UNIT 9:LOCAL NON-INTRINSIC PROPERTIES OF A SURFACE

UNIT STRUCTURE

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9.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- understand the idea of local non-intrinsic properties of a surface in $\mathbb{R}^3$.
- define normal curvature and derive its expression using the first and second fundamental forms.
- explain principal curvatures and principal directions and describe their orthogonality.
- characterize minimal surfaces using the condition on mean curvature.
- identify developable surfaces and state conditions for zero Gaussian curvature.
- describe lines of curvature and their geometric significance on a surface.

9.2 INTRODUCTION

Local non-intrinsic properties of a surface describe how the surface bends in

three-dimensional space, in contrast with intrinsic properties that depend only on distances measured along the surface. These properties depend essentially on how the surface is embedded in $\mathbb{R}^3$ and make explicit use of the normal vector to the surface.

The first fundamental form comes from the scalar products of tangent vectors and encodes lengths and angles of curves on the surface. The second fundamental form, obtained by projecting second derivatives of the position vector on the unit normal, measures how the surface curves away from its tangent plane. Using these forms, one defines normal curvature in any tangent direction, then principal curvatures and principal directions as extremal values and associated directions of the normal curvature. Minimal surfaces appear when the mean curvature vanishes, while developable surfaces are characterized by having zero Gaussian curvature and can be unfolded onto the plane. Lines of curvature are curves that follow principal directions on the surface and provide a natural orthogonal net, except at umbilic points.

9.3 NORMAL CURVATURE

Definition and Formula:

When a plane P' is drawn through a point P on a surface, it cuts the surface in a curve known as the section of the surface. If this plane P' contains the normal to the surface, the curve formed is called the *normal section*; otherwise, it is called the *oblique section*. From the figure, we see that the principal normal n to the normal section is parallel to the surface normal N. According to our convention, the vector n is taken to have the same direction as the vector N, and therefore, we can write

$n = N$.

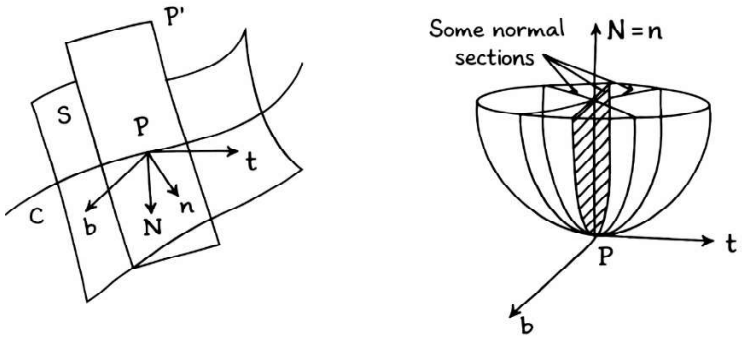


Figure 1 Sections of surfaces

In terms of Fundamental Magnitudes, the formula for Normal Curvature:

Consider $r = r(u,v)$ be an equation of surface and P be any point on it. Let κ_n be the curvature of the normal section, which will positive if the curve is concave on the side towards which N points out. We have,

$$\frac{dt}{ds} = r'' = \kappa_n n = \kappa_n N$$
$$\therefore \kappa_n = N.r''$$

Again,

$$r' = \frac{dr}{ds} = \frac{\partial r}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial r}{\partial v}\frac{\partial v}{\partial s}$$
$$= r_1 u' + r_2 v'$$

$$\therefore r'' = r_1 u'' + \frac{dr_1}{ds}u' + r_2 v'' + \frac{dr_2}{ds}v'$$

$$= r_1 u'' + r_2 v'' + \left(\frac{\partial r_1}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial r_1}{\partial v}\frac{\partial v}{\partial s}\right)u' + \left(\frac{\partial r_2}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial r_2}{\partial v}\frac{\partial v}{\partial s}\right)v'$$

$$= r_1 u + r_2 v + r_{11}u'^2 + r_{12}u'v' + r_{21}u'v' + r_{22}v'^2$$

$$= r_1 u + r_2 v + r_{11}u'^2 + 2r_{12}u'v' + r_{22}v'^2$$

$$\therefore \kappa_n = (r_1 u + r_2 v + r_{11}u'^2 + 2r_{12}u'v' + r_{22}v'^2).N$$

Again, we know that

$$r_1.N = 0, \quad r_2.N = 0, \quad r_{11}.N = L, \quad r_{12}.N = M, \quad r_{22}.N = N$$

Thus,

$$\kappa_n = N.r'' = Lu'^2 + 2Mu'v' + Nv'^2$$

$$= L\left(\frac{du}{ds}\right)^2 + 2M\frac{du}{ds}\frac{dv}{ds} + N\left(\frac{dv}{ds}\right)^2$$

$$= \frac{Ldu^2 + 2Mdudv + Ndv^2}{ds^2}$$

By, using first fundamental form, we have

$$\kappa_n = \frac{Ldu^2 + 2Mdudv + Ndv^2}{Edu^2 + 2Fdudv + Gdv^2}$$

This equation gives the curvature of the normal section, which is called normal curvature parallel to the direction (du, dv). Its reciprocal is called the Radius of normal curvature (ρ_n).

Definition: If P be a point with position vector $\boldsymbol{r}(u,v)$ on the surface $\boldsymbol{r} = \boldsymbol{r}(u,v)$, the normal curvature at P in the direction (du, dv) is equal to the curvature at P of the normal section at P parallel to that direction.

Let N be the unit normal vector to the surface at P, then the normal curvature κ_n is given by $\kappa_n = N.r''$, where r'' is the curvature vector at P. Let κ be the curvature at P of the normal section at P containing the direction (du, dv). Then

$$r'' = \kappa_n = \kappa N$$

$$\therefore N.r'' = N.(\kappa N) = \kappa$$

Hence $\kappa_n = \kappa$

Thus, we can say that the curvature at P of normal section at P containing the direction (du, dv) is equal to the normal curvature at P in the same direction.

Meusnier's Theorem:

If κ and κ_n are the curvatures of oblique and normal sections through the same tangent line and θ be the angle between these sections, then

$$\kappa_n = \kappa \cos\theta$$

Proof: Let P be a point (u, v) on the surface $r = r(u, v)$, and r'' be the curvature vector at P of the oblique section through P, containing the direction (du, dv). Then

$$r'' = \kappa n \quad (1)$$

where n is the unit principal normal vector to the oblique section at P.

Again, the unit normal vector N to the surface at Pis the unit principal normal vector of the normal section at Pparallel to the direction (du, dv). Since $\theta$is the angle between oblique and normal sections at Pthrough the same tangent line, so $\theta$is the angle between the vectors n and N.

$$n.N = cos\theta$$

Now taking dot product of both sides of (1) by N, we have

$$r''.N = \kappa n.N = \kappa cos\theta$$

Again $r''.N$ be the normal curvature at Pin the direction (du, dv) which equal to the curvature of the normal section at P parallel to be direction $(du, dv) = \kappa_n$.

Therefore, $\kappa_n = \kappa cos\theta$

9.4 PRINCIPAL DIRECTIONS

Principal Directions:

The normal sections of a surface through a given point having maximum or minimum curvatures at the point are called principal sections of the surface at that point

and the tangents to these sections are called **principal directions** at the point. There are usually two main directions at any point on a surface, and it will be proven that they are perpendicular to each other.

Principal Curvatures:

The curvatures of the principal sections of a surface through a given point i.e., the maximum and minimum curvatures at that point are called principal curvatures at that point and their corresponding radius of curvatures are called principal radius of curvatures.

Mathematical Equation of Principal Curvature:

The Normal curvature κ_n at a point P(u,v) in the direction (du,dv) is,

$$\kappa_n = \frac{Ldu^2 + 2Mdudv + Ndv^2}{Edu^2 + 2Fdudv + Gdv^2}$$

If (l,m) be the actual direction coefficients of the direction (du,dv), then

$$\kappa_n = \frac{Ll^2 + 2Mlm + Nm^2}{El^2 + 2Flm + Gm^2}$$

Where,

$$El^2 + 2Flm + Gm^2 = 1 \quad (1)$$

$$\therefore \quad \kappa_n = Ll^2 + 2Mlm + Nm^2 \quad (2)$$

The value of κ_n depends upon l, m at P, since L, M, N are fixed at that point. Hence κ_n is a function of two variable l and m. It is also related to the equation (2). We now use Lagrange's method of undetermined multipliers to find the maximum or minimum value of κ_n :

We have, $d\kappa_n = 0$

$$\Rightarrow 2Lldl + 2Mldm + 2Mmdl + 2Nmdm = 0$$

$$\Rightarrow (Ll + Mm)dl + (Ml + Nm)dm = 0 \quad (3)$$

Differentiating and simplifying (1), we get

$$(El + Fm)dl + (Fl + Gm)dm = 0 \quad (4)$$

Now, multiplying (4) by λ, we get

$$\lambda(El + Fm)dl + \lambda(Fl + Gm)dm = 0$$

Adding this equation to (3) and also equating the coefficients of dl and dm to zero, we get

$$(Ll + Mm) + \lambda(El + Fm) = 0 \quad (5)$$

$$(Ml + Nm) + \lambda(Fl + Gm) = 0 \quad (6)$$

Multiply (5) by l and (6) by m, and the adding, we get

$$(Ll^2 + 2Mlm + Nm^2) + \lambda(El^2 + 2Flm + Gm^2) = 0$$

$$\Rightarrow \kappa_n = \lambda \qquad \text{[using (1) and (2)]}$$

$$\Rightarrow \lambda = -\kappa_n$$

Using this in (5) and (6), we have

$$(Ll + Mm) - \kappa_n(El + Fm) = 0 \quad (7)$$

$$(Ml + Nm) - \kappa_n(Fl + Gm) = 0 \quad (8)$$

From (7), we get

$$(L - \kappa_n E)l + (M - \kappa_n F)m = 0 \quad (9)$$

From (8), we get

$$(M - \kappa_n F)l + (N - \kappa_n G)m = 0 \quad (10)$$

From (9) and (10), we have

$$(L - \kappa_n E)(N - \kappa_n G) = (M - \kappa_n F)(M - \kappa_n F)$$

$$\Rightarrow \kappa_n^2(EG - F^2) - \kappa_n(EN + LG - 2FM) + (N - M^2) = 0 \quad (11)$$

This is the required quadratic equation giving the maximum or minimum values of normal curvature at P. Its roots are principal curvatures of the surface at Pand are usually denoted by $\kappa_a$and κ_b. Thus, we have

$$\kappa_a + \kappa_b = \frac{EN + LG - 2FM}{EG - F^2}$$

$$\kappa_a\kappa_b = \frac{LN - M^2}{EG - F^2} = \frac{T^2}{H^2}$$

General Definitions:

(a) First curvature (J): The sum of the principal curvature $\kappa_a$and κ_b is called the first curvature at the point and defined by,

$$J = \kappa_a + \kappa_b = \frac{EN + LG - 2FM}{EG - F^2}$$

(b) Mean normal curvature or Mean curvature (μ): The arithmetic mean of the principal curvature $\kappa_a$and $\kappa_b$at a point is called the mean curvature at the point and

$$\mu = \frac{1}{2}(\kappa_a + \kappa_b) = \frac{EN + LG - 2FM}{2(EG - F^2)}$$

(c) Gaussian curvature (K): The product of the principal curvatures $\kappa_a$and κ_b at any point is called Gaussian curvature at that point and

$$K = \kappa_a\kappa_b = \frac{LN - M^2}{EG - F^2} = \frac{T^2}{H^2}$$

Elliptic, Hyperbolic and Parabolic Points:

i. **Elliptic points:** The points on the surface at which the principal curvatures κ_a and κ_b have the same signs i.e. the Gaussian curvature K is positive are called Elliptic points.

   Thus, a point is an elliptic point if $LN - M^2 > 0$

ii. **Hyperbolic points:** The points on the surface at which the Gaussian curvature K is negative are called Hyperbolic points. In this case the principal curvatures at the points are of opposite signs.

   Thus, a point is a hyperbolic point if $LN - M^2 < 0$.

iii. **Parabolic points:** The points on the surface at which the Gaussian curvature K is equal to zero are called Parabolic points.

   Thus, a point is a parabolic point if $LN - M^2 = 0$.

Equation for the Principal Direction at a Point on the Given Surface:

The direction having the maximum and minimum normal curvatures [equations (5) and (6)] are:

$$(Ll + Mm) + \lambda(El + Fm) = 0 \qquad (1)$$

$$(Ml + Nm) + \lambda(Fl + Gm) = 0 \qquad (2)$$

Eliminating λ we get

$$(Ll + Mm)(Fl + Gm) = (Ml + Nm)(El + Fm)$$

$$\Rightarrow (EM - Fl)l^2 + (EN - GL)lm + (FN - GM)m^2 = 0 \qquad (3)$$

Which gives the principal directions at the given point. Now replacing (l, m) by direction ratios (du, dv), the equation (3) reduces to

$$(EM - Fl)du^2 + (EN - GL)dudv + (FN - GM)dv^2 = 0 \qquad (4)$$

Now (4) is a quadratic equation in $\frac{du}{dv}$ therefore there are in general two principal directions at each point of the surface and the two directions given by:

$$Pdu^2 + 2Qdudv + Rdv^2 = 0$$

Are orthogonal if

$$ER - 2FQ + GP = 0 \qquad (5)$$

Comparing (4) and (5), we have

$$P = EM - FL, \qquad Q = \frac{EN - GL}{2}, \qquad R = FN - GM$$

Hence,

$$ER - 2FQ + GP = E(FN - GM) - F(EN - GL) + G(EM - FL) = 0$$

Therefore, the two-direction given by (4) are orthogonal hence the principal direction are orthogonal. The discriminant of equation (4) is

$$(EN - GL)^2 - 4(EM - FL)(FN - GM)$$

$$\equiv \frac{4(EG - F^2)}{E^2}(EM - FL)^2 + \{EN - GL - \frac{2F}{E}(EM - FL)\}$$

But $EG - F^2 > 0$, if follows that the roots of equation (4) are real and distinct, provided that the coefficients E, F, G and L, M, N are not proportional.Thus, if at a point Pthe coefficients $E, F, G$are not proportional to L, M, N then we have two real and distinct principal direction at P which are orthogonal. When$\frac{L}{E} = \frac{M}{F} = \frac{N}{G}$, the equation (4) fails to determine principal directions.

Umbilic Definition: A point on a surface is called an umbilic if at that point we get $\frac{L}{E} = \frac{M}{F} = \frac{N}{G}$.

CHECK YOUR PROGRESS

Q1. Find the curvature of a normal section of the right helicoid

$$x = ucosv, \qquad y = usinv, \qquad z = cv$$

Q2. Find the value of Gaussian curvature and Mean curvature at any point of the right helicoid $\boldsymbol{r} = (ucosv, usinv, cv)$

Q3. Find the Gaussian curvature at the point (u, v) of the anchor ring

$$x = (b + a\cos u)\cos v, \qquad y = (b + a\cos u)\sin v, \qquad z = a\sin u$$

where the domain of u, v is $0 < u < 2\pi,\ 0 < v < 2\pi$. Verify the total curvature of the whole surface is zero.

Q4. Prove that at a point of the conoid $x = ucos\,\theta, y = usin\,\theta, z = f(\theta)$, the principal curvature are given by $u^2c^2(u^2 + z'^2)\kappa^2 - u^2cz''\kappa - z'^2 = 0$; where $z' = \frac{dz}{d\theta}$ etc. and $c^2 = 1 + \frac{z'^2}{u^2}$

9.5 MINIMAL SURFACE

Definition: If the mean curvature of a surface is zero at all points, then the surface is called a minimal surface. Thus,a surface is minimal at every point of the surface if and only if

$$\mu = 0 \Rightarrow EN + GL - 2FM = 0$$

Theorem: If there is a surface of minimum passing through a closed space curve, it is necessarily a minimal surface.

Proof: Let $\boldsymbol{r} = \boldsymbol{r}(u,v)$ be the equation of a surface S bounded by a closed curve C. Consider a small displacement $\varepsilon(u,v)$ in the direction of normal to derive a surface S*.

Let $\frac{\partial \varepsilon}{\partial u} = \varepsilon_1$ and $\frac{\partial \varepsilon}{\partial v} = \varepsilon_2$ are small quantities.

Let $\boldsymbol{R}$ be the position vector of any point on S*, then

$$\boldsymbol{R} = \boldsymbol{r} + \varepsilon \boldsymbol{N}$$

$$\therefore \boldsymbol{R_1} = \boldsymbol{r_1} + \varepsilon_1 \boldsymbol{N} + \varepsilon \boldsymbol{N_1}$$

$$\boldsymbol{R_2} = \boldsymbol{r_2} + \varepsilon_2 \boldsymbol{N} + \varepsilon \boldsymbol{N_2}$$

Let E*, F*, G* are first order fundamental coefficients of S*. Therefore,

$$E^* = \boldsymbol{R_1^2} = (\boldsymbol{r_1} + \varepsilon_1 \boldsymbol{N} + \varepsilon \boldsymbol{N_1})^2$$

$$= \boldsymbol{r^2} + 2\varepsilon \boldsymbol{r_1}.\boldsymbol{N} + 2\varepsilon \boldsymbol{r_1}.\boldsymbol{N_1} + O(\varepsilon^2)$$

$$= E - 2\varepsilon \boldsymbol{L} + O(\varepsilon^2) \quad [\because \boldsymbol{r_1}.\boldsymbol{N} = \boldsymbol{0}, L = -\boldsymbol{r_1}.\boldsymbol{N_1}]$$

$$F^* = \boldsymbol{R_1}.\boldsymbol{R_2} = (\boldsymbol{r_1} + \varepsilon_1 \boldsymbol{N} + \varepsilon \boldsymbol{N_1}).(\boldsymbol{r_2} + \varepsilon_2 \boldsymbol{N} + \varepsilon \boldsymbol{N_2})$$

$$= F - 2\varepsilon \boldsymbol{M} + O(\varepsilon^2) \quad [\because M = -\boldsymbol{r_1}.\boldsymbol{N_2} = -\boldsymbol{r_2}.\boldsymbol{N_1}]$$

$$G^* = \boldsymbol{R_2^2} = (\boldsymbol{r_2} + \varepsilon_2 \boldsymbol{N} + \varepsilon \boldsymbol{N_2})^2$$

$$= G - 2\varepsilon \boldsymbol{N} + O(\varepsilon^2)$$

$$\therefore H^{*2} = E^* G^* - F^{*2}$$

$$= \left(E - 2\varepsilon \boldsymbol{L} + O(\varepsilon^2)\right)\left(G - 2\varepsilon \boldsymbol{N} + O(\varepsilon^2)\right) - \left(F - 2\varepsilon \boldsymbol{M} + O(\varepsilon^2)\right)^2$$

$$= H^2 - 4\varepsilon H^2 \left(\frac{EN - 2FM + GL}{2H^2}\right) + O(\varepsilon^2)$$

$$= H^2[1 - 4\varepsilon\mu + O(\varepsilon^2)], \quad \text{where } \mu = \text{mean curvature}$$

Hence,

$$H^* = H[1 - 4\varepsilon\mu + O(\varepsilon^2)]^{\frac{1}{2}} = H[1 - 2\varepsilon\mu] + O(\varepsilon^2)$$

Let area $A = \int_S Hdudv$

$$\therefore A^* = \int_{S^*} H^* dudv - \int_S H(1 - 2\varepsilon\mu)dudv + O(\varepsilon^2)$$

$$= A - \int_S 2\varepsilon\mu H dudv + O(\varepsilon^2)$$

$$\Rightarrow A^* - A = -\int_S 2\varepsilon\mu H dudv + O(\varepsilon^2)$$

If A remains stationary then mean curvature (μ) is zero, hence the surface is necessary a minimal surface.

9.6 DEVELOPABLE SURFACE

The envelope of a single parameter family of planes is called a developable surface. Or, the surface for which Gaussian curvature is zero is called developable surface. Thus, the surface is developable if

$$LN - M^2 = 0$$

Edge of Regression: The locus of the characteristic point is called the edge of regression of a developable surface. It is the locus of intersections of consecutive generators of the developable and is a curve lying on the developable.

Theorem: The necessary and sufficient condition for a surface to be developable is that its Gaussian curvature should be zero.

Proof: First, we want to prove the Necessary condition:

Suppose the surface is developable. Generators of the developable are nothing but the tangent to the curve. Hence, the developable is nothing but the surface generated by the tangent to the curve

$$\boldsymbol{r} = \boldsymbol{r}(s)$$

Thus, the general equation of the developable surface is given by,

$$\boldsymbol{R} = \boldsymbol{r}(s) + v\boldsymbol{t}(s) = \boldsymbol{r} + v\boldsymbol{t}$$

Differentiating partially with respect to s and v, we have

$$\frac{\partial \boldsymbol{R}}{\partial s} = \boldsymbol{R_1} = \boldsymbol{t} + \kappa v\boldsymbol{n}$$

$$\frac{\partial \boldsymbol{R}}{\partial v} = \boldsymbol{R_2} = \boldsymbol{t}$$

$$\boldsymbol{R_1} \times \boldsymbol{R_2} = -\kappa v\boldsymbol{b}$$

$$\boldsymbol{R_{11}} = \kappa\boldsymbol{n} + \kappa v(\tau\boldsymbol{b} - \kappa\boldsymbol{t}) + \kappa' v\boldsymbol{n}$$

$$\boldsymbol{R_{21}} = \boldsymbol{R_{12}} = \kappa\boldsymbol{n}$$

$$\boldsymbol{R_{22}} = 0$$

Again,

$$\boldsymbol{N} = \frac{\boldsymbol{R_1} \times \boldsymbol{R_1}}{\boldsymbol{H}} = -\frac{\kappa v\boldsymbol{b}}{\boldsymbol{H}}$$

$$L = \boldsymbol{N}.\boldsymbol{R_{11}} = -\frac{v^2\kappa^2\tau}{\boldsymbol{H}}$$

$$M = \boldsymbol{N}.\boldsymbol{R_{12}} = 0$$

$$N = \boldsymbol{N}.\boldsymbol{R_{22}} = 0$$

Hence the Gaussian curvature is

$$\boldsymbol{K} = \frac{LN - M^2}{EG - F^2} = 0$$

The condition is sufficient: Let the Gaussian curvature $\boldsymbol{K} = 0$ for the given surface. Then,

$$LN - M^2 = 0$$

Now, $LN - M^2 = (\boldsymbol{r_1}.\boldsymbol{N_1})(\boldsymbol{r_2}.\boldsymbol{N_2}) - (\boldsymbol{r_1}.\boldsymbol{N_2})(\boldsymbol{r_2}.\boldsymbol{N_1})$

$$= (\boldsymbol{r_1} \times \boldsymbol{r_2}) \cdot (\boldsymbol{N_1} \times \boldsymbol{N_2})$$

$$= \boldsymbol{H}[\boldsymbol{N}, \boldsymbol{N_1}, \boldsymbol{N_2}] = 0$$

$$\therefore \boldsymbol{K} = 0 \Rightarrow [\boldsymbol{N}, \boldsymbol{N_1}, \boldsymbol{N_2}] = 0 \qquad [\because \boldsymbol{H} \neq 0]$$

Now, $\boldsymbol{N} \neq 0$. Also, if $\boldsymbol{N_1} \times \boldsymbol{N_2} \neq 0$ then $\boldsymbol{N_1} \times \boldsymbol{N_2}$ parallel to $\boldsymbol{N}$ as $\boldsymbol{N}$ is perpendicular to both $\boldsymbol{N_1}$ and $\boldsymbol{N_2}$ Thus, if $\boldsymbol{N_1} \times \boldsymbol{N_2} \neq \boldsymbol{0}$, we cannot have $\boldsymbol{N_1} \times \boldsymbol{N_2} perpendicular\ to$ N and so we cannot have

$$\boldsymbol{N}.(\boldsymbol{N_1} \times \boldsymbol{N_2}) = 0$$

$$\therefore N.(N_1 \times N_2) = 0 \Rightarrow N_1 \times N_2 = 0$$

This implies that

i) $N_2 = 0$ or ii) $N_1 = 0$ or iii) $N_1 = \mu N_2$

Case (i): $N_2 = 0$. The equation of the tangent plane at a point $r = r(u,v)$ on the surface $r = r(u,v)$ is

$$(R - r).N = 0$$

We have,

$$\frac{\partial}{\partial v}\{(R - r).N\} = -r_2.N + (R - r).N_2 = 0, \quad [\because r_2.N = 0 \text{ and } N_2 = 0]$$

Thus, $(R - r).N$ is independent of v. Therefore, the equation of the tangent plane contains only one parameter u. Hence, the surface is the envelope of a single parameter family of planes and so it is a developable surface.

Case (ii): $N_1 = 0$. In this case the equation of tangent plane contains only one parameter v. So in this case the surface is developable.

Case (iii): $N_1 = \mu N_2$. Let us change the parameters u, v to u', v' by using the transformation

$$u = u' + v' \text{ and } v = u' - \mu v'$$

$$\therefore \frac{\partial N}{\partial u'} = \frac{\partial N}{\partial u}\frac{\partial u}{\partial u'} + \frac{\partial N}{\partial v}\frac{\partial v}{\partial u'} = N_1 + N_2$$

$$\frac{\partial N}{\partial v'} = \frac{\partial N}{\partial u}\frac{\partial u}{\partial v'} + \frac{\partial N}{\partial v}\frac{\partial v}{\partial v'} = N_1 - \mu N_2 = 0$$

Thus, here N depends only on one parameter. Therefore, the surface is developable.

9.7 LINES OF CURVATURE

A curve on a surface is called a line of curvature if the tangent at any point of it is along the principal direction at that point.

Equation of line of curvature

Let, (du, dv) is direction of a line of curvature through the point $P(u,v)$. Then (du, dv) is a principal direction at P. If κ is a principal curvature at P, then we have

$$(Ldu + Mdv) - \kappa(Edu + Fdv) = 0 \quad (1)$$

$$(Mdu + Ndv) - \kappa(Fdu + Gdv) = 0 \quad (2)$$

These equations are the equations of a line of curvature. The line of curvature through P having its direction as the principal direction corresponding to the principal curvature κ_a is obtained by putting $\kappa = \kappa_a$ in (1) or (2). Thus, the equation is

$$\frac{du}{dv} = -\frac{(\kappa_a F - M)}{(\kappa_a E - L)} \quad \text{or} \quad \frac{du}{dv} = -\frac{(\kappa_a G - N)}{(\kappa_a F - M)}$$

Similarly, we get the equations of the line of curvature corresponding to the principal curvature κ_b.

Now, eliminating κ from (1) and (2), we get

$$(EM - FL)du^2 + (EN - GL)dudv + (FN - GM)dv^2 = 0 \qquad (3)$$

Since, $E(FN - GM) - F(EN - GL) + G(EM - FL) = 0$; therefore, the two families of lines of curvature are orthogonal.

Equation (3) can be put in the following form

$$\begin{vmatrix} dv^2 & -dudv & du^2 \\ E & F & G \\ L & M & N \end{vmatrix} = 0$$

N.B.: The curvature of a line at any point is not necessarily equal to a principal curvature at that point. The principal curvature at a point on a line of curvature is equal to the normal curvature in the direction of the line of curvature. It will coincide with the curvature of the line of curvature iff the principal normal to the line of curvature coincides with the surface normal.

Centre of Curvature: The point of intersection of consecutive normals along a line pf curvature at P in called a centre of curvature of the surface.

Rodrigues' Formula: A necessaryand sufficient condition that a curve on a surface be a line of curvature is that

$$\frac{d\boldsymbol{N}}{ds} \propto \frac{d\boldsymbol{r}}{ds} \text{or} \quad d\boldsymbol{N} + \kappa d\boldsymbol{r} = 0$$

at each of its points, and κ is the normal curvature.

Proof: Let, (du, dv) is direction of a line of curvature through the point $P(u, v)$. Then (du, dv) is a principal direction at P. If κ is a principal curvature at P, then we have

$$(L - \kappa E)du + (M - \kappa F)dv = 0 \qquad (1)$$

$$(M - \kappa F)du + (N - \kappa G)dv = 0 \qquad (2)$$

In terms of derivatives of $\boldsymbol{r}$ and $\boldsymbol{N}$, putting the values of L, M, N, E, F, G in the above equations we get

$$L = -\boldsymbol{N_1}\cdot\boldsymbol{r_1}, \quad M = -\boldsymbol{N_1}\cdot\boldsymbol{r_2} = -\boldsymbol{N_2}\cdot\boldsymbol{r_1}, \quad N = -\boldsymbol{N_2}\cdot\boldsymbol{r_2}$$

$$E = \boldsymbol{r_1^2}, \quad F = \boldsymbol{r_1}\cdot\boldsymbol{r_2}, \quad G = \boldsymbol{r_2^2}$$

We have,

$$(-\boldsymbol{N_1}\cdot\boldsymbol{r_1} - \kappa\boldsymbol{r_1^2})du + (-\boldsymbol{N_2}\cdot\boldsymbol{r_1} - \kappa\boldsymbol{r_1}\cdot\boldsymbol{r_2})dv = 0$$

$$\Rightarrow (\boldsymbol{N_1}du + \boldsymbol{N_2}dv)\cdot\boldsymbol{r_1} + \kappa(\boldsymbol{r_1}du + \boldsymbol{r_2}dv)\cdot\boldsymbol{r_1} = 0$$

$$\Rightarrow (d\boldsymbol{N})\cdot\boldsymbol{r_1} + (\kappa d\boldsymbol{r})\cdot\boldsymbol{r_1} = 0$$

$$\Rightarrow (d\boldsymbol{N} + \kappa d\boldsymbol{r})\cdot\boldsymbol{r_1} = 0 \qquad (3)$$

Similarly, from (2) we get

$$(d\boldsymbol{N} + \kappa d\boldsymbol{r})\cdot\boldsymbol{r_2} = 0 \qquad (4)$$

Again, $\boldsymbol{N}.\boldsymbol{N_1} = \boldsymbol{0}$

On differentiating it, we have $2\boldsymbol{N}.d\boldsymbol{N} = 0$, hence, $d\boldsymbol{N}$ is normal to $\boldsymbol{N}$. Also, $d\boldsymbol{r}$ is a tangent vector, so the vector $d\boldsymbol{N} + \kappa d\boldsymbol{r}$ is a tangential vector to the surface. Also, $\boldsymbol{r_1}$ and $\boldsymbol{r_2}$ are tangential vectors, therefore, in order to satisfied both (3) and (4), we must have

$$d\boldsymbol{N} + \kappa d\boldsymbol{r} = 0$$

or $\frac{d\boldsymbol{N}}{ds} \propto \frac{d\boldsymbol{r}}{ds}$

which is the condition of necessary.

Sufficient Condition: Let there be curve on the surface at each point of which

$\frac{d\boldsymbol{N}}{ds} \propto \frac{d\boldsymbol{r}}{ds}$ or $d\boldsymbol{N} + \kappa d\boldsymbol{r} = 0$

where, κ is any function, now reversing the order of steps, we get the equations (3) and (4) which are

$$(L - \kappa E)du + (M - \kappa F)dv = 0$$

$$(M - \kappa F)du + (N - \kappa G)dv = 0$$

Thus, the curve is line of curvature in case κ is normal curvature to the surface.

Given, $\kappa d\boldsymbol{r} + d\boldsymbol{N} = 0 \Rightarrow \kappa d\boldsymbol{r} = -d\boldsymbol{N}$

$$\Rightarrow \kappa(\boldsymbol{r_1}du + \boldsymbol{r_2}dv) = -(\boldsymbol{N_1}du + \boldsymbol{N_2}dv)$$

Taking dot product of $(\boldsymbol{r_1}du + \boldsymbol{r_2}dv)$ both sides,

$$\kappa(\boldsymbol{r_1}du + \boldsymbol{r_2}dv)\cdot(\boldsymbol{r_1}du + \boldsymbol{r_2}dv) = -(\boldsymbol{N_1}du + \boldsymbol{N_2}dv)\cdot(\boldsymbol{r_1}du + \boldsymbol{r_2}dv)$$

$$\Rightarrow \kappa(Edu^2 + 2Fdudv + Gdv^2) = Ldu^2 + 2Mdudv + Ndv^2$$

$$\Rightarrow \kappa = \frac{Ldu^2 + 2Mdudv + Ndv^2}{Edu^2 + 2Fdudv + Gdv^2}$$

which implies that κ is a normal curvature at the point (u,v) in the direction (du,dv).

Hence, the direction at each point of the curve is a principal direction and thus the curve is a line of curvature on the surface.

Monge's theorem: A necessary and sufficient condition that a curve on a surface be a line of curvature is that the surface normals along the curve from a developable.

Proof: Let $\boldsymbol{r} = \boldsymbol{r}(u,v)$ be a surface and $\boldsymbol{r} = \boldsymbol{r}(s)$ be a curve on it. Let $\boldsymbol{N}$ be a unit vector along normal to the given surface at any point P and which is a function of s alone. Let $\boldsymbol{R}$ be the positionvector of any point on this normal, then

$$\boldsymbol{R} = \boldsymbol{r}(s) + v\boldsymbol{N}(s)$$

This equation can be taken as the function of surface generated by the normals to the given surface at points on the curve $\boldsymbol{r} = \boldsymbol{r}(s)$. Then

$$\frac{\partial \boldsymbol{R}}{\partial s} = \boldsymbol{R_1} = \frac{d\boldsymbol{r}}{ds} + v\frac{d\boldsymbol{N}}{ds} = t + v\boldsymbol{N'}$$

And $\quad \frac{\partial \boldsymbol{R}}{\partial v} = \boldsymbol{R_2} = \boldsymbol{N},\ \boldsymbol{R_{12}} = \boldsymbol{N'},\ \boldsymbol{R_{22}} = 0$

Then,

$$HM = [\boldsymbol{R_{12}}, \boldsymbol{R_1}, \boldsymbol{R_2}] = [\boldsymbol{N'}, t + v\boldsymbol{N'}, \boldsymbol{N}]$$

$$= [\boldsymbol{N'}, t, \boldsymbol{N}] + v[\boldsymbol{N'}, \boldsymbol{N'}, \boldsymbol{N}]$$

$$= [\boldsymbol{N'}, t, \boldsymbol{N}] \qquad [\because [\boldsymbol{N'}, \boldsymbol{N'}, \boldsymbol{N}] = 0, \boldsymbol{R_{22}} = 0]$$

And, $\quad HN = [\boldsymbol{R_{22}}, \boldsymbol{R_1}, \boldsymbol{R_2}] = 0 \Rightarrow \boldsymbol{N} = \boldsymbol{0}$ **since** $H \neq 0$

$$M = \frac{1}{H}[\boldsymbol{N'}, t, \boldsymbol{N}] = \frac{1}{H}[t, \boldsymbol{N}, \boldsymbol{N'}]$$

Now, the surface (1) is developable if and only if its Gaussian curvature is zero, i.e.

$$\frac{LN - M^2}{H^2} = 0$$

$$\Leftrightarrow M = 0$$

$$\Leftrightarrow [t, \boldsymbol{N}, \boldsymbol{N'}] = 0$$

Hence the normal to the surface $\boldsymbol{r} = \boldsymbol{r}(u,v)$ along the curve $\boldsymbol{r} = \boldsymbol{r}(s)$ form a

developable iff $[t, N, N'] = 0$. Therefore, in order to prove the theorem, we are now prove that

$$[t, N, N'] = 0$$

is a necessary and sufficient condition for $r = r(s)$ to be a line of curvature on the surface.

Now, we have that

$$[t, N, N'] = 0, \qquad [t, N', N\ ] = 0,\ (t \times N') \cdot N = 0$$

But, $N \neq 0$

Also, N' is perpendicular to N. So N' lies in the tangent plane to the surface $r = r(u, v)$. Now, $t \times N'$ is perpendicular to both t and N'. Therefore, $t \times N'$ is normal to the surface and parallel to N. Hence, $t \times N' \neq 0$

$$\therefore (t \times N') \cdot N = 0 \Rightarrow t \times N' = 0$$

$\Rightarrow N' = -\kappa t$, for some scalar function κ

$$\kappa \frac{dr}{ds} + \frac{dN}{ds} = 0$$

Which gives that the given curve is a line of curvature by Rodrigue's formula.

Conversely, if the given curve is a line of curvature, then by Rodrigue's formula, we have

$$\kappa \frac{dr}{ds} + \frac{dN}{ds} = 0$$

$$\Rightarrow -\kappa t = N'$$

$$\therefore [t, N, N'] = [t, N, -\kappa t] = 0$$

Hence the proof is complete.

Lines of Curvature as Parametric Curve:

Statement: The necessary and sufficient condition that parametric curves be lines of curvature $F = 0,\ M = 0$.

Proof: The parametric curves are u =constant and v =constant, therefore combined differential equation of parametric curves is given by

$$dudv = 0 \qquad (1)$$

Again, the differential equation of lines of curvature is

$$(EM - FL)du^2 + (EN - GL)dudv + (FN - GM)dv^2 = 0 \qquad (2)$$

If the lines of curvature be taken as parametric curves, then $F = 0$, since the principal directions are orthogonal. Comparing (1) and (2), we get

$$EM - FL = 0, \quad FN - GM = 0, \quad EN - GL \neq 0$$

$$\Rightarrow EM = 0, \quad GM = 0$$

$$\Rightarrow M = 0$$

Hence $F = 0,\ M = 0$ are necessary conditions for the parametric curves to be line of curvature.

Sufficient condition: If $F = 0,\ M = 0$ the equation (2) becomes

$$(EN - GL)dudv = 0$$

But $EN - GL \neq 0$

$$\therefore dudv = 0$$

Which is a differential equation of the parametric curves.

Euler's Theorem: The normal curvature κ_n at a point on a surface is given in terms of principal curvatures κ_a and κ_b by the formula

$$\kappa_n = \kappa_a \cos^2\psi + \kappa_b \sin^2\psi$$

where, ψ is the angle which is makes by the direction (du, dv) of the normal sections with the principal direction $dv = 0$.

Corollary (Dupin's Theorem): The sum of the normal curvatures at any point on a surface in two directions at right angles is constant and equal to the sum of the principal curvatures at that point.

CHECK YOUR PROGRESS

Q5. Show that the surface $e^z \cos x = \cos y$ is minimal.

Q6. Prove that the cone $\kappa xy = z\left\{\sqrt{x^2 + z^2} + \sqrt{y^2 + z^2}\right\}$ passes through a line curvature of the paraboloid $xy = az$.

Q7. Show that the principal radii of curvature of the surface $ycos\frac{z}{a} = xsin\frac{z}{a}$ are equal to $\pm\frac{x^2+y^2+a^2}{a}$. Also, find the lines of curvature.

Q8. Find the principal curvature and lines of curvature for the surface generated by the tangents to a twisted curve. Show that this surface is developable.

Q9. A developable surface is generated by the tangents to a given curve, prove that at the point Q on the tangents at P, where $PQ = r$, the principal radius of the developable is $\frac{r\sigma}{\rho}$.

2 9.8 LET US SUM UP

- The equation of the curvature of the normal section, which is called normal curvature parallel to the direction (du, dv) is

$$\kappa_n = \frac{Ldu^2 + 2Mdudv + Ndv^2}{Edu^2 + 2Fdudv + Gdv^2}$$

Its reciprocal is called the Radius of normal curvature (ρ_n).

- If P be a point with position vector $\boldsymbol{r}(u, v)$ on the surface $\boldsymbol{r} = \boldsymbol{r}(u, v)$, the normal curvature at P in the direction (du, dv) is equal to the curvature at P of the normal section at P parallel to that direction.
- The curvature at P of normal section at P containing the direction (du, dv) is equal to the normal curvature at P in the same direction i.e. $\kappa_n = \kappa$.
- The normal sections of a surface through a given point having maximum or minimum curvatures at the point are called principal sections of the surface at that point and the tangents to these sections are called **principal directions** at the point.
- The curvatures of the principal sections of a surface through a given point i.e., the maximum and minimum curvatures at that point are called principal curvatures at that point and their corresponding radius of curvatures are called principal radius of curvatures.
- The sum of the principal curvature κ_a and κ_b is called the first curvature at the point and defined by,

$$J = \kappa_a + \kappa_b = \frac{EN + LG - 2FM}{EG - F^2}$$

- The arithmetic mean of the principal curvature κ_a and κ_b at a point is called the mean curvature at the point and

$$\mu = \frac{1}{2}(\kappa_a + \kappa_b) = \frac{EN + LG - 2FM}{2(EG - F^2)}$$

- The product of the principal curvatures κ_a and κ_b at any point is called Gaussian curvature at that point and

$$K = \kappa_a \kappa_b = \frac{LN - M^2}{EG - F^2} = \frac{T^2}{H^2}$$

- A point is an elliptic point if $LN - M^2 > 0$; a point is a hyperbolic point if $LN - M^2 < 0$ and a point is a parabolic point if $LN - M^2 = 0$.
- If the mean curvature of a surface is zero at all points, then the surface is called a minimal surface. Thus, a surface is minimal at every point of the surface if and only if

$$\mu = 0 \Rightarrow EN + GL - 2FM = 0$$

- The envelope of a single parameter family of planes is called a developable surface. Or, the surface for which Gaussian curvature is zero is called developable surface. Thus, the surface is developable if

$$LN - M^2 = 0$$

- A curve on a surface is called a line of curvature if the tangent at any point of it is along the principal direction at that point. The equation of lines of curvature is

$$(EM - FL)du^2 + (EN - GL)dudv + (FN - GM)dv^2 = 0$$

9.9 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q1: The equation of the given surface is

$$\boldsymbol{r} = (u\cos v, u\sin v, cv)$$

We have,

$$E = 1, \quad F = 0, \quad G = u^2 + c^2, \quad L = 0, \quad M = -\frac{c}{H}, \quad N = 0$$

The curvature κ_n of normal section is

$$\kappa_n = \frac{Ldu^2 + 2Mdudv + Ndv^2}{Edu^2 + 2Fdudv + Gdv^2}$$

$$= \frac{-2\frac{c}{H}dudv}{du^2 + (u^2 + c^2)dv^2}$$

Ans to Q2: We have

$$E = 1, \quad F = 0, \quad G = u^2 + c^2, \quad H = u^2 + c^2, \quad L = 0, \quad M = -\frac{c}{H}, \quad N = 0$$

The Gaussian curvature K of the given surface at the point (u,v) is given by

$$K = \frac{LN - M^2}{EG - F^2} = \frac{0 - (c^2/H^2)}{H^2} = -\frac{c^2}{(u^2 + c^2)^2}$$

The Mean curvature at the point (u,v) is given by,

$$\mu = \frac{EN - 2FM + GL}{2H^2} = 0$$

Ans to Q3: The equation of the given surface is

$$\boldsymbol{r} = \big((b + a\cos u)\cos v, (+a\cos u)\sin v, a\sin u\big)$$

We have,

$$\boldsymbol{r_1} = (-a\sin u\cos v, -a\sin u\sin v, a\cos u)$$

$$\boldsymbol{r_2} = (-(b + a\cos u)\sin v, (b + a\cos u)\cos v, 0)$$

$$\therefore \boldsymbol{r_1} \times \boldsymbol{r_2} = (b + a\cos u)(-a\cos u\cos v, -a\cos u\, sinv, -a\sin u)$$

Also,

$$\boldsymbol{r_{11}} = (-a\cos u\cos v, -a\cos u\sin v, -a\sin u),$$

$$\boldsymbol{r_{12}} = (a\sin u\sin v, -a\sin u\cos v, 0),$$

$$\boldsymbol{r_{22}} = (-(b + a\cos u)\cos v, -(b + a\cos u)\sin v, 0)$$

Now,

$$E = \boldsymbol{r_1^2} = a^2, \qquad F = \boldsymbol{r_1}.\boldsymbol{r_2} = 0, \qquad G = \boldsymbol{r_2^2} = (b + a\cos u)^2,$$
$$H^2 = EG - F^2 = a^2(b + a\cos u)^2$$

$$LH = \boldsymbol{r_{11}}.(\boldsymbol{r_1} \times \boldsymbol{r_2}) = a^2(a + a\cos u)$$

$$MH = \boldsymbol{r_{12}}.(\boldsymbol{r_1} \times \boldsymbol{r_2}) = 0$$

$$NH = \boldsymbol{r_{22}}.(\boldsymbol{r_1} \times \boldsymbol{r_2}) = (b + a\cos u)^2 a\cos u$$

From these, we get

$$L = \frac{a^2(b + a\cos u)}{H} = a, \qquad M = 0, \qquad N = (b + a\cos u)\cos u$$

Hence the Gaussian curvature is:

$$K = \frac{LN - M^2}{EG - F^2} = \frac{\cos u}{a(b + \cos u)}$$

Total curvature of the whole surface is

$= \int K dS$, integrated over the whole surface S.

$$= \int_{v=0}^{2\pi}\int_{u=0}^{2\pi} KH\,du\,dv$$

$$= \int_{v=0}^{2\pi}\int_{u=0}^{2\pi} \frac{\cos u}{a(b+\cos u)}\, a(b+\cos u)\,du\,dv$$

$$= \int_{v=0}^{2\pi}\int_{u=0}^{2\pi} \cos u\,du\,dv$$

$$= 0$$

Ans to Q4: The position vector at any point of the surface is

$$\boldsymbol{r} = [u\cos\theta, u\sin\theta, f(\theta)]$$

$$\therefore \boldsymbol{r_1} = (\cos\theta, \sin\theta, 0)$$

$$\boldsymbol{r_2} = (-u\sin\theta, u\cos\theta, f'(\theta))$$

$$\boldsymbol{r_{11}} = (0,0,0)$$

$$\boldsymbol{r_{12}} = (-\sin\theta, \cos\theta, 0)$$

$$\boldsymbol{r_{22}} = (-u\cos\theta, -u\sin\theta, f'')$$

$$\therefore E = \boldsymbol{r_1^2} = \cos^2\theta + \sin^2\theta = 1$$

$$F = \boldsymbol{r_1}.\boldsymbol{r_2} = -u\cos\theta\sin\theta + u\cos\theta\sin\theta = 0$$

$$G = \boldsymbol{r_2^2} = u^2\sin^2\theta + u^2\cos^2\theta + f'^2 = u^2 + f'^2$$

$$H^2 = EG - F^2 = 1.(u^2 + f'^2) - 0 = u^2 + f'^2$$

$$\boldsymbol{N} = \frac{\boldsymbol{r_1}\times\boldsymbol{r_2}}{H} = \frac{(f'\sin\theta, -f'\cos\theta, u)}{H}$$

$$L = \boldsymbol{N}.\boldsymbol{r_{11}} = 0$$

$$M = \boldsymbol{N}.\boldsymbol{r_{12}} = \frac{2}{H}[-f'\sin^2\theta - f'\cos^2\theta + 0] = -\frac{f'}{H}$$

$$N = \boldsymbol{N}.\boldsymbol{r_{22}} = \frac{1}{H}[-uf'\cos\theta\sin\theta + uf'\cos\theta\sin\theta + uf''] = \frac{uf''}{H}$$

The principal curvatures are given by

$$H^2\kappa^2 - (EN + GL - 2FM)\kappa + (LN - M^2) = 0$$

$$\Rightarrow (u^2 + z'^2)\kappa^2 - \frac{uz''}{H}\kappa - \frac{z'^2}{H^2} = 0$$

$$\Rightarrow u^2\left(1+\frac{z'^2}{u^2}\right)(u^2+z'^2)\kappa^2-u^2\sqrt{\left(1+\frac{z'^2}{u^2}\right)}z''\kappa-z'^2=0$$

$$\Rightarrow u^2c^2(u^2z'^2)\kappa^2-u^2c\kappa z''-z'^2=0$$

Ans to Q5: Given surface is

$$e^z\cos x=\cos y\Rightarrow e^z=\frac{\cos y}{\cos x}$$

$$\Rightarrow z=\log\cos y-\log\cos x$$

i.e. the equation is in the Monge's form. The position vector $\boldsymbol{r}$ of any position (x,y,z) on the surface is

$$\boldsymbol{r}=(x,y,z)=(x,y,\log\cos y-\log\cos x)$$

$$\therefore \boldsymbol{r_1}=(1,0,tan\,x),\qquad \boldsymbol{r_2}=(0,1,-\tan y)$$

$$\boldsymbol{r_{11}}=(0,0,\sec^2 x),\qquad \boldsymbol{r_{12}}=(0,0,0),\qquad \boldsymbol{r_{22}}=(0,0,-\sec^2 y)$$

$$\boldsymbol{r_1}\times\boldsymbol{r_2}=(-\tan x\,,\tan y\,,1)$$

$$\therefore E=1+\tan^2 x=\sec^2 x$$

$$F=-\tan x\tan y$$

$$G=1+\tan^2 y=\sec^2 y$$

$$\boldsymbol{N}=\frac{\boldsymbol{r_1}\times\boldsymbol{r_2}}{H}=\frac{(-\tan x\,,tan\,y,1)}{H}$$

$$L=\boldsymbol{N}.\boldsymbol{r_{11}}=\frac{\sec^2 x}{H}$$

$$M=\boldsymbol{N}.\boldsymbol{r_{12}}=0$$

$$N=\boldsymbol{N}.\boldsymbol{r_{22}}=-\frac{\sec^2 y}{H}$$

The condition for the surface to be minimal is

$$EN-2FM+GL=0$$

Hence, we have

$$EN-2FM+GL=\frac{\sec^2 x\,(-\sec^2 y)}{H}+\frac{\sec^2 y\sec^2 x}{H}=0$$

Thus, the given surface is minimal.

Ans to Q6: From the paraboloid $z=\frac{xy}{a}$, we have

$$p=\frac{\partial z}{\partial x}=\frac{y}{a},\qquad q=\frac{\partial z}{\partial y}=\frac{x}{a},\qquad r=0,\qquad s=\frac{1}{a},\qquad t=0$$

The lines of curvature are given by

$$\begin{vmatrix} dy^2 & -dxdy & dx^2 \\ 1+p^2 & pq & 1+q^2 \\ r & s & t \end{vmatrix}=0$$

$$\Rightarrow\begin{vmatrix} dy^2 & -dxdy & dx^2 \\ 1+\frac{y^2}{a^2} & \frac{xy}{a^2} & 1+\frac{x^2}{a^2} \\ 0 & \frac{1}{a} & 0 \end{vmatrix}=0$$

$$\Rightarrow (a^2+x^2)dy^2-(a^2+y^2)dx^2=0$$

$$\Rightarrow \frac{dx}{\sqrt{a^2+x^2}}\pm\frac{dy}{\sqrt{a^2+y^2}}=0$$

Integrating, the surface on which these lines of curvature lie, is given by

$\sinh^{-1}\frac{x}{a}\pm\sinh^{-1}\frac{y}{a}=c$; where c is an arbitrary constant.

$$\Rightarrow \frac{x}{a}\sqrt{1+\frac{y^2}{a^2}}\pm\frac{y}{a}\sqrt{1+\frac{x^2}{a^2}}=\sinh c$$

$$\Rightarrow \frac{z}{y}\sqrt{1+\frac{z^2}{x^2}}\pm\frac{z}{x}\sqrt{1+\frac{z^2}{y^2}}=\kappa(constant)$$

$$\Rightarrow z\left[\sqrt{x^2+z^2}+\sqrt{y^2+z^2}\right]=\kappa xy$$

Hence the cone passes through a line of curvature of $xy=az$.

Ans to Q7: Given surface is

$$y\cos\frac{z}{a}=x\sin\frac{z}{a}$$

$$\Rightarrow z=a\tan^{-1}\frac{y}{x}$$

The parametric equations of the surface are

$$x=\cos v,\qquad y=u\sin v,\qquad z=av$$

$$\therefore \boldsymbol{r}=(u\cos v,u\sin v,av)$$

$$\boldsymbol{r_1}=(\cos v,\sin v,0);\ \boldsymbol{r_2}=(-u\sin v,u\cos v,a)$$

$$\boldsymbol{r_{11}} = (0,0,0);\ \ \boldsymbol{r_{12}} = (-\sin v, \cos v, 0);\ \ \boldsymbol{r_{22}} = (-u\cos v, -u\sin v, 0)$$

$$\therefore E = 1; F = 0; G = u^2 + a^2, H^2 = u^2 + a^2$$

$$\boldsymbol{N} = \frac{\boldsymbol{r_1} \times \boldsymbol{r_2}}{H} = \frac{(a\sin v, -a\cos v, u)}{H}$$

$$L = 0; M = -\frac{a}{H}; N = 0$$

The principal curvatures of the surface are given by

$$H^2\kappa^2 - (EN + GL - 2FM)\kappa + (LN - M^2) = 0$$

$$\Rightarrow (u^2 + a^2)\kappa^2 + \left(0 - \frac{a^2}{H^2}\right) = 0$$

$$\Rightarrow \rho^2 = \frac{(u^2 + a^2)H^2}{a^2} = \frac{(u^2 + a^2)^2}{a^2}$$

$$\Rightarrow \rho = \pm\frac{(x^2 + y^2 + a^2)}{a}$$

Which are the principal radii of curvatures.

Again, the lines of curvatures are given by

$$(EM - FL)du^2 + (EN - GL)du\,dv + (FN - GM)dv^2 = 0$$

$$\Rightarrow -\frac{a}{H}du^2 + (u^2 + a^2)\frac{a}{H}dv^2 = 0$$

$$\Rightarrow \frac{du}{\pm\sqrt{u^2 + a^2}} = dv$$

Integrating we get

$$v = \pm\sinh^{-1}\frac{u}{a} + C, \quad \text{where } C \text{ is an arbitrary constant}$$

$$\Rightarrow \tan^{-1}\frac{y}{x} = \pm\sinh^{-1}\frac{(x^2 + y^2)}{a} + C$$

Ans to Q8: Let $\boldsymbol{r} = \boldsymbol{r}(s)$ be a twisted curve and $\boldsymbol{R}$ be the position vector of any point on the surface generated by the tangents to the curve, then

$$\boldsymbol{R} = \boldsymbol{r}(s) + v\boldsymbol{t}(s)$$

$$\boldsymbol{R_1} = \boldsymbol{t};\ \ \boldsymbol{R_2} = \boldsymbol{t} + \kappa.v\boldsymbol{n};\ \ \boldsymbol{R_{11}} = \boldsymbol{0};\ \ \boldsymbol{R_{12}} = \kappa\boldsymbol{n}$$

$$\boldsymbol{R_{22}} = \kappa\boldsymbol{n} + v(\tau b - \kappa\boldsymbol{t})\kappa + \kappa'.v\boldsymbol{n}$$

$$\boldsymbol{N} = \frac{\boldsymbol{R_1} \times \boldsymbol{R_2}}{H} = \frac{\kappa.vb}{H}$$

$$E = 1;\ F = 1;\ G = 1 + v^2\kappa^2;\ H = v^2\kappa^2$$

$$L = 0;\ M = 0\ N = \kappa\tau v$$

The principal curvatures of the surface are given by

$$H^2\kappa_n^2 - (EN + GL - 2FM)\kappa_n + (LN - M^2) = 0$$

$$\Rightarrow v^2\kappa^2\kappa_n^2 - (\kappa.v\tau)\kappa_n = 0$$

$$\Rightarrow \kappa_n = 0, \qquad \kappa_n = \frac{\tau}{\kappa.v}$$

$$\therefore \kappa_a = 0;\ \kappa_b = \frac{\tau}{\kappa.v}$$

The differential equation of the lines of curvature is

$$(EM - FL)dv^2 + (EN - GL)dv\,ds + (FN - GM)ds^2 = 0$$

$$\Rightarrow ds(dv + ds) = 0$$

$$\Rightarrow ds = 0 \quad or \quad dv + ds = 0$$

Integrating, we get

$s = constant$ and $v + s = constant$; as the required equation of line of curvature.

Again, the Gaussian curvature

$$K = \frac{LN - M^2}{H^2} = 0$$

Hence the given surface is developable.

Ans to Q9: From **Q8,** the principal curvature is $\frac{\tau}{\kappa.v}$

$\therefore$ Principal radius of the developable $= \frac{\kappa.v}{\tau}$

Putting $\kappa = r, v = \frac{1}{\rho}, \tau = \frac{1}{\sigma}$, we get

Principal radius $= \frac{r\sigma}{\rho}$

O 9.10 FURTHER READING

[1] Weatherburn, C. E. 2016. *Differential Geometry of Three Dimensions*. Cambridge University Press.

[2] Graustein, W. C. 2012. *Differential Geometry*. Courier Corporation.

[3] Willmore, T. J. 2012. *An Introduction to Differential Geometry*. Dover Publications Inc.

[4] Pressley, A. 2002. *Elementary Differential Geometry*. Springer.

9.11 MODEL QUESTIONS

Q1. Find the principal directions and the principal curvatures on the surface

$$x = a(u+v), \quad y = b(u-v), \quad z = uv$$

Also find the Gaussian curvature at a point of the surface.

Q2. The surface of revolution given by,

$$x = u\cos v, \quad y = u\sin v, \quad z = a\log\{u + \sqrt{u^2 - a^2}\}$$

is generated by revolving a catenary about its directrix. Prove that it is a minimal surface. Also show that it is the only minimal surface of revolution.

Q3. Find the values of the two principal curvatures at any point of the surface of revolution $x = a\cos v,\ y = u\sin v, z = f(u)$.

Q4. Show that the equation for the principal curvatures through a point of the surface $z = f(x,y)$ is

$$H^4\kappa^2 - H\{(1+p^2)t + (1+q^2)r - 2spq\}\kappa + (rt - s^2) = 0$$

Also obtain the differential equation of the lines of curvature on the surface and deduce that at an umbilic

$$\frac{1+p^2}{r} = \frac{1+q^2}{t} = \frac{pq}{s}$$

Q5. Obtain the differential equation of the lines of curvature through a point of the surface $F(x,y,z) = 0$.

Q6. Find the first and second curvatures of the helicoid

$$x = u\cos v, y = u\sin v, z = f(u) + cv$$

Also show that the lines of curvature of this helicoid are

$$c(1+f_1^2 + uf_1f_{11})du^2 + [(u^2+c^2)uf_{11} - (1+f_1^2)u^2f_1]dudv$$
$$-c(u^2 + c^2 + u^2f_1^2)dv^2 = 0$$

Q7. Prove that the Gaussian curvature of the surface $z = f(x,y)$ is

$$(rt - s^2)(1+p^2+q^2)^{-2}$$

Q8. Find the condition that the surface of revolution $x = u\cos v, y = u\sin v, z = f(u)$ be a minimal surface.

Q9. Find the principal radii and lines of curvature through a point of the surface

$$x = f(u,v), y = \phi(u,v), z = \psi(u,v)$$

Q10. Show that the Gauss and mean curvatures on the surface

$$x = u + v, y = u - v, z = uv$$

at a point $u = 1, v = 1$ are $K = \frac{1}{16}$ and $\mu = \frac{1}{8\sqrt{2}}$

Q11. Show that the principal curvatures of the surface $xsin\, z - y\cos z = 0$ are

$$\pm 1/(x^2 + y^2 + 1)$$

Q12. Find the lines of curvature of the surface $4a^2z = x^2 + y^2$.

Q13. Prove that the surface is a minimal surface

$$x = ucos\, \alpha + \sin u \cosh v, y = v + \cos\alpha \cos u\, sinh\, v, z = sin\alpha \cos u \cosh v$$

Q14. Show that the surface $\sin az = \sinh ax \sinh ay$ is minimal.

Q15. Find the principal radii of the surface $a^2x^2 = z^2(x^2 + y^2)$ at the points where $x = y = z$.

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