'\right) + \boldsymbol{r}_2\left(\frac{\partial v}{\partial u'}du' + \frac{\partial v}{\partial v'}dv'\right)\right]^2$$

$$= (\boldsymbol{r}_1\, du + \boldsymbol{r}_2\, dv)^2$$

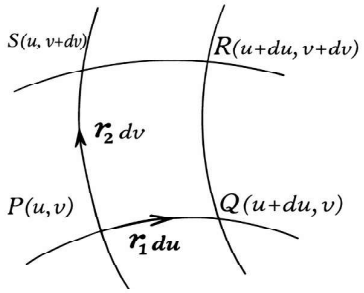
$$= \boldsymbol{r}_1^2\, du^2 + 2\boldsymbol{r}_1 \cdot \boldsymbol{r}_2\, du\, dv + \boldsymbol{r}_2^2\, dv^2$$

$$= E\, du^2 + 2F\, du\, dv + G\, dv^2.$$

Hence the result.

8.7ELEMENT OF AREA

Consider the figure with $P(u,v),\ Q(u+du,v),\ R(u+du,v+dv),\ S(u,v+dv)$ joined by parametric curves. For small positive du and dv, the figure is nearly a parallelogram.We have



$$\overrightarrow{PQ} = \text{Position vector of } Q - \text{Position vector of } P$$

$$= \boldsymbol{r}(u+du,v) - \boldsymbol{r}(u,v)$$

$$= \left[\boldsymbol{r}(u,v) + \frac{\partial \boldsymbol{r}}{\partial u}du\right] - \boldsymbol{r}(u,v) = \boldsymbol{r}_1\, du$$

Similarly,

$$\overrightarrow{PS} = \boldsymbol{r}_2\, dv$$

Therefore if dS is the area of this parallelogram, then

$$dS = |\, \boldsymbol{r}_1\, du \times \boldsymbol{r}_2\, dv\, | = |\, \boldsymbol{r}_1 \times \boldsymbol{r}_2\, |\ du\, dv$$

$$= H\, du\, dv.$$

Thus $H\, du\, dv$ is the expression for an elementary area on the surface at the point $(u\, v)$.

8.8 ANGLE BETWEEN PARAMETRIC CURVES

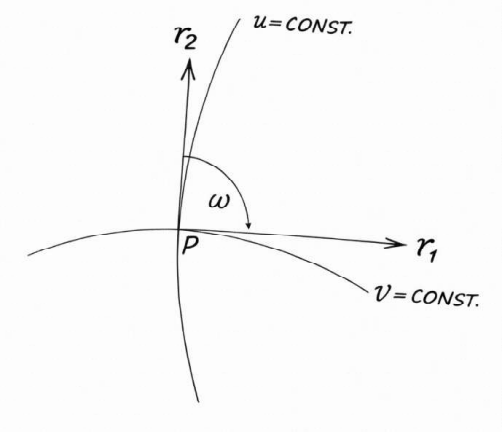
Suppose P is the point of intersection of the parametric curves $u = \text{constant}$ and $v = \text{constant}$, and let $\boldsymbol{r}$ be the position vector of P. Then the vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are along the tangents to the parametric curves $v = \text{constant}$ and $u = \text{constant}$ respectively. If $\omega\ (0 < \omega < \pi)$ is the angle between the parametric curves, then

$$\cos\omega = \frac{\mathbf{r_1}\cdot\mathbf{r_2}}{|\mathbf{r_1}|\ |\mathbf{r_2}|} = \frac{F}{\sqrt{EG}}$$

$$\sin\omega = \frac{|\mathbf{r_1}\times\mathbf{r_2}|}{|\mathbf{r_1}|\ |\mathbf{r_2}|} = \frac{H}{\sqrt{EG}} = \frac{\sqrt{EG-F^2}}{\sqrt{EG}}$$

and

$$\tan\omega = \frac{\sin\omega}{\cos\omega} = \frac{H}{F}$$



In general, the angle between the parametric curves varies from point to point. The parametric curves are said to form an orthogonal system if they cut at right angles at all points of the surface.

The necessary and sufficient condition for parametric curves to form an orthogonal system is that $F = 0$ at each point of the surface.

N.B.: If at any point ω is the angle between the parametric curves $u = \text{constant}$ and $v = \text{constant}$, then at that point

$$\cos\omega = \frac{\mathbf{r_1}\cdot\mathbf{r_2}}{|\mathbf{r_1}|\ |\mathbf{r_2}|} = \frac{F}{\sqrt{EG}}$$

From this relation we see that ω is a right angle if and only if $F = 0$.

Hence $\mathbf{r_1}\cdot\mathbf{r_2} = F = 0$ is the necessary and sufficient condition for parametric curves to form an orthogonal system.

CHECK YOUR PROGRESS

Q1. Calculate the fundamental magnitudes and the normal to the surface

$$2z = ax^2 + 2hxy + by^2$$

taking x, y as parameters.

Q2. Calculate first fundamental magnitudes for the surface

$$\mathbf{r} = [u\cos v, u\sin v, f(u)]$$

Q3. Calculate the first fundamental coefficients and the area corresponding to the domain $0 \le u \le 2\pi, 0 \le v \le 2\pi$ for the anchor ring

$$x = (b + a\cos u)\cos v, y = (b + a\cos u)\sin v, z = a\sin u.$$

Q4. Calculate E, F, G, H for the paraboloid

$$x = u, y = v, z = u^2 - v^2$$

8.9 SECOND FUNDAMENTAL FORM AND SECOND-ORDER MAGNITUDES

Suppose $\boldsymbol{r} = \boldsymbol{r}(u,v)$ is the equation of a surface and $\boldsymbol{N}$ is the unit normal vector to this surface at the point $\boldsymbol{r}(u,v)$. Then

$$\boldsymbol{N} = \frac{\boldsymbol{r_1} \times \boldsymbol{r_2}}{|\,\boldsymbol{r_1} \times \boldsymbol{r_2}\,|} = \frac{\boldsymbol{r_1} \times \boldsymbol{r_2}}{H}$$

Let

$$\boldsymbol{r_{11}} = \frac{\partial^2 \boldsymbol{r}}{\partial u^2}, \boldsymbol{r_{12}} = \frac{\partial^2 \boldsymbol{r}}{\partial u\,\partial v} = \frac{\partial^2 \boldsymbol{r}}{\partial v\,\partial u} = \boldsymbol{r_{21}}, \boldsymbol{r_{22}} = \frac{\partial^2 \boldsymbol{r}}{\partial v^2}.$$

If $L = \boldsymbol{r_{11}} \cdot \boldsymbol{N}$, $M = \boldsymbol{r_{12}} \cdot \boldsymbol{N}$, $N = \boldsymbol{r_{22}} \cdot \boldsymbol{N}$, then the quadratic differential expression in du is

$L\,du^2 + 2M\,du\,dv + N\,dv^2$ and dv is known as the second fundamental form of the surface. The functions of u and v, namely L, M and N, are called the coefficients of the second fundamental form or the fundamental magnitudes of second order. It should be noted that L, M and N represent the components of the vectors $\boldsymbol{r_{11}}, \boldsymbol{r_{12}}, \boldsymbol{r_{22}}$ respectively along the direction of the normal to the surface.

Like the relation $H^2 = EG - F^2$, it will be convenient to have a symbol for the quantity $LN - M^2$. We shall write

$$T^2 = LN - M^2$$

But it should be noted that $LN - M^2 = T^2$ is not necessarily positive just as $EG - F^2 = H^2$ is always positive.

We may express L, M, N in terms of scalar triple products of vectors. For

$$[\boldsymbol{r_{11}}, \boldsymbol{r_1}, \boldsymbol{r_2}] = \boldsymbol{r_{11}} \cdot (\boldsymbol{r_1} \times \boldsymbol{r_2}) = \boldsymbol{r_{11}} \cdot (H\boldsymbol{N}) = H(\boldsymbol{r_{11}} \cdot \boldsymbol{N}) = HL$$

Similarly

$$[\boldsymbol{r_{12}}, \boldsymbol{r_1}, \boldsymbol{r_2}] = \boldsymbol{r_{12}} \cdot (\boldsymbol{r_1} \times \boldsymbol{r_2}) = \boldsymbol{r_{12}} \cdot (H\boldsymbol{N}) = H(\boldsymbol{r_{12}} \cdot \boldsymbol{N}) = HM$$

and

$$[\boldsymbol{r_{22}}, \boldsymbol{r_1}, \boldsymbol{r_2}] = \boldsymbol{r_{22}} \cdot (\boldsymbol{r_1} \times \boldsymbol{r_2}) = \boldsymbol{r_{22}} \cdot (H\boldsymbol{N}) = H(\boldsymbol{r_{22}} \cdot \boldsymbol{N}) = HN$$

Alternative expressions for the values of L, M, N in terms of the derivatives of unit normal vector $\boldsymbol{N}$.

We know that the vectors $\boldsymbol{r_1}$ and $\boldsymbol{r_2}$ are tangential to the surface at the point $\boldsymbol{r}$.

Therefore, the unit normal vector N is perpendicular to both the vectors r_1 and r_2. So, we have

$$r_1 \cdot N = 0 \qquad (1)$$

Differentiating the relation (1) with respect to u, we get

$$r_{11} \cdot N + r_1 \cdot N_1 = 0.$$
$$\Rightarrow r_{11} \cdot N = -(r_1 \cdot N_1)$$
$$\Rightarrow L = -r_1 \cdot N_1$$

Also differentiating the relation (1) with respect to v, we have

$$r_{12} \cdot N + r_1 \cdot N_2 = 0$$
$$\Rightarrow r_{12} \cdot N = -(r_1 \cdot N_2)$$
$$\Rightarrow M = -r_1 \cdot N_2$$

Further we have

$$r_2 \cdot N = 0 (2)$$

Differentiating (2) with respect to u, we get

$$r_{12} \cdot N + r_2 \cdot N_1 = 0$$
$$\Rightarrow r_{12} \cdot N = -(r_2 \cdot N_1)$$
$$\Rightarrow M = -r_2 \cdot N_1$$

Similarly differentiating (2) with respect to v, we get

$$r_{22} \cdot N + r_2 \cdot N_2 = 0$$
$$\Rightarrow r_{22} \cdot N = -(r_2 \cdot N_2)$$
$$\Rightarrow N = -r_2 \cdot N_2$$

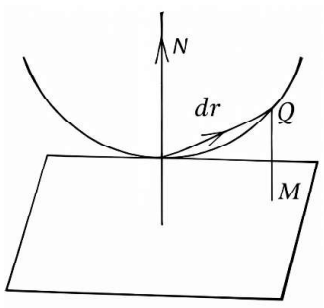
Thus we have

$$L = -r_1 \cdot N_1, M = -r_1 \cdot N_2 = -r_2 \cdot N_1, N = -r_2 \cdot N_2$$

Geometrical interpretation of the second fundamental form:

Suppose $r = r(u,v)$ is the equation of a surface and let r, $r + dr$ be the position vectors of two neighbouring points $P(u,v)$ and $Q(u+du, v+dv)$ on this surface. By Taylor's theorem, we have

$$r + dr = r + (r_1\, du + r_2\, dv) + \frac{1}{2}(r_{11}\, du^2 + 2r_{12}\, du\, dv + r_{22}\, dv^2) + \cdots$$



$\therefore$ Neglecting terms of order higher than two in du, dv

$$dr = \overrightarrow{PQ} = (r_1\, du + r_2\, dv) + \frac{1}{2}(r_{11}\, du^2 + 2r_{12}\, du\, dv + r_{22}\, dv^2)$$

The length of the perpendicular QM from Q on the tangent plane at P

= the projection of the vector $\overrightarrow{PQ}$ on the normal at P

$$= \overrightarrow{PQ} \cdot N = dr \cdot N$$

$$= (r_1\, du + r_2\, dv) \cdot N + \frac{1}{2}(r_{11}\, du^2 + 2r_{12}\, du\, dv + r_{22}\, dv^2) \cdot N.$$

$$= \frac{1}{2}(r_{11}.N\, du^2 + 2r_{12}.N du\, dv + r_{22}.N\, dv^2)$$

$[\because\ r_1 \cdot N = 0,\ r_2 \cdot N = 0,\ N$ being at right angles to r_1 and $r_2]$

$$= \frac{1}{2}(L\, du^2 + 2M\, du\, dv + N\, dv^2).$$

$$\therefore\ 2QM = L\, du^2 + 2M\, du\, dv + N\, dv^2$$

Hence, we have the following geometrical interpretation of the second fundamental form:

The value of the second fundamental form $L\, du^2 + 2M\, du\, dv + N\, dv^2$ at the point $P(u,v)$ is equal to twice the length of the perpendicular from the neighbouring point $Q(u + du, v + dv)$ on the tangent plane at P if quantities of order higher than two in du, dv are neglected.

8.10 SURFACE OF REVOLUTION

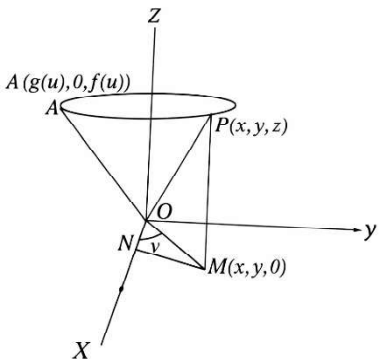
General Definition: When a plane curve is rotated around an axis in its own plane, it generates a surface called a surface of revolution.

(a) Equation of General Surface of Revolution:

Take the z-axis for the axis of revolution and the generating curve in the xz-plane be given by the parametric equations

$$x = g(u), y = 0, z = f(u)$$

Let $A(g(u),0,f(u))$ be any point on this generating curve with the position $P(x,y,z)$ after rotation of the generating curve through an angle v about the z-axis. Then



$P$is any point on the surface of revolution and $v$is called longitude of the point P. Angle between the planes $ZOX$and $ZOP$is v i.e.$\angle XOM = v$. Co-ordinates of $M$are$(x, y, 0)$.

Now,z-coordinate of $P = z$-coordinate of $A = f(u)$

$$\therefore z = f(u)$$

AlsoOM = perpendicular distance of $P$from $z-$ axis

= perpendicular distance of A from z-axis

= x-coordinate of A

$= g(u)$.

If we drop MNperpendicular from $M$to OX,

The$x$-coordinate of $P = ON = OM\cos v = g(u)\cos v$,and

the y-coordinate of $P = NM = g(u)\sin v$.

Thus, we have

$$x = g(u)\cos v, y = g(u)\sin v, z = f(u) \qquad (1)$$

as the equations of the surface of revolution. The domain of u, $v$is $0 \leq v < 2\pi$together with the range of u.

In equation (1), the parametric curves $v =$ constantare called meridian lines. These are obtained from different positions of the generating curve, or equivalently, from the intersections of the surface with planes that pass through the axis.

Similarly, the parametric curves $u =$ constantare called parallels, which are circles lying in planes parallel to the xy-plane.

In vector notation the equation of the surface of revolution may be written as

$$\mathbf{r} = (g(u)\cos v, g(u)\sin v, f(u)).$$

We have

$$\mathbf{r}_1 = (g'\cos v, g'\sin v, f').$$

$$\mathbf{r}_2 = (-g\sin v, g\cos v, 0).$$

$$\therefore \quad E = \mathbf{r}_1^2 = g'^2 + f'^2, F = \mathbf{r}_1 \cdot \mathbf{r}_2 = 0, G = \mathbf{r}_2^2 = g^2.$$

Since $F = 0$, therefore the parametric curves are orthogonal, i.e. the parallels cut the meridians orthogonally.

Also

$$H^2 = EG - F^2 = g^2(g'^2 + f'^2).$$

$$\Rightarrow H = g\sqrt{g'^2 + f'^2}$$

The unit normal to the surface is

$$\mathbf{N} = \frac{\mathbf{r}_1 \times \mathbf{r}_2}{H} = \frac{1}{H}(-gf'\cos v,\ -gf'\sin v,\ gg')$$

$$= \frac{1}{\sqrt{g'^2 + f'^2}}(-f'\cos v,\ -f'\sin v,\ g')$$

Note. Suppose $P(x,y,z)$ is any point on a surface of revolution. To write the equations of this surface, let us take the parameter u as the perpendicular distance of P from the axis, and the parameter v as the longitude of the point P. Then the equations of the surface of revolution are

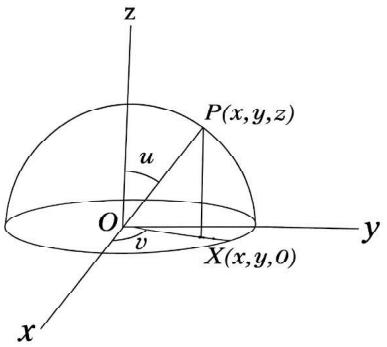
$$x = u\cos v, y = u\sin v, z = f(u).$$

(b) The Sphere:

A sphere can be considered as a surface of revolution obtained by rotating the circle

$$x = a\sin u, y = 0, z = a\cos u$$

about the z-axis.



So, putting $g(u) = a\sin u$ and $f(u) = a\cos u$ in the equations (1) of the surface of revolution, we get the equations of the sphere as

$$x = a\sin u\cos v,\ y = a\sin u\sin v,\ z = a\cos u.$$

We can also find the equations of the sphere in the following way.

Let O be the centre of the sphere and a be its radius. Take three mutually perpendicular diameters of the sphere as the coordinate axes. Let $P(x,y,z)$ be any point on the surface of the sphere.

The co-latitude u of the point P is defined as the angle between the radius OP and the z-axis, and the longitude v is defined as the angle between the plane containing P and the z-axis, and the zx-plane.

Then the coordinates of P are given by:

$$x = a\sin u\cos v,\ y = a\sin u\sin v,\ z = a\cos u.$$

The vector equation of the sphere is

$$\mathbf{r} = (a\sin u\cos v,\ a\sin u\sin v,\ a\cos u),$$

the parameters being u and v. The parametric curves $u = \text{constant}$ are the small circles called the parallels of latitude; the parametric curves $v = \text{constant}$ are great circles called the meridians of longitude. The poles $u = 0$ and $u = \pi$ are (artificial) singularities, and the domain of u, v is $0 < u < \pi,\ 0 \leq v < 2\pi$.

We have

$$\mathbf{r}_1 = (a\cos u\cos v,\ a\cos u\sin v,\ -a\sin u),$$

$$\mathbf{r}_2 = (-a\sin u\sin v,\ a\sin u\cos v,\ 0).$$

$$\therefore\ F = \mathbf{r}_1 \cdot \mathbf{r}_2 = 0.$$

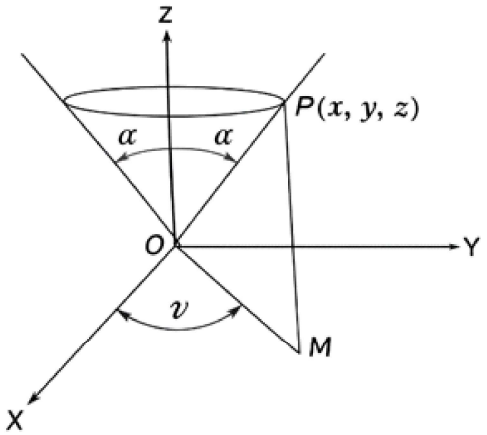
So, the parametric curves are orthogonal i.e. the latitudes and longitudes intersect orthogonally.

Note. The latitude λ of the point P on the surface of the sphere is defined as the angle between the radius through P and the xy-plane. If we take the latitude λ of P and the longitude v of P as the parameters, then the equations of the sphere are

$$x = a\cos\lambda\cos v, y = a\cos\lambda\sin v, z = a\sin\lambda.$$

(c) Right circular cone of semi-vertical angle α.

Take the z-axis as the axis of the cone. Let P be any point (x, y, z) on the surface of the cone. Let u be the distance of P from the z-axis and v be the inclination of the plane containing P and the z-axis to the zx-plane. Then



$$x = u\cos v, y = u\sin v,$$

$$z = u\cot\alpha.$$

Vector equation of the cone is

$$\mathbf{r} = (u\cos v,\ u\sin v,\ u\cot\alpha).$$

We have

$$\mathbf{r}_1 = (\cos v,\ \sin v,\ \cot\alpha),$$

$$\mathbf{r}_2 = (-u\sin v,\ u\cos v,\ 0).$$

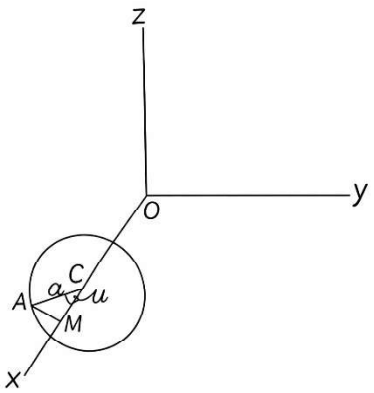
Obviously $F = \mathbf{r}_1 \cdot \mathbf{r}_2 = 0$, so the parametric curves are orthogonal.

We have

$$\mathbf{r}_1 \times \mathbf{r}_2 = (-u\cos v\cot\alpha,\ -u\sin v\cot\alpha,\ u).$$

Obviously $\mathbf{r}_1 \times \mathbf{r}_2 = 0$ only at $u = 0$. So the vertex is the only singularity.

(d) Anchor ring:



This is the surface obtained by rotating a circle of radius a about a line in its plane at a distance $b\ (> a)$ from the centre.

Take the axis of revolution as the z-axis and the generating circle in the xz-plane with its centre C on the x-axis. We have

$$CO = b > a.$$

Let A be any point on the circle and u be the angle which the radius C makes with the x-axis. The co-ordinates of A are

$$(b + a\cos u,\ 0,\ a\sin u).$$

Let $P(x, y, z)$ be the position of the point A when the generating circle has been rotated through an angle v. Then

$$x = (b + a\cos u)\cos v, y = (b + a\cos u)\sin v, z = a\sin u$$

are the equations of the anchor ring.

The domain of u, v is $0 < u < 2\pi,\ 0 < v < 2\pi$.

8.11 FAMILY OF CURVES

Suppose

$$\mathbf{r} = \mathbf{r}(u, v) \qquad (1)$$

is the equation of a surface.

Let $\phi(u, v)$ be a single-valued function of u and v which has continuous derivatives ϕ_1, ϕ_2 which do not vanish together. Then the equation

$$\phi(u, v) = c \qquad (2)$$

where c is a real parameter, gives a family of curves lying on the surface (1). For different values of c, we get different curves lying on (1). Suppose $\phi(u_0, v_0) = k$ where (u_0, v_0) is any point on the surface (1). Then $\phi(u, v) = k$ is a member of the family (2)

which passes through the point (u_0, v_0). Thus, through every point (u, v) of the surface (1) there passes one and only one member of the family (2).

Differentiating (2), we get

$$\phi_1\, du + \phi_2\, dv = 0.$$

$$\therefore \frac{du}{dv} = -\frac{\phi_2}{\phi_1}$$

Thus, $(-\phi_2, \phi_1)$ are the direction ratios of the tangent at the point (u, v) to the curve of the family (2) passing through that point.

If ϕ_1 and ϕ_2 both become zero at any point, then the tangent at that point is not well-defined. Therefore, it is necessary to place a restriction on ϕ.

The curves of the family $\phi(u, v) = \text{constant}$ are the solutions of the differential equation

$$\phi_1\, du + \phi_2\, dv = 0.$$

Conversely, every first order differential equation of the form

$$P(u, v)\, du + Q(u, v)\, dv = 0 \qquad (3)$$

where P and Q are class 1 functions which do not vanish simultaneously always defines a family of curves. From the theory of differential equations, we know that under the conditions imposed on P and Q, the differential equation (3) is integrable. Thus, there exist functions $\lambda(u, v) \neq 0$ and $\phi(u, v)$ such that $\lambda P = \phi_1$ and $\lambda Q = \phi_2$. Then equation (3) becomes

$$\frac{1}{\lambda}\,[\phi_1\, du + \phi_2\, dv] = 0$$

$$\Rightarrow \phi_1\, du + \phi_2\, dv = 0$$

whose solution is $\phi(u, v) = \text{const}$.

For the family of curves given by (3), the tangent at the point (u, v) has direction ratios $(-Q, P)$, since these are proportional to (du, dv).

CHECK YOUR PROGRESS

Q5. Show that for the surface of revolution

$$x = u\cos v, \qquad y = u\sin v, \qquad z = f(u)$$

the parametric curves form an orthogonal system and

$$ds^2 = (1 + f'^2)du^2 + u^2 dv^2$$

where f' denotes differentiation of f with respect to u.

Q6. Calculate the magnitudes for the right helicoid given by

$x = u\cos\phi,\ y = u\sin\phi,\ z = c\phi$, the parameters being u and ϕ.

2 8.12 LET US SUM UP

- The study of surfaces in differential geometry deals with understanding their shape, structure, and curvature in three-dimensional space. Fundamental forms are important tools used to describe these properties mathematically.
- The first fundamental form, also called the metric, is used to measure geometrical quantities such as lengths, angles, and areas on a surface. It is expressed in terms of the coefficients E, F, G, which depend on the parameters of the surface.
- The coefficients E, F, G are functions of the parameters and vary from point to point on the surface. They satisfy the condition $EG - F^2 > 0$, which ensures that the surface is regular and measurements are meaningful.
- The first fundamental form also helps in finding the angle between parametric curves. The parametric curves are orthogonal if and only if $F = 0$, which is an important condition in surface theory.
- The second fundamental form is used to study the curvature of the surface. It involves the coefficients L, M, N, which describe how the surface bends in space relative to the normal direction.
- The second fundamental form has a geometrical interpretation, as it represents the distance of nearby points from the tangent plane, giving information about the deviation of the surface from a plane.
- A surface of revolution is obtained by rotating a plane curve about an axis in its plane. Such surfaces have special parametric curves called meridians and parallels, which are orthogonal to each other.
- The concept of a family of curves on a surface helps in studying curves defined by parameters. The direction of tangents to these curves can be obtained using differential equations, which describe their behavior on the surface.

K 8.13 ANSWERS TO CHECK YOUR PROGRESS

Answer to Q1: The position vector of any current point on the surface is given by

$$\mathbf{r} = \left(x, y, \frac{a}{2}x^2 + hxy + \frac{b}{2}y^2\right)$$

$$\therefore \mathbf{r}_1 = (1,0, ax + hy), \mathbf{r}_2 = (0,1, hx + by)$$

$$\mathbf{r}_{11} = (0,0,a), \mathbf{r}_{12} = (0,0,h), \mathbf{r}_{22} = (0,0,b)$$

$$\mathbf{N} = \frac{\mathbf{r}_1 \times \mathbf{r}_2}{H} = \frac{(-(ax+hy), -(hx+by), 1)}{H}$$

$$E = \mathbf{r}_1 \cdot \mathbf{r}_1 = 1 + (ax+hy)^2, F = \mathbf{r}_1 \cdot \mathbf{r}_2 = (ax+hy)(hx+by)$$

$$G = \mathbf{r}_2 \cdot \mathbf{r}_2 = 1 + (hx+by)^2$$

$$H^2 = EG - F^2 = 1 + (ax+hy)^2 + (hx+by)^2$$

$$\therefore L = \mathbf{N} \cdot \mathbf{r}_{11} = \frac{a}{H}, M = \mathbf{N} \cdot \mathbf{r}_{12} = \frac{h}{H}, N = \mathbf{N} \cdot \mathbf{r}_{22} = \frac{b}{H}$$

Answer to Q2: The given surface is

$$\mathbf{r} = [u\cos v, u\sin v, f(u)].$$

$$\therefore \mathbf{r}_1 = [\cos v, \sin v, f'(u)], \mathbf{r}_2 = (-u\sin v, u\cos v, 0)$$

$$E = \mathbf{r}_1^2 = \cos^2 v + \sin^2 v + f'^2 = 1 + f'^2$$

$$F = \mathbf{r}_1 \cdot \mathbf{r}_2 = -u\cos v\sin v + u\sin v\cos v + 0 = 0$$

$$G = \mathbf{r}_2^2 = u^2\sin^2 v + u^2\cos^2 v + 0 = u^2$$

Answer to Q3: The equation of the given surface is

$$\boldsymbol{r} = ((b + a\cos u)\cos v, (b + a\cos u)\sin v, a\sin u).$$

We have

$$\boldsymbol{r_1} = (-a\sin u\cos v, -a\sin u\sin v, a\cos u),$$

$$\boldsymbol{r_2} = (-(b + a\cos u)\sin v, (b + a\cos u)\cos v, 0).$$

$$\therefore E = \boldsymbol{r_1^2} = a^2\sin^2 u\cos^2 v + a^2\sin^2 u\sin^2 v + a^2\cos^2 u = a^2\sin^2 u + a^2\cos^2 u = a^2,$$

$$F = \boldsymbol{r_1} \cdot \boldsymbol{r_2} = a(b + a\cos u)\sin u\cos v\sin v - a(b + a\cos u)\sin u\cos v\sin v + 0 = 0$$

$$G = \boldsymbol{r_2^2} = (b + a\cos u)^2(\cos^2 v + \sin^2 v) = (b + a\cos u)^2$$

Now

$$H = \sqrt{(EG - F^2)} = \sqrt{[a^2(b + a\cos u)^2]}[\because F = 0]$$

$$= \sqrt{a^2(b + a\cos u)^2}$$

$$= a(b + a\cos u)$$

Elementary area at the point $(u, v)$of the surface $= H\, du\, dv$

Hence the area of the whole surface

$$= \int_0^{2\pi}\int_0^{2\pi} H du dv$$

$$= \int_0^{2\pi}\int_0^{2\pi} a(b + a\cos u) du dv$$

$$= 2a\pi[bu + a\sin u]_0^{2\pi}$$

$$= 4ab\pi^2$$

Answer to Q4: The equation of the given surface is

$$\boldsymbol{r} = (u, v, u^2 - v^2)$$

We have

$$\boldsymbol{r_1} = (1,0,2u), \boldsymbol{r_2} = (0,1,-2v).$$

$$\therefore E = \boldsymbol{r_1^2} = 1 + 4u^2,$$

$$F = \boldsymbol{r_1} \cdot \boldsymbol{r_2} = 0 + 0 - 4uv = -4uv,$$

$$G = \boldsymbol{r_2^2} = 1 + 4v^2$$

Also

$$H = \sqrt{(EG - F^2)} = \sqrt{[(1 + 4u^2)(1 + 4v^2) - 16u^2v^2]}$$

$$= \sqrt{(1+4u^2+4v^2)}$$

Answer to Q5: The equation of the given surface is

$$\boldsymbol{r} = (u\cos v, u\sin v, f(u)).$$

We have

$$\boldsymbol{r_1} = (\cos v, \sin v, f'(u)),$$

$$\boldsymbol{r_2} = (-u\sin v, u\cos v, 0).$$

$$\therefore E = \boldsymbol{r_1^2} = \cos^2 v + \sin^2 v + [f'(u)]^2$$

$$= 1 + [f'(u)]^2 = 1 + f'^2$$

$$F = \boldsymbol{r_1} \cdot \boldsymbol{r_2} = -u\cos v\sin v + u\sin v\cos v + 0 = 0,$$

$$G = \boldsymbol{r_2^2} = u^2\sin^2 v + u^2\cos^2 v = u^2$$

Since $F = 0$, therefore the parametric curves are orthogonal.

Also

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2$$

$$= (1 + f'^2)du^2 + u^2dv^2.$$

Answer to Q6: The equation of the given surface is

$$\mathbf{r} = (u\cos\phi, u\sin\phi, c\phi).$$

Taking u as the first parameter and φ as the second parameter, we have

$$\mathbf{r}_1 = (\cos\phi, \sin\phi, 0),$$

$$\mathbf{r}_2 = (-u\sin\phi, u\cos\phi, c).$$

Therefore

$$E = \mathbf{r}_1 \cdot \mathbf{r}_1 = \cos^2\phi + \sin^2\phi + 0 = 1,$$

$$F = \mathbf{r}_1 \cdot \mathbf{r}_2 = -u\cos\phi\sin\phi + u\sin\phi\cos\phi + 0 = 0,$$

$$G = \mathbf{r}_2 \cdot \mathbf{r}_2 = u^2 \sin^2\phi + u^2 \cos^2\phi + c^2 = u^2 + c^2$$

Also

$$H = \sqrt{EG - F^2} = \sqrt{u^2 + c^2}$$

Since $F = 0$, the parametric curves are orthogonal.

We have

$$\mathbf{r}_1 \times \mathbf{r}_2 = (c\sin\phi, \ -c\cos\phi, u).$$

Therefore, the unit normal N to the surface is given by

$$N = \frac{\mathbf{r}_1 \times \mathbf{r}_2}{H} = \frac{(c\sin\phi, \ -c\cos\phi, u)}{\sqrt{u^2 + c^2}}.$$

Further

$$\mathbf{r}_{11} = (0, 0, 0),$$

$$\mathbf{r}_{12} = (-\sin\phi, \cos\phi, 0),$$

$$\mathbf{r}_{22} = (-u\cos\phi, \ -u\sin\phi, 0).$$

The second order magnitudes L, M, N are given by

$$HL = [\mathbf{r}_{11}, \boldsymbol{r}_1, \boldsymbol{r}_2] = 0,$$

$$HM = [\mathbf{r}_{12}, \boldsymbol{r}_1, \boldsymbol{r}_2] = (-\sin\phi, \cos\phi, 0) \cdot (c\sin\phi, \ -c\cos\phi, u)$$

$$= (-c\sin^2\phi - c\cos^2\phi + 0) = -c,$$

$$HN = \mathbf{r}_{22} \cdot (\boldsymbol{r}_1 \times \boldsymbol{r}_2) = (-u\cos\phi, \ -u\sin\phi, 0) \cdot (c\sin\phi, \ -c\cos\phi, u)$$

$$= -cu\sin\phi\cos\phi + cu\sin\phi\cos\phi = 0$$

$$\therefore L = 0, \qquad M = -\frac{c}{H}, N = 0$$

O 8.14 FURTHER READING

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[3] Willmore, T. J. 2012. *An Introduction to Differential Geometry*. Dover Publications Inc.

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M 8.15 MODEL QUESTIONS

Q1. Find the fundamental magnitudes and the unit normal for the surface of revolution

$$x = u\cos v,\ y = u\sin v,\ z = f(u).$$

Q2. On the surface of revolution

$$x = u\cos v,\ y = u\sin v,\ z = f(u),$$

what are the parametric curves $u = \text{constant}$ and what are the curves $v = \text{constant}$?

Q3. Prove that the metric is invariant under a parameter transformation but the coefficients E, F, G are not.

Q4. Taking x, y as parameters, calculate the fundamental magnitudes and the unit normal to the surface

$$z = \frac{1}{2}(ax^2 + 2hxy + by^2).$$

Q5. Find the metric for the paraboloid

$$\mathbf{r} = (u, v, u^2 - v^2).$$

Q6. Calculate the fundamental magnitudes and the unit normal for the conoid

$$x = u\cos v, y = u\sin v, z = f(v),$$

with u, v as parameters.

Q7. On the right helicoid given by

$$x = u\cos v,\ y = u\sin v,\ z = cv,$$

show that the parametric curves are circular helices and straight lines.

Q8. Show that the metric defined by $ds^2 = Edu^2 + 2Fdudv + Gdv^2$ is invariant under a parametric transformation.

Q9. Show that if L, M, N vanish everywhere on a surface then the surface is part of the plane.

Q10. Find the fundamental magnitudes and the unit normal for the helicoid

$$x = u\cos v, y = u\sin v, z = cv,$$

where c is constant.

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