PG MA S4-04 (DSE)

TWO YEAR MASTER'S DEGREE COURSE

(CBCS in Accordance with NEP 2020)

SEMESTER - IV

FUZZY SETS AND APPLICATIONS

MATHEMATICS

Hiranya Chandra Bhuyan School of Science and Technology

KRISHNA KANTA HANDIQUI STATE OPEN UNIVERSITY

Subject Expert

Prof. Bipan Hazarika, Gauhati University

Prof. Nazibuddin Ahmed, Gauhati University

Prof. Dipak Sarma, Cotton University

Dr. Dilip Kumar Sarma, Cotton University

Course Co-ordinator : **Dr. Harekrishna Deka,** KKHSOU

SLM Preparation Team

UNITS	CONTRIBUTORS
1, 2,3	Dr Sunmani Mudoi**,** KKHSOU
4,5	Dr Pappu Das**,** KKHSOU
6,7	Dr Jaba Kalita, PDUAM, Amjonga
8,9	Dr Harekrishna Deka **,** KKHSOU
10,11,12,13,14,15	Dr. Shahidul Islam Khan, PDUAM, Amjonga

Editorial Team

Content	**:**	**Prof. Bipan Hazarika,** Gauhati University
Structure, Format & Graphics	:	**Dr. Harekrishna Deka,** KKHSOU

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Head Office : Patgaon, Rani Gate, Guwahati-781017
City Office : Resham Nagar, Khanapara, Guwahati-781 022; Web: www.kkhsou.in

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COURSE INTRODUCTION

"Fuzzy Sets and Applications " is the fourth Discipline Specific elective course of MSc (Mathematics) programme.

The course "Fuzzy Sets and Applications " provides a structured introduction to mathematical tools for handling vagueness, imprecision, and uncertainty in real-world systems. It begins with the fundamental theory of fuzzy sets and their properties, then develops the algebra of fuzzy operations, fuzzy numbers, fuzzy relations, and fuzzy graphs. Building on these foundations, the course explores possibility theory, approximate reasoning, fuzzy logic, and inference mechanisms that support intelligent decision-making. Learners also study fuzzy control systems, uncertainty-based information measures, fuzzy decision-making models, and applications in optimization through fuzzy linear programming. The final units extend fuzziness into topological, product-induced, and metric spaces.

Unit 1: Fuzzy Set

This unit introduces the fundamental concepts of fuzzy sets, including their types, characteristics, and general properties. It lays the theoretical foundation for understanding how fuzzy sets extend classical set theory to represent imprecise information.

Unit 2: Operations on Fuzzy Sets

Learners explore the extension principle, fuzzy complements, fuzzy union and intersection, and other essential operations. The unit also covers t-norms and t-conorms, which form the algebraic backbone of fuzzy set operations.

Unit 3: Fuzzy Numbers and Arithmetic

This unit focuses on fuzzy numbers and their algebraic structures. It explains binary operations, fuzzy arithmetic, and methods for performing arithmetic operations on fuzzy quantities.

Unit 4: Fuzzy Relations

Learners study fuzzy relations, including projections, cylindrical extensions, and composition. The unit also examines min–max composition, compatibility relations, and fuzzy ordering relations.

Unit 5: Fuzzy Graph

This unit introduces fuzzy graphs and fuzzy morphisms, highlighting how graph theory can be extended to represent uncertain or graded relationships.

Unit 6: Possibility Theory

Learners are introduced to fuzzy measures, evidence theory, and probability measures. The unit explains possibility and necessity measures and shows how possibility distributions can be interpreted as fuzzy sets.

Unit 7: Approximate Reasoning

This unit covers fuzzy implications, their axioms, and criteria for selecting appropriate implication operators. It forms the basis for reasoning under uncertainty.

Unit 8: Fuzzy Logic

learners explore fuzzy propositions, fuzzy connectives, fuzzy inference mechanisms, quantifiers, and linguistic hedges. The unit also discusses inference from conditional fuzzy propositions, forming a bridge between logic and computation.

Unit 9: Fuzzy Systems and Fuzzy Control

This unit introduces fuzzy rule-based systems, fuzzification and defuzzification processes, and the principles of fuzzy control. It also discusses assumptions in fuzzy control design, fuzzy system models, and illustrative examples.

Unit 10: Uncertainty-Based Information

In this unit learners study different forms of uncertainty—simple, joint, and conditional—and concepts such as non-interactive sets, information transmission, non-specificity, credibility, fuzziness, and total uncertainty, including their roles in evidence theory.

Unit 11: Decision Making in Fuzzy Environment

This unit examines decision-making models under fuzziness, including individual, multiperson, and multicriteria decision-making. The fuzzy ranking method is also introduced.

Unit 12: Fuzzy Linear Programming

Learners learn the formulation and solution of fuzzy linear programming problems, along with special cases that arise when constraints or objectives are fuzzy.

Unit 13: Fuzzy Metric Space

The final unit covers fuzzy metric spaces, fuzzy metrization, and fuzzy continuous functions, providing a deeper understanding of distance and continuity in fuzzy settings.

Unit 14: Fuzzy Topological Space

This unit introduces fuzzy points, neighbourhood structures, level sets, and local bases, extending classical topology into the fuzzy domain.

Unit 15: Fuzzy Product and Induced Topological Space

Learners study fuzzy product spaces, fuzzy continuity, and product-induced spaces, which generalize topological concepts to multidimensional fuzzy environments.

Each unit includes some along-side boxes to help you know some of the difficult, unseen terms. Some “EXERCISES” have been included to help you apply your own thoughts. You may find some boxes marked with: “LET US KNOW”. These boxes will provide you with some additional interesting and relevant information. Again, you will get “CHECK YOUR PROGRESS” questions. These have been designed to self-check your progress of study. It will be helpful for you if you solve the problems put in these boxes immediately after you go through the sections of the units and then match your answers with “ANSWERS TO CHECK YOUR PROGRESS” given at the end of each unit.

UNIT 1: FUZZY SET

UNIT STRUCTURE

1.1 Learning Objectives
1.2 Introduction
1.3 Concepts of Fuzzy Set
1.4 Types of Fuzzy Sets
1.5 Characteristics of Fuzzy Sets
1.6 General Properties of Fuzzy Sets
1.7 Let Us Sum Up
1.8 Answers to Check Your Progress
1.9 Further Reading
1.10 Model Questions

1.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- Understand the concepts of fuzzy sets by learning about partial membership, membership functions, and the basic ideas of fuzzy set theory.
- Identify different types of fuzzy sets, such as normal, subnormal, convex, non-convex, and singleton fuzzy sets, and understand their importance.
- Explain the characteristics of fuzzy sets, including support, core, height, boundary, and α-cuts, which help in analyzing fuzzy data.
- Study the general properties of fuzzy sets, such as commutativity, associativity, distributivity, idempotency, identity, involution, and De Morgan's laws.
- Apply fuzzy set concepts to simple problems, using fuzzy set operations to handle uncertainty and make decisions.

1.2 INTRODUCTION

In many real-life situations, information is imprecise, vague, or uncertain, and classical set theory often fails to represent such uncertainty effectively. To address this limitation, the concept of fuzzy sets was introduced, allowing elements to belong to a set with varying degrees of membership rather than a strict yes-or-no classification. This approach provides a flexible and powerful mathematical framework for modeling uncertainty, ambiguity, and partial truth, which commonly arise in areas such as decision-making, control systems, pattern recognition, and artificial intelligence.

This chapter begins with the basic concepts of fuzzy sets, introducing fundamental ideas such as membership functions and degrees of belonging. It then explores the various types of fuzzy sets, highlighting how different forms are used to represent

diverse kinds of uncertainty. The discussion further extends to the characteristics of fuzzy sets, which distinguish them from classical sets, followed by an examination of their general properties, including important operations and relationships.

By the end of this chapter, the reader will gain a clear understanding of the structure, behavior, and significance of fuzzy sets, forming a strong foundation for advanced topics and practical applications in fuzzy logic and related fields.

1.3 CONCEPTS OF FUZZY SET

Fuzzy Set:

A fuzzy set is a collection of distinct elements with a varying degree of relevance. Here, the characteristic function (i.e. membership function) can take any value between 0 and 1.

Definition: If X is a universe of discourse and x be any particular element of X, then a fuzzy set $\tilde{A}$ defined on X may be written as a collection of ordered pairs

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\}$$

where, each pair of $(x, \mu_{\tilde{A}}(x))$ is called a singleton.

Membership Function: A membership function is a fundamental concept in fuzzy set theory. It is used to define how much an element belongs to a fuzzy set. Unlike classical (crisp) sets, where an element either belongs or does not belong to a set, a fuzzy set allows partial membership.

Mathematically, a membership function is denoted by

$$\mu_{\tilde{A}}(x) : X \to [0,1],$$

where $X$is the universe of discourse and $\mu_{\tilde{A}}(x)$represents the degree of membership of element $x$in the fuzzy set $\tilde{A}$. A value of 0 means no membership, 1 means full membership, and values between 0 and 1 indicate partial membership.

Note:

a. Each fuzzy set is completely and uniquely defined by one particular membership function; therefore, symbols of membership functions may also be used as labels of the associated fuzzy sets.
b. Fuzzy sets allow us to represent vague concepts expressed in natural language. The representation depends not only on the concept, but also on the context in which it is used.

Types of Membership Function:

1. **Triangular functions**:

$$\tilde{A}(x) = \mu_{\tilde{A}}(x) = \begin{cases} 0 & if \quad x \le a \\ \dfrac{x-a}{m-a} & if\ x \in (a,m) \\ \dfrac{b-x}{b-m} & if\ x \in (m,b) \\ 0 & if\ x > b \end{cases}$$

Hence, m is a model value, a and b are the lower and upper bound respectively for non-zero values of $\tilde{A}(x)$. The above notations can also be expressed as:

$$\tilde{A}(x; a, m, b) = \max\{\min\left[\frac{x-a}{m-a}, \frac{b-x}{b-m}\right], 0\}$$

2. **Γ- function**:

$$\tilde{A}(x) = \mu_{\tilde{A}}(x) = \begin{cases} 0 & if\ x \le a \\ 1 - e^{-k(x-a)^2} & if\ x > a \end{cases}$$

Or

$$\tilde{A}(x) = \mu_{\tilde{A}}(x) = \begin{cases} 0 & if\ x \le a \\ \dfrac{k(x-a)^2}{1+k(x-a)^2} & if\ x > a \end{cases}$$

3. **S-function**:

$$\tilde{A}(x) = \mu_{\tilde{A}}(x) = \begin{cases} 0 & if\ x \le a \\ 2\left(\dfrac{x-a}{b-a}\right)^2 & if\ x \in [a, m] \\ 1 - 2\left(\dfrac{x-b}{b-a}\right)^2 & if\ x \in [m, b] \\ 1 & if\ x > b \end{cases}$$

The point $m = \frac{a+b}{2}$ is known as the cross over of S-function.

4. **Trapezoidal function**:

$$\tilde{A}(x) = \mu_{\tilde{A}}(x) = \begin{cases} 0 & if\ x < a \\ \dfrac{x-a}{m-a} & if\ x \in (a, m) \\ 1 & if\ x \in (m, n) \\ \dfrac{b-x}{b-m} & if\ x \in (n, b) \\ 0 & if\ x > b \end{cases}$$

5. **Gaussian function**:

$$\tilde{A}(x) = \mu_{\tilde{A}}(x) = e^{-k(x-m)^2},\ k > 0$$

6. **Exponential like function**:

$$\tilde{A}(x) = \mu_{\tilde{A}}(x) = \frac{1}{1+k(x=m)^2}, k > 1$$

or

$$\tilde{A}(x) = \mu_{\tilde{A}}(x) = \frac{k(x=m)^2}{1+k(x=m)^2}, k > 0$$

Representation of Membership Function:

a. The fuzzy membership function for the fuzzy linguistic term “cool” relating to temperature may turn out as:

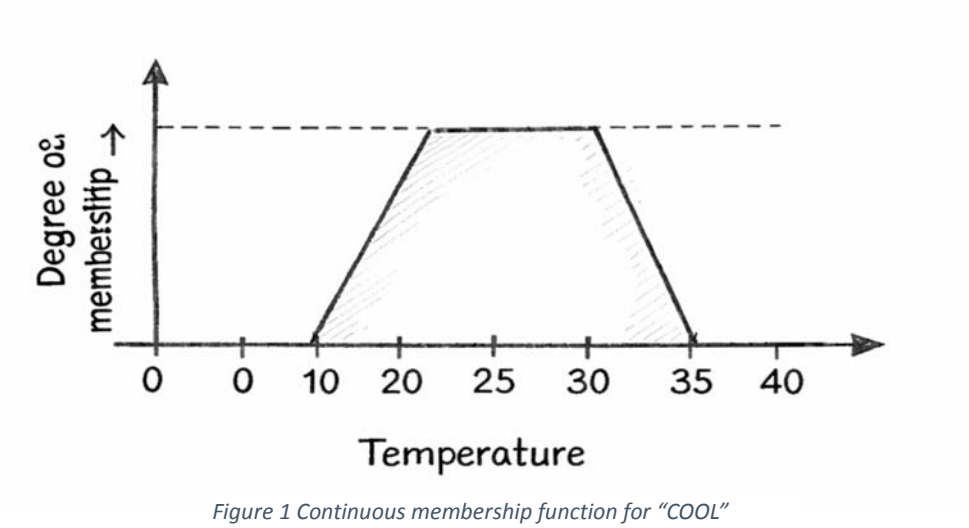


Figure 1 Continuous membership function for "COOL"

b. A membership function can express mathematically as

$$\mu_{\tilde{A}}(x) = \frac{1}{(1+x)^2}$$

The graph of the function is:

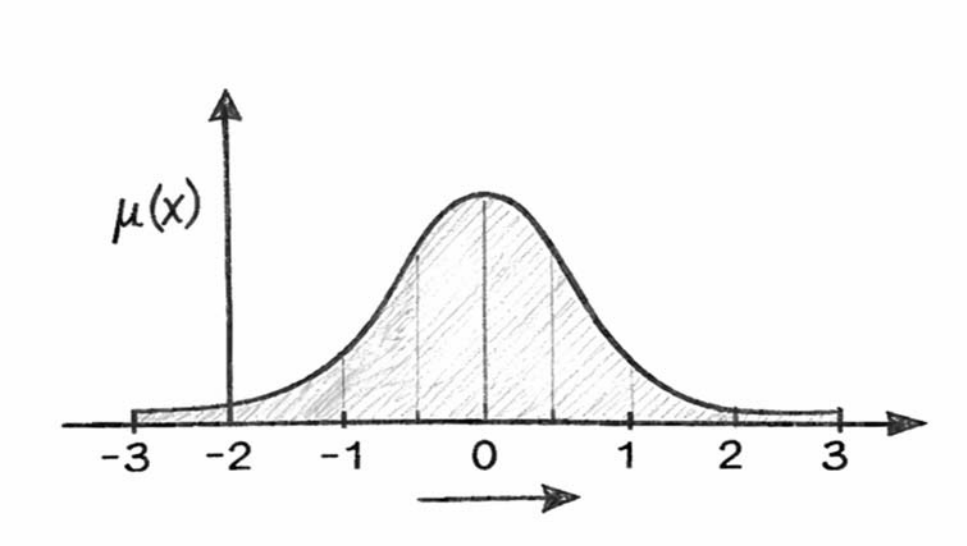


Figure 2 Continuous membership function dictated by mathematical function

c. Different types of membership function graphs are shown below. The shapes

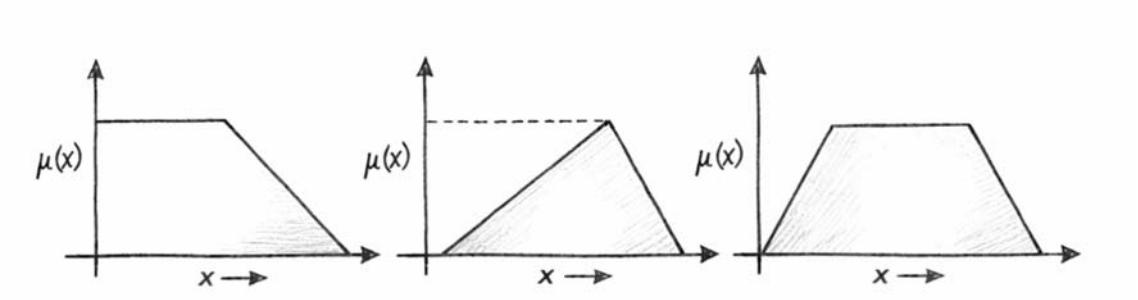


Figure 3 Different types of membership function graphs

could be triangular, trapezoidal, curved or their variations.
Let <x be the universe of discourse and <$\mu_A(x)$ (membership function) assume values in the range from 0 to 1. Then

$$A = \{(x, \mu_{\tilde{A}}(x)) : x \in X\}$$

This representation is called ordered pair representation of fuzzy set.
This ordered pair method in a different format as:

$$\tilde{A} = \frac{\mu_1}{x_1} + \frac{\mu_2}{x_2} + \cdots \ldots \ldots$$

Here the symbols / does not denote division mor doss the symbol T dente summation 'T' is used to connect the terms.

1.4 TYPES OF FUZZY SETS

1. **Interval-Valued Fuzzy sets**: A membership function based on the approach that does not assign to each element of the universal set to one real number, but a closed interval of real numbers between the identified lower and upper bounds. Fuzzy sets defined by membership function of this type are called interval-valued fuzzy sets. These sets are defined as follows

$$\tilde{A}: X \rightarrow \in ([0\ \ 1])$$

where $\in (10\ 1])$ denote the family of all closed intervals of real numbers in [0, 1].

2. **Fuzzy set of Type-2:** We can generalise interval-valued fuzzy sets by allowing their intervals to be fuzzy. Each interval now becomes an ordinary fuzzy set defined within the universal set [0,1]. The membership grades assigned to elements of the universal set by these generalised fuzzy sets are ordinary fuzzy sets. These sets are known as "fuzzy set of type-2". The membership function of this type of set having the following form

$$\tilde{A}: X \rightarrow P([0\ 1])$$

where $P([0\ 1])$ denotes the set of all ordinary fuzzy sets that can be defined with the universal set [0, 1].

3. **L-Fuzzy sets:** When we relax the requirement that the membership grades must be represented by numbers in the unit interval [0, 1] and allow them to be represented by symbols of an arbitrary set L, i.e., at least partially ordered, we obtain L-fuzzy set. membership function of L-fuzzy set have the following form

$$\tilde{A}: X \rightarrow L$$

4. **Level-2 Fuzzy sets**: The generalization of ordinary fuzzy sets involves fuzzy sets defined within a universal set whose elements are ordinary fuzzy sets.

These fuzzy sets are known as 'level-2 fuzzy sets. The membership function of level-2 fuzzy set having the following form

$$\tilde{A}: \zeta(X) \to [0,1]$$

where $\zeta(X)$ denotes the fuzzy power set of X.

1.5 CHARACTERISTICS OF FUZZY SETS

1. **Empty Fuzzy Set:** A fuzzy set $\tilde{A}$ is called empty if and only if the value of the membership function is zero for all possible members under consideration. i.e.
$$\tilde{A} = \phi \quad iff\ \mu_{\tilde{A}}(x) = 0, \forall x \in X$$
2. **Universal fuzzy set**: A fuzzy set $\tilde{A}$ is called universal fuzzy set if and only if the value of the membership function is one for all possible members under consideration. i.e.
$$\tilde{A}\ is\ universal\ \ iff\ \ \mu_{\tilde{A}}(x) = 1, \forall x \in X$$
3. **Height of a Fuzzy Set**: The height $h(\tilde{A})$ of a fuzzy set $\tilde{A}$ is the largest membership grade obtained by any element in that set, i.e.,
$$h(\tilde{A}) = \sup_{x \in X} \tilde{A}(x)$$
4. **Equality of Two Fuzzy Sets**:

   Definitions: Two fuzzy sets are said to be equal, if they have the same number of elements and their membership functions are also equal.

   Two fuzzy sets are said to be equivalent, if they have different number of elements but membership functions are equal.

   For Example: consider a non-empty set X, such that
   $$X = \{1,2,3,4\}$$
   Let, $\tilde{A} = \left\{\frac{1}{.2}, \frac{2}{.4}, \frac{3}{.6}, \frac{4}{.9}\right\}$; $\tilde{B} = \left\{\frac{1}{.3}, \frac{2}{.5}, \frac{3}{.7}, \frac{4}{1}\right\}$; $\tilde{C} = \left\{\frac{1}{.3}, \frac{2}{.6}, \frac{3}{.4}, \frac{4}{.5}\right\}$; $\tilde{D} = \left\{\frac{1}{.3}, \frac{2}{.6}, \frac{3}{.4}, \frac{4}{.5}\right\}$; $\tilde{E} = \left\{\frac{1}{.3}, \frac{2}{.6}, \frac{3}{0}, \frac{4}{0}\right\}$

   Here, $\tilde{C} = \tilde{D}$; since

   $\mu_{\tilde{C}}(1) = \mu_{\tilde{D}}(1) = .3$; $\mu_{\tilde{C}}(2) = \mu_{\tilde{D}}(2) = .6$; $\mu_{\tilde{C}}(3) = \mu_{\tilde{D}}(3) = .4$; $\mu_{\tilde{C}}(4) = \mu_{\tilde{D}}(4) = .5$

   Again, $\tilde{D} \equiv \tilde{E}$, since

   $\mu_{\tilde{D}}(1) = \mu_{\tilde{E}}(1) = .3$ and $\mu_{\tilde{D}}(2) = \mu_{\tilde{E}}(2) = .6$
5. **Union of Two Fuzzy Sets**: Let X be a non-empty set and let $\tilde{A}$ and $\tilde{B}$ be two fuzzy sets on X. Then
$$\tilde{A} \cup \tilde{B} = \sup\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\} = \sup\{\tilde{A}(x), \tilde{B}(x)\} = \max\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}$$

Thus, $(\tilde{A} \cup \tilde{B})$ contains the supremum or maximum membership functions of both the fuzzy sets.

6. **Intersection of Two Fuzzy Sets**: Let X be a non-empty set and let $\tilde{A}$ and $\tilde{B}$ be two fuzzy sets on X. Then

$$\tilde{A} \cap \tilde{B} = \inf\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\} = \inf\{\tilde{A}(x), \tilde{B}(x)\} = \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}$$

Thus, $(\tilde{A} \cap \tilde{B})$ contains the infimum or minimum membership functions of both the fuzzy sets.

For example: Let $X = \{1,2,3,4\}$

$$\tilde{A} = \{(1,.2), (2,.5), (3,.8), (4,1)\};\ \tilde{B} = \{(1,.3), (2,.6), (3,.9), (4,1)\}$$

$$\therefore\ \tilde{A} \cup \tilde{B}\ or\ \mu_{\tilde{A}\cup\tilde{B}}(x) = \{(1,.3), (2,.6), (3,.9), (4,1)\}$$

And,

$$\therefore\ \tilde{A} \cap \tilde{B}\ or\ \mu_{\tilde{A}\cap\tilde{B}}(x) = \{(1,.2), (2,.5), (3,.8), (4,1)\}$$

7. **Complement of a Fuzzy set**: The complement of a fuzzy set $\tilde{A}$ of a set X is defined by

$$\tilde{A}' = \tilde{A}^c = \bar{\tilde{A}} = 1 - \mu_{\tilde{A}}(x)$$

Example: Let, $X = \{1,2,3,4\}$ and $\tilde{A} = \{(1,.1), (2,.3), (3,.5), (4,1)\}$

Then $\tilde{A}^c = \{(1,.9), (2,.7), (3,.5), (4,0)\}$

8. **Normal Fuzzy set**: A fuzzy set $\tilde{A}$ of a set X is called normal fuzzy set if and only if $\max_{\{x\in X\}} \mu_{\tilde{A}}(x) = 1$, i.e. $\mu_{\tilde{A}}(x) = 1$ for at least one $x \in X$.

For example: Let $X = \{1,2,3,4\}$

$$\tilde{A} = \{(1,.3), (2,.5), (3,.8), (4,1)\}$$

$$\tilde{B} = \{(1,1), (2,.3), (3,.8), (4,.2)\}$$

$$\tilde{C} = \{(1,.2), (2,.4), (3,.7), (4,.8)\}$$

Here, $\mu_{\tilde{A}}(4) = 1$ and $\mu_{\tilde{B}}(1) = 1$. So $\tilde{A}$ and $\tilde{B}$ are normal.

Since, $\mu_{\tilde{C}}(x_i) \neq 1$ for $i = 1,2,3,4$; hence $\tilde{C}$ is not a normal fuzzy set.

9. **Support of a Fuzzy Set**: The support of a fuzzy set $\tilde{A}$ of the set X is a classical set defined as

$$\sup(\tilde{A}) = \{x \in X: \mu_{\tilde{A}}(x) > 0\}$$

A fuzzy set whose support is a single element in X with $\mu_{\tilde{A}}(x) = 1$ is called fuzzy singleton.

10. **α- Cut Set or α-Level Set or Cut Worthy Set**: The α-cut set of a fuzzy set $\tilde{A}$ of the set X is the following conventional (i.e. crisp) set given by

$$^{\alpha}\tilde{A} = \{x \in X: \mu_{\tilde{A}}(x) \geq \alpha\}$$

$where\ \alpha \in [0,1]$

Example: Let, $X = \{1,2,3,4\}$ and $\tilde{A} = \{(1,.2),(2,.4),(3,.6),(4,.9)\}$. Let $\alpha = .3$

Then, $\mu_{\tilde{A}}(1) = .2 < .3;\ \mu_{\tilde{A}}(2) = .4 > .3;\ \mu_{\tilde{A}}(3) = .6 > .3;\ \mu_{\tilde{A}}(4) = .9 > .3$

Hence, ${}^{\alpha}\tilde{A} = {}^{.3}\tilde{A} = \{2,3,4\}$

11. **Strong α-Cut:** A fuzzy set $\tilde{A}$ of the set X is said to be strong α-cut if $\tilde{A}$ contains only those elements whose membership function is strictly greater than α, i.e.

$$ {}^{\alpha+}\tilde{A} = \{x \in X : \mu_{\tilde{A}}(x) > \alpha\} $$

12. **Convex Fuzzy set**: A fuzzy set $\tilde{A}$ of the set X is convex if

$$ \mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\};\ \ \forall x_1, x_2 \in X; \lambda \in [0,1] $$

In other word, a fuzzy set is convex, if all α-level sets are convex.

13. **Decomposition of Fuzzy Sets**: Let X be a finite set and let $\tilde{A}$ be a fuzzy set on X. Then for any $\alpha \in [0,1]$, the decomposition of $\tilde{A}$ is defined as

$$ {}_{\alpha}\tilde{A}(x) = \alpha .\, {}^{\alpha}(\tilde{A}(x)) $$

14. **Degree of Subsethood:** Let X be a non-empty set and $\tilde{A}$ and $\tilde{B}$ are two fuzzy sets of X. Then degree of subsethood is defined as:

$$ S(\tilde{A}, \tilde{B}) = \frac{1}{|\tilde{A}|}\left\{|\tilde{A}| - \sum_{x \in X} \max\left\{0, \left(\tilde{A}(x) - \tilde{B}(x)\right)\right\}\right\} $$

and

$$ S(\tilde{B}, \tilde{A}) = \frac{1}{|\tilde{B}|}\left\{|\tilde{B}| - \sum_{x \in X} \max\left\{0, \left(\tilde{B}(x) - \tilde{A}(x)\right)\right\}\right\} $$

where,

$$ |\tilde{A}| = \sum_{x \in X} \mu_{\tilde{A}}(x) $$

and

$$ |\tilde{B}| = \sum_{x \in X} \mu_{\tilde{B}}(x) $$

15. **Cardinality of a Fuzzy set**: For a finite fuzzy set $\tilde{A}$, the cardinality $|\tilde{A}|$ is

$$ |\tilde{A}| = \sum_{x \in X} \mu_{\tilde{A}}(x) $$

And its relative cardinality is

$$ \|\tilde{A}\| = \frac{|\tilde{A}|}{10} $$

Example: Let, $X = \{1,2,3,4\}$ and $\tilde{A} = \{(1,.2),(2,.4),(3,.6),(4,.9)\}$.

Then,

$$ |\tilde{A}| = .2 + .4 + .6 + .9 = 2.1 $$

$$\|\tilde{A}\| = \frac{|\tilde{A}|}{10} = \frac{2.1}{10} = .21$$

16. **Fuzzy Cardinality**: Using fuzzy numbers, we can define the concept of fuzzy cardinality for fuzzy sets that are defined on a finite universal set. Given a fuzzy set $\tilde{A}$. defined on a finite universal set X, its fuzzy cardinality is defined by $|\tilde{A}|$ which is a fuzzy number on X, defined by

$$|\tilde{A}|(|^{\alpha}\tilde{A}|) = \alpha, \quad \forall\, \alpha \in \Lambda(\tilde{A})$$

17. **Width of a Convex Fuzzy Set:** The width of a convex fuzzy set $\tilde{A}$ with support set $S(\tilde{A})$ is defined by

$$width\,(\tilde{A}) = supremum\,\left(S(\tilde{A})\right) - infimum\,(S(\tilde{A}))$$

It should be noted that if the support set $S(\tilde{A})$ is bounded, supremum and infimum can be replaced by maximum and minimum respectively.

18. **Nucleus of a Fuzzy Set:** The nucleus of a fuzzy set $\tilde{A}$ is defined by

$$nucleus\,(\tilde{A}) = \{x \in X: \mu_{\tilde{A}}(x) = 1\}$$

If there is only one point with membership degree equal to 1, then this point is known as peak value of $\tilde{A}$.

19. **Fuzzification:** The effect of fuzzification is complimentary to that of intensification and is produced by altering the membership function as follows:

$$Fuzz_{\tilde{A}(x)} = \begin{cases} \sqrt{\dfrac{\tilde{A}(x)}{2}} & ; \quad if\ \ \tilde{A}(x) \le 0.5 \\ 1 - \sqrt{1 - \dfrac{\tilde{A}(x)}{2}} & ; otherwise \end{cases}$$

Theorem 1: Sum and difference of two convex fuzzy sets are convex.

Proof: Let $\tilde{A}$ and $\tilde{B}$ be two given convex fuzzy sets defined on X.

We want to prove that $\tilde{A} + \tilde{B}$ is convex.

Let, $x_1 + y_1 \in \tilde{A} + \tilde{B}$ and $x_2 + y_2 \in \tilde{B}$

Then $x_1 \in \tilde{A}$, $y_1 \in \tilde{B}$ and $x_2 \in \tilde{A}, y_2 \in \tilde{B}$

By the definition of convex fuzzy sets, we have

$$\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) > \min\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\}, \quad \forall x_1, x_2 \in X, \lambda \in (0,1) \qquad (1)$$

$$\mu_{\tilde{B}}(\lambda y_1 + (1-\lambda)y_2) > \min\{\mu_{\tilde{B}}(y_1), \mu_{\tilde{B}}(y_2)\}, \quad \forall y_1, y_2 \in X, \lambda \in (0,1) \qquad (2)$$

Adding, we get

$$\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) + \mu_{\tilde{B}}(\lambda y_1 + (1-\lambda)y_2) \ge \min[(\mu_{\tilde{A}}(x_1) + \mu_{\tilde{B}}(y_1)), (\mu_{\tilde{A}}(x_2) + \mu_{\tilde{B}}(y_2))]$$

$$\Rightarrow \mu_{\tilde{A}+\tilde{B}}(\lambda(x_1 + y_1) + (1-\lambda)(x_2 + y_2)) \ge \min(\mu_{\tilde{A}+\tilde{B}}(x_1 + y_1), \mu_{\tilde{A}+\tilde{B}}(x_2 + y_2))$$

$\Rightarrow \tilde{A} + \tilde{B}$ is convex

Similarly, we can say that $\tilde{A} - \tilde{B}$ is convex.

Theorem 2: Let $\tilde{A}$ and $\tilde{B}$ be two fuzzy sets defined on the universal set X. Then

$$|\tilde{A}| + |\tilde{B}| = |\tilde{A} \cup \tilde{B}| + |\tilde{A} \cap \tilde{B}|$$

Proof: We know that,

$$|\tilde{A}| = \sum_{x \in X} \mu_{\tilde{A}}(x) \qquad (1)$$

and

$$|\tilde{B}| = \sum_{x \in X} \mu_{\tilde{B}}(x) \qquad (2)$$

$$|\tilde{A} \cup \tilde{B}| = \sum \mu_{(\tilde{A} \cup \tilde{B})}(x)$$

$$= \sum [\max\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}]$$
$$= \max\left[\sum \{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}\right]$$
$$= \max[|\tilde{A}|, |\tilde{B}|] \qquad (3)$$

Similarly,

$$|\tilde{A} \cap \tilde{B}| = \min[|\tilde{A}|, |\tilde{B}|] \qquad (4)$$

Now, if $|\tilde{A}| > |\tilde{B}|$, Then

$$\max[|\tilde{A}|, |\tilde{B}|] = |\tilde{A}|$$
$$\min[|\tilde{A}|, |\tilde{B}|] = |\tilde{B}|$$

Using (3), (4) and (5), we get

$$|\tilde{A} \cup \tilde{B}| = |\tilde{A}|$$
$$|\tilde{A} \cap \tilde{B}| = |\tilde{B}|$$

Adding we get,

$$|\tilde{A} \cup \tilde{B}| + |\tilde{A} \cap \tilde{B}| = |\tilde{A}| + |\tilde{B}|$$

If $|\tilde{B}| > |\tilde{A}|$, then

$$\max[|\tilde{A}|, |\tilde{B}|] = |\tilde{B}|$$
$$\min[|\tilde{A}|, |\tilde{B}|] = |\tilde{A}|$$

Using (3), (4) and (5), we get

$$\left|\tilde{A} \cup \tilde{B}\right| = |\tilde{B}|$$

$$\left|\tilde{A} \cap \tilde{B}\right| = |\tilde{A}|$$

Adding we get,

$$\left|\tilde{A} \cup \tilde{B}\right| + \left|\tilde{A} \cap \tilde{B}\right| = \left|\tilde{A}\right| + |\tilde{B}|$$

Theorem 3: A fuzzy set $\tilde{A}$ on **R** is convex if and only if

$$\tilde{A}(\lambda x_1 + (1-\lambda)x_2) \geq \min[\tilde{A}(x_1), \tilde{A}(x_2)] \qquad \text{(i)}$$

Proof: Let us assume that $\tilde{A}$ is convex, and let $\alpha = \tilde{A}(x_1) \leq \tilde{A}(x_2)$

Then, $x_1, x_2 \in {}^{\alpha}\tilde{A}$; and clearly by definition for any $\lambda \in [0,1]$

$$\lambda x_1 + (1-\lambda)x_2 \in {}^{\alpha}\tilde{A}$$

$$\therefore\ \tilde{A}(\lambda x_1 + (1-\lambda)x_2) \geq \alpha$$

$$\Rightarrow \tilde{A}(\lambda x_1 + (1-\lambda)x_2) \geq \min(\tilde{A}(x_1), \tilde{A}(x_2))$$

Conversely, assume that $\tilde{A}$ satisfies (i). So, we want to prove that for any $\alpha \in [0,1]$, ${}^{\alpha}\tilde{A}$ is convex.

Now, for any $x_1, x_2 \in {}^{\alpha}\tilde{A}$ and for any $\lambda \in [0,1]$

$$\tilde{A}(\lambda x_1 + (1-\lambda)x_2) \geq \min(\tilde{A}(x_1), \tilde{A}(x_2))$$

$$\geq \min[\alpha, \alpha]$$

$$\Rightarrow \tilde{A}(\lambda x_1 + (1-\lambda)x_2) \geq \alpha$$

$$\Rightarrow \lambda x_1 + (1-\lambda)x_2 \in {}^{\alpha}\tilde{A}$$

$\Rightarrow {}^{\alpha}\tilde{A}$ is convex, for any $\alpha \in [0,1]$

Hence ${}^{\alpha}\tilde{A}$ is convex.

Some other operations:

i. **Product of Two Fuzzy Sets** $(\tilde{A}.\tilde{B})$**:** The product of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ is a new fuzzy set with membership function

$$\mu_{\tilde{A}.\tilde{B}}(x) = \mu_{\tilde{A}}(x)\mu_{\tilde{B}}(x)$$

Example: Let

$$\tilde{A} = \{(x_1, 0.2), (x_2, 0.8), (x_3, 0.4)\}$$
$$\tilde{B} = \{(x_1, 0.4), (x_2, 0), (x_3, 0.1)\}$$
$$\therefore\ \mu_{\tilde{A}.\tilde{B}}(x_1) = \mu_{\tilde{A}}(x_1)\mu_{\tilde{B}}(x_1) = 0.2 \times 0.4 = 0.08$$
$$\mu_{\tilde{A}.\tilde{B}}(x_2) = \mu_{\tilde{A}}(x_2)\mu_{\tilde{B}}(x_2) = 0.8 \times 0 = 0$$
$$\mu_{\tilde{A}.\tilde{B}}(x_3) = \mu_{\tilde{A}}(x_3)\mu_{\tilde{B}}(x_3) = 0.4 \times 0.1 = 0.04$$

Therefore,

$$\tilde{A}.\tilde{B} = \{(x_1, 0.08), (x_2, 0), (x_3, 0.04)\}$$

ii. Product of Fuzzy Set with a Crisp Number: Let a fuzzy set be $\tilde{A}$ and $'a'$ be any crisp number lying between 0 and 1. Then their product $a.\tilde{A}$ gives by the membership function

$$\mu_{a.\tilde{A}}(x) = a.\mu_{\tilde{A}}(x)$$

Example: Let $a = 0.3$ and

$$\tilde{A} = \{(x_1, 0.2), (x_2, 0.8), (x_3, 0.4)\}$$

Then,

$$\mu_{a.\tilde{A}}(x_1) = a.\mu_{\tilde{A}}(x_1) = 0.3 \times 0.2 = 0.06$$
$$\mu_{a.\tilde{A}}(x_2) = a.\mu_{\tilde{A}}(x_2) = 0.3 \times 0.8 = 0.24$$
$$\mu_{a.\tilde{A}}(x_3) = a.\mu_{\tilde{A}}(x_3) = 0.3 \times 0.4 = 0.12$$

Therefore,

$$a.\tilde{A} = \{(x_1, 0.06), (x_2, 0.24), (x_3, 0.12)\}$$

iii. Power of a Fuzzy Set ($\tilde{A}^{\alpha}$): The α power of a fuzzy set $\tilde{A}$ is given by the membership function:

$$\mu_{\tilde{A}^{\alpha}}(x) = \left(\mu_{\tilde{A}}(x)\right)^{\alpha}$$

Example: Let $\alpha = 2$ and

$$\tilde{A} = \{(x_1, 0.2), (x_2, 0.8), (x_3, 0.4)\}$$

Then,

$$\mu_{\tilde{A}^2}(x_1) = \left(\mu_{\tilde{A}}(x_1)\right)^2 = (0.2)^2 = 0.04$$
$$\mu_{\tilde{A}^2}(x_2) = \left(\mu_{\tilde{A}}(x_2)\right)^2 = (0.8)^2 = 0.64$$
$$\mu_{\tilde{A}^2}(x_3) = \left(\mu_{\tilde{A}}(x_3)\right)^2 = (0.4)^2 = 0.16$$

Therefore,

$$\tilde{A}^2 = \{(x_1, 0.04), (x_2, 0.64), (x_3, 0.16)\}$$

iv. Difference of Two Fuzzy Sets ($\tilde{A} \sim \tilde{B}$): The difference of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ is a new fuzzy set defined by

$$\tilde{A} \sim \tilde{B} = (\tilde{A} \cap \tilde{B}^c)$$

Example: Let

$$\tilde{A} = \{(x_1, 0.2), (x_2, 0.5), (x_3, 0.6)\}$$
$$\tilde{B} = \{(x_1, 0.1), (x_2, 0.4), (x_3, 0.5)\}$$

Then,

$$\tilde{B}^c = \{(x_1, 0.9), (x_2, 0.6), (x_3, 0.5)\}$$

Now,

$$\mu_{\tilde{A} \cap \tilde{B}^c}(x_1) = \min\{0.2, 0.9\} = 0.2,$$
$$\mu_{\tilde{A} \cap \tilde{B}^c}(x_2) = \min\{0.5, 0.6\} = 0.5,$$
$$\mu_{\tilde{A} \cap \tilde{B}^c}(x_3) = \min\{0.6, 0.5\} = 0.5.$$
$$\therefore \tilde{A} \sim \tilde{B} = \left(\tilde{A} \cap \tilde{B}^c\right) = \{(x_1, 0.2), (x_2, 0.5), (x_3, 0.5)\}$$

v. Disjunctive Sum ($\tilde{A} \oplus \tilde{B}$): The disjunctive sum of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ is a new fuzzy set defined by

$$\tilde{A} \oplus \tilde{B} = (\tilde{A}^c \cap \tilde{B}) \cup (\tilde{A} \cap \tilde{B}^c)$$

Example: Let

$$\tilde{A} = \{(x_1, 0.4), (x_2, 0.8), (x_3, 0.6)\}$$
$$\tilde{B} = \{(x_1, 0.2), (x_2, 0.6), (x_3, 0.9)\}$$

Then,

$$\tilde{A}^c = \{(x_1, 0.6), (x_2, 0.2), (x_3, 0.4)\}$$
$$\tilde{B}^c = \{(x_1, 0.8), (x_2, 0.4), (x_3, 0.1)\}$$

Now,

$$\tilde{A}^c \cap \tilde{B} = \{(x_1, 0.2), (x_2, 0.2), (x_3, 0.4)\}$$
$$\tilde{A} \cap \tilde{B}^c = \{(x_1, 0.4), (x_2, 0.4), (x_3, 0.1)\}$$

Therefore,

$$\tilde{A} \oplus \tilde{B} = \{(x_1, 0.4), (x_2, 0.4), (x_3, 0.4)\}$$

1.6 GENERAL PROPERTIES OF FUZZY SETS

Let $\tilde{A}$ be a fuzzy set defined on the reference set X. In fuzzy set theory, the membership value of any element belonging to the null set ϕ is zero, whereas the membership value of an element belonging to the universal set X is one. Similar to crisp sets, fuzzy sets also satisfy several basic set-theoretic properties. Some of the important properties of fuzzy sets are discussed below.

(i) Commutativity:

$$\tilde{A} \cup \tilde{B} = \tilde{B} \cup \tilde{A}\,; \qquad \tilde{A} \cap \tilde{B} = \tilde{B} \cap \tilde{A}$$

(ii) Associativity:

$$\tilde{A} \cup (\tilde{B} \cup \tilde{C}) = (\tilde{A} \cup \tilde{B}) \cup \tilde{C}$$

$$\tilde{A} \cap (\tilde{B} \cap \tilde{C}) = (\tilde{A} \cap \tilde{B}) \cap \tilde{C}$$

(iii) Distributivity:

$$\tilde{A} \cup (\tilde{B} \cap \tilde{C}) = (\tilde{A} \cup \tilde{B}) \cap (\tilde{A} \cup \tilde{C})$$

$$\tilde{A} \cap (\tilde{B} \cup \tilde{C}) = (\tilde{A} \cap \tilde{B}) \cup (\tilde{A} \cap \tilde{C})$$

(iv) Idempotent law:

$$\tilde{A} \cup \tilde{A} = \tilde{A}; \quad \tilde{A} \cap \tilde{A} = \tilde{A}$$

(v) Identity:

$$\tilde{A} \cup \phi = \tilde{A}; \qquad \tilde{A} \cap \phi = \phi$$

$$\tilde{A} \cup X = X; \qquad \tilde{A} \cap X = \tilde{A}$$

(vi) Transitivity: If $\tilde{A} \subset \tilde{B} \subset \tilde{C}$, then $\tilde{A} \subset \tilde{C}$

(vii) Involution:

$$(\tilde{A}^c)^c = \tilde{A}$$

(viii) De Morgan's laws:

$$\left(\tilde{A} \cup \tilde{B}\right)^c = \tilde{A}^c \cap \tilde{B}^c$$
$$\left(\tilde{A} \cap \tilde{B}\right)^c = \tilde{A}^c \cup \tilde{B}^c$$

CHECK YOUR PROGRESS

Q1) Let $X = \{1,2,\dots,10\}$. Determine cardinalities and relative cardinalities of the following sets:

a. $\tilde{A} = \{(2,.5),(4,.8),(5,1),(7,.6)\}$

b. $\tilde{B} = \{(2,.4),(3,.6),(4,.8),(5,1),(6,.8),(7,.6),(8,.4)\}$

Q2) Determine all α-level sets and all strong α-cuts for the fuzzy set for $\alpha = .3,.5,.8$

$$\tilde{A} = \{(x,\mu_{\tilde{A}}(x)) : \mu_{\tilde{A}}(x) = (1+(x-10)^2)^{-1}\}$$

Q3) Order the fuzzy sets defined by the following membership function grades (assuming $x \geq 0$) by the inclusion relation

$$\tilde{A}(x) = \frac{1}{1+10x}, \qquad \tilde{B}(x) = \left(\frac{1}{1+10x}\right)^{1/2}; \quad \tilde{C} = \left(\frac{1}{1+10x}\right)^{2}$$

Q4) Compute the scalar cardinalities and relative cardinalities of the set: $\tilde{B} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$

Q5) Calculate the degree of subsethood $S(\tilde{A},\tilde{B})$ and $S(\tilde{B},\tilde{A})$ for the fuzzy sets

a) $\tilde{A} = \frac{.2}{u} + \frac{.4}{v} + \frac{.8}{w} + \frac{.7}{x} + \frac{1}{y} + \frac{.3}{z}$

b) $\tilde{B} = \frac{x}{x+1}, \quad \forall x \in \{0,1,2,\dots,10\}$

Q6) Consider fuzzy sets $\tilde{A}$ and $\tilde{B}$ defined on the interval $X = [0,5]$ of real numbers by the membership functions $\mu_{\tilde{A}}(x) = \frac{x}{x+1}$, $\mu_{\tilde{B}}(x) = \overline{2}^{x}$.

Find the mathematical formulae of the membership grade functions of each of the following sets

a. $\tilde{A}^c,\ \tilde{B}^c$ b. $\tilde{A} \cup \tilde{B}$ c. $\tilde{A} \cap \tilde{B}$ d. $(\tilde{A} \cup \tilde{B})^c$

1.7 LET US SUM UP

- If X is a universe of discourse and x be any particular element of X, then a fuzzy set $\tilde{A}$ defined on X may be written as a collection of ordered pairs

$$\tilde{A} = \{(x,\mu_{\tilde{A}}(x)) : x \in X\}$$

- A membership function is a fundamental concept in fuzzy set theory. It is used to define how much an element belongs to a fuzzy set. Unlike classical (crisp) sets, where an element either belongs or does not belong to a set, a fuzzy set allows partial membership.
- Each fuzzy set is completely and uniquely defined by one particular membership function

- A fuzzy set $\tilde{A}$ is called empty if and only if the value of the membership function is zero for all possible members under consideration.
- A fuzzy set $\tilde{A}$ is called universal fuzzy set if and only if the value of the membership function is one for all possible members under consideration.
- Given a fuzzy set $\tilde{A}$. defined on a finite universal set X, its fuzzy cardinality is defined by $|\tilde{A}|$ which is a fuzzy number on X, defined by

$$\left|\tilde{A}\right|\left(\left|{}^{\alpha}\tilde{A}\right|\right) = \alpha, \quad \forall\, \alpha \in \Lambda(\tilde{A})$$

- In fuzzy set theory, the membership value of any element belonging to the null set ϕ is zero, whereas the membership value of an element belonging to the universal set X is one. Similar to crisp sets, fuzzy sets also satisfy several basic set-theoretic properties.

✓ 1.8 ANSWERS TO CHECK YOUR PROGRESS

Answer to Q1: a. $\left|\tilde{A}\right| = .5 + .8 + 1 + .6 = 2.9$

$$\left\|\tilde{A}\right\| = \frac{2.9}{10} = 0.29$$

b. $\left|\tilde{B}\right| = .4 + .6 + .8 + 1 + .8 + .6 + .4 = 4.6$

$$\left\|\tilde{B}\right\| = \frac{4.6}{10} = 0.46$$

Answer to Q2:

$$\tilde{A} = \{(1,.01),(2,.01),(3,.02),(4,.03),(5,.04),(6,.06),(7,.1),(8,.2),(9,.5),(10,1)\}$$

For $\alpha = .3$

$$^{.3}\tilde{B} = \{9,10\}; \qquad ^{.3+}\tilde{B} = \{9,10\}$$

For $\alpha = .5$

$$^{.5}\tilde{B} = \{9,10\}; \qquad ^{.5+}\tilde{B} = \{10\}$$

For $\alpha = .8$

$$^{.8}\tilde{B} = \{10\}; \qquad ^{.8+}\tilde{B} = \{10\}$$

Answer to Q3:

$$\tilde{A} = \frac{1}{1}, \frac{1}{11}, \frac{1}{21}, \frac{1}{31}, \frac{1}{41}, \ldots\ldots = 1, .09, .04, .03, .02, .01, \ldots\ldots$$

$$\tilde{B} = \frac{1}{\sqrt{1}}, \frac{1}{\sqrt{11}}, \frac{1}{\sqrt{21}}, \frac{1}{\sqrt{31}}, \frac{1}{\sqrt{41}}, \ldots\ldots = 1, .3, .2, .1, .1, .1, \ldots\ldots$$

$$\tilde{C} = \frac{1}{1}, \frac{1}{121}, \frac{1}{441}, \frac{1}{961}, \frac{1}{1681}, \ldots\ldots = 1, .0082, .0022, .0010, .0004, .0001, \ldots\ldots$$

We have

$$\tilde{A} = \{(0,1), (1,.09), (2,.04), (3,.03), (4,.02), (5,.01), \ldots\ldots\}$$

$$\tilde{B} = \{(0,1), (1,.3), (2,.2), (3,.1), (4,.1), (5,.11), \ldots\ldots\}$$

$$\tilde{C} = \{(0,1), (1,.0082), (2,.0022), (3,.0010), (4,.0004), (5,.0001), \ldots\ldots\}$$

Here,

$$\mu_{\tilde{A}}(x) \leq \mu_{\tilde{B}}(x);\ \mu_{\tilde{B}}(x) \geq \mu_{\tilde{C}}(x);\ \mu_{\tilde{C}}(x) \leq \mu_{\tilde{A}}(x)$$

So we have,

$$\mu_{\tilde{C}}(x) \leq \mu_{\tilde{A}}(x) \leq \mu_{\tilde{B}}(x)$$

Or $\tilde{C} \subseteq \tilde{A} \subseteq \tilde{B}$

Answer to Q4: Here, $\tilde{B} = \{(x,1), (y,1), (z,1)\}$

So, scalar cardinality is

$$\left|\tilde{B}\right| = 1 + 1 + 1 = 3$$

Relative cardinality is

$$\left\|\tilde{B}\right\| = \frac{3}{10} = 0.3$$

Answer to Q5: $\tilde{A} = \{(u,.2), (v,.4), (w,.8), (x,.7), (y,1), (z,.3)\}$

$\tilde{B} = \{(0,0), (1,.5), (2,.66), (3,.75), (4,.8), (5,.83), (6,.85), (7,.87), (8,.89), (9,.9), (10,.90)\}$

Now, $\left|\tilde{A}\right| = 3.4$ and $\left|\tilde{B}\right| = 7.95$

$$\tilde{A}(x) - \tilde{B}(x) = (.2, -.1, .14, -.05, .2, -.53, -.85, \ldots\ldots)$$

$$\tilde{B}(x) - \tilde{A}(x) = (-.2, .1, -.14, .05, -.2, .53, .85, \ldots\ldots)$$

Therefore,

$$S(\tilde{A}, \tilde{B}) = \frac{1}{\left|\tilde{A}\right|}\left\{\left|\tilde{A}\right| - \sum_{x \in X} \max\left\{\left(0, \left(\tilde{A}(x) - \tilde{B}(x)\right)\right)\right\}\right\}$$

$$= \frac{1}{3.4}[3.4 - \{.2 + 0 + .14 + 0 + .2 + 0 + 0 + \cdots + 0\}]$$

$$= 0.841$$

Similarly, $S(\tilde{B}, \tilde{A}) = 0.356$

Answer to Q6: a. we have

$$\mu_{\tilde{A}^c}(x) = 1 - \mu_{\tilde{A}}(x) = 1 - \frac{x}{x+1} = \frac{1}{x+1}$$

$$\mu_{\tilde{B}^c}(x) = 1 - \mu_{\tilde{B}}(x) = 1 - 2^{-x} = \frac{2^x - 1}{2^x}$$

b. $\mu_{\tilde{A}\cup\tilde{B}}(x) = \max\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\} = \max\left(\frac{x}{x+1}, \bar{2}^x\right)$

c. $\mu_{\tilde{A}\cap\tilde{B}}(x) = \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\} = \min\left(\frac{x}{x+1}, \bar{2}^x\right)$

d. $\mu_{(\tilde{A}\cup\tilde{B})^c}(x) = \mu_{\tilde{A}^c\cap\tilde{B}^c}(x) = \min\{\mu_{\tilde{A}^c}(x), \mu_{\tilde{B}^c}(x)\} = \min\left(\frac{1}{x+1}, \frac{2^x-1}{2^x}\right)$

1.9 FURTHER READING

- Mohan, C. An Introduction to Fuzzy Set Theory and Fuzzy Logic. Anshan Publishers, 2015.
- Lee, K. H. First Course on Fuzzy Theory and Applications. Springer International Edition, 2005.
- Yen, J., Langari, R. Fuzzy Logic - Intelligence, Control and Information Pearson Education, 1999.
- Zimmerman, H.J. Fuzzy Set Theory and its Applications. Allied Publishers Ltd., New Delhi, 1991.

1.10 MODEL QUESTIONS

Q1. Determine all α-level sets and all strong α-cuts for the following fuzzy sets:

a. $\tilde{A} = \{(3,.1),(4,.2),(5,.3),(6,.4),(7,.6),(8,.8),(10,1),(12,.8),(14,.6)\}$

b. $\tilde{C} = \{(x, \mu_{\tilde{C}}(x)) : x \in \boldsymbol{R}\}$, where $\mu_{\tilde{C}}(x) = 0$ for $x \leq 10$,
$\mu_{\tilde{C}}(x) = (1 + (x-10)^{-2})^{-1}$; for $x > 10$

Q2. Let $\tilde{A}$ and $\tilde{B}$ be two fuzzy sets defined on the universal set X. Show that

$$|\tilde{A}| + |\tilde{B}| = |\tilde{A} \cup \tilde{B}| + |\tilde{A} \cap \tilde{B}|$$

Q3. Compute the scalar cardinalities and relative cardinalities of the following sets

i. $\tilde{A} = \frac{.4}{v} + \frac{.2}{w} + \frac{.5}{x} + \frac{.4}{y} + \frac{1}{z}$

ii. $\tilde{C} = \frac{x}{x+1}, \quad \forall x \in \{0,1,2,\ldots,10\} = X$

iii. $\tilde{D}(x) = \left(1 - \frac{x}{10}\right), \quad \forall x \in \{0,1,2,\ldots,10\} = X$

Q4. Why we need fuzzy set theory?

Q5. Given

$$S(\tilde{A}, \tilde{B}) = \frac{1}{|\tilde{A}|}\left\{|\tilde{A}| - \sum_{x\in X} \max\left\{\left(0, \left(\tilde{A}(x) - \tilde{B}(x)\right)\right)\right\}\right\}$$

Then show that

$$S(\tilde{A}, \tilde{B}) = \frac{|\tilde{A} \cap \tilde{B}|}{|\tilde{A}|}$$

Q6. Let $\tilde{A} = \frac{1}{1.0} + \frac{2}{0.8} + \frac{3}{0.5} + \frac{4}{0.1}$ defined on $X = \{1,2,3,4\}$. Find all its α-cut sets. Show that A can be expressed in terms of the family composed of all of its α-cuts.

Q7. Let

$$\tilde{A} = \{(x_1, 0.3), (x_2, 0.6), (x_3, 0.8), (x_4, 0.5), (x_5, 0.7)\},$$
$$\tilde{B} = \{(x_1, 0.5), (x_2, 0.4), (x_3, 0.7), (x_4, 0.6), (x_5, 0.2)\}$$

and let $\alpha = 0.6$.

(i) Find the product $\tilde{A} \cdot \tilde{B}$.
(ii) Find the product $\alpha\tilde{A}$.
(iii) Find the power $\tilde{A}^2$.
(iv) Find the difference $\tilde{A} - \tilde{B}$.
(v) Find the disjunctive sum $\tilde{A} \oplus \tilde{B}$.

Q8. Define a fuzzy set and explain its basic concepts in detail. Discuss the membership function, types of fuzzy sets, and the main characteristics of fuzzy sets with suitable examples.

Q9. Let the universe of discourse be

$$X = \{x_1, x_2, x_3, x_4, x_5\}$$

Define the fuzzy sets:

$$\tilde{A} = \{(x_1, 0.2), (x_2, 0.6), (x_3, 0.8), (x_4, 0.5), (x_5, 0.7)\}$$
$$\tilde{B} = \{(x_1, 0.5), (x_2, 0.4), (x_3, 0.3), (x_4, 0.6), (x_5, 0.2)\}$$

Then Verify De-Morgan's Law.

*** *** ***

UNIT 2: OPERATIONS ON FUZZY SETS

UNIT STRUCTURE

2.1 Learning Objectives

2.2 Introduction

2.3 Some Important Theorems

2.4 Extension Principle

2.5 Fuzzy Complements

2.6 Other Fuzzy Operations

2.7 t-Norms & t-Conorms

2.7 Let Us Sum Up

2.9 Answers to Check Your Progress

2.10 Further Reading

2.11 Model Questions

2.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- understand and explain some important theorems related to fuzzy sets and use them in solving basic fuzzy set problems;
- describe the extension principle and apply it to extend crisp mathematical functions and relations to fuzzy sets;
- define fuzzy complements and analyze different types of fuzzy complement operators used in fuzzy logic;
- perform various fuzzy operations, including intersection, union, and difference, using appropriate fuzzy operators;
- understand the concepts of *t-norms* and *t-conorms* and explain their significance in defining fuzzy intersection and union in practical applications.

2.2 INTRODUCTION

Fuzzy set theory is a useful mathematical tool for handling information that is unclear, uncertain, or vague. Unlike classical set theory, which deals only with exact and fixed values, fuzzy set theory allows partial membership and degrees of truth. After learning the basic ideas and properties of fuzzy sets, it is important to study the main theorems and operations that help us use fuzzy sets effectively in practical situations.

This chapter introduces some important theorems of fuzzy sets that make fuzzy reasoning more systematic and logical. One of the key topics discussed is the **Extension Principle**, which helps in extending ordinary mathematical functions to fuzzy sets. Using this principle, fuzzy inputs can be converted into fuzzy outputs. This concept is widely used in areas such as engineering, decision-making, and control systems.

The chapter also explains **fuzzy complements** and other fuzzy operations, which are general forms of classical operations like union, intersection, and complement. In addition, **t-norms and t-conorms** are studied as special tools for defining intersection and union in fuzzy logic. These concepts are very important for understanding fuzzy systems and their real-life applications. Overall, this chapter provides a clear and simple foundation for further learning and practical use of fuzzy set theory.

2.3 SOME IMPORTANT THEOREMS

Theorem 1: Let $\tilde{A}, \tilde{B} \in F(X)$, where $F(X)$ is the set of all fuzzy sets on X, then, for all $\alpha \in [0,1]$

(i) $\tilde{A} \subseteq \tilde{B} \Leftrightarrow {}^{\alpha}\tilde{A} \subseteq {}^{\alpha}\tilde{B}$ and ${}^{\alpha+}\tilde{A} \subseteq {}^{\alpha+}\tilde{B}$

(ii) $\tilde{A} = \tilde{B} \Leftrightarrow {}^{\alpha}\tilde{A} = {}^{\alpha}\tilde{B}$ and ${}^{\alpha+}\tilde{A} = {}^{\alpha+}\tilde{B}$

Proof: (i) Let us first suppose $\tilde{A} \subseteq \tilde{B}$ and let $x \in {}^{\alpha}\tilde{A}$ and $x \in {}^{\alpha}\tilde{B}$.

Now, let

$$\alpha \in [0,1]$$

Then,

$$\begin{aligned} &\tilde{A}(x) \geq \alpha, \tilde{B}(x) \leq \alpha \\ &\Rightarrow x \in {}^{\alpha}\tilde{A}, x \in {}^{\alpha}\tilde{B} \\ &\Rightarrow {}^{\alpha}\tilde{A} \neq {}^{\alpha}\tilde{B} \end{aligned} \qquad (1)$$

or if

$$\begin{aligned} &\text{for } \alpha \in [0,1], y \in {}^{\alpha}\tilde{B} \Rightarrow y \in {}^{\alpha}\tilde{A} \text{ as } \tilde{A} \subseteq \tilde{B} \\ &\Rightarrow \tilde{B}(y) \geq \alpha \text{ and } \tilde{A}(y) \geq \alpha \\ &\Rightarrow y \in {}^{\alpha}\tilde{B} \text{ and } y \in {}^{\alpha}\tilde{A} \\ &\Rightarrow {}^{\alpha}\tilde{A} = {}^{\alpha}\tilde{B} \end{aligned} \qquad (2)$$

From (1) and (2),

$${}^{\alpha}\tilde{A} \neq {}^{\alpha}\tilde{B} \text{ and } {}^{\alpha}\tilde{A} = {}^{\alpha}\tilde{B}$$

We conclude that, if $\tilde{A} \subseteq \tilde{B}$, then ${}^{\alpha}\tilde{A} \subseteq {}^{\alpha}\tilde{B}$

Conversely,

Since $x_0 \in {}^{\alpha}\tilde{A} \subseteq {}^{\alpha}\tilde{B}$

for any $x_0 \in {}^{\alpha}\tilde{B}$, $x_0 \in {}^{\alpha}\tilde{A}$

$$\begin{aligned} &\Rightarrow \tilde{B}(x_0) \geq \alpha, \tilde{A}(x_0) < \alpha \\ &\Rightarrow \tilde{B}(x_0) \geq \tilde{A}(x_0) \end{aligned}$$

where x_0 is arbitrary

$$\Rightarrow \tilde{B}(x) \geq \tilde{A}(x)$$
$$\Rightarrow \tilde{A} \subseteq \tilde{B}$$

(ii) We now proceed to prove the second part using an example.
Since

$$\tilde{A} = \tilde{B} \iff \mu_{\tilde{A}}(x) = \mu_{\tilde{B}}(x), \quad \forall x \in X$$

Now, if $\alpha \in [0,1]$

$$\Leftrightarrow \mu_{\tilde{A}}(x) \geq \alpha \text{ and } \mu_{\tilde{B}}(x) \geq \alpha$$
$$\Leftrightarrow x \in {}^{\alpha}\tilde{A} \text{ and } x \in {}^{\alpha}\tilde{B}$$
$$\Leftrightarrow {}^{\alpha}\tilde{A} = {}^{\alpha}\tilde{B}$$

For example, let

$$X = \{1,2,3,4\}$$

and

$$\tilde{A} = \{(1,2), (2,4), (3,6), (4,1)\}$$
$$\tilde{B} = \{(1,2), (2,4), (3,6), (4,1)\}$$

Since,

$$\tilde{A} = \tilde{B}$$

Now, if $\alpha = .4$, so

$${}^{.4}\tilde{A} = \{2,3,4\}, \; {}^{.4}\tilde{B} = \{2,3,4\}$$

Similarly,

$${}^{.4+}\tilde{A} = \{3,4\}, \; {}^{.4+}\tilde{B} = \{3,4\}$$

i.e.,

$${}^{\alpha+}\tilde{A} = {}^{\alpha+}\tilde{B}$$

So,

$$\tilde{A} = \tilde{B} \iff {}^{\alpha}\tilde{A} = {}^{\alpha}\tilde{B} \text{ and } {}^{\alpha+}\tilde{A} = {}^{\alpha+}\tilde{B}$$

Theorem 2: Let $\tilde{A}, \tilde{B} \in F(X)$, then the following properties holds for all $\alpha, \beta \in [0,1]$

(i) ${}^{\alpha+}\tilde{A} \subseteq {}^{\alpha}\tilde{A}$

(ii) $\alpha \leq \beta \Rightarrow {}^{\alpha}\tilde{A} \geq {}^{\beta}\tilde{A} \quad and \quad {}^{\alpha+}\tilde{A} \geq {}^{\beta+}\tilde{A}$

(iii) ${}^{\alpha}(\tilde{A} \cap \tilde{B}) = {}^{\alpha}\tilde{A} \cap {}^{\alpha}\tilde{B}$ and ${}^{\alpha}(\tilde{A} \cup \tilde{B}) = {}^{\alpha}\tilde{A} \cup {}^{\alpha}\tilde{B}$

(iv) ${}^{\alpha+}(\tilde{A} \cap \tilde{B}) = {}^{\alpha+}\tilde{A} \cap {}^{\alpha+}\tilde{B}$ and ${}^{\alpha+}(\tilde{A} \cup \tilde{B}) = {}^{\alpha+}\tilde{A} \cup {}^{\alpha+}\tilde{B}$

(v) ${}^{\alpha}\bar{\tilde{A}} = \overline{{}^{(1-\alpha)+}\tilde{A}}$

Theorem 3: Let $\tilde{A}_i \in F(X), \forall i \in I$, where I is an index set. Then

$$\bigcup_{i \in I} {}^{\alpha}\tilde{A} \subseteq {}^{\alpha}\left(\bigcup_{i \in I} \tilde{A}_i\right)$$

Proof: To show

$$\bigcup_{i\in I} {}^{\alpha}\tilde{A}_i \subseteq {}^{\alpha}\left(\bigcup_{i\in I}\tilde{A}_i\right)$$

Let $x \in \bigcup_{i\in I} {}^{\alpha}\tilde{A}_i$, $\forall x \in X$and $i \in I$
iff

$$\exists i_0 \in I, \text{ such that } x \in {}^{\alpha}\tilde{A}_{i_0}$$

i.e.,

$$\tilde{A}_{i_0}(x) \geq \alpha(\text{i.e., } \tilde{A}_1(x), \tilde{A}_2(x), \dots)$$

and this inequality is satisfied if and only if

$$\sup(\tilde{A}_i(x)) \geq \alpha$$
$$\Rightarrow \left(\bigcup_{i\in I}\tilde{A}_i\right)(x) \geq \alpha \Rightarrow x \in {}^{\alpha}\left(\bigcup_{i\in I}\tilde{A}_i\right)$$

So, for $x \in \bigcup_{i\in I} {}^{\alpha}\tilde{A}_i$, we have

$$x \in {}^{\alpha}\left(\bigcup_{i\in I}\tilde{A}_i\right)$$
$$\Rightarrow \bigcup_{i\in I} {}^{\alpha}\tilde{A}_i \subseteq {}^{\alpha}\left(\bigcup_{i\in I}\tilde{A}_i\right)$$

Now, we shall prove that

$$\bigcup_{i\in I} {}^{\alpha}\tilde{A}_i \neq {}^{\alpha}\left(\bigcup_{i\in I}\tilde{A}_i\right) \qquad with\ the\ help\ of\ an\ example.$$

Let

$$\tilde{A}_i = 1 - \frac{1}{i}, \forall i \in I$$
$$\tilde{A}_i = 1 - \frac{1}{i} < 1$$

Let $\alpha = 1$

$${}^{\alpha}\tilde{A}_i = {}^{1}\left(1-\frac{1}{i}\right) = \phi$$
$$\Rightarrow \bigcup_{i\in I} {}^{\alpha}\tilde{A}_i = \bigcup_{i\in I}\phi = \phi \dots (1)$$

Again

$$\bigcup_{i\in I}\tilde{A}_i = \bigcup_{i\in I}\left(1-\frac{1}{i}\right) = X$$
$$\Rightarrow {}^{1}\left(\bigcup_{i\in I}\tilde{A}_i\right) = X \dots (2)$$

From (1) and (2), $\phi \neq X$

$$\Rightarrow \bigcup_{i\in I} {}^{\alpha}\tilde{A}_i \neq {}^{\alpha}\left(\bigcup_{i\in I} \tilde{A}_i\right)$$

whereas

$$\bigcup_{i\in I} {}^{\alpha}\tilde{A}_i \subseteq {}^{\alpha}\left(\bigcup_{i\in I} \tilde{A}_i\right)$$

Theorem 4: Let $\tilde{A} \in F(X)$, then the following properties hold

$${}^{\alpha}\tilde{A} = \bigcap_{\beta<\alpha} {}^{\beta}\tilde{A} = \bigcap_{\beta<\alpha} {}^{\beta+}\tilde{A}$$

Proof: We have to show that

$${}^{\alpha}\tilde{A} = \bigcap_{\beta<\alpha} {}^{\beta}\tilde{A} = \bigcap_{\beta<\alpha} \beta^{+}\tilde{A}$$

Since we know if $\beta < \alpha \Rightarrow {}^{\alpha}\tilde{A} \subseteq {}^{\beta}\tilde{A}$

$$\Rightarrow {}^{\alpha}\tilde{A} \subseteq \bigcap_{\beta<\alpha} {}^{\beta}\tilde{A} \qquad (1)$$

Again, for any $\varepsilon > 0$, there exists $\tilde{A}(x) \geq \alpha \Rightarrow x \in {}^{\alpha}\tilde{A}$ such that $\alpha - \varepsilon < \alpha$, then

$$x \in {}^{\beta}\tilde{A} \Rightarrow x \in {}^{\alpha-\varepsilon}\tilde{A}$$
$$\Rightarrow \tilde{A}(x) \geq \alpha - \varepsilon$$

Since ε is arbitrarily small, we can take $\varepsilon \to 0$, then

$$\tilde{A}(x) \geq \alpha \Rightarrow x \in {}^{\alpha}\tilde{A}$$

i.e.,

$$x \in \bigcap_{\beta<\alpha} {}^{\beta}\tilde{A} \Rightarrow x \in {}^{\alpha}\tilde{A}$$
$$\Rightarrow \bigcap_{\beta<\alpha} {}^{\beta}\tilde{A} \subseteq {}^{\alpha}\tilde{A} \qquad (2)$$

From (1) and (2), we have

$${}^{\alpha}\tilde{A} = \bigcap_{\beta<\alpha} {}^{\beta}\tilde{A}$$

and clearly,

$${}^{\alpha}\tilde{A} = \bigcap_{\beta<\alpha} {}^{\beta+}\tilde{A}$$

Thus,

$$^{\alpha}\tilde{A} = \bigcap_{\beta<\alpha} {}^{\beta}\tilde{A} = \bigcap_{\beta<\alpha} {}^{\beta+}\tilde{A}$$

Theorem 5 (First Decomposition Theorem): For every $\tilde{A} \in F(X)$,

$$\tilde{A} = \bigcup_{\alpha\in[0,1]} {}_{\alpha}\tilde{A}$$

Proof: For each particular $x \in X$, let $\alpha = \bar{A}(x)$
Then, we know that

$$\left(\bigcup_{\alpha\in[0,1]} {}_{\alpha}\bar{A}\right)(x) = \sup_{\alpha\in[0,1]} {}_{\alpha}\bar{A}(x)$$

$$= \max\left\{\sup_{\alpha\in[0,\alpha]} {}_{\alpha}\bar{A}(x), \sup_{\alpha\in(\alpha,1]} {}_{\alpha}\bar{A}(x)\right\} \qquad (1)$$

Now, since for each $\alpha \in (\alpha, 1]$, we have $\bar{A}(x) = \alpha < \alpha$

$$\Rightarrow \bar{A}(x) < \alpha \Rightarrow x \notin {}^{\alpha}\bar{A}$$

$$\Rightarrow {}_{\alpha}\bar{A}(x) = 0 \qquad [\because \alpha \cdot {}^{\alpha}\bar{A}(x) > \alpha]$$

Again, for each $\alpha \in [0, \alpha]$, we have

$$\bar{A}(x) = \alpha \geq \alpha$$

$$\Rightarrow \bar{A}(x) \geq \alpha$$

$$\Rightarrow x \in {}_{\alpha}\bar{A} \Rightarrow {}_{\alpha}\bar{A}(x) = \alpha$$

$$\Rightarrow \left(\bigcup_{\alpha\in[0,1]} {}_{\alpha}\bar{A}\right)(x) = \max\left\{\sup_{\alpha\in[0,\alpha]} {}_{\alpha}\bar{A}(x), \sup_{\alpha\in(\alpha,1]} {}_{\alpha}\bar{A}(x)\right\}$$

$$= \max\{\alpha, 0\} = \max\{\alpha\} = \alpha = \bar{A}(x)$$

$$\Rightarrow \left(\bigcup_{\alpha\in[0,1]} {}_{\alpha}\bar{A}\right)(x) = \bar{A}(x)$$

$$\Longrightarrow \bigcup_{\alpha\in[0,1]} {}_{\alpha}\bar{A} = \bar{A}$$

Theorem 6 (Second Decomposition Theorem): Let $\tilde{A}$ be a fuzzy set defined on X, then

$$\bigcup_{\alpha\in[0,1]} {}_{\alpha+}\tilde{A} = \tilde{A}$$

where ${}_{\alpha+}\tilde{A}$ is a special fuzzy set and ${}_{\alpha+}\tilde{A}(x) = \alpha.\,{}^{\alpha+}\tilde{A}(x)$

Proof: Let $\tilde{A}$ be a fuzzy set defined on universe of discourse X. For each particular $x \in X$, let

$$a = \tilde{A}(x) \qquad (1)$$

By definition of union and decomposition of a set, we have

$$\left[\bigcup_{\alpha\in(0,1]} {}_{\alpha+}\tilde{A}\right](x) = \sup_{\alpha\in(0,1]} {}_{\alpha+}\tilde{A}(x)$$

$$= \max\left[\sup_{\alpha\in(0,a]} {}_{\alpha+}\tilde{A}(x),\ \sup_{\alpha\in(a,1]} {}_{\alpha+}\tilde{A}(x)\right] \qquad (2)$$

Now, for each $\alpha \in [a, 1]$,

$$\tilde{A}(x) = a < \alpha \qquad (a < \alpha \leq 1)$$
$$\Rightarrow \tilde{A}(x) \leq \alpha$$
$$\Rightarrow x \notin {}^{\alpha+}\tilde{A}$$

We know that

$${}_{\alpha+}\tilde{A}(x) = \alpha \cdot {}^{\alpha+}\tilde{A}$$
$$\Rightarrow {}_{\alpha+}\tilde{A} = 0 \qquad \text{for } \alpha \in (a, 1] \qquad (3)$$

Now, for each $\alpha \in [0, a)$, i.e., $(0 \leq \alpha < a)$,

$$\tilde{A}(x) = a > \alpha$$
$$\Rightarrow \tilde{A}(x) > \alpha$$
$$\Rightarrow x \in {}^{\alpha+}\tilde{A}(x)$$

Thus,

$${}_{\alpha+}\tilde{A}(x) = \alpha \cdot {}^{\alpha+}\tilde{A}$$
$${}_{\alpha+}\tilde{A}(x) = \alpha \qquad \text{for } \alpha \in [0, a) \qquad (4)$$

Putting the values from (3) and (4) in (2), we get

$$\left[\bigcup_{\alpha\in(0,1]} {}_{\alpha+}\tilde{A}\right](x) = \max\left[\sup_{\alpha\in(0,a]} \alpha\ ,\ \sup_{\alpha\in(0,a]} 0\right]$$
$$= \max[a, 0]$$
$$= a$$
$$= \tilde{A}(x)$$

Thus,

$$\left[\bigcup_{\alpha\in(0,1]} {}_{\alpha+}\tilde{A}\right](x) = \tilde{A}(x)$$

2.4 EXTENSION PRINCIPLE

We have that any given function $f: X \to Y$, induces two functions

$$f: F(X) \to F(Y)$$

and

$$f^{-1}: F(Y) \to F(X)$$

which is defined by

$$[f(\tilde{A})](y) = \sup_{x, y=f(x)} \tilde{A}(x), \quad \forall \tilde{A} \in F(X)$$

and for all $\tilde{B} \in F(Y)$,

$$[f^{-1}(\tilde{B})](x) = \tilde{B}(f(x))$$

Theorem 1: Let $f: X \to Y$ be an arbitrary crisp function. Then, for any $\tilde{A} \in F(X)$ and $\forall\, \alpha \in (0,1]$, the following properties of f fuzzified by the extension principle hold:

i. ${}^{\alpha+}[f(\tilde{A})] = f({}^{\alpha+}\tilde{A})$

ii. $f({}^{\alpha}\tilde{A}) \subseteq {}^{\alpha}[f(\tilde{A})]$

Proof:

(i) Let $y \in Y$ such that

$$\begin{aligned} & y \in {}^{\alpha+}[f(\tilde{A})] \\ & \Leftrightarrow [f(\tilde{A})](y) > \alpha \\ & \Leftrightarrow \sup \tilde{A}(x) > \alpha \\ & \Leftrightarrow (\exists x_0 \in X)\ (y = f(x_0) \text{ and } \tilde{A}(x_0) > \alpha) \\ & \Leftrightarrow (\exists x_0 \in X)(y = f(x_0)\ and\ x_0 \in {}^{\alpha+}\tilde{A}) \Rightarrow f(x_0) \in f({}^{\alpha+}\tilde{A}) \\ & \Leftrightarrow y \in f({}^{\alpha+}\tilde{A}) \end{aligned}$$

Since y is arbitrary, therefore

$$\begin{aligned} & y \in {}^{\alpha+}[f(\tilde{A})] \Leftrightarrow y \in f({}^{\alpha+}\tilde{A}), \forall y \in Y \\ & \Rightarrow {}^{\alpha+}[f(\tilde{A})] = f({}^{\alpha+}\tilde{A}) \end{aligned}$$

(ii) Let $y \in f({}^{\alpha}\tilde{A})$. Then, $\exists\ x_0 \in {}^{\alpha}\tilde{A}$ such that $y = f(x_0)$.

Therefore,

$$\begin{aligned} & [f(\tilde{A})](y) = \sup \tilde{A}(x) \geq \tilde{A}(x_0) \geq \alpha \\ & \Rightarrow y \in {}^{\alpha}[f(\tilde{A})] \\ & \Rightarrow f({}^{\alpha}\tilde{A}) \subseteq {}^{\alpha}[f(\tilde{A})] \end{aligned}$$

Now, to show that ${}^{\alpha}[f(\tilde{A})] \neq f({}^{\alpha}\tilde{A})$, we consider the following example:

Let $X = \mathbb{N}, Y = \{a, b\}$

and

$$f(n) = \begin{cases} a, & n \leq 10 \\ b, & n > 10 \end{cases}$$

and

$$\mathrm{A}(n) = 1 - \frac{1}{n}, \forall n \in \mathbb{N}$$

Then,

$$[f(\tilde{A})](a) = \sup_{\frac{n}{a}=f(n)} \tilde{A}(n) = \frac{9}{10}$$
$$[f(\tilde{A})](b) = \sup_{\frac{n}{b}=f(n)} \tilde{A}(n) = 1$$

If $\alpha = 1$, then we observe that ${}^{1}[f(\tilde{A})] = \{b\}$, while $f({}^{1}\tilde{A}) = \phi$ $(\because {}^{1}\tilde{A} = \phi)$

Hence,

$$f({}^{\alpha}\tilde{A}) \neq {}^{\alpha}[f(\tilde{A})]$$

Theorem 2: Let $f: X \to Y$ be an arbitrary crisp function. Then, for any $\tilde{A} \in F(X)$, f fuzzified by the extension principle satisfies the equation:

$$f(\tilde{A}) = \bigcup_{\alpha \in [0,1]} f({}_{\alpha+}\tilde{A})$$

Proof: Using second decomposition theorem for $f(\tilde{A})$, we get

$$f(\tilde{A}) = \bigcup_{\alpha \in [0,1]} {}_{\alpha+}[f(\tilde{A})]$$

Now, by definition

$${}_{\alpha+}[f(\tilde{A})] = \alpha \cdot {}^{\alpha+}[f(\tilde{A})]$$

Using previous theorem, we get

$$f(\tilde{A}) = \bigcup_{\alpha \in (0,1]} \alpha . f({}^{\alpha+}\tilde{A})$$

$$f(\tilde{A}) = \bigcup_{\alpha \in (0,1]} f({}_{\alpha+}\tilde{A})$$

CHECK YOUR PROGRESS

Q1) Let $\tilde{A}$ and $\tilde{B}$ be fuzzy sets on $X = \mathbb{Z}$ given by

$$\tilde{A} = \{(1,1), (2,0.5), (3,0.3)\}, \tilde{B} = \{(1,0.8), (2,0.4), (3,0.2)\}$$

Let $f(x_1, x_2) = x_1 + x_2$. Find $f(\tilde{A}, \tilde{B})$.

Q2) Let

$$\tilde{A} = \{(1,0.9), (2,0.7)\}, \tilde{B} = \{(1,0.8), (2,0.6)\}$$

Let $f(x_1, x_2) = \max(x_1, x_2)$. Find $f(\tilde{A}, \tilde{B})$.

Q3) Show that

$${}^{\alpha+}[f(\tilde{A})] = f({}^{\alpha+}\tilde{A})$$

2.5 FUZZY COMPLEMENTS

Let $\tilde{A}$ be a fuzzy set on X. We know that $\tilde{A}(x)$ is interpreted as the degree to which $x$belongs to $\tilde{A}$. The complement $c\tilde{A}$ is defined by the function

$$c: [0,1] \to [0,1]$$

which assigns a value $c(\tilde{A}(x))$ to each membership grade $\tilde{A}(x)$ of any given fuzzy set $\tilde{A}$.

We can write

$$(c\tilde{A})(x) = c\left(\tilde{A}(x)\right), \quad \forall x \in X$$

Meaningful Fuzzy Complement:

A meaningful fuzzy complement requires the function c to satisfy certain basic conditions**:**

(i) $c(0) = 1$ and $c(1) = 0$ (Boundary condition)
(ii) For all $a, b \in [0,1]$, if $a \leq b$, then $c(a) \geq c(b)$ (Monotonicity)
(iii) c is a continuous function
(iv) $c(c(a)) = a$, $\forall a \in [0,1]$ (Involutive)

Theorem 1: Let a function $c: [0,1] \to [0,1]$ satisfy the condition of monotonicity and involutive. Then c is a continuous function and satisfies boundary conditions $c(0) = 1$ and $c(1) = 0$. Moreover, c is a bijective function.

Proof: Let $c: [0,1] \to [0,1]$ be the given function. Since the range of c is $[0,1]$, therefore, $c(0) \leq 1$ and $c(1) \geq 0$.

By monotonic properties, we have

$$c(c(0)) \geq c(1)$$

Again, by involutive property

$$0 = c\big(c(0)\big) \geq c(1)$$

Therefore, $c(1) = 0$. Further, by involutive property, we have

$$c(0) = c(c(1)) = 1$$

Hence, c satisfies the given boundary conditions.

Now to show c is a bijective function, we observe that for all $a \in [0,1]$, there exists $b = c(a) \in [0,1]$ such that

$$c(b) = c(c(a)) = a$$

$\Rightarrow c$ is onto function

Now to show c is one-one, let us assume that

$$c(a_1) = c(a_2)$$

Then

$$a_1 = c(c(a_1)) = c(c(a_2)) = a_2$$

$\Rightarrow c$ is one-one

Hence, c is a bijective function.

Threshold Type Complement: The general fuzzy complements that satisfy only the axiomatic skeleton defined by

$$c(a) = \begin{cases} 1, & for\ a \le t,\ \ a \in [0,1] \\ 0, & for\ a > t,\ \ a \in [0,1) \end{cases}$$

is known as threshold type complement. Here, t is called the threshold of c.

Sugeno Class Of Complement: It is defined by

$$c_\lambda(a) = \frac{1-a}{1+\lambda a}, \text{where } \lambda \in (1,\infty)$$

Yagor Class Of Complement: It is defined by

$$c_\omega(a) = (1-a^\omega)^{\frac{1}{w}},\ \text{where } \omega \in (0,\infty)$$

Equilibrium of a Fuzzy Complement: The equilibrium of a fuzzy complement c is the membership value in a fuzzy set $\tilde{A}$ that is equal to its membership value in the complement $c\tilde{A}$; that is, it is the value a for which $c(a) = a$.

Dual Point: A point is said to be dual point of a if $c\left({}^d a\right) - {}^d a = a - c(a)$.

Theorem 2: Every fuzzy complement has at most one equilibrium.
Proof: Let c be any arbitrary fuzzy complement. Then by definition of equilibrium, $c(a) - a = 0$, where $a \in [0,1]$.
We can show that any equation $c(a) - a = b$, where b is a real constant, must have at most one solution.
Assume a_1 and a_2 are two different solutions such that $a_1 < a_2$. Then

$$c(a_1) - a_1 = b \text{ and } c(a_2) - a_2 = b$$
$$\Rightarrow c(a_1) - c(a_2) = a_1 - a_2$$

Since c is monotonic decreasing, $c(a_1) \ge c(a_2)$ and since $a_1 < a_2$,
$\therefore c(a_1) - a_1 > c(a_2) - a_2$ which is a contradiction.
Thus, the equation has at most one solution and hence at most one equilibrium.

Theorem 3: If c is a continuous fuzzy complement having unique equilibrium e_c, then

$$a \le c(a) \text{ if } a \le e_c$$
$$a \ge c(a) \text{ if } a \ge e_c$$

Proof: Let us suppose that $a < e_c$, $a = e_c$, and $a > e_c$. Thus, c is monotonic decreasing. Therefore,

$$c(a) \ge c(e_c) \text{ for } a < e_c;\ \ c(a) = c(e_c) \text{ for } a = e_c \text{ and } c(a) \le c(e_c) \text{ for } a > e_c$$

Now since $c(e_c) = e_c$, we can write:
$c(a) \ge e_c$, $c(a) = e_c$, and $c(a) \le e_c$, respectively.
Now, since $a = e_c$, therefore, above expression can also be written as $c(a) \ge a$, $c(a) = a$ and $c(a) \le a$ respectively.

Therefore, $a \leq e_c$ implies $c(a) \geq a$ and $a \geq e_c$ implies $c(a) \leq a$. In a similar manner we can show the inverse implication.

Theorem 4: If c is a continuous fuzzy complement, then c has a unique equilibrium.
Proof: We know that e_c of a fuzzy complement c is the solution of the equation $c(a) - a = 0$, which is a special case of $c(a) - a = b$, where b is any constant such that $b \in [-1,1]$.
Now, using boundary conditions, we have

$$c(0) - 0 = 1 \text{ and } c(1) - 1 = -1$$

Now, since c is a continuous complement, then by intermediate value theorem for continuous function, we can say that for each $b \in [-1,1]$, there exists at least one a such that $c(a) - a = b$. This gives the necessary existence of an equilibrium value for a continuous function and hence c has a unique equilibrium.

Theorem 5: If a complement c has an equilibrium e_c, then ${}^d e_c = e_c$.
Proof: If $a = \mathrm{e}$, then by definition of equilibrium, we must have

$$a - c(a) = 0$$

Apply this criterion for ${}^d a$, we get

$$c({}^d a) - {}^d a = 0$$

Therefore,

$$c({}^d a) - {}^d a = a - c(a)$$

This satisfies the definition of dual point when $a = {}^d a = e_c$. Hence, the equilibrium of any complement is its own dual point.

Theorem 6: For each $a \in [0,1]$, ${}^d a = c(a)$ if $c(c(a)) = a$, i.e., when the complement is involutive.
Proof: Let us suppose

$${}^d a = c(a)$$

Then, by putting this value in the equation of dual point, we get

$$c(c(a)) - c(a) = a - c(a)$$
$$\Rightarrow c(c(a)) = a$$

Conversely, let $c(c(a)) = a$. Again, putting this value in the equation of dual point, we get

$$c({}^d a) - {}^d a = c(c(a)) - c(a)$$

For ${}^d a$, whose solution is ${}^d a = c(a)$.

Decreasing generator: The decreasing generator is a continuous and strictly decreasing function f from $[0,1]$ to $\mathbb{R}$ such that $f(1) = 0$.

Pseudo-inverse of a decreasing generator: The pseudo inverse of a decreasing generator f, denoted by f^{-1}, is a function from $\mathbb{R}$ to $[0,1]$, given by

$$f^{-1}(a) = \begin{cases} 1, & \text{for } a \in (-\infty, 0) \\ f^{-1}(a), & \text{for } a \in [0, f(0)] \\ 0, & \text{for } a \in (f(0), \infty) \end{cases}$$

where, f^{-1} is the ordinary inverse of f.

2.6 OTHER FUZZY OPERATIONS

(1) Cartesian Product

Let $\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n$ be fuzzy sets in $X_1, X_2, \ldots, X_n$. The Cartesian product of these fuzzy sets is a fuzzy set in the product space $X_1 \times X_2 \times \cdots \times X_n$, with membership function

$$\mu_{\tilde{A}_1 \times \tilde{A}_2 \times \cdots \times \tilde{A}_n}(x_1, x_2, \ldots, x_n) = \min\{(x, \mu_{\tilde{A}}(x_i)) : x = x_1, x_2, \ldots., x_n \in X_i\}$$

Here $X_i = X_1 \times X_2 \times \cdots \times X_n$

Example: Let

$$X = \{1,2,3,4\}$$
$$\tilde{A} = \{(1,.2), (2,.4), (3,.5), (4,.9)\}$$
$$\tilde{B} = \{(1,.4), (2,.3), (4,1)\}$$

Then,

$$\tilde{A} \times \tilde{B} = \{[(1,1),.2], [(1,2),.2], [(1,4),.2], [(2,1),.4], [(2,2),.3], [(2,4),.4], [(3,1),.4], [(3,2),.3], [(3,4),.5], [(4,1),.4], [(4,2),.3], [(4,4),.9]\}$$

(2) Algebraic Product: The m^{th} power of a fuzzy set $\tilde{A}$ is a fuzzy set with membership function

$$\mu_{\tilde{A}^m}(x) = [\mu_{\tilde{A}}(x)]^m$$

Example: Let $X = \{1,2,3,4\}$

$$\tilde{A} = \{(1,.2), (2,.4), (3,.5), (4,.9)\}$$

Then,

$$\tilde{A}^2 = \{(1,.04), (2,.16), (3,.25), (4,.81)\}$$

(3) Algebraic Sum: The algebraic sum $\tilde{C}$ of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ is defined as

$$\tilde{C} = \tilde{A} + \tilde{B} = \{(x, \mu_{\tilde{A}+\tilde{B}}(x)) : x \in X\}$$

where,

$$\mu_{\tilde{A}+\tilde{B}}(x) = \mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x) - \mu_{\tilde{A}}(x).\mu_{\tilde{B}}(x)$$

Example:

Let $X = \{1,2,3,4\}$

$$\tilde{A} = \{(1,.2), (2,.4), (3,.5), (4,.9)\}$$
$$\tilde{B} = \{(1,.4), (2,.3), (3,.4), (4,1)\}$$

Then,

$$\tilde{C} = \tilde{A} + \tilde{B} = \{(1,.52), (2,.58), (3,.5), (4,1)\}$$

(4) Bounded Sum: The bounded sum $\tilde{C} = \tilde{A} \oplus \tilde{B}$ is defined as

$$\tilde{C} = \{(x, \mu_{\tilde{A} \oplus \tilde{B}}(x)) : x \in X\}$$

where,

$$\mu_{\tilde{A} \oplus \tilde{B}}(x) = \min\{1, \mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x)\}$$

Example: Let

$$\tilde{A} = \{(1,.2),(2,.4),(3,.5),(4,.9)\}$$
$$\tilde{B} = \{(1,.4),(2,.3),(4,1)\}$$

Then,

$$\tilde{C} = \tilde{A} \oplus \tilde{B} = \{(1,.6),(2,.7),(3,.5),(4,1)\}$$

(5) Bounded Difference: The bounded difference of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ is defined as

$$\tilde{C} = \tilde{A} \ominus \tilde{B} = \{(x, \mu_{\tilde{A}\ominus\tilde{B}}(x)) : x \in X\}$$

where,

$$\mu_{\tilde{A}\ominus\tilde{B}}(x) = \max\{0, \mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x) - 1\}$$

Example: Let

$$\tilde{A} = \{(1,.2),(2,.4),(3,.5),(4,.9)\}$$
$$\tilde{B} = \{(1,.4),(2,.3),(4,1)\}$$

Then

$$\tilde{C} = \tilde{A} \ominus \tilde{B} = \{(1,0),(2,0),(3,0),(4,.9)\}$$

(6) Algebraic Product of Two Different Fuzzy Sets: The algebraic product is defined as

$$\tilde{C} = \tilde{A} \cdot \tilde{B} = \{(x, \mu_{\tilde{A}}(x).\mu_{\tilde{B}}(x)) : x \in X\}$$

(7) Compositions of Relations:

Let $\tilde{R}$ be a fuzzy relation on $X \times Y$ and $\tilde{S}$ be a fuzzy relation on $Y \times Z$. Then $\tilde{R} \circ \tilde{S}$ is a fuzzy relation on $X \times Z$. The fuzzy max–min composition is defined as

$$\mu_{\tilde{R}\circ\tilde{S}}(x,z) = \max_{y\in Y}\{\min(\mu_{\tilde{R}}(x,y), \mu_{\tilde{S}}(y,z))\}$$

Example:

Let $X = \{x_1, x_2, x_3\}, Y = \{y_1, y_2\}, Z = \{z_1, z_2, z_3\}$

Relation R:

$$\begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 0.5 & 0.1 \\ 0.2 & 0.9 \\ 0.8 & 0.6 \end{bmatrix} \end{array}$$

Relation S:

$$\begin{array}{c} \\ y_1 \\ y_2 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.6 & 0.4 & 0.7 \\ 0.5 & 0.8 & 0.9 \end{bmatrix} \end{array}$$

Then,

$$\mu_{\tilde{R}\circ\tilde{S}}(x,z) = \max_{y\in Y}(\min(\mu_{\tilde{R}}(x,y), \mu_{\tilde{S}}(y,z)))$$
$$= \max_{y\in Y}(\mu_{\tilde{R}}(x_1,y_1), \mu_{\tilde{S}}(y_1,z_1), \min(\mu_{\tilde{R}}(x_1,y_2), \mu_{\tilde{S}}(y_2,z_1)))$$

$$\mu_{\tilde{R}\circ\tilde{S}}(x_1, z_3) = \max[\min(0.5,0.7), \min(0.1,0.9)] = \max(0.5,0.1) = 0.5$$
$$\mu_{\tilde{R}\circ\tilde{S}}(x_2, z_1) = \max[\min(0.2,0.6), \min(0.9,0.5)] = \max(0.2,0.5) = 0.5$$
$$\mu_{\tilde{R}\circ\tilde{S}}(x_2, z_2) = \max[\min(0.2,0.4), \min(0.9,0.8)] = \max(0.2,0.8) = 0.8$$
$$\mu_{\tilde{R}\circ\tilde{S}}(x_2, z_3) = \max[\min(0.2,0.7), \min(0.9,0.9)] = \max(0.2,0.9) = 0.9$$
$$\mu_{\tilde{R}\circ\tilde{S}}(x_3, z_1) = \max[\min(0.8,0.6), \min(0.6,0.5)] = \max(0.6,0.5) = 0.6$$
$$\mu_{\tilde{R}\circ\tilde{S}}(x_3, z_2) = \max[\min(0.8,0.4), \min(0.6,0.8)] = \max(0.4,0.6) = 0.6$$
$$\mu_{\tilde{R}\circ\tilde{S}}(x_3, z_3) = \max[\min(0.8,0.7), \min(0.6,0.9)] = \max(0.7,0.6) = 0.7$$

Then $\tilde{R} \circ \tilde{S}$, by max–min composition gives

$$\tilde{R} \circ \tilde{S} = \begin{matrix} \\ x_1 \\ x_2 \\ x_3 \end{matrix} \begin{matrix} \begin{matrix} z_1 & z_2 & z_3 \end{matrix} \\ \begin{bmatrix} 0.5 & 0.4 & 0.5 \\ 0.5 & 0.8 & 0.9 \\ 0.6 & 0.6 & 0.7 \end{bmatrix} \end{matrix}$$

CHECK YOUR PROGRESS

Q4) Show that a fuzzy complement c satisfying monotonicity and involutive property must satisfy boundary conditions.
Q5) Let $\tilde{A} = \{(1,0.2), (2,0.4)\}$, $\tilde{B} = \{(1,0.5), (2,0.3)\}$.
Find the algebraic sum $\tilde{C} = \tilde{A} + \tilde{B}$.

2.7 t-NORMS & t-CONORMS

Definition: t-norms are two valued functions from $[0,1] \times [0,1]$ to $[0,1]$, i.e., fuzzy intersection (t-norms) is a binary operation on the unit interval which satisfies the following conditions:
(i) $t(0,0) = 0$
$t(1, \mu_{\tilde{A}}(x)) = t(\mu_{\tilde{A}}(x),1) = \mu_{\tilde{A}}(x)$ [Bounded condition or closure property]
(ii) $t\big(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\big) \leq t\big(\mu_{\tilde{C}}(x), \mu_{\tilde{D}}(x)\big)$ iff $\mu_{\tilde{A}}(x) \leq \mu_{\tilde{C}}(x)$, $\mu_{\tilde{B}}(x) \leq \mu_{\tilde{D}}(x)$
$\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x) \leq \mu_{\tilde{C}}(x)$
(iv) $t(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = t(\mu_{\tilde{B}}(x), \mu_{\tilde{A}}(x))$ [Commutativity property]
(v) $t(t(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)), \mu_{\tilde{C}}(x)) = t(t(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)), \mu_{\tilde{C}}(x)))$ [Associativity property]

The function t defines a general class of intersection operators for fuzzy sets. The operators belonging to this class of t-norms are in particular associative and therefore it is possible to compute the membership values for the intersection of more than two fuzzy sets by recursive applying t-norm operator.
Three additional properties of t-norms are as follows:
(vi) t is a continuous function [Continuity]
(vii) $t(a,a) \leq a$ [Sub-idempotency property]
(viii) If $\mu_{A_1}(x) < \mu_{A_2}(x)$and $\mu_{B_1}(x) < \mu_{B_2}(x)$, then
$t(\mu_{A_1}(x), \mu_{B_1}(x)) < t(\mu_{A_2}(x), \mu_{B_2}(x))$ [Strict monotonicity]

In following examples of t-norms that are frequently used as fuzzy intersection where each $\mu_A(x)$or $\mu_B(x) \in [0,1]$:

Standard Intersection

Let

$$a = \mu_A(x), b = \mu_B(x)$$
$$t(a,b) = \min(a,b)$$

Algebraic Product

$$t(a,b) = a \cdot b$$

Bounded Difference

$$t(a,b) = \max\{0, a+b-1\}$$

Drastic Intersection

$$t(a,b) = \begin{cases} a, & \text{where } b = 1 \\ b, & \text{where } a = 1 \\ 0, & \text{otherwise} \end{cases}$$

Definition: t-conorms or s-norms are associative, commutative and monotonic two-place functions which map from $[0,1] \times [0,1]$into $[0, 1]$which satisfy the following conditions:

(i) $s(1,1) = 1, s(\mu_{\tilde{A}}(x),0) = s(0,\mu_{\tilde{A}}(x)) = \mu_{\tilde{A}}(x)$ [Bounded condition]

(ii) $s(\mu_{\tilde{A}}(x),\mu_{\tilde{B}}(x)) \leq s\big(\mu_{\tilde{C}}(x),\mu_{\tilde{D}}(x)\big)$if $\mu_{\tilde{A}}(x) \leq \mu_{\tilde{C}}(x)$, $\mu_{\tilde{B}}(x) \leq \mu_{\tilde{D}}(x)$ [Monotonicity]

(iii) $s(s(\mu_{\tilde{A}}(x),\mu_{\tilde{B}}(x)),\mu_{\tilde{C}}(x)) = s(\mu_{\tilde{A}}(x),s(\mu_{\tilde{B}}(x),\mu_{\tilde{C}}(x)))$ [Associativity property]

Relation Between t-Norms and t-Conorms

The t-norms and s-norms are related to each other by the following relation:

$$t(\mu_{\tilde{A}}(x),\mu_{\tilde{B}}(x)) = 1 - s(1-\mu_{\tilde{A}}(x),1-\mu_{\tilde{B}}(x))$$

and

$$s(\mu_{\tilde{A}}(x),\mu_{\tilde{B}}(x)) = 1 - t(1-\mu_{\tilde{A}}(x),1-\mu_{\tilde{B}}(x))$$

so that any (t-conorms or s-norms) can be generalized from a (t-norm) through this transformation.

Theorem 1: The standard fuzzy intersection is the only idempotent t-norm.

Proof: By definition, we have

$$\min(\mu_{\tilde{A}}(x),\mu_{\tilde{A}}(x)) = \mu_{\tilde{A}}(x), \forall\, \mu_{\tilde{A}}(x) \in [0,1],\ x \in X.$$

Assume that there exists a t-norm such that

$$t(\mu_{\tilde{A}}(x),\mu_{\tilde{A}}(x)) = \mu_{\tilde{A}}(x).$$

Then, for any $\mu_{\tilde{A}}(x),\mu_{\tilde{B}}(x) \in [0,1]$,

If $\mu_{\tilde{A}}(x) \leq \mu_{\tilde{B}}(x)$, then

$$\mu_{\tilde{A}}(x) = t\big(\mu_{\tilde{A}}(x),\mu_{\tilde{A}}(x)\big) \leq t\big(\mu_{\tilde{A}}(x),\mu_{\tilde{B}}(x)\big) \quad \text{[by monotonicity]}$$

and

$$t\big(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\big) \leq t(\mu_{\tilde{A}}(x), 1)$$
$$\Rightarrow \mu_{\tilde{A}}(x) = t\big(\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(x)\big) \leq t\big(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\big) \leq t(\mu_{\tilde{A}}(x), 1) = \mu_{\tilde{A}}(x)$$
$$\Rightarrow t\big(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\big) = \mu_{\tilde{A}}(x)\min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}.$$

Similarly,

$$t\big(\mu_{\tilde{B}}(x), \mu_{\tilde{A}}(x)\big) = \mu_{\tilde{B}}(x) = \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}.$$
$$\Rightarrow t\big(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\big) = \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\} = \mu_{\tilde{A}}(x) \cap \mu_{\tilde{B}}(x)$$
$$= \tilde{A} \cap \tilde{B} = \mu_{\tilde{A}\cap\tilde{B}}(x)$$
$$\Rightarrow t\big(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\big) = \mu_{\tilde{A}\cap\tilde{B}}(x)$$

Hence proved.

Theorem 2: Let $\tilde{A}$ and $\tilde{B}$ are two fuzzy sets of a set X. Let $\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x) \in [0,1]$, $\forall x \in X$, then show that

$$t_w\big(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\big) \leq t(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) \leq \mu_{\tilde{A}\cap\tilde{B}}(x)$$

Proof: The boundary conditions and monotonicity in t-norms are defined as

$$t(\mu_{\tilde{A}}(x),1) = \mu_{\tilde{A}}(x)$$

and

$$t(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) \leq t(\mu_{\tilde{A}}(x),1), \forall x \in X$$

which implies that

$$t(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) \leq \mu_{\tilde{A}}(x)$$

Also, the commutativity in t-norms is

$$t(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = t(\mu_{\tilde{B}}(x), \mu_{\tilde{A}}(x)) \leq t(\mu_{\tilde{B}}(x),1) = \mu_{\tilde{B}}(x)$$

Thus,

$$t\big(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\big) \leq \mu_{\tilde{A}}(x) \quad \text{and} \quad t(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) \leq \mu_{\tilde{B}}(x)$$
$$\Rightarrow t(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) \leq \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\} = \mu_{(\tilde{A}\cap\tilde{B})}(x)$$
$$\Rightarrow t\big(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\big) \leq \mu_{(\tilde{A}\cap\tilde{B})}(x) \qquad (1)$$

Again, by boundary condition, we have

$$t\big(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\big) = \mu_{\tilde{A}}(x), \quad \text{if} \quad \mu_{\tilde{B}}(x) = 1$$
$$t\big(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\big) = \mu_{\tilde{B}}(x), \quad \text{if} \quad \mu_{\tilde{A}}(x) = 1$$

Since $t(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) \in [0,1]$,

$$\Rightarrow t(\mu_{\tilde{A}}(x),0) = t(0, \mu_{\tilde{A}}(x)) = 0$$

Also, by monotonicity

$$0 = t(\mu_{\tilde{A}}(x),0) \leq t(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x))$$
$$\Rightarrow 0 = t(0, \mu_{\tilde{B}}(x)) = t(\mu_{\tilde{A}}(x),0) \leq t(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x))$$
$$\Rightarrow t_w\big(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\big) \leq t\big(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\big) \qquad (2)$$

Hence, from (1) and (2), we have

$$t_w(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) \leq t(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) \leq \mu_{(\tilde{A}\cap\tilde{B})}(x).$$

CHECK YOUR PROGRESS

Q6) Verify that the standard intersection $t(a, b) = \min(a, b)$ is idempotent.
Q7) Let $t(a, b) = ab$ be a t-norm. Find the corresponding t-conorm $s(a, b)$ using the relation between t-norm and t-conorm.

2.8 LET US SUM UP

- The extension principle was introduced as a fundamental concept that allows classical mathematical functions and relations to be extended to fuzzy sets, enabling the transformation of crisp inputs into fuzzy outputs.
- We learned about fuzzy complements, which generalize the idea of classical complements, and we examined their axiomatic properties such as boundary conditions, monotonicity, continuity, and involution.
- Different types of fuzzy complements, including threshold type, Sugeno class, and Yager class complements, were discussed along with concepts like equilibrium and dual points.
- Various fuzzy operations such as algebraic sum, algebraic product, bounded sum, bounded difference, and Cartesian product were studied, showing how they extend classical set operations.
- The concept of composition of fuzzy relations using the max–min method was explained, which plays an important role in fuzzy reasoning and decision-making.
- The unit introduced t-norms (triangular norms) as generalized operators for fuzzy intersection, satisfying properties like commutativity, associativity, monotonicity, and boundary conditions.
- Similarly, t-conorms (s-norms) were studied as generalized operators for fuzzy union, along with their defining properties.
- The relationship between t-norms and t-conorms was established using complement operations, showing how one can be derived from the other.

2.9 ANSWERS TO CHECK YOUR PROGRESS

Answer to Q1: Possible sums are:

$1 + 1 = 2$, membership = $\min(1,0.8) = 0.8$

$1 + 2 = 3$, membership = $\min(1,0.4) = 0.4$

$2 + 1 = 3$, membership = $\min(0.5,0.8) = 0.5 \rightarrow$ max = 0.5

$2 + 2 = 4$, membership = $\min(0.5,0.4) = 0.4$

$3 + 1 = 4$, membership = $\min(0.3,0.8) = 0.3 \rightarrow$ max = 0.4

$$f(\tilde{A}, \tilde{B}) = \{(2,0.8), (3,0.5), (4,0.4)\}$$

Answer to Q2: $max(1,1) = 1 \Rightarrow min = 0.8$

$$max(1,2) = 2 \Rightarrow min = 0.6$$
$$max(2,1) = 2 \Rightarrow min = 0.7 \Rightarrow max = 0.7$$
$$max(2,2) = 2 \Rightarrow min = 0.6$$

$$f(\tilde{A}, \tilde{B}) = \{(1,0.8), (2,0.7)\}$$

Answer to Q3: Let $y \in {}^{\alpha+}[f(\tilde{A})]$.
Then,

$$[f(\tilde{A})](y) > \alpha$$

So,

$$\sup_{x:f(x)=y} \tilde{A}(x) > \alpha$$

This implies there exists $x_0 \in X$ such that

$$y = f(x_0) \text{ and } \tilde{A}(x_0) > \alpha$$

Thus,

$$x_0 \in {}^{\alpha+}\tilde{A} \Rightarrow y \in f({}^{\alpha+}\tilde{A})$$

Hence,

$${}^{\alpha+}[f(\tilde{A})] \subseteq f({}^{\alpha+}\tilde{A})$$

Similarly, reverse inclusion can be shown.
Therefore,

$${}^{\alpha+}[f(\tilde{A})] = f({}^{\alpha+}\tilde{A})$$

Answer to Q4: Given that:

i. Monotonicity: if $a \leq b \Rightarrow c(a) \geq c(b)$
ii. Involutive: $c(c(a)) = a$

Since $c: [0,1] \rightarrow [0,1]$, we have

$$c(0) \leq 1, c(1) \geq 0$$

By monotonicity,

$$c(c(0)) \geq c(1)$$

Using involutive property,

$$c(c(0)) = 0 \Rightarrow 0 \geq c(1) \Rightarrow c(1) = 0$$

Again,

$$c(0) = c(c(1)) = 1$$

Hence,

$$c(0) = 1, c(1) = 0$$

Answer to Q5: We have,

$$\mu_{\tilde{A}+\tilde{B}}(x) = \mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x) - \mu_{\tilde{A}}(x)\mu_{\tilde{B}}(x)$$

For $x = 1$:

$$\mu_{\tilde{A}+\tilde{B}}(x) = 0.2 + 0.5 - (0.2 \times 0.5) = 0.7 - 0.1 = 0.6$$

For $x = 2$:

$$\mu_{\tilde{A}+\tilde{B}}(x) = 0.4 + 0.3 - (0.4 \times 0.3) = 0.7 - 0.12 = 0.58$$

Thus,

$$\tilde{C} = \{(1,0.6), (2,0.58)\}$$

Answer to Q6: We want to check idempotency; we have:

$$t(a, a) = a$$

Using definition:

$$t(a, a) = \min(a, a) = a$$

Thus, standard intersection is idempotent.

Answer to Q7: From the relation:

$$s(a, b) = 1 - t(1 - a, 1 - b)$$

Given:

$$t(a, b) = ab$$

$$\therefore\ t(1 - a, 1 - b) = (1 - a)(1 - b)$$

Now,

$$\begin{aligned} s(a, b) &= 1 - (1 - a)(1 - b) \\ &= 1 - (1 - a - b + ab) \\ &= a + b - ab \end{aligned}$$

Thus,

$$s(a, b) = a + b - ab$$

2.10 FURTHER READING

1. Mohan, C. An Introduction to Fuzzy Set Theory and Fuzzy Logic. Anshan Publishers, 2015.

2. Lee, K. H. First Course on Fuzzy Theory and Applications. Springer International Edition, 2005.

3. Yen, J., Langari, R. Fuzzy Logic - Intelligence, Control and Information Pearson Education, 1999.

4. Zimmerman, H.J. Fuzzy Set Theory and its Applications. Allied Publishers Ltd., New Delhi, 1991.

2.11 MODEL QUESTIONS

Q1. Describe generators of t-norms.

Q2. Let $\tilde{A}$ and $\tilde{B}$ be fuzzy sets defined on the universal set $X = Z$whose membership functions are given by

$$\tilde{A} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{2}\right\} \text{ and } \tilde{B} = \left\{2, \frac{1}{3}, \frac{1}{4}, \frac{1}{3}\right\}$$

Let a function $f: X \times X \to X$be defined for all $x_1, x_2 \in X$by

$$f(x_1, x_2) = x_1 \cdot x_2$$

Find $f(\tilde{A}, \tilde{B})$.

Q3. Find whether each of the following functions is an increasing generator. If it is, find and plot the fuzzy complement and t-conorm generated by it.

(a) $g(a) = \sin(a)$

(b) $g(a) = t(a)$

(c) $g(a) = 1 + a$

Q4. Show that the following operations satisfy the law of excluded middle and the law of contradiction:

(a) $t_{\min}$, $c(a) = 1 - a$

(b) $t(a, b) = \min(1, a + b)$, $c(a) = 1 - a$

Q5. Show that the operation on fuzzy sets satisfies the De-Morgan's law

$$t_{\max}, t_{\min}, c(a) = 1 - a$$

Q6. Show that bounded difference t-norm is weaker than the standard t-norm.

Q7. Define a fuzzy complement. Verify whether the function

$$c(a) = \frac{1 - a}{1 + a}$$

satisfies the axioms of a meaningful fuzzy complement.

Q8. Let $\tilde{A} = \{(1,0.2), (2,0.5), (3,0.7)\}$and
$\tilde{B} = \{(1,0.6), (2,0.4), (3,0.8)\}$.
Find:
(i) Algebraic sum $\tilde{A} + \tilde{B}$
(ii) Algebraic product $\tilde{A} \cdot \tilde{B}$
(iii) Bounded sum $\tilde{A} \oplus \tilde{B}$

Q9. Let $\tilde{A}$ and $\tilde{B}$ be fuzzy sets. Prove that

$$\mu_{A\cap B}(x) \leq \mu_A(x), \mu_{A\cap B}(x) \leq \mu_B(x)$$

Q10. Using the extension principle, find the image of a fuzzy set

$$\tilde{A} = \{(1,0.3), (2,0.6), (3,1)\}$$

under the function

$$f(x) = x^2$$

Q11. Explain the relation between t-norms and t-conorms using complement functions.

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UNIT 3: FUZZY NUMBERS AND ARITHMETIC

UNIT STRUCTURE

3.1 Learning Objectives
3.2 Introduction
3.3 Fuzzy Numbers
3.4 Algebraic Operations with Fuzzy Numbers
3.5 Binary operations of two Fuzzy numbers
3.6 Fuzzy Arithmetic
3.7 Arithmetic Operations on Fuzzy numbers in the form of α-Cut Sets
3.8 Let Us Sum Up
3.9 Answers to Check Your Progress
3.10 Further Reading
3.11 Model Questions

3.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- understand the concept and properties of fuzzy numbers and their representation.
- perform algebraic and binary operations on fuzzy numbers.
- explain the principles and applications of fuzzy arithmetic.
- apply arithmetic operations on fuzzy numbers using the α-cut method.
- analyze and interpret results obtained from operations on fuzzy numbers in practical situations.

3.2 INTRODUCTION

This unit introduces the basic idea of fuzzy numbers and explains how calculations can be done using them. Unlike ordinary real numbers, fuzzy numbers are used when the information is not exact or is uncertain, which often happens in real-life situations. The chapter starts by explaining what fuzzy numbers are and how they are represented, helping learners understand how uncertainty can be expressed in mathematical form.

In this unit we discuss algebraic and binary operations on fuzzy numbers, such as addition, subtraction, multiplication, and division, in a clear and step-by-step manner. It also explains fuzzy arithmetic, which allows calculations to be performed while keeping the uncertainty of the values. Finally, arithmetic operations using the α-cut method are introduced, where fuzzy numbers are converted into intervals to make

calculations easier. Overall, this unit helps learners understand how fuzzy numbers can be used in practical fields like decision-making, engineering, and applied sciences.

3.3 FUZZY NUMBERS

Fuzzy numbers and spaces spread around a central point are often represented using bell-shaped fuzzy sets. Consider the fuzzy sets that are defined on the set **R** of real numbers. The membership grade of these functions has the form

$$\mu_{\tilde{A}}: R \to [0,1] \text{ or } \tilde{A}: R \to [0,1]$$

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x))\}$$

Clearly, they have a quantitative meaning and under certain conditions, can be viewed as fuzzy numbers or fuzzy intervals.

For act as a fuzzy number, a fuzzy set $\tilde{A}$ on R must possess at least the following three properties:

(i) $\tilde{A}$ must be normal fuzzy set i.e., $\mu_{\tilde{A}}(x) = 1$ for at least one $x \in \tilde{A}$

(ii) $\alpha\tilde{A}$ must be closed interval for every $\alpha \in]0,1]$, i.e., $\alpha \neq 0$

(iii) The support of $\tilde{A}$, ${}^{0+}\tilde{A}$ must be bounded, i.e., a fuzzy number must be 1 (for normal fuzzy set).

For a fuzzy number, its support must be bounded, and all its α-cuts (for $\alpha \neq 0$) should be closed intervals. This ensures that arithmetic operations on fuzzy numbers can be defined using the usual operations on closed intervals. Furthermore, each α-cut for $\alpha \in [0,1]$ is a closed interval, which means that every fuzzy number is a convex fuzzy set. A typical example of a fuzzy number is the idea of 'approximately 5' or 'around 5', as illustrated in the figure below.

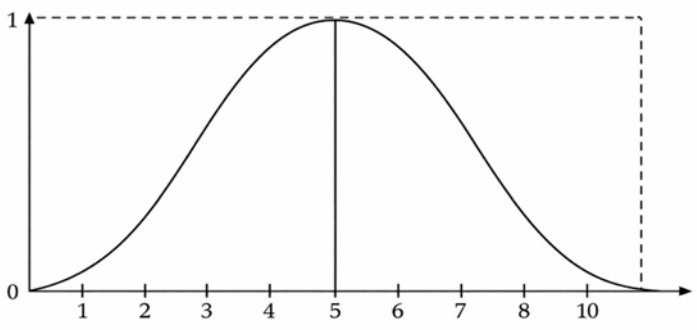


Figure 1 Bell curve around 5.

A fuzzy number describes a value that is not exact and gives more flexibility than a crisp value. It can also represent concepts like middle age. In crisp terms, we must choose a fixed point where a person changes from ‘young’ to ‘middle-aged’, and then from ‘middle-aged’ to ‘old’.

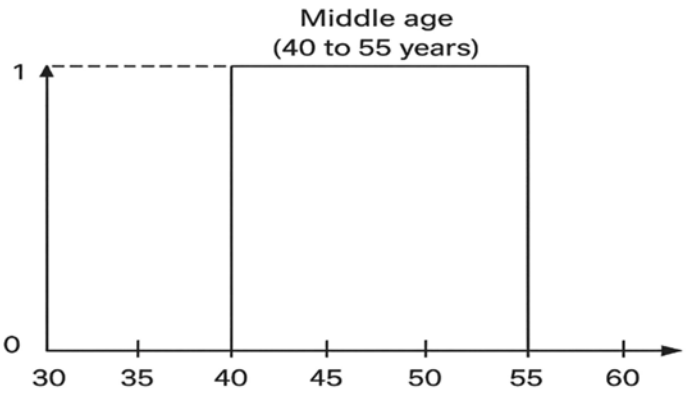


Figure 2 Crisp Set Depicting the Middle-Aged Category

Fig. 2 shows one such set where middle age is defined by the characteristic function

$$m_{tall}(x) = 1[\text{Age} \geq 40 \wedge \text{Age} \leq 55]$$

This, of course, means that $39\frac{3}{4}$ years is not middle aged and that $55\frac{1}{4}$ years old is.

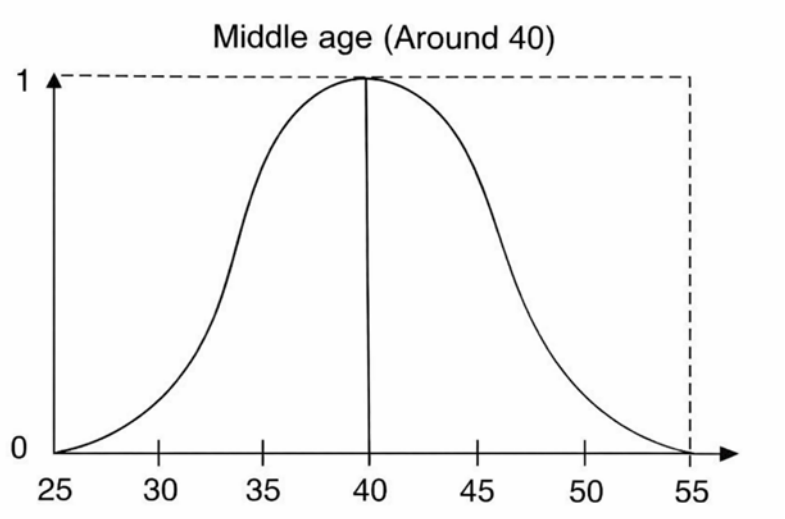


Figure 3 Fuzzy set representation of middle age

Figure 3 shows how the concept of ‘middle age’ can be represented as a fuzzy number. In this fuzzy set, 40 years is taken as the most typical middle age. A bell-shaped curve is used to spread this idea on both sides of 40. This means people between 30 and 50 years have a high degree of being middle-aged. Even at the edges, such as age 50, a person is still considered middle-aged to some extent.

3.4 ALGEBRAIC OPERATIONS WITH FUZZY NUMBERS

Definition 1: A fuzzy number $\widetilde{M}$ is a convex normalized fuzzy set of the real line R such that

(i) there exists exactly one $x_0 \in R$ with $\mu_{\widetilde{M}}(x_0) = 1$

(ii) $\mu_{\widetilde{M}}(x)$ is piecewise continuous.

Here x_0 is called the mean value of $\widetilde{M}$.

Example: The following fuzzy sets are fuzzy numbers

(i) Approximately 5 = $[(3,.2),(4,.6),(5,1),(6,.6),(7,.1)]$

(ii) Approximately 8 = $[(6,.3),(7,.6),(8,1),(9,.7),(10,.2)]$

(iii) Approximately 10 = $[(8,.3),(9,.7),(10,1),(11,.7),(12,.3)]$

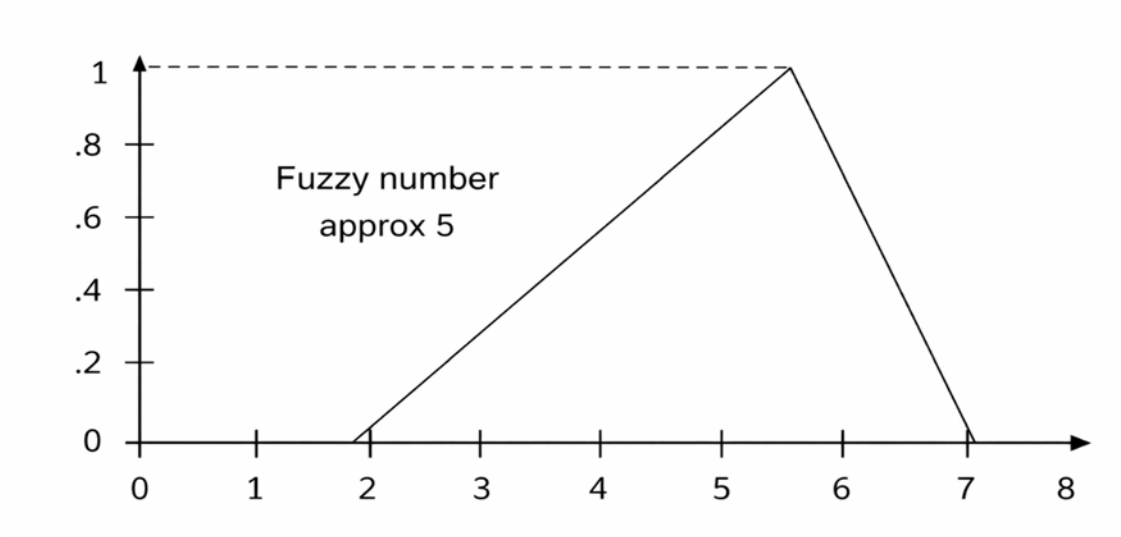


Figure 4 Fuzzy numbers

On the other hand, if we take a set as

$$\tilde{A} = [(3,.7),(4,1),(5,1),(6,.6)]$$

Then, it is not a fuzzy number because $\mu_{\tilde{A}}(4) = 1, \mu_{\tilde{A}}(5) = 1$, i.e., a strict convex set does not exist.

Similarly, the set $[(3,.8),(4,1),(5,1),(6,.7)]$ is not a fuzzy number.

Definition 2: A binary operation $*$ in R is called increasing if for

$$x_1 > y_1 \text{ and } x_2 > y_2$$
$$x_1 * x_2 > y_1 * y_2$$

Also, it is said to be decreasing if for

$$x_1 > y_1 \text{ and } x_2 > y_2$$
$$x_1 * x_2 < y_1 * y_2$$

Example:

$f(x,y) = x \oplus y$, is an increasing operation

$f(x,y) = x \odot y$, is an increasing operation on R^+

$f(x,y) = \ominus (x \oplus y)$, is a decreasing operation

Hence, the operations $\oplus, \ominus, \odot$ are the extended version of '+' and '−' on normal algebraic operations.

CHECK YOUR PROGRESS

Q1) For fuzzy triangular numbers, show that

$$(a,b,c) - (d,e,f) = (a-f,\ b-e,\ c-d).$$

Q2) Let $\tilde{M}$ and $\tilde{N}$ be two fuzzy numbers whose membership functions are continuous and surjective from $\mathbb{R}$ to $[0,1]$. Let $*: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous increasing binary operation. Define

$$\tilde{M} \circledast \tilde{N} = \{(z, \mu_{\tilde{M} \circledast \tilde{N}}(z)) : z \in \mathbb{R}\},$$

where

$$\mu_{\tilde{M} \circledast \tilde{N}}(z) = \sup_{z = x*y} \min\{\mu_{\tilde{M}}(x), \mu_{\tilde{N}}(y)\}.$$

Show that $\tilde{M} \circledast \tilde{N}$ is a fuzzy number.

3.5 BINARY OPERATIONS OF TWO FUZZY NUMBERS

If $\tilde{M}$ and $\tilde{N}$ are two fuzzy numbers whose membership functions are continuous and surjective from R to $[0,1]$ and '*' is a continuous increasing (or decreasing) binary operation, then $\tilde{M} \circledast \tilde{N}$ is a fuzzy number whose membership function is continuous and surjective from R to $[0,1]$, i.e.,

$$\tilde{M} \circledast \tilde{N} = \{(x, \mu_{\tilde{M} \circledast \tilde{N}}(x)) : x \in R\}$$

where $\mu_{\tilde{M} \circledast \tilde{N}}(x)$ is defined on the basis of $\mu_{\tilde{M}}$ and $\mu_{\tilde{N}}$ as

$$\mu_{\tilde{M} \circledast \tilde{N}}(z) = \sup_{z = x*y} \min\{\mu_{\tilde{M}}(x), \mu_{\tilde{N}}(y)\}$$

$*: R \times R \to R$(crisp numbers)

$\circledast: R \times R \to R$(fuzzy numbers)

Definition:

If $\tilde{M}$ and $\tilde{N} \in F(R)$, where $F(R)$ is the collection of all fuzzy sets of real numbers with $\mu_{\tilde{M}}(x), \mu_{\tilde{N}}(x)$ continuous membership functions, then, by application of the extension principle for the binary operation $\circledast: R \times R \to R$, the membership function of the fuzzy number $\tilde{M} \circledast \tilde{N}$ is given by

$$\mu_{\tilde{M} \circledast \tilde{N}}(z) = \sup_{z = x*y} \min\{\mu_{\tilde{M}}(x), \mu_{\tilde{N}}(y)\}$$

Properties of Extended Operation $\circledast$

(i) '*' the extended operation $\circledast$ is commutative.

Example:

$$x * y = y * x$$

$$\widetilde{M}(x) \circledast \widetilde{N}(x) = \widetilde{N}(x) \circledast \widetilde{M}(x)$$

(ii) '*' the extended operation $\circledast$ is associative.

Example:

$$x * (y * z) = (x * y) * z$$
$$\widetilde{M} \circledast (\widetilde{N} \circledast \widetilde{P}) = (\widetilde{M} \circledast \widetilde{N}) \circledast \widetilde{P}$$

3.6 FUZZY ARITHMETIC

Fuzzy arithmetic is based on the following two properties of fuzzy numbers:

i. Each fuzzy set and therefore each fuzzy number can fully and uniquely be represented by its α-cuts.

ii. α-cut of each fuzzy number are closed intervals of real numbers for all real $\alpha \in [0,1]$.

The above two properties enable us to define arithmetic operations on fuzzy numbers in terms of arithmetic operations on their α-cuts (i.e., arithmetic operations on closed intervals). Let '*' denote any of the four arithmetic operations on closed intervals, say addition (+), subtraction (−), multiplication (·) and division (/), then

$$[a,b] * [d,e] = \{f * g : a \le f \le b,\ d \le g \le e\}$$

is a general property of all arithmetic operations on closed intervals except that $[a,b]/[d,e]$ is not defined when $0 \in [d,e]$, i.e., the result of an arithmetic operation on closed intervals is again a closed interval. So we have the following properties:

- $[a,b] + [d,e] = [a+d,\ b+e]$
- $[a,b] - [d,e] = [a-e, b-d]$
- $[a,b] \cdot [d,e] = [\min(ad, ae, bd, be), \max(ad, ae, bd, be)]$
- $[a,b]/[d,e]$, if $[d,e] \neq 0$, then

$$[a,b]/[d,e] = \left[\min\left(\frac{a}{d}, \frac{a}{e}, \frac{b}{d}, \frac{b}{e}\right), \max\left(\frac{a}{d}, \frac{a}{e}, \frac{b}{d}, \frac{b}{e}\right)\right]$$

Example: Consider $[2,5]$ and $[1,3]$

$$[2,5] + [1,3] = [2+1, 5+3] = [3,8]$$
$$[2,5] - [1,3] = [2-3, 5-1] = [-1,4]$$
$$[2,5] \cdot [1,3] = [\min(2,6,5,15), \max(2,6,5,15)] = [2,15]$$
$$[-1,1] \cdot [-2,-5] = [\min(2,.5,-2,-.5), \max(2,.5,-2,-.5)] = [-2,2]$$
$$[2,5]/[1,3] = [2,5]\left[\frac{1}{1}, \frac{1}{3}\right] = \left[min\left(2, \frac{2}{3}, 5, \frac{5}{3}\right), \max\left(2, \frac{2}{3}, 5, \frac{5}{3}\right)\right] = [2/3, 5]$$

ARITHMETIC OPERATIONS ON CLOSED INTERVALS:

Let $\tilde{A} = [a_1, a_2]$, $\tilde{B} = [b_1, b_2]$, $\tilde{C} = [c_1, c_2]$

$0 = [0,0]$, $\mathrm{I} = [1,1]$

Then

(i) $\tilde{A} + \tilde{B} = \tilde{B} + \tilde{A}$ and $\tilde{A} \cdot \tilde{B} = \tilde{B} \cdot \tilde{A}$ [Commutativity]

(ii) $(\tilde{A}+\tilde{B})+\tilde{C}=\tilde{A}+(\tilde{B}+\tilde{C})$and $(\tilde{A}\cdot\tilde{B})\cdot\tilde{C}=\tilde{A}\cdot(\tilde{B}\cdot\tilde{C})$ [Associativity]
(iii) $\tilde{A}=0+\tilde{A}=\tilde{A}+0$and $\tilde{A}=\tilde{A}\cdot 1=1\cdot\tilde{A}$ [Identity]
(iv) $\tilde{A}\cdot(\tilde{B}+\tilde{C})=\tilde{A}\cdot\tilde{B}+\tilde{A}\cdot\tilde{C}$ [Distributivity]
(v) If $b,c\geq 0$, $\forall b\in\tilde{B}$, $\forall c\in\tilde{C}$, then

$$\tilde{A}\cdot(\tilde{B}+\tilde{C})=\tilde{A}\cdot\tilde{B}+\tilde{A}\cdot\tilde{C}$$

Also, if $A=[a,a]$, then

$$a\cdot(\tilde{B}+\tilde{C})=a\cdot\tilde{B}+a\cdot\tilde{C}$$

(vi) $0\in\tilde{A}-\tilde{A}$, then $1\in\tilde{A}\cdot\tilde{A}$
(vii) If $\tilde{A}\subseteq\tilde{E}$and $\tilde{B}\subseteq\tilde{F}$, then

$$\tilde{A}+\tilde{B}\subseteq\tilde{E}+\tilde{F}$$
$$\tilde{A}\cdot\tilde{B}\subseteq\tilde{E}\cdot\tilde{F}$$
$$\tilde{A}/\tilde{B}\subseteq\tilde{E}/\tilde{F}$$

Proof (v):

$$\tilde{A}(\tilde{B}+\tilde{C})=\tilde{A}\cdot\tilde{B}+\tilde{A}\cdot\tilde{C}$$

Let L.H.S.:

$$\tilde{A}(\tilde{B}+\tilde{C})=(a(b+c):a\in\tilde{A},\ b\in\tilde{B},\ c\in\tilde{C})$$
$$=(ab+ac:a\in\tilde{A},b\in\tilde{B},c\in\tilde{C})$$
$$=\tilde{A}\cdot\tilde{B}+\tilde{A}\cdot\tilde{C}$$

Hence,

$$\tilde{A}(\tilde{B}+\tilde{C})=\tilde{A}\cdot\tilde{B}+\tilde{A}\cdot\tilde{C}$$

Case 1: If $a_1\geq 0$

$$\tilde{A}(\tilde{B}+\tilde{C})=[a_1(b_1+c_1),\ a_2(b_2+c_2)]$$
$$=[(a_1b_1+a_1c_1),(a_2b_2+a_2c_2)]$$
$$=[a_1b_1,\ a_2b_2]+[a_1c_1,\ \ a_2c_2]$$
$$=\tilde{A}\cdot\tilde{B}+\tilde{A}\cdot\tilde{C}$$

Example

$$\tilde{A}=[2,3],\tilde{B}=[3,1],\tilde{C}=[4,2]$$
$$\tilde{A}\cdot\tilde{B}=[2,3]\cdot[3,1]=[2,9]$$
$$\tilde{A}\cdot\tilde{C}=[2,3]\cdot[4,2]=[4,12]$$
$$\tilde{B}+\tilde{C}=[3,1]+[4,2]=[7,3]$$
$$\tilde{A}(\tilde{B}+\tilde{C})=[2,3][7,3]=[6,21]$$
$$\tilde{A}\cdot\tilde{B}+\tilde{A}\cdot\tilde{C}=[2,9]+[4,12]=[6,21]$$
$$\Rightarrow\tilde{A}(\tilde{B}+\tilde{C})=\tilde{A}\cdot\tilde{B}+\tilde{A}\cdot\tilde{C}$$

Case 2: If $a_1\leq 0$ and $a_2\leq 0$

$$-a_2\geq 0,(-\tilde{A})=[-a_2,-a_1]$$
$$(-\tilde{A})(\tilde{B}+\tilde{C})=(-\tilde{A})\cdot\tilde{B}+(-\tilde{A})\cdot\tilde{C}$$

Hence,

$$\tilde{A}(\tilde{B}+\tilde{C})=\tilde{A}\cdot\tilde{B}+\tilde{A}\cdot\tilde{C}$$

Case 3: If $a_1 \leq 0$ and $a_2 \geq 0$

$$\tilde{A}(\tilde{B} + \tilde{C}) = [a_1(b_1 + c_1), a_2(b_2 + c_2)]$$
$$= [(a_1.b_1 + a_1.c_1), (a_2.b_2 + a_2.c_2)]$$

$$= \tilde{A}.\tilde{B} + \tilde{A}.\tilde{C}$$

To show that distribution does not hold in general, let

$\tilde{A} = [0,1], \tilde{B} = [1,2], \tilde{C} = [-2,-1]$, then

$\tilde{A} \cdot \tilde{B} = [0,1] \cdot [1,2] = [0,2]$

$\tilde{A} \cdot \tilde{C} = [0,1] \cdot [-2,-1] = [-2,0]$

$\tilde{B} + \tilde{C} = [1,2] + [-2,-1] = [-1,1]$

and

$$\tilde{A} \cdot (\tilde{B} + \tilde{C}) = [0,1] \cdot [-2,2] \subset \tilde{A} \cdot \tilde{B} + \tilde{A} \cdot \tilde{C}$$
$$\Rightarrow \tilde{A} \cdot (\tilde{B} + \tilde{C}) \subset \tilde{A} \cdot \tilde{B} + \tilde{A} \cdot \tilde{C}$$

3.7 ARITHMETIC OPERATIONS ON FUZZY NUMBERS IN THE FORM OF α-CUT SETS

There are two main methods to develop fuzzy arithmetic. One uses interval arithmetic, and the other uses the extension principle, which extends operations on real numbers to fuzzy numbers.

Method – I: In this method, we assume that the fuzzy numbers are represented by the membership functions. Let $\tilde{A}$ and $\tilde{B}$ denote the fuzzy numbers and let $+, -, \times, \div$ denote any of the four basic arithmetic operations. Then we define a fuzzy set R, $\tilde{A} * \tilde{B}$ and the α-cut of this by

$$^{\alpha}(\tilde{A} * \tilde{B}) = {}^{\alpha}\tilde{A} * {}^{\alpha}\tilde{B} \text{ for any } \alpha \in [0,1]$$

Thus,

$$\tilde{A} * \tilde{B} = \bigcup_{\alpha \in [0,1]} {}^{\alpha}(\tilde{A} * \tilde{B})$$

Since $^{\alpha}(\tilde{A} * \tilde{B})$ is a closed interval for each $\alpha \in [0,1]$, $\tilde{A}, \tilde{B}$ is also a fuzzy number.

Example: Let us consider the following two triangular shaped fuzzy numbers

$$\tilde{A}(x) = \begin{cases} 0, & \text{for } x \leq -1 \text{ and } x > 3 \\ \dfrac{x+1}{2}, & \text{for } -1 < x \leq 1 \\ \dfrac{3-x}{2}, & \text{for } 1 < x \leq 3 \end{cases}$$

$$\tilde{B}(x) = \begin{cases} 0, & \text{for } x \le 1 \text{ and } x > 5 \\ \dfrac{x-1}{2}, & \text{for } 1 < x \le 3 \\ \dfrac{5-x}{2}, & \text{for } 3 < x \le 5 \end{cases}$$

Now, we have to find the sum, difference, product and division of $\tilde{A}(x)$ and $\tilde{B}(x)$.
For this, we first find α-cuts of above fuzzy numbers

$$^{\alpha}\tilde{A} = [2\alpha - 1,\ 3 - 2\alpha]$$
$$^{\alpha}\tilde{B} = [2\alpha + 1,\ 5 - 2\alpha]$$
$$^{\alpha}(\tilde{A} + \tilde{B}) = \alpha\tilde{A} + \alpha\tilde{B} = [2\alpha - 1{,}3 - 2\alpha] + [2\alpha + 1{,}5 - 2\alpha] = [4\alpha, 8 - 4\alpha]$$
$$^{\alpha}(\tilde{A} - \tilde{B}) = \alpha\tilde{A} - \alpha\tilde{B} = [2\alpha - 1, 3 - 2\alpha] - [2\alpha + 1, 5 - 2\alpha] = [4\alpha - 6, 2 - 4\alpha]$$
$$^{\alpha}(\tilde{A}.\tilde{B}) = \alpha\tilde{A} \cdot \alpha\tilde{B} = [2\alpha - 1, 3 - 2\alpha]\,[2\alpha + 1, 5 - 2\alpha]$$
$$= \begin{cases} \min[4\alpha^2 - 1,\ -4\alpha^2 + 12\alpha - 5,\ -4\alpha^2 + 4\alpha + 3,\ 4\alpha^2 - 16\alpha + 15]] \\ \max[4\alpha^2 - 1,\ -4\alpha^2 + 12\alpha - 5,\ -4\alpha^2 + 4\alpha + 3,\ 4\alpha^2 - 16\alpha + 15]] \end{cases}$$
$$\Rightarrow {}^{\alpha}(\tilde{A}.\tilde{B}) = \begin{cases} [-4\alpha^2 + 12\alpha - 5,\ 4\alpha^2 - 16\alpha + 15], & \text{for } \alpha \in (0{,}0.5] \\ [4\alpha^2 - 1,\ 4\alpha^2 - 16\alpha + 15], & \text{for } \alpha \in (0.5{,}1] \end{cases}$$

$$^{\alpha}(\tilde{A}/\tilde{B}) = \begin{cases} \left[\dfrac{2\alpha - 1}{2\alpha + 1},\ \dfrac{3 - 2\alpha}{2\alpha + 1}\right], & \text{for } \alpha \in (0{,}0.5] \\ \left[\dfrac{2\alpha - 1}{5 - 2\alpha},\ \dfrac{3 - 2\alpha}{3 - 2\alpha}\right], & \text{for } \alpha \in (0.5{,}1] \end{cases}$$

$$(\tilde{A} + \tilde{B})(x) = \begin{cases} 0, & \textit{for } x \le 0 \textit{ and } x > 8 \\ \dfrac{x}{4}, & \textit{for } 0 < x \le 4 \\ \dfrac{8 - x}{4}, & \textit{for } 4 < x \le 8 \end{cases}$$

$$(\tilde{A} - \tilde{B})(x) = \begin{cases} 0, & \textit{for } x \le -6 \textit{ and } x > 2 \\ \dfrac{x + 6}{4}, & \textit{for } -6 < x \le -2 \\ \dfrac{2 - x}{4}, & \textit{for } -2 < x \le 2 \end{cases}$$

$$(\tilde{A} \cdot \tilde{B})(x) = \begin{cases} 0, & \textit{for } x \le -5 \textit{ and } x \ge 15 \\ \dfrac{3 - \sqrt{4 - x}}{2}, & \textit{for } -5 < x < 0 \\ \dfrac{1 + \sqrt{x}}{2}, & \textit{for } 0 \le x < 3 \\ \dfrac{\left[4 - \left(1 + \sqrt{x}\right)\right]}{2}, & \textit{for } 3 \le x < 15 \end{cases}$$

$$(\tilde{A}/\tilde{B})(x) = \begin{cases} 0, & \text{for } x \le -1 \text{ and } x \ge 3 \\ \dfrac{x+1}{2-2x}, & \text{for } -1 < x < 0 \\ \dfrac{5x+1}{2x+2}, & \text{for } 0 \le x < 1/3 \\ \dfrac{3-x}{2x+2}, & \text{for } 1/3 \le x < 3 \end{cases}$$

CHECK YOUR PROGRESS

Q3) Let $\tilde{A} = \{(1,0.5), (2,1), (3,0.6)\}$, $\tilde{B} = \{(1,0.7), (2,0.9)\}$
Define $f(x,y) = x + y$. Find $f(\tilde{A} \times \tilde{B})$.
Q4) Let $X = \mathbb{N} \times \mathbb{N}$ and $\tilde{A}, \tilde{B}$ be two fuzzy numbers defined by

$$\tilde{A} = \{(1,0.5), (2,0.7), (3,1), (4,0.6)\}$$
$$\tilde{B} = \{(1,0.6), (2,0.8), (3,1), (4,0.5)\}$$

Define $f: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that

$$f(x,y) = x + y, x \in \tilde{A}, y \in \tilde{B}.$$

Find the image $f(\tilde{A} \times \tilde{B})$ using the extension principle.
Q5) Let $\tilde{A}$ and $\tilde{B}$ be two fuzzy numbers. If

$${}^{\alpha}\tilde{A} = [a_1, a_2] \text{ and } {}^{\alpha}\tilde{B} = [b_1, b_2],$$

show that

$${}^{\alpha}(\tilde{A} * \tilde{B}) = [\, a_1 b_1,\ a_2 b_2\,],$$

assuming all values are non-negative.

3.8 LET US SUM UP

- A fuzzy number is a special type of fuzzy set defined on the real line that represents imprecise numerical values and is characterized by properties such as normality, convexity, and bounded support.
- A fuzzy number must satisfy certain conditions, including having at least one element with full membership (normality), having α-cuts as closed intervals, and possessing bounded support.
- Fuzzy numbers are often represented using membership functions, commonly in bell-shaped or triangular forms, to describe approximate values.
- Algebraic operations on fuzzy numbers extend classical operations like addition, subtraction, multiplication, and division, allowing computations under uncertainty.
- Binary operations on fuzzy numbers are defined using the extension principle, ensuring that the result of operations between fuzzy numbers is also a fuzzy number.
- Fuzzy arithmetic is based on the concept of α-cuts, where each fuzzy number can be represented as a family of closed intervals corresponding to different α-levels.

- Arithmetic operations on fuzzy numbers are performed by applying interval arithmetic to their α-cuts, which simplifies computations while preserving uncertainty.
- The result of arithmetic operations on fuzzy numbers is also a fuzzy number, maintaining properties such as convexity and continuity under suitable conditions.
- Properties like commutativity, associativity, identity, and distributivity (in certain cases) are extended from classical arithmetic to fuzzy arithmetic on intervals.
- Fuzzy arithmetic provides a useful framework for handling imprecise data in real-life applications such as decision-making, engineering, and applied sciences.

✓ 3.9 ANSWERS TO CHECK YOUR PROGRESS

Answer to Q1: Let $\tilde{A} = (a, b, c)$and $\tilde{B} = (d, e, f)$be two triangular fuzzy numbers. The α-cuts of $\tilde{A}$and $\tilde{B}$are given by

$$A_\alpha = [\, a + \alpha(b-a),\ c - \alpha(c-b)\,], B_\alpha = [\, d + \alpha(e-d),\ f - \alpha(f-e)\,], \alpha \in [0,1].$$

Using interval arithmetic,

$$A_\alpha - B_\alpha = [\, (a + \alpha(b-a)) - (f - \alpha(f-e)),\ (c - \alpha(c-b)) - (d + \alpha(e-d))\,].$$

$$= [\, (a-f) + \alpha((b-a) + (f-e)),\ (c-d) - \alpha((c-b) + (e-d))\,].$$

This interval represents a triangular fuzzy number with parameters $(a-f,\ b-e,\ c-d)$.

Hence,

$$(a, b, c) - (d, e, f) = (a-f,\ b-e,\ c-d).$$

Answer to Q2: Given that $\mu_{\tilde{M}}$ and $\mu_{\tilde{N}}$ are continuous and surjective on $[0,1]$, the function

$$\min\{\mu_{\tilde{M}}(x), \mu_{\tilde{N}}(y)\}$$

is continuous.

Since $*$ is a continuous increasing binary operation, the relation $z = x * y$ preserves continuity. Hence, the supremum

$$\mu_{\tilde{M} \circledast \tilde{N}}(z) = \sup_{z = x*y} \min\{\mu_{\tilde{M}}(x), \mu_{\tilde{N}}(y)\}$$

is continuous in z.

Further, because both $\mu_{\tilde{M}}$ and $\mu_{\tilde{N}}$ take all values in $[0,1]$, the resulting membership function also takes all values in $[0,1]$, i.e., it is surjective.

Therefore, $\tilde{M} \circledast \tilde{N}$ is a fuzzy number.

Answer to Q3: Using extension principle:

$$\mu_C(z) = \sup \min(\mu_A(x), \mu_B(y))$$

Now, we have

$$(1,1) \rightarrow 2 \rightarrow min(0.5,0.7) = 0.5$$
$$(1,2) \rightarrow 3 \rightarrow min(0.5,0.9) = 0.5$$
$$(2,1) \rightarrow 3 \rightarrow min(1,0.7) = 0.7$$
$$(2,2) \rightarrow 4 \rightarrow min(1,0.9) = 0.9$$
$$(3,1) \rightarrow 4 \rightarrow min(0.6,0.7) = 0.6$$
$$(3,2) \rightarrow 5 \rightarrow min(0.6,0.9) = 0.6$$

Taking sup we get:

$$C = \{(2,0.5), (3,0.7), (4,0.9), (5,0.6)\}$$

Answer to Q4: Given

$$\tilde{A} = \{(1,0.5), (2,0.7), (3,1), (4,0.6)\}, \qquad \tilde{B} = \{(1,0.6), (2,0.8), (3,1), (4,0.5)\}$$

and $f(x, y) = x + y$.

Using the extension principle, the membership function of $z = x + y$ is defined by

$$\mu_{f(\tilde{A}\times\tilde{B})}(z) = \max_{x+y=z} \min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)).$$

we have, for:

$z = 2$: $\mu(2) = \min(0.5,0.6) = 0.5$

$z = 3$: $\mu(3) = \max\{\min(0.5,0.8), \min(0.7,0.6)\} = \max\{0.5,0.6\} = 0.6$

$z = 4$: $\mu(4) = \max\{\min(0.5,1), \min(0.7,0.8), \min(1,0.6)\} = \max\{0.5,0.7,0.6\} = 0.7$

$z = 5$: $\mu(5) = \max\{\min(0.5,0.5), \min(0.7,1), \min(1,0.8), \min(0.6,0.6)\} = 0.8$

$z = 6$: $\mu(6) = \max\{\min(0.7,0.5), \min(1,1), \min(0.6,0.8)\} = \max\{0.5,1,0.6\} = 1$

$z = 7$: $\mu(7) = \max\{\min(1,0.5), \min(0.6,1)\} = \max\{0.5,0.6\} = 0.6$

$z = 8$: $\mu(8) = \min(0.6,0.5) = 0.5$

$$\therefore f(\tilde{A} \times \tilde{B}) = \{(2,0.5), (3,0.6), (4,0.7), (5,0.8), (6,1), (7,0.6), (8,0.5)\}.$$

Answer to Q5: Given

$$^{\alpha}\tilde{A} = [a_1, a_2],\ ^{\alpha}\tilde{B} = [b_1, b_2],$$

by the definition of multiplication of fuzzy numbers in terms of α-cuts,

$$^{\alpha}(\tilde{A} * \tilde{B}) = {}^{\alpha}\tilde{A} * {}^{\alpha}\tilde{B}.$$

Since both intervals are non-negative, the product of intervals is

$$[a_1, a_2] * [b_1, b_2] = [\, a_1 b_1,\ a_2 b_2\,].$$

Therefore,

$$^{\alpha}(\tilde{A} * \tilde{B}) = [\, a_1 b_1,\ a_2 b_2\,].$$

3.10 FURTHER READING

1. Mohan, C. An Introduction to Fuzzy Set Theory and Fuzzy Logic. Anshan Publishers, 2015.

2. Lee, K. H. First Course on Fuzzy Theory and Applications. Springer International Edition, 2005.

3. Yen, J., Langari, R. Fuzzy Logic - Intelligence, Control and Information Pearson Education, 1999.

4. Zimmerman, H.J. Fuzzy Set Theory and its Applications. Allied Publishers Ltd., New Delhi, 1991.

3.11 MODEL QUESTIONS

Q1. Let $X = \mathbb{R} \times \mathbb{R}$and define fuzzy numbers

$$\tilde{A} = \{(1,0.5), (2,0.7), (3,1), (4,0.6)\}$$
$$\tilde{B} = \{(1,0.4), (2,0.8), (3,1), (5,0.3)\}$$

Let $f(x,y) = x + y$. Find $f(\tilde{A} \times \tilde{B})$using the extension principle.

Q2. Let, $\tilde{A} = \{(1,0.3), (2,1), (3,0.4)\}$ and $\tilde{B} = \{(2,0.7), (3,1), (4,0.2)\}$; then show that

$$\tilde{A} \odot \tilde{A} = \{(2,0.3), (3,0.3), (4,0.7), (6,1), (8,0.2), (9,0.4), (12,0.2)\}$$

Q3. Let $\tilde{A} = (a,b,c)$, $\tilde{B} = (d,e,f)$ be triangular fuzzy numbers.
Prove that:

i) $\tilde{A} - \tilde{B} = (a - f,\ b - e,\ c - d)$

ii) $\tilde{A} \cdot \tilde{B} = (ad,\ be,\ cf)$(under positivity conditions)

Q4. Show that $0 \in \tilde{B} - \tilde{B}$ and $1 \in \tilde{B}/\tilde{B}$ for $0 \notin \tilde{B}$, where $\tilde{B}$ is a closed interval.

Q5. For the fuzzy triangular members, show that

$$(a,b,c) + (d,e,f) = (a + d, b + e, c + f)$$

Q6. Let $\tilde{A} = \{(1,0.4), (2,0.9), (3,1)\}, \tilde{B} = \{(2,0.5), (3,1), (4,0.6)\}$
Using max–min composition, find: $\tilde{A} \oplus \tilde{B}$

Q7. Let $\mathrm{X} = \mathrm{N} \times \mathrm{N}$ and $\tilde{A}, \tilde{B}$ be two fuzzy numbers defined by

$$\tilde{A} = \{(1,0.6), (2,0.8), (3,1), (4,0.6)\}$$
$$\tilde{B} = \{(0,0.5), (1,0.7), (2,0.9), (3,1), (4,0.4)\}$$

Define $f: \mathrm{N} \times \mathrm{N} \to \mathrm{N}$ such that $f(x,y) = z,\ x \in \tilde{A}, y \in \tilde{B}$. Find the image $f(\tilde{A} \times \tilde{B})$ using the extension principle.

Q8. Let $\tilde{A} = \{(1,0.5), (2,1), (3,0.6)\}$

Define $f(x) = x^2$. Find the fuzzy set $f(\tilde{A})$ using the extension principle.

Q9. Let $\tilde{A}$ and $\tilde{B}$ be fuzzy numbers with α-cuts

$$A_\alpha = [a_1(\alpha), a_2(\alpha)], B_\alpha = [b_1(\alpha), b_2(\alpha)]$$

Show that

$$(\tilde{A} + \tilde{B})_\alpha = [a_1(\alpha) + b_1(\alpha),\ a_2(\alpha) + b_2(\alpha)]$$

and prove that the result is again a fuzzy number.

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UNIT 4: FUZZY RELATIONS

UNIT STRUCTURE

4.1 Learning Objectives
4.2 Introduction
4.3 Projections and Cylindrical Fuzzy Relations
4.4 Composition
4.5 Properties of Min-Max Composition
4.6 Binary Relation on a Single Set
4.7 Compatibility Relation
4.8 Fuzzy Ordering Relation
4.9 Let Us Sum Up
4.10 Answers to Check Your Progress
4.11 Further Reading
4.12 Model Questions

4.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- learn the concept of fuzzy relation and distinguish it from a crisp relation.
- learn basic operations including union, intersection, complement, and composition of fuzzy relations.
- learn the concepts of projections and cylindrical fuzzy relations
- learn relational properties such as reflexivity, symmetry, transitivity, and equivalence within fuzzy contexts.
- identify and classify fuzzy equivalence and compatibility relations.
- learn the fuzzy ordering relation.

4.2 INTRODUCTION

This unit presents the fundamental principles, operations, characteristics, and practical uses of fuzzy relations. It begins with essential definitions and various forms of representation, then moves on to discuss how fuzzy relations can be operated upon, combined, and applied in inference processes. The unit places particular emphasis on relational properties such as reflexivity, symmetry, and transitivity, as these are vital for constructing fuzzy equivalence relations and clustering frameworks. It also introduces fuzzy relational equations, which serve as an important mechanism for modelling uncertain environments and addressing real-world problems.

General Definitions:

Definition (1): Let $X, Y \subset \boldsymbol{R}$ be universal set, then

$$R = \{((x,y), \mu_R(x,y)) : (x,y) \in X \times Y\}$$

is called a Fuzzy Relation from X to Y.

For Example: Let $X = Y = \boldsymbol{R}$, i.e., set of real numbers and $\tilde{A} =$ 'considerably larger than'.

The membership function of the fuzzy relation which is of course a fuzzy set on $X \times Y$ can be defined as follows:

$$\mu_{\tilde{A}}(x,y) = \begin{cases} 0, & x \le y \\ x - y/10y, & y < x \le 11y \\ 1, & x > 11y \end{cases}$$

or

$$\mu_{\tilde{A}}(x,y) = \begin{cases} 0, & x \le y \\ [1 + (y-x)^{-1}]^{-1}, & x > y \end{cases}$$

For discrete support, fuzzy relations can also be defined by matrices.

Let $\tilde{A} =$ "x is considerably larger than y". Then

	y_1	y_2	y_3	y_4
x_1	.8	1	.1	.7
x_2	0	.8	0	0
x_3	.9	1	.7	.8

And $\tilde{B} =$ "y is very close to x".

	y_1	y_2	y_3	y_4
x_1	.4	0	.9	.6
x_2	.9	.4	.5	.7
x_3	.3	0	.8	.5

Definition (2): Let $X, Y \subset \boldsymbol{R}$, and $X = \{(x, \mu_X(x)) : x \in X\}$; $Y = \{(y, \mu_Y(y)) : y \in Y\}$ be two fuzzy sets. Then fuzzy relation $\tilde{R}(X,Y)$ is defined as follows:

$\tilde{R} \equiv \tilde{R}(X,Y) = \{((x,y), \mu_{\tilde{R}}(x,y)) : (x,y) \in X \times Y\}$ is a fuzzy relation on X and Y if

$$\mu_{\tilde{R}}(x,y) \le \mu_X(x), \quad \forall\ (x,y) \in X \times Y$$

$$\mu_{\tilde{R}}(x,y) \le \mu_Y(y), \quad \forall\ (x,y) \in X \times Y$$

or

$$\mu_{\tilde{R}}(x,y) \le \min(\mu_X(x), \mu_Y(y))$$

Remarks:

- The above definition of fuzzy relation is very useful in defining a graph.
- Fuzzy relations are obviously the fuzzy sets in the product spaces.

Definition (3): Let $\tilde{R}$ and $\tilde{Z}$ be two fuzzy relations in the same product spaces. Then the union and intersection of $\tilde{R}$ and $\tilde{Z}$ is defined as follows:

$$\tilde{R} \cup \tilde{Z} = \{[(x,y), \mu_{\tilde{R}\cup\tilde{Z}}(x,y)]: (x,y) \in X \times Z\}$$

where

$$\mu_{\tilde{R}\cup\tilde{Z}}(x,y) = max\{\mu_{\tilde{R}}(x,y), \mu_{\tilde{Z}}(x,y): (x,y) \in X \times Z\}$$

and

$$\tilde{R} \cap \tilde{Z} = \{[(x,y), \mu_{\tilde{R}\cap\tilde{Z}}(x,y)]: (x,y) \in X \times Z\}$$

where

$$\mu_{\tilde{R}\cap\tilde{Z}}(x,y) = min\{\mu_{\tilde{R}}(x,y), \mu_{\tilde{Z}}(x,y): (x,y) \in X \times Z\}$$

For example: Let $\tilde{R}$ and $\tilde{Z}$ be the two fuzzy relations. The union of $\tilde{R}$ and $\tilde{Z}$ which can be interpreted as "x considerably larger than y" and "x is very close to y", i.e., "x considerably larger or very close to y", is then given by the following two matrix of the relations

"x considerably larger than y"

$$\tilde{R} =$$

	y_1	y_2	y_3	y_4
x_1	.8	1	.1	.7
x_2	0	.8	0	0
x_3	.9	1	.7	.8

"y is very close to x"

$$\tilde{Z} =$$

	y_1	y_2	y_3	y_4
x_1	.4	0	.9	.6
x_2	.9	.4	.5	.7
x_3	.3	0	.8	.5

Then

$$\tilde{R} \cup \tilde{Z} =$$

	y_1	y_2	y_3	y_4
x_1	.8	1	.9	.7
x_2	.9	.8	.5	.7
x_3	.9	1	.8	.8

and

$$\tilde{R} \cap \tilde{Z} =$$

	y_1	y_2	y_3	y_4
x_1	.4	0	.1	.6
x_2	0	.4	0	0
x_3	.3	0	.7	.5

4.3 PROJECTIONS AND CYLINDRICAL FUZZY RELATIONS

The projection and cylindrical extensions of fuzzy relations is defined as follows:
Let $\tilde{R} = \{((x,y), \mu_{\tilde{R}}(x,y)) : (x,y) \in X \times Y\}$ be the fuzzy binary relation, then the first projection of $\tilde{R}$ is defined as

$$\tilde{R}^{(1)} = \{(x, \max_y \mu_{\tilde{R}}(x,y)) : (x,y) \in X \times Y\}$$

and the second projection of $\tilde{R}$ is defined as

$$\tilde{R}^{(2)} = \{(y, \max_x \mu_{\tilde{R}}(x,y)) : (x,y) \in X \times Y\}$$

Also, the total projection is defined as

$$\tilde{R}^{(T)} = \max_x \max_y \{\mu_{\tilde{R}}(x,y)) : (x,y) \in X \times Y\}$$

And the cylindrical extension represents the greatest fuzzy relation obtained from either the first or second projection. In other words, it is the largest possible fuzzy relation on the extended Cartesian product whose membership grades remain consistent with the given projection.

Example: Let $\tilde{\boldsymbol{R}}(\boldsymbol{X}, \boldsymbol{Y})$ be a fuzzy relation defined by the following relation matrix. Then the first, second, total projection and cylindrical extensions are given below:

y_1	y_2	y_3	y_4	y_5	First

						Projection $\tilde{R}^{(1)}$
x_1	.8	0	.3	.4	0	.8
x_2	0	1	.5	0	.5	1
x_3	1	.4	.6	.8	.6	1
Second projection $\tilde{R}^{(2)}$	1	1	.6	.8	.6	

$max = 1$

Total projection$= 1$

The cylindrical extension of $\tilde{R}^{(1)}$ is

.8	.8	.8	.8	.8
1	1	1	1	1
1	1	1	1	1

The cylindrical extension of $\tilde{R}^{(2)}$ is

1	1	.6	.8	.6
1	1	.6	.8	.6
1	1	.6	.8	.6

Example: Let $\tilde{R}$ be a fuzzy relation between two sets

$X = \{New\ York, Paris\}$ and $Y = \{Beijing, New\ York, London\}$

Which represent the relation “very far”. Then

$$\tilde{R}(X,Y) \equiv \tilde{R} = \frac{1}{[N.Y., Beijing]} + \frac{0}{[N.Y., N.Y.]} + \frac{.6}{[N.Y., London]} + \frac{.9}{[Paris, Beijing]} + \frac{.7}{[Paris, N.Y.]} + \frac{.3}{[Paris, London]}$$

or $\tilde{R}$ is represented as in matrix form by

	New York	Paris
Beijing	1	.9
New York	0	.7
London	.6	.3

Some Definitions:

1. **Domain of a Fuzzy Relation:** If $\tilde{R}(X,Y) \equiv \tilde{R}$ is a binary relation of X and Y, then its domain is a fuzzy set on X, denoted by dom. $\tilde{R}$ and defined by (i.e., its membership functions are)

$$dom.\tilde{R}(x) = \max_{y \in Y} \tilde{R}(x,y) \quad \text{for each } x \in X$$

i.e., each element of set X belongs to the domain of $\tilde{R}$ to the degree equal to the strength of its strongest relation to any member of the set Y.

Remark: Domain of $\tilde{R}(x,y)$ is the row wise maximum membership grade in $\tilde{R}$.

2. **Range of a Fuzzy Relation:** The range of $\tilde{R}(x,y)$ is a fuzzy set on Y denoted by $ran.\tilde{R}$ and defined as

$$ran.\tilde{R}(y) = \max_{x\in X} \tilde{R}(x,y) \quad \text{for each } y \in Y$$

i.e., each element of set Y belongs to an element of X equal to the degree of that elements membership in the range of $\tilde{R}$.

Remark: Range of $\tilde{R}(x,y)$ is the columnwise maximum membership grade in $\tilde{R}$.

3. **Height of a Fuzzy Relation:** The height of a fuzzy relation $\tilde{R}(X,Y)$ is a number $h(\tilde{R})$, defined by

$$h(\tilde{R}) = \max_{y\in Y} \max_{x\in X} \tilde{R}(x,y)$$

i.e., $h(\tilde{R})$ is the largest membership grade attained by any pair (x,y) in $\tilde{R}$.

Remark: $h(\tilde{R})$ is also called the total projection.

4. **Inverse of a Fuzzy Relation:** The inverse of a fuzzy relation $\tilde{R}(X,Y)$ which is denoted by $\tilde{R}^{-1}(Y,X)$ is a relation on $Y \times X$ and is defined by

$$\tilde{R}^{-1}(y,x) = \tilde{R}(x,y), \quad \forall\ x \in X, y \in Y$$

Clearly,

$$\left(\tilde{R}^{-1}\right)^{-1} = \tilde{R} \qquad \text{for the fuzzy relation.}$$

CHECK YOUR PROGRESS

Q 1. Define the projection of a fuzzy relation.
Q 2. What is the relationship between a fuzzy relation and its projections?
Q 3. State the difference between first projection and second projection of a fuzzy relation.
Q 4. Let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$, and the fuzzy relation

$$R = \{(x_1,y_1)/0.8, (x_1,y_2)/0.4, (x_2,y_1)/0.6, (x_2,y_2)/0.2\}.$$

Compute the first projection $\tilde{R}^{(1)}$.
Q 5. Using the same relation as above, compute the second projection $\tilde{R}^{(2)}$.
Q 6. For the fuzzy set $A = \{x_1/0.7, x_2/0.5\}$, find its cylindrical extension. Assume $Y = \{y_1, y_2\}$.

4.4 COMPOSITION

Let $\tilde{P}(X,Y)$ and $\tilde{Q}(Y,Z)$ with a common set Y, then their composition is denoted as $\tilde{P}(X,Y) \circ \tilde{Q}(Y,Z)$ and is defined as $\tilde{R}(X,Z)$ on $X \times Z$ where

$$\tilde{R}(x,z) = [\tilde{P} \circ \tilde{Q}](x,z) = \max_{y\in Y} \min[\tilde{P}(x,y), \tilde{Q}(y,z)], \quad \forall\ x \in X, z \in Z$$

The above composition which is based on the standard t-norms and t-conorms, is often referred to as the max.min composition.

For composition

$$\left[\tilde{P}(X,Y) \circ \tilde{Q}(Y,Z)\right]^{-1} = \tilde{Q}^{-1}(Z,Y) \circ \tilde{P}^{-1}(Y,X)$$

and

$$\left[\tilde{P}(X,Y) \circ \tilde{Q}(Y,Z)\right] \circ \tilde{R}(Z,W) = \tilde{P}(X,Y) \circ \left[\tilde{Q}(Y,Z) \circ \tilde{R}(Z,W)\right] \quad \text{[i.e., associative]}$$

However, the standard composition is not commutative, because $\tilde{Q}(Y,Z) \circ \tilde{P}(X,Y)$ is not well defined when $X \neq Z$. Even if $X = Z$ and $\tilde{Q}(Y,Z) \circ \tilde{P}(X,Y)$ are well defined, we have

$$\tilde{P}(X,Y) \circ \tilde{Q}(Y,Z) \neq \tilde{Q}(Y,Z) \circ \tilde{P}(X,Y)$$

The composition of binary fuzzy relation can be performed in terms of membership matrices of the relations as follows:

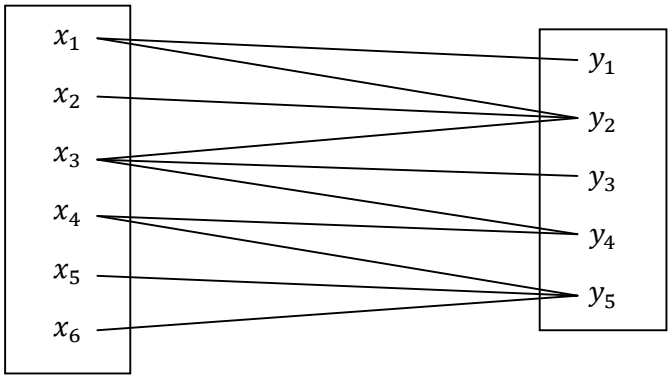


Fig. 1: Segittal Diagram

	y_1	y_2	y_3	y_4	y_5
x_1	.9	1	0	0	0
x_2	0.4	0	0	0	0
x_3	0	.5	1	.2	0
x_4	0	0	0	1	.4
x_5	0	0	0	0	.5
x_6	0	0	0	0	.2

Fig. 2: Membership Matrix

Properties of Composition

1. Standard composition is associative i.e.

$$\left[\tilde{P}(x,y)\circ\tilde{Q}(y,z)\right]\circ\tilde{R}(z,w)=\tilde{P}(x,y)\circ\left[\tilde{Q}(y,z)\circ\tilde{R}(z,w)\right]$$

2. The inverse of the standard composition is equal to the reverse composition of inverse relation.

3. Standard composition is not commutative.

4. If $\tilde{P}=[p_{ik}],\tilde{Q}=\left[q_{kj}\right]$ and $\tilde{R}=\left[r_{ij}\right]$, then

$$\tilde{R}=\tilde{P}\circ\tilde{Q}$$

where $\left[r_{ij}\right]=[p_{ik}]\circ\left[q_{kj}\right]$

and

$$r_{ij}=\max_k\min(p_{ik},\ q_{kj})$$

Relational Join: Let $\tilde{P}(X,Y)$ and $\tilde{Q}(Y,Z)$ be two fuzzy relations, then their relational join is denoted by $\tilde{P}*\tilde{Q}$ and is defined by a relation $\tilde{R}(X,Y,Z)$ as

$$\tilde{R}(X,Y,Z)=\left[\tilde{P}*\tilde{Q}\right](x,y,z)=\min\left[\tilde{P}(x,y),\tilde{Q}(y,z)\right],\qquad\forall\ x\in X,y\in Y,z\in Z$$

Let $\tilde{P}=[p_{ik}],\tilde{Q}=\left[q_{kj}\right]$ and $\tilde{R}=\left[r_{ij}\right]$ be the membership matrices of binary relations such that $\tilde{R}=\tilde{P}\circ\tilde{Q}$, then

$$\left[r_{ij}\right]=[p_{ik}]\circ\left[q_{kj}\right]$$

where

$$r_{ij}=\max_k\min[p_{ik},q_{kj}]$$

Example: Let $\tilde{P}=\begin{bmatrix}.3 & .5 & .8\\ 0 & .7 & 1\\ .4 & .6 & .5\end{bmatrix},\ \tilde{Q}=\begin{bmatrix}.9 & .5 & .7 & .7\\ .3 & .2 & 0 & .9\\ 1 & 0 & .5 & .5\end{bmatrix}$

Then by using $r_{11}=\max[\min[p_{11},q_{11}],\min[p_{12},q_{21}],\min[p_{13},q_{31}]]$, we can find

$$\tilde{P}\circ\tilde{Q}=\begin{bmatrix}.3 & .5 & .8\\ 0 & .7 & 1\\ .4 & .6 & .5\end{bmatrix}\circ\begin{bmatrix}.9 & .5 & .7 & .7\\ .3 & .2 & 0 & .9\\ 1 & 0 & .5 & .5\end{bmatrix}=\begin{bmatrix}r_{11} & r_{12} & r_{13} & r_{14}\\ r_{21} & r_{22} & r_{23} & r_{24}\\ r_{31} & r_{32} & r_{33} & r_{34}\end{bmatrix}$$

Where r_{ij} can be found as follows:

$$\begin{aligned}r_{11}&=\max[\min(p_{11},q_{11}),\quad\min(p_{12},q_{21}),\quad\min(p_{13},q_{31})]\\&=\max[\min(0.3,0.9),\quad\min(0.5,0.3),\quad\min(0.8,1)]\\&=\max[0.3,\ \ 0.3,\ \ 0.8]=0.8\\r_{12}&=\max[\min(p_{11},q_{12}),\quad\min(p_{12},q_{22}),\quad\min(p_{13},q_{32})]\\&=\max[\min(0.3,0.5),\quad\min(0.5,0.2),\quad\min(0.8,0)]\\&=\max[0.3,\ \ 0.2,\ \ 0]=0.3\\r_{13}&=\max[\min(p_{11},q_{13}),\quad\min(p_{12},q_{23}),\quad\min(p_{13},q_{33})]\\&=\max[\min(0.3,0.7),\quad\min(0.5,0),\quad\min(0.8,0.5)]\\&=\max[0.3,\ \ 0,\ \ 0.5]=0.5\\r_{14}&=\max[\min(p_{11},q_{14}),\quad\min(p_{12},q_{24}),\quad\min(p_{13},q_{34})]\\&=\max[\min(0.3,0.7),\quad\min(0.5,0.9),\quad\min(0.8,0.5)]\end{aligned}$$

$$= \max[0.3,\ 0.5,\ 0.5] = 0.5$$

$$r_{21} = \max[\min(p_{21}, q_{11}),\ \min(p_{22}, q_{21}),\ \min(p_{23}, q_{31})]$$
$$= \max[\min(0, 0.9),\ \min(0.7, 0.3),\ \min(1, 1)]$$
$$= \max[0,\ 0.3,\ 1] = 1$$

$$r_{22} = \max[\min(p_{21}, q_{12}),\ \min(p_{22}, q_{22}),\ \min(p_{23}, q_{32})]$$
$$= \max[\min(0, 0.5),\ \min(0.7, 0.2),\ \min(1, 0)]$$
$$= \max[0,\ 0.2,\ 0] = 0.2$$

$$r_{23} = \max[\min(p_{21}, q_{13}),\ \min(p_{22}, q_{23}),\ \min(p_{23}, q_{33})]$$
$$= \max[\min(0, 0.7),\ \min(0.7, 0),\ \min(1, 0.5)]$$
$$= \max[0,\ 0,\ 0.5] = 0.5$$

$$r_{24} = \max[\min(p_{21}, q_{14}),\ \min(p_{22}, q_{24}),\ \min(p_{23}, q_{34})]$$
$$= \max[\min(0, 0.7),\ \min(0.7, 0.9),\ \min(1, 0.5)]$$
$$= \max[0,\ 0.7,\ 0.5] = 0.7$$

$$r_{31} = \max[\min(p_{31}, q_{11}),\ \min(p_{32}, q_{21}),\ \min(p_{33}, q_{31})]$$
$$= \max[\min(0.4, 0.9),\ \min(0.6, 0.3),\ \min(0.5, 1)]$$
$$= \max[0.4,\ 0.3,\ 0.5] = 0.5$$

$$r_{32} = \max[\min(p_{31}, q_{12}),\ \min(p_{32}, q_{22}),\ \min(p_{33}, q_{32})]$$
$$= \max[\min(0.4, 0.5),\ \min(0.6, 0.2),\ \min(0.5, 0)]$$
$$= \max[0.4,\ 0.2,\ 0] = 0.4$$

$$r_{33} = \max[\min(p_{31}, q_{13}),\ \min(p_{32}, q_{23}),\ \min(p_{33}, q_{33})]$$
$$= \max[\min(0.4, 0.7),\ \min(0.6, 0),\ \min(0.5, 0.5)]$$
$$= \max[0.4,\ 0,\ 0.5] = 0.5$$

$$r_{31} = \max[\min(p_{31}, q_{14}),\ \min(p_{32}, q_{24}),\ \min(p_{33}, q_{34})]$$
$$= \max[\min(0.4, 0.7),\ \min(0.6, 0.9),\ \min(0.5, 0.5)]$$
$$= \max[0.4,\ 0.6,\ 0.5] = 0.6$$

Hence $\tilde{P} \circ \tilde{Q} = \begin{bmatrix} 0.8 & 0.3 & 0.5 & 0.5 \\ 1 & 0.2 & 0.5 & 0.7 \\ 0.5 & 0.4 & 0.5 & 0.6 \end{bmatrix}$

Remark: It is a fact that the relational join produces a ternary relation from the binary relations, is a major difference from the composition because it results in another binary relation.

4.5 PROPERTIES OF MIN-MAX COMPOSITION

1. **Associativity:** The max-min composition is associative, i.e.,
$$(\tilde{R}_1 \circ \tilde{R}_2) \circ \tilde{R}_3 = \tilde{R}_1 \circ (\tilde{R}_2 \circ \tilde{R}_3)$$
2. **Reflexivity:** Let $\tilde{R}$ be a fuzzy relation on $X \times X$, then

(i) $\tilde{R}$ is said to be reflexive if

$$\mu_{\tilde{R}}(x,x) = 1, \qquad \forall\, x \in X$$

(ii) $\tilde{R}$ is said to be $\in$ reflexive if

$$\mu_{\tilde{R}}(x,x) \geq \in, \qquad \forall\, x \in X$$

(iii) $\tilde{R}$ is said to be weekly reflexive if

$$\left.\begin{matrix}\mu_{\tilde{R}}(x,y) \leq \mu_{\tilde{R}}(x,x) \\ \mu_{\tilde{R}}(y,x) \leq \mu_{\tilde{R}}(x,x)\end{matrix}\right\}, \qquad \forall\, x,y \in X$$

3. **Symmetry:**

(i) A fuzzy relation $\tilde{R}$ is called symmetric if

$$\tilde{R}(x,y) = \tilde{R}(y,x)$$

(ii) A relation $\tilde{R}$ is called antisymmetric if for

$$\left.\begin{matrix}x \neq y, either\ \mu_{\tilde{R}}(x,y) \neq \mu_{\tilde{R}}(y,x) \\ or\ \mu_{\tilde{R}}(x,y) = \mu_{\tilde{R}}(y,x) = 0\end{matrix}\right\}, \qquad \forall\, x,y \in X$$

(iii) A relation $\tilde{R}$ is called perfectly antisymmetric if for $x \neq y$, wherever

$$\mu_{\tilde{R}}(x,y) > 0, \text{ then } \mu_{\tilde{R}}(y,x) = 0, \ \forall\, x,y \in X.$$

4. If $\tilde{R}_1$ is reflexive and $\tilde{R}_2$ is an arbitrary fuzzy relation from $\tilde{R}$, then

$$\tilde{R}_2 \subset \tilde{R}_1 \circ \tilde{R}_2 \ \text{ and } \ \tilde{R}_2 \subset \tilde{R}_2 \circ \tilde{R}_1$$

5. If $\tilde{R}$ is reflexive, then $\tilde{R} \subset \tilde{R} \circ \tilde{R}$.
6. If $\tilde{R}_1$ and $\tilde{R}_2$ are symmetric, then $\tilde{R}_1 \circ \tilde{R}_2$ is symmetric if $\tilde{R}_1 \circ \tilde{R}_2 = \tilde{R}_2 \circ \tilde{R}_1$.
7. If $\tilde{R}$ is symmetric, then each power of $\tilde{R}$ is symmetric.
8. **Transitive:** A fuzzy relation $\tilde{R}$ is called max-min transitive if $\tilde{R} \circ \tilde{R} \subseteq \tilde{R}$.
9. If $\tilde{R}$ is symmetric and transitive, then $\mu_{\tilde{R}}(x,y) \leq \mu_{\tilde{R}}(y,x)$ for all $x,y \in X$.
10. If $\tilde{R}$ is reflexive and transitive, then $\tilde{R} \circ \tilde{R} = \tilde{R}$.
11. If $\tilde{R}_1$ and $\tilde{R}_2$ are transitive and $\tilde{R}_1 \circ \tilde{R}_2 = \tilde{R}_2 \circ \tilde{R}_1$, then $\tilde{R}_1 \circ \tilde{R}_2$ is transitive.

4.6 BINARY RELATION ON A SINGLE SET

A binary relation $\tilde{R}$ on a single set X is denoted by $\tilde{R}(X^2) \equiv R(X \times X)$ is a subset of $X \times X$ or X^2. These types of relations are used in directed graphs.

Transitive Closure:

The transitive closure of a fuzzy relation is the generalization that the elements of the transitive closure have the smallest possible membership grades that still allow the first two components to be met. The transitive closure $\tilde{R}_T(X,X)$ of a relation $\tilde{R}(X,X)$ is determined by simple algorithm that consists of the following three steps:

Step-(i): $\tilde{R}' = \tilde{R} \cup (\tilde{R} \circ \tilde{R})$

Step-(ii): If $\tilde{R}' \neq \tilde{R}$, make $\tilde{R} = \tilde{R}'$ and go to step (i).

Step-(iii): Stop, $\tilde{R}' = \tilde{R}_T$

The above algorithm is applicable to both crisp and fuzzy relations. However, the composition in step (i) must be compatible with definition of transitivity.

Equivalence Relation: A fuzzy binary relation, i.e., reflexive, symmetric and transitive is known as a fuzzy equivalence relation.

CHECK YOUR PROGRESS

Q.7: Let $\tilde{R}_1 = \begin{bmatrix} 1 & 0 & .7 \\ .3 & .2 & 0 \\ 0 & .5 & 1 \end{bmatrix}$, $\tilde{R}_2 = \begin{bmatrix} .6 & .6 & 0 \\ 0 & .6 & .1 \\ 0 & .1 & 0 \end{bmatrix}$, $\tilde{R}_3 = \begin{bmatrix} 1 & 0 & .7 \\ 0 & 1 & 0 \\ .7 & 0 & 1 \end{bmatrix}$

(i) Find the domain, range, height in above relations $\tilde{R}_1,\ \tilde{R}_2,\ \tilde{R}_3$.

(ii) Find inverses of $\tilde{R}_1,\ \tilde{R}_2,\ \tilde{R}_3$.

(iii) Find $\tilde{R}_1 \circ \tilde{R}_2,\ \tilde{R}_2 \circ \tilde{R}_3,\ \tilde{R}_1 \circ \tilde{R}_3$.

Q.8: Show that a fuzzy relation $\tilde{R}$ is called max-min transitive if $\tilde{R} \circ \tilde{R} \subseteq \tilde{R}$.

Q.9: If $\tilde{R}$ is reflexive, then show that $\tilde{R} \subset \tilde{R} \circ \tilde{R}$.

Q.10: If $\tilde{R}$ is symmetric, then show that each power of $\tilde{R}$ is symmetric.

4.7 COMPATIBILITY RELATION

A binary relation $\tilde{R}(X,Y)$ which is reflexive and symmetric is called a compatible or tolerance relation.

If $\tilde{R}$ is a fuzzy compatible relation, then compatibility classes are defined in terms of a specified membership degree α. An α-compatible class is a set-$\tilde{A}$ of X such that $\tilde{R}(x,y) \geq \alpha,\ \forall\, x,y \in \tilde{A}$.

For example: Consider a fuzzy relation $\tilde{R}(X,X)$ defined on $X = \{x_1, x_2, x_3\}$ by the following membership matrix

$$\begin{bmatrix} 1 & 0.7 & 0.5 \\ 0.7 & 1 & 0.6 \\ 0.5 & 0.6 & 1 \end{bmatrix}$$

Since all the entries on the diagonal are 1, so the relation is reflexive.
Since the matrix is symmetric, so the relation is symmetric also.
So, the relation $\tilde{R}$ is a compatible relation.

4.8 FUZZY ORDERING RELATION

A crisp binary relation $R(X,X)$ which is reflexive, antisymmetric and transitive is called a partial ordering relation. It is denoted by the symbol '$\leq$'. Thus $x \leq y$ denotes

$$< x,y > \in \tilde{R}$$

And signifies that x precedes y.

Some fundamental concepts associated with partial ordering

(i) If $x \in X$ and $x \leq y$ (x is fixed and y is varying) $\forall\, y \in X$, then x is called the first member (or minimum of X) w.r.t. the relation '$\leq$'.

(ii) If $x \in X$ and $y \leq x\ \ \forall\, y \in X$, then x is called the last (or maximum) element of X w.r.t. the relation '$\leq$'.

(iii) If $x \in X$ and $y \leq x \Rightarrow x = y$, then x is called a maximal member of X w.r.t. the relation.

Theorem: For any fuzzy relation R on X^2, the fuzzy relation

$$R_{T(i)} = \bigcup_{n=1}^{\infty} R^{(n)}$$

Is the smallest i-transitive closure of R.

Proof:

$$R_{T(i)} \circ^i R_{T(i)} = \left(\bigcup_{n=1}^{\infty} R^n\right) \circ^i \left(\bigcup_{m=1}^{\infty} R^m\right)$$

$$= \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \left(R^{(n)} \circ^i R^{(m)}\right)$$

$$= \bigcup_{n,m=1}^{\infty} R^{(n+m)} \subseteq \bigcup_{k=1}^{\infty} R^{(k)}$$

$$= R_{T(i)}$$

which means that $R_{T(i)}$ is i-transitive.

Consider now a fuzzy relation S that is i-transitive and contain $R (R \subseteq S)$, then

$$R^{(2)} = R \circ^i R \subseteq S \circ^i S \subseteq S$$

and moreover, if

$$R^{(n)} \subseteq S, \text{ then } R^{(n+1)} = R \circ^i R^{(n)} \subseteq S \circ^i S \subseteq S$$

Hence, $R^{(k)} \subseteq S$, for any $k \in N$ and therefore

$$R_{T(i)} = \bigcup_{k=1}^{\infty} R^{(k)} \subseteq S$$

i.e., $R_{T(i)}$ is the smallest i-transitive fuzzy relation containing R.

Theorem: Let R be a reflexive fuzzy relation on X^2, where $|X| = n \geq 2$, then

$$R_{T(i)} = R^{(n-1)}$$

Proof: Since R is reflexive, $E \subseteq R$ and

$$R = E \circ^i R \subseteq R \circ^i R = R^{(2)}$$

Thus

$$R^{(m)} \subseteq R^{(m+1)} \quad \text{for any } m \in N$$

Now, we prove that $R^{(n-1)} = R^{(n)}$ for any $x, y \in X$ if $x = y$

then $R^{(n)}(x,y) = \sup i[R(x,z_1), R(z_1,z_2), \dots, R(z_{n-1},y)]$

Since $|x| = n$, then sequence $x = z_0, z_1, \dots, z_{n-1}, z_n = y$ of $n+1$ elements must contain atleast two equal elements. Assume $z_r = z_s$, where $r < s$, then

$$\begin{aligned} & i[R(x,z_1), \dots, R(z_{r-1},z_r), \dots, R(z_s,z_{s+1}), \dots, R(z_{n-1},y)] \\ & \leq i[R(x,z_1), \dots, R(z_{r-1},z_r), R(z_s,z_{s+1}), \dots, R(z_{n-1},y)] \\ & \leq R^{(k)}(x,y) \qquad (k \leq n-1) \\ & \leq R^{(n-1)}(x,y) \end{aligned}$$

Hence $R^{(n)}(x,y) \leq R^{(n-1)}(x,y)$ for any $x, y \in X$

$$\Rightarrow R^{(n)} \subseteq R^{(n-1)}$$

$$\Rightarrow R_{T(i)} = R^{(n-1)}$$

4.9 LET US SUM UP

- Fuzzy relations generalize crisp relations by assigning a degree of membership to each pair of elements, and these graded connections are expressed using membership functions or relation matrices.
- The cylindrical projection represents the largest possible fuzzy relation that can be formed on the extended Cartesian product while still preserving the membership values dictated by the given projection.
- The unit describes how fuzzy relations are combined through compositions based on t-norms and t-conorms, highlighting that while max–min composition is associative, it does not satisfy commutativity.
- A fuzzy binary relation that is reflexive, symmetric and transitive is called fuzzy equivalence relation.
- A binary relation $\tilde{R}(X,Y)$, that is reflexive and symmetric is called a compatible relation
- A crisp binary relation $R(X,X)$ which is reflexive, antisymmetric and transitive is called a partial ordering relation. It is denoted by the symbol '≤'. Thus $x \leq y$ denotes

  $$< x, y > \in \tilde{R}$$

  and signifies that x precedes y.

4.10 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q 1. Do it yourself.

Ans to Q 2. A fuzzy relation is a fuzzy subset of a Cartesian product of crisp sets. If R is a fuzzy relation on $X \times Y$, then its projections describe how R induces fuzzy subsets on the individual sets X and Y.

Ans to Q 3. Do it yourself.

Ans to Q 4. For x_1

$$\mu_{\tilde{R}^{(1)}}(x_1) = \max\{0.8, 0.4\} = 0.8$$

For x_2

$$\mu_{\tilde{R}^{(1)}}(x_2) = \max\{0.6, 0.2\} = 0.6$$

Therefore

$$\tilde{R}^{(1)} = \{x_1/0.8, x_2/0.6\}$$

Ans to Q 5. Do it yourself.

Ans to Q 6. Do it yourself.

Ans to Q 7. (i) Domain of $\tilde{R}_1 = \{1, .3, 1\}$

Range of $\tilde{R}_1 = \{1, .5, 1\}$

Height of $\tilde{R}_1 = 1$

Do it yourself for $\tilde{R}_2$ and $\tilde{R}_3$.

(ii) $\tilde{R}_1^{-1} = \begin{bmatrix} 1 & .3 & 0 \\ 0 & .2 & .5 \\ .7 & 0 & 1 \end{bmatrix}$

Do it yourself for $\tilde{R}_2$ and $\tilde{R}_3$.

(iii) $\tilde{R}_1 \circ \tilde{R}_2 = \begin{bmatrix} .6 & .6 & 0 \\ .3 & .3 & .1 \\ 0 & .5 & .1 \end{bmatrix}$

Do it yourself for $\tilde{R}_2 \circ \tilde{R}_3,\ \tilde{R}_1 \circ \tilde{R}_3$.

Ans to Q 8. Let

$$R = \begin{bmatrix} .2 & 1 & .4 \\ 0 & .6 & .3 \\ 0 & 1 & .3 \end{bmatrix}$$

Now, firstly we shall prove that R is transitive.

$$R \circ R = \begin{bmatrix} .2 & 1 & .4 \\ 0 & .6 & .3 \\ 0 & 1 & .3 \end{bmatrix} \circ \begin{bmatrix} .2 & 1 & .4 \\ 0 & .6 & .3 \\ 0 & 1 & .3 \end{bmatrix}$$

$$= \begin{bmatrix} \max(.2, 0, 0) & \max(.2, .6, .4) & \max(0, .3, .3) \\ \max(0, 0, 0) & \max(0, .6, .3) & \max(0, .3, .3) \\ \max(0, 0, 0) & \max(0, .6, .3) & \max(0, .3, .3) \end{bmatrix}$$

$$R \circ R = \begin{bmatrix} .2 & .6 & .3 \\ 0 & .6 & .3 \\ 0 & .6 & .3 \end{bmatrix} \qquad (1)$$

$$(R \circ R) \cup R = \begin{bmatrix} .2 & .6 & .3 \\ 0 & .6 & .3 \\ 0 & .6 & .3 \end{bmatrix} \circ \begin{bmatrix} .2 & 1 & .4 \\ 0 & .6 & .3 \\ 0 & 1 & .3 \end{bmatrix} = \begin{bmatrix} .2 & 1 & .4 \\ 0 & .6 & .3 \\ 0 & 1 & .3 \end{bmatrix} = R'$$

Here,

$$(R \circ R) \cup R = R'$$

Thus, $R' = R_T$, the closure membership matrix.
So R is transitive.
Now, we shall prove that $\tilde{R} \circ \tilde{R} \subseteq \tilde{R}$

$$\mu_{\tilde{R} \circ \tilde{R}}(x) \leq \mu_{\tilde{R}}(x) \qquad \text{(by (1))}$$

So,

$$\tilde{R} \circ \tilde{R} \subseteq \tilde{R}$$

Hence proved.

Ans to Q 9. Do it yourself.

Ans to Q 10. Let

$$\tilde{R} = \begin{bmatrix} .7 & .5 & .3 \\ .5 & .6 & .8 \\ .3 & .8 & .4 \end{bmatrix}$$

Here, $\tilde{R}$ is symmetric.
Now, we have to show that each power of $\tilde{R}$ is also symmetric.

$$\tilde{R} \circ \tilde{R} = \begin{bmatrix} .7 & .5 & .3. \\ .5 & .6 & .8 \\ .3 & .8 & .4 \end{bmatrix} \circ \begin{bmatrix} .7 & .5 & .3. \\ .5 & .6 & .8 \\ .3 & .8 & .4 \end{bmatrix}$$

$$= \begin{bmatrix} \max(.7, .5, .3) & \max(.5, .5, .3) & \max(.3, .5, .3) \\ \max(.5, .5, .3) & \max(.5, .6, .8) & \max(.3, .6, .4) \\ \max(.3, .5, .3) & \max(.3, .6, .4) & \max(.3, .8, .4) \end{bmatrix}$$

$$\tilde{R} \circ \tilde{R} = \begin{bmatrix} .7 & .5 & .5 \\ .5 & .8 & .6 \\ .5 & .6 & .8 \end{bmatrix}$$

Here, we observe that $\tilde{R} \circ \tilde{R}$ is symmetric.
Now,

$$\left(\tilde{R} \circ \tilde{R}\right) \circ \tilde{R} = \begin{bmatrix} .7 & .5 & .5 \\ .5 & .8 & .6 \\ .5 & .6 & .8 \end{bmatrix} \circ \begin{bmatrix} .7 & .5 & .3 \\ .5 & .6 & .8 \\ .3 & .8 & .4 \end{bmatrix}$$

$$= \begin{bmatrix} \max(.7, .5, .3) & \max(.5, .5, .5) & \max(.3, .5, .4) \\ \max(.5, .5, .3) & \max(.5, .6, .6) & \max(.3, .8, .4) \\ \max(.5, .5, .3) & \max(.5, .6, .8) & \max(.3, .6, .4) \end{bmatrix}$$

$$= \begin{bmatrix} .7 & .5 & .5 \\ .5 & .6 & .8 \\ .5 & .8 & .6 \end{bmatrix}$$

Thus, we observe that $\left(\tilde{R} \circ \tilde{R}\right) \circ \tilde{R}$ is also symmetric.
Hence we can say that if $\tilde{R}$ is symmetric, then each power of $\tilde{R}$ is symmetric.

4.11 FURTHER READINGS

1. Mohan, C. An Introduction to Fuzzy Set Theory and Fuzzy Logic. Anshan Publishers, 2015.
2. Lee, K. H. First Course on Fuzzy Theory and Applications. Springer International Edition, 2005.
3. Yen, J., Langari, R. Fuzzy Logic - Intelligence, Control and Information Pearson Education, 1999.
4. Zimmerman, H.J. Fuzzy Set Theory and its Applications. Allied Publishers Ltd., New Delhi, 1991.

4.12 MODEL QUESTIONS

Q 1. Let $X = \{x_1, x_2\}, Y = \{y_1, y_2, y_3, y_4\}$ and

$R =$

	y_1	y_2	y_3	y_4
x_1	.1	.2	.4	.8
x_2	.4	.8	1	.2

be a fuzzy relation on $X \times Y$. Find cylindrical extension of second projection $\tilde{R}^{(2)}$ of R.

Q 2. Let $\tilde{R}_1 = \begin{bmatrix} 1 & .2 & 0 & 0 \\ 0 & 0 & .4 & .3 \\ 1 & .2 & 0 & 0 \\ 0 & 0 & .4 & .3 \end{bmatrix}$, $\tilde{R}_2 = \begin{bmatrix} .3 & .6 & 0 & 1 \\ .7 & 0 & 1 & .5 \\ .5 & 0 & 0 & .2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

(i) Find the domain, range and height of the above relations.

(ii) Find inverse of $\tilde{R}_1$ and $\tilde{R}_2$.

(iii) Find $\tilde{R}_1 \circ \tilde{R}_2$

Q 3. Let $\tilde{R}(x, x)$ be a fuzzy relation defined on X, defined by membership matrix

$$\tilde{R} = \begin{bmatrix} .7 & .5 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & .4 & 0 & 0 \\ 0 & 0 & .8 & 0 \end{bmatrix}$$

then find $\tilde{R} \circ \tilde{R}$.

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UNIT 5: FUZZY GRAPH

UNIT STRUCTURE

5.1 Learning Objectives
5.2 Introduction
5.3 Fuzzy Graphs
5.4 Types of Fuzzy Graphs
5.5 Fuzzy Morphisms
5.6 Isomorphism of Fuzzy Graphs
5.7 Let Us Sum Up
5.8 Answers to Check Your Progress
5.9 Further Reading
5.10 Model Questions

5.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- Define fuzzy graphs and distinguish them from crisp graphs using membership functions.
- Identify and explain types of fuzzy graphs, such as complete, bipartite, strong, and weak fuzzy graphs.
- Understand fuzzy morphisms, including homomorphisms and isomorphisms between fuzzy graphs.
- Describe operations on fuzzy graphs, like union, intersection, and join.
- Represent fuzzy graphs using matrices and compute basic properties.

5.2 INTRODUCTION

Fuzzy graphs extend classical graph theory to handle uncertainty in relationships between objects. In standard graphs, edges either exist or not, but fuzzy graphs assign degrees of membership to vertices and edges, ranging from 0 to 1. This allows modeling real-world situations where connections are not absolute, such as social networks with varying friendship strengths or traffic networks with partial connectivity. The concept originates from fuzzy set theory, introduced by Lotfi Zadeh, and was adapted to graphs by Azriel Rosenfeld. A fuzzy graph consists of a vertex set with membership function $\sigma: V \rightarrow$ and an edge set with membership function $\mu: V \times V \rightarrow$, where $\mu(x, y) \leq min(\sigma(x), \sigma(y))$ for all x, y in V. This condition ensures compatibility between vertex and edge strengths. Fuzzy graphs find applications in decision-making, optimization, and clustering under imprecise data.

Consider a simple example with vertices representing cities and edges showing road connectivity degrees. If $\sigma(A) = 0.8, \sigma(B) = 0.9$, and $\mu(A, B) = 0.7$, the road between A and B has high but not full strength. Visual representations use lines with thickness

or labels indicating membership values. For instance, a graph with three vertices u, v, w where $\sigma(u) = 1, \sigma(v) = 0.8, \sigma(w) = 0.6, \mu(u,v) = 0.8, \mu(v,w) = 0.5, \mu(u,w) = 0$ can be drawn with solid line for $u - v$, dashed for $v - w$, and no line for $u - w$.
Fuzzy graphs generalize crisp graphs, where all memberships are 0 or 1. They enable analysis of partial truths, making them suitable for artificial intelligence and operations research. Basic properties include the degree of a vertex $d(v) = \sum_{u \neq v} \mu(u,v)$, order as $\sum \sigma(v)$, and size as $\sum \mu(e)$. Strong edges have $\mu(e) = min(\sigma(u), \sigma(v))$, forming the underlying crisp graph.

5.3 FUZZY GRAPHS

A fuzzy graph $G = (\sigma, \mu)$ on a crisp graph $H = (V, E)$ has σ assigning membership to vertices and μ to possible edges, satisfying the compatibility condition $\mu(x,y) \leq min(\sigma(x), \sigma(y))$. If $\mu(x,y) > 0$, the edge is effective. The level graph G_t consists of vertices with $\sigma(v) \geq t$ and edges with $\mu(e) \geq t$, for t in. This creates a family of crisp graphs as t varies.
We know that any relation between two sets X and Y is called binary relation, denoted by $R(X,Y)$.
When $X \neq Y$, then binary relation $R(X,Y)$ is called bipartite graph and when $X = Y$, then binary relation $R(X,Y)$ is called directed graph or digraph.
If each of sets X, Y is represented by a set of nodes in the diagram, then nodes corresponding to one set are different from nodes representing the other set. Also elements of $X \times Y$ with non-zero membership grades in $R(X,Y)$ are represented by the lines connecting the respective nodes, such lines are labelled with the value of the membership grade $\mu_R(x,y)$. Such type of graph is called sagettal diagrams.
$\tilde{H}(x_i, x_j)$ is a fuzzy subgraph of a graph $\tilde{G}(x_i, x_j)$ if

$$\mu_{\tilde{H}}(x_i, x_j) \subseteq \mu_{\tilde{G}}(x_i, x_j), \qquad \forall\, x_i, x_j \in N \times N$$

$\tilde{H}(x_i, x_j)$ spans the graph $\tilde{G}(x_i, x_j)$ if the node sets of $\tilde{H}(x_i, x_j)$ and $\tilde{G}(x_i, x_j)$ are equal. i.e., if they differ only by their arc weight.
Fuzzy subgraphs arise when both $\sigma' \leq \sigma$ and $\mu' \leq \mu$ pointwise. A subgraph is strong if $\mu'(x,y) = min(\sigma'(x), \sigma'(y))$ for all effective edges.
In applications, fuzzy graphs model communication networks where signal strength varies. Path strength helps find reliable routes. Fuzzy cycles and trees are defined similarly, with cycles having non-zero strength closed paths.

CHECK YOUR PROGRESS

Q 1. Define a fuzzy graph and state its compatibility condition.

Q 2. What is a strong edge in a fuzzy graph?

Q 3. Give an example of a fuzzy subgraph.

5.4 TYPES OF FUZZY GRAPHS

Fuzzy graphs are classified based on structure and properties.
A **fuzzy simple graph** has no loops or multiple edges. Loops occur if $\mu(v,v) > 0$, pseudo-graphs allow multiples.
Regular fuzzy graphs have constant degree $d(v) = k$ for all v with $\sigma(v) > 0$.
Complete fuzzy graphs connect every pair with $\mu(x,y) = min(\sigma(x), \sigma(y))$.
Bipartite fuzzy graphs partition vertices into U, W with edges only between them, $\mu(u,w) \leq min(\sigma(u), \sigma(w))$.

5.5 FUZZY MORPHISMS

Let $\tilde{R}(X,X)$ and $\tilde{Q}(Y,Y)$ be two crisp binary relations defined on sets X and Y respectively, then a function $f: X \to Y$ is said to be homomorphism from $(X, \tilde{R})$ to $(Y, \tilde{Q})$ if

$$\langle x_1, x_2 \rangle \in \tilde{R} \text{ implies } \langle f(x_1), f(x_2) \rangle \in \tilde{Q}$$

A homomorphism implies that for every two elements of set X which are related under the relation $\tilde{R}$, their homomorphic images $f(x_1), f(x_2)$ in Y is related under the relation $\tilde{Q}$.
A homomorphism from (X, R_1) to (Y, R_2) is said to be strong homomorphism if it satisfies the following implications:

$$(x_1, x_2) \in R_1 \Rightarrow \big(h(x_1), h(x_2)\big) \in R_2 \ \ \forall\, x_1, x_2 \in X$$

and

$$(y_1, y_2) \in R_2 \Rightarrow (x_1, x_2) \in X \ \ \forall\, y_1, y_2 \in Y$$

where $x_1 \in h^{-1}(y_1)$ and $x_2 \in h^{-1}(y_2)$.

If $Y \subset X$ then a homomorphism $f: X \to Y$ is called endomorphism.
A homomorphism $f: (X, R_1) \to (X, R_2)$ is called an automorphism.

5.6 ISOMORPHISM OF FUZZY GRAPHS

If $f: X \to Y$ be a homomorphism from $(X, \tilde{R})$ to $(Y, \tilde{Q})$ and if f is completely specified one-to-one and onto then it is called an isomorphism.
Example 1: Let $G_1 = (V_1, \sigma_1, \mu_1)$ and $G_2 = (V_2, \sigma_2, \mu_2)$ be two fuzzy graphs, where σ denotes the fuzzy vertex membership function and μ denotes the fuzzy edge membership function.
Let $V_1 = \{a, b\}$, $V_2 = \{x, y\}$
Define vertex memberships:

$$\sigma_1(a) = 0.8, \sigma_1(b) = 0.6$$
$$\sigma_2(x) = 0.9, \sigma_2(y) = 0.7$$

Edge memberships:

$$\mu_1(a,b) = 0.5, \mu_2(x,y) = 0.6$$

Define a mapping

$$f: V_1 \to V_2 \text{ by } f(a) = x, f(b) = y$$

Verification:

$$\sigma_1(v) \le \sigma_2\big(f(v)\big), \qquad \forall\, v \in V_1$$
$$\mu_1(u,v) \le \mu_2\big(f(u), f(v)\big), \qquad \forall\, u, v \in V_1$$

Both the conditions are satisfied as

$$0.8 \le 0.9, 0.6 \le 0.7, 0.5 \le 0.6$$

Hence, f is a fuzzy graph morphism.

Example 2: Let $G = (\sigma, \mu)$ be a fuzzy graph on $S = \{a, b\}$ with $\sigma(a) = 0.7, \sigma(b) = 0.9, \mu(a,b) = 0.6$. Let $G' = (\sigma', \mu')$ be a fuzzy graph on the same set $S' = \{a, b\}$ with $\sigma'(a) = 0.8, \sigma'(b) = 0.9, \mu'(a,b) = 0.7$ Define $h: S \to S'$ by $h(a) = a,\ h(b) = b$

Then

$$\sigma(a) = 0.7 \le 0.8 = \sigma'\big(h(a)\big)$$
$$\sigma(b) = 0.9 \le 0.9 = \sigma'\big(h(b)\big)$$
$$\mu(a,b) = 0.6 \le 0.7 = \mu'\big(h(a), h(b)\big)$$

So $h: G \to G'$ is a fuzzy graph morphism.

Example 3: Let $G = (\sigma, \mu)$ on $S = \{x, y, z\}$ with $\sigma(x) = 0.5, \sigma(y) = 0.6, \sigma(z) = 0.4, \mu(x,y) = 0.4, \mu(y,z) = 0.3, \mu(x,z) = 0.$

Let $G' = (\sigma', \mu')$ on $S' = \{u, v\}$ with $\sigma'(u) = 0.7, \sigma'(v) = 0.6, \mu'(u,v) = 0.5$

Define $h: S \to S'$ by $h(x) = u,\ h(y) = v,\ h(z) = v.$

Check the conditions:

Vertices: $\sigma(x) = 0.5 \le 0.7 = \sigma'(u),\ \sigma(y) = 0.6 \le 0.6 = \sigma'(v),\ \sigma(z) = 0.4 \le 0.6 = \sigma'(v).$

Edges: $\mu(x,y) = 0.4 \le 0.5 = \mu'(u,v)$

$$\mu(y,z) = 0.3 \le \mu'(v,v)$$

set $\mu'(v,v)$ at least 0.3 (e.g. 0.3) to satisfy this.

$$\mu(x,z) = 0 \le \mu'(u,v)$$

With $\mu'(v,v) \ge 0.3$, all inequalities hold, hence h is a fuzzy graph morphism.

Example 4: Consider two fuzzy graphs each having three vertices.

Let

$$G_1 = \{V_1, \sigma_1, \mu_1\}$$

$$V_1 = \{a, b, c\}$$

$$\sigma_1(a) = 0.7, \sigma_1(b) = 0.6, \sigma_1(c) = 0.7$$

$$\mu_1(a,b) = 0.4, \mu_1(b,c) = 0.5, \mu_1(c,a) = 0.4$$

$$G_2 = \{V_2, \sigma_2, \mu_2\}$$

$$V_2 = \{x, y, z\}$$

$$\sigma_2(a) = 0.6, \sigma_2(b) = 0.7, \sigma_2(c) = 0.7$$

$$\mu_2(x, y) = 0.5, \mu_2(y, z) = 0.4, \mu_2(z, x) = 0.4$$

Define a bijection

$$f(a) = y, f(b) = x, f(c) = z$$

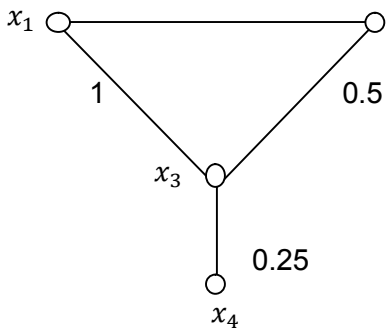
Since vertex and edge membership grades are preserved under f

Thus

$$G_1 \cong G_2$$

CHECK YOUR PROGRESS

Q 4. Consider the following fuzzy graph 1 x_2



(i) Given an example of a spanning graph $\tilde{G}$.

(ii) Give all paths from x_1 to x_4 and determine their μ lentgh.

(iii) Is the graph a forest or a tree? If not make it.

Q 5. Let

$\tilde{G}(x_i, x_j) =$

$\{[(x_1, x_2), .3], [(x_1, x_3), .6], [(x_1, x_1), 1], [(x_2, x_1), .4], [(x_3, x_1), .2], [(x_3, x_2), .5], [(x_4, x_3), .8]\}$

Determine the graph $\tilde{G}$ and its spanning graph.

Q 6. Let $G_1 = (V_1, \sigma_1, \mu_1)$ and $G_2 = (V_2, \sigma_2, \mu_2)$ be two fuzzy graphs and $V_1 = \{a, b\}$, $V_2 = \{x, y\}$ be vertices with

$$\sigma_1(a) = 0.9, \sigma_1(b) = 0.7$$
$$\sigma_2(x) = 0.8, \sigma_2(y) = 0.6$$
$$\mu_1(a, b) = 0.7, \mu_2(x, y) = 0.8$$

Define a mapping

$f: V_1 \to V_2$ by $f(a) = y, f(b) = x$

Verify f is a fuzzy graph morphism.

Q 7. Consider two fuzzy graphs $G_1 = \{V_1, \sigma_1, \mu_1\}$ and $G_2 = \{V_2, \sigma_2, \mu_2\}$ and $V_1 = \{a, b, c\}$, $V_2 = \{x, y, z\}$ with

$$\sigma_1(a) = 0.8, \sigma_1(b) = 0.7, \sigma_1(c) = 0.8$$

$$\mu_1(a, b) = 0.5, \mu_1(b, c) = 0.6, \mu_1(c, a) = 0.5$$

$$\sigma_2(a) = 0.7, \sigma_2(b) = 0.8, \sigma_2(c) = 0.8$$

$$\mu_2(x, y) = 0.6, \mu_2(y, z) = 0.5, \mu_2(z, x) = 0.5$$

Define a bijection $f: V_1 \to V_2$ by

$$f(a) = z, f(b) = x, f(c) = y$$

Verify f is a fuzzy graph isomorphism.

5.7 LET US SUM UP

- This unit introduces the concept of fuzzy graphs as an extension of classical graph theory, incorporating degrees of membership to represent uncertainty and imprecision in vertices and edges.
- It explains the basic definitions and terminology of fuzzy graphs, including fuzzy vertices, fuzzy edges, fuzzy sub graphs, order, size, degree, and strength of edges.
- It discusses fuzzy morphisms, including homomorphisms and isomorphisms between fuzzy graphs
- Various operations on fuzzy graphs—union, intersection, complement, and Cartesian product are presented with emphasis on their structural behaviour.

5.8 ANSWERS TO CHECK YOUR PROGRESS

Answer to Q 1. Let V be a non-empty finite set of vertices. A fuzzy graph G is an ordered pair $G = (\sigma, \mu)$, where σ and μ are fuzzy vertex membership and fuzzy edge membership functions respectively such that μ is symmetric i.e., $\mu(u, v) = \mu(v, u),\ \forall\, u, v \in V$.

The fuzzy edge membership function μ is said to be compatible with the fuzzy vertex membership function σ if for all $u, v \in V$,

$$\mu(u, v) \leq \min\{\sigma(u), \sigma(v)\}$$

Answer to Q 2. Let $G = (\sigma, \mu)$ be a fuzzy graph on a vertex set V. An edge (u, v) in G is called a strong edge if $\mu(u, v) = \min\{\sigma(u), \sigma(v)\}$.

Answer to Q 3. Let $G = (V, \sigma, \mu)$ be a fuzzy graph and $V = \{v_1, v_2, v_3\}$ with

$$\sigma(v_1) = 0.8, \sigma(v_2) = .6, \sigma(v_3) = 0.5$$

$$\mu(v_1, v_2) = 0.5, \mu(v_2, v_3) = 0.4, \mu(v_3, v_1) = 0.3$$

and $V' = \{v_1, v_2\} \subset V$ with $\sigma'(v_1) = 0.6, \sigma'(v_2) = 0.5, \mu'(v_1, v_2) = 0.4$

Since $\sigma'(v_i) \leq \sigma(v_i), \ \ i = 1, 2$

and $\mu'(v_1, v_2) \leq \mu(v_1, v_2)$.

Hence the fuzzy graph $G' = (V', \sigma', \mu')$ is a fuzzy subgraph of $G = (V, \sigma, \mu)$.

Answer to Q 4. (i) It is a spanning graph

$\tilde{H}(x_i, x_j)$, where $\mu_{\tilde{H}}(x_i, x_j) \subseteq \mu_{\tilde{G}}(x_i, x_j)$

μ length $\Rightarrow$ the μ distance or length $d(x_i, x_j)$ between two nodes (x_i, x_j) is the smallest μ-length of any path from x_i to x_j, $\forall \ x_i, x_j \in \tilde{G}$.

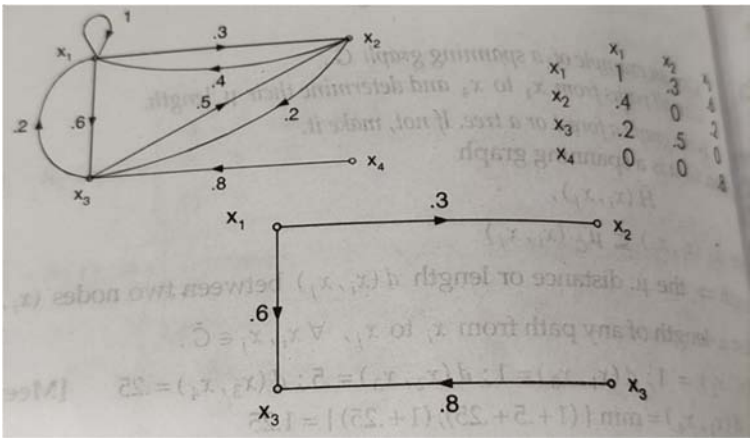
(ii) $d(x_1, x_2) = 1; d(x_1, x_3) = 1; d(x_2, x_3) = .5; d(x_3, x_4) = .25$

$$d(x_1, x_4) = \min\{(1 + .5 + .25), (1 + .25)\} = 1.25.$$

(iii) It is neither a forest nor a tree.

There is a closed loop, degree of $x_4 = 1$, degree of $x_3 = 3$, i.e., here we see that two edges are of odd degree. So, it is not a tree.

Answer to Q 5.



$$\tilde{H}(x_i, x_j) = \{[(x_1, x_2), .3], [(x_1, x_3), .6], [(x_3, x_4), .8]\}$$

Answer to Q 6. Do it yourself.

Answer to Q 7. Do it yourself.

5.9 FURTHER READINGS

1. Mohan, C. An Introduction to Fuzzy Set Theory and Fuzzy Logic. Anshan Publishers, 2015.
2. Lee, K. H. First Course on Fuzzy Theory and Applications. Springer International Edition, 2005.
3. Yen, J., Langari, R. Fuzzy Logic - Intelligence, Control and Information Pearson Education, 1999.
4. Zimmerman, H.J. Fuzzy Set Theory and its Applications. Allied Publishers Ltd., New Delhi, 1991.

5.10 MODEL QUESTIONS

1. Let $G = (V, \sigma, \mu)$ be a fuzzy graph with $V = \{a, b, c\}$, vertex memberships
$\sigma(a) = 0.7,\ \sigma(b) = 0.5,\ \sigma(c) = 0.9,$
and edge memberships
$\mu(a, b) = 0.4,\ \mu(b, c) = 0.5,\ \mu(a, c) = 0.6.$

(a) Verify whether G satisfies $\mu(x, y) \leq \min\{\sigma(x), \sigma(y)\}$ for all edges.
(b) Compute the degree of each vertex defined by
$deg(a) = \mu(a, b) + \mu(a, c), deg(b) = \mu(a, b) + \mu(b, c), deg(c) = \mu(a, c) + \mu(b, c)$

2. A fuzzy graph $G = (V, \sigma, \mu)$ has $V = \{1,2,3,4\}$ with
$\sigma(1) = 0.8,\ \sigma(2) = 0.6,\ \sigma(3) = 0.7,\ \sigma(4) = 0.5.$
The only non-zero edge memberships are
$\mu(1,2) = 0.5,\ \mu(2,3) = 0.4,\ \mu(3,4) = 0.3,\ \mu(1,3) = 0.6.$
(a) Compute the total degree $td(v) = \sigma(v) + deg(v)$ of each vertex.
(b) Find the vertex with maximum total degree.

3. Let $G_1 = (V_1, \sigma_1, \mu_1)$ and $G_2 = (V_2, \sigma_2, \mu_2)$ be fuzzy graphs with
$V_1 = \{x, y\},\ \sigma_1(x) = 0.6,\ \sigma_1(y) = 0.9,\ \mu_1(x, y) = 0.5;$
$V_2 = \{u, v\},\ \sigma_2(u) = 0.7,\ \sigma_2(v) = 0.8,\ \mu_2(u, v) = 0.6.$
Define $f: V_1 \to V_2$ by $f(x) = u,\ f(y) = v.$
(a) Check numerically whether f is a fuzzy graph morphism by verifying
$\sigma_1(p) \leq \sigma_2(f(p))$ for $p \in \{x, y\}$ and
$\mu_1(x, y) \leq \mu_2(f(x), f(y)).$
(b) If any inequality fails, adjust one membership value minimally (specify which and by how much) to make f a fuzzy graph morphism.

4. Consider two fuzzy graphs $G_1 = \{V_1, \sigma_1, \mu_1\}$ and $G_2 = \{V_2, \sigma_2, \mu_2\}$ and $V_1 = \{a, b, c\}$, $V_2 = \{x, y, z\}$ with

$$\sigma_1(a) = 0.9, \sigma_1(b) = 0.8, \sigma_1(c) = 0.9$$

$$\mu_1(a, b) = 0.6, \mu_1(b, c) = 0.7, \mu_1(c, a) = 0.6$$

$$\sigma_2(a) = 0.8, \sigma_2(b) = 0.9, \sigma_2(c) = 0.9$$

$$\mu_2(x, y) = 0.7, \mu_2(y, z) = 0.6, \mu_2(z, x) = 0.6$$

Define a bijection $f: V_1 \to V_2$ by

$$f(a) = x, f(b) = y, f(c) = z$$

Verify f is a fuzzy graph isomorphism.

5. Let $G = (V, \sigma, \mu)$, $V = \{v_1, v_2\}$ and $\sigma(v_1) = \sigma(v_2) = 0.7$, $\mu(v_1, v_2) = 0.6$. Define $f(v_1) = v_2$ and $f(v_2) = v_1$. Verify f is a fuzzy graph automorphism.

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UNIT 6: POSSIBILITY THEORY

UNIT STRUCTURE

6.1 Learning Objectives
6.2 Introduction
6.3 Fuzzy measures
6.4 Evidence theory
6.5 Probability measure
6.6 Probability and necessity measure
6.7 Possibility distributions as Fuzzy sets
6.8 Let Us Sum Up
6.9 Answers to Check Your Progress
6.10 Further Reading
6.11 Model Questions

6.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- Understand the basics of possibility,
- Understandthe concept of fuzzy measures,
- Learn about evidence theory,
- Differentiate between probability and possibility measures,
- Understand possibility and necessity measures,
- Relate possibility distribution to fuzzy.

6.2 INTRODUCTION

Possibility theory,introduced by Lofti Zadeh in 1978,is a mathematical framework, designed to handle uncertainty and imprecision in a way that complements traditional probability theory.Unlike probability theory,which relies on precise numerical values to describe the likelihood of events,possibility theory provides a more flexible approach by focusing on the degree of possibility of an event occurring.This is particularly useful in scenarios where information is incomplete,vague,or conflicting and traditional probabilistic models are not sufficient.This unit provides an overview of the key concepts in possibility theory including fuzzy measures, evidencetheory, probability measures, possibility and necessity measures, and the relationship between possibility distributions and fuzzy sets.Through these topics,we will gain a deeper understanding of how possibility theory can be applied in various fields that deal with understand or imprecise information.

6.3 FUZZY MEASURES

Consider a universal set X and a non empty family $\mathcal{C}$ of subsets of X.Then a fuzzy measure on $(X, \mathcal{C})$ is a function

$$g: \mathcal{C} \longrightarrow [0,1]$$

that satisfies the following requirements:

(1) $g(\phi) = 0$ and $g(X) = 1$

(2) If $A, B \in \mathcal{C}$ and $A \subseteq B$,then $g(A) \leq g(B)$.

(3) For any increasing sequence $A_1 \subset A_2 \subset \cdots$ in $\mathcal{C}$, if $\bigcup_{i=1}^{\infty} A_i \in \mathcal{C}$,then $\lim_{i\to\infty} g(A_i) = g(\bigcup_{i=1}^{\infty} A_i)$

(4) For any decreasing sequence $A_1 \supset A_2 \supset \cdots$ in $\mathcal{C}$, if $\bigcap_{i=1}^{\infty} A_i \in \mathcal{C}$,then $\lim_{i\to\infty} g(A_i) = g(\bigcap_{i=1}^{\infty} A_i)$

Note:

A function which satisfy the conditions (1),(2) and either (3) or (4) is called semicontinuous fuzzy measures.They are either continuous from below (if satisfy condition (3)) or continuous from above (if satisfy condition (4)).

Example :

(1) Consider a set $X = \{x_1, x_2\}$ representing two possible events.Let's assign the fuzzy measure g as follows:

$g(\phi) = 0, g(\{x_1\}) = 0.5, g(\{x_2\}) = 0.5, g(\{x_1, x_2\}) = 1$

This measure doesnot follow the traditional additive properly but captures the possibility of each event individually and in combination.

(2) Suppose D be the set of possible diagnoses of an ill patient:

$$D = \{pneumania, branchitis, emphysema, common\ cold\ \}$$

Let $g(branchitis) = 0.75$

$$g(pneumania\ \cup\ emphysema) = 0.45$$

$$g(common\ cold\ ) = 0.75$$

Hence g represents the fuzzy measure and the values indicate the degree to which the patients's symptoms match each diagnosis.This representation captures the uncertainty in the diagnosis,where multiple diagnoses are possible with varying degrees of likelihood.

Properties of Fuzzy measures:

A fuzzy measure is

(1) Additive if for any $E, F \in \mathcal{C}$ such that $E \cap F = \phi$,we have

$$g(E \cup F) = g(E) + g(F)$$

(2) Supermodular if for any $E, F \in \mathcal{C}$,we have

$$g(E \cup F) + g(E \cap F) \geq g(E) + g(F)$$

(3) Submodular if for any $E, F \in \mathcal{C}$,we have

$$g(E \cup F) + g(E \cap F) \leq g(E) + g(F)$$

(4) Superadditive if for any $E, F \in \mathcal{C}$ such that $E \cap F = \phi$,we have

$$g(E \cup F) \geq g(E) + g(F)$$

(5) Subadditive if for any $E, F \in \mathcal{C}$ such that $E \cap F = \phi$,we have

$$g(E \cup F) \leq g(E) + g(F)$$

CHECK YOUR PROGRESS 1

Q1. What is the difference between a fuzzy measure and a classical probability measure?
Q2. Let $X = \{a, b, c\}$ be a universal set. Define a fuzzy measure μ on X such that $\mu(\{a\}) = 0.4, \mu(\{b\}) = 0.3$ and $\mu(\{c\}) = 0.2$.Calculate $\mu(\{a, b\})$ if μ satisfies the monotonicity property.
Q3. If g is a fuzzy measure on a set X,and $A \subseteq X$,is it always true that $g(A) \leq g(X)$?

Do you know?

(i) For any two sets A and B,
$A \cap B \subseteq A$ and $A \cap B \subseteq A$

Therefore from the monotonicity of fuzzy measures it follows that every fuzzy measure g satisfies the inequality
$g(A \cap B) \leq min[g(A), g(B)]$ for any three sets $A, B, A \cap B \in \mathcal{C}$.

(ii) Similarly,$A \cup B \supseteq A$ and $A \cup B \supseteq B$for any two sets A and B.

Therefore every fuzzy measure g satisfies the inequality
$g(A \cup B) \geq max[g(A), g(B)]$ for any three sets $A, B, A \cup B \in \mathcal{C}$

6.4 EVIDENCE THEORY

There are special forms of fuzzy measures. A form associated with preconceived notions is called a belief measure.A form associated with information that is possible is called a plausibility measure. Evidence theory is based on these two dual non additive measures: belief measure and plausibility.

Belief measure:

Let X be a universal set.If we assume X to be finite, then a belief measure is a function $Bel: \mathcal{P}(X) \longrightarrow [0,1]$ such that

(i) $Bel(\phi) = 0, Bel(X) = 1$

(ii) $Bel(A_1 \cup A_2 \cup \cdots \cup A_n) \geq$
$\sum_j Bel(A_j) - \sum_{j<k} Bel(A_j \cap A_k) + \cdots + (-1)^{n+1} Bel(A_1 \cap A_2 \cap \cdots \cap A_n)$ (6.1)
for all possible families of subsets of X.

If X is infinite, then the Bel function is required to be continuous from above.

Note:

(i) For each $A \in \mathcal{P}(X), Bel(A)$ is interpreted as the degree of belief that a given element of X belongs to set A.

(ii) $Bel(A) + Bel(A^c) \leq 1$.

Theorem: If $A \subseteq B$,then prove that $Bel(A) \leq Bel(B)$.

Proof: Given $A \subseteq B$

Suppose $C = B - A$

$\therefore A \cup C = B$ and $A \cap C = \phi$

$\Longrightarrow Bel(A \cup C) = Bel(B)$ and $\Longrightarrow Bel(A \cap C) = Bel(\phi) = 0$

Now according to the definition of Belief measure

$$Bel(A \cup C) \geq Bel(A) + Bel(C) - Bel(A \cap C)$$
$$\Longrightarrow Bel(A \cup C) \geq Bel(A) + Bel(C)$$
$$\Longrightarrow Bel(B) \geq Bel(A) + Bel(C)$$

and consequently

$$Bel(A) \leq Bel(B).$$

Do you Know?

(i) Probability measures are special cases of belief measure for which the equality in inequality (6.1) is always satisfied.

(ii) Due to the inequality (6.1), belief measures are called superadditive.

Plausibility measure:

A plausibility measure is a function

$Pl: \mathcal{P}(X) \longrightarrow [0,1]$ such that

(i) $Pl(\phi) = 0, Pl(X) = 1$

(ii) $Pl(A_1 \cap A_2 \cap \cdots \cap A_n) \leq \sum_j Pl(A_j) - \sum_{j<k} Pl(A_j \cup A_k) + \cdots + (-1)^{n+1} Pl(A_1 \cup A_2 \cup \cdots \cup A_n)$ (6.2)

for all possible families of subsets of X,when X is finite universal set.

When X is infinite, function Pl is also required to be continuous from below.

Note:

(i) $Pl(A) + Pl(A^c) \geq 1$ for any $A \in \mathcal{P}(X)$.

(ii) Associated with each belief measure is a plausibility measures defined by the equations

$$Pl(A) = 1 - Bel(\bar{A})$$

and $Bel(A) = 1 - Pl(\bar{A})$,for all $A \in \mathcal{P}(X)$.

Do you Know?

(i) Belief measures and plausibility measures are mutually dual.

(ii) Due to inequality (6.2), plausibility measures are called subadditive.

Basic probability assignment:

A function $m: \mathcal{P}(X) \longrightarrow [0,1]$ such that

(i) $m(\phi) = 0$

(ii) $\sum_{A \in \mathcal{P}(X)} m(A) = 1$ is called a basic probability assignment.

The measurement $m(A)$ is the degree of belief that a specific element x of the universal set X belongs to the set A,but not to any specific subset of A.

Note:

(i) Basic probability assignments are not fuzzy measures.

(ii) For a given basic assignment$m$,a belief measure and a plausibility measure are uniquely determined for all set $A \in \mathcal{P}(X)$ by the formulas

$$Bel(A) = \sum_{B|B \subseteq A} m(B)$$
$$Pl(A) = \sum_{B|A \cap B \neq \phi} m(B)$$

(iii) For a given belief measure Bel,the corresponding basic probability assignment m is determined for all $A \in \mathcal{P}(X)$ by the formula

$$m(A) = \sum_{B|B\subseteq A} (-1)^{|A-B|} Bel(B)$$

(iv) If a plausibility measure is given, it can be converted to the associated belief measure and then we can apply the above formula.

(v) Each of the three functions m, Bel and Pl,is sufficient to determine the other two.

Do You Know?

(i) $m(A)$ characterize the degree of evidence or belief that the element in question belongs to the set A alone,$Bel(A)$ represents the total evidence or belief that the element belongs to A as well as to the various special subsets of A. $Pl(A)$ represents not only the total evidence or belief that the element in question belongs to the set A or to any of its subsets,but also additional evidence or belief associated with sets that overlap with A.

(ii) $. Pl(A) \geq Bel(A) \forall A \in \mathcal{P}(X)$.

Example: In an assessment of vertical bounce in a short-range jet aircraft, an expert engineer provided measures of evidence as follows:

Focal Element	Expert		
	$m(A_i)$	$Bel(A_i)$	$Pl(A_i)$
ϕ	0	0	0
O	0.4	0.4	0.8
B	0.2	0.2	0.6
$O \cup B$	0.4	1	1

Here O denotes oscillations caused by other phenomena and B denotes oscillations characteristic of the vertical bounce.

The degrees of belief and plausibility for this evidence set are calculated as follows:

Here $Bel(O) = m(O) = 0.4$

$$Bel(B) = m(B) = 0.2$$

We know that

$$Bel(A) = \sum_{B|B\subseteq A} m(B)$$

$$\therefore Bel(O \cup B) = m(O) + m(B) + m(O \cup B)$$
$$= 0.4 + 0.2 + 0.4 = 1$$

Also,$Pl(A) = \sum_{B|A\cap B\neq\phi} m(B)$

$$\therefore Pl(O) = m(O \cap O) + m\big(O \cap (O \cup B)\big) = 0.4 + 0.4 = 0.8$$
$$Pl(B) = m(B \cap B) + m\big(B \cap (O \cup B)\big) = 0.2 + 0.4 = 0.6$$

and $Pl(O \cup B) = m\big((O \cup B) \cap O\big) + m\big((O \cup B) \cap B\big) + m\big((O \cup B) \cap (O \cup B)\big)$

$$= 0.4 + 0.2 + 0.4 = 1$$

Thus, the engineer believes the evidence supporting set O(other oscillations) is atleast 0.4 and possibly as high as 0.8(Plausibility) and believes the evidence supporting set

B(vertical bounce) is atleast 0.2 and possibly as high as 0.6 (plausibility). Finally, the evidence supporting either of these sets $O \cup B$ is full or complete.

Note:

(i) A set $A \in \mathcal{P}(X)$, such that $m(A) > 0$ is called focal element of m.

(ii) If $\mathcal{F}$ is a family of all focal elements, then$(\mathcal{F}, m)$ is called body of evidence.

(iii) If $m(\mathrm{X}) = 1$ and $m(\mathrm{A}) = 0, \forall A \neq X$,it is referred as total ignorance.

(iv) A basic assignment m is said to be a simple support function focused at a set A if $m(A) = S, m(X) = 1 - S$ and $m(B) = 0, \forall B \neq A$.

(v) If a belief measure $'Bel'$ is given, then the basic assignment m can be constructed by setting

$$m(A) = \sum_{B \subseteq A} (-1)^{|A-B|} Bel(B)$$

where $|A - B|$ denotes the cardinality of $(A - B)$.

Dempster's rule of Combination:

Evidence obtained in the same context from two independent sources and expressed by two basic assignments m_1 and m_2 on $\mathcal{P}(X)$ must be appropriately combined to obtain a joint basic assignment $m_{1,2}$.The standard way of combining evidence is expressed by the formula

$$m_{1,2}(A) = \frac{\sum_{B \cap C = A} m_1(B).m_2(C)}{1 - \sum_{B \cap C = \phi} m_1(B).m_2(C)}$$

for all $A \neq \phi$ and $m_{1,2}(\phi) = 0$.

Example: Assume that an old painting was discovered which strongly resembles paintings by Raphael. Such a discovery is likely to generate various questions regarding the status of the painting. Assume the following three questions

(i) Is the discovered painting a genuine painting by Raphael?

(ii) Is the discovered painting a product of one of Raphael's many disciples?

(iii) Is the discovered painting a counterfeit?

Solution: Let R, D and C denote subset of our universal set X -the set of all paintings which contain the set of all paintings by Raphael, the set of all paintings by disciples of Raphael, and the set of all counterfeits of Raphael's paintings respectively.

Assume now that two experts performed careful examinations of the painting and subsequently provided us with basic assignments m_1 and m_2 specified in the following table. These are the degrees of evidence which each expert obtained by the examination and which the various claims that the painting belongs to one of the sets of our concern.

Focal Elements	Expert 1		Expert 2		Combined Evidence	
	m_1	Bel_1	m_2	Bel_2	$m_{1,2}$	$Bel_{1,2}$
R	.05	.05	.15	.15	.21	.21
D	0	0	0	0	.01	.01
C	.05	.05	.05	.05	.09	.09
$R \cup D$	.15	.2	.05	.2	.12	.34

$R \cup C$	.1	.2	.2	.4	.2	.5
$D \cup C$	.05	.1	.05	.1	.06	.16
$R \cup D \cup C$	.6	1	.05	.1	.31	1

Now we apply Dempster's rule to m_1 and m_2,we get

$$m_{1,2}(R) = \frac{\sum_{B \cap C = R} m_1(B).m_2(C)}{1 - \sum_{B \cap C = \phi} m_1(B).m_2(C)}$$

Now,

$$\sum_{B \cap C = \phi} m_1(B).m_2(C) = m_1(R).m_2(D) + m_1(R).m_2(C) + m_1(R).m_2(R \cup C)$$
$$+m_1(D).m_2(R) + m_1(D).m_2(C) + m_1(D).m_2(R \cup C) + m_1(C).m_2(R) + m_1(C).m_2(R)$$
$$+m_1(C).m_2(R \cup D) + m_1(R \cup D).m_2(C) + m_1(R \cup C).m_2(D) + m_1(D \cup C).m_2(R)$$
$$= 0.03$$

Also, $\sum_{B \cap C = R} m_1(B).m_2(C) = m_1(R).m_2(R) + m_1(R).m_2(R \cup D) + m_1(R).m_2(R \cup C)$
$$+m_1(R).m_2(R \cup D \cup C) + m_1(R \cup D).m_2(R) + m_1(R \cup D).m_2(R \cup C)$$
$$+ m_1(R \cup C).m_2(R) + m_1(R \cup C).m_2(R)$$
$$+m_1(R \cup C).m_2(R) + m_1(R \cup C).m_2(R \cup D) + m_1(R \cup D \cup C).m_2(R)$$
$$= 0.2037$$
$$\therefore m_{1,2}(R) = \frac{0.2037}{0.97} = 0.21.$$

Also,

$$m_{1,2}(D) = \frac{\begin{bmatrix} m_1(D).m_2(D) + m_1(D).m_2(R \cup D) + m_1(D).m_2(D \cup C) + \\ m_1(D).m_2(R \cup D \cup C) + m_1(R \cup D).m_2(D) + m_1(R \cup D).m_2(D \cup C) \\ m_1(D \cup C).m_2(D) + m_1(D \cup C).m_2(R \cup D) + m_1(R \cup D \cup C).m_2(D) \end{bmatrix}}{0.97}$$

$$= 0.01$$

$$m_{1,2}(R \cup C) = \frac{\begin{matrix} m_1(R \cup C).m_2(R \cup C) + m_1(R \cup C).m_2(R \cup D \cup C) + \\ m_1(R \cup D \cup C).m_2(R \cup C) \end{matrix}}{1 - 0.03}$$

$$= 0.2$$

$$m_{1,2}(R \cup D \cup C) = \frac{m_1(R \cup D \cup C).m_2(R \cup D \cup C)}{1 - 0.03}$$
$$= 0.31$$

Similarly, we can calculate for the remaining focal elements $C, R \cup D$ and $D \cup C$

$$m_{1,2}(C) = 0.9$$
$$m_{1,2}(R \cup D) = 0.12$$

$$m_{1,2}(D \cup C) = 0.06$$

The joint basic assignment can now be used to calculate the joint belief $Bel_{1,2}$ and joint plausibility $P_{1,2}$.

Now,

$$Bel_{1,2}(R) = m_{1,2}(R) = 0.21$$
$$Bel_{1,2}(D) = m_{1,2}(D) = 0.01$$
$$Bel_{1,2}(C) = m_{1,2}(C) = 0.09$$
$$Bel_{1,2}(R \cup D) = m_{1,2}(R) + m_{1,2}(D) + m_{1,2}(R \cup D) = 0.34$$
$$Bel_{1,2}(R \cup C) = m_{1,2}(R) + m_{1,2}(C) + m_{1,2}(R \cup C) = 0.5$$
$$Bel_{1,2}(D \cup C) = m_{1,2}(D) + m_{1,2}(C) + m_{1,2}(D \cup C) = 0.16$$

and $Bel_{1,2}(R \cup D \cup C) = m_{1,2}(R) + m_{1,2}(D) + m_{1,2}(C) + m_{1,2}(R \cup D) + m_{1,2}(R \cup C)$
$$+m_{1,2}(D \cup C) + m_{1,2}(R \cup D \cup C)$$
$$= 1$$

Basic assignment on the Cartesian product

Consider a basic assignment m defined on the Cartesian product $Z = X \times Y$;i.e.,

$$m: \mathcal{P}(X \times Y) \longrightarrow [o, 1]$$

In this case, each focal element of m is a binary relation R on $X \times Y$.Let R_X denote the projection of R on X.Then

$$R_X = \{x \in X : \langle x, y\rangle \in R \; for \; some \; y \in Y\}$$

Similarly,

$R_Y = \{y \in Y : \langle x, y\rangle \in R \; for \; some \; x \in Y\}$ defines the projection of R on Y.

The projection m_X of m on X can define by the formula

$m_X(A) = \sum_{R|A=R_X} m(R)$ for all $A \in \mathcal{P}(X)$

Similarly,$m_Y(B) = \sum_{R|B=R_Y} m(R)$ for all $B \in \mathcal{P}(Y)$defines the projection of m on Y.

Note:

Let m_X and m_Y be called marginal basic assignments and let $(\mathcal{F}_x, m_x)$ and $(\mathcal{F}_Y, m_Y)$ be the associated marginal bodies of evidence.Two marginal bodies of evidence $(\mathcal{F}_x, m_x)$ and $(\mathcal{F}_Y, m_Y)$ are said to be non interactive iff for all $A \in \mathcal{F}_x$ and all $B \in \mathcal{F}_Y$

$$m(A \times B) = m_X(A).\, m_Y(A)$$

and $m(R) = 0$ for all $R \neq A \times B$.

Example: Consider the body of evidence given in the following table. Focal elements are subsets of the Cartesian product $X \times Y$,where $X = \{1,2,3\}$ and $Y = \{a, b, c\}$;they are defined in the table by their characteristic functions.To emphasize that each focal element is in fact,a binary relation on $X \times Y$,they are labelled as $R_1, R_2, \cdots, R_{12}$.

$X \times Y$

	1a	1b	1c	2a	2b	2c	3a	3b	3c	$m(R_i)$
$R_1 =$	0	0	0	0	1	1	0	1	1	0.0625
$R_2 =$	0	0	0	1	0	0	1	0	0	0.0625
$R_3 =$	0	0	0	1	1	1	1	1	1	0.125
$R_4 =$	0	1	1	0	0	0	0	1	1	0.0375
$R_5 =$	0	1	1	0	1	1	0	0	0	0.075

$R_6 =$	0	1	1	0	1	1	0	1	1	0.075
$R_7 =$	1	0	0	0	0	0	1	0	0	0.375
$R_8 =$	1	0	0	1	0	0	0	0	0	0.075
$R_9 =$	1	0	0	1	0	0	1	0	0	0.075
$R_{10} =$	1	1	1	0	0	0	1	1	1	0.075
$R_{11} =$	1	1	1	1	1	1	0	0	0	0.15
$R_{12} =$	1	1	1	1	1	1	1	1	1	0.15

Now, marginal basic assignments are

$$m_X(\{2,3\}) = m(R_1) + m(R_2) + m(R_3) = 0.0625 + 0.0625 + 0.125 = 0.25$$
$$m_X(\{1,2\}) = m(R_5) + m(R_8) + m(R_{11}) = 0.075 + 0.075 + 0.15 = 0.3$$
$$m_X(\{a\}) = m(R_2) + m(R_7) + m(R_8) + m(R_9) = 0.0625 + 0.375 + 0.075 + 0.075 = 0.25$$
$$m_X(\{a,b,c\}) = m(R_3) + m(R_{10}) + m(R_{11}) + m(R_{12}) = 0.125 + 0.075 + 0.15 + 0.15 = 0.5$$

Similarly,

$$m_X(\{1,3\}) = 0.15$$
$$m_X(\{2,3\}) = 0.3$$
$$m_X(\{b,c\}) = 0.5$$

We can easily verify that the joint basic assignment m is uniquely determined in this case by the marginal basic assignments. The marginal bodies of evidence are thus non-interactive.

For example,

$$R_1 = \{2b, 2c, 3b, 3c\} = \{2,3\} \times \{b,c\}$$
$$\therefore m(R_1) = m_X\{2,3\}.m_Y\{b,c\}$$
$$= 0.25 \times 0.25$$
$$= 0.0625$$

Similarly, $R_{10} = \{1a, 1b, 1c, 3a, 3b, 3c\} = \{1,3\} \times \{a,b,c\}$

$$m(R_{10}) = m_X\{1,3\}.m_Y\{a,b,c\}$$
$$= 0.15 \times 0.15$$
$$= 0.075$$

and so on.

Note:

(i) In addition to the basic probability assignment, belief measure, and plausibility measure, it is also sometimes convenient to use the function

$$Q(A) = \sum_{B|A\subseteq B} m(B)$$

which is called a commonality function.

(ii) For each $A \in \mathcal{P}(X)$,the value $Q(A)$ represents the total portion of belief that can move freely to every point of A.Each commonality function defined on the power set of X is a unique representation of a body of evidence on the power set.

(iii) Conversion formulas in the Evidence theory

	m	Bel	Pl	Q

$m(A) =$	$m(A)$	$\sum_{B\subseteq A}(-1)^{\lvert A-B\rvert}Bel(B)$	$\sum_{B\subseteq A}(-1)^{\lvert A-B\rvert}(1 - Pl(\bar{B}))$	$\sum_{A\subseteq B}(-1)^{\lvert B-A\rvert}Q(B)$
$Bel(A) =$	$\sum_{B\subseteq A} m(B)$	$Bel(A)$	$1 - Pl(\bar{A})$	$\sum_{B\subseteq \bar{A}}(-1)^{\lvert B\rvert}Q(B)$
$Pl(A) =$	$\sum_{B\cap A\neq\phi} m(B)$	$1 - Bel(\bar{A})$	$Pl(A)$	$\sum_{\phi\neq B\subset \bar{A}}(-1)^{\lvert B\rvert+1}Q(B)$
$Q(A) =$	$\sum_{A\subseteq B} m(B)$	$\sum_{B\subset A}(-1)^{\lvert B\rvert}Bel(\bar{B})$	$\sum_{\phi\neq B\subset A}(-1)^{\lvert B\rvert+1}Pl(B)$	Q(A)

CHECK YOUR PROGRESS 2

Q1. Prove that $Bel(A) + Bel(\bar{A}) \leq 1$.
Q2. Prove that $Pl(A) + Pl(\bar{A}) \geq 1$.
Q3. Suppose$X = \{a, b, c\}$.If the basic probability assignments of subsets of X are as follows
$m(\{a\}) = 0.2, m(\{b\}) = 0.3, m(\{c\}) = 0.4, m(\{a, b\}) = 0.1,$
$m(\{a, c\}) = 0, m(\{b, c\}) = 0, m(\{a, b, c\}) = 0.$
Then calculate the belief and plausibility measures for the proposition $\{a, b\}$.
Q4. Let $X = \{a, b, c, d\}$.Given the belief measures
$Bel(\{b\}) = 0.1, Bel(\{a, b\}) = 0.2, (\{b, c\}) = 0.3, Bel(\{b, d\}) = 1.$
Determine the corresponding basic assignment.
Q5. Suppose two experts,E_1 and E_2 are called to evaluate the efficiency and performance of a compound generator. In the following table their individual assessments of over compounded (OC),flat compounded(FC) and under compunded(UC) generates with their corresponding evidence measures.
Calculate $Bel_1(OC \cup FC), Bel_2(FC \cup UC), m_{12}(FC \cup UC)$.

Experts	Elements (A_i)						
	OC	FC	UC	$OC \cup FC$	$OC \cup UC$	$FC \cup UC$	$OC \cup FC \cup UC$
Expert 1,$m_1(A_i)$	0	0.05	0.05	0.15	0.05	0.10	0.60
Expert 1,$m_1(A_i)$	0	0.15	0.05	0.05	0.05	0.20	0.50

6.5 PROBABILITY MEASURE

A probability measure is a real –valued function defined on a set of events in a probability space that satisfies measure properties such as countable additivity. A probability measure,Pro,is required to satisfy the equation

$Pro(AUB) = Pro(A) + Pro(B)$ for all sets $A, B \in \mathcal{P}(X)$ such that $A \cap B = \phi$.

Theorem: A belief measure Bel on a finite power set $\mathcal{P}(X)$ is a probability measure iff the associated basic probability assignment is given by

$m(\{x\}) = Bel(\{x\})$ and $m(A) = 0$ for all subsets of X that are not singletons.

Proof: We assume that Bel is a probability measure.

Case I: For the empty set ϕ,the theorem trivially holds, since$m(\phi) = 0$,by the definition of basic assignment $'m'$.

Case II: Let $A \neq \phi$ and assume that $A = \{x_1, x_2, \cdots, x_n\}$.

$\because Bel$is a probability measure, so by repeated application of the additivity axiom, we obtain

$$Bel(A) = Bel(\{x_1\}) + Bel(\{x_2, x_3, \cdots, x_n\})$$
$$= Bel(\{x_1\}) + Bel(\{x_2\}) + Bel(\{x_3, x_4, \cdots, x_n\})$$
$$= Bel(\{x_1\}) + Bel(\{x_2\}) + \cdots Bel(\{x_n\})$$

$\because Bel(\{x\}) = m(\{x\})$ for any $x \in X$

$$\therefore Bel(A) = \sum_{i=1}^{n} m(\{x_i\})$$

Hence Bel is defined in terms of a basic assignment that focuses only on singletons.

Conversely, we assume that a basic assignment m is given such that $\sum_{x \in X} m(\{x\}) = 1$.

Then for any sets $A, B \in \mathcal{P}(X)$ such that $A \cap B = \phi$,we have

$$Bel(A) + Bel(B) = \sum_{x \in A} m(\{x\}) + \sum_{x \in B} m(\{x\})$$
$$= \sum_{x \in A \cup B} m(\{x\})$$
$$= Bel(A \cup B)$$

Hence Belief measure is a probability measure.

Probability distribution function:

The probability measure on finite sets are represented by a function $p: X \longrightarrow [0,1]$ such that $p(x) = m(\{x\})$.

This function is called a probability distribution on X.$p = \langle p(x): x \in X \rangle$ be referred to as a probability distribution on X.

Note:

(i) Belief and plausibility measures merge under the additivity axiom of probability measures.

Thus, for all $A \in \mathcal{P}(X)$

$$Bel(A) = Pl(A) = \sum_{x \in A} p(x)$$

If we denote the probability measure by Pro,then

$Pro(A) = \sum_{x \in A} p(x), \forall A \in \mathcal{P}(X)$

(ii) Within probability measures, total ignorance is expressed by the uniform probability distribution

$$p(x) = \frac{1}{|X|} (= m\{x\}) \forall A \in X$$

(iii) When a probability distribution p is defined on the Cartesian product $X \times Y$,it is called a joint probability distribution.

(iv) Projections p_x and p_y of p on X and Y are defined by the formulas
$p_X(x) = \sum_{y \in Y} p(x,y)$ for each $x \in X$
and $p_Y(y) = \sum_{x \in X} p(x,y)$ for each $y \in Y$

(v) Projections p_x and p_y are called marginal probability distributions.

(vi) The marginal distributions are called noninteractive with respect to p iff
$p(x,y) = p_x . p_y$ for all $x \in X$ and all $y \in Y$.

(vii) Two conditional probability distributions,$p_{X|Y}$ and $p_{Y|X}$ are defined in terms of a joint distribution p by the formulas

$$p_{X|Y}(x|y) = \frac{p(x,y)}{p_Y(y)}$$

and $p_{Y|X}(y|x) = \frac{p(x,y)}{p_X(x)} \forall x \in x$ and $y \in Y$.

(viii) The two marginal distributions are called independent iff

$$p_{X|Y}(x|y) = p_X(x)$$

$p_{Y|X}(y|x) = p_Y(y)$ for all $x \in X$ and all $y \in Y$.

DO YOU KNOW?

(i) The value $p_{X|Y}(x|y)$ represents the probability that x occurs provided that y is known. Similarly,$p_{Y|X}(y|x)$ designates the probability of y given x.

(ii) Joint probability distribution is defined by

$$p(x,y) = p_{X|Y}(x|y) . p_Y(y) = p_{Y|X}(y|x) . p_X(x)$$

(iii) The marginal distributions are independent if and only they are noninteractive.

Example: Two quality control experts are assessing the source of scratches on media exiting a new laser printer. Suppose$W$ and M denote the scratches from wiper arm and the other media respectively. The experts provide their assessments of evidence as follows:

Focal elements	Expert 1,m_1	Expert 2,m_2
W	0.6	0.3
M	0.4	0.7
$W \cup M$	0	0

Determine the beliefs, plausibilities and probabilities for each non null focal element.

Solution: Here $m_1(W) = 0.6, m_1(M) = 0.4, m_1(W \cup M) = 0$

$m_2(W) = 0.3, m_2(M) = 0.4, m_2(W \cup M) = 0$

Now $bel_1(W) = m_1(W) = 0.6$

$$bel_1(M) = m_1(M) = 0.4$$

and $bel_1(W \cup M) = m_1(W) + m_1(M) + m_1(W \cup M)$

$$= 0.6 + 0.4 + 0 = 1$$

Again $pl_1(W) = m_1(W) + m_1(W \cup M) = 0.6 + 0 = 0.6$

$$pl_1(M) = m_1(M) + m_1(W \cup M) = 0.4 + 0 = 0.4$$

$$pl_1(W \cup M) = m_1(W) + m_1(W \cup M) + m_1(M)$$
$$= 0.6 + 0 + 0.4 = 1$$

$$\because bel_1(W) = pl_1(W) = 0.6$$
$$bel_1(M) = pl_1(M) = 0.4$$

and $bel_1(W \cup M) = pl_1(W \cup M) = 1$

$\therefore$ These are probabilities.

Hence $p_1(W) = 0.6, p_1(M) = 0.4$ and $p_1(W \cup M) = p_1(W) + p_1(M) = 1$.

Similarly, we can find for the second expert

$p_2(W) = 0.3, p_2(M) = 0.7$

and $p_2(W \cup M) = p_2(W) + p_2(M) = 0.3 + 0.7 = 1$.

CHECK YOUR PROGRESS 3

Q1. Two experts E_1 and E_2 are evaluating the likelihood of software project succeeding based on two factors: the competence of the development team(C) and the clarity of project requirements (R).The experts provide their assessments of evidence as follows:

$m_1(C) = 0.6, m_1(R) = 0.4, m_1(C \cup R) = 0$

$m_2(C) = 0.5, m_2(R) = 0.5, m_2(C \cup R) = 0$

Calculate the beliefs, plausibilities and probabilities for Experts E_1 and E_2 assessments.

6.6 POSSIBILITY AND NECESSITY MEASURE

A family of subsets of a universal set is nested if they can be ordered so that one is contained in other. If the focal elements of a body of evidence $(\mathcal{F}, m)$ are nested, the corresponding belief and plausibility measures are said to be consonant. A special branch of evidence theory that deals only with bodies of evidence whose focal elements are nested is referred to as possibility theory. Special counterparts of belief measures and plausibility measures in possibility theory are called necessity measures and possibility measures respectively. A consonant belief measure is called necessity measure and a consonant plausibility measure is called possibility measure.

Theorem: If a finite body of evidence $(\mathcal{F}, m)$ is nested, then for all $A, B \in \mathcal{P}(X)$

(i) $Bel(A \cap B) = min[Bel(A), Bel(B)]$,

(ii) $Pl(A \cup B) = max[Pl(A), Pl(B)]$

Proof: Suppose that $\mathcal{F} = \{A_1, A_2, \cdots, A_n\}$ and $A_i \subset A_j$ for $i < j$.

Let us consider two arbitrary subsets A and B of X.Let i_1 be the largest integer i such that $A_i \subset A$,and let i_2 be the largest integer i such that $A_i \subset B$.

Then $A_i \subset A$ and $A_i \subset B$ iff $i \leq i_1$ and $i \leq i_2$

$\Rightarrow A_i \subset A \cap B$ iff $i \leq min(i_1, i_2)$

Thus (i)$Bel(A \cap B) = \sum_{i=1}^{min(i_1,i_2)} m(A_i)$

$$= min\left[\sum_{i=1}^{i_1} m(A_i), \sum_{i=1}^{i_2} m(A_i),\right]$$
$$= min[Bel(A), Bel(B)]$$

(ii)We know

$$Pl(A \cup B) = 1 - Bel(\overline{AUB})$$
$$= 1 - Bel(\bar{A} \cap \bar{B})$$

From (1), we get

$$Bel(\bar{A} \cap \bar{B}) = min[Bel(\bar{A}), Bel(\bar{B})]$$

$$\therefore\ Pl(A \cup B) = 1 - min[Bel(\bar{A}), Bel(\bar{B})]$$
$$= max[1 - Bel(\bar{A}), 1 - Bel(\bar{B})]$$
$$= max[Pl(A), Pl(B)] \forall A, B \in \mathcal{P}(X)$$

Note:

(i) If necessity measures and possibility measures are denoted 'Nec' and 'Pos' respectively, then for every $A, B \in \mathcal{P}(X)$.

We have

$$Nec(A \cap B) = min[Nec(A), Nec(B)]$$
$$Pos(A \cup B) = max[Pos(A), Pos(B)]$$

(ii) If Nec is a fuzzy measure on $(X, \mathcal{C})$,then Nec is called a necessity measure iff

$$Nec\left(\bigcap_{k\in K} A_k\right) = inf_{k\in K} Nec(A_k)$$

for any family $\{A_k . k \in K\}$ in $\mathcal{C}$ such that $\bigcap_{k\in K} A_k \in \mathcal{C}$ where K is an arbitrary index set.

(iii) If Pos is a fuzzy measure on $(X, \mathcal{C})$,then Pos is called a possibility measure iff

$$Pos\left(\bigcup_{k\in K} A_k\right) = sup_{k\in K} Pos(A_k)$$

for any family $\{A_k . k \in K\}$ in $\mathcal{C}$ such that $\bigcup_{k\in K} A_k \in \mathcal{C}$ where K is an arbitrary index set.

Do you know?

(i) Necessity measures are special belief measures and possibility measures are special plausibility measures.

$$Nec(A) + Nec(\bar{A}) \leq 1$$
$$Pos(A) + Pos(\bar{A}) \geq 1$$

and $Nec(A) = 1 - Pos(\bar{A})$.

(ii) $min[Nec(A), Nec(\bar{A})] = 0$

$$max[Pos(A), Pos(\bar{A})] = 1$$

Theorem: For every $A \in \mathcal{P}(X)$,any necessity measure, Nec, on $\mathcal{P}(X)$ and the associated possibility measure,Pos,satisfy the following implications:

(i) $Nec(A) > 0 \Rightarrow Pos(A) = 1$;

(ii) $Pos(A) < 1 \Longrightarrow Nec(A) = 0.$

Proof:

(i) Let $Nec(A) > 0$ for some $A \in \mathcal{P}(X)$.

$$\because min[Nec(A), Nec(\bar{A})] = 0$$

$\therefore Nec(\bar{A}) = 0.$

Now, $Pos(A) = 1 - Nec(\bar{A})$

$$= 1 - 0$$

$= 1.$

(ii) Let $Pos(A) < 1$for some $A \in \mathcal{P}(X)$.

Since $max[Pos(A), Pos(\bar{A})] = 1$

$\therefore Pos(\bar{A}) = 1.$

Now, $Nec(A) = 1 - Pos(\bar{A})$

$$= 1 - 1$$

$= 0.$

Theorem: Every possibility measure Pos on a finite power set $\mathcal{P}(X)$ is uniquely determined by a possibility distribution function $r: X \longrightarrow [0,1]$ via formula $Pos(A) = \max_{x \in A} \sqcap (x)$ for each $A \in \mathcal{P}(X)$.

Proof: We have the theorem by induction on the cardinality of the set.

If $|A| = 1$,then $A = \{x\}$,where $x \in X$.

Then the result is obvious.

Assume that the given equation is satisfied for $|A| = n - 1$ and let $A = \{x_1, x_2, \cdots, x_n\}$.

Now,

$$Pos(A) = Pos(\{x_1, x_2, \cdots, x_n\})$$

$$= Pos(\{x_1, x_2, \cdots, x_{n-1}\} \cup \{x_n\})$$

$= \max[Pos\{x_1, x_2, \cdots, x_{n-1}\} \cup Pos\{x_n\}][\because Pos(A \cup B) = max[Pos(A), Pos(B)]$

$= max[\max[Pos\{x_1\}, Pos\{x_2\}, \cdots, Pos\{x_{n-1}\}], Pos\{x_n\}]]$

$$= \max_{x \in A} r(x)$$

Note:

(i) If r is a possibility distribution on $X = \{x_1, x_2, \cdots, x_n\}$.Then the n tuple $r = \langle r_1, r_2, \cdots, r_n \rangle$ where $r_i = r(x_i)$ for all $x_i \in X$ is called a possibility distribution associated with the function r.The number of components in a possibility distribution is called its length.

(ii) Let ${}^n\mathcal{R}$ denote the set of all ordered possibility distributions of length n and let $\mathcal{R} = \cup_{n \in N} {}^n\mathcal{R}$

For any two possibility distributions, ${}^1r = \langle {}^1r_1, {}^1r_2, \cdots, {}^1r_n \rangle \in {}^n\mathcal{R}$ and ${}^2r = \langle {}^2r_1, {}^2r_2, \cdots, {}^2r_n \rangle \in {}^n\mathcal{R}$,for some $n \in N$,we define

, ${}^1r \leq {}^2r$ iff ${}^1r_i \leq {}^2r_i \forall i \in \mathbb{N}_n$.

The ordering on $\mathcal{R}$ is partial and forms a lattice whose join $' \vee '$ and $' \wedge '$ are defined respectively as

$${}^ir \vee {}^jr = \langle max({}^ir_1, {}^jr_1), max({}^ir_2, {}^jr_2), \cdots, max({}^ir_n, {}^jr_n) \rangle$$

and

$${}^ir \wedge {}^jr = \langle min({}^ir_1, {}^jr_1), min({}^ir_2, {}^jr_2), \cdots, min({}^ir_n, {}^jr_n) \rangle$$

for all ${}^ir, {}^jr \in {}^n\mathcal{R}$.

For each $n \in N$,$\langle {}^{n}\mathcal{R}, \leq \rangle$ be called a lattice of possibility distributions of length n.

(iii) If $X = \{x_1, x_2, \cdots, x_n\}$ and Pos is a possibility measure defined on $\mathcal{P}(X)$ in terms of its basic assignments m,then all the focal elements are nested.
Suppose that $A_1 \subset A_2 \subset \cdots A_n (= X)$,
then $m(A) = 0, \forall A = A_i$ and $\sum_{i=1}^{n} m(A_i) = 1$.

Now $m = (m_1, m_2, \cdots, m_n)$ where $m_i = m(m_1, m_2, \cdots, m_n)$,where $m_i = m(A_i)$ is called a basic distribution of length n.

(iv) Let ${}^{n}\mathcal{M}$ denote the set of all basic distributions with n components and let

$$\mathcal{M} = \bigcup_{n \in N} {}^{n}\mathcal{M}$$

Clearly $r_i = r(x_i) = Pos(\{x_i\}) = Pl(\{x_i\}) \forall x_i \in X$
Now, we know $Pl(A) = \sum_{B|A \cap B \neq \phi} m(B)$

$$\therefore r_i = Pl(\{x_i\}) = \sum_{k=i}^{n} m(A_k) = \sum_{k=i}^{n} m_k$$

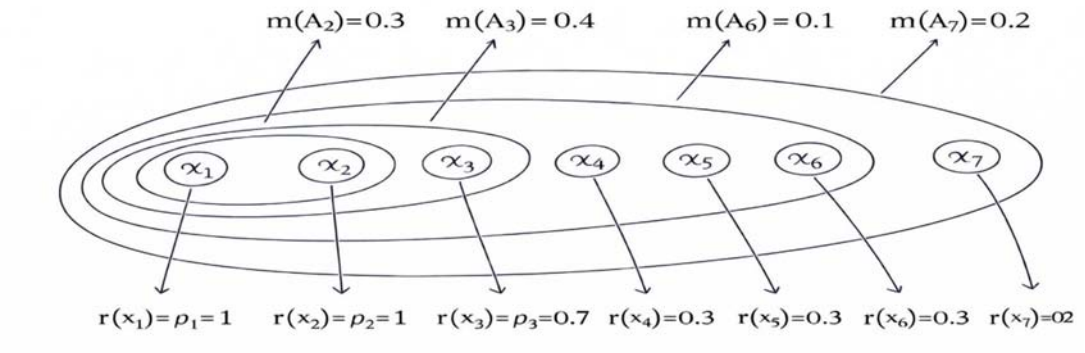
Thus

$$\begin{aligned} r_1 &= m_1 + m_2 + \cdots + m_i + \cdots + m_n \\ r_2 &= \quad m_2 + \cdots + m_i + \cdots + m_n \\ &\cdots\cdots\cdots \\ r_i &= \quad m_i + \cdots + m_n \\ &\cdots\cdots\cdots \\ r_n &= \quad m_n \end{aligned}$$

Solving these equations for $m_i (i \in \mathbb{N})$,we obtain
$m_i = r_i - r_{i+1} \forall i \in \mathbb{N}_n$ with $r_{n+1} = 0$.
It is clear that there is a one-one correspondence between R and M and it is defined by

$$t: R \longrightarrow M$$

which is one-one and onto.
Example: Obtain the possibility distribution and degree of possibility for the set $A = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$ where the basic distributions is represented in the following diagram



Solution: Here $m(A_1) = m(A_4) = m(A_5) = 0$

$$\therefore m = \big(m(A_1), m(A_2), \cdots, m(A_7)\big)$$
$$= (0,0.3,0.4,0,0,0.1,0.2)$$

We obtain the possibility distribution

$$r = (\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6, \rho_7) = (1,1,0.7,0.3,0.3,0.3,0.2)$$

Degree of possibility for A is given by

$$\prod(A) = max(\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6, \rho_7)$$
$$= max(1,1,0.7,0.3,0.3,0.3,0.2)$$

$= 1.$

CHECK YOUR PROGRESS 4

Q1. What is the difference between possibility and necessity measures?

Q2. Give an example of how possibility and necessity measures are used in real life applications?

Q3.A patient complaining of a severe headache is examined by two doctors. The diagnosis of doctor 1 is that the patient has either meningitis (M) with probability 0.99 or a brain tumour (BT) with probability 0.01.Doctor 2 agrees with doctor 1 that the probability of a brain tumour (BT) is 0.01,but disagrees with doctor 1 on the meningitis ;instead he feels that there is a probability of 0.99 that the patient just has concussion (C).Calculate the combined evidence measure $m_{12}(BT)$ for a brain tumour.

DO YOU KNOW?

Possibility measures and necessity measures differ from probability measures in that they donot require the addition axiom and can handle non additive uncertainty.

6.7 POSSIBILITY DISTRIBUTION AS FUZZY SETS

In the realm of fuzzy logic and possibility theory, probability distributions play a crucial role in representing and manipulating uncertain information. Belief structures that are nested are called consonant. A problem in equating consonant belief structures with fuzzy sets is that combination of two consonant belief functions using Dempster's rule of combination in general doesnot necessarily lead to a consonant result. Hence since Dempster's rule is essentially a conjunction operation, the intersection of two fuzzy sets interpreted as consonant belief structures maynot result in a valid fuzzy set.

Consider the following example.

Example: Suppose we have a universe $X = \{x_1, x_2, x_3, x_4, x_5\}$ and we have evidence provided by two experts. The accompanying table provides the experts degrees of belief about specific subsets of the universe X.

Focal Element	Expert 1,$m_1(A_i)$	Expert 1,$m_1(A_i)$
$A_1 = \{x_1, x_2, x_3\}$	0.7	
$A_2 = X$	0.3	
$B_1 = \{x_3, x_4, x_5\}$		0.8
$B_2 = X$		0.2

∵ $A_1 \subset A_2$ and $B_1 \subset B_2$,so we have two consonant belief structures represented by A and B.

∴ Using Dempster's rule of combination, we have for any set D on the universe X.

$$m(D) = \frac{1}{1-k} \sum_{A_i \cap B_j = D} m_1(A_i).m_2(B_j)$$

where $k = \sum_{A_i \cap B_j = \phi} m_1(A_i).m_2(B_j)$

Since there are two focal elements in each experts' belief structures,we will have $2^2 = 4$ belief structures in the combined evidence case, which will denote as m.

∵ for any two focal elements A and B,$A \cap B \neq \phi$

∴ $k = 0$.

Therefore,

$D_1 = A_1 \cap B_1 = \{x_3\}$,$m(D_1) = m(A_1).m(B_1) = 0.56$

$D_2 = A_1 \cap B_2 = \{x_1, x_2, x_3\}$,$m(D_2) = m(A_1).m(B_2) = 0.14$

$D_3 = A_2 \cap B_1 = \{x_3, x_4, x_5\}$,$m(D_3) = m(A_2).m(B_1) = 0.14$

and $D_4 = A_2 \cap B_2 = X$,$m(D_4) = 0.06$

∵ For the focal elements D_i,$D_1 \subset D_2 \subset D_4$ and $D_1 \subset D_3 \subset D_4$,but we don't have $D_2 \subset D_3$ or $D_3 \subset D_2$.

Hence the combined case is not consonant.

Yager has developed a method that combine consonant possibility measures where result is also a consonant possibility measure(1993).However,this procedure is very lengthy to describe.Another interpretation of a possibility distribution as a fuzzy set was proposed by Zadeh(1978).He defined a possibility distribution as a fuzzy restriction that acts as an elastic constraint on the values that may be assigned to a variable.In this case the possibility distribution represents the degrees of membership for some linguistic variable,but the membership values are strictly monotonic as they are for an ordered possibility distribution.For example,let $\underset{\smile}{A}$ be a fuzzy on a universe X,and let the membership value,μ be a variable on X that assigns a possibility that an element of x is in $\underset{\smile}{A}$.

∴ we get $Pos(x) = \mu_{\underset{\smile}{A}}(x)$.

Note:

Zadeh points out that the possibility distribution is nonprobabilistic and is used primarily in natural language applications.

Example: Let A be a fuzzy set defined on the universe of columns needed to support a building. Suppose there are 10 columns altogether and use start taking columns away until the building collapses; we record the number of columns at the time the building collapses. Let A be the fuzzy set defined by the number of columns possibly needed out of 10 total, just before the structure fails.The structure most certainly needs alleast

three columns to stand. After that the number of columns required for the building to stand is a fuzzy issue but there is a possibility it may need more than three.
The following fuzzy set may represent the possibility:

$$\underset{\sim}{A} = \left\{\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{0.9}{4} + \frac{0.6}{5} + \frac{0.3}{6} + \frac{0.1}{7} + \frac{0}{8} + \frac{0}{9} + \frac{0}{10}\right\}$$

A probability distribution on the same universe may look something like the following table, where u is the no of columns prior to collapse and $P(u)$ is the probability that u is the number of columns at collapse:

u:	1	2	3	4	5	6	7	8	9	10
$P(u)$:	0	0	0.1	0.5	0.3	0.1	0	0	0	0

As seen, although it is possible that one column will sustain the building, it is not probable.Hence,neither does a high degree of possibility imply a high probability nor does a low degree of probability imply a low degree of possibility.

CHECK YOUR PROGRESS 5

Q1. What is a possibility distribution in the context of fuzzy set
Q2. In a fuzzy set, the possibility distribution for the variable x is given by

$$\Pi(x) = \begin{cases} 0.1 & if \ \ x = 1 \\ 0.3 & if \ \ x = 2 \\ 0.7 & if \ \ x = 3 \\ 1.0 & if \ \ x = 4 \\ 0.5 & if \ \ x = 5 \end{cases}$$

What is the highest degree of possibility in the set.

Do You Know?

(i) Belief and plausibility measures overlap when they both become probability measures.

(ii) Possibility,necessity and probability measures donot overlap with one another except for one special measure:the measure of one focal element that is a singleton.

(iii) Possibility,necessity and probability measures become equal when one element of the universal set is assigned a value of unity,and all other elements in the universe are assigned a value of zero.This measures represents perfect evidence.

6.8 LET US SUM UP

- Fuzzy measure theory considers generalized measures in which the additive property is replaced by the weaker property of monotonicity.
- Fuzzy measures are important in possibility theory because they provide a way to assess the degree of possibility of various events or outcomes,which may not always be easily quantifiable in terms of traditional probabilities.

- Evidence theory also known as Dempster –shafer theory is a framework for reasoning under uncertainty that is particularly useful when dealing with partial and conflicting information.
- A belief measure is a quantity that expresses the degree of support or evidence for a collection of elements defined by one or more of the crisp sets existing on the power set of a universe.
- Plausibility measure of A is defined as the complement of the belief of the complement of A.
- $Bel(A) + Bel(A^c) \leq 1, \forall A \in \mathcal{P}(X)$
  $Pl(A) + Pl(A^c) \leq 1$
- $Bel(A) = 1 - Pl(\bar{A})$
  $Pl(A) = 1 - Bel(\bar{A})$,$\forall A \in \mathcal{P}(X)$
- For a given basic assignment m,
  $Bel(A) = \sum_{B|B\subseteq A} m(B)$
  and $Pl(A) = \sum_{B|A\cap B\neq\phi} m(B)$,$\forall A \in \mathcal{P}(X)$
  Also $m(A) = \sum_{B|B\subseteq A}(-1)^{|A-B|} Bel(B)$, $\forall A \in \mathcal{P}(X)$
- For all $A \in \mathcal{P}(X)$, $Pl(A) \geq Bel(A)$.
- Dempster's rule of combination for combining two basic probability assignments m_1 and m_2 obtained from two independent sources in the same context is
  $m_{1,2}(A) = \frac{\sum_{B\cap C=A} m_1(B).m_2(C)}{1-\sum_{B\cap C=\phi} m_1(B).m_2(C)} \quad \forall A \neq \phi$
  and $m_{1,2}(\phi) = 0$.
- A probability measure 'Pro' is a real valued function defined on a set of events such that
  $Pro(AUB) = Pro(A) + Pro(B)$, $\forall A, B \in \mathcal{P}(X)$ such that $A \cap B = \phi$.
- A Belief measure Bel on a finite power set $\mathcal{P}(X)$ is a probability measure iff the associated basic probability assignment is $m(\{x\}) = Bel(\{x\})$ and $m(A) = 0$,for all subsets A of X which are not singletons.
- Probability distribution on X is a function $p: X \longrightarrow [0,1]$ such that $p(x) = m(\{x\})$.
- Consonant belief and plausibility measures are referred to as necessity and plausibility measures respectively.

✓ 6.9 ANSWERS TO CHECK YOUR PROGRESS

Check your progress 1:

Answer to Q1:

A classical probability measure satisfies the property of additivity.But a fuzzy measure doesnot necessarily satisfy additivity.Instead it satisfies subadditivity.i.e., the measure of the union of disjoint sets is less than or equal to the sum of their individual measures,reflecting the possibility of interactions between sets.

Answer to Q2:
Here $X = \{a, b, c\}$ and $\mu(\{a\}) = 0.4, \mu(\{b\}) = 0.3$.
Since μ satifies the monotonicity property,
$\mu(\{a, b\}) \geq \mu(\{a\}) = 0.4$ and $\mu(\{a, b\}) \geq \mu(\{b\}) = 0.3$
$\therefore\ \mu(\{a, b\}) \geq 0.4.$

Answer to Q3:
Yes,it is always true that $g(A) \leq g(X)$ for any subset A of X.This follows from the monotonicity property of fuzzy measures.

Check your progress 2:

Answer to Q1:

Let $A \cup \bar{A} = X$ and $A \cap \bar{A} = \phi$

Now $Bel(A \cup \bar{A}) = Bel(X) = 1$

Again $Bel(A \cup \bar{A}) \geq Bel(A) + Bel(\bar{A}) - Bel(A \cap \bar{A})$

$$\geq Bel(A) + Bel(\bar{A}) - Bel(\phi)$$

$\because\ Bel(\phi) = 0$ and $Bel(A \cup \bar{A}) = 1,$

$\therefore\ Bel(A) + Bel(\bar{A}) \leq 1.$

Answer to Q2:

We know that $Pl(X) = 1 - Bel(\bar{X})$

$$= 1 - Bel(\phi)$$

$$= 1 - 0 = 1$$

and $Pl(\phi) = 1 - Bel(\phi) = 1 - Bel(X) = 0.$
Now,$A \cap \bar{A} = \phi$ and $A \cup \bar{A} = X$
$\Longrightarrow Pl(A \cap \bar{A}) = 0$ and $Pl(A \cup \bar{A}) = 1$
Again $Pl(A \cap \bar{A}) \leq Pl(A) + Pl(\bar{A}) - Pl(A \cup \bar{A})$
$\Longrightarrow 0 \leq Pl(A) + Pl(\bar{A}) - 1$
$\Longrightarrow Pl(A) + Pl(\bar{A}) \geq 1$

Answer to Q 3:
Here $X = \{a, b, c\}$ and basic probability assignment of subsets of X are given by
$m(\{a\}) = 0.2, m(\{b\}) = 0.3, m(\{c\}) = 0.4$
$m(\{a, b\}) = 0.1, m(\{a, c\}) = 0, m(\{b, c\}) = 0$
$m(\{a, b, c\}) = 0$
We know

$$Bel(A) = \sum_{B|B \subseteq A} m(B)$$

and $Pl(A) = \sum_{B|A\cap B\neq\phi} m(B)$

$\therefore\ Bel(\{a,b\}) = m(\{a\}) + m(\{b\}) + m(\{a,b\})$
$= 0.2 + 0.3 + 0.4$
$= 0.6$
and $Pl(\{a,b\}) = m(\{a\}) + m(\{b\}) + m(\{a,b\}) + m(\{a,c\}) + m(\{b,c\}) + m(\{a,b,c\})$
$= 0.6$

Answer to Q 4:
Here $X = \{a,b,c,d\}$
$Bel(\{b\}) = 0.1, Bel(\{a,b\}) = 0.2$
$Bel(\{b,c\}) = 0.3, Bel(\{b,d\}) = 1$
We know
$m(A) = \sum_{B\subseteq A}(-1)^{|A-B|} Bel(B)$
$\therefore\ m(\{b\}) = Bel(\{b\}) = 0.1$
$m(\{a,b\}) = Bel(\{a,b\}) - Bel(\{b\})$
$= 0.2 - 0.1$
$= 0.1$
$m(\{b,c\}) = Bel(\{b,c\}) - Bel(\{b\})$
$= 0.2$
$m(\{b,d\}) = Bel(\{b,d\}) - Bel(\{b\})$
$= 0.9$

Answer to Q 5:

$$\boldsymbol{bel_1(OC \cup FC)} = \sum_{\boldsymbol{B\subseteq(OC\cup FC)}} \boldsymbol{m_1(B)}$$
$$= \boldsymbol{m_1(OC \cup FC) + m_1(OC) + m_1(FC)}$$
$$= \boldsymbol{0.15 + 0 + 0.05}$$
$$= \boldsymbol{0.20}$$
$$\boldsymbol{bel_2(FC \cup UC)} = \sum_{\boldsymbol{B\subseteq(FC\cup UC)}} \boldsymbol{m_2(B)}$$
$$= \boldsymbol{m_2(FC \cup UC) + m_2(FC) + m_2(UC)}$$
$$= \boldsymbol{0.20 + 0.15 + 0.05}$$
$$= \boldsymbol{0.40}$$

Also, we know

$\boldsymbol{m_{12}(A) = \frac{\sum_{B\cap C=A} m_1(B)m_2(C)}{1-\sum_{B\cap C=\phi} m_1(B)m_2(C)}}$, **for** $\boldsymbol{A \neq \phi}$

Now,

$$\boldsymbol{m_{12}(FC \cup UC)}$$
$$= \frac{\boldsymbol{m_1(FC\cup UC)m_2(FC\cup UC) + m_1(FC\cup UC)m_2(FC\cup OC\cup UC) + m_1(FC\cup OC\cup UC)m_2(FC\cup UC)}}{\boldsymbol{1 - \Big[m_1(FC)m_2(OC) + m_1(FC)m_2(UC) + m_1(FC)m_2(OC\cup UC) + m_1(OC)m_2(FC) + m_1(OC)m_2(UC) + m_1(OC)m_2(FC\cup UC) + m_1(UC)m_2(FC) + m_1(UC)m_2(OC) + m_1(UC)m_2(FC\cup OC) + m_1(FCUOC)m_2(UC) + m_1(FCUUC)m_2(OC) + m_1(OCUUC)}}$$
$$= \boldsymbol{0.20}$$

Check your progress 3:

Answer to Q1:

Here the experts E_1 and E_2 provide their assessment of evidence as follows

$m_1(C) = 0.6, m_1(R) = 0.4, m_1(C \cup R) = 0$

$m_2(C) = 0.6, m_2(R) = 0.4, m_2(C \cup R) = 0$

$\therefore Bel_1(C) = \sum_{B \subseteq C} m_1(B) = m_1(C) = 0.6$

$\therefore Bel_1(R) = \sum_{B \subseteq C} m_1(B) = m_1(R) = 0.4$

$Bel_1(C \cup R) = m_1(C) + m_1(R) + m_1(C \cup R) = 0.6 + 0.4 = 1$

Again

$Pl_1(C) = \sum_{B \cap C \neq \phi} m_1(B) = m_1(C) + m_1(C \cup R) = 0.6$

$Pl_1(R) = \sum_{B \cap R \neq \phi} m_1(B) = m_1(R) + m_1(C \cup R) = 0.4$

And

$$Pl_1(C \cup R) = \sum_{B \cap (C \cup R) \neq \phi} m_1(B) = m_1(C) + m_1(R) + m_1(C \cup R) = 1$$

Here $Bel_1(\boldsymbol{C}) = Pl_1(C)$, $Bel_1(R) = Pl_1(R)$, $Bel_1(C \cup R) = Pl_1(C \cup R)$

$\therefore$ Probabilities for Expert $E_i's$ assessment are

$p_1(C) = 0.6, p_1(R) = 0.4$ and $Pl_1(C \cup R) = p_1(C) +, p_1(R) = 1.$

Check your progress 4:

Answer to Q. No.1:

Possibility measure (Pos) represents the degree of possibility of an event, while necessity measure (Nec) represents the degree of necessity or certainty of an event.

Answer to Q. No.2:

Possibility and necessity measures are used in decision making under uncertainty, such as in risk assessment, financial modelling, and medical diagnosis. For example, in medical diagnosis, possibility measures can be used to represent the degree of possibility of a patient having a particular disease, while necessity measures can be used to represent the degree of certainty of the diagnosis.

Answer to Q. No.3:

Here $m_1(M) = 0.99, m_1(BT) = 0.01, m_1(C) = 0$

$$m_2(M) = 0, m_2(BT) = 0.01, m_2(C) = 0.99$$

$$m_{12}(BT) = \frac{m_1(BT)\, m_2(BT))}{1 - [m_1(M)m_2(BT) + m_1(M)m_2(C) + m_1(BT)m_2(M) + m_1(BT)m_2(C) + m_1(C)m_2(M) + m_1(Cm_2(BT))]}$$

$$= \frac{(0.01) \times (0.01)}{1 - \{0.99(0.01 + 0.99) + 0.01(0 + 0.99) + 0(0 + 0.01)\}}$$

$$= 1$$

Check your progress 4:

Answer to Q. No.1:

In the realm of fuzzy logic and possibility theory, probability distributions play a crucial role in representing and manipulating uncertain information. It describes the degree of possibility that a certain event occurs based on imprecise information, rather than randomness.

Answer to Q. No.2:

The highest degree of possibility in the given fuzzy set is 1 as the possibility distribution is 1 when $x = 4$.Here the value of the possibility distribution indicates the degree to which an element belongs to a set.

6.10 FURTHER READINGS

- An Introduction to Fuzzy set theory and Fuzzy Loic;C.Mohan, Ahshan Publishers,2015.
- Fuzzy set theory and its Applications;H.J.Zimmermans,Allied Publishers Ltd.New Delhi,1991.
- Fuzzy logic with Engineering Applications;Timothy J.Ross,John Wiley & Sons Ltd.
- Fuzzy sets and Fuzzy Logic;G.J.Klir,Bo Yuan

6.11 MODEL QUESTIONS

Q1. Let $X = \{a, b, c, d\}$.Given basic assignments $m(\{a, b, c\}) = 0.5$, $m(\{a, b, d\}) = 0.2$ and $m(X) = 0.3$.Determine the corresponding belief and plausibility.

Q2. Let basic probability assignments m_1 and m_2 on $X = \{a, b, c, d\}$ which are obtained from two independent sources, be defined as follows: $m_1(\{a, b\}) = 0.2$,$m_1(\{a, c\}) = 0.3$, $m_1(\{b, d\}) = 0.5$,$m_2(\{a, d\}) = 0.2$, $m_2(\{b, c\}) = 0.5$, $m_2(\{a, b, c\}) = 0.3$.
Calculate the combined basic probability assignment $m_{1,2}$ by using Dempster rule of combination.

Q3. Let $X = \{a, b, c, d\}$.Given the belief measure $Bel(\{b\}) = 0.1$, $Bel(\{a, b\}) = 0.2$, $Bel(\{b, c\}) = 0.3$, $Bel(\{b, d\}) = 1$.
Determine the corresponding basic assignment.

Q4. Let $X = \{a, b, c, d\}$.Calculate $m_x(X)$ and $m_y(\{b, c\})$ if the basic assignments are given by $m(\{a, b, c\}) = 0.5$, $m(\{a, b, d\}) = 0.2$ and $m(X) = 0.3$.

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UNIT 7: APPROXIMATE REASONING

UNIT STRUCTURE

7.1 Learning Objectives
7.2 Introduction
7.3 Fuzzy Implications
7.4 Axioms of Fuzzy Implications
7.5 Selection of Fuzzy Implications
7.6 Let Us Sum Up
7.7 Answers to Check Your Progress
7.8 Further Readings
7.9 Model Questions

7.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- understand fuzzy implications,
- apply axioms of fuzzy implications,
- select an appropriate fuzzy implication for approximate reasoning under each particular situation,
- solve problems using fuzzy implications.

7.2 INTRODUCTION

In real world scenarios, we often encounter situations where the information is imprecise, uncertain or vague. Traditional binary logic, with its strict true or false values, may not be sufficient to handle such situations. Approximate reasoning, a key aspect of fuzzy logic, provides a framework for dealing with uncertainty and imprecision. This unit will delve into the concept of approximate reasoning, focusing on fuzzy implications, their axioms and selection. We will explore how fuzzy implications can be used to model uncertain relationship and make decision in various applications.

7.3 FUZZY IMPLICATIONS

A fuzzy implication $\mathcal{I}$ of two fuzzy propositions p and q is a function of the form

$$\mathcal{I}: [0,1] \times [0,1] \longrightarrow [0,1]$$

which for any truth values a and b of given fuzzy propositions p , q respectively defined the truth value. $\mathcal{I}(a,b)$ of the conditional proposition
"if p then q".

This function should be an extension of the classical implication, $p \Longrightarrow q$, from the restricted domain $\{0,1\}$ to the full domain $[0,1]$ of truth values in fuzzy logic.

Fuzzy implications as extensions of the classical logic implication:

S –implications: One way of defining $\mathcal{I}$ in classical logic is to use the logic formula

$$\mathcal{I}(a,b) = \bar{a} \vee b \quad \forall a,b \in \{0,1\}$$

When extending the formula to fuzzy logic, we interpret the disjunction and negation as a fuzzy union (t –conorm) and a fuzzy complement respectively. This results in defining $\mathcal{I}$ in fuzzy logic by the formula

$$\mathcal{I}(a,b) = U(C(a),b)$$

Fuzzy implications that are obtained from above formula are referred to as S –implications. The following are examples of four well – known S –implications.

(i) Kleene – Dienes implication

When we choose the standard fuzzy union, we obtain a function $\mathcal{I}_b$ defined for all $a,b \in [0,1]$ by the formula

$$\mathcal{I}_b = max(1-a,b)$$

Which is called a Kleene – Dienes implication.

(ii) Reichenbach implication

Choosing the algebraic sum as a fuzzy union, we obtain

$$\mathcal{I}_b = 1 - a + ab$$

which is called a Reichenbach implication.

(iii) Lukasiewicz implication

The choice of the bounded sum $u(a,b) = min(1, a+b)$ results in the function

$$\mathcal{I}_b = min(1,1-a+b)$$

which is called a Lukasiewicz implication.

(iv) Largest S –implication

When we choose the drastic fuzzy union U_{max}, which is the largest t –conorm, we obtain

$$\mathcal{I}_b = \begin{cases} b & when \quad a = 1 \\ 1-a & when\ b = 0 \\ 1 & otherwise \end{cases}$$

which is the Largest S –implication. $\mathcal{I}$ in fuzzy logic is then defined by the extended formula

$\mathcal{I}(a,b) = sup\{x \in [0,1]: i(a,x) \leq b\}\ \forall a,b \in [0,1]$

where i denotes a continuous fuzzy intersection.

The fuzzy implications that are characterized by the above formula are called R –implicaions

R- implications:

Another way of defining $\mathcal{I}$ in classical logic is to employ the formula

$\mathcal{I}(a,b)$ =max $\{x \in \{0,1\}: a \wedge x \leq b\}\ \forall a,b \in \{0,1\}$

Interpreting the conjunction in this formula as a fuzzy intersection (t –norm). $\mathcal{I}$ in fuzzy logic is then defined by the extended formula

$$\mathcal{I}(a,b) = \sup\{x \in [0,1] / \ i(a,x) \leq b\}\ \forall a,b \in [0,1]$$

Where i denotes a continuous fuzzy intersection. The fuzzy implications that are characterised by above formula are called R −implications. The following are example of four well-known R −implications.

1. **Gödel implication:** Choosing the standard fuzzy intersection, we obtain
$$\mathcal{J}_g(a,b) = \sup\{x \,/\, \min(a,x) \le b\} = \begin{cases} 1 & when \quad a \le b \\ b & when \quad a > b \end{cases}$$
Which is called Gödel implication.
2. **Goguen implication:** Choosing $i(a,b) = ab$, we obtain
$$\mathcal{J}_\Delta(a,b) = \sup\{x/\ a \le b\} = \begin{cases} 1, & when \quad a \le b \\ \dfrac{b}{a}, & when \quad a > b \end{cases}$$
Which is called a Goguen implication.
3. **Lukasiewicz implication:** The choice of $i(a,b) = \max(0, a+b-1)$ (bounded difference) results in the Lukasiewicz implication.
$$\mathcal{J}_a(a,b) = \sup\{x/\max(0, a+x-1) \le b\}$$
$$= \min(1, 1-a+b)$$
4. **Largeot R −implication:** Another implication, defined by
$$\mathcal{J}_{LR}(a,b) = \begin{cases} b & when \quad a = 1 \\ 1 & otherwise \end{cases}$$
Is actually the limit of all $R-$ implications. It serves as the least upper bound of the class of $R-$ implications. For this reason, it is categorized as the largest $R-$ implication.

QL-implications:

One way of defining $\mathcal{J}$ in classical logic is to use the logic formula
$$\mathcal{J}(a,b) = \bar{a} \vee (a \wedge b)$$
The extension of this equation in fuzzy logic is
$$\mathcal{J}(a,b) = u(c(a), i(a,b))$$
Where u, i, c are required to satisfy the De Margan laws, $i.e.$, u and i are dual with respect to c.

The fuzzy implications based on the above formula are called QL-implications, as they were originally employed in quantum logic. The following are four examples of QL-implications, in all of which the standard fuzzy complement is assumed.

1. **Zadeh Implication:** When i and u are the standard min and max operators, we obtain
$$\mathcal{J}_m(a,b) = max[1 - a, min(a,b)]$$
which is sometimes called the Zadeh implication.
2. When i is the algebraic product and u is the algebraic sum, we obtain
$$\mathcal{J}_p(a,b) = 1 - a + a^2 b$$
3. When i is the bounded difference and u is the bounded sum, we obtain the Kleene–Dienes implication $\mathcal{J}_\delta$.
4. When $i = i_{min}$ and $u = u_{max}$, we obtain
$$\mathcal{J}_q(a,b) = \begin{cases} b & when \quad a = 1 \\ 1-a & when \quad a \ne 1, b \ne 1 \\ 1 & when \quad a \ne 1, b \ne 1 \end{cases}$$

Do you know?

- The extension of the equation
$$\mathcal{T}(a,b) = (\bar{a} \wedge \bar{b}) \vee b$$
in fuzzy logic are
$$\mathcal{T}(a,b) = u(i(c(a),c(b)),b)$$
- Lukasiewicz implication is both S −implication and R −implication.

Theorem 7.3.1:

Let u_1 and u_2 be t −conorms such that $u_1(a,b) \leq u_2(a,b) \forall a,b \in [0,1]$, and let f_1, f_2 be S −implications based on the same fuzzy complement c and u_1, u_2 respectively. Then $\mathcal{T}_1(a,b) \leq \mathcal{T}_2(a,b) \forall a,b \in [0,1]$.

Proof:

For all $a,b \in [0,1]$,

$$\mathcal{T}_1(a,b) = u_1(c(a),b)$$

and

$$\mathcal{T}_2(a,b) = u_2(c(a),b)$$

$$\therefore \quad u_1(c(a),b) \leq u_2(c(a),b) \Rightarrow \mathcal{T}_1(a,b) \leq \mathcal{T}_2(a,b)$$

Theorem 7.3.2:

Let i_1 and i_2 be t −norms such that $i_1(a,b) \leq i_2(a,b) \forall a,b \in [0,1]$, and let $\mathcal{T}_1, \mathcal{T}_2$ be R −implications based on i_1, i_2 respectively. Then $\mathcal{T}_1(a,b) \geq \mathcal{T}_2(a,b) \forall a,b \in [0,1]$.
Proof:

Since $i_1(a,b) \leq i_2(a,b) \forall a,b \in [0,1]$, we have:

$$i_1(a,x_0) \leq i_2(a,x_0) \leq b \qquad \forall x_0 \in \{x / i_2(a,x) \leq b\}$$

Then $x_0 \in \{ x \mid i_1(a,x) \leq b \}$.

and, consequently

$$\{ x \mid i_2(a,x) \leq b \} \subseteq \{ x \mid i_1(a,x) \leq b \}$$

Hence:

$$\mathcal{T}_2(a,b) = sup\{ x \mid i_2(a,x) \leq b \} \leq sup\{ x \mid i_1(a,x) \leq b \} = \mathcal{T}_1(a,b)$$

$$\Rightarrow \mathcal{T}_2(a,b) \leq \mathcal{T}_1(a,b), \forall a,b \in [0,1]$$

Do you know?

- $\mathcal{T}_{LS}$ is the largest S −implication and $\mathcal{T}_b$ is the smallest S −implication. Furthermore

$$\mathcal{T}_b \leq \mathcal{T}_r \leq \mathcal{T}_a \leq \mathcal{T}_{LS}$$

- $\mathcal{T}_g$ is the smallest R –implication and

$$\mathcal{T}_g \leq \mathcal{T}_\Delta \leq \mathcal{T}_a \leq \mathcal{T}_{LR}$$

CHECK YOUR PROGRESS 1

Q1. Write two common fuzzy implication operators.
Q2. Which fuzzy implication satisfies the property:

$$\mathcal{T}(0, b) = 1, \quad \forall\, b \in [0,1]$$

Q3. Compute the fuzzy implication using Łukasiewicz implication for $a = 0.6, b = 0.4$.

7.4 AXIOMS OF FUZZY IMPLICATIONS

All fuzzy implications are obtained by generalizing the implication operator of classical logic. That is, they collapse to the classical implication when truth values are restricted to 0 and 1. Identifying various properties of the classical implication and generalizing them appropriately leads to the following properties (reasonable axioms of fuzzy implications):

Axiom 1 (Monotonicity in first argument):

$$a \leq b \Rightarrow \mathcal{T}(a, x) \geq \mathcal{T}(b, x)$$

This means that the truth value of fuzzy implication increases as the truth value of the antecedent decreases.

Axiom 2 (Monotonicity in second argument):

$$a \leq b \Rightarrow \mathcal{T}(x, a) \leq \mathcal{T}(x, b)$$

This means that the truth value of fuzzy implications increases as the truth value of the consequent increases.

Axiom 3 (Dominance of falsity):

$$\mathcal{T}(0, a) = 1$$

This means that falsity implies everything.

Axiom 4 (Neutrality of truth):

$$\mathcal{T}(1, b) = b$$

This means that truth does not imply anything.

Axiom 5 (Identity):

$$\mathcal{T}(a, a) = 1$$

This means that fuzzy implications are true whenever the truth values of the antecedent and consequent are equal.

Axiom 6 (Exchange Property):

$$\mathcal{I}(a,\mathcal{I}(b,x)) = \mathcal{I}(b,\mathcal{I}(a,x))$$

This is a generalization of the equivalence of $a \to (b \to x)$ and $b \to (a \to x)$ in classical logic.

Axiom 7 (Boundary Condition):

$$\mathcal{I}(a,b) = 1 \; iff \; a \leq b$$

This means that fuzzy implications are true if and only if the consequent is at least as true as the antecedent.

Axiom 8 (Contraposition):

$$\mathcal{I}(a,b) = \mathcal{I}(c(b),c(a))$$

for a fuzzy complement c. This means that fuzzy implications are equally true when the antecedent and consequent are exchanged and negated.

Axiom 9 (Continuity):

$\mathcal{I}$ is a continuous function. This property ensures that small changes in the truth values of the antecedent or consequent do not produce large changes in the truth values of fuzzy implications.

Theorem 7.4.1:

A function$\mathcal{I} : [0,1]^2 \to [0,1]$ satisfies Axioms $1-9$ of fuzzy implications for a particular fuzzy complement c if and only if there exists a strict increasing continuous function $\varphi : [0,1] \to [0,\infty)$ such that $\varphi(0) = 0$.

$$\mathcal{I}(a,b) = \varphi^{-1}(\varphi(1) - \varphi(a) + \varphi(b)), \; \forall\, a,b \in [0,1]$$

$$c(a) = \varphi^{-1}(\varphi(1) - \varphi(a)), \; \forall\, a \in [0,1]$$

where φ⁻¹ is the generalized inverse of φ.

Do you know?

- Lukasiewicz implication is the only fuzzy implication that satisfies Axioms $1-9$ for the given standard fuzzy complement.

Example 1:

Consider the function $\varphi(a) = ln(1 + a), \; a \in [0,1]$.
Its pseudo-inverse is

$$\varphi^{-1}(a) = \begin{cases} e^a - 1 & when \quad 0 \leq a \leq \ln 2 \\ 1 & when \quad otherwise \end{cases}$$

And the fuzzy complement generated by φ is given by

$$c(a) = \frac{1 - a}{1 + a}, \quad \forall\, a \in [0,1]$$

The resulting fuzzy implication is

$$\mathcal{T}(a,b) = \min\left(\frac{1 - a + 2b}{1 + a}\right) \qquad \text{for all} \quad a, b \in [0,1]$$

Example 2:

Consider $\varphi(a) = ln(1 + \lambda a)$, where $\lambda > 0$.

Then

$$\varphi^{-1}(a) = \begin{cases} e^a - 1 & when \quad 0 \le a \le \ln 2 \\ 1 & when \quad otherwise \end{cases}$$

And we obtain the Sugeno clan of fuzzy compliments

$$c_\lambda(a) = \frac{1 - a}{1 + \lambda\, a}$$

And the parameterized clan of fuzzy implications,

$$\mathcal{T}_\lambda(a,b) = min\left[1, \frac{1 - a + b + \lambda b}{1 + \lambda a}\right]$$

Fuzzy implications in this class, which can be extended to $\lambda > -1$ by using $f(a) = -ln(1 + \lambda a)$ for $\lambda \in (-1,0)$, are called pseudo-Łukasiewicz implications.

CHECK YOUR PROGRESS 2

Q1. For the function

$$\varphi(a) = \frac{2a}{1 + a}, \quad a \in [0,1]$$

find the pseudo-inverse and fuzzy complement.

Q2. Give an example of a pseudo-Łukasiewicz implication associated with the Yager class of fuzzy complements.

$$C_w(a) = (1 - a^w)^{\left\{\frac{1}{w}\right\}}$$

7.5 SELECTION OF FUZZY IMPLICATIONS

To select an appropriate fuzzy implication for approximate reasoning under each particular situation is a difficult problem. In this section, we discuss some of the issues involved in dealing with this problem and examine some theoretical results pertaining to it.

Let us begin with the generalized modus ponens. According to the fuzzy inference rule, given a fuzzy proposition of the form

$\tilde{p}$: if X is A, then Y is B (1)

and a fact 'X is A", we conclude that 'Y is B" by the compositional rule of inference

$$B' = A' \overset{i}{\circ} R$$

where, $\overset{i}{\circ}$ is the sup$-i$ composition for a t −norm i.

In classical modus ponens, the proposition 'X is A' $(i.e., A' = A)$ leads to the conclusion 'Y is B' by (1).

The generalized modus ponens should coincide with the classical one when $A' = A$.

The composition rule of inference in this case is given by:

$$B = A \overset{i}{\circ} R$$

or

$$\tilde{B}(y) = sup_{\{x \in X\}}\, i\,[\,\tilde{A}(x), \mathcal{T}(\tilde{A}(x), \tilde{B}(y))\,] \quad (2)$$

This equation provides a criterion for selecting an appropriate fuzzy implication $\mathcal{T}$ $i.e.$ any fuzzy implication suitable for approximate reasoning based on the generalized modus ponens should satisfy this condition (2) for arbitrary fuzzy sets $\tilde{A}$ and $\tilde{B}$.

Theorem 7.5.1:

Let A be a normal fuzzy set. For any continuous t −norm i and the associated w −operator, let $\mathcal{T} = w_i, i.e.$,

$$\mathcal{T}(A(x), B(y)) = w_i(A(x), B(y)), \ \forall\, x \in X, y \in Y$$

Then

$$B(y) = sup_{\{x \in X\}}\, i\,[\,A(x), \mathcal{T}(A(x), B(y))\,]$$

Proof:

We know that

$$i(a, w_i(a, b)) \leq b, \ \forall\, a, b \in [0,1]$$

Therefore, we have

$$i(A(x), w_i(A(x), B(y))) \leq B(y), \ \forall\, x \in X, y \in Y$$

Since A is normal, there exists $x_0 \in X$ such that $A(x_0) = 1$. Hence,

$$i(A(x_0), w_i(A(x_0), B(y))) = w_i(1, B(y)) = B(y)$$

Thus,

$$sup_{\{x\in X\}}\, i(A(x), w_i(A(x), B(y))) \;=\; B(y)$$

Theorem 7.5.2:

Let the range of the membership function A in

$$B(y) \;=\; sup_{\{x\in X\}}\, i\,[\,A(x), f(A(x), B(y))\,] \qquad (3)$$

cover the whole interval $[0,1]$. Then the Wu implication is given by

$$\mathcal{T}_{wu}(a,b) = \begin{cases} 1 & a \le b \\ \min(1-a, b) & a > b \end{cases}$$

and it satisfies the above equation (3) for any t −norm i.

Proof:

For all $y \in Y$,

$$sup_{\{x\in X\}}\, i\,[\,A(x), \mathcal{T}_{wu}(A(x), B(y))\,] \;=\; sup_{\{a\in[0,1]\}}\, i\,[\,a, \mathcal{T}_{wu}(a, B(y))\,]$$
$$= \max\{\, sup_{\{a\le B(y)\}}\, i\big[a, \mathcal{T}_{wu}\big(a, B(y)\big)\big], sup_{\{a>B(y)\}}\, i\big[a, \mathcal{T}_{wu}\big(a, B(y)\big)\big]\}$$
$$= \max\left\{B(y), \sup_{\mathrm{a}>B(\mathrm{y})}\, i[a, \min(1-a, B(y))]\right\}$$
$$= B(y)$$

Note: Some known results regarding the outcome of the expression $sup_{\{x\in X\}}\, i\,[\,A(x), \mathcal{T}(A(x), B(y))\,]$ are summarized below.

Table 7.1: Generalized modus Poners

Name	**Standard intersection**	**Algebraic product**	**Bounded difference**	**Drastic intersection**
Early Zadeh $\mathcal{T}_m$	$\max\left[\frac{1}{2}, B\right]$	$\max\left[\frac{1}{4}, B\right]$	B	B
Gaines-Rescher $\mathcal{T}_s$	B	B	B	B
Gödel, $\mathcal{T}_g$	B	B	B	B
Goguen, $\mathcal{T}_\Delta$	$B^{\frac{1}{2}}$	B	B	B
Kleene-Dienes, $\mathcal{T}_b$	$\max\left[\frac{1}{2}, B\right]$	$\max\left[\frac{1}{4}, B\right]$	B	B
Lukasiewicz, $\mathcal{T}_a$	$\frac{1}{2}(1+B)$	$\frac{1}{4}(1+B)^2$	B	B

Reichenbach, $\mathcal{T}_r$	$\frac{1}{2-B}$	$\max\left[B, \frac{1}{4-4\min\left(B,\frac{1}{2}\right)}\right]$	B	B
Willmott, $\mathcal{T}_{wi}$	$\max\left[\frac{1}{2}, B\right]$	$\max\left[\frac{1}{4}, B\right]$	B	B
Wu, $\mathcal{T}_{wu}$	B	B	B	B
$\mathcal{T}_{ss}$	B	B	B	B
$\mathcal{T}_{sg}$	B	B	B	B
$\mathcal{T}_{gg}$	B	B	B	B

- Fuzzy implication suitable for approximate reasoning based upon the generalized modus tollens should satisfy the equation

$$C\big(A(x)\big) = \sup_{y\in Y} i[C(B(y), \mathcal{T}\big(A(x), B(x)\big)] \quad (4)$$

for all $x \in X$

If the ranges of both membership function A and B are assumed to be $[0,1]$, then some results regarding the outcome of the right-hand side of equation (4) for some fuzzy implications and some t –norms are summarized in **Table 7.2**.

Table 7.2 (Generalized modus tollers)

Name	Standard intersection	Algebraic Product	Bounded difference	Drastic intersection
Early Zadeh $\mathcal{T}_m$	$\max\left[\frac{1}{2}, \bar{A}\right]$	$\max\left[\frac{1}{4}, \bar{A}\right]$	$\bar{A}$	$\bar{A}$
Gaines-Rescher $\mathcal{T}_s$	$\bar{A}$	$\bar{A}$	$\bar{A}$	$\bar{A}$
Gödel, $\mathcal{T}_g$	$\max\left[\frac{1}{2}, \bar{A}\right]$	$\max\left[\frac{1}{4}, \bar{A}\right]$	$\bar{A}$	$\bar{A}$
Goguen, $\mathcal{T}_\Delta$	$\frac{1}{1+A}$	$\max\left[\frac{1}{4A}, \bar{A}\right]$	$\bar{A}$	$\bar{A}$
Kleene-Dienes, $\mathcal{T}_b$	$\max\left[\frac{1}{2}, \bar{A}\right]$	$\max\left[\frac{1}{4}, \bar{A}\right]$	$\bar{A}$	$\bar{A}$
Lukasiewicz, $\mathcal{T}_a$	$1-\frac{A}{2}$	$\frac{(A-2)^2}{4}$	$\bar{A}$	$\bar{A}$
Reichenbach, $\mathcal{T}_r$	$\frac{1}{1+A}$	$\begin{cases}\frac{1}{4A(x)}, & A(x)\geq\frac{1}{2}\\ \bar{A}(x), & A(x)<2\end{cases}$	$\bar{A}$	$\bar{A}$
Willmott, $\mathcal{T}_{wi}$	$\max\left[\frac{1}{2}, \bar{A}\right]$	$\max\left[\frac{1}{4}, \bar{A}\right]$	$\bar{A}$	$\bar{A}$
Wu, $\mathcal{T}_{wu}$	$\bar{A}$	$\bar{A}$	$\bar{A}$	$\bar{A}$
$\mathcal{T}_{ss}$	$\bar{A}$	$\bar{A}$	$\bar{A}$	$\bar{A}$
$\mathcal{T}_{sg}$	$\bar{A}$	$\bar{A}$	$\bar{A}$	$\bar{A}$

$\mathcal{T}_{gg}$	$\max\left[\frac{1}{2}, \bar{A}\right]$	$\max\left[\frac{1}{4}, \bar{A}\right]$	$\bar{A}$	$\bar{A}$

For the generalized hypothetical syllogism, the counterpart of equation (3) and (4) is the equation

$$\mathcal{T}\big(A(x), C(z)\big) = \sup_{y \in Y} i[\mathcal{T}\big(A(x), B(y)\big), \mathcal{T}\big(B(y), C(z)\big)] \quad (5)$$

for all $x \in X$ and $z \in Z$.

In Table 7.3, we indicate for some fuzzy implications and some t –norms whether equation (5) is satisfied (Y) or not (N). It is again assumed that the ranges of all membership functions involved are $[0,1]$.

Table 7.3: Generalized hypothetical syllogisms

Name	Standard intersection	Algebraic product	Bounded difference	Drastic intersection
Early Zadeh $\mathcal{T}_m$	N	N	N	N
Gaines-Rescher $\mathcal{T}_s$	Y	Y	Y	Y
Gödel, $\mathcal{T}_g$	Y	Y	Y	Y
Goguen, $\mathcal{T}_\Delta$	N	Y	Y	Y
Kleene-Dienes, $\mathcal{T}_b$	N	N	Y	Y
Lukasiewicz, $\mathcal{T}_a$	N	N	Y	Y
Reichenbach, $\mathcal{T}_r$	N	N	N	N
Willmott, $\mathcal{T}_{wi}$	N	N	N	N
Wu, $\mathcal{T}_{wu}$	Y	Y	Y	Y
$\mathcal{T}_{ss}$	Y	Y	Y	Y
$\mathcal{T}_{sg}$	Y	Y	Y	Y
$\mathcal{T}_{gg}$	Y	Y	Y	Y
$\mathcal{T}_{gs}$	Y	Y	Y	Y

CHECK YOUR PROGRESS 3

Q1. What is Generalized Modus Tollens (GMT) in fuzzy reasoning?

Q2. What is Generalized Hypothetical Syllogism (GHS)?

Q3. Let $A(x) = 0.8,\ A'(x) = 0.7$ and $B(y) = 0.9$.

If an implication is defined as:

$$\mathcal{T}_G(a,b) = 1, if\ a \leq b;$$
$$= b, otherwise$$

Compute,

$$B'(y) = min\ (A'(x), \mathcal{T}_G(A(x), B(y)))$$

Q4. Given $A(x) = 0.6, A'(x) = 0.4$ and $B(y) = 0.9$. Using generalized modus ponens with the Łukasiewicz implication, compute $B'(y)$.

7.6 LET US SUM UP

- A function $\mathcal{T} : [0,1] \times [0,1] \to [0,1]$ is said to be a fuzzy implication if for any possible truth values a, b of given fuzzy propositions p, q, we can define the truth value $\mathcal{T}(a,b)$ of the conditional statement 'if p then q'.
- S −implications are characterized by $\mathcal{T}(a,b) = u(c(a),b), \forall\, a,b \in [0,1]$, where u and c denote a fuzzy union and a fuzzy complement respectively.
- Fuzzy implications characterized by $\mathcal{T}(a,b) = sup\{x \in [0,1] : i(a,x) \leq b\}$ are called R −implications, where i is a continuous fuzzy intersection (t −norm).
- The two implications $\mathcal{T}(a,b) = u(c(a), i(a,b))$ and $\mathcal{T}(a,b) = u(i(c(a),c(b)),b)$ are called QL-implications, where the t −norm i and t −conorm u are dual with respect to the complement c.
- Lukasiewicz implication is both an S −implication and an R −implication.
- The truth value of fuzzy implications increases as the truth value of the antecedent decreases.
- If u_1, u_2 are t −conorms such that $u_1(a,b) \leq u_2(a,b), \forall\, a,b \in [0,1]$, and $\mathcal{T}_1, \mathcal{T}_2$ are S −implications based on the same fuzzy complement c and u_1, u_2 respectively, then $\mathcal{T}_1(a,b) \leq \mathcal{T}_2(a,b), \forall\, a,b \in [0,1]$.

7.7 ANSWERS TO CHECK YOUR PROGRESS

Check your progress 1:
Answer to Q1:

(i) Lukasiewicz implication:

$$\mathcal{T}(a,b) = min(1, 1 - a + b)$$

(ii) Gödel implication

$$\mathcal{T}(a,b) = \begin{cases} 1 & if \quad a \leq b \\ b & if \quad a > b \end{cases}$$

Answer to Q2:
The property $\mathcal{T}(0,b) = 1$ is satisfied by the Lukasiewicz implication.

$$\mathcal{T}(0,b) = min(1, 1 - 0 + b) = min(1, 1 + b) = 1$$

Answer to Q3:
Using Lukasiewicz implication:

$$\mathcal{T}(a,b) = min(1, 1 - a + b) = min(1, 1 - 0.6 + 0.4) = 0.8$$

Check your progress 2:
Answer to Q1:
Given

$$\varphi(a) = \frac{2a}{1 + a}, \quad a \in [0,1]$$

Then its pseudo inverse is

$$\varphi^{-1}(a) = \begin{cases} \frac{a}{2-a} & when \quad 0 \le a \le 1 \\ 1 & otherwise \end{cases}$$

and the fuzzy complement

$$C(a) = \frac{1-a}{1+3a} \qquad \text{for all} \quad a \in [0,1]$$

Answer to Q2:

The required pseudo-Lukasiewicz implication is defined by:

$$\mathcal{T}_w(a,b) = min\,[\,1, (1 - a^w + b^w)^{\{\frac{1}{w}\}}\,]$$

where $w > 0$. This class of fuzzy implications is based on $f(a) = a^w$, and $\mathcal{T}_w$ increases with increasing w.

Check your progress 3:

Answer to Q1:

Modus Tollens in classical logic:

If A then B, and if not B then not A.

In fuzzy logic, Generalized Modus Tollens is represented by:

$$c(A(x)) = sup_{\{y \in Y\}}\, i\,[\,c(B(y)), \mathcal{T}(A(x), B(y))\,], \quad \forall\, x \in X$$

Answer to Q2:

In classical logic, GHS is:

If $A \to B$ and $B \to C$, then $A \to C$.

In fuzzy logic, Generalized Hypothetical Syllogism is represented by:

$$\mathcal{T}(A(x), C(z)) = sup_{\{y \in Y\}}\, i\,[\,\mathcal{T}(A(x), B(y)), \mathcal{T}(B(y), C(z))\,],$$

for all $x \in X$ and $z \in Z$.

Answer to Q3:

Given,

$$A(x) = 0.8, \qquad A'(x) = 0.7, \qquad B(y) = 0.9$$

Since,

$$A(x) = 0.8 \le 0.9 = B(y)$$

Thus,

$$\mathcal{T}_a(A(x), B(y)) = 1$$

Therefore,

$$B'(y) = min(0.7, 1) = 0.7$$

Answer to Q4:

Given,

$$A(x) = 0.6, \qquad A'(x) = 0.4, \qquad B(y) = 0.9$$

Using Lukasiewicz implication,

$$\mathcal{T}(a,b) = \mathcal{T}(0.6,0.9) = min(1, 1 - a + b) = min(1, 1 - 0.6 + 0.9) = 1$$

Hence,

$$B'(y) = min(0.4, 1) = 0.4$$

7.8 FURTHER READING

1. Fuzzy Sets and Fuzzy Logic — G. J. Klir, Bo Yuan
2. Fuzzy Sets and Their Applications — S. K. Pundir, R. Pundir

7.9 MODEL QUESTIONS

Q1. Prove that the following fuzzy implications satisfy the generalized modus ponens, modus tollens, and hypothetical syllogism:
a) $\mathcal{I}_s$ b) $\mathcal{I}_{wu}$ c) $\mathcal{I}_{ss}$ d) $\mathcal{I}_{sg}$
Q2. Let f be a function defined by, $f(a) = e^a$, $\forall\, a \in [0,1]$. Find the fuzzy intersection, fuzzy union, fuzzy implication and fuzzy complement generated by f.
Q3. Prove that Axiom 7 implies Axiom 3 and Axiom 4.
Q4. Let i and u be the standard fuzzy intersection and fuzzy union respectively. Suppose that $\mathcal{I}$ is a fuzzy implication satisfying Axiom 1 and Axiom 2. Prove that for any $a, b, c \in [0,1]$,

$$\mathcal{I}(a, i(b, c)) = i(\mathcal{I}(a, b), \mathcal{I}(a, c))$$

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UNIT 8: FUZZY LOGIC

UNIT STRUCTURE

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8.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- understand the basic concept of fuzzy logic and how it differs from classical (crisp) logic.
- explain the meaning of fuzzy propositions and represent statements using fuzzy truth values.
- describe different types of fuzzy connectives such as AND, OR, and NOT used in fuzzy reasoning.
- understand the concept and working principle of fuzzy inference systems.
- explain the role and significance of fuzzy quantifiers such as *most, many, few,* etc. in fuzzy logic.
- understand the concept of linguistic hedges and their use in modifying fuzzy sets.
- perform inference from conditional fuzzy propositions using fuzzy reasoning rules.Understand possibility and necessity measures.

8.2 INTRODUCTION

In classical logic, statements are considered to be either true or false. This type of reasoning is known as binary or crisp logic, where the truth value of a proposition is restricted to two possibilities: 0 (false) **or** 1 (true)**.** However, many situations in real life cannot be described accurately using such strict true–false distinctions. For example, statements such as *"the weather is hot," "the road is long,"* or *"the student is intelligent"* are not absolutely true or false; instead, they may be partially true to different degrees**.**

To deal with such situations involving uncertainty, vagueness, and imprecision, the concept of fuzzy logic was introduced. Fuzzy logic provides a mathematical framework for representing and reasoning with approximate information rather than precise values. In fuzzy logic, the truth value of a statement can take any value between 0 and 1, representing the degree of truth.

Fuzzy logic plays an important role in many fields such as artificial intelligence, control systems, decision-making, pattern recognition, and expert systems. It allows machines and systems to handle problems that involve human-like reasoning and linguistic descriptions, which are often difficult to express using classical mathematical models.

In this unit, we discuss the basic concepts of fuzzy logic and the logical structures used in fuzzy reasoning. We begin with the ideas of fuzzy logic and fuzzy propositions, which extend classical propositions by allowing graded truth values. We then study fuzzy connectives, which generalize logical operations such as AND, OR, and NOT. The unit also introduces the concept of fuzzy inference, a method used to derive conclusions from fuzzy rules.

Further, we explore fuzzy quantifiers (such as *most, many,* and *few*) and linguistic hedges (such as *very* and *somewhat*), which help represent human language in fuzzy systems. Finally, we study the process of inference from conditional fuzzy propositions, which forms the basis of reasoning in fuzzy logic systems.

Thus, this unit provides a foundation for understanding how fuzzy logic can be used to model uncertainty and approximate reasoning in real-world problems.

8.3 FUZZY LOGIC AND FUZZY PROPOSITION

Fuzzy logic is an extension of classical logic that allows reasoning withuncertain, vague, or imprecise information. In many real-life situations, statements cannot be described simply as completely true or completely false. Instead, they may be partially true to some degree. Fuzzy logic provides a mathematical framework to deal with such situations.

The fundamental difference between a classical proposition and a fuzzy proposition lies in the range of their truth values. A classical proposition can have only two truth values, namely true or false. In contrast, in fuzzy logic the truth value of a fuzzy proposition is not limited to these two possibilities; instead, it can take any value between 0 and 1, representing different degrees of truth or falsity. Thus, the truth of a fuzzy proposition is a matter of degree rather than a strict true or false condition.

Let $\bar{V}(\tilde{A})$be a normalized fuzzy set representing the truth value of a fuzzy proposition $\tilde{A}$.It is written as

$\bar{V}(\tilde{A}) = \{(v_i, \mu_i): i = 1,2,\cdots,n, v_i \in [0,1]\}$ where μ_i is possible truth values (between 0 and 1) and v_i is membership degree associated with v_i.

This set represents a possibility distribution of truth values.

In fuzzy logic, the negation of a proposition is obtained using the extension principle.

If$\bar{V}(\tilde{A}) = \{(v_i, \mu_i): i = 1,2,\cdots,n, v_i \in [0,1]\}$,then the truth value of not$\tilde{A}$ is

$$\bar{V}(not\ \tilde{A}) = \{(1 - v_i, \mu_i): i = 1,2,\cdots,n, v_i \in [0,1]\}$$

In particular “False” is interpreted as “not true”, that is

$$\bar{V}(\text{False}) = \{(1 - v_i, \mu_i): i = 1,2,\cdots,n, v_i \in [0,1]\}$$

For example: Let us consider the term true and false respectively, defined as the following possibility distributions.

$$\bar{V}(\text{true}) = \{(.5,.6),(.6,.7),(.7,.8),(.8,.9),(.9,1),(1,1)\}$$

$$\bar{V}(\text{False}) = \bar{V}(\text{not true}) = \{(.5,.6),(.4,.7),(.3,.8),(.2,.9),(.1,1),(0,1)\}$$
$$\bar{V}(\text{very true}) = \{(.5,.6),(.6,.7),(.7,.8),(.8,.9),(.9,1),(1,1)\}$$
$$\bar{V}(\text{very false}) = \{(.5,.6),(.4,.7),(.3,.8),(.2,.9),(.1,1),(0,1)\}$$

Fuzzy logic is essentially considered as an application theory of logic. Hence, the logical operators "and"," or" and "not" are defined accordingly.

Definition: For numerical truth values $\bar{v}(\tilde{A})$ and $\bar{v}(\tilde{B})$,the logical operations and, or and nor, implied are defined as

(1) $\bar{v}(\tilde{A}) \wedge \bar{v}(\tilde{B}) = \bar{v}(\tilde{A} \text{ and } \tilde{B}) = \{(v, min\{\mu_{\tilde{A}}(v), \mu_{\tilde{B}}(v)\})\}$

(2) $\bar{v}(\tilde{A}) \vee \bar{v}(\tilde{B}) = \bar{v}(\tilde{A} \text{ or } \tilde{B}) = \{(v, max\{\mu_{\tilde{A}}(v), \mu_{\tilde{B}}(v)\})\}$

(3) $\neg\bar{v}(\tilde{A}) = nor\bar{v}(\tilde{A}) = \{v, 1 - \mu_{\tilde{A}}(v)\}$

(4) $\bar{v}(\tilde{A}) \Longrightarrow \bar{v}(\tilde{B}) \equiv \bar{v}(\tilde{A} \Longrightarrow \tilde{B}) = \neg\bar{v}(\tilde{A}) \vee \bar{v}(\tilde{B})$
$= \{(v, max\{1 - \mu_{\tilde{A}}(v), \mu_{\tilde{B}}(v)\})\}$

For Example:

(1) $\bar{v}(\tilde{A}) = \{(.6,.7),(.7,.8),(.8,.9),(.9,1),(1,1)\}$
$\bar{v}(\tilde{B}) = \{(.5,.6),(.6,.7),(.9,1)\}$

In fuzzy logic, the AND operation is defined as

$$A \wedge B$$

Membership is calculated by

$$\min(\mu_A(v), \mu_B(v))$$

So, for each truth value v ,we find membership in A membership in B and then we take the minimum. If a value is missing in one set, its membership is taken as 0**.**

Then for all possible v:

$$0.5,0.6,0.7,0.8,0.9,1$$

Now, we calculate minimum for each value as follows

v	$\mu_{\tilde{A}}(v)$	$\mu_{\tilde{B}}(v)$	$min\{\mu_{\tilde{A}}(v), \mu_{\tilde{B}}(v)\}$
0.5	0	0.6	0
0.6	0.7	0.7	0.7
0.7	0.8	0	0
0.8	0.9	0	0
0.9	1	1	1
1	1	0	0

$$\bar{v}(\tilde{A}) \wedge \bar{v}(\tilde{B}) = \{(v, min\{\mu_{\tilde{A}}(v), \mu_{\tilde{B}}(v)\})\}$$

$$\therefore \bar{v}(\tilde{A}) \wedge \bar{v}(\tilde{B}) = \{(.5,0), (.6,.7), (.7,0), (.8,0), (.9,1), (1,0)\}$$

(2) Let $\bar{v}(\tilde{A}) = True = \{(.5,.6), (.6,.7), (.7,.8), (.8,.9), (.9,1), (1,1)\}$

The negation of a fuzzy truth value is obtained by $1 - v$.

Thus the truth scale is reversed and we obtain

$\neg\bar{v}(A)$

$= \{(0,1), (0.1,1), (0.2,1), (0.3,1), (0.4,1), (0.5,0.4), (0.6,0.3), (0.7,0.2), (0.8,0.1), (0.9,0), (1,0)\}$ which represents False.

Truth table:

Zadeh extended the usual **Boolean logic** (which has only two truth values: **True (T)** and **False (F)**) to handle situations where the truth of a statement is **not completely known**.

So, he introduced additional truth values.

1. Four truth values in Zadeh's logic

The four possible truth values are:

1. **True (T)** → The statement is completely true.
2. **False (F)** → The statement is completely false.
3. **Unknown (T + F)** → We do not know whether it is true or false. It could be either.
4. **Undecided (⊖)** → The truth cannot yet be determined.

Zadeh suggested truth tables for the determination of truth values for operation using a four valued logic including the truth values, True, False, Undecided and Unknown. "Unknown is then interpreted as 'True or False'$(T + F)$ and "Undecided" is denoted by $\ominus$.

Extended the normal Boolean logic with truth values (1) and False (00 to a fuzzy three valued logic (True $= T$,False$= F$,Unknown$= T + F$) with a universe of truth values being two valued (True and False), we obtain the following table.

$\wedge$	T	F	$T + F$
T	T	F	$T + F$
F	F	F	F
$T + F$	$T + F$	F	$T + F$

$\vee$	T	F	$T + F$
T	T	T	T
F	T	F	$T + F$
$T + F$	T	$T + F$	$T + F$

$\tilde{A}$	$\neg\tilde{A}$ or $\sim \tilde{A}$
T	F
F	T
$T + F$	$F + T$

For example:

We have a set

$$v = \{0, 0.1, 0.2, \ldots, 1\}$$

This represents **degrees of truth** (from completely false = 0 to completely true = 1).

Let $v = \{0, .1, .2, \cdots, 1\}$ be the universal such that
True= $\{(.8, .9), (.9, 1), (1, 1)\}$
More or less true= $\{(.6, .2), (.7, .4), (.8, .7), (.9, 1), (1, 1)\}$
and almost true= $\{(.8, .9), (.9, 1), (1, .8)\}$
Let $\tilde{A}$ = "More or less true", be the ith row and $\tilde{B}$ = "almost true".

In fuzzy logic:

$$\tilde{A} \vee \tilde{B} = \max(\tilde{A}, \tilde{B})$$

Now, we compute

x	Ã (More or less true)	B̃ (Almost true)	Max (OR result)
0.6	0.2	-	0.2
0.7	0.4	-	0.4
0.8	0.7	0.9	0.9
0.9	1	1	1
1	1	0.8	1

So,

$$\tilde{A} \vee \tilde{B} = \{(.6, .2), (.7, .4), (.8, .9), (.9, 1), (1, 1)\}$$

The approximate right value of this equation gives

True= $\{(.8, .9), (.9, 1), (1, 1)\}$

i.e., "More or less true ∨ Almost true ≈ True".

8.4 FUZZYCONNECTIVES

Fuzzy logic similar to crisp logic support the following connectives:

(i) Negation: ~
(ii) Disjunction: ∨
(iii) Conjunction: ∧
(iv) Implication: ⟹

Let $\tilde{p}$, $\tilde{q}$ are fuzzy propositions and $T(\tilde{p}), T(\tilde{q})$ are their truth values, then we have the following table of fuzzy connectives.

Symbol	Connective	Usage	Definition
~	Negation	$\tilde{p}$	$1 - T(\tilde{p})$
∨	Disjunction	$\tilde{p} \vee \tilde{q}$	$\max\big(T(\tilde{p}), T(\tilde{q})\big)$
∧	Conjunction	$\tilde{p} \wedge \tilde{q}$	$\min\big(T(\tilde{p}), T(\tilde{q})\big)$
$\Longrightarrow$	Implication	$\tilde{p} \Longrightarrow \tilde{q}$	$\sim\tilde{p} \vee \tilde{q} = \max\big(1 - T(\tilde{p}), T(\tilde{q})\big)$

Examples:

(i) Negation:
If $T(p) = 0.7$, then $T(\sim p) = 1 - 0.7 = 0.3$.

(ii) Disjunction:
If $T(p) = 0.6, T(q) = 0.8$, then Result = max (0.6, 0.8) = 0.8.

(iii) Conjunction:
If $T(p) = 0.6, T(q) = 0.8$, then Result = min (0.6, 0.8) = 0.6.

(iv) Implication:

If $T(p) = 0.7, T(q) = 0.5$, then $\max(1 - 0.7, 0.5) = \max(0.3, 0.5) = 0.5$

IF–THEN Rule:

The implication ($\Longrightarrow$) represents the IF –THEN statement as
IF x is $\tilde{A}$, THEN y is $\tilde{B}$ and is equivalent to

$$\max\left(\min\left(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}^{c}(y)\right), 1 - \mu_{\tilde{A}}(x)\right)$$

$$\tilde{R} = \left(\tilde{A}^c \times \tilde{B}\right) \vee \left(\tilde{A} \times \tilde{Y}\right) \approx \left(\tilde{A} \times \tilde{B}\right) \vee \left(\tilde{A}^c \times \tilde{Y}\right)$$

The membership function of $\tilde{R}$ is given by

$$\mu_{\tilde{R}}(x, y) = \max\left(\min\left(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}^{c}(y)\right), 1 - \mu_{\tilde{A}}(x)\right)$$

Compound Rule (IF–THEN–ELSE):

Also, for the compound implication IF x is $\tilde{A}$, THEN y is $\tilde{B}$ ELSE y is $\tilde{C}$,the relation $\tilde{R}$ is equivalent to

$$\tilde{R} = \left(\tilde{A} \times \tilde{B}\right) \vee \left(\tilde{A}^c \times \tilde{C}\right)$$

The membership function of $\tilde{R}$ is given by

$$\mu_{\tilde{R}}(x, y) = \max\left(\min\left(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)\right), \min(1 - \mu_{\tilde{A}}(x), \mu_{\tilde{C}}(x)\right)$$

Example 1: Let $\tilde{p}$: Ravi is efficient, $\mathrm{T}(\tilde{\mathrm{p}}) = 0.8$
$\tilde{q}$: Sita is efficient, $\mathrm{T}(\tilde{\mathrm{q}}) = 0.65$
Then, determine $T(\tilde{p}^c), T(\tilde{p} \wedge \tilde{q}), T(\tilde{p} \vee \tilde{q})$ and $\tilde{p} \Longrightarrow \tilde{q}$.
Solution:

(i) $\tilde{p}^c$: Ravi is not efficient.

$$T(\tilde{p}^c) = 1 - \mathrm{T}(\tilde{\mathrm{p}}) = 1 - 0.8 = 0.2$$

(ii) $\tilde{p} \wedge \tilde{q}$: Ravi and Sita both are efficient

$$T(\tilde{p} \wedge \tilde{q}) = \min\big(T(\tilde{p}), T(\tilde{q})\big)$$

$$= \min(0.8,0.65)$$
$$= 0.65$$

(iii) $\tilde{p} \vee \tilde{q}$: Either Ravi or Sita is efficient

$$T(\tilde{p} \vee \tilde{q}) = \max\big(T(\tilde{p}), T(\tilde{q})\big)$$
$$= \max(0.8,0.65)$$
$$= 0.8$$

(iv) $\tilde{p} \Longrightarrow \tilde{q}$: If Ravi is efficient, then Sita is efficient

$$T(\tilde{p} \Longrightarrow \tilde{q}) = \max\big(1 - T(\tilde{p}), T(\tilde{q})\big)$$
$$= \max(0.2,0.65)$$
$$= 0.65$$

Example 2: Let $X = \{a, b, c, d\}, Y = \{1,2,3,4\}$
and $\tilde{A} = \{(a, 0), (b, 0.8), (c, 0.6), (d, 1)\}$; $\tilde{B} = \{(1,0.2), (2,1), (3,0.8), (4,0)\}$
$\tilde{C} = \{(1,0), (2,0.4), (3,1), (4,0.81)\}$

Determine the implication relations:

(a) IF x is $\tilde{A}$, THEN y is $\tilde{B}$

(b) IF x is $\tilde{A}$, THEN y is $\tilde{B}$ ELSE y is $\tilde{C}$.

Solution:

(a) IF x is $\tilde{A}$, THEN y is $\tilde{B}$ is equivalent to $\tilde{R} = (\tilde{A} \times \tilde{B}) \vee (\tilde{A}^c \times \tilde{Y})$

where $\mu_{\tilde{R}}(x, y) = \max\Big(\min\big(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}^{c}(y)\big), 1 - \mu_{\tilde{A}}(x)\Big)$

Complement of $\tilde{A}$:

$\tilde{A}^c = \{(a, 1), (b, 0.2), (c, 0.4), (d, 0)\}$

Now,

$$\tilde{A} \times \tilde{B} = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.2 & 0.8 & 0.8 & 0 \\ 0.2 & 0.6 & 0.6 & 0 \\ 0.2 & 1 & 0.8 & 0 \end{bmatrix} \end{array}$$

Also,

$$\tilde{A}^c \times \tilde{Y} = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0.2 & 0.2 & 0.2 & 0.2 \\ 0.4 & 0.4 & 0.4 & 0.4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Here,$\tilde{Y}$ is the universe of discourse could be viewed as $\{(1,1),(2,1),(3,1),(4,1)\}$ a fuzzy set of all of whose elements x,we have $\mu(x)=1$.
Therefore,

$$\tilde{R}=\begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0.2 & 0.8 & 0.8 & 0.2 \\ 0.4 & 0.6 & 0.6 & 0.4 \\ 0.2 & 0.1 & 0.8 & 0 \end{bmatrix} \end{array}$$

which represents

"IF x is $\tilde{A}$, THEN y is $\tilde{B}$".

(b) Here,we want to find

$$\tilde{R}=(\tilde{A}\times\tilde{B})\vee(\tilde{A}^c\times\tilde{C})$$

where $\mu_{\tilde{R}}(x,y)=\max\left(\min\left(\mu_{\tilde{A}}(x),\mu_{\tilde{B}}(y)\right),\min(1-\mu_{\tilde{A}}(x),\mu_{\tilde{C}}(x)\right)$

Now,

$$\tilde{A}\times\tilde{B}=\begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.2 & 0.8 & 0.8 & 0 \\ 0.2 & 0.6 & 0.6 & 0 \\ 0.2 & 1 & 0.8 & 0 \end{bmatrix} \end{array}$$

Also,

$$\tilde{A}^c\times\tilde{C}=\begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0.4 & 1 & 0.8 \\ 0 & 0.2 & 0.2 & 0.2 \\ 0 & 0.4 & 0.4 & 0.4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Hence,
$\tilde{R}=\max\left((\tilde{A}\times\tilde{B}),(\tilde{A}^c\times\tilde{C})\right)$ gives

$$\tilde{R}=\begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0.4 & 1 & 0.8 \\ 0.2 & 0.8 & 0.8 & 0.2 \\ 0.2 & 0.6 & 0.6 & 0.4 \\ 0.2 & 1 & 0.8 & 0 \end{bmatrix} \end{array}$$

which represents

"IF x is $\tilde{A}$, THEN y is $\tilde{B}$ ELSE y is $\tilde{C}$."

CHECK YOUR PROGRESS 1

Q1. Let $\tilde{p}$: Amit is hardworking, $\mathrm{T}(\tilde{\mathrm{p}}) = 0.7$

$\tilde{q}$: Neha is hardworking, $\mathrm{T}(\tilde{\mathrm{q}}) = 0.5$

Find:

$T(\tilde{p}^c), T(\tilde{p} \wedge \tilde{q}), T(\tilde{p} \vee \tilde{q})$ and $\tilde{p} \Longrightarrow \tilde{q}$.

Q2. Let $X = \{p, q, r\}, Y = \{1,2,3\}$

Define fuzzy sets

$$\tilde{A} = \{(p, 0.3), (q, 0.7), (r, 1)\};$$
$$\tilde{B} = \{(1,0.5), (2,1), (3,0.2)\};$$
$$\tilde{C} = \{(1,0.1), (2,0.6), (3,1)\}$$

Determine the implication relations:

(a) IF x is $\tilde{A}$, THEN y is $\tilde{B}$

(b) IF x is $\tilde{A}$, THEN y is $\tilde{B}$ ELSE y is $\tilde{C}$.

8.5 FUZZY INFERENCE

In the traditional logic,the main branch of reasoning are tautologies.Then instance,the modus ponens is

$$\left(\tilde{A} \wedge \left(\tilde{A} \Longrightarrow \tilde{B}\right)\right) \Longrightarrow \tilde{B}$$

Or

Premises	$\tilde{A}$ is true
Implication	If $\tilde{A}$,then $\tilde{B}$
Conclusion	$\tilde{B}$ is true

where $\tilde{A}$ and $\tilde{B}$ are propositions or statements.

The above version of modus ponens is then called "Generalized Modus Ponens".

Generalized Modus Ponens (GMP):

Premise:x is $\tilde{A}'$ (approximate A)

Rule: If x is,then y is $\tilde{B}$

Conclusion: y is $\tilde{B}'$(approximate B)

For Example : Let $\tilde{A}$,$\tilde{A}'$ and $\tilde{B}$,$\tilde{B}'$ be fuzzy relations.Then the Generalized Modus Ponens stated as

Premises	x is $\tilde{A}'$	$x \in \tilde{A}'$
Implication	If x is,then y is $\tilde{B}$	If $x \in \tilde{A}'$,then $y \in \tilde{B}$
Conclusion	y is $\tilde{B}'$	$y \in \tilde{B}'$

For Example:

Premises	Today's temperature is warm
Implication	If temperature is hot, then ice cream sales are high
Conclusion	Ice cream sales are moderately high

Fuzzy Restriction:

Definition: Let $\tilde{R}(x)$, $\tilde{R}(x,y)$ and $\tilde{R}(y)$ $\forall x \in X, y \in Y$ be fuzzy relations in $X, X \times Y$ and Y respectively, which acts as fuzzy restrictions on X, (x,y) and , y respectively.

Let $\tilde{A}$ and $\tilde{B}$ denote particular fuzzy sets in X and $X \times Y$,then the corresponding rule of inference asserts that the solution of the relational assignments equations

$\tilde{R}(x) = \tilde{A}$, $\tilde{R}(x,y) = \tilde{B}$ is given by

$$\tilde{R}(y) = \tilde{A} o \tilde{B}$$

Rule of Inference :

If we know $\tilde{R}(x) = \tilde{A}$, $\tilde{R}(x,y) = \tilde{B}$,then we can find the restriction on y by $\tilde{R}(y) = \tilde{A} o \tilde{B}$.

This is called **max–min composition**.

For example: Let $X = \{1,2,3,4\}$ be the set in the universe and $A = \{little\} = \{(1,1), (2,.6), (3,.2), (4,0)\}$ and

$\tilde{R}$ =approximately equal be a fuzzy relation defined by the membership matrix.

$$B \equiv \tilde{R}(x,y) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0.5 & 0 & 0 \\ 0.5 & 1 & 0.5 & 0 \\ 0 & 0.5 & 1 & 0.5 \\ 0 & 0 & 0.5 & 1 \end{bmatrix} \end{array}$$

where $x, y \in X$
Then,by formula

$$\tilde{R}(y) = \tilde{A} o \tilde{B}$$
$$= [1,0.6,0.2,0] o \begin{bmatrix} 1 & 0.5 & 0 & 0 \\ 0.5 & 1 & 0.5 & 0 \\ 0 & 0.5 & 1 & 0.5 \\ 0 & 0 & 0.5 & 1 \end{bmatrix}$$
$$= [1,0.6,0.5,0.2]$$
$$= \{(1,1), (2,0.6), (3,0.5), (4,0.2)\}$$

or $\tilde{R}(y)$ =max min $\{\mu_{\tilde{A}}(x), \mu_{\tilde{R}}(x, y)\}$

Note: Fuzzy restriction uses known information about x and a relation between x and y to infer information about y using max–min composition.
Example 3: Apply the fuzzy modus ponens rule to deduce rotation is quite slow when it is given that:

(i) If the temperature is high,then the rotation is slow.
(ii) The temperature is very high.

Let $\tilde{H}$(high),$V\tilde{H}$ (very high),$\tilde{S}(slow)$ and $Q\tilde{S}$(quite slow) indicate the associated fuzzy sets as follows:
For X =the set of temperatures $= \{30,40,50,60,70,80,90,100\}$

$$\tilde{H} = \{(70,1), (80,1), (90,0.3)\}$$
$$V\tilde{H} = \{(90,0.9), (100,1)\}$$

and Y =the set of rotations per minute= $\{10,20,30,40,50,60\}$

$$Q\tilde{S} = \{(10,1), (20,0.8)\}$$
$$\tilde{S} = \{(30,0.8), (40,1), (50,0.6)\}$$

Solution: The implication is converted into a fuzzy relation:

$$\tilde{R}(x, y) = \max\left(\tilde{H} \times \tilde{S}, \tilde{H}^c \times Y\right)$$

Now,

$$\tilde{H} \times \tilde{S} = \begin{array}{c} \\ 30 \\ 40 \\ 50 \\ 60 \\ 70 \\ 80 \\ 90 \\ 100 \end{array} \begin{array}{c} \begin{array}{cccccc} 10 & 20 & 30 & 40 & 50 & 60 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 1 & 0.6 & 0 \\ 0 & 0 & 0.8 & 1 & 0.6 & 0 \\ 0 & 0 & 0.3 & 0.3 & 0.3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\tilde{H}^c \times Y = \begin{array}{c} \\ 30 \\ 40 \\ 50 \\ 60 \\ 70 \\ 80 \\ 90 \\ 100 \end{array} \begin{array}{c} \begin{array}{cccccc} 10 & 20 & 30 & 40 & 50 & 60 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0.7 & 0.7 & 0.7 & 0.7 & 0.7 & 0.7 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

and

$$\tilde{R}(x,y) = \begin{array}{c} \\ 30 \\ 40 \\ 50 \\ 60 \\ 70 \\ 80 \\ 90 \\ 100 \end{array} \begin{array}{c} \begin{array}{cccccc} 10 & 20 & 30 & 40 & 50 & 60 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0.8 & 1 & 0.6 & 0 \\ 0 & 0 & 0.8 & 1 & 0.6 & 0 \\ 0.7 & 0.7 & 0.7 & 0.7 & 0.7 & 0.7 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

To deduce rotation is quite slow,we use the following composition rule.

$$Q\tilde{S} = V\tilde{H} o\, \tilde{R}(x,y)$$

$$= [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0.9 \quad 1]\, o \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0.8 & 1 & 0.6 & 0 \\ 0 & 0 & 0.8 & 1 & 0.6 & 0 \\ 0.7 & 0.7 & 0.7 & 0.7 & 0.7 & 0.7 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= [r_{11} \quad r_{12} \quad r_{13} \quad r_{14} \quad r_{15} \quad r_{16}]$$

Where

$$r_{11} =$$

$\max\{min(0,1), min(0,1), min(0,1), min(0,1), min(0,0), min(0,0), min(.9,.7) min(1,1)\}$

$= \max\{0,0,0,0,0,0,.7,1\} = 1$

$$r_{12} =$$

$\max\{min(0,1), min(0,1), min(0,1), min(0,1), min(0,0), min(0,0), min(.9,.7) min(1,1)\}$

$= \max\{0,0,0,0,0,0,.7,1\} = 1$

$$r_{13} =$$

$\max\{min(0,1), min(0,1), min(0,1), min(0,1), min(0,.8), min(0,.8), min(.9,.7) min(1,1)\}$

$= \max\{0,0,0,0,0,0,.7,1\} = 1$

$$r_{14} =$$

$\max\{min(0,1), min(0,1), min(0,1), min(0,1), min(0,1), min(0,1), min(.9,.7) min(1,1)\}$

$= \max\{0,0,0,0,0,0,.7,1\} = 1$

$$r_{15} =$$

$\max\{min(0,1), min(0,1), min(0,1), min(0,1), min(0,.6), min(0,.6), min(.9,.7) min(1,1)\}$

$=\max\{0,0,0,0,0,0,.7,1\} = 1$

$$r_{15} =$$

$\max\{min(0,1), min(0,1), min(0,1), min(0,1), min(0,0), min(0,0), min(.9,.7) min(1,1)\}$
$=\max\{0,0,0,0,0,0,.7,1\} = 1$

$$\Longrightarrow Q\tilde{S} = [1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1]$$

Now, We use the composition:

$$Q\tilde{S} = V\tilde{H} o\, \tilde{R}(x,y)$$

$$= [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0.9 \quad 1] o \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0.8 & 1 & 0.6 & 0 \\ 0 & 0 & 0.8 & 1 & 0.6 & 0 \\ 0.7 & 0.7 & 0.7 & 0.7 & 0.7 & 0.7 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= [1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1]$$

8.6 FUZZY PROPOSITION

A fuzzy proposition is a statement whose truth is not just true or false but can be partly true**.**

Therefore,given $\tilde{p}$ to be a fuzzy proposition,$T(\tilde{p})$ represent the truth values which lies in the interval $[0,1]$.

In fuzzy logic,truth values are multivalued such as absolutely true,partly true,absolutely false,very true, and so on and are numerically equivalent to $[0,1]$.The fuzzy propositions are associated with fuzzy sets.The fuzzy membership values associated with the fuzzy set $\tilde{A}$ for $\tilde{p}$ is trated as the fuzzy truth values $T(\tilde{p})$.

i.e., $T(\tilde{p}) = \mu_{\tilde{A}}(x)$,where $0 \leq \mu_{\tilde{A}}(x) \leq 1$.

For example:

$\tilde{p}$: Ravi is honest.

$T(\tilde{p}) = 0.8$, if Ravi is partly honest (mostly true).

$T(\tilde{p}) = 1$, if Ravi is completely honest (absolutely true).

Note: The main difference between classical propositions and fuzzy propositions lies in their truth values. A classical proposition is either completely true or completely false, whereas a fuzzy proposition can have varying degrees of truth. In fuzzy logic, the truth of a proposition is determined by its degree of membership, which is represented by a value in the interval $[0,1]$.

Types of Fuzzy Propositions:

Fuzzy propositions can be grouped into four categories. The first category is Unconditional and Unqualified Propositions.

1. Unconditional and Unqualified Fuzzy Propositions

A fuzzy proposition of this type has the canonical form:

$\tilde{p}$: v is $\tilde{F}$ (1)

Where v is a variable that takes values from a universal set X and $\tilde{F}$is a fuzzy set on X, representing a fuzzy predicate such as *tall*, *expensive*, *low*, *normal*, etc.
The value v belongs to the fuzzy set $\tilde{F}$with membership grade $\tilde{F}(v)$. This membership grade is interpreted as the degree of truth of the fuzzy proposition $\tilde{p}$:

$$T(\tilde{p}) = \tilde{F}(v)$$

Example
Let v represent the air temperature at a location on Earth, measured in degrees Fahrenheit. Suppose the fuzzy predicate high temperature is defined by a membership function (for example, assigning values such as 75°F, 80°F, 88°F certain degrees of membership).
Then the fuzzy proposition becomes:

$\tilde{p}$:temperature(v) is high

The truth value $T(\tilde{p})$depends on the actual temperature value v, and the definition of the fuzzy set high, given by its membership function.
Thus, an unconditional fuzzy proposition simply states a property of a variable, and its truth is determined by the membership grade assigned by the corresponding fuzzy set.
Example: If $v = 85$,then $F(v) = F(85) = .75 \Longrightarrow T(\tilde{p}) = .75$

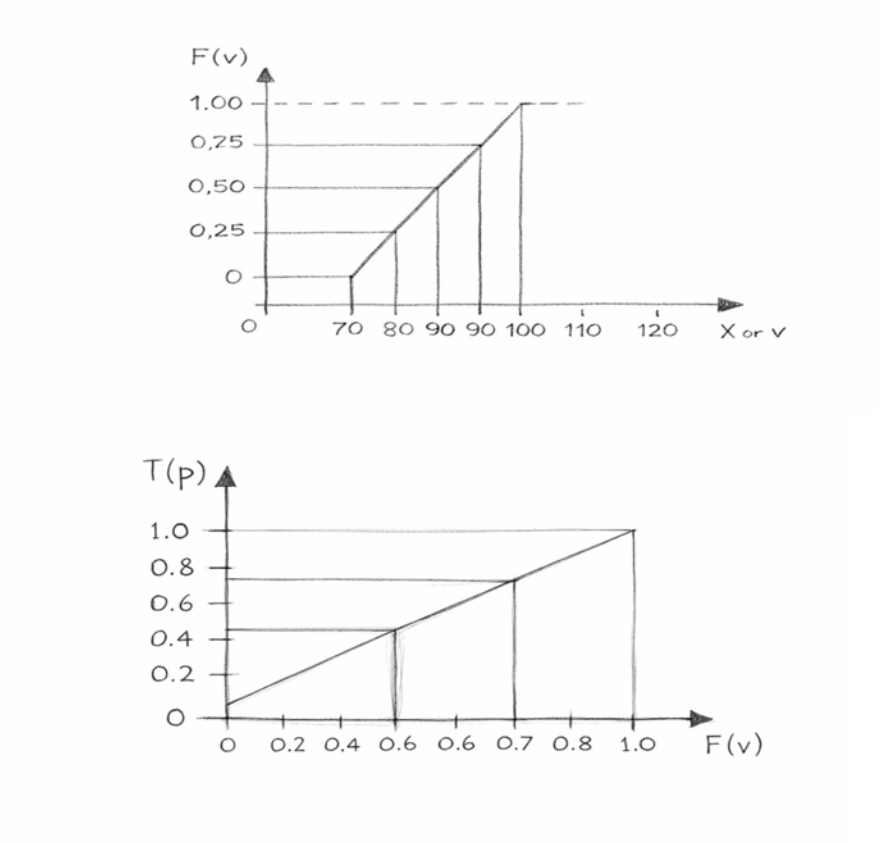


Fig: Components of Fuzzy Proposition: "$\tilde{p}$:temperature(v) is high."

2. Unconditional and Qualified Proposition:

A proposition $\tilde{p}$ of this type is expressed in one of the following canonical forms:

$$\tilde{p}: v \text{ is } \tilde{F} \text{ is } S \qquad (2)$$

or

$$\tilde{p}: \text{probability}\{v \text{ is } \tilde{F}\} \text{ is } \tilde{p} \qquad (3)$$

Here, v is a variable taking values from a universal set, and $\tilde{F}$ is a fuzzy set representing a fuzzy property (as defined earlier). The expression probability $\{v$ is $\tilde{F}\}$ denotes the probability of the fuzzy event "v is $\tilde{F}$". In equation (2), S is a fuzzy truth quantifier (such as *very true*, *mostly true*), while in equation (3), $\tilde{p}$ represents a fuzzy probability quantifier.

3. Conditional and Unqualified propositions

The proposition of this type is expressed by the canonical form

p: If x is A,then y is $\tilde{B}$ (4)

where $x$and $y$are variables taking values in the sets $X$and Y, respectively, and $\tilde{A}$ and $\tilde{B}$ are fuzzy sets defined on $X$and Y.

Such propositions may also be interpreted as the statement

$(x, y) \in \tilde{R}$, (5)

where $\tilde{R}$ is a fuzzy relation on $\tilde{A}$and $\tilde{B}$. The relation $\tilde{R}$ is a fuzzy set on $X \times Y$, specified for each $x \in X$and $y \in Y$ by the formula

$$\tilde{R}(x, y) = \Im\left(\tilde{A}(x), \tilde{B}(x)\right)$$

where $\Im$ is a binary operation on $[0,1]$ representing a suitable fuzzy implication.

$\Im$ is defined as

$$\Im(a, b) = min\{1, 1 - a + b\}$$

4. Conditional and Qualified Propositions

A proposition of this type can be represented in one of two canonical forms:

1. Rule-based form
   p:If x is $\tilde{A}$,then y is $\tilde{B}$ is S.

2. Probability-based form

$$p: \text{pro}\{x \text{ is } \tilde{A}: y \text{ is } \tilde{B}\} \text{ is } P,$$

where pro$\{x$ is $\tilde{A}: y$ is $\tilde{B}\}$ denotes the conditional probability of $y$being $\tilde{B}$given that x is $\tilde{A}$.

This method is combination of previous types of propositions.

8.7 FUZZY QUANTIFIERS

Fuzzy logic propositions are often expressed using **fuzzy quantifiers**. These quantifiers help describe quantities in an imprecise or approximate manner. There are **two main types** of fuzzy quantifiers:

(i) Absolute quantifiers
(ii) Relative quantifiers

It must be noted that the Absolute quantifiers are defined over the set of real numbers $\mathbb{R}$. and Relative quantifiers are defined over the interval $[0, 1]$.
For example:

Absolute quantifier	Relative quantifier
Round about 350	almost
Much greater than 8	about
Some where around 30	most

Types of Fuzzy Quantifiers

1. First Kind (Absolute Quantifiers)

These are defined on real numbers and are described using linguistic terms such as about 10, much more than 100, at least about 5 and so on.

2. Second Kind (Relative Quantifiers)

These are defined on the interval $[0, 1]$and are expressed using terms such as almost all, about half, most and so on.

Absolute Quantifiers or Fuzzy Quantifier of First Kind

A common form of a fuzzy proposition is:

$\tilde{p}$: " There are Q elements in I such that v(i) is F ". (1)

where v is a variable that for each individual $'i'$ in a given set $I(\text{index set})$,assumes a value $v(i)$, is a fuzzy set defined on the set of values of variable v, and Q is a fuzzy number on R.

Example:

“There are about 10 students in a class whose fluency in English is high.”

Here $v(i)$represents the fluency level of student i and F represents the fuzzy concept high fluency.

The above proposition (1) can also be written as:

p':There are Q .E.'s (2)

where E is a fuzzy set defined as:

$E(i) = F(v(i)), \forall i \in I$ (3)

aThus, E represents the degree to which each element satisfies the property F.

Thus, the original statement becomes:

"There are about 10 high-fluency English-speaking students in the class."

The proposition p' may be rewritten as

p': W is Q (4)

Where W is a variable taking values in R that represents the scalar cardinality of fuzzy set E,i.e., $W = |E|$.

The size (cardinality) of fuzzy set $E$is:

$$| E |= \sum_{i \in I} E(i) = \sum_{i \in I} F(v(i))$$

The truth value of the proposition is given by:

$T(p) = T(p') = Q(| E |)$ (5)

Another Form of Fuzzy quantifier of 1st kind:

Fuzzy quantifiers can also appear in propositions like:

p:There are Q_i elements such that $v_1(i)$ is F_1 and $v_2(i)$ is F_2 (6)

where the variables $v_1$and $v_2$take values from the sets $X_1$and X_2, respectively. The fuzzy sets $F_1$and $F_2$are defined on $X_1$and X_2, respectively. The quantifier $Q$is a fuzzy number defined on $\mathbb{R}$, and $I$is an index set representing distinct observations of the variables $v_1$and v_2.

Relative Quantifiers or Fuzzy Quantifier of Second Kind:

Relative quantifiers also called *quantifiers of the second kind* include expressions such as "almost half," "about half," and similar terms. They are represented by fuzzy numbers in the interval $[0,1]$.

A proposition involving such a quantifier can be expressed as:

p: "Among the elements $i$in $I$for which $v_1(i)$ satisfies F_1, there are Q of the elements in $I$for which $v_2(i)$ satisfies F_2."

Here, $Q$is a fuzzy number on the closed interval $[0, 1]$.

8.8 LINGUISTIC HEDGES

The terms by which other linguistic terms are modified are known as Linguistic Hedges. Linguistic terms such as very, more or less, fairly or extremely are examples of linguistic hedges.

They are mainly used in fuzzy logic to modify fuzzy predicates (like "young", "tall"), fuzzy truth values and fuzzy probabilities.

Example: A basic statement is: "X is young"

Now "X is young" may be modified by the hedge "very" is any of the following three ways:

(i) X is very young

(ii) X is young is very true

(iii) X is very young is very true

General Form:

A fuzzy proposition:

$p: x$ is F (for example, "Ram is young").

After applying hedge H:

$Hp: x$ is HF (for example, "Ram is very young"),

where HF is the fuzzy predicate obtained by applying hedge H to F.

So:

"young" → fuzzy predicate F

and "very young" → modified fuzzy predicate HF.

Note: The linguistic hedges are not applicable to the crisp predicate, truth, values or probabilities. In classical (crisp) logic, something is either **true or false**. There is no "very true" or "fairly pregnant" in a meaningful logical sense. So, hedges only make sense when truth is **graded**—that is, in fuzzy logic.

That's why phrases like**: very horizontal**, **very rectangular**, **very pregnant**, **very teenage** are considered not meaningful in this formal sense.

Hedges as functions on $[0, 1]$:

In fuzzy logic, every statement has a **truth value** in $[0, 1]$. A hedge is modeled as a **unary operation** on $[0, 1]$.

A hedge Hcorresponds to a function $h: [0,1] \rightarrow [0,1]$.

Examples:

The hedge "very" is interpreted as $h(a) = a^2$ and

The hedge "fairly "is interpreted as $h(a) = \sqrt{a}$.

Some important definitions:

Definition 8.8.1: The unary operations that represent the linguistic hedges is called a modifier.

Definition 8.8.2: Let $\tilde{F}$ be a fuzzy predicate on X and h is a modifier on linguistic hedge H.Then modifier fuzzy predicate HF is defined as

$$HF(x) = h(F(x)), \forall x \in X.$$

Definition 8.8.3:

Let a modifier h be an increasing bijection, then

If h(a) < a, ∀ a ∈ [0,1], then the modifier is called a strong modifier.

Definition 8.8.4:

If h(a) > a, ∀ a ∈ [0,1], then the modifier is called a weak modifier.

Definition 8.8.5:

The modifier for which h(a) = a is called an identity modifier.

Example:

We consider three propositions:

p_1: "Aman is young"

p_2: "Aman is very young"

p_3: "Aman is fairly young"

Let the linguistic hedges "very" and "fairly" be represented by the strong modifier: $h(a) = a^2$, ∀ a ∈ [0,1], and weak modifier: $h(a) = \sqrt{a}$, ∀ a ∈ [0,1].

Now, we assume that Aman is 26 years old, and according to the fuzzy set young,

$$\text{young}(26) = 0.8.$$

Then very young

$$\text{very young}(26) = (0.8)^2 = 0.64.$$

and fairly young

$$\text{fairly young}(26) = \sqrt{0.8} \approx 0.89.$$

Truth values:

Proposition	Meaning	Truth value
p_1	Aman is young	$T(p_1) = 0.8$
p_2	Aman is very young	$T(p_2) = 0.64$
p_3	Aman is fairly young	$T(p_3) = 0.89$

Properties of modifiers:

The function h(modifier) usually satisfies:

1. **Boundary conditions**

$$h(0) = 0, h(1) = 1.$$

2. **Continuity**

h is continuous on $[0, 1]$.

3. **Inverse relation** If $h$is strong, then $h^{-1}$is weak, and vice versa. Example: if $h(a) = a^2$(strong), then $h^{-1}(a) = \sqrt{a}$(weak).
4. **Composition** If $g$and $h$are modifiers, then:
   - $g \circ h$and $h \circ g$are also modifiers.
   - If both are strong, their composition is also strong (and similarly for weak).

8.9 INFERENCE FROM CONDITIONAL FUZZY PROPOSITION

In this section, we describe the generalization of three classical inference rules say modus ponens, modus tollens and Hypothetical syllogism.

If variables $X$and $Y$are related by a function:

$$y = f(x)$$

If we know $X = x$,then we can infer $Y = f(x)$.

Similarly, If $X \in \tilde{A}$,then $Y \in \tilde{B}$ where

$B = \{y \in Y: y = f(x), x \in A\}$.

Now instead of a function, we consider a general relation R between $X$and Y.

So,$(x, y) \in R$ then,

If $X \in A'$,We infer $Y \in B'$ where $B' = \{y \in Y: (x, y) \in R\}$

Similarly, knowing that $X \in \tilde{A}$,we can infer that $\psi \in \tilde{B}$, where

$$\tilde{B} = \{y \in Y: \langle x, y\rangle \in R, x \in \tilde{A}\}$$

Let this inference may be expressed equally well in terms of characteristic function $\chi_{\tilde{A}}$,$\chi_{\tilde{B}}$,$\chi_{\tilde{R}}$ of sets $\tilde{A}$,$\tilde{B}$ and $\tilde{R}$ respectively by the equation

$\chi_{\tilde{B}}(y) = \sup_{x \in X} min\{\chi_{\tilde{A}}(x), \chi_{\tilde{R}}(x, y)\}, \forall y \in Y$

(1)

Let us assume that $\tilde{R}$ is a fuzzy relation on X, Y and $\tilde{A}'$,$\tilde{B}'$ are fuzzy sets on X and Y respectively.

Then,if R and $\tilde{A}'$ are given,we can obtain $\tilde{B}'$ by the equation

$$\tilde{B}'(y) == \sup_{x \in X} min\{\tilde{A}'(x), R(x, y)\}$$

(2)

i.e., $\tilde{B}' \equiv \tilde{A}' o R$

which is called compositional rule of inference.

Generalized Modus Ponens:

Consider the case when relation is embedded in a single conditional fuzzy proposition.

A relation R that is embedded in fuzzy conditional proposition p of the form

p: If χ is $\tilde{A}$,then ψ is $\tilde{B}$.

is determined for all $x \in X$,$y \in Y$ by the formula

$$R(x, y) = \zeta\left(\tilde{A}(x), \tilde{B}(y)\right)$$

(3)

where, ζ denotes fuzzy implication. Using (3) and another given proposition q of the form

q: χ is $\tilde{A}'$

We may conclude that $\psi \in \tilde{B}'$ by (2).

This procedure is called a generalized Modus Ponens.

Then

Rule: If χ is $\tilde{A}$,then ψ is $\tilde{B}$.

Fact: χ is $\tilde{A}'$,

Conclusion: ψ is $\tilde{B}'$ (4)

Generalized Modus Tollens:

Another inference rule in fuzzy logic, which is a generalized modus Tollens, is expressed by the following procedure, such that

Rule: If χ is $\tilde{A}$,then ψ is $\tilde{B}$,

Fact: ψ is $\tilde{B}$,

Conclusion: χ is $\tilde{A}'$.

In this case, the compositional rule of inference has the following form

$$\tilde{A}'(x) = \sup_{y \in X} min\{\tilde{B}'(x), R(x,y)\} \quad (5)$$

Generalized of Hypothetical Syllogism

The generalized hypothetical syllogism is expressed by the following procedure

Rule 1: If χ is $\tilde{A}$, then ψ is $\tilde{B}$

Rule 2: If ψ is $\tilde{B}$,then Z is $\tilde{C}$

Conclusion: If χ is $\tilde{A}$,then Z is $\tilde{C}$.

Here χ, ψ and Z are variables in X, Y and Z respectively and $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ are fuzzy sets on X, Y and Z respectively.

The fuzzy relations are determined as

$$R_1(x,y) = \zeta\left(\tilde{A}(x), \tilde{B}(y)\right),$$

$$R_2(x,y) = \zeta\left(\tilde{B}(y), \tilde{C}(z)\right),$$

$$R_3(x,y) = \zeta\left(\tilde{A}(x), \tilde{C}(z)\right) \quad \forall\, x \in X, y \in Y, z \in Z.$$

Then, generalized hypothetical syllogism holds if

$$R_3(x,z) = \sup_{y \in X} min\{R_1(x,y), R_2(x,y)\}$$

i.e., $R_3 \equiv R_1 o R_2$.

CHECK YOUR PROGRESS 2

Q3. Let set of values of variables χ and ψ be $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2\}$ respectively. Let us take a proposition

p: "If χ is $\tilde{A}$ then ψ is $\tilde{B}$", where

$\tilde{A} = \frac{.5}{x_1} + \frac{1}{x_2} + \frac{.6}{x_3}$ and $\tilde{B} = \frac{1}{y_2} + \frac{.4}{y_2}$

Then given a fact expressed by the proposition "χ is $\tilde{A}'$" where

$$\tilde{A}' = \frac{.6}{x_1} + \frac{.9}{x_2} + \frac{.7}{x_3}.$$

Using generalized modus ponens, derive a conclusion in the form "ψ is $\tilde{B}'$".

Q4. Let $Z = \{z_1, z_2\}$ and X and Y are given by $X = \{X_1, X_2, X_3\}$ and $Y = \{Y_1, Y_2\}$.

Moreover let

$\tilde{A} = \frac{.5}{x_1} + \frac{1}{x_2} + \frac{.6}{x_3}$, $\tilde{B} = \frac{1}{y_1} + \frac{.4}{y_2}$,$\tilde{C} = \frac{.2}{z_1} + \frac{1}{z_2}$

and $\zeta(a,b) = \begin{cases} 1, & if \quad a \le b \\ b & if \quad a > b \end{cases}$

Find the resultant $R_3 = R_1 o R_2$.

Q5. Let $X = \{1,2,3,4\}$ and $Y = \{1,2,3,4,5,6\}$ be two universe of discourse.

Define

$\tilde{A} = \left[\frac{0.6}{2} + \frac{1}{3} + \frac{0.2}{4}\right]$;$\tilde{B} = \left[\frac{0.4}{2} + \frac{1}{3} + \frac{0.8}{4} + \frac{0.3}{5}\right]$

and $\tilde{C} = \left[\frac{0.3}{1} + \frac{0.5}{2} + \frac{0.6}{3} + \frac{0.6}{4} + \frac{0.5}{5} + \frac{0.3}{6}\right]$

Apply the fuzzy modus ponens rule and find the relation

(i) If x is $\tilde{A}$,then y is $\tilde{B}$

(ii) If x is $\tilde{A}$,then y is $\tilde{B}$ else y is $\tilde{C}$.

8.10LET US SUM UP

- A fuzzy truth value is given as a normalized fuzzy set $\bar{V}(\tilde{A}) = \{(v_i, \mu_i)\}$, where each $v_i \in [0,1]$is a possible truth value and $\mu_i$is its membership degree. This set forms a *possibility distribution* over truth values. Using the extension principle, the negation of a fuzzy proposition is obtained by replacing each truth value $v_i$with $1 - v_i$, while keeping the same membership degrees. Thus, $\bar{V}(\text{not } \tilde{A}) = \{(1 - v_i, \mu_i)\}$. In particular, "False" is interpreted as "not true" using the same transformation.
- For numerical truth values $\bar{v}(\tilde{A})$ and $\bar{v}(\tilde{B})$,the logical operations and, or and nor, implied are defined as

$$\bar{v}(\tilde{A}) \wedge \bar{v}(\tilde{B}) = \bar{v}(\tilde{A}) \text{ and } \bar{v}(\tilde{B}) = \bar{v}(\tilde{A} \text{ and } \tilde{B}) = \{(v, min\{\mu_{\tilde{A}}(v), \mu_{\tilde{B}}(v)\})\}$$

$$\bar{v}(\tilde{A}) \vee \bar{v}(\tilde{B}) = \bar{v}(\tilde{A}) \text{ or } \bar{v}(\tilde{B}) = \bar{v}(\tilde{A} \text{ or } \tilde{B}) = \{(v, max\{\mu_{\tilde{A}}(v), \mu_{\tilde{B}}(v)\})\}$$

$$\urcorner\bar{v}(\tilde{A}) = nor\bar{v}(\tilde{A}) = \{v, 1 - \mu_{\tilde{A}}(v)\}$$

$$\bar{v}(\tilde{A}) \Longrightarrow \bar{v}(\tilde{B}) \equiv \bar{v}(\tilde{A} \Longrightarrow \tilde{B}) = \urcorner\bar{v}(\tilde{A}) \vee \bar{v}(\tilde{B})$$

$$= \{(v, max\{1 - \mu_{\tilde{A}}(v), \mu_{\tilde{B}}(v)\})\}$$

- **IF–THEN Rule:**
  The implication ($\Longrightarrow$) represents the IF –THEN statement as
  IF x is $\tilde{A}$, THEN y is $\tilde{B}$ and is equivalent to

$$\max\left(\min\left(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}^{c}(y)\right), 1 - \mu_{\tilde{A}}(x)\right)$$

$$\tilde{R} = \left(\tilde{A}^c \times \tilde{B}\right) \vee \left(\tilde{A} \times \tilde{Y}\right) \approx \left(\tilde{A} \times \tilde{B}\right) \vee \left(\tilde{A}^c \times \tilde{Y}\right)$$

  The membership function of $\tilde{R}$ is given by

$$\mu_{\tilde{R}}(x, y) = \max\left(\min\left(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}^{c}(y)\right), 1 - \mu_{\tilde{A}}(x)\right)$$

- **Compound Rule (IF–THEN–ELSE):**
  Also, for the compound implication IF x is $\tilde{A}$, THEN y is $\tilde{B}$ ELSE y is $\tilde{C}$,the relation $\tilde{R}$ is equivalent to

$$\tilde{R} = \left(\tilde{A} \times \tilde{B}\right) \vee \left(\tilde{A}^c \times \tilde{C}\right)$$

  The membership function of $\tilde{R}$ is given by

$$\mu_{\tilde{R}}(x, y) = \max\left(\min\left(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)\right), \min\left(1 - \mu_{\tilde{A}}(x), \mu_{\tilde{C}}(x)\right)\right)$$

- Let $\tilde{R}(x), \tilde{R}(x, y)$ and $\tilde{R}(y)$ $\forall x \in X, y \in Y$ be fuzzy relations in $X, X \times Y$ and Y respectively, which acts as fuzzy restrictions on X, (x, y) and , y respectively.
  Let $\tilde{A}$ and $\tilde{B}$ denote particular fuzzy sets in X and $X \times Y$,then the corresponding rule of inference asserts that the solution of the relational assignments equations
  $\tilde{R}(x) = \tilde{A}$, $\tilde{R}(x, y) = \tilde{B}$ is given by

$$\tilde{R}(y) = \tilde{A} o \tilde{B}$$

- A fuzzy proposition is a statement whose truth is not just true or false but can be partly true.
  Therefore, given $\tilde{p}$ to be a fuzzy proposition,$T(\tilde{p})$ represent the truth values which lies in the interval $[0,1]$.

- Fuzzy logic propositions are often expressed using **fuzzy quantifiers**. These quantifiers help describe quantities in an imprecise or approximate manner. There are **two main types** of fuzzy quantifiers:

  - Absolute quantifiers
  - Relative quantifiers

- The terms by which other linguistic terms are modified are known as Linguistic Hedges. Linguistic terms such as very, more or less, fairly or extremely are examples of linguistic hedges.

✓ 8.11 ANSWERS TO CHECK YOUR PROGRESS

Check your progress 1:

Answer to Q1: Left as an exercise.

Answer to Q2: Left as an exercise.

Answer to Q3:

We first obtain the relation R using the Luckssiewicz implementation as

$$\zeta(a,b) = min\{1,1-a+b\}$$

$$R = \frac{1}{x_1y_1} + \frac{.9}{x_1y_2} + \frac{1}{x_2y_1} + \frac{.4}{x_2y_2} + \frac{1}{x_3y_2} + \frac{.8}{x_3y_2}$$

Now, by composition rule of inference, we obtain

$$\tilde{B}' = \tilde{A}' o R$$

$$= [.6 \quad .9 \quad .7] o \begin{bmatrix} 1 & .9 \\ 1 & .4 \\ 1 & .8 \end{bmatrix}$$

$$= [.9 \quad .7]$$

$$\Longrightarrow \tilde{B}' = \frac{.9}{y_1} + \frac{.7}{y_2}$$

Because $\tilde{B}'(y) \equiv \sup_{x \in X} min\{\tilde{A}'(x), R(x,y)\}$

Thus, we conclude that "ψ is $\tilde{B}$''.

$$\tilde{B}' = \frac{.9}{y_1} + \frac{.7}{y_2}$$

Answer to Q 4:

Given that $\tilde{A} = \frac{.5}{x_1} + \frac{1}{x_2} + \frac{.6}{x_3}$, $\tilde{B} = \frac{1}{y_1} + \frac{.4}{y_2}$,$\tilde{C} = \frac{.2}{z_1} + \frac{1}{z_2}$

$$R_1(x,y) = \begin{bmatrix} 1 & .4 \\ 1 & .4 \\ 1 & .4 \end{bmatrix}, R_2(y,z) = \begin{bmatrix} .2 & 1 \\ .2 & 1 \end{bmatrix}$$

Now, $R_3(x,z) = R_1(x,y) o R_2(y,z)$

$$= \begin{bmatrix} 1 & .4 \\ 1 & .4 \\ 1 & .4 \end{bmatrix} o \begin{bmatrix} .2 & 1 \\ .2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} .2 & 1 \\ .2 & 1 \\ .2 & 1 \end{bmatrix}$$

$\therefore \quad R_3 = \frac{.2}{x_1 z_1} + \frac{1}{x_1 z_2} + \frac{.2}{x_2 z_1} + \frac{1}{x_2 z_2} + \frac{.2}{x_3 z_1} + \frac{1}{x_3 z_2}$.

Answer to Q5:

Given that $\tilde{A} = \left[\frac{0.6}{2} + \frac{1}{3} + \frac{0.2}{4}\right]; \tilde{B} = \left[\frac{0.4}{2} + \frac{1}{3} + \frac{0.8}{4} + \frac{0.3}{5}\right]$

and $\tilde{C} = \left[\frac{0.3}{1} + \frac{0.5}{2} + \frac{0.6}{3} + \frac{0.6}{4} + \frac{0.5}{5} + \frac{0.3}{6}\right]$

(i) The matrix of the membership function $\mu_{\tilde{R}}(x,y)$ of the implication if x is $\tilde{A}$,then y is $\tilde{B}$ is given by

$$\tilde{A} \times \tilde{B} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.4 & 0.6 & 0.6 & 0.3 & 0 \\ 0 & 0.4 & 1 & 0.8 & 0.3 & 0 \\ 0 & 0.2 & 0.2 & 0.2 & 0.2 & 0 \end{bmatrix} \end{array}$$

Also, we may obtain

$$\tilde{A}^c \times \tilde{Y} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0.4 & 0.4 & 0.4 & 0.4 & 0.4 & 0.4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0.8 & 0.8 & 0.8 & 0.8 & 0.8 & 0.8 \end{bmatrix} \end{array}$$

Therefore, $\tilde{R} = \boldsymbol{max}\left(\tilde{A} \times \tilde{B}, \tilde{A}^c \times \tilde{Y}\right)$

$$= \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0.4 & 0.4 & 0.6 & 0.6 & 0.4 & 0.4 \\ 0 & 0.4 & 1 & 0.8 & 0.3 & 0 \\ 0.8 & 0.8 & 0.8 & 0.8 & 0.8 & 0.8 \end{bmatrix} \end{array}$$

(ii) When the logical conditional implication is of the compound form

If x is $\tilde{A}$,then y is $\tilde{B}$ else y is $\tilde{C}$

Then the equivalent fuzzy relation $\tilde{R}$ is expressed as

$$\tilde{R} \equiv \tilde{A} \times \tilde{B} \cup \left(\tilde{A}^c \times \tilde{C}\right)$$

With membership function $\mu_{\tilde{R}}(x,y)$ given by the following formula

$\mu_{\tilde{R}}(x,y) = \max\left(\left(\mu_{\tilde{A}}(x) \wedge \mu_{\tilde{B}}(y)\right), (1 - \mu_{\tilde{A}}(x) \wedge \mu_{\tilde{C}}(x)\right)$

Therefore, we have the new relation

$$\tilde{R} \equiv \tilde{A} \times \tilde{B} \cup \left(\tilde{A}^c \times \tilde{C}\right)\left(\tilde{A}^c \times \tilde{C}\right) : \left(\tilde{A}^c \times \tilde{C}\right)$$

$$= \begin{matrix} \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \begin{bmatrix} 0.3 & 0.5 & 0.6 & 0.6 & 0.5 & 0.3 \\ 0.3 & 0.4 & 0.6 & 0.6 & 0.4 & 0.3 \\ 0 & 0.4 & 1 & 0.8 & 0.3 & 0 \\ 0.3 & 0.5 & 0.6 & 0.6 & 0.5 & 0.3 \end{bmatrix}$$

Finally

$$\tilde{R} = \begin{matrix} \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \begin{bmatrix} 0.3 & 0.5 & 0.6 & 0.6 & 0.5 & 0.3 \\ 0.3 & 0.4 & 0.6 & 0.6 & 0.4 & 0.3 \\ 0 & 0.4 & 1 & 0.8 & 0.3 & 0 \\ 0.3 & 0.5 & 0.6 & 0.6 & 0.5 & 0.3 \end{bmatrix}$$

8.12 FURTHER READING

- George J.Klir and Bo Yuan,Fuzzy Sets and Fuzzy Logic Theory and Applications,Prentice Hall P.T.R Upper Saddle River,New Jersey 07458,1995. Cole, E. (2011). *Network security bible*. John Wiley & Sons.
- Sudhir K.Pundir and Rimple Pundir,Fuzzy Sets and Their Applications,PragatiPrakashan,Ninth Edition,2018.

8.13 MODEL QUESTIONS

Q1. Define two fuzzy sets $\tilde{A}$ and $\tilde{B}$ such that
$\tilde{A} = \left[\frac{0}{-2} + \frac{0.5}{-1} + \frac{1.0}{0} + \frac{0.5}{1} + \frac{0}{2}\right]; \tilde{B} = \left[\frac{0}{0} + \frac{0.5}{1} + \frac{1.0}{2} + \frac{0.5}{3} + \frac{0}{5}\right]$
Construct the relation "If $\tilde{A}$ then $\tilde{B}$" using the following

(i) $\mu_{\tilde{R}}(x, y) = min[\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)]$

(ii) $\mu_{\tilde{R}}(x, y) = \mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)$

Q2. Let us define the input on the universe of discourse $X = [0,50,100,150,200]$ and $Y = [0,50,100,150,200]$. Define two fuzzy sets $\widetilde{w}$ and $\widetilde{M}$ as follows

$$\widetilde{w} = \left[\frac{1}{0} + \frac{0.9}{50} + \frac{0.3}{100} + \frac{0}{150} + \frac{0}{200}\right] \subset X$$

$$\widetilde{M} = \left[\frac{0}{0} + \frac{0.4}{50} + \frac{1}{100} + \frac{0.4}{150} + \frac{0}{200}\right] \subset X$$

and

$$\tilde{S} = \left[\frac{0}{0} + \frac{0}{50} + \frac{0.5}{100} + \frac{0.9}{150} + \frac{1}{200}\right] \subset Y$$

Then construct the propositions 'If $\tilde{w}$,then not $\tilde{S}'$. Also find $\tilde{M} o \tilde{R}$.

Q3. Calculate the nonspecificity and fuzziness for the following fuzzy sets defined on the interval $[0,10]$:

(a) $A_1(x) = \frac{x}{10}$

(b) $A_2(x) = 2^{-x}$

(c) $A_2(x) = \begin{cases} \frac{x}{5} & for\ x \leq 5 \\ 2 - \frac{x}{5} & for\ x \geq 5 \end{cases}$

Q4. Consider the following basic assignments defined on $X = \{a, b, c\}$:

$m_1(\{c\}) = 0.2, m_1(\{b, c\}) = 0.3, m_1(X) = 0.5;$

$m_1(\{a\}) = 0.5, m_1(\{b\}) = 0.2, m_1(\{c\}) = 0.3;$

$m_1(\{c\}) = 0.2, m_1(\{a, b\}) = 0.3, m_1(X) = 0.5;$

Calculate $N(m_i), S(m_i)$ and $N_S(m_i)$ for each $i = 1,2,3$.

*** *** ***

UNIT 9: FUZZY SYSTEMS AND FUZZY CONTROL

UNIT STRUCTURE

9.1 Learning Objectives
9.2 Introduction
9.3 Fuzzy rule -based system
9.4 Fuzzification and Defuzzification
9.5 Fuzzy Control
9.6 Assumption in a Fuzzy control system design
9.7 Design of Fuzzy controllers
9.8 Fuzzy Control systems Models
9.9 Some Examples of Fuzzy systems
9.10 Let Us Sum Up
9.11 Answers to Check Your Progress
9.12 Further Reading
9.13 Model Questions

9.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- understand the basic concept and structure of a fuzzy rule-based system.
- explain the processes of fuzzification and defuzzification used in fuzzy systems.
- understand the concept and working principle of fuzzy control.
- identify the assumptions involved in designing a fuzzy control system.
- describe the steps involved in the design of fuzzy controllers.
- understand different models of fuzzy control systems and their characteristics.
- analyse how fuzzy logic is applied in practical systems through examples of fuzzy systems.

9.2 INTRODUCTION

In many real-life situations, systems have to operate under conditions of uncertainty, vagueness, and incomplete information. Classical control techniques often require precise mathematical models of the system, which are sometimes difficult or impossible to obtain. To overcome these limitations, fuzzy logic provides an effective approach for designing intelligent systems that can handle imprecise data and human-like reasoning. One of the most important applications of fuzzy logic is in the development of fuzzy control systems.

A fuzzy control system uses the principles of fuzzy logic to model and control complex processes. Instead of relying solely on exact numerical values, fuzzy control systems use linguistic variables and if–then rules similar to the way humans make decisions. For example, a human operator may control the temperature of a room using rules

such as: *"If the room is very hot, then decrease the heater power significantly."* Such reasoning can be effectively represented using fuzzy rule-based systems**.**

A fuzzy rule-based system forms the core of a fuzzy controller. It consists of a set of fuzzy rules**, a** fuzzification process that converts crisp input values into fuzzy values, an inference mechanism that processes the rules, and a defuzzification process that converts the fuzzy output back into a crisp value for practical use.

In this unit, we study the main components and design principles of fuzzy control systems. First, we introduce the concept of a fuzzy rule-based system, which represents knowledge in the form of fuzzy if–then rules. We then discuss fuzzification and defuzzification, which are essential steps for transforming crisp inputs into fuzzy sets and converting the fuzzy results back into numerical outputs.

The unit also explains the concept of fuzzy control**,** the assumptions involved in designing a fuzzy control system, and the procedure for designing fuzzy controllers. Further, we examine different models of fuzzy control systems and their characteristics. Finally, the unit presents examples of fuzzy systems that illustrate how fuzzy logic is applied in practical situations.

Thus, this unit provides an understanding of how fuzzy logic can be used to design intelligent and flexible control systems capable of dealing with complex and uncertain environments.

9.3 FUZZY RULE-BASED SYSTEM

A Fuzzy Rule-Based System (FRBS) is a fundamental component of fuzzy logic that is used to model complex systems involving uncertainty, vagueness, and imprecision.

The core idea of a fuzzy rule-based system is to represent knowledge in the form of linguistic rules, typically expressed as IF–THEN statements.

IF (set of conditions) are satisfied THEN (a set of consequent) can be inferred.

Fuzzy IF-THEN rules are coded in Mathematically as:

"IF (x_1 is $\tilde{A}_1, x_2$ is $\tilde{A}_2, \cdots, x_n$ is $\tilde{A}_n$),THEN (y_1 is $\tilde{B}_1, y_2$ is $\tilde{B}_2, \cdots, y_n$ is $\tilde{B}_n$)"

Where x_i,y_j are linguistic variables, which takes the values of fuzzy sets $\tilde{A}_i$ and $\tilde{B}_j$ respectively.

Example:

IF temperature is high and humidity is moderate

THEN cooling level should be strong

The vague terms high, moderate and strong are fuzzy sets qualifying the variables temperature, humidity and cooling level respectively.

Definition 9.3.1: A collection of rules referring a particular system is known as a fuzzy rule base.

A fuzzy rule base is simply a collection of fuzzy IF–THEN rules that describe how a system should behave.

Each rule has a consequent (output fuzzy set), written as $C_1, C_2, \ldots, C_n$.

When we combine all the rules, the final conclusion Ccan be formed in two ways:

1. When the rules are combined using AND (conjunction)

This means all rules must be satisfied together.

Conclusion:

$$C = C_1 \cap C_2 \cap \cdots \cap C_n$$

Membership function:

For every output value $y$in the universe Y:

$\mu_C(y) = \min\left(\mu_{C_1}(y), \mu_{C_2}(y), \ldots, \mu_{C_n}(y)\right), \forall y \in Y$

2. When the rules are combined using OR (disjunction)

This means any rule being satisfied contributes to the conclusion.

Conclusion:

$$C = C_1 \cup C_2 \cup \cdots \cup C_n$$

Membership function:

For every output value $y$in the universe Y:

$\mu_C(y) = \max(\mu_{C_1}(y), \mu_{C_2}(y), \ldots, \mu_{C_n}(y)), \forall y \in Y$

Fuzzy Controllers:

Fuzzy controllers are a type of expert system used to control and manage different processes. They can differ from one another depending on the type of problem they are designed to solve.

A general fuzzy controller consists of the following modules:

(a) A fuzzy rule base
(b) A fuzzy inference engine
(c) Fuzzification
(d) Defuzzification

9.4 FUZZIFICATION AND DEFUZZIFICATION

In a fuzzy logic system, real-world problems are handled using approximate reasoning rather than precise numerical computation. However, most real-life inputs and outputs are in the form of exact (crisp) values. To bridge this gap between crisp data and fuzzy reasoning, two important processes are used: fuzzification and defuzzification.

Fuzzification is the first step in a fuzzy system, where crisp input values are transformed into fuzzy sets using appropriate membership functions. This allows the system to interpret numerical data in terms of linguistic variables such as *low*, *medium*, and *high*. On the other hand, after processing the fuzzy information through a rule base and inference mechanism, the resulting fuzzy output must be converted back into a single crisp value. This process is known as defuzzification.

Together, fuzzification and defuzzification form essential components of any fuzzy inference system. They ensure smooth interaction between real-world numerical data and the fuzzy logic framework, enabling effective modeling and control of complex systems.

Methods of Defuzzification

(1) Centre of Area (COA) Method

It is also known as centroid or centre of gravity method.

This method finds the crisp value x^*corresponding to the **centre of the area** under the aggregated membership function.

For a continuous membership function

$x^* = \frac{\int \mu(x) x dx}{\int \mu(x) dx}$

For a discrete membership function

$x^* = \frac{\sum_{i=1}^{n} x_i \mu(x_i)}{\sum_{i=1}^{n} \mu(x_i)}$

where,n is the number of elements in the sample,x_i's are the elements and $\mu(x_i)$ is its membership function.

(2) Centre of Sums (COS) Method

In the COA method, overlapping areas of fuzzy sets are counted once. In the COS method, overlapping areas are counted twice.

This method constructs the final membership function by taking the algebraic sum of the membership functions of all contributing fuzzy sets $\tilde{A}_1, \tilde{A}_2, ...$, etc.

The defuzzified value x^* is given by

$$x^* = \frac{\sum_{i=1}^{n} x_i - \sum_{k=1}^{n} \mu_{\lambda_k}(x_i)}{\sum_{i=1}^{n} \sum_{k=1}^{n} \mu_{\lambda_k}(x_i)}$$

(3) Mean of Maxima (MOM) Method

This method selects the crisp value(s) corresponding to the maximum membership grade of the fuzzy set.If multiple values achieve the maximum membership, the mean of those values is taken.

The equation of the defuzzified value $x^*$is given by

$$x^* = \frac{\sum_{x_i \in M} x_i}{|M|}$$

where,M ={x_i: $\mu(x_i)$ is equal to the height of fuzzy set}.

$h(\tilde{A})$= height (maximum membership value) of the fuzzy set

and $|M|$ = number of elements in M.

For continuous membership functions:

$$M = \{x \in [-c, c] : \mu(x) = h(\tilde{A})\}$$

(4) Centre of Maxima (COM)Method

In this method,the defuzzified value $d_{CM}(\mu)$ is defined as the average of the smallest value and the largest value of the variables for which $\mu(x)$ is the height $h(\mu)$ of μ.

For continuous membership functions:

$$d_{CM}(\mu) = \frac{\inf M + \sup M}{2}$$

where

$$M = \{x \in [-c, c] : \mu(x) = h(\mu)\}$$

For discrete membership functions:

$$d_{CM}(\mu) = \frac{\min(x_k \in M) + \max(x_k \in M)}{2}$$

where $M = \{x_k : \mu(x_k) = h(\mu)\}$

(5) Weighted Average Method

In weighted average method,we use the following formula

$$x^* = \frac{\sum \mu_c(x).\bar{\bar{x}}}{\sum \mu_c(x)}$$

Where Σ denotes the algebraic sum and where $\bar{x}$ is the centroid of each symmetric membership function.

Example 9.4.1: Let $\tilde{A}_1$,$\tilde{A}_2$ and $\tilde{A}_3$ are three fuzzy sets as shown in the following figures (1), (2) and (3). Figure (4) illustrates of the fuzzy sets.

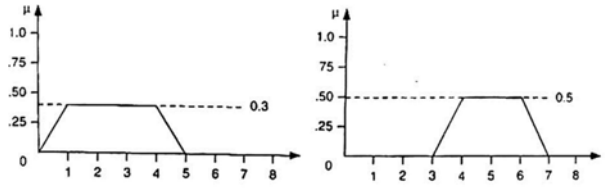


Fig. 1. Fuzzy set $\tilde{A}_1$

Fig. 2. Fuzzy set $\tilde{A}_2$

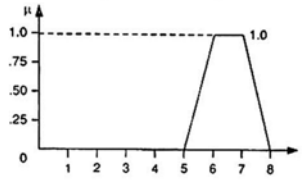


Fig. 3. Fuzzy set $\tilde{A}_3$

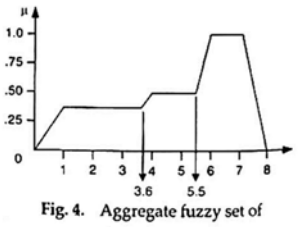


Fig. 4. Aggregate fuzzy set of $\tilde{A}_1, \tilde{A}_2$ & $\tilde{A}_3$

Using the (1) centroid method, (2) centre of sums method,(3) mean of maxima method,defuzzified $\tilde{A}_1$,$\tilde{A}_2$ and $\tilde{A}_3$.

Solution :

(1) Centroid method: To find x^*,the centroid,we see the aggregate fuzzy sets as shown in Fig.(4) and (5).We divide the aggregated output into areas for better understanding.

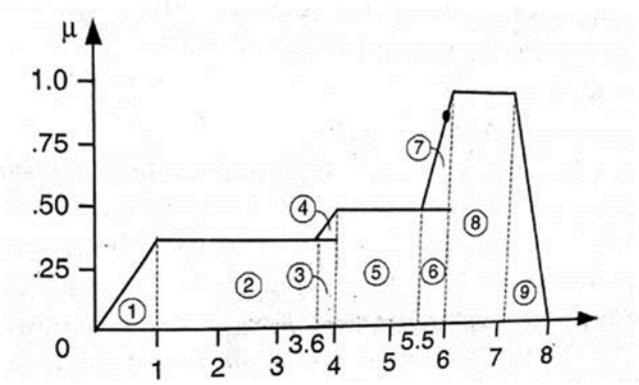


Fig.5. Aggregated fuzzy sets of $\tilde{A}_1$,$\tilde{A}_2$ and $\tilde{A}_3$ viewed as area segments.

Computation of x^*

Area segment no.	Area A	$\bar{x}$	$A\bar{x}$
1	$\frac{1}{2} \times 0.3 \times 1 = 0.15$	0.67	0.1005
2	$2.6 \times 0.3 = 0.78$	2.3	1.748
3	$0.3 \times 0.4 = 0.12$	3.8	0.456

4	$\frac{1}{2} \times 0.4 \times 0.2 = 0.04$	3.8667	0.1546
5	$1.5 \times 0.5 = 0.75$	4.75	3.5625
6	$0.5 \times 0.5 = 0.25$	5.75	1.4375
7	$\frac{1}{2} \times 0.5 \times 0.5 = 0.125$	5.833	0.729
8	$1 \times 1 = 1$	6.5	6.5
9	$\frac{1}{2} \times 1 \times 1 = 0.5$	7.33	3.665

Then, we can find $x^* = \frac{\sum A\bar{x}}{\sum \bar{x}}$

This gives the defuzzified value using the centroid method.

(2) Centre of sums method:In this method,the overlapping area is counted once but twice.Now,making use of the aggregated fuzzy set shown in Fig.(4),the centre of sum x^* is given by

$$x^* = \frac{\frac{1}{2} \times 0.3 \times (3+5) \times 2.5 + \frac{1}{2} \times 0.5 \times (4+2) \times 5 + \frac{1}{2} \times 1 \times (3+1) \times 6.5}{\frac{1}{2} \times 0.3 \times (3+5) + \frac{1}{2} \times 0.5 \times (4+2) + \frac{1}{2} \times (3+5)}$$

$$= 2.3$$

This is the **centre of sums** defuzzified value.

Also,the area covered by fuzzy sets $\tilde{A}_1$,$\tilde{A}_2$,$\tilde{A}_3$(Fig.(1),(2),(3)) are respectively,given by $\frac{1}{2} \times 0.3 \times (3+5)$, $\frac{1}{2} \times 0.5 \times (4+2)$ and $\frac{1}{2} \times 1 \times (3+1)$.

(3) Mean of Maxima method: Since,the aggregate fuzzy set is a continuous set,therefore x^*,the mean of maxima iscomputed at $x^* = 6.5$.
Also,$M = [X \in \{6,7\}: \mu(x) = 1]$
And the height of the aggregated fuzzy set is 1.
This is the **mean of maxima** defuzzified value.

9.5 FUZZY CONTROL

Fuzzy control systems and expert systems share a common goal: both try to imitate human thinking, experience, and decision-making. Kickert and Mamdani described this idea by saying that the purpose of fuzzy control is to embed the knowledge and experience of a skilled human operator directly into the controller design.

In fuzzy control, we start with a set of linguistic rules—simple sentences that describe how a human operator would control the system. These rules use words like *high*, *low*, *fast*, or *slow*, and each word is represented by a fuzzy set. From these rules, a complete control algorithm is built.

The biggest advantage of fuzzy control is that it allows us to use human intuition, practical experience, rules of thumb, and heuristics. Another benefit is that fuzzy control does not require a precise mathematical model of the system being controlled.

Difference Between Expert Systems and Fuzzy Logic Control

1. Fuzzy Logic Control (FLC) is rule-based**.** It uses a collection of fuzzy IF–THEN rules to make decisions.
2. FLC is mainly used for controlling industrial processes**.** Examples include electric power plants, chemical plants, cement factories, and similar production systems. Its application area is usually narrower than that of general expert systems.
3. FLC rules are created directly by the designers**.** Unlike expert systems, where rules are often extracted from human experts through interviews or knowledge-acquisition methods, FLC designers explicitly write the rules themselves.

9.6 ASSUMPTIONS IN A FUZZY CONTROL SYSTEM DESIGN

When designing a fuzzy control system, engineers usually rely on six basic assumptions. These assumptions help guide the design process and make it practical to implement fuzzy logic in real systems.

1. The plant is observable and controllable

This means that the important variables of the system—such as the state, input, and output—can be **measured, observed, or calculated**. If we cannot observe what the system is doing, we cannot control it properly.

2. There is useful expert knowledge available

We assume that some form of **expert knowledge** exists. This knowledge may come from:

(a) linguistic rules used by human operators,
(b) engineering common sense,
(c) intuition and experience,
(d) input/output data collected from the system, or
(e) an analytical model that can be converted into fuzzy rules. From this knowledge, we can extract fuzzy IF–THEN rules.

3. A solution exists

We assume that it is actually possible to design a fuzzy controller that can control the system effectively.

4. The goal is a "good enough" solution

The control engineer is not necessarily looking for the mathematically optimal solution. Instead, the aim is to find a solution that works well enough in practice, even if it is not perfect.

5. The controller will be designed using the best available knowledge

The fuzzy controller is built using whatever information we currently have, and the design is expected to work within an acceptable level of accuracy.

6. Stability and optimality are still open issues

In fuzzy control theory, the problems of ensuring stability and finding optimal solutions are not fully solved. These remain active research areas.

Structure of fuzzy production rules

A fuzzy production rule system is built using four main components:

1. A set of rules

These rules represent the policies, strategies, and heuristics used by an expert human decision-maker. Example: *IF temperature is high THEN reduce heater power.*

2. A set of input data

These are the measurements or observations taken just before making a decision. For example, current temperature, pressure, speed, etc.

3. A method to evaluate actions

The system must be able to check whether a proposed action matches the rules and fits the current input data. This step determines how well an action satisfies the fuzzy rules.

4. A method to generate and select actions

The system needs a way to:

(a) produce possible control actions,
(b) compare them, and
(c) decide when to stop searching for better alternatives. This ensures that the controller chooses a reasonable and effective action.

9.7 DESIGN OF FUZZY CONTROLLERS

Designing a fuzzy logic controller involves a sequence of steps that gradually convert expert knowledge or intuitive reasoning into a working control system. The process can be understood as follows:

Steps in Designing a Simple Fuzzy Logic Control System

1. **Identify the variables of the plant** First determine all the important quantities involved in the system:
   (i) Input variables (what the controller senses),
   (ii) State variables (internal conditions of the plant),
   (iii) Output variables (what the controller must produce to influence the plant).
2. **Divide each variable's range into fuzzy subsets**
   For every variable, take its full range (called the *universe of discourse*) and split it into several fuzzy sets such as *Low*, *Medium*, *High*, etc. Each fuzzy subset is given a linguistic label, and together they cover the entire range of the variable.
3. **Define membership functions for each fuzzy subset**
   For every fuzzy set, assign a membership function that indicates how strongly a value belongs to that set. These functions may be triangular, trapezoidal, Gaussian, etc.
4. **Create the rule base**

Establish fuzzy rules that connect input fuzzy sets to output fuzzy sets. Example: *IF temperature is High AND pressure is Medium THEN valve opening is Small.* These rules form the core knowledge of the controller.

5. **Choose scaling factors for inputs and outputs**

To simplify processing, normalize all variables to a standard interval such as [0, 1] or [-1, 1]. Scaling ensures that the fuzzy controller works consistently regardless of the original units.

6. **Fuzzify the inputs**

Convert the crisp input values into fuzzy values using the membership functions defined earlier.

7. **Apply fuzzy reasoning to evaluate each rule**

For every rule in the rule base, determine how strongly it applies based on the fuzzified inputs. This produces a fuzzy output contribution from each rule.

8. **Combine (aggregate) all fuzzy outputs**

Merge the outputs of all rules into a single fuzzy set that represents the controller's overall decision.

9. **Defuzzify to obtain a crisp output**

Convert the aggregated fuzzy output back into a single numerical value that can be applied to the plant.

A simple fuzzy logic control system is shown in the following block diagram.

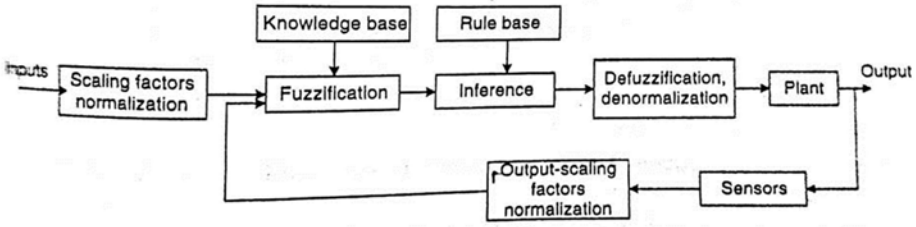


Fig. 12

Features of Simple Fuzzy Logic Control Systems

A simple fuzzy logic control system has the following features:

1. **Fixed and uniform scaling factors** The input and output scaling values remain constant.
2. **A flat, single-layer rule base** All rules are independent, non-interactive, and have equal certainty (usually taken as 1).
3. **Fixed membership functions** The shapes and parameters of the membership functions do not change during operation.
4. **A limited number of rules** The number of rules grows exponentially as the number of input variables increases.
5. **Fixed meta-knowledge** The methods for reasoning, combining rules, and defuzzifying outputs are predetermined and do not adapt.
6. **Low-level control without hierarchy** There is no multi-level or hierarchical structure; all decisions are made at a single level.

General Fuzzy Logic Controllers

The principal design elements in general fuzzy logic system are as follows:

1. Fuzzification Strategies

(i) How input values are converted into fuzzy values.
(ii) Interpretation of the fuzzifier used.

2. Knowledge Base

This includes:

(i) Discretization/normalization of the universe of discourse.
(ii) Fuzzy partitions of input and output spaces.
(iii) Ensuring completeness of partitions (no gaps or missing regions).
(iv) Choosing appropriate membership functions for all fuzzy sets.

3. Rule Base

This involves:

(i) Selecting the input (state) and output (control) variables.
(ii) Determining the source of the rules (expert knowledge, heuristics, data).
(iii) Choosing the types of rules (e.g., Mamdani, Sugeno).
(iv) Ensuring consistency, interaction, and completeness of the rule set.

4. Decision-Making Logic

This includes:

(i) Defining the fuzzy implication used in rules.
(ii) Interpreting logical connectives like AND, OR, NOT.
(iii) Designing the inference mechanism that processes the rules.

5. Defuzzification Strategies

(i) Selecting the method (centroid, maximum, weighted average, etc.).
(ii) Understanding how the chosen method interprets the fuzzy output.

9.8 FUZZY CONTROL SYSTEM MODELS

Fuzzy control systems can be described using fuzzy relational equations, which express how the system's states and inputs evolve over time. The following models show how fuzzy relations are used to represent different types of discrete-time systems.

1. First-Order Discrete Fuzzy System

Consider a simple discrete system with one input u. In state-space form, its fuzzy model is written as:

$$x_{k+1} = x_k \circ u \circ \tilde{R}, k = 1,2, \ldots$$

Here:

(i) $x_k$is the current state,
(ii) $x_{k+1}$is the next state,
(iii) $\tilde{R}$is the fuzzy system transfer relation,
(iv) $\circ$ denotes the standard fuzzy composition.

This equation means that the next state is obtained by combining the current state and the input through the fuzzy relation $\tilde{R}$.

2. General p-th Order Discrete Fuzzy System (Single Input)

For a higher-order system (order p), the fuzzy model becomes:

$$x_{k+p} = x_k \circ x_{k+1} \circ \cdots \circ x_{k+p-1} \circ u_{k+p-1} \circ \tilde{R}, k \in \mathbb{N}$$

The output is:

$$y_{k+p} = x_{k+p}$$

This means:

(i) The future state x_{k+p}depends on all previous $p$states and the input.
(ii) The system has one input and one output.
(iii) The output is simply the final state.

3. Second-Order System with Full State Feedback

A second-order system that uses full state feedback (i.e., both current and next state are used to compute the control input) is modeled as:

$$u_k = x_k \circ x_{k+1} \circ \tilde{R}, k \in \mathbb{N}$$

This means the control input $u_k$is determined by combining:

(i) the current state x_k,
(ii) the next state x_{k+1},
(iii) and the fuzzy relation $\tilde{R}$.

4. General p-th Order System with Full State Feedback

For a general p-th order system with one input and one output, and with full state feedback, the fuzzy relational equation is:

$$x_{k+p} = x_k \circ x_{k+1} \circ \cdots \circ x_{k+p-1} \circ u_{k+p-1} \circ \tilde{R}, k \in \mathbb{N}$$

This is similar to the earlier p-th order model but emphasizes that **all previous states and the input are used together** to determine the next state.

9.9 SOME EXAMPLES OF FUZZY SYSTEMS

Fuzzy logic has been successfully applied in many real-world control problems. The following examples illustrate how fuzzy controllers work in practical systems.

(1) Greg Viot's Cruise Controller

Greg Viot developed a fuzzy logic–based cruise control system designed to keep a vehicle moving at a desired constant speed. The controller uses:

(a) Two fuzzy inputs:
  (i) *Speed difference* (difference between desired and actual speed)
  (ii) *Acceleration*
(b) One fuzzy output:

(i) *Throttle control* (how much the accelerator should be pressed)

These inputs and outputs are processed through fuzzy rules to determine how the throttle should be adjusted.

Sample Fuzzy Rule Base

A portion of the fuzzy rule base $\tilde{R}$used in the cruise controller is shown below

Rule	Description
1	If (speed difference is NL) and (acceleration is ZE) then (throttle control is PL)
2	If (speed difference is ZE) and (acceleration is NL) then (throttle control is PL)
3	If (speed difference is NM) and (acceleration is ZE) then (throttle control is PM)
4	If (speed difference is NS) and (acceleration is PS) then (throttle control is PS)
5	If (speed difference is PS) and (acceleration is NS) then (throttle control is NS)
6	If (speed difference is PL) and (acceleration is ZE) then (throttle control is NL)
7	If (speed difference is ZE) and (acceleration is NS) then (throttle control is PS)
8	If (speed difference is ZE) and (acceleration is NM) then (throttle control is PM)

where NL – Negative large; PM – Positive medium; ZE -Zero;
NS- Negative small; PL- Positive large; PS- Positive small;
NM- Negative medium.

(2) Air Conditioner Controller

This fuzzy system controls the temperature of a room using:

(i) A dial that determines whether warm/hot air or cool/cold air is supplied.
(ii) **A** thermometer that measures the room temperature (e.g., 7°C).

The dial works as follows:

(a) Turning the dial positive → warm or hot air is supplied.
(b) Turning the dial negative → cool or cold air is supplied.
(c) Setting the dial to zero → no air is supplied.

A fuzzy controller can interpret vague temperature descriptions like *too cold*, *slightly warm*, or *comfortable* and adjust the airflow smoothly.

3) Fuzzy Logic Control of Blood Pressure During Anesthesia

During anesthesia, maintaining the correct mean arterial pressure (MAP) is crucial because it reflects the depth of anesthesia.

However, anesthesia is a biological process, which means:

(a) It is nonlinear
(b) It changes over time
(c) Its parameters vary from patient to patient

Because of this complexity, fuzzy logic is well-suited for controlling MAP.

Design Characteristics

(i) The fuzzy controller was developed through an iterative process.
(ii) Membership functions and linguistic rules were refined by trial and error.
(iii) The controller adjusts drug infusion or other control actions to maintain stable blood pressure.

This example shows how fuzzy logic can handle uncertain, time-varying medical systems.

(4) Aircraft Landing Control Problem

In this example, fuzzy logic is used to simulate the final descent and landing of an aircraft.

Desired Behavior

The downward velocity should be proportional to the square of the height.
(i) At high altitudes, the aircraft should descend quickly.
(ii) As the aircraft gets closer to the ground, the downward velocity should decrease.
(iii) When the height becomes extremely small, the downward velocity should approach zero.

Why This Matters

(a) The aircraft descends efficiently from high altitude.
(b) It slows down smoothly near the ground.
(c) It touches down gently, preventing damage and ensuring passenger safety.

Fuzzy logic helps achieve this smooth, human-like landing behavior.

CHECK YOUR PROGRESS

Q1. What is a fuzzy rule base?
Q2. What is the role of scaling factors in fuzzy controllers?
Q3. Explain the steps involved in designing a fuzzy logic controller.
Q4. Describe the application of fuzzy logic in cruise control.
Q5. Why is fuzzy logic suitable for real-world control systems?

9.10 LET US SUM UP

- The rule base consists of a collection of IF–THEN rules that express expert knowledge in linguistic form, allowing the controller to make decisions under uncertainty.
- Defuzzification converts the aggregated fuzzy output into a crisp numerical value that can be applied to the physical system, with common methods including centroid, mean of maxima, and weighted average.
- Fuzzy control system models can represent first-order, second-order, and p-th order dynamic systems using fuzzy relational equations and standard composition operations.
- Fuzzy logic is widely applied in real-world systems because it handles nonlinearity, uncertainty, and imprecision more effectively than classical control methods.
- Greg Viot's fuzzy cruise controller uses fuzzy rules based on speed difference and acceleration to adjust throttle smoothly and maintain a desired vehicle speed.

- Fuzzy air-conditioner controllers interpret vague temperature descriptions and adjust heating or cooling levels gradually, resulting in more comfortable and energy-efficient temperature regulation.
- Fuzzy logic is used in anesthesia to control mean arterial pressure, where biological processes are nonlinear and time-varying, making fuzzy controllers more suitable than rigid mathematical models.
- Aircraft landing control systems use fuzzy rules to relate altitude to desired downward velocity, ensuring a smooth and safe descent that mimics human pilot behavior.

✓ 9.11 ANSWERS TO CHECK YOUR PROGRESS

Check your progress 1:

Answer to Q1:

A fuzzy rule base is a collection of IF–THEN rules that describe how the system should behave. Each rule connects fuzzy input conditions to fuzzy output actions.

Answer to Q2:

Scaling factors normalize input and output variables to a standard range like [0,1] or [–1,1]. This ensures consistent fuzzy processing regardless of the original units.

Answer to Q3:

A fuzzy controller is designed through the following steps:

1. Identify variables — inputs, states, and outputs of the system.
2. Define fuzzy subsets — partition each variable's range into fuzzy sets like Low, Medium, High.
3. Assign membership functions — choose shapes such as triangular or Gaussian.
4. Construct the rule base — create IF–THEN rules linking inputs to outputs.
5. Choose scaling factors — normalize variables to standard intervals.
6. Fuzzify inputs — convert crisp values into fuzzy values.
7. Apply fuzzy inference — evaluate rules and compute fuzzy outputs.
8. Aggregate outputs — combine results of all rules.
9. Defuzzify — convert the final fuzzy output into a crisp control action.

This process converts human reasoning into a mathematical control system.

Answer to Q 4:

Greg Viot's fuzzy cruise controller maintains a vehicle at a desired speed. It uses two fuzzy inputs:

(i) Speed difference

(ii) Acceleration

and one fuzzy output:

(i) Throttle control

A rule base determines how the throttle should change depending on whether the car is too slow, too fast, accelerating, or decelerating. This allows smooth, human-like speed control even under varying road conditions.

Answer to Q 5:

Because real systems often involve uncertainty, imprecision, and nonlinear behavior. Fuzzy logic handles vague concepts like *slightly hot* or *very fast* and converts them into meaningful control actions.

9.12 FURTHER READING

- George J.Klir and Bo Yuan,Fuzzy Sets and Fuzzy Logic Theory and Applications,Prentice Hall P.T.R Upper Saddle River,New Jersey 07458,1995. Cole, E. (2011). *Network security bible*. John Wiley & Sons.
- Sudhir K.Pundir and Rimple Pundir,Fuzzy Sets and Their Applications Pragati Prakashan,Ninth Edition,2018.

9.13 MODEL QUESTIONS

Q1. Explain how fuzzy logic is used in air-conditioner control.
Q2. Discuss the use of fuzzy logic in controlling blood pressure during anesthesia.
Q3. Explain the fuzzy logic approach to aircraft landing control.
Q4. Explain the fuzzy control model of a discrete first-order system.

*** *** ***

UNIT 10: UNCERTAINTY BASED INFORMATION

UNIT STRUCTURE

10.1 Learning Objectives
10.2 Introduction
10.3 Uncertainty Based information
10.4 Simple, Joint and Conditional uncertainties
10.5 Non-interactive sets
10.6 Information Transmission
10.7 On- specifying of Fuzzy sets
10.8 Degree of Validity or Credibility
10.9 Fuzziness
10.10 Total Uncertainty
10.11 Fuzziness in Evidence Theory
10.12 Let Us Sum Up
10.13 Answers to Check Your Progress
10.14 Further Reading
10.15 Model Questions

10.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- Understand the concept of uncertainty,
- Understand decision – making under uncertainty,
- Analyze how uncertainty impacts choices in real world applications (e.g. medicine ,finance, autonomous system),
- Understand simple,Joint and conditional,
- Understand the concept of information,
- Understand the concept of fuzzinesss,
- Understand the nature of total uncertainty,
- Decision – making under total uncertainty..

10.2 INTRODUCTION

Uncertainty based on information refers to information that is designed to reduce or quantify uncertainty in various problem – solving situation.It acknowledges that in many real –world scenarios information is often incomplete,imprecise,vague,ambiguous or contradictory,rather than perfectly certain.The core idea is that information gains its value by helping us diminish our lack of knowledge or confidence about something:
Here's breakdown of key aspects:

1.The nature of uncertainty : Uncertainty is not a single concept;it manifests in different forms:
(a) Non-specificity
(b) Fuzzinesss
(c) Contradiction
(d) Aleatory uncertainty
(e) Epistemic uncertainty
2. Information as uncertainty Reduction: In the context of uncertainty,information valuable because it reduces uncertainty.The more uncertainty a piece of information eliminates,the more it is considered.This concept is formalized in various mathematical theories,most notably :
(a) Classical Information Theory.
(b) Fuzzy set theory
(c) Dempster – Shafer Theory.
3. Applications of uncertainty Based Information:
The ability to manage and quantify uncertainty is crucial in many fields especially in Artificial Intelligence (AI):

(a) Decision –Making under uncertainty : AI systems often operate in environments where information is incomplete or noisy.Techniques like probabilistic reasoning,Bayesian networks and Markov Decision processes allow AI to make informed decisions by considering the likelihood of different outcomes.
(b) Medical Diagnosis: AI models analyzing medical images or patient data deal with significant uncertainty due to system varability.Uncertainty based approaches help in providing more reliable diagnoses and treatment recommendations.
(c) Natural Language Processing (NLP):Human language in inherently ambiguous.NLP systems use uncertainty – based methods to infer user intent,resolve ambiguities and understand context.
(d) Robotics: Robots navigating real –world environments face uncertainty from noisy sensors,unpredictable movements and dynamic surroundings.Uncertainty quantification helps robots make safe and effective decisions.
(e) Fraud Detection: In financial systems,AI models need to distinguish between genuine and fraudulent transactions amidst uncertain patterns,often employing probabilistic models.
(f) Predictive Modeling :Understanding and quantifying the uncertainty in predictions (e.g. in financial forecasting,weather forecasting) is vital for assessing their reliability and making robust plans.

In essence,uncertainty – based information is about acknowledging the imperfections in our knowledge and developing formalisms and techniques to represent,measure and ultimately reduce that uncertainty to enable more intelligent and robust systems.

10.3 UNCERTAINTY – BASED INFORMATION

The purpose of the generalized inform theory is to capture properties of uncertainty – based information formalized within any feasible mathematics frame work.

10.3.1 TYPES UNCERTAINTY

There are three types of uncertainty recognized. These are

(i) Nonspecificity or imprecision : Non specificity uncertainty is connected with sizes (cardinalities) of relevant sets of alternatives.

(ii) Fuzziness or vagueness: In this type of uncertainty, results are imprecise from boundaries of fuzzy sets.

(iii) Strife or Discord: This type of uncertainty expresses conflicts among the various sets of alternatives.

10.3.2 NONSPECIFICITY OF CRISP SETS

In 1928, Hartley derived a function from the class of functions denoted by $U(\tilde{A})$ and define $U(\tilde{A}) = c\log_b|\tilde{A}|$

where $|\tilde{A}|$ denotes the cardinality of a finite nonempty set $\tilde{A}$ and $b > 1, c > 0$ are constants. The choice of values of constants b and c determines the unit in which uncertainty is measured.

If $b = 2$ and $c = 1$,then unit of uncertainty is called bits and we obtain

$$U(\tilde{A}) = \log_2|\tilde{A}| \qquad (1)$$

One bit of uncertainty is equivalent to the total uncertainty regarding the truth or falsity of one preposition and the function U is called Harley function.

Remark: Hartley function U defined on a nonempty subsets of a given finite universal set X of the form $U: P(X) - \{0\} \to \mathbb{R}^+$ and the range of U is $0 \le U(\tilde{A}) \le \log_2|X|$.

10.3.3 MEANING OF MEASURE OF UNCERTAINTY

The meaning of uncertainty measured by the Hartley function is depends on the meaning of the set A.

1. Predictive Uncertainty : If $\tilde{A}$ is a set of predictive states of a variable from the set X of all states defined for the variable,then $U(\tilde{A})$ is called the measure of predictive uncertainty.
2. Diagnostic Uncertainty : If $\tilde{A}$ is a set of possible disease of a patient determined from relevant medical evidence,then $U(\tilde{A})$ is called the measure of diagnostic uncertainty.
3. Retrodictive Uncertainty : If $\tilde{A}$ is a set of possible to an unsettled historical questions,the measure $U(\tilde{A})$ is called the measure of retrodictive uncertainty.
4. Perscriptive Uncertainty: If $\tilde{A}$ is a set of policies of a variable,the measure of $U(\tilde{A})$ is called the measure of perscriptive uncertainty.

10.3.3 MEANING OF MEASURE OF UNCERTAINTY

The amount of uncertainty based information $I(\tilde{A}, \tilde{B})$ produced by the action relevant to the situation is equal to the amount of reduced uncertainty defined as the difference $U(\tilde{A}) - U(\tilde{B})$ i.e., $I(\tilde{A}, \tilde{B}) = \log_2 \frac{|\tilde{A}|}{|\tilde{B}|}$

when the action eliminates all alternatives except one (i.e. when $|\tilde{B} = 1|$),we obtain

$$I(\tilde{A}, \tilde{B}) = \log_2 |\tilde{A}| = U(\tilde{A})$$

Therefore, $U(\tilde{A})$ may also be viewed as the amount of information needed to characterize one element of the set $\tilde{A}$.

10.4 SIMPLE, JOINT AND CONDITIONAL UNCERTAINTY

Consider two universal sets X and Y and assume that a relation $R \subseteq X \times Y$ describes a set of possible joint alternatives in some situation of interest. Assume further that the domain and range of R are sets $R_X \subseteq X$ and $R_Y \subseteq Y$,respectively. Then three distinct Hartley functions are applicable, defined on the power sets of X,Y and $X \times Y$.To identify clearly which universal set is involved in each case, it is useful (and a common practice) to write $U(X), U(Y), U(X \times Y)$ instead of $U(R_X), U(R_Y)$ and $U(R)$ respectively.

1. Simple Uncertainty : The functions

$$U(X) = \log_2 |R_X|$$

and

$$U(Y) = \log_2 |R_Y|$$

are called the simple uncertainties

2. Joint Uncertainty : The function denoted by $U(X, Y)$ and defined by

$$U(X, Y) = \log_2 |R|$$

is called joint uncertainty.

3. Conditional Uncertainty : The functions defined as

$$U(X|Y) = \log_2 \frac{|R|}{|R_Y|}$$

and

$$U(Y|X) = \log_2 \frac{|R|}{|R_X|}$$

are called conditional uncertainties.

Observe that the ratio $\frac{|R|}{|R_Y|}$ in $U(X|Y)$ represents the average number of elements of X that are possible alternatives under condition that an element of Y has already been selected.This means that $U(X|Y)$ measures the average nonspecificity regarding alternatives choices from X for all particular choices from Y.Function $U(Y|X)$ clearly has a similar meaning with the roles of sets X and Y exchanged.Observe also that the conditional uncertainties can expressed in terms of the joint uncertainty and two simple uncertainties:

$$U(X|Y) = U(X, Y) - U(Y) \qquad (2)$$

$$U(Y|X) = U(X, Y) - U(X) \qquad (3)$$

Furthermore,

$$U(X) - U(Y) = U(X|Y) - U(Y|X)$$

10.5 NON-INTERACTIVE SETS

If alternatives from X do not depend on selections from Y and vice versa, then $R = X \times Y$ and the sets X and Y are called noninteractive.Then, clearly

$$U(X|Y) = U(X)$$
$$U(Y|X) = U(Y)$$

and $$U(X,Y) = U(X) + U(Y)$$

In all other cases, when sets X and Y are interactive, these equations become the inequalities:

$$U(X|Y) < U(X)$$
$$U(Y|X) < U(Y)$$

and $$U(X,Y) < U(X) + U(Y)$$

10.6 INFORMATION TRANSMISSION

The following symmetric function,which is usually referred to as information transmission,is a useful indicator of the strength of constraint between sets X and Y:

$$T(X,Y) = U(X) + U(Y) - U(X,Y)$$

when the sets are noniteractive, $T(X,Y) = 0$,otherwise $T(X,Y) > 0$.Using (2) and (3), $T(X,Y)$ can be expressed in terms of the conditional uncertainties

$$T(X,Y) = U(X) - U(X|Y)$$

$$T(X,Y) = U(Y) - U(Y|X)$$

Example 1: Let x and y be two variables whose values are taken from sets $X = \{low, medium, high\}$ and $Y = \{1,2,3,4\}$ respectively.Variables are constrained by the relation R,expressed by the matrix

$$\begin{array}{c} \\ low \\ medium \\ high \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Find the values of $U(X), U(Y), U(X,Y), U(X|Y), U(Y|X)$ and $T(X,Y)$.
Solution: We have

$$R = \begin{array}{c} \\ low \\ medium \\ high \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Here, we see that low values of x does not constrain y at all, the medium value of x constrains y partially and the high value of x constraints it totally.
Therefore,

$$U(X) = log_2|X| = \log_2 3 = 1.6$$
$$U(X) = log_2|Y| = \log_2 4 = 2$$
$$U(X,Y) = \log_2|R| = \log_2 7 = 2.8$$
$$U(X|Y) = U(X,Y) - U(Y) = 2.8 - 2 = 0.8$$
$$U(Y|X) = U(X,Y) - U(X) = 2.8 - 1.6 = 1.2$$
$$T(X,Y) = U(X) + U(Y) - U(X,Y)$$
$$= 1.6 + 2 - 2.8$$
$$= 0.8$$

Remark:The Hartley function (1) is applicable only a finite set.However,this form may be appropriately modified to infinite sets R (or more generally,R^k for some natural number k).Given a measurable and Lebesgue – integrable subset A of R (R^k),a meaningful counterpart of (1) for infinite sets take the form

$$U(A) = log[1 + \mu(A)]$$

where $\mu(A)$ is the measure of A defined by the Lebesgue integral of the characteristic function of A.For example,when A is an interval $[a,b]$ of R,then $\mu(A) = b - a$ and

$$U([a,b]) = log[1 + (b - a)]$$

10.7 NONSPECIFICITY OF FUZZY SETS

For any nonempty fuzzy set A defined on a finite universal set X,the generalized Hartley function has the form

$$U(A) = \frac{1}{h(A)} \int_0^{h(A)} \log_2|\alpha_A| d\alpha \qquad (4)$$

Where $|\alpha_A|$ denotes the cardinality of the α –cut of A and $h(A)$ is the height of A.Observe that $U(A)$,which measures nonspecificity of A,is a weighted average of values of the Hartley function of all distinct α –cuts of the normalized counterpart of A,defined by $\frac{A(x)}{h(A)}$ for all $x \in A$.Each weight is a difference between the values of α of a given α –cut and the immediate proceeding α –cut.For any $A, B \in F(X) - \{\phi\}$,if $\frac{A(x)}{h(A)} = \frac{B(x)}{h(B)}$ for all $x \in X$,then $U(A) = U(B)$.

Remark: Let A be a nonempty fuzzy set defined on R such that α –cuts α_A of A are infinite sets.Then

$$U(A) = \frac{1}{h(A)} \int_0^{h(A)} \log[1 + \mu(\alpha_A)] d\alpha \qquad (5)$$

where α_A is a measurable and Lebesgue integrable function and $\mu(\alpha_A)$ is the measure of α_A defined by the Lebesgue integral of the characteristic function of α_A.

Example 2: Let A be a fuzzy set defined by

$$A = \left\{ \begin{matrix} (1,.1),(2,.1),(3,.3),(4,.4),(5,.6),(6,.7),(7,.9), \\ (8,1),(9,1),(10,1),(11,.9,)(12,.7),(13,.6),(14,.4),(15,1) \end{matrix} \right\}$$

and $A(x) = 0$ for all $x > 15$.Calculate the nonspecificity of A.

Solution:We know that nonspecificity of a fuzzy set A is denoted by $U(A)$ is defined as

$$U(A) = \frac{1}{h(A)} \int_0^{h(A)} \log_2 |\alpha_A| d\alpha$$

Thus,

$$U(A) = \frac{1}{1} \int_0^1 \log_2 |\alpha_A| d\alpha$$

$$= \int_0^{0.1} \log_2 15 \, d\alpha + \int_{0.1}^{0.3} \log_2 12 \, d\alpha + \int_{0.3}^{0.4} \log_2 11 \, d\alpha + \int_{0.4}^{0.6} \log_2 9 \, d\alpha$$

$$+ \int_{0.6}^{0.7} \log_2 7 \, d\alpha + \int_{0.7}^{0.9} \log_2 5 \, d\alpha + \int_{0.9}^{1} \log_2 3 \, d\alpha$$

$$= 0.1 \log_2 15 + 0.2 \log_2 12 + 0.1 \log_2 11 + 0.2 \log_2 9 + 0.1 \log_2 7$$

$$+0.2 \log_2 5 + 0.1 \log_2 3$$

$$\approx 2.99$$

CHECK YOUR PROGRESS

Q1) Consider the fuzzy sets A_1, A_2 and A_3 defined on $\mathbb{R}$ whose membership functions are given as below:

$$A_1(x) = \begin{cases} \frac{x}{2} & for\ x \in [0,2] \\ 2 - \frac{x}{2} & for\ x \in [2,4] \\ 0 & otherwise \end{cases}$$

$$A_2(x) = \begin{cases} x - 1 & for\ x \in [1,2] \\ 3 - x & for\ x \in [2,3] \\ 0 & otherwise \end{cases}$$

and

$$A_3(x) = \begin{cases} \frac{x-1}{2} & for\ x \in [1,2] \\ \frac{3-x}{2} & for\ x \in [2,3] \\ 0 & otherwise \end{cases}$$

Determine the nonspecificity of these fuzzy sets.

10.8 DEGREE OF VALIDITY OR CREDIBILITY

The height of a fuzzy set is called the degree of validity or credibility of the information expressed by the fuzzy set.
In general, given a normal fuzzy set $A \in F(X)$, all fuzzy set B in the infinite family

$$\mathcal{B}_A = \{B \in F(X) - \{\phi\} |\ B(x) = h(B)A(x)\ for\ all\ x \in X\}$$

have the same nonspecificity,regardless whether X is finite or infinite and we apply (4) or (5) respectively.Although the sets in $\mathcal{B}_A$ are not distinguished by the nonspecificity,they are clearly distinguished by their height $h(B)$.For each fuzzy set $B \in \mathcal{B}_A$,its height may be viewed as the degree of validity or credibility of information expressed by the fuzzy set.From this point of view,information expressed by the empty set has no validity.

10.9 FUZZINESS

The second type of uncertainty that involves fuzzy sets (but not crisp sets) is fuzziness (or vagueness).In general,a measure of fuzziness is a function

$$f: F(X) \to \mathbb{R}^+$$

where $F(X)$ denotes the set of all fuzzy subsets of X (fuzzy power set).For each fuzzy set A,this function assigns a nonnegative real number $f(A)$ that expresses the degree to which the boundary of A is not sharp.

Definition 10.9.1

A function $f: F(X) \to \mathbb{R}^+$ is called a sensible measure of fuzziness if it satisfies the following three conditions:

(i) $f(A) = 0 \Leftrightarrow A$ is a crisp set.

(ii) $f(A)$ attains its maximum $\Leftrightarrow A(x) = 0.5$ for all $x \in X$.

(iii) $f(A) \leq f(B)$ when set A is undoubtedly sharpen than the set B,which means that $A(x) \leq B(x)$ when $B(x) \leq 0.5$
and $A(x) \geq B(x)$ when $B(x) \geq 0.5$ for all $x \in X$.

The measure of fuzziness $f(A)$ is defined as follows:

$$f(A) = \sum_{x \in X}[1 - |2A(x) - 1|]$$

and the range of function f is $[0, |X|]$.

If $X = [a, b]$, then fuzziness $f(A)$ of a fuzzy set A on X is

$$f(A) = \int_a^b [1 - |2A(x) - 1|]dx$$
$$= (b - a) - \int_a^b |2A(x) - 1|dx$$

Example: Calculate the fuzziness of the fuzzy set A,where

$$A = \{(1,0.1), (2,0.1), (3,0.3), (4,0.4), (5,0.6), (6,0.7), (7,0.9), (8,1), (9,1),$$
$$(10,1), (11,0.9), (12,0.7), (13,0.6), (14,0.4), (15,0.1)\}$$

Solution : We have

$A(1) = 0.1, A(2) = 0.1, A(3) = 0.3, A(4) = 0.4, A(5) = 0.6, A(6) = 0.7,$
$A(7) = 0.9, A(8) = 1, A(9) = 1, A(10) = 1, A(11) = 0.9, A(12) = 0.7,$
$A(13) = 0.6, A(14) = 0.4, A(15) = 0.1.$

The fuzziness of the set A,

$$f(A) = \sum_{x \in X}[1 - |2A(x) - 1|]$$
$$= \sum_{x \in X} 1 - \sum_{x \in X}[2A(x) - 1]$$
$$= 15 - (0.8 + 0.8 + 0.4 + 0.2 + 0.2 + 0.4 + 0.8 + 1 + 1 + 1 + 0.8 + 0.4 + 0.2 + 0.2 + 0.8)$$

$= 15 - 9$
$= 6$

CHECK YOUR PROGRESS

Q2. Calculate the fuzziness for the sets A_1, A_2 and A_3 are defined within the interval $[0,4]$ as:

$$A_1(x) = \begin{cases} \frac{x}{2} & for\ x \in [0,2] \\ 2 - \frac{x}{2} & for\ x \in [2,4] \\ 0 & otherwise \end{cases}$$

$$A_2(x) = \begin{cases} x - 1 & for\ x \in [1,2] \\ 3 - x & for\ x \in [2,3] \\ 0 & otherwise \end{cases}$$

and

$$A_3(x) = \begin{cases} \frac{x-1}{2} & for\ x \in [1,2] \\ \frac{3-x}{2} & for\ x \in [2,3] \\ 0 & otherwise \end{cases}$$

10.10 TOTAL UNCERTAINTY

There are two types of uncertainty,nonspecificity and strife,coexist in evidence theory and both are measured in the same units,it is reasonable to consider the possibility of adding their individual measures to form a measure of total uncertainty.

The total uncertainty, N_λ is defined by the equation

$$N_\lambda(m) = N(m) + S(m)$$

where $N(m) = \sum_{A \in F} m(A) \log_2 |A|$,

(F, m) is an arbitrary body of evidence.

and $S(m) = N(m) - \sum_{A \in F} m(A) \log_2 \sum_{B \in F} m(B) |A \cap B|$

Therefore $N_\lambda = 2 \sum_{A \in F} \log_2 |A| - \sum_{A \in F} m(A) \log_2 \sum_{B \in F} m(B) |A \cap B|$

$$= \sum_{A \in F} m(A) \log_2 \frac{|A|^2}{\sum_{B \in F} m(B) |A \cap B|}$$

Notes 1: The range of total uncertainty N_λ is $[0, \log_2|X|]$, which is the same as the range of each of its components.

2. The minimum $N_\lambda(m) = 0$ is obtained $m(\{x\}) = -1$ for some $x \in X$.

3. The maximum $N_\lambda(m) = log_2|X|$ is not unique.

4. Two significant maxima are obtained for $m(X) = 1$ (when $S(m) = 0$) and for $m(\{x\}) = \frac{1}{X}$ for all $x \in X$ (when $N(m) = 0$).

10.11 FUZZINESS IN EVIDENCE

Evidence theory normally operates with focal elements that are crisp subsets of a given universal set.However,the theory can be fuzzified,as can any other mathematical theory.Focal elements in fuzzified evidence theory are fuzzy sets,each of which has some degree of fuzziness.Given a body of evidence (m, F) in which elements of F are fuzzy sets,it is reasonable to express the total degree of fuzziness,$F(m)$ of this body by the weighted average of the individual degree of fuzziness $f(A)$,of all focal elements $(A \in F)$,each weighted by $m(A)$.
That is,

$$F(m) = \sum_{A \in F} m(A) f(A)$$

Where $f(A) = \sum_{x \in X}[1 - |2A(x) - 1|]$.

10.12 LET US SUM UP

- The measurement of uncertainty in terms of classical set theory was first discussed by Hartley in 1928.
- Hartley derived a function from the class of functions denoted by $U(\tilde{A})$ and defined as

  $$U(\tilde{A}) = c \log_b |\tilde{A}|$$

  where $|\tilde{A}|$ denotes the cardinality of a finite nonempty set $\tilde{A}$ and $b > 1, c > 0$ are constants.
  If $b = 2$ and $c = 1$, then the unit of uncertainty is called bits and we get the following Hartley function

  $$U(\tilde{A}) = \log_2 |\tilde{A}|$$
- Hartley function U defined on a nonempty subsets of a given finite universal set X is of the form

  $$U: P(X) - \{\phi\} \to \mathbb{R}^+$$

  And the range of U is $0 \le U(\tilde{A}) \le log_2|X|$.
- The amount of uncertainty – based information $I(\tilde{A}, \tilde{B})$ produced by the action relevant to the situation is equal to the amount of reduced uncertainty defined as the difference $U(\tilde{A}) - U(\tilde{B})$,

$$\text{i.e.,} I\left(\tilde{A}-\tilde{B}\right)=\log_2\frac{|\tilde{A}|}{|\tilde{B}|}$$

- Simple Uncertainty: The functions

  $$U(X)=\log_2|R_X| \text{ and } U(Y)=\log_2|R_Y|$$

  are called the simple uncertainties.
- Joint Uncertainty : The function

  $$U(X,Y)=\log_2|R|$$

  is called a joint uncertainty.
- Conditional Uncertainty:The function defined as

  $$U(X|Y)=\log_2\frac{|R|}{|R_Y|}$$

  $$\text{and } U(Y|X)=\log_2\frac{|R|}{|R_X|}$$

  are called conditional uncertainties.
- Conditional uncertainties can also be expressed in terms of the joint uncertainty and two simple uncertainties.

  $$U(X|Y)=U(X,Y)-U(Y)$$
  $$U(Y|X)=U(X,Y)-U(X)$$
- Let A be a measurable and Lebesgue integrable subset of R,then

  $U(A)=log\left(1+\mu(A)\right)$ where $\mu(A)$ is the measure of A.
- Generalized Hartley function is given by

  $$U(A)=\frac{1}{h(A)}\int_0^{h(A)}\log_2|\alpha_A|\,d\alpha$$

  where $|\alpha_A|$ denotes the cardinality of the α −cut of set A and $h(A)$ is the height of A.
- Two fuzzy sets are said to be equal when their normalization have the same nonspecificity measured by the function U.
- If A is an infinite subsets of $\mathbb{R}$,then

  $$U(A)=\frac{1}{h(A)}\int_0^{h(A)}\log[1+|\alpha_A|]\,d\alpha$$
- A measure of fuzziness is a function defined as follows:

  $$f: F(X)\to R^+$$

  where $F(X)$ denotes the set of all fuzzy subsets of X.For each fuzzy set A,the image $f(A)$ expresses the degree to which the boundary is not sharp.
- The measure of fuzziness $f(A)$ is defined as follows:

  $$f(A)=\sum_{x\in X}[1-|2A(x)-1|]$$

  and the range of function f is $[0,|X|]$.

  When $X=[a,b]$ and A is a fuzzy set on X,then

$$f(A) = \int_a^b [1 - |2A(x) - 1|]\ dx$$

Summary of Basic Uncertainty Measure

Uncertainty	Uncertainty Measure	Uncertainty Type
Classical set theory	$U(\tilde{A}) = \log_2\|\tilde{A}\|$	Nonspecificity
Fuzzy set theory	$U(\tilde{A}) = \frac{1}{h(A)}\int_0^{h(A)} \log_2\|\alpha_A\| d\alpha$	Nonspecificity
Evidence theory	$\mathrm{N(m)} = \sum_{\mathrm{A\in F}} \mathrm{m(A)}\log_2\|\mathrm{A}\|$	Nonspecificity
Evidence theory	$S(m) = -\sum_{A\in F} m(A)\log_2 \sum_{B\in F} m(B)\frac{\|A\cap B\|}{\|A\|}$	Strif
Evidence theory	$\mathrm{N_{\lambda=}} \sum_{\mathrm{A\in F}} \mathrm{m(A)}\log_2\left(\frac{\|\mathrm{A}\|^2}{\sum_{\mathrm{B\in F}} \mathrm{m(B)}\|\mathrm{A\cap B}\|}\right)$	Total fuzziness
Fuzziness set theory	$f(A) = \sum_{x\in X}[1 - \|2A(x) - 1\|]$	Fuzziness
Fuzzified evidence theory	$F(m) = \sum_{A\in F} m(A)f(A)$	Fuzziness

✓ 10.13 ANSWERS TO CHECK YOUR PROGRESS

Answer to Q1:

Observe that the set A_3 is not normal can be normalized replacing $A_3(x)$ with $2A_3(x)$ for each $x \in \mathbb{R}$.

$$A_1(x) = \begin{cases} \frac{x}{2} & for\ x \in [0,2] \\ 2 - \frac{x}{2} & for\ x \in [2,4] \\ 0 & otherwise \end{cases}$$

α –cuts of A_1 is

$$\alpha_{A_1} = [2\alpha, 4 - 2\alpha]$$

So that $\mu(\alpha_{A_1}) = (4 - 2\alpha) - 2\alpha = 4 - 4\alpha$.

Thus $U(A_1) = \int_0^1 log[1 + \mu(\alpha_{A_1})]\, d\alpha$

$$= \int_0^1 log(1+4-4\alpha)\, d\alpha$$

$$= \int_0^1 log(5-4\alpha)\, d\alpha$$

$$= log(5-4\alpha)\int_0^1 d\alpha - \int_0^1 \left\{\frac{d}{d\alpha} log(5-4\alpha) \int d\alpha\right\} d\alpha$$

$$= [\alpha . log(5-4\alpha)]_0^1 - \int_0^1 \left(\frac{-4-\alpha}{5-4\alpha}\right) d\alpha$$

$$= (1log1 - 0.log5) - \int_0^1 \frac{(5-4\alpha)-5}{5-4\alpha} d\alpha$$

$$= -\int_0^1 d\alpha + 5\int_0^1 \frac{1}{5-4\alpha} d\alpha$$

$$= -[\alpha]_0^1 - \frac{5}{4}\int_0^1 \frac{-4}{5-4\alpha} d\alpha$$

$$= -1 - \frac{5}{4}[log(5-4\alpha)]_0^1$$

$$= \frac{5}{4} log5 - 1$$

The fuzzy set A_2 is defined as

$$A_2(x) = \begin{cases} x-1 & for\ x \in [1,2] \\ 3-x & for\ x \in [2,3] \\ 0 & otherwise \end{cases}$$

α –cuts of A_2 is

$\alpha_{A_2} = [\alpha + 1, 3-\alpha]$ so that $\mu(\alpha_{A_2}) = 2 - 2\alpha$.

Therefore

$$U(A_2) = \int_0^1 log[1 + \mu(\alpha_{A_2})]\, d\alpha$$
$$= \int_0^1 log(1+2-2\alpha) d\alpha$$
$$= \int_0^1 log(3-2\alpha) d\alpha$$
$$= \frac{3}{2} log3 - 1$$

The fuzzy set A_3 is defined as

$$A_2(x) = \begin{cases} \frac{x-1}{2} & for\ x \in [1,2] \\ \frac{3-x}{2} & for\ x \in [2,3] \\ 0 & otherwise \end{cases}$$

α –cuts of A_3 is

$$\alpha_{A_3} = [2\alpha + 1, 3 - 2\alpha]$$

so that $\mu(\alpha_{A_3}) = (3 - 2\alpha) - (2\alpha + 1) = 2 - 4\alpha.$

Therefore,

$$U(A_3) = 2\int_0^{0.5} log[1 + \mu(\alpha_{A_3})]\, d\alpha$$

$$= 2\int_0^{0.5} log(3 - 4\alpha) d\alpha$$

$$= 2\left(\frac{3}{4} log 3 - \frac{1}{2}\right)$$

Answer to Q2: The set A_1 is defined within the interval$[0,4]$ as

$$A_1(x) = \begin{cases} \frac{x}{2} & for\ x \in [0,2] \\ 2 - \frac{x}{2} & for\ x \in [2,4] \\ 0 & otherwise \end{cases}$$

Therefore $f(A_1) = \int_0^4 [1 - |2A_1(x) - 1|]\, dx$

$$= \int_0^4 dx - \int_0^4 |2A(x) - 1|\, dx$$

$$= 4 - \int_0^2 |2A(x) - 1|\, dx - \int_2^4 |2A(x) - 1|\, dx$$

$$= 4 - \int_0^2 |x - 1|\, dx - \int_0^2 |3 - x|\, dx$$

$$= 4 - \int_0^1 (1 - x) dx - \int_1^2 (x - 1) dx - \int_2^3 (3 - x) dx - \int_3^4 (x - 3) dx$$

$$= 4 + \left[\frac{(1-x)^2}{2}\right]_0^1 - \left[\frac{(x-1)^2}{2}\right]_1^2 + \left[\frac{(3-x)^2}{2}\right]_2^3 - \left[\frac{(x-3)^2}{2}\right]_3^4$$

$$= 4 - 0.5 - 0.5 - 0.5 - 0.5$$

$$= 2$$

The set A_2 is defined within the interval$[0,4]$ as

$$A_2(x) = \begin{cases} x - 1 & for\ x \in [1,2] \\ 3 - x & for\ x \in [2,3] \\ 0 & otherwise \end{cases}$$

Therefore

$$f(A_2) = \int_0^4 [1 - |2A_2(x) - 1|]\, dx$$

$$= \int_0^4 dx - \int_0^4 |2A_2(x) - 1| dx$$

$$= 4 - \int_0^1 |2A_2(x) - 1| dx - \int_1^2 |2A_2(x) - 1| dx - \int_2^3 |2A_2(x) - 1| dx - \int_3^4 |2A_2(x) - 1| dx$$

$$= 4 - \int_0^1 dx - \int_1^2 |2x - 3| dx - \int_2^3 |5 - 2x| dx - \int_3^4 dx$$

$$= 4 - 1 - \int_1^{\frac{3}{2}} |2x-3|dx - \int_{\frac{3}{2}}^{2} |2x-3|dx - \int_2^{\frac{5}{2}} |5-2x|dx - \int_{\frac{5}{2}}^{3} |5-2x|dx - 1$$

$$= 2 + \int_1^{\frac{3}{2}} (2x-3)dx - \int_{\frac{3}{2}}^{2} (2x-3)dx - \int_2^{\frac{5}{2}} (5-2x)dx + \int_{\frac{5}{2}}^{3} (5-2x)dx$$

$$= 2 + [x^2 - 3x]_1^{\frac{3}{2}} - [x^2 - 3x]_{\frac{3}{2}}^{2} - [5x - x^2]_2^{\frac{5}{2}} + [5x - x^2]_{\frac{5}{2}}^{3}$$

$$= 2 + \left(-\frac{1}{4}\right) - \frac{1}{4} - \frac{1}{4} - \frac{1}{4}$$

$$= 1$$

The set A_3 is defined within the interval $[0,4]$ as

$$A_3(x) = \begin{cases} \dfrac{x-1}{2} & for\ x \in [1,2] \\ \dfrac{3-x}{2} & for\ x \in [2,3] \\ 0 & otherwise \end{cases}$$

Therefore

$$f(A_3) = \int_0^4 [1 - |2A_3(x) - 1|]\, dx$$

$$= \int_0^4 dx - \int_0^4 |2A_3(x) - 1| dx$$

$$= 4 - \int_0^1 |2A_3(x) - 1| dx - \int_1^2 |2A_3(x) - 1| dx - \int_2^3 |2A_3(x) - 1| dx - \int_3^4 |2A_3(x) - 1| dx$$

$$= 4 - \int_0^1 dx - \int_1^2 |x-2| dx - \int_2^3 |2-x| dx - \int_3^4 dx$$

$$= 4 - 1 + \int_1^2 (x-2)dx + \int_2^3 (2-x)dx - 1$$

$$= 2 + \left[\frac{x^2}{2} - 2x\right]_1^2 + \left[2x - \frac{x^2}{2}\right]_2^3$$

$$= 2 + \left[(2-4) - \left(\frac{1}{2} - 2\right)\right] + \left[\left(6 - \frac{9}{2}\right) - (4-2)\right]$$

$$= 2 - \frac{1}{2} - \frac{1}{2}$$

$$= 1$$

10.14 FURTHER READING

- George J.Klir and Bo Yuan,Fuzzy Sets and Fuzzy Logic Theory and Applications,Prentice Hall P.T.R Upper Saddle River,New Jersey 07458,1995. Cole, E. (2011). *Network security bible*. John Wiley & Sons.
- Sudhir K.Pundir and Rimple Pundir,Fuzzy Sets and Their Applications,Pragati Prakashan,Ninth Edition,2018.

10.15 MODEL QUESTIONS

Q1. Calculate the nonspecificity and fuzziness for each of the following sets:

(a) $A = \{(a, 0.3), (b, 0.5), (c, 0.9), (d, 1), (e, 0.7), (f, 0.4), (g, 0.2)\}$

(b) Fuzzy relation R defined by the matrix:

$$R = \begin{bmatrix} 1 & 0.7 & 0 & 0.5 & 0 \\ 0.5 & 1 & 0.3 & 0.9 & 0.4 \\ 0 & 0.5 & 1 & 0 & 0.9 \end{bmatrix}$$

Q2. Calculate the nonspecificity and fuzziness of each relation involved in the procedure for obtaining transitive closure of relation R,defined by the matrix

$$R = \begin{bmatrix} 0.7 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0.8 & 0 \end{bmatrix}$$

Q3. Calculate the nonspecificity and fuzziness for the following fuzzy sets defined on the interval $[0,10]$:

(a) $A_1(x) = \frac{x}{10}$

(b) $A_2(x) = 2^{-x}$

(c) $A_2(x) = \begin{cases} \frac{x}{5} & for\ x \leq 5 \\ 2 - \frac{x}{5} & for\ x \geq 5 \end{cases}$

Q4. Consider the following basic assignments defined on $X = \{a, b, c\}$:

$m_1(\{c\}) = 0.2,$ $m_1(\{b, c\}) = 0.3,$ $m_1(X) = 0.5;$

$m_1(\{a\}) = 0.5,$ $m_1(\{b\}) = 0.2,$ $m_1(\{c\}) = 0.3;$

$m_1(\{c\}) = 0.2,$ $m_1(\{a, b\}) = 0.3,$ $m_1(X) = 0.5;$

Calculate $N(m_i)$,$S(m_i)$ and $N_S(m_i)$ for each $i = 1,2,3$.

*** *** ***

UNIT 11: DECISION MAKING IN FUZZY ENVIRONMENT

UNIT STRUCTURE

11.1 Learning Objectives
11.2 Introduction
11.3 Individual Decision Making
11.4 Multiperson Decision Making
11.5 Multicriteria Decision Making
11.6 Fuzzy Ranking Method
11.7 Let Us Sum Up
11.8 Answers to Check Your Progress
11.9 Further Reading
11.10 Model Questions

11.1 LEARNING OBJECTIVES

After reading this unit, students will be able to:

- Understand the fundamental concepts of fuzzy set theory and how it enhances decision-making under ambiguity.
- Gain insight into modelling uncertain data using fuzzy numbers, membership functions, and linguistic variables.
- Develop the ability to formulate and solve decision-making problems using fuzzy logic tools such as fuzzy multicriteria decision making, individual decision making, multiperson decision making, and fuzzy ranking methods.
- Apply fuzzy decision-making techniques to real-life scenarios in fields such as engineering, business, healthcare, and environmental management.
- Build skills in selecting appropriate fuzzy methods based on the nature of the problem, interpreting fuzzy results, and justifying decisions made under uncertainty.

11.2 INTRODUCTION

Decision making is a fundamental aspect of human activity, often involving complex problems that are characterized by uncertainty, ambiguity, and imprecision. Traditional decision-making models assume that all information is known, well-defined, and crisp. However, in real-world scenarios, information is often vague, incomplete, or expressed in linguistic terms such as "high risk", "moderate cost", or "low priority". This is where fuzzy set theory becomes a valuable tool for supporting decision making in uncertain or imprecise environments.

A fuzzy environment refers to situations in which:

- The data or criteria used for making decisions are imprecise or vague.
- The preferences of decision-makers are expressed in qualitative terms.

- There is uncertainty due to lack of exact numerical data.

Fuzzy logic, introduced by Lotfi Zadeh in 1965, provides a framework for reasoning with such imprecise information. Unlike classical binary logic, where variables are either true or false, fuzzy logic allows variables to have degrees of truth between 0 and 1.

Fuzzy decision making refers to applying fuzzy set theory and fuzzy logic to the process of making decisions. This approach helps decision-makers to:

- Model uncertainty and vagueness in data and preferences.
- Use linguistic variables and fuzzy rules to evaluate alternatives.
- Derive more realistic and human-like conclusions.

Key Concepts for Decision-Making:

- **Fuzzy Sets:** Represent imprecise concepts by assigning a membership value between 0 and 1 to each element.
- **Linguistic Variables:** Variables described using words ($e.g.$, "low", "medium", "high") instead of numbers.
- **Fuzzy Rules:** IF–THEN rules used to capture expert knowledge or decision criteria.
- **Aggregation and Defuzzification:** Combining fuzzy outputs and converting them to crisp decisions.

Fuzzy decision making is widely used in areas such as:

- Risk assessment and management
- Multicriteria decision analysis (MCDA)
- Supply chain management
- Healthcare decision support.
- Environmental planning.

11.3 INDIVIDUAL DECISION MAKING

Let us introduce the sets:

D = The set of fuzzy decisions represented by its membership function $\mu_{D(x)}$.

G = The set of fuzzy goals with membership function $\mu_{G(x)}$.

C = The set of constraints with membership function $\mu_{C(x)}$.

where x is an element of a set of alternatives A.

$$D = G \cap C \quad (1)$$

and

$$\mu_{D(x)} = min\{\mu_{G(x)}, \mu_{C(x)}\},\ x \in A \quad (2)$$

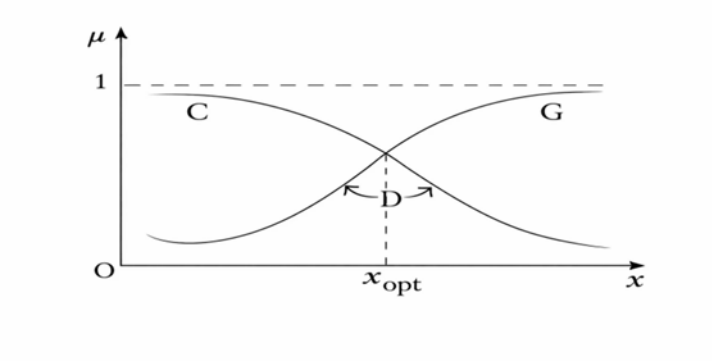


Figure 11.1 Fuzzy goal G, Constraint C and Decision D

Usually, the decision makers want to have a crisp result which requires defuzzification of D. It is natural to adopt for that purpose the value x_{opt} from the set of alternatives A with the highest degree of membership, $i.e.$, $max\{\mu_{D(x)} : x \in A\}$ in the set D. It is called the optimal decision and is expressed by:

$$x_{opt} = \{x \mid \max \mu_{D(x)} = \max \min(\mu_{G(x)}, \mu_{C(x)})\} \quad (3)$$

Formulas $(1) - (3)$ can be generalized for decision-making problems with more goals and constraints. For instance, for three goals G_1, G_2, G_3 and two constraintsC_1, C_2, one obtains the decision

$$D = G_1 \cap G_2 \cap G_3 \cap C_1 \cap C_2$$

with membership function

$$\mu_{D(x)} = min\{\mu_{G_1}(x), \mu_{G_2}(x), \mu_{G_3}(x), \mu_{C_1}(x), \mu_{C_2}(x)\}$$

and the optimal decision is given by

$$x_{opt} = \{x \mid \mu_{D(x)} \; is \; maximum\}$$

Example 1:

In a private university run by trust, the board of management want to pay an attractive dividend to the shareholders, but on the other hand, for some reasons, it should be modest. Assume that on the set of alternatives$A = \{x \mid 0 \leq x \leq 8\}$the membership functions $\mu_{G(x)}$ and $\mu_{C(x)}$ are given by:

$$\mu_G(x) = \begin{cases} 0, & for \quad 0 < x \leq 1 \\ \dfrac{x-1}{4}, & for \quad 1 \leq x \leq 5 \\ 1, & for \quad 5 \leq x \leq 8 \end{cases}$$

$$\mu_C = \begin{cases} 1, & for \quad 0 < x \leq 2 \\ -\dfrac{x-6}{4}, & for \quad 2 \leq x \leq 6 \\ 0, & for \quad 6 \leq x \leq 8 \end{cases}$$

Then,

(a) Draw the graphs of $\mu_{G(x)}$ and $\mu_{C(x)}$ and find geometrically the decision set D.

(b) Find geometrically the optimal decision x_{opt}.

Solution:

Assume that the fuzzy set goal G (attractive dividend) is defined on the set of alternatives

$$A = \{x \mid 0 \leq x \leq 8\}$$

as

$$\mu_G(x) = \begin{cases} 0, & for \quad 0 < x \leq 1 \\ \dfrac{x-1}{4}, & for \quad 1 \leq x \leq 5 \\ 1, & for \quad 5 \leq x \leq 8 \end{cases}$$

and the fuzzy set constraint C (modest dividend) is given on A by:

$$\mu_C = \begin{cases} 1, & for \quad 0 < x \leq 2 \\ -\dfrac{x-6}{4}, & for \quad 2 \leq x \leq 6 \\ 0, & for \quad 6 \leq x \leq 8 \end{cases}$$

(a) The graph of $\mu_{G(x)}$ and $\mu_{C(x)}$ is shown in Figure 11.2.

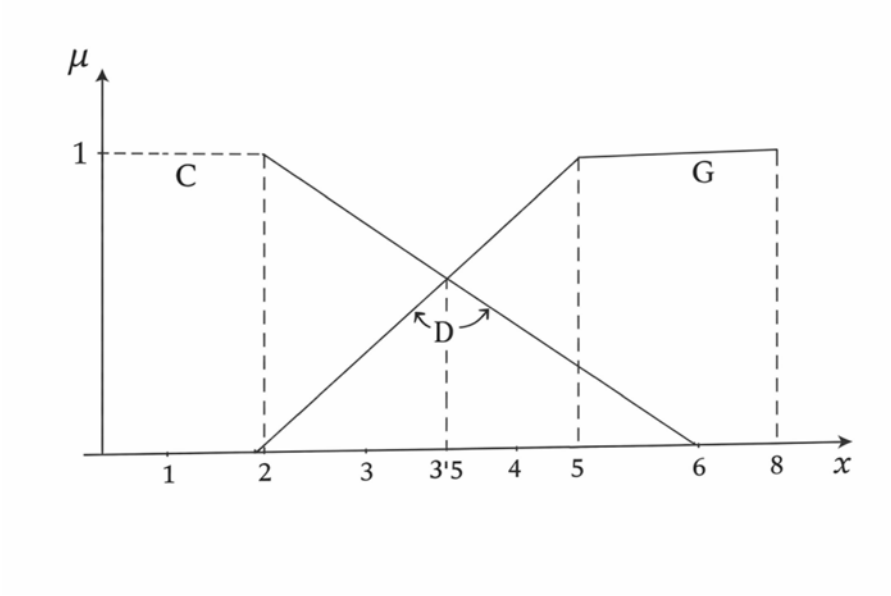


Figure 11.2

From the figure, we get:

$$D = G \cap C$$
$$= \{(3.5, 0.625)\}$$

(b) The optimal decision is: $x_{opt} = 3.5$, the abscissa of the intersection point of the straight lines , $u = \frac{x-1}{4}$ and $\mu = -\frac{x-6}{4}$

Example 2:

A company wants to fill a position. There are five candidates $x_i, i = 1,2,3,4,5$, who form the set of alternatives $A = \{x_1, x_2, x_3, x_4, x_5\}$. The hiring committee has three objectives (goals) which the candidates have to satisfy:

(i) Experience
(ii) Computer Knowledge
(iii) Young Age

Also, the committee has a constraint that the salary offered should be modest. After a serious discussion, each candidate is evaluated from the point of view of the goals and constraint. Finally, the committee constructs the following fuzzy sets on the set of alternatives.

$$G_1 = \{(x_1, 0.8), (x_2, 0.6), (x_3, 0.3), (x_4, 0.7), (x_5, 0.5)\}$$

$$G_2 = \{(x_1, 0.7), (x_2, 0.6), (x_3, 0.8), (x_4, 0.2), (x_5, 0.3)\}$$
$$G_3 = \{(x_1, 0.7), (x_2, 0.8), (x_3, 0.5), (x_4, 0.5), (x_5, 0.4)\}$$
$$C = \{(x_1, 0.4), (x_2, 0.7), (x_3, 0.6), (x_4, 0.5), (x_5, 0.9)\}$$

where G_1 represents experience, G_2 computer knowledge, G_3 young age, and C gives the readiness of the candidates to accept a modest salary.

Solution:

First, we find

$$\mu_{D(x)} = min\{\mu_{G_1}(x), \mu_{G_2}(x), \mu_{G_3}(x), \mu_C(x)\}$$

for all $x \epsilon A$, where D is the fuzzy decision set.

$$\mu_D(x_1) = min\{0.8, 0.7, 0.7, 0.4\} = 0.4$$
$$\mu_D(x_2) = min\{0.6, 0.6, 0.8, 0.7\} = 0.6$$
$$\mu_D(x_3) = min\{0.3, 0.8, 0.5, 0.6\} = 0.3$$
$$\mu_D(x_4) = min\{0.7, 0.2, 0.5, 0.5\} = 0.2$$
$$\mu_D(x_5) = min\{0.5, 0.3, 0.4, 0.9\} = 0.3$$

Thus, the decision set is:

$$D = \{(x_1, 0.4), (x_2, 0.6), (x_3, 0.3), (x_4, 0.2), (x_5, 0.3)\}$$

and

$$x_{opt} = \{x_i / \max \mu_D(x_i) \ is \ maximum\}$$
$$= x_2$$

Since candidate x_2 has the highest membership value (0.6), he is selected for the job.

CHECK YOUR PROGRESS

***Q* 1**.John must choose one of the three jobs offered to him by three different companies C_1, C_2, and C_3. The salaries (in rupees per month) are given below:

Company	C_1	C_2	C_3
Salary	40000	35000	30000

Suppose the constraints 'interesting job' and 'job within close driving distance' are presented by the fuzzy sets:

$$C_1 = \{(C_1, 0.5), (C_2, 0.7), (C_3, 0.8)\}$$
$$C_2 = \{(C_1, 0.3), (C_2, 0.8), (C_3, 1)\}$$

on the set of alternatives: $A = \{C_1, C_2, C_3\}$and the goal G of a high salary is given by the membership function:

$$\mu_G(x) = \begin{cases} 0, & for \quad 0 < x \leq 25000 \\ \dfrac{x - 25000}{20000}, & for \quad 25000 < x < 45000 \\ 1, & for \quad x \geq 45000 \end{cases}$$

On the universal set A, determine which company best satisfies the goal and constraints.

11.4 MULTIPERSON DECISION MAKING

When decisions are made by more than one person, the process is called multiperson decision making. The differences between individual and multiperson decision making are:

(i) The goals of individual decision makers may differ, so that each decision maker places a different ordering on the alternatives.

(ii) The decision makers may have different information bases for taking decisions.

A fuzzy model for group decision making was proposed by Blin and Whinston. In this model, each member of the group of individual decision makers is assumed to have a preference ordering which is reflexive $(a \sim a)$, antisymmetric $(a \sim b, b \nsim a)$, and transitive$(a \sim b, b \sim c \Rightarrow a \sim c)$. This ordering may be partial or total. The model allows individual decision makers to possess different aims. To deal with multiplicity of opinions in a group, a social preference function is defined as:

$$S : X \times X \rightarrow [0,1]$$

defined as,

$S(x_i, x_j)$ =degree of group preference of x_i overx_j ($i.e$ how many degree x_i over x_j is preferred with respect to x_j in any given group)

A simple form is given by:

$$S(x_i, x_j) = \frac{N(x_i, x_j)}{n}$$

where $N(x_i, x_j)$ is the number of persons preferring x_i over $x_j$and n is the total number of decision makers.

Aggregating $S(x_i, x_j)$ by a suitable function

$S = \cup \alpha.^{\alpha} S, \qquad \alpha \in [0,1]$

We get the result.

Example1:

Assume that there are 5 decision makers P_1, P_2, P_3, P_4 and P_5 for 4 alternatives a, b, c, d. The ordering for each decision maker is as follows:

$$P_1 = (a, b, c, d)$$
$$P_2 = P_5 = (c, a, d, b)$$
$$P_3 = (b, a, c, d)$$
$$P_4 = (a, b, d, c)$$

Now construct the preference table

$$\begin{matrix} & a & b & c & d \\ a & N(a,a) & N(a,b) & N(a,c) & N(a,d) \\ b & N(b,a) & N(b,b) & N(b,c) & N(b,d) \\ c & N(c,a) & N(c,b) & N(c,c) & N(c,d) \\ d & N(d,a) & N(d,b) & N(d,c) & N(d,d) \end{matrix}$$

Where $N(x_i, x_j)$ =number of persons preferring x_i to x_j. Since there is no self-preference, so $N(a,a) = N(b,b) = N(c,c) = N(d,d) = 0$.

$$N(a,b) = P_1, P_2, P_5, P_4 = 4 \Rightarrow N(b,a) = 1$$

Similarly, $N(a,c) = 3, N(c,a) = 2, N(a,d) = 5, N(d,a) = 0, N(b,c) = 3, N(c,b) = 2, N(b,d) = 2, N(d,b) = 3, N(c,d) = 4, N(d,c) = 1.$

Thus,

$$N(x_i, x_j) = \begin{matrix} \\ a \\ b \\ c \\ d \end{matrix} \begin{matrix} \begin{matrix} a & b & c & d \end{matrix} \\ \begin{bmatrix} 0 & 4 & 3 & 5 \\ 1 & 0 & 3 & 2 \\ 2 & 2 & 0 & 4 \\ 0 & 3 & 1 & 0 \end{bmatrix} \end{matrix}$$

By the formula $S(x_i, x_j) = \frac{N(x_i, x_j)}{n}$ where $n =$ number of decision makers, the table for $S(x_i, x_j)$ is as follows

$$S(x_i, x_j) = \begin{matrix} \\ a \\ b \\ c \\ d \end{matrix} \begin{matrix} \begin{matrix} a & b & c & d \end{matrix} \\ \begin{bmatrix} 0 & 0.8 & 0.6 & 1 \\ 0.2 & 0 & 0.6 & 0.4 \\ 0.4 & 0.4 & 0 & 0.8 \\ 0 & 0.6 & 0.2 & 0 \end{bmatrix} \end{matrix}$$

Take $\alpha \in [0,1]$. For $\alpha = 1$, $.^1 S = (a,d)$

$\alpha = 0.8$, $.^{0.8} S = \{(a,b),(c,d),(a,d)\}$

The total ordering $.^{0.8} O$ that are compatible with the pairs in the crisp relations $.^{0.8} S$ are

$$.^{0.8} O = \{(a,b,c,d),(a,c,b,d),(a,c,d,b)\}$$

For $\alpha = 0.6$, $.^{0.6} S = \{(a,c),(b,c),(d,b),(a,b),(c,d),(a,d)\}$

and

$$.^{0.6} O = \{(a,d,b,c),(a,c,d,b)\}$$

Now,

$$.^{0.8} O \bigcap .^{0.6} O = \{(a,c,d,b)\}$$

Therefore, the most preferred ordering is, (a,c,d,b).

In this process, it is necessary that each decision maker orders the alternatives.

In the second method, given by Shimura, pairwise comparison of each alternative is taken. A numerical scale is used for relative grading and

$f(x_i, x_j)$ = comparative grade of x_i with respect to x_j

and

$F(x_i, x_j)$ = relative preference grade of x_i with respect to x_j

$$= \frac{f(x_i, x_j)}{\max\{f(x_i, x_j), f(x_j, x_i)\}}$$

$$= \min\left\{1, \quad \frac{f(x_i, x_j)}{f(x_j, x_i)}\right\}$$

Then, the overall relative preference grade of x_i with respect to all other alternatives is:

$$p(x_i) = min\, F(x_i, x_j)$$

The preference ordering of alternatives is given by the numerical ordering of these grades $p(x_i)$.

Example 2:

The attractiveness grades are defined as:

$f(x_i, x_j)$	attractiveness of x_i with respect to x_j
1	little attractive

3	moderately attractive
5	strongly attractive
7	very strongly attractive
9	extremely attractive
2,4,6,8	intermediate values between levels

Attractiveness grades for five alternatives $a, b, c, d,$ and e are given in the following table.

$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array}\begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} f(a,a)=1 & f(a,b)=7 & f(a,c)=9 & f(a,d)=3 & f(a,e)=8 \\ f(b,a)=3 & f(b,b)=1 & f(b,c)=3 & f(b,d)=2 & f(b,e)=4 \\ f(c,a)=1 & f(c,b)=1 & f(c,c)=1 & f(c,d)=3 & f(c,e)=5 \\ f(d,a)=2 & f(d,b)=7 & f(d,c)=7 & f(d,d)=1 & f(d,e)=7 \\ f(e,a)=2 & f(e,b)=6 & f(e,c)=8 & f(e,d)=3 & f(e,e)=1 \end{bmatrix} \end{array}$$

Relative preference grades $F(x_i, x_j)$ are as follows:

$$F(b,e)=\frac{f(b,e)}{\max\{f(b,e),f(e,b)\}}=\frac{4}{\max\{4,6\}}=\frac{4}{6}=0.66$$

So, by the formula find all $F(x_i, x_j)$ and we get the table for relative preference grades:

$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array}\begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0.43 & 1 & 1 & 0.29 & 0.66 \\ 0.11 & 0.33 & 1 & 0.43 & 0.625 \\ 0.67 & 1 & 1 & 1 & 1 \\ 0.25 & 1 & 1 & 0.43 & 1 \end{bmatrix} \end{array}$$

Then overall relative preference grades of x_i with respect to all other alternative is given by

$$p(x_i)=\min\{F(x_i,x_j)\}$$
$$=minimum\ of\ each\ row$$

$i.ep(a)=1$, $p(b)=0.29$, $p(c)=0.11$, $p(d)=0.67$, $p(e)=0.25$.

So, the order of preference is (a, d, b, e, c).

11.5 MULTICRITERIA DECISION MAKING

If there are multiple criteria to determine an appropriate decision, the process is called multicriteria decision making. Each criterion induces an ordering of alternatives, and we determine the most preferable one. This type of problem is similar to multiperson decision making. One depends on persons, while the other depends on criteria. The difference is that ordering represents either preferences of different people or ratings based on different criteria.

Let, $C=\{C_1, C_2, \ldots, C_m\}$be the set of criteria used to construct ordering for a set of alternatives$X=\{x_1, x_2, \ldots, x_n\}$. It is noted that the set C must always be finite. Here we assume the set X is also finite.

Let the information regarding ordering of each alternative with respect to different criteria is given by a matrix:

$$R = \begin{array}{c} \\ C_1 \\ C_2 \\ . \\ . \\ . \\ C_m \end{array} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & \dots\dots & x_n \end{array} \\ \begin{bmatrix} r_{11} & r_{12} & r_{13} & \dots\dots & r_{1n} \\ r_{21} & r_{22} & r_{23} & \dots\dots & r_{2n} \\ r_{31} & r_{32} & r_{33} & \dots\dots & r_{3n} \\ \dots\dots. & \dots\dots\dots & \dots\dots\dots & \dots\dots. & \dots\dots.. \\ \dots\dots. & \dots\dots\dots & \dots\dots\dots & \dots\dots. & \dots\dots.. \\ r_{m1} & r_{m2} & r_{m3} & \dots\dots. & r_{mn} \end{bmatrix} \end{array}$$

where r_{ij} =degree to which criteria C_i is satisfied by an alternative x_j.

The matrix R is a representation of relation in $C \times X$.

Case-I: If the entries$r_{ij} \in [0,1]$, $i.e.$ they are fuzzy and each entry r_{ij} expresses the degree to which criterion C_i is satisfied by alternative$x_j (i \in N_m, j \in N_j)$, R may be viewed as a matrix representation of a fuzzy relation on $C \times X$.

Case-II: If r_{ij} are not in$[0,1]$, then we have to convert them in fuzzy by a suitable formula and then we get matrix R.

Let $S = \{s_{ij}\}$ is the matrix of $C \times X$ in which s_{ij} are not in $[0,1]$. then by formula we can convert s_{ij} into r_{ij} is

$$r_{ij} = \frac{s_{ij} - \min\{s_{ij}\}}{\max\{s_{ij}\} - \min\{s_{ij}\}} \qquad \text{for all } i, j \in N]$$

$$= \frac{s_{ij} - minimum\ of\ the\ row}{maximum\ of\ the\ row - minimum\ of\ the\ row}$$

Therefore Case-I is special case of Case-II.

By finding a global criterion r_j for each $x_j \in X$, we convert into single criterion decision problem

$$r_j = h(r_{1j}, r_{2j}, \dots, r_{nj})$$

This function $h(r)$ may be maximize, minimize, add and multiply. A frequently employed aggregating operator is the weighted average

$$r_j = \frac{\sum_{i=1}^{m} w_i r_{ij}}{\sum_{i=1}^{m} w_i}, \quad j \in N$$

where $w_1, w_2, \dots, w_m$ are weights that indicate the relative important of criteria $C_1, C_2, \dots., C_m$. A class of possible weighted aggregations is given by the formula

$$r_j = h\left(r_{1j}^{w_1}, r_{2j}^{w_2}, \dots., r_{mj}^{w_m}\right)$$

where h is an aggregation operator and $w_1, w_2, \dots, w_m$ are weights.

Consider now a more general situation in which the entries of matrix R are fuzzy numbers r_{ij} on R^+, and weights are specified in terms of fuzzy numbers $\overline{w_i}$ on $[0,1]$. Then using the operations of fuzzy addition and fuzzy multiplication, we can calculate the weighted average $\overline{r_j}$ by the formula,

$$\overline{r_j} = \sum_{i=1}^{m} \overline{w_i}.\overline{r_{ij}}$$

Since fuzzy numbers are not linearly ordered, a ranking method is needed to order the resulting fuzzy numbers $\overline{r_1}, \overline{r_2}, \dots, \overline{r_n}$.

Now we explain this process with the following example.

Example:

Let there are four alternatives $X = \{x_1, x_2, x_3, x_4\}$ with the help of five criteria $C = \{C_1, C_2, C_3, C_4, C_5\}$. We have to order the alternatives. Let $w_1 = 0.5, w_2 = 0.4, w_3 =$

$0.6, w_4 = 0.8$ and $w_5 = 0.2$ be the weights for the criteria $C_1, C_2, C_3 C_4$ and C_5 respectively.

The relation matrix $C \times X$ is given by

$$S = \begin{array}{c} \\ C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} 1 & 7 & 2 & 5 \\ 3 & 5 & 8 & 2 \\ 2 & 2 & 3 & 4 \\ 9 & 3 & 5 & 3 \\ 8 & 2 & 4 & 7 \end{bmatrix} \end{array}$$

Now we use formula

$$r_{ij} = \frac{s_{ij} - minimum\ of\ i^{th} row}{maximum\ of\ 3^{rd}\ row - minimum\ of\ 3^{rd} row}$$

$$= \frac{4-2}{4-2}$$

$$= 1$$

So,

$$R = \begin{array}{c} \\ C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0.17 & 0.67 \\ 0.17 & 0.5 & 1 & 0 \\ 0 & 0 & 0.5 & 1 \\ 1 & 0 & 0.34 & 0 \\ 1 & 0 & 0.34 & 0.84 \end{bmatrix} \end{array}$$

Now global criteria

$$r_j = \frac{\sum w_i r_{ij}}{\sum w_i}$$

So

$$r_1 = \frac{\sum_{i=1}^{5} w_i r_{i1}}{\sum_{i=1}^{5} w_i}$$

$$= \frac{w_1 r_{11} + w_2 r_{21} + w_3 r_{31} + w_4 r_{41} + w_5 r_{51}}{w_1 + w_2 + w_3 + w_4 + w_5}$$

$$= \frac{0.5 \times 0 + 0.17 \times 0.4 + 0.6 \times 0 + 0.8 \times 1 + 0.2 \times 1}{0.5 + 0.4 + 0.6 + 0.8 + 0.2}$$

$$= 0.4272 \qquad \text{for} \quad x_1$$

Similarly, $r_2 = 0.28$ for x_2

$r_3 = 0.45$ for x_3

$r_4 = 0.4412$ for x_4

So, the order is (x_3, x_4, x_1, x_2).

11.6 FUZZY RANKING METHOD

In many decision problems, the final scores of alternatives are represented in fuzzy number. In order to express a crisp preference of alternative, we need a method for constructing a crisp total ordering. Since $(R, min, \max)$ is not linearly independent, so fuzzy numbers are not directly comparable. To illustrate the problem of total ordering we have three methods:

1. The first method is based on hamming distance on the set R of all fuzzy numbers. We know that the formula for Hamming distance for A and B is

$$d(A,B) = \int |A(x) - B(x)| dx$$

for any given fuzzy numbers A and B, which we want to compare, we first determine their least upper bound $Max(A,B)$ in the lattice. Then we calculate the Hamming distances $d(M(A,B),A)$ and $d(Max(A,B),B)$ and define

$$A \leq B \text{ if } d(Max(A,B),A) \geq d(M(A,B),B)$$

2. Second method is based on α −cuts. For comparing fuzzy numbers A and B, select a particular value $\alpha \in [0,1]$ and determine the α −cuts ${}^{\alpha}A = [a_1, a_2]$ and ${}^{\alpha}B = [b_1, b_2]$. Then we define $A \leq B$ if $a_2 \leq b_2$.

This definition is of course, dependent on the chosen value of α. It is usually required that $\alpha > 0.5$.

3. The third method is based on the extension principle. This method can be employed for ordering several fuzzy numbers, say $A_1, A_2, \ldots, A_n$. The basic idea to construct a fuzzy set P on $\{A_1, A_2, \ldots, A_n\}$, called a priority set, such as $P(A_i)$ is the degree to which A_i is ranked as the greatest fuzzy number. Using the extension principle, P is defined for each $i \in N_n$ by the formula

$$P(A_i) = \sup_{k \in N_n} \min A_k(r_k)$$

where the supremum is taken over all vectors $(r_1, r_2, \ldots, r_n) \in R^n, r_i \geq r_j$ for all $j \in N_n$

Example:

Let A and B two fuzzy numbers whose triangular type membership function and $Max(A,B)$are given in the figure 11.3. Then $Max(A,B)$ is a fuzzy number whose membership function indicated in the figure in bold.

A is triangle CAD

Equation of CA, $y = x - 1$

Equation of AD, $y = -x + 3$

B is triangle OBE

Equation ofOB, $y = \frac{x}{3}$

Equation of BE, $y = -x + 4$

1. Hamming distance: The Hamming distance $d(M(A,B),A)$ and $d(Max(A,B),B)$ are expressed by the area in the figure that are hatched horizontally and vertically respectively.

$$d(Max(A,B),A)$$
$$= \int_1^{1.5} |A(x) - A(x)|dx + \int_{1.5}^{2} |B(x) - A(x)|dx + \int_2^{2.25} |B(x) - A(x)|dx$$
$$+ \int_{2.25}^{3} |B(x) - A(x)|dx + \int_3^4 |B(x) - A(x)|dx$$
$$= 0 + \int_{1.5}^{2} \left|\frac{x}{3} - (x-1)\right| dx + \int_2^{2.25} \left|\frac{x}{3} - (-x+3)\right| dx + \int_{2.25}^{3} \left|\frac{x}{3} - (-x+3)\right| dx$$
$$+ \int_3^4 |4 - x - 0|dx$$
$$= \frac{1}{12} + \frac{1}{4} + \frac{3}{8} + \frac{1}{2}$$
$$= 1$$

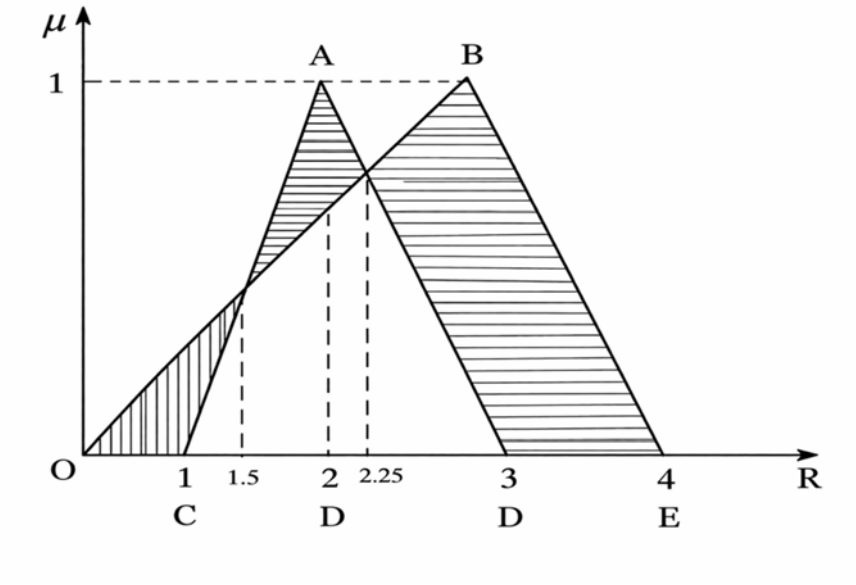


Figure-11.3 1: Ranking of fuzzy members

$$d(Max(A,B),B)$$
$$= \int_0^1 |0 - B(x)|dx + \int_1^{1.5} |B(x) - A(x)|dx + \int_{1.5}^{2} |B(x) - B(x)|dx$$
$$+ \int_2^{2.25} |B(x) - B(x)|dx + \int_{2.25}^{3} |B(x) - B(x)|dx + \int_3^4 |B(x) - B(x)|dx$$
$$= \int_0^1 \frac{x}{3} dx + \int_1^{1.5} \left|x - 1 - \frac{x}{3}\right| dx$$
$$= 0.25$$

Since $d(Max(A,B),A) > d(Max(A,B),B)$, therefore we may conclude that, according to the first ranking method,

$$A \leq B$$

2. $\alpha - cut$ method:

Here $A = [x-1, -x+3]$ and $B = \left[\frac{x}{3}, 4-x\right]$

$\therefore .^{\alpha}A = [\alpha + 1, 3 - \alpha]$ and $.^{\alpha}B = [3\alpha, 4 - \alpha]$

We see that $3-\alpha < 4-\alpha$ for all $\alpha \in [0,1]$

$\Rightarrow A < B$

3. Extension principle method:

We construct the priority fuzzy set P on $\{A, B\}$ as follows:

$$P(A) = \sup_{r_1 \geq r_2} \min[A(r_1), B(r_2)]$$

$$P(B) = \sup_{r_2 \geq r_1} \min[A(r_1), B(r_2)]$$

We see that from 1.5 to 2.25, figure A is upper that B and from 0 to 1.5 and from 2.25 to 4, B is upper than A. Therefore

$$P(A) = 0.75 \text{ and } P(B) = 1$$

$$\Rightarrow P(B) > P(A)$$

$$\Rightarrow B > A$$

Hence, we see that in all three methods, the result is $B > A$.

Hence for ranking A and B we can choose any one of the above methods.

11.7 LET US SUM UP

- **Individual Decision Making:** If D be the set of fuzzy decisions represented by its membership function $\mu_{D(x)} = min\{\mu_{G(x)}, \mu_{C(x)}\}$, $x \in A$, where A is the set of alternatives, G is the set of fuzzy goals, and C is the set of constraints then optimal decision is expressed by

$$x_{opt} = \{ x \mid \underbrace{max}_{x} \mu_D(x) \} = \underbrace{max}_{x} \underbrace{min}_{\mu} \{\mu_G(x), \mu_C(x)\}$$

- **Multiperson Decision making:** In this method, we have decision makers and their choice of alternatives. Let $X = \{x_1, x_2, \ldots, x_n\}$ be the set of alternatives. Then define social preference function

$S : X \times X \rightarrow [0,1]$ by

$$S(x_i, x_j) = \frac{N(x_i, x_j)}{n}$$

where $N(x_i, x_j)$ is the number of persons preferring x_i to x_j and n is the total number of decision makers.

Aggregating $S(x_i, x_j)$ by a suitable function $S = U_\alpha.^\alpha S$, $\alpha \in [0,1]$ we get the required result.

In the second method, $f(x_i, x_j)$ denotes the attractiveness grade given by the individual to x_i with respect to x_j . When they are converted to relative preference grades $F(x_i, x_j)$ by using the following formula:

$$F(x_i, x_j) = min\{1, \frac{f(x_i, x_j)}{f(x_j, x_i)}\}$$

where $F(x_i, x_j) \in [0,1]$ for all pairs $(x_i, x_j) \in X^2$.
Then we find the overall relative preference grades $p(x_i)$ of x_i with respect to all other alternatives in X given by:

$$p(x_i) = min\, F(x_i, x_j), \qquad for\ all\ x_j \in X$$

The preference ordering of alternatives in X is induced by the numerical ordering of these grades $p(x_i)$.

- **Multicriteria Decision making:** Let $X = \{x_1, x_2, \ldots, x_m\}$ and $C = \{c_1, c_2, \ldots, c_m\}$ be a set of alternatives and a set of criteria characterizing a decision situation respectively. Then the information in multicriteria decision making can be expressed as a matrix S

$$S = \begin{array}{c} \\ C_1 \\ C_2 \\ . \\ . \\ . \\ C_m \end{array} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & \ldots\ldots & x_n \end{array} \\ \begin{bmatrix} s_{11} & s_{12} & s_{13} & \ldots\ldots & s_{1n} \\ s_{21} & s_{22} & s_{23} & \ldots\ldots & s_{2n} \\ s_{31} & s_{32} & s_{33} & \ldots\ldots & s_{3n} \\ \ldots\ldots. & \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots. & \ldots\ldots.. \\ \ldots\ldots. & \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots. & \ldots\ldots.. \\ s_{m1} & s_{m2} & s_{m3} & \ldots\ldots. & s_{mn} \end{bmatrix} \end{array}$$

If all $s_{ij} \notin [0,1]$, then we use the formula:

$$r_{ij} = \frac{s_{ij} - \min of\ the\ i^{th}\ row}{maximum\ of\ i^{th}\ row - minimu\ of\ i^{th}\ row}$$

to convert s_{ij} into r_{ij} such that $r_{ij} \in [0,1]$.

Then find global criteria r_j for each $x_j \in X$ by the formula:

$$r_j = \frac{\sum_{i=1}^{m} w_i r_{ij}}{\sum_{i=1}^{m} w_i}$$

where $w_1, w_2, \ldots, w_m$ are weights that indicate the relative importance of criteria $c_1, c_2, \ldots, c_m$.

- **Fuzzy ranking Method:** Sometimes, in fuzzy decision-making problem, the final sores of alternatives are represented in terms of fuzzy numbers. Then we need a method foe constructing a crip total ordering from fuzzy numbers. To ordering fuzzy numbers we use three methods:
    1. Method using Hamming distance
    2. α –cut method
    3. Extension Principle method

11.8 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q.1.

The set of alternatives is $\{C_1, C_2, C_3\}$, the goals G of a high salary is given by the membership function

$$\mu_G(x) = \begin{cases} 0, & for \quad 0 < x \leq 25000 \\ \dfrac{x - 25000}{20000}, & for \quad 25000 < x < 45000 \\ 1, & for \quad x \geq 45000 \end{cases}$$

On the universal set R_+ and constraints (i) interesting job and job within close driving distance are presented by the fuzzy sets

$$C_1 = \{(c_1, 0.5), (c_2, 0.7), (3, 0.8)\}$$

$$C_2 = \{(c_1, 0.3), (c_2, 0.8), (c_3, 1)\}$$

Now $\mu_G(40000) = 0.75$, $\mu_G(35000) = 0.5$ and $\mu_G(30000) = 0.25$

Hence the fuzzy set goal G on the universe R_+ is now substituted by the fuzzy set goal G_a on the set of the alternatives.

$$G_a = \{(c_1, 0.75), (c_2, 0.5), (c_3, 0.25)\}$$

$$\therefore \quad D = G_a \cap C_1 \cap C_2$$

$$= \{(c_1, 0.3), (c_2, 0.5), (c_3, 0.25)\}$$

The maximum membership value is 0.5, hence John has to take the job wit6h company C_2.

11.9 FURTHER READING

1. George J. Klir and Bo Yuan, Fuzzy Sets and Fuzzy Logic: Theory and Applications, Prentice Hall PTR, Upper Saddle River, New Jersey 07458 (1995).
2. Sudhir K. Pundir and Rimple Pundir, Fuzzy Sets and Their Applications, Pragati Prakashan (Meerut), 2008.
3. Zimmerman, H.J., Fuzzy Set Theory and its Applications, Allied Publishers Ltd., New Delhi, 1991.

11.10 MODEL QUESTIONS

Q 1. Let the set of alternatives be R_+ and the goal G and the constraint C are expressed by their membership functions

$$\mu_G(x) = \begin{cases} 0, & for \quad 0 \leq x < 10 \\ 1 + \{(x - 10)^{-2}\}^{-1}, & for \quad x \geq 10 \end{cases}$$

and

$$\mu_C(x) = \{1 + (x-15)^4\}^{-1} \qquad for \quad x \in R_+$$

(a) Sketch $\mu_G(x)$ and $\mu_C(x)$ and the decision set D

(b) Find geometrically x_{opt}.

***Q* 2**. Assume that Noor must choose one of four jobs offered to her by four different companies C_1, C_2, C_3 and C_4. The salaries are presented by the table:

Company	C_1	C_2	C_3	C_4
Salary	95000	91000	85000	83000

The constraints C_1: interesting job and C_2: near home are represented by the fuzzy set given by the table:

Company	C_1	C_2	C_3	C_4
$.^{\mu}\, C_1$	0.7	0.8	0.3	0.5
$.^{\mu}\, C_2$	0.6	0.7	0.5	0.4

and the goal of a high salary is

$$\mu_G(x) = \begin{cases} 0, & for \quad 0 < x \leq 80000 \\ \dfrac{x-80000}{20000}, & for \quad 80000 < x < 100000 \\ 1, & for \quad x \geq 100000 \end{cases}$$

on the universe R_+. Which company satisfies best the goal and constraints?

***Q* 3**. Consider four travel packages P_1, P_2, P_3 and P_4 from which we want to choose one. Their costs are 1,00,000; 80,000; 1,20,000 and 1,50,000 respectively. Their travel times in days are $8, 4, 5$ and 7 respectively. Assume that they are viewed as interesting with the degrees $0.5, 0.4, 0.3$ and 0.9 respectively. Define fuzzy set of acceptable cost as

$$\mu_{C_1}(x) = \begin{cases} 1, & for \quad 0 < x \leq 75000 \\ \dfrac{x-75000}{75000}, & for \quad 75000 < x < 150000 \\ 0, & for \quad x \geq 150000 \end{cases}$$

And define fuzzy set of acceptable travel times as

$$\mu_{C_2}(x) = \begin{cases} 0, & for \quad 0 < x \leq 3 \\ \dfrac{x-3}{5}, & for \quad 3 < x < 8 \\ 1, & for \quad x \geq 8 \end{cases}$$

Then determine the fuzzy set of interesting travel packages whose costs and travel times are acceptable and use this set to choose one of the four travel packages.

$\boldsymbol{Q}$ **4**. Assume that each individual of a group of five judges has a total preference ordering $P_i\ (i \in N_5)$ on four figure skaters a, b, c, d. The orderings are: $P_1 = (a, b, d, c)$, $P_2 = (a, c, d, b), P_3 = (b, a, c, d), P_4 = (a, d, b, c)$. Use fuzzy multiperson decision making to determine the group decision.

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UNIT 12: FUZZY LINEAR PROGRAMMING

UNIT STRUCTURE

12.1. Learning Objectives

12.2. Introduction

12.3. General (Classical) Linear Programming Problem

12.4. Fuzzy Linear Programming problem

12.5. Special Cases of Fuzzy Linear Programming

12.5.1. Symmetric Fuzzy Linear Programming

12.5.2. Fuzzy Linear Programming with Triangular fuzzy numbers

12.6. Let Us Sum Up

12.7. Answer to Check Your Progress

12.8. Further Reading

12.9. Model Questions

12.1 LEARNING OBJECTIVES

After reading this unit reader will be able to

- understand the limitations of classical linear programming when applied to real-world problems with vague or imprecise information.
- gain knowledge of fuzzy set theory, particularly membership functions and fuzzy numbers, and how these concepts integrate with optimization models.
- evaluate the strengths and limitations of fuzzy linear programming and compare it with traditional optimization methods.

12.2 INTRODUCTION

Fuzzy linear programming (FLP) is an extension of classical linear programming that addresses decision-making problems under uncertainty and imprecision. In many real-world scenarios such as resource allocation, production planning and transportation, exact values of coefficients, objectives or constraints may not be available. This uncertainties often arise due to human judgement, incomplete data, or fluctuating environments. Traditional linear programming assumes precise and well-defined data, which limits its effectiveness in such situations.

FLP incorporates the principles of fuzzy set theory, introduced by Lotfi A. Zadeh, to model this vagueness using fuzzy numbers and membership functions. Instead of using fixed numerical values, FLP allows parameters to expressed in linguistic terms (e.g. “approximately 100” or “high demand”) and modeled mathematically through fuzzy sets.

In FLP, the goal is to find an optimal or satisfactory solution that balances various fuzzy constraints and objectives. This is often achieved by transforming fuzzy

problems into equivalent deterministic (crisp) forms using techniques like α-cuts, ranking indices or possibility distributions.

By integrating fuzzy logic with optimization, FLP provides a flexible and realistic framework for solving complex problems where uncertainty is a critical factor, making it valuable in engineering, economics, logistics and management science.

12.3 GENERAL (CLASSICAL) LINEAR PROGRAMMING

A general linear programming problem can be stated as follows:
Find $x_1, x_2, \ldots, x$ which optimize the linear function

$$z = c_1x_1 + c_2x_2 + \ldots\ldots + c\ x \quad --(1)$$

subject to the constraints

$$a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_1\ x \ \ (\leq, =, \geq)\ b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_2\ x \ \ (\leq, =, \geq)\ b_2$$
$$\vdots$$
$$a\ _1x_1 + a\ _2x_2 + \ldots\ldots + a \quad x \ \ (\leq, =, \geq)\ b \quad --(2)$$

and non-negative restrictions $x_i \geq 0$ for all $i = 1,2, \ldots\ldots, n.$
where all $a_i\ 's, b_i's$ and $c_i's$ are constants and $x_i's$ are variables. In the conditions given by (2) there may be any of the three signs $\leq, =, \geq$.
The function z given by (1) is called the objective function and the conditions given by (2) are termed as the constraints of the linear programming problem.

The above LPP programming problem may also be stated in matrix form as follows:

Optimize $z = Cx$

subject to

$$Ax\ (\leq, =, \geq)\ b$$

and

$$x \geq 0.$$

where

$$A = [a_{ij}]$$

is the matrix of coefficients of order $m \times n$.

$$C = (c_1, c_2, \ldots, c_n)$$

is a row vector known as price vector.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

is a column vector of variables.
and

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

is a column vector called the requirement vector.

12.3.1 Solution of a linear programming problem

In general, we use the following two methods for solution of a linear programming problems:

1. Graphical method.
2. Simplex method.

12.3.2 Computational procedure of the simplex method for the solution of a maximization L.P.P.:

Step 1. If the problem is of minimization, then convert it into the maximization problem by letting

$$-z = z' = -c_1x_1 - c_2x_2 - \cdots - c_nx_n$$

Step 2. Make all the b_i's positive

Step 3. Convert the constraints into equations by introducing the non-negative slack or surplus variables. Also introduce artificial variables for the constraints where surplus variables are inserted and which do not form the column of unit matrix.

Step 4. Find initial basic feasible solution. If in a L.P.P., artificial variables are also introduced, then we follow Big M-method.

Step 5. Construct starting simplex table as follows:

B.V.	C_B	C_j	c_1	c_2	⋯	c_n	0	0	⋯	0	Minimum Ratio
		X_B	x_1	x_2	⋯	x_n	s_1	s_2	⋯	s_m	
s_1	0	b_1	a_{11}	a_{12}	⋯	a_{1n}	1	0	⋯	0	
s_2	0	b_2	a_{21}	a_{22}	⋯	a_{2n}	0	1	⋯	0	
⋮	⋮	⋮	⋮	⋮	⋯	⋮	⋮	⋮	⋯	⋮	
s_m	0	b_m	a_{m1}	a_{m2}	⋯	a_{mn}	0	0	⋯	1	

(where $s_1, s_2, \ldots, s_m$ are slack variables.)

Step 6. Test of starting basic feasible solution for optimality: find

$$\Delta_j = C_B x_j - C_j$$

If $\Delta_j \geq 0$ for each j, the solution under test is optimal. If one or more $\Delta_j$are negative, the solution under test is not optimal.

Step 7. Find in-coming and out-going vectors: the incoming vector will be taken x_k if

$$\Delta_k = \min \Delta_j$$

The out-going vector s_p is taken corresponding to the value of $r$such that

$$\frac{x_{B_r}}{a_{rk}} = \min\left\{\frac{x_{B_i}}{a_{ik}} \mid a_{ik} > 0\right\}$$

when x_k is the in-coming vector

Step 8. When x_k is the incoming vector and s_p the out-going vector, then a_{pk} is called key element. We mark this element in [ ].

In order to bring s_r in place x_k there should be unity at the position [ ]. If it is not 1, then divide all the element of this row by this key element a_{rk}.

Thus subtract appropriate multipliers of this row (containing key element) from all the other row and obtain zero at all other positions of this column x_k. In this way we construct a new simplex table and get a improved basic feasible solution.

Step 9. We again test the improved B.F.S. for optimality as in step 6. If this solution is not optimal then we repeat step 7 and 8 until an optimal solution is finally obtained.

Slack variable : If a constraint has a sign ≤, then in order to make it an equality we have to add some thing positive to the left-hand side which is called slack variable.

Surplus variable : The positive variables which are subtracted from the left-hand sides of the constraints have ≥sign to convert them into equalities are called the surplus variables.

Example 1: Solve the linear programming problem

Maximize $Z = 3x_1 + 5x_2 + 4x_3$

subject to

$$2x_1 + 3x_2 \leq 8$$
$$2x_2 + 5x_3 \leq 10$$
$$3x_1 + 2x_2 + 4x_3 \leq 15$$

and $x_1, x_2, x_3 \geq 0$.

Solution: The problem is a problem of maximization and all the b_i's are already positive. Since all the constraints have '≤' sign therefore we convert into equalities by introducing slack variables s_1, s_2, s_3as follows:

$$2x_1 + 3x_2 + 0x_3 + s_1 = 8$$
$$0x_1 + 2x_2 + 5x_3 + s_2 = 10$$
$$3x_1 + 2x_2 + 4x_3 + s_3 = 15$$

Taking $x_1 = x_2 = x_3 = 0$, we get $s_1 = 8, s_2 = 10, s_3 = 15$

which is the starting B.F.S.

Now we construct the following simplex table :

B	C_B	C_j	3	5	4	0	0	0	
		X_B	x_1	x_2	x_3	s_1	s_2	s_3	Ratio
s_1	0	8	2	[3]	0	1	0	0	8/3 (min)→

s_2	0	10	0	2	5	0	1	0	10/2
s_3	0	15	3	2	4	0	0	1	15/2
$Z = C_{BX_B} = 0$		$\Delta j = C_{BX_j} - C_j$	-3	-5 ↑	-4	0 ↓	0	0	$\frac{X_B}{X_3}$
x_2	5	8/3	2/3	1	0	1/3	0	0	--
s_2	0	14/3	-4/3	0	[5]	-2/3	1	0	14/15(min)→
s_3	0	29/3	5/3	0	4	-2/3	0	1	29/15
Z=40/3		Δj	1/3	0	-4 ↑	5/3	0 ↓	0	$\frac{X_B}{X_1}$
x_2	5	8/3	2/3	1	0	1/3	0	0	4
x_3	4	14/15	-4/15	0	1	-2/15	1/5	0	--
s_3	0	89/15	[41/15]	0	0	-2/15	-4/5	1	89/41(min)→
Z=256/45		Δ_j	-11/15 ↑	0	0	17/15	4/5	0 ↓	
x_2	5	50/41	0	1	0	15/41	8/41	-10/41	
x_3	4	62/41	0	0	1	-6/41	5/41	4/41	
x_1	3	89/41	1	0	0	-2/41	-12/41	15/41	
$Z = 5 \times \frac{50}{41} + 4 \times \frac{62}{41} + 3 \times \frac{89}{41}$		Δ_j	0	0	0	45/41	24/41	11/41	

Since all $\Delta_j \geq 0$, so the solution

$$x_1 = \frac{89}{41}, x_2 = \frac{50}{41}, x_3 = \frac{62}{41}$$

is optimal and

$$\text{Max } Z = 5 \times \frac{50}{41} + 4 \times \frac{62}{41} + 3 \times \frac{89}{41} = \frac{765}{41}.$$

12.3.3 Big M–method

The Big M–method is a mathematical technique used in linear programming to solve optimization problems that include artificial variables. It is typically used when converting a problem into standard form for the simplex method, but where some constraints do not easily lend themselves to an initial feasible solution (for example, ≥ or = constraints). We apply Big M–method to find an initial basic feasible solution when one isn't readily apparent, especially in cases involving artificial variables.

How it Works

1. Convert constraints into equalities using slack, surplus and artificial variables.
   - For "≥" constraints, subtract surplus variable and add an artificial variable.
   - For "=" constraints, directly add an artificial variable.
2. Add a large penalty M (a very large positive number) to the objective function for each artificial variable:
   - If maximizing : Subtract $M \times$(artificial variable)
   - If minimizing : Add $M \times$(artificial variable)
3. Solve using the Simplex method, treating M as a very large constant. The goal is to drive artificial variables to zero, thereby finding a feasible solution to the original problem.

Example 2: Solve the following L.P. problem

Max $Z = x_1 + 2x_2 + 3x_3 - x_4$

subject to

$$x_1 + 2x_2 + 3x_3 = 15$$
$$2x_1 + x_2 + 5x_3 = 20$$
$$x_1 + 2x_2 + x_3 + x_4 = 10$$

and $x_1, x_2, x_3, x_4 \geq 0$.

Solution: We have three constraints and in order to get an identity matrix of order 3×3we need two more unit vectors (columns of the unit matrix) as one unit vector (column of identity matrix, formed by coefficients of x_4) is already present in the constraints. Thus we need only two artificial variables in the first two constraints. Introducing the artificial variables $a_1$and $a_2$in the first two constraints and assigning large negative price $-M$to the artificial variables in the objective functions.

Max $Z = x_1 + 2x_2 + 3x_3 - x_4 - Ma_1 - Ma_2$

such that

$$x_1 + 2x_2 + 3x_3 + 0x_4 + a_1 = 15$$
$$2x_1 + x_2 + 5x_3 + 0x_4 + a_2 = 20$$
$$x_1 + 2x_2 + x_3 + 1x_4 = 10$$

and $x_1, x_2, x_3, x_4, a_1, a_2 \geq 0$.

Taking $x_1 = x_2 = x_3 = 0$, we get $x_4 = 10, a_1 = 15, a_2 = 20$,

which is the starting B.F.S. All computation work is done in the following table —

		C_j	1	2	3	-1	-M	-M	Minimum Ratio (X_B/x_3)
B	C_B	X_B	x_1	x_2	x_3	x_4	a_1	a_2	
a_1	-M	15	1	2	3	0	1	0	5
a_2	-M	20	2	1	[5]	0	0	1	4 (min)→
x_4	-1	10	1	2	1	1	0	0	10
Z = -35M - 10		$\Delta_j = C_b X_j - C_j$	-3M-2	-3M-4	-8M-4 ↑	0	0	0 ↓	X_B/x_2
a_1	-M	3	-1/5	[7/5]	0	0	1		15/7(min)→
x_3	3	4	2/5	1/5	1	0	0		4
x_4	-1	6	3/5	9/5	0	1	0		10/3
Z = -3M + 6		Δ_j	(M-2)/5	(-7M+16)/5 ↑	0	0	0 ↓		X_B/x_1
x_2	2	15/7	-1/7	1	0	0			25/3
x_3	3	25/7	3/7	0	1	0			
x_4	-1	15/7	[6/7]	0	0	1			5/2 (min)→
Z = 90/7		Δ_j	-6/7 ↑	0	0	0 ↓			
x_2	2	5/2	0	1	0				
x_3	3	5/2	0	0	1				
x_1	1	5/2	1	0	0				
Z = 15		Δj	0	0	0				

Here no $\Delta_j < 0$, so this solution is optimal. the optimal solution is

$$x_1 = \frac{5}{2}, x_2 = \frac{5}{2}, x_3 = \frac{5}{2}, x_4 = 0$$

and Max $Z = 15$.

12.4 Fuzzy Linear programming Problem

The most general fuzzy linear programming problem is formulated as follows:

$$\max Z = \sum_{j=1}^{n} C_j X_j$$

subject to

$$\sum_{j=1}^{m} A_{ij} x_j \leq B_i \quad (i \in N_m)$$

$$X_j \geq 0 \quad (j \in N_n)$$

where A_{ij}, B_i, C_j are fuzzy numbers and x_j are variables whose states are fuzzy numbers $(i \in N_m, j \in N_n)$; the operations addition and multiplication are operations of fuzzy arithmetic and $\leq$ denotes the ordering of fuzzy numbers.

12.5 SPECIAL CASES OF FUZZY LINEAR PROGRAMMING

We consider following two special cases of fuzzy linear programming problems:

Case (I): Fuzzy linear programming problems in which only the right-hand side numbers $B_i$are fuzzy numbers:

$$\max Z = \sum_{j=1}^{n} C_j x_j$$

subject to

$$\sum_{j=1}^{m} a_{ij} x_j \leq B_i \quad (i \in N_m)$$

$$x_j \geq 0 \quad (j \in N_n)$$

Case (II): Fuzzy linear programming problems in which the right-hand side numbers $B_i$and the coefficients $A_{ij}$of the constraints matrix are fuzzy triangular numbers.

$$\max Z = \sum_{j=1}^{n} C_j x_j$$

subject to

$$\sum_{j=1}^{m} A_{ij} x_j \leq B_i \quad (i \in N_m)$$

$$x_j \geq 0 \; (j \in N_n)$$

12.5.1 Symmetric Fuzzy Linear Programming

Let us discuss now fuzzy linear programming problem of type (1). In this case, fuzzy numbers $B_i(\, i \in N_m\,)$ typically have the form

$$B_i(x) = \begin{cases} 1 \text{when } x \leq b_i \\ \dfrac{b_i + p_i - x}{p_i} \text{when } b_i < x < b_i + p_i \\ 0 \text{ when } x \geq b_i + p_i \end{cases}$$

where $x \in \mathbb{R}$. For each vector $\bar{x} = (x_1, x_2, \ldots, x_n)$, we first calculate the degree $D_i(\bar{x})$, to which $\bar{x}$satisfies the i^{th}constraint $(\, i \in N_m\,)$by the formula

$$D_i(\bar{x}) = B_i\left(\sum_{j=1}^{m} a_{ij} x_j\right)$$

These degrees are fuzzy sets on $\mathbb{R}^n$, and their intersection, $\bigcap_{i=1}^{m} D_i$ is a fuzzy feasible set.

Next, we determine the fuzzy set of optimal values. This is done by calculating the lower and upper bounds of the optimal values first. The lower bound of the optimal values $z_l$is obtained by solving the standard linear programming problem

$$\max z = \bar{c}\bar{x}$$

subject to

$$\sum_{j=1}^{m} a_{ij}\, x_j \leq b_i (i \in N_m)$$
$$x_j \geq 0 \qquad (j \in N_n)$$

The upper bound of the optimal values, z_u, is obtained by solving a similar linear programming problem in which each $b_i$is replaced with $b_i + p_i$:

$$\max z = \bar{c}\bar{x}$$

subject to

$$\sum_{j=1}^{m} a_{ij}\, x_j \leq b_i + p_i \qquad (i \in N_m)$$
$$x_j \geq 0 (j \in N_n)$$

Then the fuzzy of optimal values, G, which is fuzzy subset of $\mathbb{R}$, is defined by

$$G(x) = \begin{cases} 1 \text{when } z_u \leq \bar{c}\bar{x} \\ \dfrac{\bar{c}\bar{x} - z_l}{z_u - z_l} \text{when } z_l \leq \bar{c}\bar{x} \leq z_u \\ 0 \text{ when } \bar{c}\bar{x} \leq z_l \end{cases}$$

Now, the problem (1) becomes the following classical optimization problem

$$\max \lambda$$

subject to

$$\lambda(z_u - z_l) - \bar{c}\bar{x} \leq -z_l$$
$$\lambda p_i + \sum_{j=1}^{m} a_{ij}\, x_j \leq b_i + p_i (i \in N_m)$$
$$\lambda, x_j \geq 0 (j \in N_n)$$

Example: Solve the following fuzzy linear programming problem

Max $z = 0.5x_1 + 0.2x_2$

subject to

$$x_1 + x_2 \leq B_1$$
$$2x_1 + x_2 \leq B_2$$
$$x_1, x_2 \geq 0$$

where

$$B_1(x) = \begin{cases} 1 & \text{when } x \leq 300 \\ \dfrac{400 - x}{100} & \text{when } 300 < x < 400 \\ 0 & \text{when } x \geq 400 \end{cases}$$

and

$$B_2(x) = \begin{cases} 1 & \text{when } x \leq 400 \\ \dfrac{500 - x}{100} & \text{when } 400 < x < 500 \\ 0 & \text{when } x \geq 500 \end{cases}$$

Solution: First we find the lower bound $z_l$of the objective function by solving the classical L.P.P. corresponding to the F.L.P.P.

$$\max z_l = 0.5x_1 + 0.2x_2$$

subject to

$$x_1 + x_2 \leq 300$$
$$2x_1 + x_2 \leq 400$$
$$x_1, x_2 \geq 0$$

Introducing slack variables $s_1$and s_2, we can write

$$\max z_l = 0.5x_1 + 0.2x_2 + 0.s_1 + 0.s_2$$

subject to

$$x_1 + x_2 + s_1 = 300$$
$$2x_1 + x_2 + s_2 = 400$$
$$x_1, x_2, s_1, s_2 \geq 0$$

Now we construct the first simplex table:

		C_j	0.5	0.2	0	0	Minimum Ratio (X_B/x_j)
B	C_B	X_B	x_1	x_2	s_1	s_2	
s_1	0	300	1	1	1	0	300
s_2	0	400	[2]	1	0	1	200 (min)→
$Z_l = 0$		Δ_j	-0.5	-0.2	0	0	

			↑			↓	
s_1	0	100	0	1/2	1	-1/2	
x_1	0.5	200	1	1/2	0	1/2	
$Z_l = 100$		Δ_j	0	0.05	0	0.25	

All $\Delta_j \geq 0$, so the optimal solution is

$$x_1 = 200,\ x_2 = 0 \text{and} z_l = 100$$

Next, we find upper bound z_u of objective function by solving the following L.P.P

$$\max z_u = 0.5x_1 + 0.2x_2$$

subject to

$$x_1 + x_2 \leq 400$$
$$2x_1 + x_2 \leq 500$$
$$x_1, x_2 \geq 0$$

Introducing slack variables s_1' and s_2', we can write

$$\max z_u = 0.5x_1 + 0.2x_2 + 0.s_1' + 0.s_2'$$

such that

$$x_1 + x_2 + s_1' = 400$$
$$2x_1 + x_2 + s_2' = 500$$
$$x_1, x_2, s_1', s_2' \geq 0$$

Now we construct the second simplex table:

		C_j	0.5	0.2	0	0		Minimum
B	C_B	X_B	x_1	x_2	s_1'	s_2'		Ratio (X_B/x_j)
s_1'	0	400	1	1	1	0		400
s_2'	0	500	[2]	1	0	1		250 (min)→
Zu=0		Δ_j	-0.5 ↑	-0.2	0	0 ↓		
s_1'	0	150	0	1/2	1	-1/2		
x_1	0.5	250	1	1/2	0	1/2		
Zu=125		Δ_j	0	0.05	0	0.25		

All $\Delta_j \geq 0$, so the optimal solution is

$$x_1 = 250,\ x_2 = 0 \text{ and } z_u = 125$$

Thus the fuzzy L.P.P becomes

$$\max z = \lambda$$

subject to constraints

$$(125 - 100)\lambda - 0.5x_1 - 0.2x_2 \leq -100$$
$$\Rightarrow -25\lambda + 0.5x_1 + 0.2x_2 \geq 100$$

and

$$100\lambda + x_1 + x_2 \leq 400$$
$$100\lambda + 2x_1 + x_2 \leq 500$$
$$\lambda, x_1, x_2 \geq 0$$

We solve this L.P.P by Big M–method.

The first constraint is a "≥" constraint, so we introduce a surplus variable S and artificial variable A. For second and third constraints we introduce slack variables S_1 and S_2.

Thus the L.P.P becomes

$$\max z = 1\lambda + 0x_1 + 0x_2 + 0S - MA + 0S_1 + 0S_2$$

such that

$$-25\lambda + 0.5x_1 + 0.2x_2 - S + A = 100$$
$$100\lambda + x_1 + x_2 + S_1 = 400$$
$$100\lambda + 2x_1 + x_2 + S_2 = 500$$

and

$$\lambda, x_1, x_2, S, A, S_1, S_2 \geq 0$$

Now, we construct the 3rd simplex table:

		C_j	1	0	0	0	-M	0	0	
B	C_B	X_B	λ	x_1	x_2	S	A	S_1	S_2	Ratio
A	-M	100	-25	[0.5]	0.2	-1	1	0	0	200→
S_1	0	400	100	1	1	0	0	1	0	400
S_2	0	500	100	2	1	0	0	0	1	250
λ=-100M		Δ_j	25M-1	-M/2 ↑	-M/5	M	0 ↓	0	0	
x_1	0	200	-50	1	0.4	-2		0	0	
S_1	0	200	150	0	0.6	2		1	0	4/3

S_2	0	100	[200]	0	0.2	4		0	1	1/2→
λ=0		Δ_j	-1 ↑	0	0	0		0	0 ↓	
x_1	0	225	0	1	0.45	-1		0		
S_1	0	125	0	0	0.45	-1		1		
λ	1	1/2	1	0	0.001	0.02		0		
λ=0.5		Δ_j	0	0	0.001	0.02	0	0	0	

Since all Δ_j are non-negative, so obtained solution is optimal. Hence, the solution of fuzzy L.P.P is given by

$$x_1 = 225,\ x_2 = 0,\ \lambda = 0.5$$

and maximum profit is given by

$$\max z = 0.5x_1 + 0.2x_2$$
$$= 0.5 \times 225$$
$$= 112.5$$

CHECK YOUR PROGRESS

Q.1. Assume that a company makes two types of products. A Product P_1 has a Rs 0.40 per unit profit and product P_2 has a Rs 0.30 per unit profit. Each unit of product P_1 requires twice as many labours hours as product P_2. Total available labours hours are at least 500 hours per day and may possible to extended to 600 hours per day due to special arrangements for overtime work. The supply of material is atleast sufficient for 400 units of both P_1 and P_2 products per day but may possible be extended to 500 units per day according to previous experience. Find how many units of product P_1 and P_2 should be made per day to maximize the total profit.

12.5.2 Linear Programming with Triangular Fuzzy Numbers

Let us consider now the more general problem of fuzzy linear programming defined by (2). In this case, we assume that all fuzzy numbers are triangular. Any triangular fuzzy A can be represented by three real numbers s, l, r whose meanings are defined in fig. 12.1. Using this representation, we write $A = \langle s, l, r \rangle$.

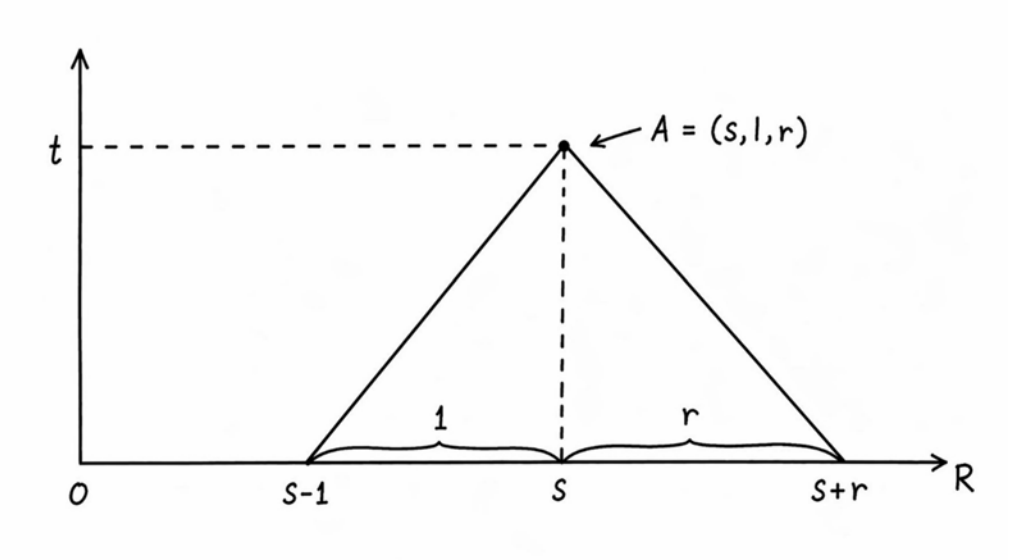


Fig 12.1: Triangular fuzzy numbers employed in (2).

Problem (2) can be rewritten as

$$\max z = \sum_{j=1}^{m} C_j x_j$$

subject to constraints

$$\sum_{j=1}^{m} (a_{ij}, l_{ij}, r_{ij}) x_j \leq (t_i, u_i, v_i), (i \in N_m)$$

and

$$x_j \geq 0 \qquad (j \in N_n)$$

where $A_{ij} = (s_{ij}, l_{ij}, r_{ij})$ and $B_i = (t_i, u_i, v_i)$ are fuzzy numbers. We consider the operations summation and multiplication on fuzzy numbers and the partial order $\leq$ is defined by

$$A \leq B \iff \max(A, B)$$

It is easy to prove that for any two triangular fuzzy numbers $A = (s_1, l_1, r_1)$ and $B = (s_2, l_2, r_2)$,

$$A \leq B \iff s_1 \leq s_2,\ s_1 - l_1 \leq s_2 - l_2,\ s_1 + r_1 \leq s_2 + r_2$$

Moreover

$$(s_1, l_1, r_1) + (s_2, l_2, r_2) = (s_1 + s_2,\ l_1 + l_2,\ r_1 + r_2)$$

and

$$(s_1, l_1, r_1)x = (s_1x,\ l_1x,\ r_1x)$$

for any non-negative real numbers x. Then the above problem can be rewritten as

$$\max z = \sum_{j=1}^{n} C_j\, x_j$$

subject to constraints

$$\sum_{j=1}^{n} s_{ij}\, x_j \le t_i$$
$$\sum_{j=1}^{n} (s_{ij} - l_{ij})x_j \le t_i - u_i$$
$$\sum_{j=1}^{n} (s_{ij} + r_{ij})x_j \le t_i + v_i (i \in N_m)$$

and

$$x_j \ge 0 (j \in N_n)$$

However, since all numbers involved are real numbers, this is a classical L.P.P.

Example 1 Solve the following fuzzy linear programming problem:

Max $z = 5x_1 + 4x_2$

subject to constraints

$$(4,2,1)x_1 + (5,3,1)x_2 \le (24,5,8)$$
$$(4,1,2)x_1 + (1,5,1)x_2 \le (12,6,3)$$
$$x_1, x_2 \ge 0$$

Solution: We have $\max z = 5x_1 + 4x_2$

subject to

$$\sum_{j=1}^{2} s_{ij}\, x_j \le t_i \qquad (1)$$
$$\sum_{j=1}^{2} (s_{ij} - l_{ij})x_j \le t_i - u_i \qquad (2)$$
$$\sum_{j=1}^{2} (s_{ij} + r_{ij})x_j \le t_i + v_i \qquad (3)$$

From (1),

$$s_{i1}x_1 + s_{i2}x_2 \le t_i$$

For $i = 1$,

$$s_{11}x_1 + s_{12}x_2 \leq t_1$$

Here $s_{11} = 4,\ l_{11} = 2,\ r_{11} = 1;\ s_{12} = 5,\ l_{12} = 3,\ r_{12} = 1$

$$s_{21} = 4, l_{21} = 1, r_{21} = 2; s_{22} = 1, l_{22} = 0.5, r_{22} = 1$$

Therefore, we get

$$4x_1 + 5x_2 \leq 24$$

For $i = 2$,

$$s_{21}x_1 + s_{22}x_2 \leq t_2$$
$$\Rightarrow 4x_1 + 1x_2 \leq 12$$

From (2), $(s_{i1} - l_{i1})x_1 + (s_{i2} - l_{i2})x_2 \leq t_i - u_i$

For $i = 1$,

$$(s_{11} - l_{11})x_1 + (s_{12} - l_{12})x_2 \leq t_1 - u_1$$
$$\Rightarrow (4-2)x_1 + (5-3)x_2 \leq 24 - 5$$
$$\Rightarrow 2x_1 + 2x_2 \leq 19$$

For $i = 2$,

$$(s_{21} - l_{21})x_1 + (s_{22} - l_{22})x_2 \leq t_2 - u_2$$
$$\Rightarrow (4-1)x_1 + (1-5)x_2 \leq 12 - 6$$
$$\Rightarrow 3x_1 + 0.5x_2 \leq 6$$

From (3),

$$(s_{i1} + r_{i1})x_1 + (s_{i2} + r_{i2})x_2 \leq t_i + v_i$$

For $i = 1$,

$$(s_{11} + r_{11})x_1 + (s_{12} + r_{12})x_2 \leq t_1 + v_1$$
$$\Rightarrow (4+1)x_1 + (5+1)x_2 \leq 24 + 8$$
$$\Rightarrow 5x_1 + 6x_2 \leq 32$$

For $i = 2$,

$$(s_{21} + r_{21})x_1 + (s_{22} + r_{22})x_2 \leq t_2 + v_2$$
$$\Rightarrow (4+2)x_1 + (1+1)x_2 \leq 12 + 3$$
$$\Rightarrow 6x_1 + 2x_2 \leq 15$$

Since all constraints have $\leq$sign, so we introduce slack variables $s_1, s_2, s_3, s_4, s_5, s_6$and we have

$$\max z = 5x_1 + 4x_2 + 0.s_1 + 0.s_2 + 0.s_3 + 0.s_4 + 0.s_5 + 0.s_6$$

such that

$$4x_1 + 5x_2 + s_1 + 0.s_2 + 0.s_3 + 0.s_4 + 0.s_5 + 0.s_6 = 24$$
$$4x_1 + x_2 + 0.s_1 + 1.s_2 + 0.s_3 + 0.s_4 + 0.s_5 + 0.s_6 = 12$$
$$2x_1 + 2x_2 + 0.s_1 + 0.s_2 + 1.s_3 + 0.s_4 + 0.s_5 + 0.s_6 = 19$$
$$3x_1 + 0.5x_2 + 0.s_1 + 0.s_2 + 0.s_3 + 1.s_4 + 0.s_5 + 0.s_6 = 6$$
$$5x_1 + 6x_2 + 0.s_1 + 0.s_2 + 0.s_3 + 0.s_4 + 1.s_5 + 0.s_6 = 32$$
$$6x_1 + 2x_2 + 0.s_1 + 0.s_2 + 0.s_3 + 0.s_4 + 0.s_5 + 1.s_6 = 15$$
$$x_1, x_2, s_1, s_2, s_3, s_4, s_5, s_6 \geq 0$$

Now, we construct the following simplex table to solve the problem :

		C_j	5	4	0	0	0	0	0	0	Minimum Ratio
B	C_B	X_B	x_1	x_2	s_1	s_2	s_3	s_4	s_5	s_6	
s_1	0	24	4	5	1	0	0	0	0	0	6
s_2	0	12	4	1	0	1	0	0	0	0	3
s_3	0	19	2	2	0	0	1	0	0	0	9.5
s_4	0	6	3	0.5	0	0	0	1	0	0	2 (min)→
s_5	0	32	5	6	0	0	0	0	1	0	6.4
s_6	0	15	6	2	0	0	0	0	0	1	2.5
Z=0		Δ_j	-5 ↑	-4	0	0	0	0 ↓	0	0	
s_1	0	16	1	-26/6	1	0	0	4/3	0	0	
s_2	0	4	0	1/3	0	1	0	-4/3	0	0	12
s_3	0	15	0	5/3	0	0	1	-2/3	0	0	9
x_1	5	2	1	1/6	0	0	0	1/3	0	0	12
s_5	0	22	0	31/6	0	0	0	-5/3	1	0	4.25
s_6	0	3	0	[1]	0	0	0	-2	0	1	3 (min)→
Z=10		Δ_j	0	-19/6 ↑	0	0	0	5/3	0	0 ↓	

s_1	0	3	0	0	1	0	0	22/3	0	-26/6	
s_2	0	3	0	0	0	1	0	-2/3	0	-1/3	
s_3	0	10	0	0	0	0	1	8/3	0	-5/3	
x_1	5	3/2	1	0	0	0	0	2/3	0	-1/6	
x_5	0	13/2	0	0	0	0	0	26/3	1	-31/6	
x_2	4	3	0	1	0	0	0	-2	0	1	
Z=39/2		Δ_j	0	0	0	0	0	10/3	0	19/6	

Since all $\Delta_j \geq 0$, therefore the required optimal solution is

$$x_1 = \frac{3}{2}, x_2 = 3$$

and

$$\max Z = \frac{39}{2} = 19.5$$

CHECK YOUR PROGRESS

Q.2. Solve the following fuzzy linear programming problems:

(a) $\max z = 6x_1 + 5x_2$

subject to constraints

$$(5,3,2)x_1 + (6,4,2)x_2 \leq (25,6,9)$$
$$(5,2,3)x_1 + (3,1.5,1)x_2 \leq (13,7,4)$$

and $x_1, x_2 \geq 0$.

(b) $\max z = 5x_1 + 6x_2$

subject to constraints

$$(3,2,4)x_1 + (4,2.5,5.5)x_2 \leq (18,16,19)$$
$$(2,1,3)x_1 + (1,0.5,2)x_2 \leq (7,6,9)$$

and $x_1, x_2 \geq 0$.

12.6 LET US SUM UP

- The most general fuzzy linear programming problem is formulated as follows :

$$\max Z = \sum_{j=1}^{n} C_j X_j$$

subject to the constraints

$$\sum_{j=1}^{n} A_{ij} X_j \leq B_i \qquad (i \in N_m)$$
$$X_j \geq 0 \quad (j \in N_n)$$

where A_{ij}, B_i, C_j are fuzzy members and X_j are variables whose states are fuzzy numbers $(i \in N_m, j \in N_n)$.

- **Symmetric Fuzzy Linear Programming:** The fuzzy linear programming problems in which only the right-hand side numbers B_i are fuzzy numbers:

$$\max Z = \sum_{j=1}^{n} C_j x_j$$

subject to the constraints

$$\sum_{j=1}^{m} a_{ij} x_j \leq B_i \qquad (i \in N_m)$$
$$x_j \geq 0 \qquad (j \in N_n)$$

Here the fuzzy numbers $b_i (i \in N_m)$ have the form

$$B_i(x) = \begin{cases} 1, & \text{when } x \leq b_i \\ \dfrac{b_i + p_i - x}{p_i}, & \text{when } b_i < x < b_i + p_i \\ 0, & \text{when } x \geq b_i + p_i \end{cases}$$

where $x \in R$. To solve symmetric fuzzy linear programming problem, first we calculate the lower bound z_l and upper bound z_u of the optimal values of z. The lower bound of the optimal values z_l is obtained by solving the classical L.P.P.

$$\max z = \bar{C}\bar{x}$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad (i \in N_m)$$
$$x_j \geq 0 \qquad (j \in N_n)$$

The upper bound of the optimal values, z_u is obtained by solving the classical L.P.P.

$$\max z = \bar{C}\bar{x}$$

subject to

$$\sum_{j=1}^{n} a_{ij}\, x_j \le b_i + p_i \qquad (i \in N_m)$$
$$x_j \ge 0 \qquad (j \in N_n)$$

Using z_l and z_u, the given symmetric fuzzy linear programming problem becomes the following classical optimization problem

$$\max \lambda$$

subject to

$$\lambda(z_u - z_l) - \bar{C}\bar{x} \le -z_l$$
$$\lambda \mathrm{p_i} + \sum_{j=1}^{n} a_{ij}\, x_j \le b_i + p_i \qquad (i \in N_m)$$
$$\lambda, x_j \ge 0 \quad (j \in N_n)$$

- **Linear Programming with Triangular Fuzzy numbers:** A fuzzy linear programming with triangular fuzzy number can be written as

$$\max z = \sum_{j=1}^{n} C_j\, x_j$$

subject to

$$\sum_{j=1}^{m} (s_{ij}, l_{ij}, r_{ij}) x_j \le (t_i, u_i, v_i) \qquad (i \in N_m)$$

and

$$x_j \ge 0 \qquad (j \in N_n)$$

where (s_{ij}, l_{ij}, r_{ij}) and (t_i, u_i, v_i) are fuzzy numbers. With the help of operations on fuzzy numbers the above problem converted to the following classical L.P.P

$$\max z = \sum_{j=1}^{n} C_j\, x_j$$

subject to

$$\sum_{j=1}^{n} s_{ij}\, x_j \le t_i$$
$$\sum_{j=1}^{n} (s_{ij} - l_{ij}) x_j \le t_i - u_i$$
$$\sum_{j=1}^{n} (s_{ij} + r_{ij}) x_j \le t_i + v_i \qquad (i \in N_m)$$

and

$$x_j \geq 0 \quad (j \in N_n).$$

✓ 12.7 ANSWERS TO CHECK YOUR PROGRESS

Answer to Q.1

Let x_1 and x_2 denote the number of units of products P_1 and P_2 made per day respectively. Then the problem can be formulated as the following fuzzy linear programming problem

$$\max z = 0.4x_1 + 0.3x_2$$

subject to constraints

$$x_1 + x_2 \leq B_1 \text{(Material)}$$
$$2x_1 + x_2 \leq B_2 \text{(Labours hours)}$$

where

$$B_1(x) = \begin{cases} 1, & \text{when } x \leq 400 \\ \dfrac{500 - x}{100}, & \text{when } 400 < x < 500 \\ 0, & \text{when } x \geq 500 \end{cases}$$

and

$$B_2(x) = \begin{cases} 1, & \text{when } x \leq 500 \\ \dfrac{600 - x}{100}, & \text{when } 500 < x < 600 \\ 0, & \text{when } x \geq 600 \end{cases}$$

Now, we calculate lower bound z_l of the objective function by solving the classical L.P.P corresponding to the F.L.P.P is given by

$$\max z_l = 0.4x_1 + 0.3x_2$$

subject to

$$x_1 + x_2 \leq 400$$
$$2x_1 + x_2 \leq 500$$
$$x_1, x_2 \geq 0$$

Introducing slack variables s_1 and s_2, we obtain

$$\max z_l = 0.4x_1 + 0.3x_2 + 0.s_1 + 0.s_2$$

such that

$$x_1 + x_2 + 1.s_1 + 0.s_2 = 400$$
$$2x_1 + x_2 + 0.s_1 + 1.s_2 = 500$$

$$x_1, x_2, s_1, s_2 \geq 0$$

To find $z_l$we construct the following simplex table :

		C_j	0.4	0.3	0	0	Minimum
B	C_B	X_B	x_1	x_2	s_1	s_2	Ratio
s_1	0	400	1	1	1	0	400
s_2	0	500	[2]	1	0	1	250 (min)→
$Z_l = 0$		Δ_j	-0.4 ↑	-0.3	0	0 ↓	
s_1	0	150	0	[1/2]	1	-1/2	75→
x_1	0.4	250	1	1/2	0	1/2	125
$Z_l = 100$		Δ_j	0	-0.1 ↑	0	0.2	
x_2	0.3	300	0	1	2	-1	
x_1	0.4	100	1	0	-1	1	
$Z_l = 130$		Δ_j	0	0	0.2	0.1	

Since all $\Delta_j \geq 0$, so optimal solution is $x_1 = 100$, $x_2 = 300$ and $z_l = 130$.

Next, we calculate upper bound z_u of the objective function by solving the classical L.P.P corresponding to the F.L.P.P is given by

$$\max z_u = 0.4x_1 + 0.3x_2$$

subject to constraints

$$x_1 + x_2 \leq 500$$
$$2x_1 + x_2 \leq 600$$
$$x_1, x_2 \geq 0$$

Introducing slack variables s_1' and s_2', we obtain

$$\max z_u = 0.4x_1 + 0.3x_2 + 0.s_1' + 0.s_2'$$

such that

$$x_1 + x_2 + 1.s_1' + 0.s_2' = 500$$
$$2x_1 + x_2 + 0.s_1' + 1.s_2' = 600$$
$$x_1, x_2, s_1', s_2' \geq 0$$

To solve the above L.P.P we construct the following simplex table :

		C_j	0.4	0.3	0	0	Minimum Ratio
B	C_B	X_B	x_1	x_2	$s_1{}'$	s_2'	
$s_1{}'$	0	500	1	1	1	0	500
s_2'	0	600	[2]	1	0	1	300 (min)→
$Z_u = 0$		Δ_j	-0.4 ↑	-0.3	0	0 ↓	
$s_1{}'$	0	200	0	[1/2]	1	-1/2	400→
x_1	0.4	300	1	1/2	0	1/2	600
$Z_u = 120$		Δ_j	0	-0.1 ↑	0	0.2	
x_2	0.3	400	0	1	2	-1	
x_1	0.4	100	1	0	-1	1	
$Z_u = 160$		Δ_j	0	0	0.2	0.1	

Since all $\Delta_j \geq 0$, therefore optimal solution is $x_1 = 100$, $x_2 = 400$ and $z_u = 160$.
Then, the given fuzzy linear programming problem becomes

$$\max z = \lambda$$

subject to constraints

$$(160 - 130)\lambda - (0.4x_1 + 0.3x_2) \leq -130$$
$$100\lambda + x_1 + x_2 \leq 500$$
$$100\lambda + 2x_1 + x_2 \leq 600$$

and

$$x_1, x_2, \lambda \geq 0$$

We solve this L.P.P by Big M–method. The first constraint is a "≥" constraint, so we introduce a surplus variable $S$and artificial variable A. For second and third constraints, we introduce slack variables $S_1$and S_2. Thus the L.P.P becomes

$$\max z = 1.\lambda + 0.x_1 + 0.x_2 + 0.S - M.A + 0.S_1 + 0.S_2$$

such that

$$-30\lambda + 0.4x_1 + 0.3x_2 - S + A = 130$$
$$100\lambda + x_1 + x_2 + S_1 = 500$$
$$100\lambda + 2x_1 + x_2 + S_2 = 600$$

Now, we construct the following simplex table :

		C_j	1	0	0	0	-M	0	0	Minimum Ratio
B	C_B	X_B	λ	x_1	x_2	S	A	S_1	S_2	
A	-M	130	-30	0.4	0.3	-1	1	0	0	325
S_1	0	500	100	1	1	0	0	1	0	500
S_2	0	600	100	[2]	1	0	0	0	1	300→
Z=-130M		Δ_j	30M-1	-0.4M ↑	-0.3M	M	0	0	0 ↓	
A	-M	10	-50	0	[0.1]	-1	1	0	-2	100→
S_1	0	200	50	0	0.5	0	0	1	-0.5	400
x_1	0	300	50	1	0.5	0	0	0	0.5	600
Z=-10M		Δ_j	50M-1	0	-0.1M ↑	M	0	0	0.2 M	
x_2	0	100	-500	0	1	-10	10	0	-2	
S_1	0	250	[300]	0	0	5	-5	1	0.5	1/2→
x_1	0	250	300	1	0	5	5	0	-0.5	5/6
x_2	0	350	0	0	1	-10/6	10/6	5/3	0.5	
λ	1	1/2	1	0	0	1/60	-1/60	1/300	0.5/300	
x_1	0	100	0	1	0	0	0	-1	-1	
Z=1/2		Δ_j	0	0	0	1/60	M-1/60	1/300	0.5/300	

Since all $\Delta_j \geq 0$, therefore the solution of fuzzy L.P.P is given by

$$x_1 = 100,\ x_2 = 350,\ \lambda = \frac{1}{2}$$

and maximum profit is given by

$$\max z = 0.4 \times 100 + 0.3 \times 350 = 145$$

Answer to Q. 2

The given fuzzy linear programming problem can be written as

$$\max z = 6x_1 + 5x_2$$

subject to

$$\sum_{j=1}^{2} s_{ij}\, x_j \le t_i \qquad (1)$$

$$\sum_{j=1}^{2} (\, s_{ij} - l_{ij})x_j \le t_i - u_i \qquad (2)$$

$$\sum_{j=1}^{2} (\, s_{ij} + r_{ij})x_j \le t_i + v_i \qquad (3)$$

Here $s_{11} = 5,\ l_{11} = 3,\ r_{11} = 2;\ \ s_{12} = 6,\ l_{12} = 4,\ r_{12} = 2$

$s_{21} = 5,\ l_{21} = 2,\ r_{21} = 3;\ \ s_{22} = 2,\ l_{22} = 1.5,\ r_{22} = 1$

and $t_1 = 25,\ u_1 = 6,\ v_1 = 9$

$t_2 = 13,\ u_2 = 7,\ v_2 = 4$

From (1),

$$s_{i1}x_1 + s_{i2}x_2 \le t_i$$

For $i = 1$,

$$s_{11}x_1 + s_{12}x_2 \le t_1$$
$$\Rightarrow 5x_1 + 6x_2 \le 25$$

For $i = 2$,

$$s_{21}x_1 + s_{22}x_2 \le t_2$$
$$\Rightarrow 5x_1 + 2x_2 \le 13$$

From (2),

$$(s_{i1} - l_{i1})x_1 + (s_{i2} - l_{i2})x_2 \le t_i - u_i$$

For $i = 1$,

$$(s_{11} - l_{11})x_1 + (s_{12} - l_{12})x_2 \le t_1 - u_1$$
$$\Rightarrow 2x_1 + 2x_2 \le 19$$

For $i = 2$,

$$(s_{21} - l_{21})x_1 + (s_{22} - l_{22})x_2 \le t_2 - u_2$$
$$\Rightarrow 3x_1 + 0.5x_2 \le 6$$

From (3),

$$(s_{i1} + r_{i1})x_1 + (s_{i2} + r_{i2})x_2 \le t_i + v_i$$

For $i = 1$,

$$(s_{11} + r_{11})x_1 + (s_{12} + r_{12})x_2 \le t_1 + v_1$$

$$\Rightarrow 7x_1 + 8x_2 \leq 34$$

For $i = 2$,

$$(s_{21} + r_{21})x_1 + (s_{22} + r_{22})x_2 \leq t_2 + v_2$$
$$\Rightarrow 8x_1 + 3x_2 \leq 17$$

Introducing slack variables $s_1, s_2, s_3, s_4, s_5, s_6$, we can write

$$\max z = 6x_1 + 5x_2 + 0.s_1 + 0.s_2 + 0.s_3 + 0.s_4 + 0.s_5 + 0.s_6$$

such that

$$5x_1 + 6x_2 + s_1 + 0.s_2 + 0.s_3 + 0.s_4 + 0.s_5 + 0.s_6 = 25$$
$$5x_1 + 2x_2 + 0.s_1 + s_2 + 0.s_3 + 0.s_4 + 0.s_5 + 0.s_6 = 13$$
$$2x_1 + 2x_2 + 0.s_1 + 0.s_2 + s_3 + 0.s_4 + 0.s_5 + 0.s_6 = 19$$
$$3x_1 + 0.5x_2 + 0.s_1 + 0.s_2 + 0.s_3 + s_4 + 0.s_5 + 0.s_6 = 6$$
$$7x_1 + 8x_2 + 0.s_1 + 0.s_2 + 0.s_3 + 0.s_4 + s_5 + 0.s_6 = 34$$
$$8x_1 + 3x_2 + 0.s_1 + 0.s_2 + 0.s_3 + 0.s_4 + 0.s_5 + s_6 = 17$$
$$x_1, x_2, s_1, s_2, s_3, s_4, s_5, s_6 \geq 0$$

The solution of this LPP can be obtained by using following simplex table:

		C_j	6	5	0	0	0	0	0	0	Minimum Ratio
B	C_B	X_B	x_1	x_2	s_1	s_2	s_3	s_4	s_5	s_6	
s_1	0	25	5	6	1	0	0	0	0	0	5
s_2	0	13	5	2	0	1	0	0	0	0	13/5
s_3	0	19	2	2	0	0	1	0	0	0	19/2
s_4	0	6	[3]	0.5	0	0	0	1	0	0	2 (min)→
s_5	0	34	7	8	0	0	0	0	1	0	34/7
s_6	0	17	8	3	0	0	0	0	0	1	17/8
Z=0		Δ_j	-6 ↑	-5	0	0	0	0 ↓	0	0	
s_1	0	15	0	31/6	1	0	0	-5/3	0	0	90/31
s_2	0	3	0	7/6	0	1	0	-5/3	0	0	18/7
s_3	0	15	0	5/3	0	0	1	-2/3	0	0	9
x_1	6	2	1	1/6	0	0	0	1/3	0	0	12
s_5	0	20	0	41/6	0	0	0	-7/3	1	0	120/41

s_6	0	1	0	[5/3]	0	0	0	-8/3	0	1	3/5→
Z=12		Δ_j	0	-4 ↑	0	0	0	+2	0	0 ↓	
s_1	0	199/10	0	0	1	0	0	33/5	0	-31/10	199/66≈3.01
s_2	0	23/10	0	0	0	1	0	1/5	0	-7/10	23/2≈11.5
s_3	0	14	0	0	0	0	1	2	0	-1	7
x_1	6	19/10	1	0	0	0	0	3/5	0	-1/10	19/6≈3.16
s_5	0	159/10	0	0	0	0	0	[129/15]	1	-41/10	159/86≈1.84→
x_2	5	3/5	0	1	0	0	0	-8/5	0	13/5	---
Z=72/5=14.4		Δ_j	0	0	0	0	0	-22/5 ↑	0	12/5	
s_1	0	331/43	0	0	1	0	0	0	99/43	2/43	
s_2	0	912/430	0	0	0	1	0	0	-1/43	-178/430	
s_3	0	443/43	0	0	0	0	1	0	-15/43	-2/43	
x_1	6	340/430	1	0	0	0	0	0	-9/129	8/43	
s_4	0	159/86	0	0	0	0	0	1	5/43	-123/258	
x_2	5	153/43	0	1	0	0	0	0	8/43	-35/215	
Z=969/43		Δ_j	0	0	0	0	0	0	66/129	13/43	

Since all $\Delta_j \geq 0$, so obtained solution is optimal and solution is

$$x_1 = \frac{340}{43}, x_2 = \frac{153}{43}$$

and

$$\max z = \frac{969}{43} \approx 22.53.$$

(b) Left as an exercise.

12.8 FURTHER READING

1. Sudhir K. Pundir and Rimple Pundir, *Fuzzy Sets and Their Applications*, Pragati Prakashan (Meerut), 2008.
2. George J. Klir and Bo Yuan, *Fuzzy Sets, Fuzzy Logic Theory and Applications*, Prentice Hall P.T.R., Upper Saddle River, New Jersey – 07458 (1995).
3. Zimmermann, H.J., *Fuzzy Set Theory and its applications*, Allied Publishers Ltd., New Delhi, 1991.

12.9 MODEL QUESTIONS

1. Solve the following fuzzy linear Programming problems:

   (a) $\max z = 5x_1 + 7.5x_2$
   subject to the constraints

   $$(1,2,1)x_1 + (2,3,1)x_2 \leq (5,6,2)$$
   $$(2,3,1)x_1 + (1,2,1)x_2 \leq (4,6,2)$$

   and $x_1, x_2 \geq 0$.

   (b) $\max z = 6x_1 + 4x_2 + x_3$
   subject to constraints

   $$(3,6,8)x_1 + (-1,2,4)x_2 + (5,10,12)x_3 \leq (16,18,26)$$
   $$x_1, x_2, x_3 \geq 0.$$

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UNIT 13: FUZZY METRIC SPACES

UNIT STRUCTURE

13.1 Learning Objectives
13.2 Introduction
13.3 Fuzzy Metric Spaces: Definitions and Examples
13.4 Properties of Fuzzy Metric Spaces
13.5 Fuzzy Metrization of the Topologies Induced by Fuzzy Metrics
13.6 Complete Fuzzy Metric Spaces
13.7 Let Us Sum Up
13.8 Answers to Check Your Progress
13.9 Further Reading
13.10 Model Questions

13.1 LEARNING OBJECTIVES

The learning objective of studying fuzzy metric spaces typically include:

- Understanding the specific axioms that a fuzzy set must satisfy to be considered a fuzzy metric space on a given set, often involving a continuous t-norm.
- Distinguishing between different types of fuzzy metric spaces.
- Comparing fuzzy metric spaces with classical metric spaces.
- Understanding how fuzzy metrics generate a topology.
- Understanding convergence and completeness in fuzzy metric spaces.
- Defining and understanding fuzzy continuity.
- Relating fuzzy continuity to fuzzy open sets.
- Exploring properties of fuzzy continuous function.

13.2 INTRODUCTION

Fuzzy set theory, introduced by Lotfi A. Zadeh in 1965, revolutionized our way of thinking about imprecision and uncertainty. Unlike classical sets where an element either belongs or does not belong to a set (crisp 0 or 1), fuzzy sets allow for degrees of membership, represented by values in the interval $[0,1]$. This simple yet profound idea opened doors to modelling phenomena where boundaries are inherently vague.

Building upon this foundation, the concept of fuzzy metric space emerged as a natural generalization of the classical metric space. In the classical metric space, the distance between two points is a precise, non-negative real number. For example, the distance between two cities on a map is a single fixed value. However, in many real-world scenarios, the “distance” or “similarity” between objects might not be so absolute. Consider the “distance” between two colours, or the “similarity” between two medical

symptoms; these are often subjective and involve degrees of resemblance rather than exact measures.

A fuzzy metric space attempts to capture this inherent vagueness. Instead of a single numerical distance, the "distance" between two points in a fuzzy set is, more precisely, a degree of nearness. This degree of nearness is typically a value in the interval $[0,1]$, where a higher value indicates a greater degree of closeness or similarity.

The need for fuzzy metric spaces arises from the limitations of classical metrics in addressing problems characterized by:

- Uncertainty and vagueness
- Approximation and estimation
- Subjectivity

While the formal definition involves specific axioms and a continuous t-norm (a generalization of multiplication used in fuzzy logic), the core idea is that for any two points x and y in the space, and for any "distance" $t > 0$, there is a value $M(x, y, t)$ representing the degree of nearness between x and y with respect to t.

- A higher value of $M(x, y, t)$ means that x and y are "closer" for the specific distance t.
- As t increases, $M(x, y, t)$ is typically non-decreasing, meaning that for larger tolerances, points are considered "closer" with a higher degree.
- If $x = y$, then $M(x, y, t)$ should be 1 for all $t > 0$, indicating perfect nearness.

The introduction of fuzzy metric spaces has opened up new avenues of research, particularly in:

- Topology,
- Fixed Point Theory
- Applications

In essence, fuzzy metric spaces provide a powerful and flexible framework for extending classical metric concepts to situations where precision is not always possible or desirable, allowing for a more nuanced and realistic mathematical modelling of real-world phenomena.

13.3 FUZZY METRIC SPACE: DEFINITIONS AND EXAMPLES

Definition 13.3.1: A binary operation $*: [0,1] \times [0,1] \to [0,1]$ is a continuous triangular norm if for all $a, b, c, e \in [0,1]$, the following conditions hold:

(a) $a * b = b * a$ (commutativity)

(b) $a * 1 = a$

(c) $(a * b) * c = a * (b * c)$ (associativity)

(d) $a * b \leq c * e$ whenever $a \leq c$ and $b \leq e$

In the sequel, we shall refer to triangular norm as a t −norm.

Example 1. Define $a * b = ab$ for all $a, b \in [0,1]$. Note that ab is the usual multiplication in $[0,1]$ for all $a, b \in [0,1]$.

It follows that $*$ is a continuous t −norm.

Example 2. Define $a * b = min\{a, b\}$, for all $a, b \in [0,1]$.

Then $*$ is a continuous t −norm.

Definition 15.3.2 (Fuzzy Metric Space).

The 3 −tuple $(X, M, *)$ is said to be a fuzzy metric space, where X is an arbitrary set, $*$ is a continuous t −norm and M is a fuzzy set on $X \times X \times [0, \infty)$ satisfying the following conditions:

$\boldsymbol{C1}$. $M(x, y, 0) = 0, \ \forall\, x, y \in X$.

$\boldsymbol{C2}$. $M(x, y, t) = 1 \ iff \ x = y, \ \forall\, x, y \in X \ and \ \forall\, t > 0$.

$\boldsymbol{C3}$. $M(x, y, t) = M(y, x, t), \ \forall\, x, y \in X \ and \ \forall\, t > 0$.

$\boldsymbol{C4}$. $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s), \ \forall\, x, y, z \in X \ and \ \forall\, t, s > 0$.

$\boldsymbol{C5}$. $M(x, y, \cdot)\colon [0, \infty) \to [0,1]$ is continuous.

Remarks:

1. Condition $C5$ means that for each $x, y \in X$, there is a continuous function $M(x, y, \cdot)\colon [0, \infty) \to [0,1]$, $t \mapsto M(x, y, t)$.
2. $M(x, y, t)$ can be thought of as the degree of nearness between x and y with respect to $t \geq 0$.

The following definition is a modification of **Definition 15.3.2**. This modification is necessary since the topology induced by the fuzzy metric in **Definition 15.3.2** is not Hausdorff.

Definition 13.3.3. The 3 −tuple $(X, M, *)$ is said to be a **fuzzy metric space**, where X is an arbitrary set, $*$ is a continuous t −norm and M is a fuzzy set on $X \times X \times (0, \infty)$ satisfying the following conditions:

$\boldsymbol{C1}$. $\forall x, y \in X$ and $\forall\, t > 0$, $M(x, y, t) > 0$.

$\boldsymbol{C2}$. $\forall\, x, y \in X$ and $\forall\, t > 0$, $M(x, y, t) = 1 \ \ iff \ \ x = y$

$\boldsymbol{C3}$. $\forall\, x, y \in X$ and $\forall\, t > 0$, $M(x, y, t) = M(y, x, t)$

$\boldsymbol{C4}$. . $\forall\, x, y, z \in X$ and $\forall\, s, t > 0$, $M(x, y, t + s) \geq M(x, y, t) * M(y, z, s)$

$\boldsymbol{C5}$. . $\forall\, x, y, z \in X$, $M(x, y, t)\colon (0, \infty) \to [0,1]$ is continuous.

In the sequel the fuzzy set M as in **Definition 15.3.3** will be referred to as a fuzzy metric.

Lemma 15.3.4. $M(x, y, .)$ is nondecreasing for all x, y in X.

Proof: Suppose $M(x, y, t) > M(x, y, s)$ for some $0 < t < s$. Then

$$M(x, y, t) * M(y, y, s - t) \leq M(x, y, s) < M(x, y, t)$$

By **Definition 13.3.3,** we have $M(y,y,s-t)=1$. Thus

$$M(x,y,t)<M(x,y,s)<M(x,y,t)$$

a contradiction. Thus $M(x,y,.)$ is a non-decreasing for all x,y in X.

Remarks:

1. Let $(X,M,*)$ be a fuzzy metric space and let $x,y\in X,\ t>0,\ 0<r<1$. Then if $M(x,y,t)>1-r$, we can find t_0 with $0<t_0<t$, such that
$$M(x,y,t_0)>1-r$$

2. For any $r_1>r_2$, we can find an r_3 such that $r_1*r_3\geq r_2$ and for any r_4 we can find an r_5 such that $r_5*r_5\geq r_4$ where $r_1,r_2,r_3,r_4,r_5\in(0,1)$.

Example 3. Let $X=\mathbb{R}$. Define $a*b=ab$ for all $a,b\in[0,1]$and $M(x,y,t)=\left[\exp\left(\left(\frac{|x-y|}{t}\right)\right)\right]^{-1}$for all $x,y\in X$and $t\in(0,\infty)$. Then show that $(X,M,*)$is a fuzzy metric space.

Solution.

C1. From the definition of M,

$$M(x,y,t)>0 \text{ for all } x,y\in\mathbb{R} \text{ and for all } t>0.$$

C2. Assume $x=y$, then $|\,x-y\,|=0$and hence

$$M(x,y,t)=\left[\exp\left(\left(\frac{|\,x-y\,|}{t}\right)\right)\right]^{-1}=1.$$

Conversely, let

$$\begin{aligned}&M(x,y,t)=1\\ \Rightarrow&\left[\exp\left(\left(\frac{|\,x-y\,|}{t}\right)\right)\right]^{-1}=1\\ \Rightarrow&\frac{|\,x-y\,|}{t}=0\\ \Rightarrow&x=y\end{aligned}$$

Therefore

$$M(x,y,t)=1\ \Leftrightarrow\ x=y.$$

C3. We have $|\,x-y\,|=|\,y-x\,|$for all $x,y\in\mathbb{R}$. It follows for all $x,y\in\mathbb{R}$and for all $t>0$,

$$M(x,y,t)=M(y,x,t).$$

C4. For all $x,y,z\in\mathbb{R}$and for all $t,s>0$, we have

$$\begin{aligned}&|\,x-z\,|\leq\frac{t+s}{t}|\,x-y\,|+\frac{t+s}{s}|\,y-z\,|\\ \Rightarrow&\frac{|\,x-z\,|}{t+s}\leq\frac{|\,x-y\,|}{t}+\frac{|\,y-z\,|}{s}\end{aligned}$$

$$\Rightarrow \exp\left(\frac{|\,x-z\,|}{t+s}\right) \leq \exp\left(\frac{|\,x-y\,|}{t}\right) \cdot \exp\left(\frac{|\,y-z\,|}{s}\right),$$

since $\exp(x)$is an increasing function for $x > 0$.

$$\Rightarrow \left[\exp\left(\frac{x-z}{t+s}\right)\right]^{-1} \geq \left[\exp\left(\frac{|x-y|}{t}\right)\right]^{-1} * \left[\exp\left(\frac{|y-z|}{s}\right)\right]^{-1}$$

Thus

$$M(x,z,t+s) \geq M(x,y,t) * M(y,z,s).$$

C5. Take a sequence $\{t_n\}$in $(0,\infty)$such that the sequence $\{t_n\}$converges to $t \in (0,\infty)$, where $(0,\infty)$is equipped with the usual metric, i.e.

$$\lim_{n\to\infty} |\,t_n - t\,| = 0.$$

Without the loss of generality, fix $x, y \in X$. Since the function $\exp(x)$is continuous on $\mathbb{R}$, therefore

$$\exp\left(\frac{|\,x-y\,|}{t_n}\right) \to \exp\left(\frac{|\,x-y\,|}{t}\right)$$

as $\{t_n\}$converges to t, with respect to the usual metric. Therefore

$$M(x,y,\cdot)\colon (0,\infty) \to [0,1]$$

is continuous. Hence $(\mathbb{R}, M, *)$is a fuzzy metric space.

Example 4. Let (X,d)be a classical metric space. Define $a * b = ab$ for all $a, b \in [0,1]$and

$$M(x,y,t) = \frac{kt^n}{kt^n + md(x,y)}, k, m, n \in \mathbb{N}.$$

Then $(X, M, *)$is a fuzzy metric space.

Remarks:

1. Every metric space induces a fuzzy metric space.
2. **Example 3** and **Example 4** hold even with the continuous t-norm

$$a * b = \min\{a, b\}.$$

In particular, taking $k = m = n = 1$in Example 4, we get

$$M(x,y,t) = \frac{t}{t + d(x,y)}.$$

We shall call this metric induced by a metric d, the standard fuzzy metric and $M_d$denotes a standard fuzzy metric induced by the metric d.

Example 5. Let $X = \mathbb{N}$. Define $a * b = ab$and for all $t > 0$, let

$$M(x,y,t) = \begin{cases} \frac{x}{y}, & \text{if } x \leq y, \\ \frac{y}{x}, & \text{if } y \leq x. \end{cases}$$

Show that $M$is a fuzzy metric.

Solution:

C1. From the definition of M,

$M(x,y,t) > 0$ for all $x, y \in \mathbb{N}$ and for all $t > 0$.

C2. Assume that $x = y$. Then $\frac{x}{y} = \frac{y}{x} = 1$. Hence $M(x, y, t) = 1$.

Conversely, assume that $M(x, y, t) = 1$. Then either $\frac{x}{y} = 1$ or $\frac{y}{x} = 1$, i.e. $x = y$.

Thus $M(x, y, t) = 1 \Leftrightarrow x = y$.

C3. For all $x, y \in X$ and for all $t > 0$, clearly

$$M(x, y, t) = M(y, x, t).$$

C4. To prove that

$$M(x, z, t + s) \geq M(x, y, t) * M(y, z, s),$$

we consider the following cases:

Case (I). If $x = y = z$, then

$$M(x, y, t) = M(y, z, s) = M(x, z, t + s) = 1.$$

It follows that

$$M(x, z, t + s) = M(x, y, t) * M(y, z, s).$$

Case (II). Let $x \neq y = z$. Without loss of generality, we may assume that $x < y$ and $y = z$. Then

$$M(x, y, t) = \frac{x}{y}, \qquad M(y, z, t) = 1.$$

Also, we have

$$M(x, z, t + s) = \frac{x}{z}.$$

So

$$M(x, y, t) * M(y, z, t) = \frac{x}{y} * 1 = \frac{x}{y} = \frac{x}{z} \qquad [\because y = z]$$
$$= M(x, z, t + s).$$

Case (III). Let $x = y \neq z$. Without loss of generality, we may assume that $x = y$and $y < z$. Then

$$M(x, y, t) = 1, M(y, z, t) = \frac{y}{z}, M(x, z, t + s) = \frac{x}{z}.$$

Therefore, $M(x, y, t) * M(y, z, s) = 1 * \frac{y}{z} = \frac{y}{z} = \frac{x}{z} = M(x, z, t + s)$

Case (IV). Let $x \neq y \neq z$. Without loss of generality, we may assume that $x < y < z$. Then

$$M(x, z, t) = \frac{x}{z}, \quad M(y, z, s) = \frac{y}{z}, \quad M(x, y, t + s) = \frac{x}{y}$$

We have

$$y < z \Rightarrow y^2 < z^2 \Rightarrow \frac{1}{y^2} > \frac{1}{z^2}$$

$$\Rightarrow \frac{xy}{z^2} < \frac{xy}{y^2}$$
$$\Rightarrow \frac{x}{z} * \frac{y}{z} < \frac{x}{y}$$
$$\Rightarrow M(x,z,t) * M(y,z,s) < M(x,y,t+s)$$

Case (V). Note that $M(x,y,t)$ is independent of t $i.e.$ $M(x,y,t)$ is a constant in terms of t. For $s,t>0$ we have $M(x,y,t) = M(x,y,s)$. Thus $M(x,y,.)$ is continuous. Therefore $(X,M,*)$ is a fuzzy metric space.

Remark:

1. The fuzzy metric in the above example (Example 5) is independent of t. Such a fuzzy metric is referred to as a stationary fuzzy metric.

CHECK YOUR PROGRESS

***Q* 1**. Show that there is no metric d on X satisfying $M(x,y,t) = \frac{t}{t+d(x,y)}$, where M is defined by

$$M(x,y,t) = \begin{cases} \frac{x}{y}, & if \ \ x \le y \\ \frac{y}{x} & if \ \ y \le x \end{cases}$$

Also show that $M$is not a fuzzy metric on $X$with the t-norm defined by

$$a * b = \min\{a,b\}.$$

13.4 PROPERTIES OF FUZZY METRIC SPACES

Definition 13.4.1 (Open Ball). Let $(X,M,*)$be a fuzzy metric space. We define the open ball $B(x,r,t)$with centre $x \in X$and radius r, $0<r<1$, $t>0$, as

$$B(x,r,t) = \{y \in X : M(x,y,t) > 1-r\}.$$

Definition 13.4.2 (Closed Ball). Let $(X,M,*)$be a fuzzy metric space. We define the closed ball $B[x,r,t]$with centre $x \in X$and radius r, $0<r<1$, $t>0$, as

$$B[x,r,t] = \{y \in X : M(x,y,t) \ge 1-r\}.$$

Theorem 13.4.3. Let $B(x,r_1,t)$and $B(x,r_2,t)$be open balls with the same centre $x \in X$and $t>0$ with radius $0<r_1<1$and $0<r_2<1$respectively. Then either

$$B(x,r_1,t) \subseteq B(x,r_2,t) \text{or} B(x,r_2,t) \subseteq B(x,r_1,t).$$

Proof. Let $x \in X$and $t>0$. If $r_1 = r_2$, then the theorem holds. Next, we assume that $r_1 \neq r_2$. Without loss of generality, $0<r_1<r_2<1$. Then

$$1-r_2 < 1-r_1.$$

Let $y \in B(x,r_1,t)$, then

$$M(x,y,t) > 1-r_1$$
$$\Rightarrow M(x,y,t) > 1-r_2$$
$$\Rightarrow y \in B(x,r_2,t).$$

This shows that $B(x,r_1,t) \subseteq B(x,r_2,t)$. By assuming that $0<r_2<r_1<1$, we can show $B(x,r_2,t) \subseteq B(x,r_1,t)$.

Definition 13.4.4. A subset $A$of a fuzzy metric space $(X, M,*)$is said to be open if given any point $a \in A$, there exists $0 < r < 1$ and $t > 0$such that

$$B(a, r, t) \subseteq A.$$

Theorem 13.4.5. Every open ball in a fuzzy metric space $(X, M,*)$ is an open set.

Proof: Consider an open ball $B(x, r, t)$ and let $y \in B(x, r, t)$. Then

$$M(x, y, t) > 1 - r.$$

Since $M(x, y, t) > 1 - r$, therefore we can find a t_0, $0 < t_0 < t$, such that

$$M(x, y, t_0) > 1 - r.$$

Let $r_0 = M(x, y, t_0) > 1 - r$. Since $r_0 > 1 - r$, therefore we can find an s, $0 < s < 1$, such that

$$r_0 > 1 - s > 1 - r.$$

Now for a given $r_0$and $s$such that $r_0 > 1 - s$, we can find r_1, $0 < r_1 < 1$, such that

$$r_0 * r_1 \geq 1 - s.$$

Now consider the ball $B(y, 1 - r_1, t - t_0)$. We claim

$$B(y, 1 - r_1, t - t_0) \subseteq B(x, r, t).$$

Let $z \in B(y, 1 - r_1, t - t_0)$. Then

$$M(y, z, t - t_0) > r_1.$$

Therefore

$$\begin{aligned} M(x, z, t) &\geq M(x, y, t_0) * M(y, z, t - t_0) \\ &\geq r_0 * r_1 \\ &\geq 1 - s \\ &> 1 - r. \end{aligned}$$

Thus $z \in B(x, r, t)$and hence

$$B(y, 1 - r_1, t - t_0) \subseteq B(x, r, t),$$

which implies that $B(x, r, t)$is an open set.

Theorem 13.4.6. Every fuzzy metric space is Hausdorff.

Proof: Let $(X, M,*)$be the given fuzzy metric space. Let x, ybe two distinct points of X. Then

$$0 < M(x, y, t) < 1.$$

Let $M(x, y, t) = r$for some r, $0 < r < 1$. For each r_0, $r < r_0 < 1$, we can find $r_1$such that

$$r_1 * r_1 > r_0.$$

Now consider the open balls $B\left(x, 1 - r_1, \frac{t}{2}\right)$ and $B\left(y, 1 - r_1, \frac{t}{2}\right)$.

Clearly,

$$B(x, 1 - r_1, \frac{t}{2}) \cap B(y, 1 - r_1, \frac{t}{2}) = \Phi.$$

For if there exists $z \in B\left(x, 1 - r_1, \frac{t}{2}\right) \cap B\left(y, 1 - r_1, \frac{t}{2}\right)$, then

$$\begin{aligned} r = M(x, y, t) &\geq M\left(x, z, \frac{t}{2}\right) * M\left(z, y, \frac{t}{2}\right) \\ &> r_1 * r_1 > r_0 > r, \end{aligned}$$

which is a contradiction. Therefore $(X, M,*)$ is Hausdorff.

13.5 FUZZY MATRIZATION OF THE TOPOLOGIES INDUCED BY FUZZY MATRICS

Let $(X, M, *)$be a fuzzy metric space. Define
$T_M = \{A \subset X: x \in A$ if and only if there exists $t > 0$ and $r, 0 < r < 1$ such that $B(x, r, t) \subset A\}$.
(i) Clearly Φ and X belong to T_M.
(ii) Let $\{A_i: i \in I, I$ is an index set be a family of sets in $T_M$and $A = \bigcup_{i\in I} A_i$. We shall show that $A \in T_M$. Let $a \in A$. Then $a \in \bigcup_{i\in I} A_i$

$$\Rightarrow a \in A_i \text{ for some } i \in I.$$

Since $A_i \in T_M$, there exists $t > 0$and r, $0 < r < 1$, such that

$$B(a, r, t) \subset A_i.$$

Hence

$$B(a, r, t) \subset A_i \subset \bigcup_{i\in I} A_i = A.$$

This shows that $A \in T_M$.
(iii) Let $A_1, A_2, \cdots, A_n \in T_M$and $U = \bigcap_{i=1}^{n} A_i$. We shall show that $U \in T_M$.

Let $a \in U = \bigcap_{i=1}^{n} A_i$. Then $a \in A_i$for all $i \in I$.

Then there exists $t_i > 0$ and r_i, $0 < r_i < 1$, such that

$$B(a, r_i, t_i) \subset A_i.$$

Let

$$r = \min\{r_i: i \in I\} \text{and} t = \max\{t_i: i \in I\}.$$

Then $r \leq r_i$for all $i \in I$. Therefore $1 - r \geq 1 - r_i$for all $i \in I$.

Also $t > 0$. So

$$B(a, r, t) \subset A_i \text{ for all } i \in I.$$
$$\Rightarrow B(a, r, t) \subset \bigcap_{i=1}^{n} A_i$$
$$\Rightarrow B(a, r, t) \subset U.$$

This shows that $U \in T_M$. Thus T_M is a topology on X induced by metric space $(X, M, *)$.
Theorem 13.5.1. Let (X, d) be a metric space and

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}$$

be the corresponding standard fuzzy metric on X. Then the topology $T_d$induced by the metric $d$and the topology T_{M_d}induced by fuzzy metric $M_d$are same, i.e.,

$$T_d = T_{M_d}.$$

Proof: Suppose $A \in T_d$. Then there exists $\varepsilon > 0$such that

$$B(x, \varepsilon) \subset A \text{ for every } x \in A.$$

For a fixed $t > 0$, we obtain that

$$M_d(x, y, t) = \frac{t}{t + d(x, y)} > \frac{t}{t + \varepsilon}.$$

Let $1 - r = \frac{t}{t+\varepsilon}$. Then $M_d(x, y, t) > 1 - r$. It follows that

$$B(x, r, t) \subset A.$$

Hence $A \in T_{M_d}$. Therefore

$$T_d \subseteq T_{M_d} \tag{1}$$

Conversely, suppose that $A \in T_{M_d}$. Then there exists $0 < r < 1$and $t > 0$ such that

$$B(x, r, t) \subset A \text{ for every } x \in A.$$

We obtain that

$$M(x, y, t) = \frac{t}{t + d(x, y)} > 1 - \varepsilon(\text{say})$$
$$\Rightarrow t > (1 - \varepsilon)t + (1 - \varepsilon)d(x, y)$$
$$\Rightarrow d(x, y) < \frac{\varepsilon t}{1 - \varepsilon}$$

Let

$$r = \frac{\varepsilon t}{1-\varepsilon}, \text{ where } 0 < \varepsilon < 1.$$

Then $d(x, y) < r$ and therefore $B(x, \varepsilon) \subset A$. Hence $A \in T_d$.
This implies that

$$T_{M_d} \subseteq T_d \tag{2}$$

From (1) and (2), we obtain

$$T_d = T_{M_d}.$$

13.6 COMPLETE FUZZY METRIC SPACES

Definition 13.6.1. A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is a Cauchy sequence if for each $\varepsilon > 0$, $t > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$M(x_n, x_m, t) > 1 - \varepsilon$$

for all $n, m \geq n_0$.
Remark: Let (X, d) be a metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence in (X, d) if and only if it is a Cauchy sequence in $(X, M_d, *)$.
Definition 13.6.2. A sequence $\{x_n\}$in X is convergent to $x \in X$ if

$$\lim_n M(x_n, x, t) = 1 \text{ for each } t > 0.$$

Definition 13.6.3. A fuzzy metric space in which every Cauchy sequence is convergent is called a complete fuzzy metric space.
Theorem 13.6.4. Let $(X, M, *)$be a fuzzy metric space. Then for a sequence $\{x_n\}$ in X, $x_n$converges to x if and only if $M(x_n, x, t)$ converges to 1 for $t > 0$.
Proof: Fix $t > 0$. Suppose that x_n converges to x. Then for $0 < r < 1$, there exists $n_0 \in \mathbb{N}$ such that

$$x_n \in B(x, r, t) \text{for all } n \geq n_0.$$

It follows that

$$M(x_n, x, t) > 1 - r$$
$$\Rightarrow 1 - M(x_n, x, t) < r.$$

Therefore

$$M(x_n, x, t) \text{ converges to } 1.$$

Conversely, let

$$\lim_{n\to\infty} M(x_n, x, t) = 1 \text{ for each } t > 0.$$

Then for $0 < r < 1$, there exists $n_0 \in \mathbb{N}$such that

$$1 - M(x_n, x, t) < r \text{ for all } n \geq n_0.$$
$$\Rightarrow M(x_n, x, t) > 1 - r \text{ for all } n \geq n_0.$$
$$\Rightarrow x_n \in B(x, r, t) \text{ for all } n \geq n_0.$$

Hence $x_n$converges to x.

Theorem 13.6.5. Let $(X_1, M_1, *)$ and $(X_2, M_2, *)$ be two fuzzy metric spaces. For $(x_1, x_2), (y_1, y_2) \in X_1 \times X_2$, $t > 0$, let us define

$$M: (X_1 \times X_2) \times (X_1 \times X_2) \times (0, \infty) \to [0,1]$$

by

$$M((x_1, x_2), (y_1, y_2), t) = M_1(x_1, y_1, t) * M_2(x_2, y_2, t).$$

Then M is a fuzzy metric on $X_1 \times X_2$.

Proof:

C1. Since $M_1(x_1, y_1, t) > 0$ and $M_2(x_2, y_2, t) > 0$, therefore

$$M_1(x_1, y_1, t) * M_2(x_2, y_2, t) > 0$$
$$\Rightarrow M((x_1, x_2), (y_1, y_2), t) > 0.$$

C2. Suppose for all $t > 0$,

$M((x_1, x_2), (y_1, y_2), t) = 1$
$\Leftrightarrow M_1(x_1, y_1, t) * M_2(x_2, y_2, t) = 1$
$\Leftrightarrow M_1(x_1, y_1, t) = 1$ and $M_2(x_2, y_2, t) = 1$
$\Leftrightarrow x_1 = y_1$ and $x_2 = y_2$(as M_1 and M_2 are fuzzy metrics on X_1 and X_2 respectively)
$\Leftrightarrow (x_1, x_2) = (y_1, y_2)$.

C3. We have

$$M\big((x_1, x_2), (y_1, y_2), t\big) = M_1(x_1, y_1, t) * M_2(x_2, y_2, t)$$
$$= M_1(y_1, x_1, t) * M_2(y_2, x_2, t)$$
$$= M((y_1, y_2), (x_1, x_2), t)$$

C4. Since $(X_1, M_1, *)$ and $(X_2, M_2, *)$ are fuzzy metric spaces, therefore

$$M_1(x_1, z_1, t + s) \geq M_1(x_1, y_1, t) * M_1(y_1, z_1, s) \quad \text{and}$$

$$M_2(x_2, z_2, t + s) \geq M_2(x_2, y_2, t) * M_2(y_2, z_2, s)$$

for all $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in X_1 \times X_2$ and $s, t > 0$.

Now,

$$M\big((x_1, x_2), (z_1, z_2), t + s\big) = M_1(x_1, z_1, t + s) * M_2(x_2, z_2, t + s)$$

$$\geq M_1(x_1, y_1, t) * M_1(y_1, z_1, s) * M_2(x_2, y_2, t) * M_2(y_2, z_2, s)$$

$$= \{M_1(x_1, y_1, t) * M_2(x_2, y_2, t)\} * \{M_1(y_1, z_1, s) * M_2(y_2, z_2, s)\}$$

$$= M\big((x_1, x_2), (y_1, y_2), t\big) * M\big((y_1, y_2), (z_1, z_2), s\big)$$

C5. Since $M_1(x_1, y_1, \cdot): (0, \infty) \to [0,1]$ and $M_2(x_2, y_2, \cdot): (0, \infty) \to [0,1]$ are continuous t −norms, therefore

$$M\big((x_1, x_2), (y_1, y_2), \cdot\big): (0, \infty) \to [0,1]$$

is also continuous t −norm.

Hence $(X_1 \times X_2, M, *)$ is a fuzzy metric space.

Theorem 13.6.6. Let $(X_1, M_1, *)$ and $(X_2, M_2, *)$ be fuzzy metric spaces. Define $M: (X_1 \times X_2) \times (X_1 \times X_2) \times (0, \infty) \to [0,1]$ by

$$M\big((x_1, x_2), (y_1, y_2), t\big) = M_1(x_1, y_1, t) * M_2(x_2, y_2, t)$$

Then $(X_1 \times X_2, M, *)$ is complete fuzzy metric space if and only if both spaces are complete

Proof: Suppose $(X_1, M_1, *)$ and $(X_2, M_2, *)$ are complete fuzzy spaces. Let $\{a_n\}$ be a Cauchy sequence in $X_1 \times X_2$, where

$$a_n = \left(x_1^{(n)}, x_2^{(n)}\right), \quad a_m = \left(x_1^{(m)}, x_2^{(m)}\right)$$

Also,

$$M(a_n, a_m, t) \text{ converges to } 1 \quad \text{for } t > 0$$

$$\Rightarrow M\left(\left(x_1^{(n)}, x_2^{(n)}\right), \left(x_1^{(m)}, x_2^{(m)}\right), t\right) \text{ converges to 1 for t} > 0$$

$$\Rightarrow M_1\left(x_1^{(n)}, x_1^{(m)}, t\right) * M_2\left(x_2^{(n)}, x_2^{(m)}, t\right) \text{ converges to 1 for t} > 0$$

$$\Rightarrow M_1\left(x_1^{(n)}, x_1^{(m)}, t\right) \; converges\; to\; 1 \quad \text{and} \quad M_2\left(x_2^{(n)}, x_2^{(m)}, t\right) \; converges\; to\; 1$$

$\Rightarrow \{x_1^{(n)}\}$ and $\{x_2^{(n)}\}$ are Cauchy sequences in $(X_1, M_1, *)$ and $(X_2, M_2, *)$ respectively.

Since $(X_1, M_1, *)$ and $(X_2, M_2, *)$ are complete fuzzy metric spaces, therefore, there exist $x_1 \in X_1$ and $x_2 \in X_2$ such that
$M_1\left(x_1^{(n)}, x_1, t\right)$ converges to 1 and $M_2\left(x_2^{(n)}, x_2, t\right) \; converges\; to\; 1$ for each $t > 0$.

Let $a = (x_1, x_2) \in X_1 \times X_2$. Then for each $t > 0$. Let $a = (x_1, x_2)$. Then $a \in X_1 \times X_2$. It follows that $M(a_n, a, t)$ converges to 1 for each $t > 0$. This shows that $(X, M, *)$ is complete.

Conversely, suppose that $(X, M, *)$ is complete. We shall show that $(X_1, M_1, *)$ and $(X_2, M_2, *)$ are complete.

Let $\{x_1^{(n)}\}$ and $\{x_2^{(n)}\}$ be Cauchy sequences in $(X_1, M_1, *)$ and $(X_2, M_2, *)$ respectively. Then

$M_1\left(x_1^{(n)}, x_1^{(m)}, t\right)$ converges to 1 and $M_2\left(x_2^{(n)}, x_2^{(m)}, t\right)$ converges to 1 for each $t > 0$.

It follows that

$M\left(\left(x_1^{(n)}, x_2^{(n)}\right), \left(x_1^{(m)}, x_2^{(m)}\right), t\right) = M_1\left(x_1^{(n)}, x_1^{(m)}, t\right) * M_2\left(x_2^{(n)}, x_2^{(m)}, t\right)$ converges to 1.

Let $x^{(n)} = \left(x_1^{(n)}, x_2^{(n)}\right)$ in $X_1 \times X_2$ for $n \geq 1$. Then $\{x^{(n)}\}$ is a Cauchy sequence in X.

Since $(X, M, *)$ is complete, there exists $x \in X_1 \times X_2 = X$ such that

$$M\left(x^{(n)}, x, t\right) \text{ converges to } 1.$$

Since $x \in X_1 \times X_2$, there exist $x_1 \in X_1$ and $x_2 \in X_2$ such that $x = (x_1, x_2)$. Thus,

$$\lim_n M\left(x^{(n)}, x, t\right) = 1$$

$$\Rightarrow \lim_n M\left(\left(x_1^{(n)}, x_2^{(n)}\right), (x_1, x_2), t\right) = 1$$

$$\Rightarrow \lim_n \left[M_1\left(x_1^{(n)}, x_1, t\right) * M_2\left(x_2^{(n)}, x_2, t\right)\right] = 1$$

$$\Rightarrow \lim_n M_1\left(x_1^{(n)}, x_1, t\right) = 1 \quad \text{and} \quad \lim_n M_2\left(x_2^{(n)}, x_2, t\right) = 1$$

Hence,

$$x_1^{(n)} \to x_1 \text{ in } X_1 \quad \text{and} \quad x_2^{(n)} \to x_2 \text{ in } X_2.$$

Hence $(X_1, M_1, *)$ and $(X_2, M_2, *)$ are complete.

Definition 13.6.7 Let $(X, M, *)$ be a fuzzy metric space. A collection of sets $\{F_n\}_{n \in I}$ in X is said to have fuzzy diameter zero if for each $r, t > 0$, $0 < r < 1$, there exists $n \in I$ such that

$$M(x, y, t) > 1 - r \quad \text{for all } x, y \in F_n.$$

Remark: A non-empty subset F of a fuzzy metric space X has fuzzy diameter zero if and only if F is a singleton set, where $F = F_n$ for all $n \geq 1$.

Theorem 13.6.8 A necessary and sufficient condition that a fuzzy metric space $(X, M, *)$ be complete is that every nested sequence of nonempty closed sets $\{F_n\}_{n=1}^{\infty}$ with fuzzy diameter zero has nonempty intersection.

Proof:

First suppose that every nested sequence of nonempty closed sets $\{F_n\}_{n=1}^{\infty}$ with fuzzy diameter zero has nonempty intersection. We claim that $(X, M, *)$ is complete.

Let $\{x_n\}$ be a Cauchy sequence in X. Take

$$A_n = \{x_n, x_{n+1}, x_{n+2}, \ldots\}$$

and

$$F_n = \mathrm{Cl}(\mathrm{A_n}).$$

Then we claim that $\{F_n\}$ has fuzzy diameter zero.

For given $s, t > 0$, $0 < s < 1$, we can find an r such that

$$(1-r) * (1-r) * (1-r) > (1-s).$$

Since $\{x_n\}$ is a Cauchy sequence, for $r, t > 0$, $0 < r < 1$, there exists $n_0 \in \mathbb{N}$ such that

$$M\left(x_m, x_n, \frac{t}{3}\right) > 1-r \quad \text{for all } m, n \geq n_0.$$

Therefore,

$$M\left(x, y, \frac{t}{3}\right) > 1-r \quad \text{for all } x, y \in A_{n_0}.$$

Let $x, y \in F_{n_0}$. Then there exist sequences $\{x'_n\}$ and $\{y'_n\}$ in A_{n_0} such that x'_n converges to x and y'_n converges to y.

Hence,

$$x'_n \in B\left(x, r, \frac{t}{3}\right), \quad y'_n \in B\left(y, r, \frac{t}{3}\right)$$

for sufficiently large n.

Now,

$$M(x, y, t) \geq M\left(x, x'_n, \frac{t}{3}\right) * M\left(x'_n, y'_n, \frac{t}{3}\right) * M\left(y'_n, y, \frac{t}{3}\right)$$

$$> (1-r) * (1-r) * (1-r) > 1-s.$$

Therefore,

$$M(x, y, t) > 1-s \quad \text{for all } x, y \in F_{n_0}.$$

Thus $\{F_n\}$ has fuzzy diameter zero. Hence by hypothesis,

$$\bigcap_{n=1}^{\infty} F_n \neq \emptyset.$$

Take $x \in \bigcap_{n=1}^{\infty} F_n$. Then for $r, t > 0$, $0 < r < 1$, there exists n_1 such that

$$M(x_m, x, t) > 1-r \quad \text{for all } n \geq n_1.$$

Therefore, for each $t > 0$,

$$M(x_n, x, t) \text{ converges to } 1 \quad \text{as } n \to \infty.$$

Hence $\{x_n\}$ converges to x. Therefore $(X, M, *)$ is a complete fuzzy metric space.

Conversely, suppose that $(X, M, *)$ is a complete fuzzy metric space and $\{F_n\}_{n=1}^{\infty}$ is a nested sequence of nonempty closed sets with fuzzy diameter zero.

Let $x_n \in F_n$, $n = 1,2,3, \ldots$.

Since $\{F_n\}$ has fuzzy diameter zero, for $r, t > 0$, $0 < r < 1$, there exists $n_0 \in \mathbb{N}$ such that

$$M(x, y, t) > 1 - r \quad \text{for all } x, y \in F_{n_0}.$$

Therefore,

$$M\left(x_n, x_m, \frac{t}{3}\right) > 1 - r \quad \text{for all } n, m \geq n_0.$$

Since $x_n \in F_n \subseteq F_{n_0}$, $x_m \epsilon F_m \subset F_{n_0}$, $\{x_n\}$ is a Cauchy sequence and $(X, M, *)$ is a complete fuzzy metric space and hence $\{x_n\}$ converges to x for some $x \in X$.

Now, for each fixed n, $x_k \in F_n$ for all $k \geq n$. Therefore, $x \epsilon Cl(F_n) = F_n$ for every n and hence $x \epsilon \bigcap_{n=1}^{\infty} F_n$. This completes our proof.

Remark: The element $x \in \bigcap_{n=1}^{\infty} F_n$ is unique. For if there are two elements , $y \in \bigcap_{n=1}^{\infty} F_n$, then foe each fixed $t > 0$, $M(x, y, t) > 1 - \frac{1}{n}$ for each n because $\{F_n\}_{n=1}^{\infty}$ has fuzzy diameter zero. This implies that $M(x, y, t) = 1$ and hence $x = y$.

CHECK YOUR PROGRESS

***Q* 2.** Let I^x =set of fuzzy points $\lambda_x^{\alpha}(y) = \begin{cases} \alpha & if\ x = y \\ 0 & otherwise \end{cases}$ and $(I^x, M_d, *)$ be induced fuzzy metric space. If the fuzzy sequence $\langle \lambda_{x_n}^{\alpha} \rangle$ in I^x fuzzy converges to λ_x^{α}, then $\langle \lambda_{x_n}^{\alpha} \rangle$ is a fuzzy Cauchy sequence.

***Q* 3.** A sequence $\langle \lambda_{x_n}^{\alpha} \rangle$ in a fuzzy metric space $(I^x, M_d, *)$ is a fuzzy Cauchy sequence if and only if the sequence $< x_n >$ is Cauchy sequence in (X, d), where I^x = set of fuzzy points $\lambda_x^{\alpha}: X \to I$ defined as $\lambda_x^{\alpha}(y) = \begin{cases} \alpha & if\ x = y \\ 0 & otherwise \end{cases}$.

***Q* 4.** The fuzzy metric space $(I^R, M_{|.|}, *)$ is complete, where I^R =set of fuzzy points on R and $M_{|.|}$ is induced fuzzy metric of metric $d: R \times R \to [0, \infty)$ defined as $d(x, y) = |x - y|$.

13.7 LET US SUM UP

- **Continuous Triangular Norm:** A binary operation $*: [0,1] \times [0,1] \to [0,1]$ is a continuous triangular norm if for all $a, b, c, e \in [0,1]$ the following conditions hold:
  (a) $a * b = b * a$
  (b) $a * 1 = a$
  (c) $(a * b) * c = a * (b * c)$
  (d) $a * b \leq c * e$ *whenever* $a \leq c$ *and* $b \leq e$.

- **Fuzzy Metric Space:** The $3-$tuple $(X, M, *)$ is said to be a fuzzy metric space, where X is an arbitrary set, $*$ is continuous $t-$norm and M is a fuzzy set on $X \times X \times (0, \infty)$, satisfying the following conditions:

C1. $\forall$ x, y $\in$ X and $\forall\, t > 0, M(x, y, t) > 0$.
C2. $\forall\, x, y \in X$ and $\forall\, t > 0, M(x, y, t) = 1 \Leftrightarrow x = y$.
C3. $\forall\, x, y \in X$ and $\forall\, t > 0, M(x, y, t) = M(y, x, t)$.
C4. $\forall\, x, y, z \in X$ and $\forall\, t, s > 0$,

$$M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$$

C5. $\forall\, x, y \in X, M(x, y, .) : (0, \infty) \to [0,1]$ is continuous.

- $M(x, y, .) : (0, \infty) \to [0,1]$ is non-decreasing for all $x, y \in X$.
- **Open Ball:** Let $(X, M, *)$ be a fuzzy metric space. We define the open ball $B(x, r, t)$ with centre $x \in X$ and radius $r, 0 < r < 1, t > 0$, as

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}$$

- **Closed Ball:** Let $(X, M, *)$ be fuzzy metric space. We define the closed ball $B[x, r, t]$ with centre x and radius r, $0 < r < 1$, $t > 0$, as

$$\mathrm{B[x, r, t]} = \{\mathrm{y} \in \mathrm{X} : \mathrm{M(x, y, t)} \geq 1 - \mathrm{r}\}$$

- A subset A of a fuzzy metric space $(\mathrm{X}, \mathrm{M}, *)$ is said to be open if given any point $\mathrm{a} \in \mathrm{A}$ there exists $0 < r < 1$ and $t > 0$ such that

$$\mathrm{B(a, r, t)} \subseteq \mathrm{A}$$

- Every open ball in a fuzzy metric space $(\mathrm{X}, \mathrm{M}, *)$ is an open set.
- Every fuzzy metric space is Hausdorff.
- **Fuzzy metrization of the topology induced by fuzzy metric:** Let $(\mathrm{X}, \mathrm{M}, *)$ be a fuzzy metric space. Define $T_M = \{\mathrm{A} \subseteq \mathrm{X} : \mathrm{x} \in \mathrm{A}$ iff there exists $\mathrm{t} > 0\ and\ r, 0 < r < 1\ such\ that\ B(x, r, t) \subseteq A\}$. Then T_M is a topology on X induced by the fuzzy metric M.
- Let (X, d) be a metric space and $M_d(x, y, t) = \frac{t}{t + d(x, y)}$ be corresponding standard fuzzy metric on X. Then the topology T_d induced by the metric d and the topology $T_{\{M_d\}}$ induced by the fuzzy metric M_d are same, $i.e., T_d = T_{\{M_d\}}$.
- **Cauchy sequence:** A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is a Cauchy sequence if for $\varepsilon > 0, t > 0$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geq n_0$.
- **Convergence sequence:** A sequence $\{x_n\}$ in X is convergent to $x \in X$ if $\lim_n M(x_n, x, t) = 1$ for each $t > 0$.
- **Complete fuzzy metric space:** A fuzzy metric space in which every Cauchy sequence is convergent is called complete fuzzy metric space.
- Let $(X, M, *)$ be a fuzzy metric space. Then for a sequence $\{x_n\}$ in X, x_n converges to x iff $M(x_n, x, t)$ converges to 1 for $t > 0$.
- Let $(X_1, M_1, *)$ and $(X_2, M_2, *)$ be fuzzy metric spaces. We define $M : (X_1 \times X_2) \times (X_1 \times X_2) \times (0, \infty) \to [0,1]$ by

$$M((x_1, x_2), (y_1, y_2), t) = M_1(x_1, y_1, t) * M_2(x_2, y_2, t).$$

Then $(X_1 \times X_2, M, *)$ is complete fuzzy metric space if $(X_1, M_1, *)$ and $(X_2, M_2, *)$ are complete fuzzy metric spaces.

- Let $(X, M, *)$ be a fuzzy metric space. A collection of sets $\{F_n\}_{n \epsilon I}$ in X is said to have fuzzy diameter zero if for each $r, t > 0, 0 < r < 1$ there exists $n \in I$ such that

$$M(x, y, t) > 1 - r\ for\ all\ x, y \in F_n.$$

- A necessary sufficient condition that a fuzzy metric space $(X, M, *)$ be complete is that every nested sequence of nonempty closed set $\{F_n\}$ with fuzzy diameter zero has nonempty intersection.

✓ 13.8 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q.1. Suppose that there is a metric d on X that induces fuzzy metric

$$M(x,y,t) = \frac{t}{t + d(x,y)} \text{ for all } t > 0$$

$$\Rightarrow \{t + d(x,y)\}M(x,y,t) = t$$

$$\Rightarrow d(x,y) = \frac{t\{1 - M(x,y,t)\}}{M(x,y,t)}$$

$(m1)$ Assume $x = y$. Then this implies that $M(x,y,t) = 1$.

Therefore $d(x,y) = \frac{t(1-1)}{1} = 0$.

Conversely, assume $d(x,y) = 0$. Then

$$\frac{t\{1 - M(x,y,t)\}}{M(x,y,t)} = 0$$

$$\Rightarrow M(x,y,t) = 1 \; since \; t > 0$$

$$\Rightarrow x = y$$

$(m2)$ We have $d(x,y) = \frac{t\{1-M(x,y,t)\}}{M(x,y,t)}$

$$= t\frac{\{1 - M(y,x,t)\}}{M(y,x,t)}$$

$$= d(x,y)$$

$(m3)$ We now show that the triangular inequality does not hold. Let $t = 2, a = 1, y = 2$ and $z = 3$. Then

$$d(x,y) = \frac{t\{1 - M(x,y,t)\}}{M(x,y,t)}$$

$i.e.$

$$d(1,2) = \frac{2.\{1 - d(1,2,2)\}}{d(1,2,2)}$$

$$= \frac{2\left(1 - \frac{1}{2}\right)}{\frac{1}{2}}$$

$$= 2$$

$$d(1,3) = \frac{2\left(1 - \frac{1}{3}\right)}{\frac{1}{3}} = 4$$

$$d(2,3) = \frac{2\left(1-\frac{2}{3}\right)}{\frac{2}{3}} = 1$$

Therefore $d(1,3) > d(1,2) + d(2,3)$

Thus (X, d) is not a metric space.

Second Part: We now prove that M is not a fuzzy metric with the continuous $t-$norm defined by $a * b = min\{a, b\}$

$C1. M(x, y, t) > 0 \; for \; all \; x, y \in X \; and \; t > 0$

$C2.$ Assume that $x = y$. Then $d(x, y) = 0$ and

$$M(x, y, t) = \frac{t}{t + d(x,y)} = \frac{t}{t+0} = 1$$

Conversely, assume that $M(x, y, t) = 1$

$$\Rightarrow \frac{t}{t+d(x,y)} = 1$$
$$\Rightarrow t + d(x, y) = t$$
$$\Rightarrow d(x, y) = 0$$
$$\Rightarrow x = y$$

$C3.$ It is obvious that $M(x, y, t) = M(y, x, t)$ as $d(x, y) = d(y, x)$

$C4.$ To show the equality $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ does not hold. Choose $t = 2, x = 1, y = 2 \; and \; z = 3.$ Then $M(1,2,2) * M(2,3,2) = \frac{1}{2} * \frac{2}{3} = \min\left\{\frac{1}{2}, \frac{2}{3}\right\} = \frac{1}{2}$

and $M(1,3,2+2) = M(1,3,4) = \frac{1}{3}$

Therefore, $M(1,3,2+2) < M(1,2,2) * M(2,3,2)$

Thus M is not a fuzzy metric.

Ans to Q.2

We have by definition,

$$M_d(\lambda_{x_n}^{\alpha}, \lambda_{x_m}^{\alpha}, t + \varepsilon) \geq M(\lambda_{x_n}^{\alpha}, \lambda_{x}^{\alpha}, t) * M(\lambda_{x}^{\alpha}, \lambda_{x_m}^{\alpha}, \varepsilon) \text{ for each } t > 0 \text{ and } \varepsilon > 0.$$

Also, we have $M_d\left(\lambda_{x_n}^{\alpha}, \lambda_{x}^{\alpha}, t\right) = \frac{t}{t + d(x_n,x)} \to 1$ and $M_d\left(\lambda_{x}^{\alpha}, \lambda_{x_m}^{\alpha}, \varepsilon\right) = \frac{\varepsilon}{\varepsilon + d(x,x_m)} \to 1$

Thus $M_d(\lambda_{x_n}^{\alpha}, \lambda_{x_m}^{\alpha}, t + \varepsilon) \to 1$ and ε is arbitrary small

$\Rightarrow < x_n >$ is a fuzzy Cauchy sequence.

Ans to Q.3

Suppose that $< x_n >$ is a Cauchy sequence, so by definition,

$$\lim_{n,m\to\infty} M(\lambda_{x_n}^{\alpha}, \lambda_{x_m}^{\alpha}, t) = 1 \text{ for } t > 0$$

$\Leftrightarrow$ for each small real number $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $1 - M\left(\lambda_{x_n}^{\alpha}, \lambda_{x_m}^{\alpha}, t\right) < \frac{\varepsilon}{\varepsilon + t} \; for \; all \; n, m > k$

$\Leftrightarrow M(\lambda_{x_n}^{\alpha}, \lambda_{x_m}^{\alpha}, t) > 1 - \frac{\varepsilon}{\varepsilon + t}$ $for\ all\ n, m > k$

$\Leftrightarrow \frac{t}{t + d(x_n, x_m)} > 1 - \frac{\varepsilon}{\varepsilon + t}$ $for\ all\ n, m > k$

$\Leftrightarrow d(x_n, x_m) < \epsilon$ for all $n, m > k$

$\Leftrightarrow < x_n >$ is a Cauchy sequence in (X, d)

Ans to Q.4

Left as an exercise.

13.9 FURTHER READING

1. Alias B. Khakfa and Maha Waleed, On a Fuzzy Metric space and Fuzzy Convergence, Proyecciones Journal of Mathematics 40(5), 1279–1291, 2021.
2. Maggie Aphane, On some Results of Analysis in Metric Spaces and Fuzzy Metric Spaces, A Dissertation for the degree of Master of Science in Mathematics, University of South Africa, 2000.
3. A. George and P. Veeramani, On some results in fuzzy metric spaces, Fuzzy Sets and Systems, 64, 395–399, 1994.

13.10 MODEL QUESTIONS

***Q* 1**. Define a fuzzy metric space $(X, M, *)$ and explain the significance of t –norm $*$.

***Q* 2**. Prove that if $(X, M, *)$ is a fuzzy metric space, then for any $x, y \in X$, $M(x, y, t)$ is a non-decreasing function for all $t > 0$.

***Q* 3**. Give an example of a fuzzy metric space induced by a standard metric space (X, d).

***Q* 4**. Consider the fuzzy metric space $(\mathbb{R}, M, *)$ where $a * b = \min\{a, b\}$, $M(x, y, t) = \frac{1}{t + |x - y|}$ for $t > 0$ and $M(x, y, 0) = 0$. Show that this space is complete.

***Q* 5**. Let $(X, M, *)$ be a fuzzy metric space. Define the topology T_M induced by the fuzzy metric M on X. Prove that for any $x \in X$ and any $t \in (0,1)$, the set

$B(x, r, t) = \{ y \in X \mid M(x, y, t) > r$ for some $r \in (0,1)$ forms a basis for T_M.

***Q* 6**. Consider a fuzzy metric space $(X, M, *)$ where $X = \mathbb{R}$, $M(x, y, t) = \frac{t}{t + |x - y|}$ for all $x, y \in \mathbb{R}$ and $t > 0$ and $*$ is the product t –norm $(i.e., a * b = ab)$. Let T_M be the topology induced by the fuzzy metric M. Show that a sequence $< x_n >$ in X converges

to $x \in X$ with respect to T_M if and only if $< x_n >$ converges to x in the usual Euclidean topology on $\mathbb{R}$.

***Q* 7. .** Let $(X, M, *)$ be a fuzzy metric space. Prove that the fuzzy metric M is continuous with respect to product topology on $X \times X \times (0, \infty)$ and usual topology on $[0,1]$, where the product topology on $X \times X \times (0, \infty)$ is generated by the topology T_M on X and the usual topology on $(0, \infty)$.

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UNIT 14: FUZZY TOPOLOGICAL SPACE

UNIT STRUCTURE

14.1 Learning Objectives
14.2 Introduction
14.3 Fuzzy Topological Space: Definitions and Examples
14.4 Closure and Interior of Fuzzy Sets
14.5 Boundary of Fuzzy Sets
14.6 Base and Subbase for Fuzzy Topological Space
14.7 Fuzzy Points
14.8 Neighbourhood
14.9 Let Us Sum Up
14.10 Answers to Check Your Progress
14.11 Further Reading
14.12 Model Questions

14.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- define fuzzy topological space.
- describe fuzzy topological spaces and understand how they generalize classical topological spaces.
- identify fuzzy open sets, fuzzy closed sets, neighbourhood, closure, interior, boundary of fuzzy sets.
- understand fuzzy bases and subbases in fuzzy topological spaces.
- define fuzzy points.

14.2 INTRODUCTION

Fuzzy topological spaces are a generalization of classical topological spaces that incorporate the concept of fuzziness introduced by Lotfi A. Zadeh in 1965 through fuzzy set theory. This extension allow us to deal with uncertainty and partial membership, which classical topology cannot adequately handle.

In classical topology, a set either belongs to an open set or not- there is no in-between. However, many real-world problems require a more nuanced treatment. For example, consider the concept of “tall people”. There is no sharp boundary where a person becomes “tall”, rather there is gradual transition. Fuzzy topology is ideal for modeling such gradual or imprecise concepts.

Fuzzy topological spaces have found applications in Artificial Intelligence, Decision-Making, Pattern recognition, Control systems and Geographic information (GIS).

Fuzzy topological spaces provide a flexible mathematical framework for modeling uncertainty and vagueness, we gain tools that are more aligned with the complexities of real world scenarios.

14.3 FUZZY TOPOLOGICAL SPACE: DEFINITIONS AND EXAMPLES

Definition 14.3.1 On a set X, a fuzzy topology is a family $T = \{\mu: \mu$ is a fuzzy set in $X\}$ of subsets (i.e. $T \subseteq I^X$) that satisfies the following axioms:

(i) $\bar{0}, \bar{1} \in T$

(ii) $\mu_1, \mu_2 \in T \Rightarrow \mu_1 \cap \mu_2 \in T$

(iii) If $\{\mu_i: i \in I, I$ is an index set$\} \subseteq T$, then

$$\bigcup_{i \in I} \mu_i \in T$$

T is described as a fuzzy topology for X and the pair (X, T) is named as a fuzzy topological space or in short fts. The members of T are defined as T −open fuzzy set. If the complement of μ, denoted by μ^c, is T −open, then an element $\mu \in I^X$ is said to be a closed fuzzy set.

Example 1: Let $X = \{a, b\}$. Let μ be a fuzzy set on X defined as $\mu(a) = 0.5, \mu(b) = 0.4$. Then $T = \{\bar{0}, \mu, \bar{1}\}$ is a fuzzy topology on X, where $\bar{0}(a) = 0 \ \forall \ a \in X$ and $\bar{1}(a) = 1 \ \forall \ a \in X$.

Example 2: Let X be a non-empty set. Then $I = \{\bar{0}, \bar{1}\}$ is a fuzzy topology on X, i.e. (X, I) is a fuzzy topological space. The fuzzy topological space (X, I) is called indiscrete fuzzy topological space.

Example 3: Let $X = \{x, y, z\}$ and $\alpha = (0.8, 0.9, 0.7), \beta = (0.6, 0.5, 0.4), \delta = (0.3, 0.3, 0.2)$ be fuzzy sets on X. Let $T = \{\bar{0}, \alpha, \beta, \delta, \bar{1}\}$. Then

$$\alpha \cap \beta = (0.8, 0.9, 0.7) \cap (0.6, 0.5, 0.4) = (0.6, 0.5, 0.4) \in T$$

Similarly,

$$\alpha \cap \delta, \beta \cap \delta \in T$$

Also

$$\alpha \cup \beta = (0.8, 0.9, 0.7) \cup (0.6, 0.5, 0.4) = (0.8, 0.9, 0.7) \in T$$

Similarly,

$$\alpha \cup \delta, \beta \cup \delta \in T$$

And

$$\alpha \cup \beta \cup \delta = (0.8, 0.9, 0.7) \cup (0.6, 0.5, 0.4) \cup (0.3, 0.3, 0.2)$$
$$= (0.8, 0.9, 0.7)$$

$$= \alpha \in T$$

Also, we have $\bar{0}, \bar{1} \in T$

Therefore (X, T) is a topological space.

CHECK YOUR PROGRESS

Q 1. Consider X be an abstract set equipped with the family T of fuzzy sets, where

$$T = \left\{\mu : \mu \geq \frac{1}{2}\ \forall x \in X\right\} \cup \left\{\mu' : \mu' is\ constant < \frac{1}{2}\right\}$$

Show that (X, T) is a topological space.

14.4 CLOSURE AND INTERIOR OF FUZZY SETS

Definition 14.4.1 (Closure of fuzzy set)

Let (X, T) be a fuzzy topological space. Let μ be a fuzzy set on X. Then closure of μ is defined as the smallest closed fuzzy set that containing μ, i.e.

$$cl(u) = \inf\{\alpha : \mu \leq \alpha, \alpha^c \in T\}$$

Obviously μ is always T −closed.

Example 1: Let A, B, C be fuzzy sets on I defined as

$$A(x) = \begin{cases} 0, & if\ \ 0 \leq x \leq \frac{1}{2} \\ 2x - 1, & if\ \frac{1}{2} \leq x \leq 1 \end{cases}$$

$$B(x) = \begin{cases} 1, & if\ 0 \leq x \leq \frac{1}{4} \\ -4x + 2, & if\ \frac{1}{4} \leq x \leq \frac{1}{2} \\ 0, & if\ \frac{1}{2} \leq x \leq 1 \end{cases}$$

and

$$C(x) = \begin{cases} 0, & if\ \ 0 \leq x \leq \frac{1}{4} \\ \frac{4x - 1}{3}, & if\ \frac{1}{4} \leq x \leq 1 \end{cases}$$

Then $T = \{\bar{0}, A, B, A \cup B, \bar{1}\}$ is a fuzzy topology on I. Then $Cl(A) = B^c$, $Cl(B) = A^c$, $Cl(A \cup B) = \bar{1}$.

Remark: A fuzzy set μ is closed⇔ $Cl(\mu) = \mu$.

Theorem 14.4.2 Let (X, T) be a fuzzy topological space. Then for any two fuzzy sets μ, λ on X,

(i) $\mu \leq Cl(\mu)$

(ii) $Cl\big(Cl(\mu)\big) = Cl(\mu)$

(iii) $Cl(\mu \cup \lambda) = Cl(\mu) \cup Cl(\lambda)$

(iv) $Cl(\bar{0}) = \bar{0}$

(v) $\mu \leq \lambda \Rightarrow Cl(\mu) \leq Cl(\lambda)$

Proof: (i) From the definition of closure of a set μ,

$$Cl(\mu) = \inf\{\alpha : \alpha \; is \; closed \; and \; \alpha \geq \mu\}$$

Thus $\mu \leq Cl(\mu)$.

(ii) Since $Cl\big(Cl(\mu)\big)$ is the small closed set containing $\bar{\mu}$ and $Cl(\mu)$ is itself a closed set, therefore

$$Cl\big(Cl(\mu)\big) = Cl(\mu)$$

(iii) Since union of two closed sets is closed, therefore $Cl(\mu) \cup Cl(\lambda)$ is closed and

$$Cl(\mu) \cup Cl(\lambda) \geq \mu \cup \lambda$$
$$\Rightarrow Cl\big(Cl(\mu) \cup Cl(\lambda)\big) \geq Cl(\mu \cup \lambda)$$
$$\Rightarrow Cl(\mu) \cup Cl(\lambda) \geq Cl(\mu \cup \lambda) \qquad (1)$$

Again

$$\mu \cup \lambda \geq \mu$$
$$\Rightarrow Cl(\mu \cup \lambda) \geq Cl(\mu) \qquad (2)$$

Similarly

$$Cl(\mu \cup \lambda) \geq Cl(\lambda) \qquad (3)$$

From (2) and (3), we get

$$Cl(\mu \cup \lambda) \geq Cl(\mu) \cup Cl(\lambda) \qquad (4)$$

From (1) and (4), we obtain

$$Cl(\mu \cup \lambda) = Cl(\mu) \cup Cl(\lambda)$$

(iv) Since $Cl(\bar{1}) \in T$, i.e. $\bar{1}$ is T –open, therefore its complement, $\bar{1}^c = \bar{0}$ is closed. So we can write

$$Cl(\bar{0}) = \bar{0}$$

(v) Proof is obvious.

Definition 14.4.3 The fuzzy topology generated by closure operator is denoted by (X, T_X), where

$$T_X = \{\mu \in [0,1]^X : Cl(\bar{1} - \mu) = \bar{1} - \mu\}$$

Definition 14.4.4 (Interior of Fuzzy Set)

Let (X, T) be a fuzzy topological space. The interior μ^0 of a fuzzy set μ on X is defined as

$$\mu^0 = \sup\{\alpha : \alpha \leq \mu, \alpha \in T\}$$

Clearly, μ^0 is T-open.

Remark A fuzzy set μ on X is open $\Leftrightarrow \mu^0 = \mu$

Example 2: From Example 1, we see that

$$(A^c)^0 = B, \qquad (B^c)^0 = A \ \ and \ \ [(A \cup B)^c]^0 = \bar{0}$$

Theorem 14.4.5 Assume (X, T) be a fuzzy topological space. Then

(i) $\bar{0}^0 = \bar{0}, \bar{1}^0 = \bar{1}$

(ii) $\mu^0 \leq \mu$

(iii) $(\mu^0)^0 = \mu^0$

(iv) $(\mu \cap \lambda)^0 = \mu^0 \cap \lambda^0$ for any fuzzy sets μ and λ on X.

(v) $\mu \leq \lambda \Rightarrow \mu^0 \leq \lambda^0$

Proof: (i) Since $\bar{0}, \bar{1} \in T$, i.e. $\bar{0}$ and $\bar{1}$ are T-open set, therefore $\bar{0}^0 = \bar{0} \ \ and \ \ \bar{1}^0 = \bar{1}$

(ii) From the definition of interior of a fuzzy set μ,

$$\mu^0 = \sup\{\alpha : \alpha \leq \mu, \alpha \in T\}$$

Therefore

$$\mu^0 \leq \mu$$

(iii) Since μ^0 itself open and $(\mu^0)^0$ is the greatest open set contained in μ^0, therefore

$$(\mu^0)^0 = \mu^0$$

(iv) We have

$$\mu \cap \lambda \leq \mu \ \text{ and } \ \mu \cap \lambda \leq \lambda$$
$$\Rightarrow (\mu \cap \lambda)^0 \leq \mu^0 \ \text{ and } \ (\mu \cap \lambda)^0 \leq \lambda^0$$
$$\Rightarrow (\mu \cap \lambda)^0 \leq \mu^0 \cap \lambda^0 \qquad (1)$$

On the other hand,

$$\mu^0 \leq \mu \ \text{ and } \ \lambda^0 \leq \lambda$$
$$\Rightarrow \mu^0 \cap \lambda^0 \leq \mu \cap \lambda \ \text{ and } \ \mu^0 \cap \lambda^0 \ \text{ is open}$$

Since $(\mu \cap \lambda)^0$ is the largest open set contain in $\mu \cap \lambda$, thus

$$\mu^0 \cap \lambda^0 \subseteq (\mu \cap \lambda)^0 \qquad (2)$$

From (1) and (2),

$$(\mu \cap \lambda)^0 = \mu^0 \cap \lambda^0$$

(v) Proof is obvious.

CHECK YOUR PROGRESS

Q 2. If α and β are two fuzzy sets in a fuzzy topological space (X, T), then

$$(\alpha \cup \beta)^0 \geq \alpha^0 \cap \beta^0$$

Q 3. For fuzzy set μ and λ in a fts (X, T),

$$Cl(\mu \cap \lambda) \leq Cl(\mu) \cap Cl(\lambda)$$

14.5 BOUNDARY OF FUZZY SETS

Definition 14.5.1 (Boundary of fuzzy sets)
Let λ be a fuzzy set in a fts (X, T). Then the fuzzy boundary of λ is defined as

$$Bd(\lambda) = Cl(\lambda) \cap Cl(\lambda^c)$$

Obviously, $Bd(\lambda)$ is a fuzzy closed set.
Remark In classical topology, for an arbitrary set A of a topological space X, we have

$$A \cup Bd(A) = Cl(A)$$

but in fuzzy topology, we have $\lambda \cup Bd(\lambda) \leq Cl(\lambda)$, for any arbitrary fuzzy set λ in X.

Example 1: Suppose $X = \{a, b\}$ and let $\mu = \{(a, 0.3), (b, 0.6)\}$ be a fuzzy set on X, then $T = \{\bar{0}, \bar{1}, A\}$ is a fuzzy topology on X.
Let $\lambda = \{(a, 0.6), (b, 0.7)\}$ be fuzzy set on X.
Then
$Cl(\lambda) = \bar{1}$ and $Cl(\lambda^c) = \mu^c$
Now

$$Bd(\lambda) = Cl(\lambda) \cap Cl(\lambda^c) = \mu^c$$

and

$$\lambda \cup Bd(\lambda) = \{(a, 0.6), (b, 0.7)\} \cup \mu^c$$
$$= \{(a, 0.7), (b, 0.7)\}$$

Hence

$$Cl(\lambda) \neq \lambda \cup Bd(\lambda)$$

Theorem 14.5.2 For fuzzy sets λ and μ in a fuzzy topological space (X, T), the following conditions hold:
(i) $Bd(\lambda) = Bd(\lambda^c)$
(ii) If λ is a T-closed, then $Bd(\lambda) = \lambda^c$.
(iii) If λ is a T-open, then $Bd(\lambda) = \lambda^c$.
(iv) Let $\lambda \leq \mu$, if μ is T-closed, then $Bd(\lambda) \leq \mu$.
(v) $\big(Bd(\lambda)\big)^c = \lambda^0 \cup (\lambda^c)^0$.

Proof: (i)

$$Bd(\lambda) = Cl(\lambda) \cap Cl(\lambda^c)$$
$$= Cl(\lambda^c) \cap Cl(\lambda)$$
$$= Cl(\lambda^c) \cap Cl((\lambda^c)^c)$$
$$= Bd(\lambda^c)$$

(ii) Let λ is a T-closed. Then $Cl(\lambda) = \lambda$.
We have to prove $Bd(\lambda) \leq \lambda$

$$\begin{aligned} Bd(\lambda) &= Cl(\lambda) \cap Cl(\lambda^c) \\ &\leq Cl(\lambda) \\ &= \lambda \quad [\because \ Cl(\lambda) = \lambda] \end{aligned}$$

Hence

$$Bd(\lambda) \leq \lambda$$

(iii) Let λ be a T-open. Then λ^c is T-closed$\Rightarrow Cl(\lambda^c) = \lambda^c$.
We have to prove $Bd(\lambda) \leq \lambda^c$.

$$\begin{aligned} Bd(\lambda) &= Cl(\lambda) \cap Cl(\lambda^c) \\ &\leq Cl(\lambda^c) \\ &= \lambda^c \quad [\because \ Cl(\lambda^c) = \lambda^c] \end{aligned}$$

Hence

$$Bd(\lambda) \leq \lambda^c$$

(iv) Let $\lambda \leq \mu$. Then $Cl(\lambda) \leq Cl(\mu)$. Let μ be a T-closed set. Then $Cl(\mu) = \mu$. We have to prove $Bd(\lambda) \leq \mu$.

$$\begin{aligned} Bd(\lambda) &= Cl(\lambda) \cap Cl(\lambda^c) \\ &\leq Cl(\mu) \cap Cl(\lambda^c) \quad [\because \ Cl(\lambda \leq Cl(\mu)] \\ &\leq Cl(\mu) = \mu \quad [\because \ Cl(\mu) = \mu] \end{aligned}$$

Hence

$$Bd(\lambda) \leq \mu.$$

(v)

$$\begin{aligned} \big(Bd(\lambda)\big)^c &= [Cl(\lambda) \cap Cl(\lambda^c)]^c \\ &= \big(Cl(\lambda)\big)^c \cup \big(Cl(\lambda^c)\big)^c \\ &= \lambda^0 \cup (\lambda^c)^0 \end{aligned}$$

Note: The converse of (ii) and (iii) of Theorem 14.5.2 is in general not true as is shown by the following example.

Example 2: Let $X = \{a, b, c\}$ be a set and T, the fuzzy topology generated by fuzzy sets $\mu = \{(a, .4), (b, .7), (c, .2)\}, \nu = \{(a, .6), (b, .9), (c, .1)\}$ and $\omega = \{(a, .5), (b, .7), (c, .3)\}$.
Then
T
$= \{\bar{0}, \mu, \nu, \omega, \{(a, .4), (b, .8), (c, .1)\}, \{(a, .4), (b, .7), (c, .2)\}, \{(a, .5), (b, .7), (c, .1)\}, \{(a, .4), (b, .7), (c, .1)\},$
$\{(a, .5), (b, .8), (c, .3)\}, \{(a, .5), (b, .8), (c, .1)\}, \{(a, .6), (b, .9), (c, .2)\}, \{(a, .5), (b, .8), (c, .2)\},$
$\{(a, .6), (b, .9), (c, .3)\}, \{(a, .5), (b, .7), (c, .2)\}, \bar{1}\}$

Choose

$$\lambda = \{(a, .5), (b, .7), (c, .9)\}$$

and

$$\psi = \{(a, .5), (b, .3), (c, .3)\}$$

Then
(i) $Bd(\lambda) = \{(a, .5), (b, .3), (c, .7)\} \leq \lambda$, but λ is not T-closed
(ii) $Bd(\psi) = \{(a, .5), (b, .3), (c, .7)\} \leq \psi^c$, but ψ is not T-open.

Theorem 14.5.3 Let λ be a fuzzy set in a fts (X, T). Then

(i) $Bd(\lambda) = Cl(\lambda) = Cl(\lambda) - \lambda^0$
(ii) $Bd(\lambda^0) \leq Bd(\lambda)$
(iii) $Bd\big(Cl(\lambda)\big) \leq Bd(\lambda)$
(iv) $\lambda^0 \leq \lambda - Bd(\lambda)$

Proof: (i) Since $[Cl(\lambda^c)]^c = [(\lambda^c)^c]^0 = \lambda^0$
Therefore

$$\begin{aligned} Bd(\lambda) &= Cl(\lambda) \cap Cl(\lambda^c) \\ &= Cl(\lambda) - [Cl(\lambda^c)]^c \\ &= Cl(\lambda) - \lambda^0 \end{aligned}$$

(ii)

$$\begin{aligned} Bd(\lambda^0) &= Cl(\lambda^0) - (\lambda^0)^0 \\ &= Cl(\lambda^0) - \lambda^0 \\ &= Cl(\lambda^0) \cap (\lambda^0)^c \\ &= Cl(\lambda^0) \cap \big(Cl(\lambda^c)\big) \\ &\leq Cl(\lambda) \cap \big(Cl(\lambda^c)\big) \\ &= Bd(\lambda) \end{aligned}$$

(iii)

$$\begin{aligned} Bd\big(Cl(\lambda)\big) &= Cl\big(Cl(\lambda)\big) \cap Cl\big(Cl(\lambda)\big)^c \\ &= Cl\big(Cl(\lambda)\big) - \Big[Cl\big(Cl(\lambda)\big)^c\Big]^c \\ &= Cl\big(Cl(\lambda)\big) - [Cl(\lambda)]^0 \\ &= Cl(\lambda) - [Cl(\lambda)]^0 \\ &= Cl(\lambda) - \lambda^0 \\ &= Bd(\lambda) \end{aligned}$$

(iv)

$$\begin{aligned} \lambda - Bd(\lambda) &= \lambda \cap [Bd(\lambda)]^c \\ &= \lambda \cap [Cl(\lambda) \cap Cl(\lambda^c)]^c \\ &= \lambda \cap \Big[\big(Cl(\lambda)\big)^c \cup \big(Cl(\lambda^c)\big)^c\Big] \\ &= \lambda \cap [(\lambda^c)^0 \cup \lambda^0] \\ &= [\lambda \cap (\lambda^c)^0] \cup [\lambda \cap \lambda^0] \\ &= [\lambda \cap (\lambda^c)^0] \cup \lambda^0 \\ &\geq \lambda^0 \\ \therefore \quad \lambda^0 &\leq \lambda - Bd(\lambda) \end{aligned}$$

Theorem 14.5.4 Let λ and μ be fuzzy sets in fts (X, T). Then $Bd(\lambda \cup \mu) \leq Bd(\lambda) \cup Bd(\mu)$.
Proof:

$$\begin{aligned} Bd(\lambda \cup \mu) &= Cl(\lambda \cup \mu) \cap Cl(\lambda \cup \mu)^c \\ &\leq [Cl(\lambda) \cup Cl(\mu)] \cap Cl(\lambda^c \cap \mu^c) \\ &= [Cl(\lambda) \cup Cl(\mu)] \cap [Cl(\lambda^c) \cap Cl(\mu^c)] \\ &= Cl(\lambda) \cap [Cl(\lambda^c) \cap Cl(\mu^c)] \cup Cl(\mu) \cap [Cl(\lambda^c) \cap Cl(\mu^c)] \end{aligned}$$

$$= \left[[Cl(\lambda^c) \cap Cl(\mu^c)] \cap Cl(\mu^c)\right] \cup \left[[Cl(\lambda^c) \cap Cl(\mu^c)] \cap Cl(\lambda^c)\right]$$
$$= [Bd(\lambda) \cap Cl(\mu^c)] \cup [Bd(\mu) \cap Cl(\lambda^c)]$$
$$\leq Bd(\lambda) \cup Bd(\mu)$$

Note: In Theorem 13.5.4, the equality does not hold. In example 2, if we choose $\alpha = \{(a,0),(b,0),(c,.2)\}$ and $\beta = \{(a,.4),(b,.8),(c,.1)\}$, then

$$Bd(\alpha) \cup Bd(\beta) = \{(a,.6),(b,.2),(c,.9)\} \neq \{(A,.6),(b,.2),(c,.8)\} = Bd(\alpha \cup \beta)$$

CHECK YOUR PROGRESS

Q 4. For any fuzzy sets λ and μ in fts (X,T), prove that

$$Bd(\lambda \cap \mu) \leq [Bd(\lambda \cap Cl(\mu)] \vee [Bd(\mu) \cap Cl(\lambda)]$$

Q 5. For any fuzzy set λ in fts (X,T), prove that $Bd\big(Bd(\lambda)\big) \leq Bd(\lambda)$

14.6 BASE AND SUBBASE OF FUZZY TOPOLOGICAL SPACE

Definition 14.6.1 (Base) Let (X,T) be a fuzzy topological space and B be a subfamily of T. Then B is called a base of T if every member of T can be represent as supremum of members of B.

Example 1 Let $X = \{x,y\}$ and $\alpha,\beta,\nu \in I^X$ such that $\alpha = \{(x,.3),(y,.6)\}, \beta = \{(x,.7),(y,.5)\}, \nu = \{(x,.7),(y,.6)\}$. Then $T = \{\bar{0},\bar{1},\alpha,\beta,\nu\}$ is a fuzzy topology on X. Let $B = \{\bar{0},\alpha,\beta,\bar{1}\}$. Then $\bar{0},\alpha,\beta,\bar{1} \in T$ and

$$\nu = \{(x,.7),(y,.6)\}$$
$$= \{(x,.3),(y,.6)\} \cup \{(x,.7),(y,.5)\}$$
$$= \alpha \cup \beta \in T$$

Hence B is a base for T.

Definition 14.6.2 (Subbase) Suppose (X,T) is a fuzzy topological space and S is a non-empty subset of T. Then S is called a subbase of T if the finite intersection of members of S forms a base for T.

Example 2 Assume $X = \{x,y\}$ and $\alpha,\beta,\nu,\delta \in I^X$, where $\alpha = \{(x,.2),(y,.6)\}, \beta = \{(x,.3),(y,.5)\}, \gamma = \{(x,.2),(y,.5)\}$ and $\delta = \{(x,.3),(y,.6)\}$. Then $T = \{\bar{0},\bar{1},\alpha,\beta,\nu,\delta\}$ form a topological space on X. Also $S = \{\alpha,\beta\}$ is a subbase and $B = \{\bar{0},\bar{1},\alpha,\beta,\gamma\}$ is a base for T.

14.7 FUZZY POINT

Definition 14.7.1 (Fuzzy Subspace) Let (X,T) be a fts. Suppose A is a subset of X. Then (A,T_A) is called a fuzzy subspace of (X,T), where $T_A=\{B_A: B\in T\}, B=\{(x,\mu_B(x): x\in X\}$ and $B_A=\{\big(x,\mu_{B|A}(x): x\in A\big\}$.

Definition 14.7.2 A fuzzy point P in X is a special fuzzy set with membership function defined by

$$P(x)=\begin{cases}\lambda, & if\ \ x=y\\ 0, & if\ \ x\neq y\end{cases}$$

where $0<\lambda\leq 1$. P is said to have support y, value λ and is denoted by P_y^λ or $P(y,\lambda)$.
Let μ be a fuzzy set in X. Then $P_y^\alpha<\mu\Leftrightarrow\alpha\leq\mu(y)$. In particular, $P_y^\alpha<P_z^\beta\Leftrightarrow y=z,\alpha\leq\beta$. The complement of the fuzzy point P_x^λ is denoted by either $P_x^{1-\lambda}$ or $\big(P_x^\lambda\big)^c$.

Definition 14.7.3 The fuzzy point P_x^λ is said to be contained in a fuzzy set μ, denoted by

$$P_x^\lambda\in\mu\Leftrightarrow\lambda<\mu(x)$$

Every fuzzy set μ can be expressed as the union of all fuzzy points which belong to μ, i.e. if $\mu(x)$ is not zero for $x\in X$, then

$$\mu(x)=\sup\{\lambda: P_x^\lambda<\lambda\leq\mu(x)\}$$

Definition 14.7.4 Two fuzzy sets μ,λ in X are said to be intersecting if and only if there exists a point $x\in X$ such that $(\mu\cap\lambda)(x)\neq 0$. For such a case, we say that μ and λ intersect at x.
Let $\mu,\lambda\in I^X$. Then $\mu=\lambda$ if and only if $P\in\mu\Leftrightarrow P\in\lambda$ for every fuzzy point P in X.
Theorem 14.7.5 Let $\{\mu_j: j\in J\}$ be a family of fuzzy sets in X, P_x^a and P_y^b be fuzzy points in X. Then we have the following:
(i) $P_x^a\in\cup\{\mu_j: j\in J\}\Leftrightarrow$ there exists $j\in J$ such that $P_x^a\in\mu_j$
(ii) If $P_x^a\in\cap\{\mu_j: j\in J\}$, then for every $j\in J$, we have $P_x^a\in\mu_j$
(iii) $P_x^a\in P_y^b\Leftrightarrow x=y$ and $a\leq b$
(iv) If $P_x^a\in P_y^b$ and for $j\in J, P_y^b\in\mu_j$, then $P_x^a\in\cap\{\mu_j: j\in J\}$

Proof: (i) Let $P_x^a\in\cap\{\mu_j: j\in J\}$. Then

$$a\leq\left(\bigcup_{j\in J}\mu_j\right)(x)$$
$$\Rightarrow a\leq\mu_j(x)\ \text{ for some }\ j\in J$$
$$\Rightarrow P_x^a\in\mu_j\ \text{ for some }\ j\in J$$

Conversely, let $P_x^a\in\mu_j$ for some j

$$\Rightarrow a\leq\mu_j(x)$$
$$\Rightarrow a\leq\max\{\mu_j(x): j\in J\}$$
$$\Rightarrow a\leq\left(\bigcup_{j\in J}\mu_j\right)(x)$$

$$\Rightarrow P_x^a \leq \bigcup\{\mu_j : j \in J\}$$

(ii) Let $P_x^a \in \cap\{\mu_j : j \in J\}$. Then

$$a \leq \left(\bigcap_{j\in J} \mu_j\right)(x)$$

$$\Rightarrow a \leq \mu_j(x) \text{ for every } j \in J$$

$$\Rightarrow P_x^a \in \mu_j \text{ for every } j \in J$$

Conversely,

$P_x^a \in \mu_j$ for every $j \in J$

$\Rightarrow a \leq \mu_j(x)$ for every $j \in J$

$$\Rightarrow a \leq \min\{\mu_j(x): j \in J\}$$

$$\Rightarrow a \leq \left(\bigcap_{j\in J} \mu_j\right)(x)$$

$$\Rightarrow P_x^a \in \bigcap\{\mu_j : j \in J\}$$

(iii) We have $P_x^a \in P_y^b \Leftrightarrow a \leq P_y^b(x) = \begin{cases} b, & if \ x = y \\ 0, & if \ x \neq y \end{cases}$

If $x \neq y$, then $P_y^b(x) = 0 \geq a$, which is a contradiction because $a > 0$.

Thus $P_x^a \in P_y^b \Leftrightarrow x = y$ and $a \leq b$.

(iv) Let $P_x^a \in P_y^b \in \mu_j$ for every j.

$$\Rightarrow a \leq \mu_j(x) \text{ for every } j$$

$$\Rightarrow a \leq \min\{\mu_j(x): j \in J\}$$

$$\Rightarrow a \leq \left(\bigcap_{j} \mu_j\right)(x)$$

$$\Rightarrow P_x^a \in \bigcap\{\mu_j : j \in J\}$$

Theorem 14.7.6 A subfamily B of T is a base for the fts (X, T) if and only if for all $\mu \in T$ and for every fuzzy point P in μ, $\exists\ \lambda \in B$ such that $P \in \lambda \leq \mu$.

Proof: Assume that B is a base for T, i.e. for every $\mu \in T$ is a union of members of B. Let $\mu \in T$ and $P_x^\alpha \in \mu$.

$$\mu \in T \Rightarrow \mu = \bigcup_{i\in I}\{\mu_i : \mu_i \in B\}$$

and

$$P_x^\alpha \in \mu \Rightarrow P_x^\alpha \in \bigcup_{i\in I}\{\mu_i : \mu_i \in B\}$$

$$\Rightarrow P_x^\alpha \in \mu_i \leq \mu \text{ for some } i$$

Conversely, assume that for each $\mu \in T$ and for each $P_x^\alpha \in \mu$, $\exists\ \mu_i$ such that

$$P_x^\alpha \in \mu_i \leq \mu$$

Let $\mu \in T$. We have to prove μ can be written as a union of members of B.

Consider any arbitrary $P_x^\alpha \in \mu$. By hypothesis there exists $\mu_x \in B$ such that

$$P_x^\alpha \in \mu_x \leq \mu$$

Thus

$$\bigcup_{P_x^\alpha \in \mu} \mu_x \leq \mu \qquad (1)$$

Since x is arbitrary, therefore

$$\mu \subseteq \bigcup_{P_x^\alpha \in \mu} \mu_x \qquad (2)$$

From (1) and (2), we get

$$\mu = \bigcup_{P_x^\alpha \in \mu} \mu_x$$

14.8 NEIGHBOURHOOD

Definition 14.8.1 A fuzzy point P_x^λ is said to be quasi-coincident with μ, denoted by P_{xqu}^λ if $\lambda > \mu^c(x)$ or $\lambda + \mu(x) > 1$.

Definition 14.8.2 A fuzzy set μ in (X, T) is called a neighbourhood of fuzzy point P_x^λ if and only if there exists $\beta \in T$ such that $P_x^\lambda \in \beta \leq \mu$. A neighbourhood μ is said to be open if and only if μ is open. The family consisting of all the neighbourhoods of P_x^λ is called the system of neighbourhoods of P_x^λ.

Definition 14.8.3 A fuzzy set μ in (X, T) is called a Q-neighbourhood of fuzzy point P_x^λ if and only if there exists a $\lambda \in T$ such that $P_{xq\lambda}^\lambda \leq \mu$. The family consisting of all the Q-neighbourhoods of P_x^λ is called the system of Q-neighbourhoods of P_x^λ.

Theorem 14.8.4 Let μ and λ be two fuzzy sets on X. Then $\mu \leq \lambda$ if and only if μ and λ^c are not quasi-coincident; in particular, $P_x^\alpha \in \mu$ if and only if P_x^α is not quasi-coincident with μ_c.

Proof: Let $\mu \leq \lambda$. Then $\mu(x) \leq \lambda(x)$ for all x.

$$\Rightarrow \mu(x) - \lambda(x) \leq 0$$

Now

$$\mu(x) + \lambda^c(x) = \mu(x) + 1 - \lambda(x)$$
$$= \mu(x) - \lambda(x) + 1$$
$$\leq 1 \text{ as } \mu(x) - \lambda(x) \leq 0$$
$$\Rightarrow \mu \text{ and } \lambda^c \text{ are not quasi-coincident.}$$

Conversely, let μ and λ^c are not quasi-coincident.

Now

$$\mu(x) - \lambda(x) = \mu(x) + 1 - \lambda(x) - 1$$
$$= \mu(x) + \lambda^c(x) - 1$$
$$\leq 1 - 1 = 0$$

Therefore

$$\mu(x) - \lambda(x) \leq 0$$
$$\mu(x) \leq \lambda(x) \text{ for all } x \in X$$
$$\Rightarrow \mu \leq \lambda$$

In particular,

$$P_x^\alpha \in \mu \Leftrightarrow \alpha \leq \mu(x)$$
$$\Leftrightarrow \alpha + \mu^c(x) \leq \mu(x) + \mu^c(x)$$
$$\Leftrightarrow \alpha + \mu^c(x) \leq 1$$
$$\Leftrightarrow P_x^\alpha \text{ is not quasi-coincident with } \mu^c$$

14.9 LET US SUM UP

- **Fuzzy Topological Space:** On a set X, a fuzzy topology is a family $T = \{\mu: \mu$ is a fuzzy set in $X\}$ of subsets (i.e. $T \subseteq I^X$) that satisfies the following axioms:
  (i) $\bar{0}, \bar{1} \in T$
  (ii) $\mu_1, \mu_2 \in T \Rightarrow \mu_1 \cap \mu_2 \in T$
  (iii) If $\{\mu_i: i \in I, I$ is an index set$\} \subseteq T$, then $\cup_{i\in I} \mu_i \in T$
  T is described as a fuzzy topology for X and the pair (X, T) is named as a fuzzy topological space.

- **Closure of fuzzy set:** Let (X, T) be a fuzzy topological space. Let μ be a fuzzy set on X. The closure of μ is defined as the smallest closed fuzzy set that containing μ, i.e.
  $$Cl(\mu) = \inf\{\alpha: \mu \leq \alpha, \alpha^c \in T\}$$

- A fuzzy set μ is closed $\Leftrightarrow Cl(\mu) = \mu$.
- Let (X, T) be a fuzzy topological space. Then for any two fuzzy sets μ, λ on X,
  (i) $\mu \leq Cl(\mu)$
  (ii) $Cl\big(Cl(\mu)\big) = Cl(\mu)$
  (iii) $Cl(\mu \cup \lambda) = Cl(\mu) \cup Cl(\lambda)$
  (iv) $Cl(\bar{0}) = \bar{0}$
  (v) $\mu \leq \lambda \Rightarrow Cl(\mu) \leq Cl(\lambda)$
- **Interior of fuzzy set:** Let (X, T) be a fuzzy topological space. The interior μ^0 of a fuzzy set μ on X defined as
  $$\mu^0 = \sup\{\alpha: \alpha \leq \mu, \alpha \in T\}$$
  Clearly μ^0 is T-open.
- A fuzzy set μ on X is open $\Leftrightarrow \mu^0 = \mu$.
- Assume (X, T) be a fuzzy topological space. Then
  (i) $\bar{0}^0 = \bar{0}, \bar{1}^0 = \bar{1}$
  (ii) $\mu^0 \leq \mu$
  (iii) $(\mu^0)^0 = \mu^0$
  (iv) $(\mu \cap \lambda)^0 = \mu^0 \cap \lambda^0$ for any fuzzy sets μ and λ on X.
  (v) $\mu \leq \lambda \Rightarrow \mu^0 \leq \lambda^0$

- **Boundary of fuzzy sets:** Let λ be a fuzzy set in a fts (X,T). Then the fuzzy boundary of λ is defined as
$$Bd(\lambda) = Cl(\lambda) \cap Cl(\lambda^c)$$
- $Bd(\lambda)$ is a fuzzy closed set for any fuzzy set λ in a fts (X,T).
- For fuzzy sets λ and μ in a fuzzy topological space (X,T), the following conditions hold:
(i) $Bd(\lambda) = Bd(\lambda^c)$
(ii) If λ is a T-closed, then $Bd(\lambda) \leq \lambda$.
(iii) If λ is a T-open, then $Bd(\lambda) = \lambda^c$.
(iv) Let $\lambda \leq \mu$. If μ is T-closed, then $Bd(\lambda) \leq \mu$.
(v) $(Bd\ \lambda)^c = \lambda^0 \cup (\lambda^c)^0$
- Let λ be a fuzzy set in a fts (X,T). Then
(i) $Bd(\lambda) = Cl(\lambda) - \lambda^0$
(ii) $Bd(\lambda^0) \leq Bd(\lambda)$
(iii) $Bd\left(Cl(\lambda)\right) \leq Bd(\lambda)$
(iv) $\lambda^0 \leq \lambda - Bd(\lambda)$
- Let λ and μ be fuzzy sets in fts (X,T). Then $Bd(\lambda \cup \mu) \leq Bd(\lambda) \cup Bd(\mu)$.
- **Base:** Let (X,T) be a fts and B be a subfamily of T. Then B is called a base of T if every member of T can be represent as supremum of members of B.
- **Subbase:** Suppose (X,T) is a fts and S is a non-empty subset of T. Then S is called a subbase of T if the finite intersection of members of S forms a base for T.
- **Fuzzy point:** A fuzzy point P in X is a special fuzzy set with membership function defined by
$$P(x) = \begin{cases} \lambda, & if\ \ x = y \\ 0, & if\ \ x \neq y \end{cases}$$
where $0 < \lambda \leq 1$. P is said to have support y, value λ and is denoted by P_y^λ or $P(y,\lambda)$.
    - Let $\{\mu_j : j \in J\}$ be a family of fuzzy sets in X, P_x^a and P_y^b be fuzzy points in X. Then we have
    (i) $P_x^a \in \cup\{\mu_j : j \in J\}$ ⇔there exists $j \in J$ such that $P_x^a \in \mu_j$.
    (ii) If $P_x^a \in \cap\{\mu_j : j \in J\}$, then for every $j \in J$, we have $P_x^a \in \mu_j$.
    (iii) $P_x^a \in P_y^b \Leftrightarrow x = y$ and $a \leq b$
    (iv) If $P_x^a \in P_y^b$ and for $j \in J, P_y^b \in \mu_j$, then $P_x^a \in \cap\{\mu_j : j \in J\}$
    - A subfamily B of T is a basis for the fts (X,T) if and only if for all $\mu \in T$ and for every fuzzy point P in $\mu, \exists\ \lambda \in B$ such that $P \in \lambda \leq \mu$.
    - **Neighbourhood:** A fuzzy set μ in fts (X,T) is called neighbourhood of fuzzy point P_x^λ if and only if there exists $\beta \in T$ such that $P_x^\lambda \in \beta \leq \mu$.

✓ 14.10 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q 1.

(i) From the definition of T

$$T = \left\{\mu: \mu(x) \geq \frac{1}{2} \ \forall\, x \in X\right\} \cup \left\{\mu': \mu' \text{ is constant} < \frac{1}{2}\right\}$$
$$= T_1 \cup T_2 \ \ (say)$$

and $\bar{0}, \bar{1} \in T$.

(ii) Let $\mu_1, \mu_2 \in T$

Case-I: If $\mu_1, \mu_2 \in T_1$, then

$$(\mu_1 \cap \mu_2)(x) = \inf\{\mu_1(x), \mu_2(x)\}$$
$$\geq \frac{1}{2}$$

Therefore

$$\mu_1 \cap \mu_2 \in T$$

Case-II: If $\mu_1 \in T_1$ and $\mu_2 \in T_2$, then

$$(\mu_1 \cap \mu_2)(x) = \inf\{\mu_1(x), \mu_2(x)\}$$
$$< \frac{1}{2} \ \ as \ \ \mu_2(x) < \frac{1}{2}$$

Therefore

$$\mu_1 \cap \mu_2 \in T$$

Case-III: If $\mu_1, \mu_2 \in T_2$, then

$$(\mu_1 \cap \mu_2)(x) = \inf\{\mu_1(x), \mu_2(x)\}$$
$$< \frac{1}{2}$$

Therefore

$$\mu_1 \cap \mu_2 \in T$$

(iii) Let $\{\mu_i: i \in I, I$ is an index set$\}$ be a family of sets in T. Then

$$\left(\bigcup_{i\in I} \mu_i\right)(x) = \sup_{i\in I}\{\mu_i(x)\}$$
$$\geq \frac{1}{2}$$

Therefore

$$\bigcup_{i\in I} \mu_i \in T$$

Hence, we conclude that (X, T) is a fuzzy topological space.

Ans to Q 2. We have

$$\alpha^0 \leq \alpha, \qquad \beta^0 \leq \beta$$

$\Rightarrow \alpha^0 \cup \beta^0 \leq \alpha \cup \beta$, i.e. $\alpha^0 \cup \beta^0$ is a T-open set contained in $\alpha \cup \beta$.

$(\alpha \cup \beta)^0$ is the supremum of T-open sets contained in $\alpha \cup \beta$, therefore

$$(\alpha \cup \beta)^0 \geq \alpha^0 \cup \beta^0$$

Ans to Q 3. We have

$$\mu \cap \lambda \leq \mu$$
$$\Rightarrow Cl(\mu \cap \lambda) \leq Cl(\mu)$$

Similarly

$$Cl(\mu \cap \lambda) \leq Cl(\lambda)$$

Thus

$$Cl(\mu \cap \lambda) \leq Cl(\mu) \cap Cl(\lambda)$$

Ans to Q 4.

$$Bd(\lambda \cap \mu) = Cl(\lambda \cap \mu) \cap Cl((\lambda \cap \mu)^c)$$
$$= [Cl(\lambda) \cap Cl(\mu)] \cap [Cl(\lambda^c) \cup Cl(\mu^c)]$$
$$= [\{Cl(\lambda) \cap Cl(\mu)\} \cap Cl(\lambda^c)] \cup [\{Cl(\lambda) \cap Cl(\mu)\} \cap Cl(\mu^c)]$$
$$= [Bd(\lambda \cap Cl(\mu)] \cup [Cl(\lambda) \cap Bd(\mu)]$$

Ans to Q 5.

$$Bd\big(Bd(\lambda)\big) = Cl\big(Bd(\lambda)\big) \cap \big(Bd(\lambda)\big)^c$$
$$\leq Cl\big(Bd(\lambda)\big)$$

$= Bd(\lambda)$ as $Bd(\lambda)$ is a T-closed.

14.11 FURTHER READINGS

1. Tapati Das, Fuzzy Topological Space, A Thesis for the degree of Master of Science in Mathematics, National Institute of Technology, Rourkela, 2013.
2. C. L. Chang, Fuzzy Topological Spaces, Journal of Mathematical Analysis and Applications, 24, 182-190, 1968.
3. M. Athar and B. Ahmad, Fuzzy Boundary and Fuzzy Semiboundary, Advances in Fuzzy Systems, 1-9, 2008, doi: 10.1155/2008/586893.

14.12 MODEL QUESTIONS

1. Let (X,T) be a fts. Let α and β be two fuzzy sets in X. Is $(\alpha \cup \beta)^0 = \alpha^0 \cup \beta^0$ true? Justify your answer.

2. Let λ and μ be two fuzzy sets in a fts (X,T). Is $Cl(\lambda \cap \mu) = Cl(\lambda) \cap Cl(\mu)$ true? Justify your answer.

3. Let λ be a fuzzy set in a fts (X,T). Then prove that
(a) $(\lambda^c)^0 = [Cl(\lambda)]^c$
(b) $Cl(\lambda^c) = (\lambda^0)^c$

4. Let λ and μ be fuzzy sets in a fts (X, T). Then prove that $Bd(\lambda \cup \mu) \leq Bd(\lambda) \cup Bd(\mu)$. Give an example to show that the equality does not hold.

5. Show that a fuzzy set μ is T-open if and only if for each fuzzy set λ contained in μ, μ is a neighbourhood of λ.

6. Let μ be a fuzzy set in a fts (X, T). Show that
(a) μ is T-open if and only if $\mu^0 = \mu$
(b) μ is T-closed if and only if $Cl(\mu) = \mu$.

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UNIT 15: FUZZY PRODUCT AND INDUCED TOPOLOGICAL SPACE

UNIT STRUCTURE

15.1 Learning Objectives
15.2 Introduction
15.3 Fuzzy Continuous Map: Definitions and Examples
15.4 Product Fuzzy Topological Spaces
15.5 Induced Fuzzy Topological Spaces
15.6 Let Us Sum Up
15.7 Answers to Check Your Progress
15.8 Further Reading
15.9 Model Questions

15.1 LEARNING OBJECTIVES

After studying this unit, the learner will be able to:

- understand the concept of continuity in the context of fuzzy topological space.
- differentiate between classical continuous functions and fuzzy continuous functions.
- understand the concept of product fuzzy topological spaces.
- define the product of two or more fuzzy topological spaces.
- identify the role of projections in constructing product spaces.
- construct product fuzzy topologies.
- understand how a fuzzy topology can be induced on a subset or through a mapping.
- understand similarities and differences between classical and fuzzy induced topologies.

15.2 INTRODUCTION

In classical topology, a function $f: X \to Y$ between topological spaces is said to be continuous if the preimage of every open set in Y is open in X. Fuzzy continuous maps generalize this idea by incorporating fuzzy sets where elements have degrees of membership ranging from 0 to 1.

A function $f: X \to Y$ between two fuzzy topological spaces is called fuzzy continuous if for every fuzzy open set V in Y, the preimage $f^{-1}(V)$is a fuzzy open set in X.

A product fuzzy topological space is a generalization of the product of classical topological spaces to the realm of fuzzy topology, where sets have degrees of membership.

Let (X, T_X) and (Y, T_Y) be fuzzy topological spaces. Their product fuzzy topological space is defined on the cartesian product $X \times Y$ with a topology T generated by:

Fuzzy basic open sets of the form $u(x,y) = \min\{u_1(x), u_2(y)\}$, where $u_1 \in T_X$ and $u_2 \in T_Y$.

An induced fuzzy topological space arises when a fuzzy topology is defined in such a way that it reflects or preserves the structure of the original fuzzy topology.

Induced subspace fuzzy topology: Given a fuzzy topological space (X, T) and a subset $A \subset X$, the induced fuzzy topology T_A on A is defined by:

- For each fuzzy set $u \in T$, define its restriction to A, denoted by $u|_A$ as

$$u|_A(x) = \begin{cases} u(x), & \text{if } x \in A \\ 0, & \text{otherwise.} \end{cases}$$

- The collection $T_A = \{u|_A : u \in T\}$ forms the induced fuzzy topology on A.

Induced fuzzy topology via function: If $f : X \to Y$ is a function and (Y, T_Y) is a fuzzy topological space, then we can induce a fuzzy topology T_X on X such that f becomes fuzzy continuous. The induced fuzzy topology T_X is the coarsest topology making f fuzzy continuous:

$$T_X = \{f^{-1}(u) : u \in T_Y\}.$$

15.3 FUZZY CONTINUOUS MAP: DEFINITIONS AND EXAMPLES

Definition 15.3.1 Let f be a function from X to Y. Let B be a fuzzy set in Y with membership function $\mu_B(y)$. Then the inverse of B, written as $f^{-1}[B]$, is a fuzzy set in X whose membership function is defined by

$$\mu_{f^{-1}[B]}(x) = \mu_B(f(x)) \text{ for all } x \in X.$$

Conversely, let A be a fuzzy set in X with membership function $\mu_A(x)$. The image of A, written as $f[A]$, is a fuzzy set in Y whose membership function is given by

$$\mu_{f[A]}(y) = \begin{cases} \sup\{\mu_A(z) : z \in f^{-1}[y]\}, & \text{if } f^{-1}[y] \neq \phi \\ 0, & \text{otherwise} \end{cases}$$

for all $y \in Y$, where $f^{-1}[y] = \{x \mid f(x) = y\}$.

Theorem 15.3.2 Let f be a function from X to Y, then

(a) $f^{-1}[B^c] = \{f^{-1}[B]\}^c$ for any fuzzy set B in Y.

(b) $f[A^c] \supset \{f[A]\}^c$ for any fuzzy set A in X.

(c) $B_1 \subseteq B_2 \Rightarrow f^{-1}[B_1] \subseteq f^{-1}[B_2]$; where B_1, B_2 are fuzzy sets in Y.

(d) $A_1 \subseteq A_2 \Rightarrow f[A_1] \subseteq f[A_2]$; where A_1 and A_2 are fuzzy sets in X.
(e) $f[f^{-1}[B]] \subseteq B$ for any fuzzy set B in Y.
(f) $A \subseteq f^{-1}[f[A]]$ for any fuzzy set A in X.
(g) Let f be a function from X to Y and g be a function from Y to Z. Then

$$(g \circ f)^{-1}[C] = f^{-1}[g^{-1}[C]]$$

for any fuzzy set C in Z, where $g \circ f$ is the composition of g and f.

Proof. (a) For all x in X,

$$\begin{aligned}\mu_{f^{-1}[B^c]}(x) &= \mu_{B^c}[f(x)] \\ &= 1 - \mu_B[f(x)] \\ &= 1 - \mu_{f^{-1}[B]}(x) \\ &= \mu_{\{f^{-1}[B]\}^c}(x).\end{aligned}$$

(b) For each $y \in Y$, if $f^{-1}[y]$ is not empty, then

$$\begin{aligned}\mu_{f[A^c]}(y) &= \sup\{\mu_{A^c}(z): z \in f^{-1}[y]\} \\ &= \sup\{1 - \mu_A(z): z \in f^{-1}[y]\} \\ &= 1 - \inf\{\mu_A(z): z \in f^{-1}[y]\}.\end{aligned}$$

and

$$\begin{aligned}\mu_{\{f[A]\}^c}(y) &= 1 - \mu_{f[A]}(y) \\ &= 1 - \sup\{\mu_A(z): z \in f^{-1}[y]\}\end{aligned}$$

therefore

$$\begin{aligned}&\mu_{f[A^c]}(y) \geq \mu_{\{f[A]\}^c}(y) \\ &\Rightarrow f[A^c] \supset \{f[A]\}^c \text{ for any subset } A \text{ in } X.\end{aligned}$$

(c) Let B_1 and B_2 be two fuzzy sets in Y such that $B_1 \subseteq B_2$. Then for all $x \in X$,

$$\mu_{f^{-1}[B_1]}(x) = \mu_{B_1}[f(x)] \text{ and } \mu_{f^{-1}[B_2]}(x) = \mu_{B_2}[f(x)].$$

Since $B_1 \subseteq B_2$, therefore

$$\begin{aligned}&\mu_{f^{-1}[B_1]}(x) \leq \mu_{f^{-1}[B_2]}(x) \text{ for all } x \in X. \\ &\Rightarrow f^{-1}[B_1] \subseteq f^{-1}[B_2].\end{aligned}$$

(d) Let A_1 and A_2 be two fuzzy sets in X such that $\mathrm{A}_1 \subseteq \mathrm{A}_2$. Then for any y ∈ Y,

$$\mu_{\mathrm{f}[\mathrm{A}_1]}(\mathrm{y}) = \sup_{\mathrm{z} \in \mathrm{f}^{-1}[\mathrm{y}]} \{\mu_{\mathrm{A}_1}(\mathrm{z})\} \text{ and } \mu_{\mathrm{f}[\mathrm{A}_2]}(\mathrm{y}) = \sup_{\mathrm{z} \in \mathrm{f}^{-1}[\mathrm{y}]} \{\mu_{\mathrm{A}_2}(\mathrm{z})\}.$$

Since $\mathrm{A}_1 \subseteq \mathrm{A}_2$, therefore

$$\begin{aligned}&\mu_{\mathrm{f}[\mathrm{A}_1]}(\mathrm{y}) \leq \mu_{\mathrm{f}[\mathrm{A}_2]}(\mathrm{y}) \text{for all y} \in \mathrm{Y}. \\ &\Rightarrow \mathrm{f}[\mathrm{A}_1] \subseteq \mathrm{f}[\mathrm{A}_2].\end{aligned}$$

(e) If $f^{-1}[y]$ is non-empty,

$$\begin{aligned}\mu_{f[f^{-1}[B]]}(y) &= \sup\{\mu_{f^{-1}[B]}(z)\} \\ &= \sup\{\mu_B(f(z))\}\end{aligned}$$

$$= \mu_B(y).$$

If $f^{-1}[y]$ is empty, then

$$\mu_{f[f^{-1}[B]]}(y) = 0 \leq \mu_B(y).$$

Thus

$$\mu_{f[f^{-1}[B]]}(y) \leq \mu_B(y) \text{ for all } y \in Y$$
$$\Rightarrow f[f^{-1}[B]] \subseteq B.$$

(f)

$$\mu_{f^{-1}[f[A]]}(x) = \mu_{f[A]}[f(x)]$$
$$= \sup_{z \in f^{-1}[f(x)]} \{\mu_A(z)\}$$
$$\geq \mu_A(x) \text{ for all } x \in X.$$

Thus, $f^{-1}[f[A]] \supseteq A$.

(g) For all $x \in X$,

$$\mu_{(g \circ f)^{-1}[C]}(x) = \mu_C[g(f(x))]$$
$$= \mu_{g^{-1}[C]}[f(x)]$$
$$= \mu_{f^{-1}[g^{-1}[C]]}(x)$$

Thus $(g \circ f)^{-1}[C] = f^{-1}[g^{-1}[C]]$.

Definition 15.3.3 (F-continuous)
A function f from a fts (X, T) to a fts (Y, U) is F-continuous if the inverse of each U-open fuzzy set is T-open, i.e., $f^{-1}(u) \in T$ whenever $u \in U$.

Clearly, if f is an F-continuous function from X onto $Y$and $g$is an F-continuous function from Yonto Z, then the composition $g \circ f$ is an F-continuous function from X onto Z, for

$$(g \circ f)^{-1}[v] = f^{-1}[g^{-1}[v]]$$

for each fuzzy set v in Z and using the F-continuity of g and f it follows that if v is open so is $(g \circ f)^{-1}[v]$.

Example 1. The identity function $I_X: (X, T) \to (X, T)$ is F-continuous on a fuzzy topological space (X, T) is F-continuous.
Let $X = [0,1]$ and define a fuzzy topology T on X by

$$T = \{u: X \to [0,1] \mid u(x) \leq x \text{ for all } x \in X\}.$$

Then for any fuzzy open set $u \in T$,

$$I_X^{-1}(u)(x) = u\big(I_X(x)\big) = u(x) \leq x.$$

Then therefore $I_X^{-1}(u) \in T$. Thus, I_X is F-continuous.

Example 2. Let $X = [0,1]$ and T is defined as $T = \{u: X \to [0,1] \mid u(x) \leq \sqrt{x}$ for all $x \in X\}$.
Define $f: X \to X$ by $f(x) = x^2$. Then for any $u \in T$, the preimage $f^{-1}(u)(x) = u(f(x)) = u(x^2)$. Since $u(x) \leq \sqrt{x^2} = |x|$ and $x \in [0,1]$, therefore

$$u(x^2) \leq x$$
$$\Rightarrow f^{-1}(u)(x) \leq x$$
$$\Rightarrow f^{-1}(u) \in T.$$

Hence f is F-continuous.
Theorem 15.3.4 If (X,T) and (Y,U) are fuzzy topological spaces and f is a function on X to Y, then the conditions below are related as follows:
(a) and (b) are equivalent; (c) and (d) are equivalent, (a) implies (c).

(a) The function f is F-continuous.
(b) The inverse of every closed fuzzy set is closed.
(c) For every fuzzy set u in X, the inverse of every neighbourhood of $f(u)$ is a neighbourhood of u.
(d) For each fuzzy set u in X and each neighbourhood V of $f(u)$, there is a neighbourhood W of u such that $f[W] \subseteq V$.

Proof: (a) ⇒ (b). Let λ be a U-closed in Y.

$$\Rightarrow \lambda^c \text{ is U-open in } Y.$$

Since f is F-continuous, therefore

$$f^{-1}(\lambda^c) \text{ is T-open in } X.$$
$$\Rightarrow [f^{-1}(\lambda)]^c \text{ is T-open in } X.$$
$$\Rightarrow f^{-1}(\lambda) \text{ is T-closed in } X.$$

(b) ⇒ (a). Let inverse image of every closed fuzzy set in $Y$is closed in Xunder f.
Let u be a U-open fuzzy set in Y.

$$\Rightarrow u^c \text{ is U-closed in } Y.$$

By the given condition

$$f^{-1}(u^c) \text{ is T-closed in } X.$$
$$\Rightarrow [f^{-1}(u)]^c \text{ is T-closed in } X.$$
$$\Rightarrow f^{-1}(u) \text{ is T-open in } X.$$

Hence f is continuous.

(a) ⇒ (c). If f is F-continuous, u is a fuzzy set in X and V is a neighbourhood of $f(u)$, then V contains an open neighbourhood W of $f(u)$ such that

$$f(u) \subseteq W \subseteq V,$$

which implies

$$f^{-1}[f(u)] \subseteq f^{-1}(W) \subseteq f^{-1}(V).$$

But $u \subseteq f^{-1}[f(u)]$and $f^{-1}(W)$is T-open. Consequently, $f^{-1}(V)$is a neighbourhood of u.

(c) ⇒ (d) Let Vbe a neighbourhood of $f(u)$ and $f^{-1}(V)$ is a neighbourhood of u for each fuzzy set u in X. Then there exists a T-open set Wsuch that

$$u \subseteq W \subseteq f^{-1}(V).$$

Therefore

$$f(W) \subseteq f(f^{-1}(V)) \subseteq V. [\because f(f^{-1}(V)) \subseteq V]$$

(d) ⇒ (c) Let V be a neighbourhood of $f(u)$. Then there is a neighbourhood $W$of u such that

$$f(W) \subset V$$
$$\Rightarrow f^{-1}[f(W)] \subseteq f^{-1}(V).$$

Since $W \subseteq f^{-1}[f(W)]$, therefore $W \subseteq f^{-1}(V)$. Thus $f^{-1}(V)$ is a neighbourhood of u.

CHECK YOUR PROGRESS

Q.1. Let (X, T)be a fuzzy topological space. Then every constant function from (X, T)into another fuzzy topological space is F-continuous if and only if Tcontains all constant fuzzy sets in X.

15.4 PRODUCT FUZZY TOPOLOGICAL SPACE

Definition 15.4.1 Let (X, T) be a fuzzy topological space. A subfamily B of T is called a base if each member of T can be expressed as the union of some members of B.

Definition 15.4.2 Let (X, T) be a fuzzy topological space. A subfamily $S$of $T$is called a subbase for Tif the family of finite intersections of members of Sforms a base for T.

Let $\{X_\alpha\}, \alpha \in I$ be a family of spaces. Let

$$X = \prod_{\alpha \in I} X_\alpha$$

be the usual product space and let p_α be the projection from X onto X_α.

Further assume that each X_α is a fts with fuzzy topology T_α. Let $A \in T_\alpha$. Then $P_\alpha^{-1}(A)$

is a fuzzy set in X. The family of fuzzy sets

$$S = \{P_\alpha^{-1}(A) : A \in T_\alpha, \alpha \in I\}$$

is now used to generate a fuzzy topology T for X in the following manner:

Let B be the family of finite intersections of members of S. Let T be the family of all unions of members of B. It is clear that T is indeed a fuzzy topology for X, where B is a base and S a subbase.

Definition 15.4.3 Given a family of fts $\{(X_\alpha, T_\alpha): \alpha \in I\}$, the fuzzy topology T defined as above is called the product fuzzy topology for

$$X = \prod_{\alpha \in I} X_\alpha$$

and (X, T) is called the product fts.

Example 1. Let $X = \{a, b\}$ with fuzzy topology $T_X = \{\bar{0}, \bar{1}, u_X\}$ where $u_X(a) = 1, u_X(b) = 0.5$. Let $Y = \{1,2\}$ with fuzzy topology $T_Y = \{\bar{0}, \bar{1}, u_Y\}$ where $u_Y(1) = 0.8, u_Y(2) = 0.6$. Then $X \times Y = \{(a,1), (a,2), (b,1), (b,2)\}$ and product fuzzy topology on $X \times Y$ generated from fuzzy basis element of the form $u_X \times u_Y$ is

$$(u_X \times u_Y)(x, y) = \min\{u_X(x), u_Y(y)\}.$$

So for example,

$$(u_X \times u_Y)(a,1) = \min\{u_X(a), u_Y(1)\} = \min\{1,0.8\} = 0.8$$
$$(u_X \times u_Y)(a,2) = \min\{u_X(a), u_Y(2)\} = \min\{1,0.6\} = 0.5$$
$$(u_X \times u_Y)(b,1) = \min\{u_X(b), u_Y(1)\} = \min\{0.5,0.8\} = 0.5$$
$$(u_X \times u_Y)(b,2) = \min\{u_X(b), u_Y(2)\} = \min\{0.5,0.6\} = 0.5$$

Theorem 15.4.4: Let (X, T) be the product fts of the family of fts's $\{(X_\alpha, T_\alpha): \alpha \in I\}$. Then

(a) for each $\alpha \in I$, the projection P_α is F-continuous.

(b) the product fuzzy topology is the smallest fuzzy topology for X such that (a) is true.

(c) Let (Y, U) be a fts and let f be a function from Y to X. Then f is F-continuous iff for every $\alpha \in I$, $P_\alpha \circ f$ is F-continuous.

Proof: (a) Let (X, T) be the product fts of the family of fts's $\{(X_\alpha, T_\alpha): \alpha \in I\}$ and for each α, $P_\alpha: X = \prod_{\alpha \in I} X_\alpha \to X_\alpha$ be the projection map, then T is generated by the base B with subbase

$$S = \{P_\alpha^{-1}[A]: A \in T_\alpha, \alpha \in I\}$$

So by definition of product fuzzy topology Ton X, $P_\alpha^{-1}(A) \in T$ for each $A \in T_\alpha$. Hence P_α is a F-continuous.

(b) By definition of product fuzzy topology T on $X = \prod_{\alpha \in I} X_\alpha$, T is the family of all union of members of base $B$and $B$is the family of finite intersections of members of subbase

$$S = \{P_\alpha^{-1}[A]: A \in T_\alpha \text{ and } \alpha \in I\}$$

Therefore the product fuzzy topology is the smallest fuzzy topology for X such that (a) is true.

(c) Let $f: Y \to X$ be a F-continuous function, where (Y, U) is a fts. Since $P_\alpha: \prod_{\alpha \in I} X_\alpha \to X_\alpha$ is a F-continuous function, therefore composition $P_\alpha \circ f$ is also F-continuous. Conversely, let $P_\alpha \circ f: Y \to X_\alpha$ be a F-continuous function. Then for any $A \in T_\alpha$,

$$(P_\alpha \circ f)^{-1}(A) = (f^{-1} \circ P_\alpha^{-1})(A)$$

is U-open. Therefore $f^{-1}[P_\alpha^{-1}(A)], A \in T_\alpha, \alpha \in I$ is a family of U-open fuzzy sets in Y.

Since every member of $T$is the union of finite intersections of members of the family $\{P_\alpha^{-1}(A): A \in T_\alpha, \alpha \in I\}$ and f^{-1} preserves union and intersection, it follows that f^{-1} maps T-open fuzzy sets onto U-open fuzzy sets. Hence f is F-continuous.

Definition 15.4.5: A fts (X, T) is said to be C_{II} if there exists a countable base B for T.

Theorem 15.4.6 Let $\{(X_\alpha, T_\alpha): \alpha = 1,2, \ldots\}$ be a countable family of C_{II} fts's. Then the product fts (X, T) is also C_{II}.

Proof: Let (X_α, T_α) be C_{II} fts's, for each $\alpha = 1,2,3, \ldots$. Let B_α be a countable base for T_α and $U = \{P_\alpha^{-1}[A]: A \in B_\alpha\}$. Let B be the family of all finite intersections of members of U. Then B is a countable subfamily of T. We shall show that B is a base for T.

Let $F \in T$. By definition of product fuzzy topology F is the union of fuzzy set of the form

$$\bigcap_{i=1}^{n} P_{\alpha_i}^{-1}[A_i] \text{where } A_i \in T_{\alpha_i}.$$

Since B_{α_i} is a base for T_{α_i}, therefore

$$A_i = \bigcup_{j \in J_i} B_{ji}, B_{ji} \in B_{\alpha_i}.$$

Consequently

$$P_{\alpha_i}^{-1}[A_i] = P_{\alpha_i}^{-1}\left[\bigcup_{j_i \in J_i} B_{ji}\right] = \bigcup_{j_i \in J_i} P_{\alpha_i}^{-1}[B_{ji}]$$

Therefore

$$F = \bigcap_{i=1}^{n} P_{\alpha_i}^{-1}[A_i] = \bigcap_{i=1}^{n} \bigcup_{j_i \in J_i} P_{\alpha_i}^{-1}[B_{ji}]$$

is the union of finite intersections of members of U. It follows that $B$is a base for T.

CHECK YOUR PROGRESS

Q.2. Let $\{(X_\alpha, T_\alpha): \alpha \in I\}$ be an uncountable family of C_{II} spaces such that

(a) none is indiscrete and

(b) in each fts (X_α, T_α), for any $F \in T_\alpha$ and $F \neq \phi$, there exists a point $x \in X_\alpha$ such that $\mu_F(x) = 1$, where μ_F is the membership function of F, then the product fts (X, T) is not C_{II}.

15.5 INDUCED FUZZY TOPOLOGICAL SPACES

Definition 15.5.1 A function $f: X \to \mathbb{R}$ is lower continuous (respectively upper semicontinuous) if and only if, for each $n \in \mathbb{R}$, $\{x \in X \mid f(x) \leq n\}$ is closed (respectively $\{x \in X \mid f(x) \geq n\}$is closed).If $A: X \to \mathbb{R}$is upper semicontinuous, then $1 - A$ is lower semicontinuous.

Definition 15.5.2 (Induced Fuzzy Topology) Let (X, T) be a topological space. Then the set of lower semicontinuous functions from (X, T) to closed unit interval $[0, 1]$ being a fuzzy topology is called an induced fuzzy topology on (X, T), denoted by $\omega(T)$ and $(X, \omega(T))$ is called an induced fuzzy topological space.

Let Abe a fuzzy set in X. $\sigma_r(A) = \{x: A(x) \geq r\}$is called the strong r-cut and $\omega_r(A) = \{x: A(x) > r\}$is called weak r-cut. If $E$is a subset of X, let $\chi_E$denote the characteristic function of E. Then Amay be represented in the following form:

$$A = \bigcup_{r \in [0,1]} r\, \chi_{\sigma_r(A)} = \bigcup_{r \in [0,1]} r\, \chi_{\omega_r(A)}.$$

The formula above is called the decomposition theorem or resolution of identity.

Lemma 15.5.3 Let $A_i (i \in I)$be fuzzy sets in $X$and $B_i (i \in I)$be subsets of X, then for $r \in [0,1]$, the following relations hold:

(i) $\sigma_r\left(\bigcup_{i \in I} A_i\right) = \bigcup_{i \in I} \sigma_r(A_i)$;

(ii) $\omega_r\left(\bigcap_{i \in I} A_i\right) = \bigcap_{i \in I} \omega_r(A_i)$;

(iii) $\sigma_r\left(\bigcap_{i \in I} A_i\right) \subseteq \bigcap_{i \in I} \sigma_r(A_i)$;

(iv) $\omega_r\left(\bigcup_{i \in I} A_i\right) \supseteq \bigcup_{i \in I} \omega_r(A_i)$;

(v) $\chi_{\bigcup_{i \in I} B_i} = \bigcup_{i \in I} \chi_{B_i}$;

(vi) $\chi_{\bigcap_{i \in I} B_i} = \bigcap_{i \in I} \chi_{B_i}$;

(vii) $\sigma_s(r\chi_{B_i}) = B_i$ for $0 \leq s < r$;

(viii) $\omega_s(r\chi_{B_i}) = B_i$ for $0 < s \leq r$.

Lemma 15.5.4 Let $f: X \to Y$be a function, let $A$and $A_i (i \in I)$be fuzzy set in $Y$and let Bbe a subset of Y. Then the following relations holds:
(i) $f^{-1}[\sigma_r(A)] = \sigma_r(f^{-1}[A])$;
(ii) $f^{-1}[\omega_r(A)] = \omega_r(f^{-1}[A])$;
(iii) $f^{-1}[r\chi_B] = r\chi_{f^{-1}[B]}$;
(iv) $f^{-1}\left[\bigcup_{i\in I} A_i\right] = \bigcup_{i\in I} f^{-1}[A_i]$.

Lemma 15.5.5 Let (X, T) be a topological space, let $(X, \omega(T))$ be its induced space, let A be a fuzzy set in X and let Bbe a subset of X, then the following statements holds:
(i) A is open in $(X, \omega(T))$ iff for each $r \in [0,1]$, $\sigma_r(A)$ is open in (X, T);
(ii) A is closed in $(X, \omega(T))$ iff for each $r \in [0,1]$, $\omega_r(A)$is closed in (X, T);
(iii) B is open in (X, T) iff χ_B is open in $(X, \omega(T))$;
(iv) B is closed in (X, T) iff χ_B is closed in $(X, \omega(T))$.

Theorem 15.5.6 A mapping $f: (X, \omega(T)) \to (Y, \omega(U))$ is F-continuous if and only if $f: (X, T) \to (Y, U)$ is continuous.
Proof: Suppose f is F-continuous. Let A be U-open. Then

$$f^{-1}A = \{x \in X \mid \chi_A(f(x)) = 1\}$$
$$= \{x \in X \mid (f^{-1}\chi_A)(x) > \frac{1}{2}\}$$
$$= \sigma_{1/2}(f^{-1}\chi_A)$$

Since A is U-open, so by lemma 14.5.5 (iii), χ_A is open in $(X, \omega(U))$. Also f is F-continuous, $f^{-1}(\chi_A)$ is open in $(X, \omega(T))$. Again, by lemma 14.5.5 (i),

$$\sigma_{1/2}(f^{-1}\chi_A)$$

is T-open in X. Hence f is continuous.
Conversely, suppose f is continuous and Ban $\omega(U)$-open in Y. For any $r \in (0,1)$,

$$\sigma_r(f^{-1}B) = \{x \in X \mid B(f(x)) > r\}$$
$$= f^{-1}(B^{-1}(r, \infty))$$

Since B is lower semicontinuous and $f$is continuous, therefore

$$f^{-1}(B^{-1}(r, \infty))$$

is open in (X, T), i.e.

$$\sigma_r(f^{-1}B)$$

is open in (X, T). Therefore, by lemma 14.5.5 (i), $f^{-1}B$ is open in $(X, \omega(T))$. Hence f is F-continuous.

Lemma 15.5.7 Let $(X, \omega(T))$ be an induced fts, let A be a fuzzy set in X and B be a subset of X. Then following relations hold:
(i) $\sigma_r(A^o) \subseteq (\sigma_r(A))^o$;

(ii) $\omega_r(cl(A)) \supseteq cl(\omega_r(A))$;
(iii) $cl(r\chi_B) = r\chi_{cl(B)}$;
(iv) $(r\chi_B)^o = r\chi_{B^o}$.
Proof: (i) Let

$$x \in \sigma_r(A^o)$$
$$\Rightarrow A^o(x) > r \qquad (1)$$

Again

$$A^o \leq A$$
$$\Rightarrow A^o(x) \leq A(x) \qquad (2)$$

From (1) and (2), we obtain

$$A(x) > r$$
$$\Rightarrow x \in \sigma_r(A)$$

Therefore $\sigma_r(A^o) \subseteq \sigma_r(A)$.
Since $(\sigma_r(A))^o$is the largest open set such that

$$(\sigma_r(A))^o \subseteq \sigma_r(A),$$

therefore

$$\sigma_r(A^o) \subseteq (\sigma_r(A))^o.$$

(ii) Let

$$x \in \omega_r(A)$$
$$\Rightarrow A(x) > r$$

Again

$$A \leq cl(A)$$
$$\Rightarrow A(x) \leq cl(A)(x)$$

Therefore $cl(A)(x) > r$

$$\Rightarrow x \in \omega_r(cl(A))$$

Thus $\omega_r(A) \subseteq \omega_r(cl(A))$.
Since $cl(\omega_r(A))$ is the smallest closed set such that

$$\omega_r(A) \subseteq cl(\omega_r(A)),$$

therefore

$$\omega_r(cl(A)) \supseteq cl(\omega_r(A)).$$

(iii) Since $r\chi_{cl(B)}$ is closed and $r\chi_B \leq r\chi_{cl(B)}$, therefore

$$cl(r\chi_B) \leq r\chi_{cl(B)} \qquad (1)$$

Conversely, assume C is closed in $(X, \omega(T))$ with $C \geq r\chi_B$. Then $\omega_r(C) \supseteq B$, therefore $\omega_r(C) \supseteq cl(B)$as $\omega_r(C)$is closed.

$$C \geq r\chi_{\omega_r(C)} \geq r\chi_{cl(B)}$$

Hence

$$cl(r\chi_B) \geq r\chi_{cl(B)} \qquad (2)$$

From (1) and (2), we get

$$cl(r\chi_B) = r\chi_{cl(B)}.$$

(iv) If $r = 0$, it is clear. Suppose $r > 0$. It is obvious that

$$r\chi_B \leq (r\chi_B)^o \qquad (1)$$

Conversely, assume $C$is open in $(X, \omega(T))$with $C \leq r\chi_B$.
Then $\sigma_r(C) \subseteq B$ for each s with $0 \leq s < r$. Since $\sigma_s(C)$ is open, therefore $\sigma_s(C) \subseteq B^o$ and $s\chi_{\sigma_s(C)} \leq r\chi_{B^o}$.
We have

$$C = \bigcup_{0\leq s<r} s\,\chi_{\sigma_s(C)} \leq r\chi_{B^o}.$$

Hence

$$(r\chi_B)^o \leq r\chi_{B^o} \qquad (2)$$

From (1) and (2), we obtain

$$(r\chi_B)^o = r\chi_{B^o}.$$

Lemma 15.5.8 Let $(X, \omega(T))$be an induced fts and let $\{B_r\}_{r\in[0,1]}$be a family of subsets of Xsatisfying $B_s \supseteq B_t$for $s < t$. Then the following relations hold:
(i) $\sigma_s(\cup_r r\,\chi_{B_r}) = \bigcup_{r>s} B_r$;
(ii) $\omega_s(\cup_r r\,\chi_{B_r}) = \bigcap_{r<s} B_r$;
(iii) $cl(\cup_r r\,\chi_{B_r}) = \cup_r r\,\chi_{cl(B_r)}$;
(iv) $(\cup_r r\,\chi_{B_r})^o = \cup_r r\,\chi_{B_r^o}$.
Proof: (i)

$$\begin{aligned}\sigma_s\left(\bigcup_r r\,\chi_{B_r}\right) &= \{x: \sup_r r\chi_{B_r}(x) > s\}\\ &= \bigcup_{r>s}\{x: r\,\chi_{B_r}(x) > s\}\\ &= \bigcup_{r>s} B_r\,.\end{aligned}$$

(ii) Assume $x \notin \bigcap_{r<s} B_r$. Then there is some $r_0 < s$ such that $x \notin B_{r_0}$. In view of monotone of the family $\{B_r\}$, we have $x \notin B_r$ for each r with $r_0 < r$.
Then

$$\sup_r r\chi_{B_r}(x) \leq r_0 < s,$$

hence $x \notin \omega_s(\cup_r r\,\chi_{B_r})$. Therefore

$$\omega_s\left(\bigcup_r r\,\chi_{B_r}\right) \subseteq \bigcap_{r<s} B_r \qquad (1)$$

Conversely, assume $x \in \omega_s\left(\bigcup_r r\,\chi_{B_r}\right)$. Then

$$\sup_r r\chi_{B_r}(x) < s.$$

Choose some $r_0$so that

$$\sup_r r\chi_{B_r}(x) < r_0 < s.$$

Then $x \notin B_{r_0}$, hence $x \notin \bigcap_{r<s} B_r$.
Therefore

$$\bigcap_{r<s} B_r \subseteq \omega_s\left(\bigcup_r r\,\chi_{B_r}\right) (2)$$

From (1) and (2), we get

$$\bigcap_{r<s} B_r = \omega_s\left(\bigcup_r r\,\chi_{B_r}\right).$$

(iii) Applying (ii) to the family $\{cl(B_r)\}$, we get

$$\omega_s\left(\bigcup_r r\,\chi_{cl(B_r)}\right) = \bigcap_{r<s} cl(B_r)$$

for each s. This is a closed set in (X, T) and hence $\bigcup_r r\,\chi_{cl(B_r)}$ is closed in $(X, \omega(T))$.
Since

$$\begin{aligned} cl\left(\bigcup_r r\,\chi_{B_r}\right) &\geq \bigcup_r r\,\chi_{cl(B_r)}, \\ &= \bigcup_r r\,\chi_{cl(B_r)} \\ &\geq \bigcup_r r\,\chi_{B_r}, \end{aligned}$$

by definition of closures, we get

$$cl\left(\bigcup_r r\,\chi_{B_r}\right) = \bigcup_r r\,\chi_{cl(B_r)}.$$

(iv) Obviously

$$\left(\bigcup_r r\,\chi_{B_r}\right)^o \geq \bigcup_r r\,\chi_{B_r^o}.$$

Assume $A = \bigcup_r r\,\chi_{B_r}$. Then

$$A^o = \bigcup_r r\,\chi_{\sigma_r(A^o)} \leq \bigcup_r r\,\chi_{(\sigma_r(A))^o}.$$

By (i), $\sigma_r(A) = \bigcup_{s>r} B_s \subset B_r$ and hence

$$(\sigma_r(A))^o \subset B_r^o.$$

Therefore

$$A^o \leq \bigcup_r r\,\chi_{(\sigma_r(A))^o} \leq \bigcup_r r\,\chi_{B_r^o}.$$

i.e.

$$\left(\bigcup_r r\,\chi_{B_r}\right)^o \leq \bigcup_r r\,\chi_{B_r^o}.$$

Hence

$$\left(\bigcup_r r\,\chi_{B_r}\right)^o = \bigcup_r r\,\chi_{B_r^o}.$$

Theorem 15.5.9 Let $(X, \omega(T))$be an induced fts and let Abe a fuzzy set in X. Then the following relations hold:
(i) $cl(A) = \cup_r r\,\chi_{cl(\sigma_r(A))} = \cup_r r\,\chi_{cl(\omega_r(A))}$;
(ii) $A^o = \cup_r r\,\chi_{(\sigma_r(A))^o} = \cup_r r\,\chi_{(\omega_r(A))^o}$.
Proof: (i) From decomposition theorem,

$$cl(A) = \bigcup_r r\,\chi_{\sigma_r(cl(A))} = \bigcup_r r\,\chi_{\omega_r(cl(A))}$$
$$= \bigcup_r r\,\chi_{cl(\sigma_r(A))} = \bigcup_r r\,\chi_{cl(\omega_r(A))},$$

by using lemma 14.5.8 (iii).
(ii) From decomposition theorem,

$$A^o = \bigcup_r r\,\chi_{\sigma_r(A^o)} = \bigcup_r r\,\chi_{\omega_r(A^o)}$$
$$= \bigcup_r r\,\chi_{(\sigma_r(A))^o} = \bigcup_r r\,\chi_{(\omega_r(A))^o},$$

using lemma 14.5.8 (iv).

CHECK YOUR PROGRESS

Q.3. Let $(X, \omega(T))$ be an induced space and let A be a fuzzy set in X. Then

$$\sigma_r(A^o) \subset (\sigma_r(A))^o.$$

Is equality hold for the inclusion. Justify your answer.

15.6 LET US SUM UP

- **F-continuous:** A function $f$from a fts $(X, T)$to a fts (Y, U) is F-continuous if the inverse of each U-open fuzzy set is T-open, i.e. $f^{-1}(u) \in T$ whenever $u \in U$.
- If f is an F-continuous from X onto Y and g is an F-continuous from Y onto Z, then $g \circ f$ is an F-continuous from X onto Z.
- If (X, T) and (Y, U) are fuzzy topological spaces and f is a function on X to Y, then

  (a) f is F-continuous if and only if the inverse of every closed fuzzy set is closed,

  (b) if the function f is F-continuous, then for every fuzzy set $u$in X, the inverse of every neighbourhood of $f(u)$is a neighbourhood of u.
- **Product fuzzy topological space:** Let $\{X_\alpha : \alpha \in I\}$ be a family of spaces. Let $X = \prod_{\alpha \in I} X_\alpha$ be the usual product space and let P_α be the projection from X onto X_α. Let $A \in T_\alpha$. Then $P_\alpha^{-1}(A)$ is a fuzzy set in X. The family of fuzzy sets $S = \{P_\alpha^{-1}(A) : A \in T_\alpha, \alpha \in I\}$ generate a fuzzy topology T for X in the following manner: Let B be the family of finite intersections of members of S. Let T be the family of all unions of members of B. Then T is a fuzzy topology on X with base B and subbase S, then (X, T) is called the product fuzzy topological space.
- Let (X, T)be the product fts of the family of fts's $\{(X_\alpha, T_\alpha) : \alpha \in I\}$. Then

  (a) for each $\alpha \in I$, the projection P_α is F-continuous,

  (b) the product fuzzy topology is the smallest fuzzy topology for X such that (a) is true,

  (c) Let (Y, U) be a fuzzy topological space and let f be a function from Y to X. Then $f$is F-continuous iff for every $\alpha \in I$, $P_\alpha \circ f$ is F-continuous.

- A fts (X, T) is said to be C_{II} if there exists a countable base B for T.
- Let $\{(X_\alpha, T_\alpha) : \alpha = 1,2, \ldots\}$ be a countable family of C_{II} fts's. Then the product fts (X, T) is C_{II}.
- **Lower and Upper semicontinuous:** A function $f : X \to \mathbb{R}$ is lower semicontinuous (respectively, upper semicontinuous) if and only if for each $r \in \mathbb{R}$, $\{x \in X \mid f(x) \leq r\}$ is closed (respectively $\{x \in X \mid f(x) \geq r\}$ is closed).
- If $f : X \to \mathbb{R}$is upper semicontinuous, then $1 - f$ is lower semicontinuous.
- **Induced fuzzy topology:** Let (X, T) be a topological space. Then the set of all lower semicontinuous functions from (X, T) to closed unit interval $[0, 1]$, being a

fuzzy topology is called an induced fuzzy topology on (X, T) denoted by $\omega(T)$ and $(X, \omega(T))$ is called an induced fts.

- Let A be a fuzzy set in X. Then $\sigma_r(A) = \{x: A(x) > r\}$ is called the strong r-cut and $\omega_r(A) = \{x: A(x) \geq r\}$ is called the weak r-cut.
- Let (X, T)be a topological space, let $(X, \omega(T))$be its induced space, let Abe a fuzzy set in $X$and let Bbe a subset of X. Then the following statements holds:

  (i) $A$is open in $(X, \omega(T))$iff for each $r \in [0,1]$, $\sigma_r(A)$is open in (X, T);

  (ii) $A$is closed in $(X, \omega(T))$iff for each $r \in [0,1]$, $\omega_r(A)$is closed in (X, T);

  (iii) $B$is open in (X, T)iff $\chi_B$is open in $(X, \omega(T))$;

  (iv) $B$is closed in (X, T)iff $\chi_B$is closed in $(X, \omega(T))$.

- A mapping $f: (X, \omega(T)) \to (Y, \omega(U))$ is F-continuous iff $f: (X, T) \to (Y, U)$ is continuous.
- Let $(X, \omega(T))$ be an induced fts, let A be a fuzzy set in $X$and B be a subset of X. Then

  (i) $\sigma_r(A^o) \subseteq (\sigma_r(A))^o$;

  (ii) $\omega_r(cl(A)) \supseteq cl(\omega_r(A))$;

  (iii) $cl(r\chi_B) = r\chi_{cl(B)}$;

  (iv) $(r\chi_B)^o = r\chi_{B^o}$.

- Let $(X, \omega(T))$ be an induced fts and let $\{B_r\}_{r\in[0,1]}$ be a family of subsets of X satisfying $B_s \supseteq B_t$ for $s < t$. Then the following relations hold:

  (i) $\sigma_s\left(\cup_r r\,\chi_{B_r}\right) = \bigcup_{r>s} B_r$;

  (ii) $\omega_s\left(\cup_r r\,\chi_{B_r}\right) = \bigcap_{r<s} B_r$;

  (iii) $cl\left(\cup_r r\,\chi_{B_r}\right) = \cup_r r\,\chi_{cl(B_r)}$;

  (iv) $\left(\cup_r r\,\chi_{B_r}\right)^o = \cup_r r\,\chi_{B_r^o}$.

- Let $(X, \omega(T))$be an induced fts and let Abe a fuzzy set in X. Then the following relations hold:

  (i) $cl(A) = \cup_r r\,\chi_{cl(\sigma_r(A))} = \cup_r r\,\chi_{cl(\omega_r(A))}$;

(ii) $A^o = \bigcup_r r\,\chi_{(\sigma_r(A))^o} = \bigcup_r r\,\chi_{(\omega_r(A))^o}$.

✓ 15.7 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q.1

Suppose that every constant function from (X, T) into any fuzzy topological space is fuzzy continuous and consider the fuzzy topology U on $[0, 1]$ defined by

$$U = \{\bar{0}, \bar{1}, I_{[0,1]}\}.$$

Let k be any real number, $0 \leq k \leq 1$. The constant function $f: X \to [0,1]$ defined by $f(x) = k$ is F-continuous and so $f^{-1}(I_{[0,1]}) \in T$. But for $x \in X$,

$$f^{-1}(I_{[0,1]})(x) = I_{[0,1]}(f(x)) = I_{[0,1]}(k) = k,$$

whence the constant fuzzy set k in X belong to T.

Conversely, suppose that T contains all constant fuzzy sets in X and consider a constant function $f: (X, T) \to (Y, U)$ defined by $f(x) = y_0$. If $v \in U$, then for all $x \in X$ we have

$$f^{-1}(v)(x) = v(f(x)) = v(y_0),$$

so that $f^{-1}(v)$ is a constant fuzzy set in $X$and hence, a member of T. Thus f is F-continuous.

Ans. to Q.2

By definition of product fuzzy topology, T has a base B_0 with members of the form

$$B_0 = \bigcap_{i=1}^{n} P_{\alpha_i}^{-1}[A_{\alpha_i}], A_{\alpha_i} \in T_{\alpha_i}.$$

Then by (b), $P_\alpha[B_0] = X_\alpha$, for $\alpha \neq \alpha_i, i = 1,2,\ldots,n$ and $B_0 \neq \phi$. Indeed, let $x_i \in X_{\alpha_i}$ such that $\mu_{A_{\alpha_i}}(x_i) = 1, i = 1,2,\ldots,n$. Let $x_\alpha \in X_\alpha$. Consider the subset

$$S = \{x_1\} \times \cdots \times \{x_n\} \times \{x_\alpha\} \times \prod_{\beta \neq \alpha_i, i=1,2,\ldots,n; \beta \neq \alpha} X_\beta$$

of X. Then $\mu_{B_0}(s) = 1$for all $s \in S$. Therefore, $\mu_{P_\alpha[B_0]}(x_\alpha) = 1$. Since x_α is arbitrary chosen, it follows that $P_\alpha[B_0] = X_\alpha$.

Now if (X, T) is C_{II}, then T has a countable base $B = \{B_i\}, i = 1,2,\ldots$. For each $B_i \neq \phi$ of B, there exists a member $B_0 \neq \phi$ of B_0 such that $B_i \supset B_0$, say

$$B_0 = \bigcap_{i=1}^{n} P_{\alpha_i}^{-1}[A_{\alpha_i}], A_{\alpha_i} \in T_{\alpha_i}.$$

Then $P_\alpha[B_i] \supset P_\alpha[B_0] = X_\alpha$ for $\alpha \neq \alpha_i, i = 1,2,\ldots,n$. Therefore $P_\alpha[B_i] = X_\alpha$, for $\alpha \neq \alpha_i, i = 1,2,\ldots,n$. Do this for all members of B. It follows that there exists a countable subset $K$of the index set I such that for any $\alpha \in I - K$, $P_\alpha[B_i] = X_\alpha$ for all $B_i \neq \phi$. Since I is uncountable, there exists $\alpha_0 \in I - K$. Hence $P_{\alpha_0}[B_i] = X_{\alpha_0}$for all B_i. By assumption,

$X_{\alpha_0}$is not indiscrete; therefore, there exists an open fuzzy set $A_0, A_0 \neq \phi$in X_{α_0}. $P_{\alpha_0}^{-1}[A_0]$ is an open fuzzy set in T. By definition of base, there exists a member $B_i \neq \phi$ of B such that $P_{\alpha_0}^{-1}[A_0] \supseteq B_i$. Therefore

$$P_{\alpha_0}[P_{\alpha_0}^{-1}[A_0]] \supseteq P_{\alpha_0}[B_i] = X_{\alpha_0}.$$

On the other hand $P_{\alpha_0}[P_{\alpha_0}^{-1}[A_0]] = A_0$. This is a contradiction. Hence product fts (X, T) is not C_{II}.

Ans. to Q.3

Let $(X, \omega(T))$ be an induced space, then for fuzzy set A, $\sigma_r(A^o) = (\sigma_r(A))^o$ does not hold.

For example, let (X, T) be a topological space where $X = [0,1]$ with usual topology. Define the fuzzy set A in X by

$$A(x) = \begin{cases} \frac{1}{m}, & x = \frac{n}{m} \text{ (irreducible)} \\ 1, & x \text{ is irrational or zero.} \end{cases}$$

Put $r = 0$, then $\sigma_r(A) = X$and $(\sigma_r(A))^o = X$. But $A^o = 0$, thus $\sigma_r(A^o) = \phi$. Therefore

$$\sigma_r(A^o) \subset (\sigma_r(A))^o.$$

15.8 FURTHER READING

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4. Michael D. Weiss, Fixed point, Separation and Induced Topologies for fuzzy sets, Journal of Mathematical Analysis and Applications 50, 142–150, 1975.
5. Wang Gepimg and Hu Lanfang, On Induced Fuzzy Topological Space, Journal of Mathematical Analysis and Applications 108, 495–506, 1985.

15.9 MODEL QUESTIONS

Q.1. Define F-continuous function in a fuzzy topological space. Provide an example to illustrate the concept.

Q.2. Prove that the composition of two F-continuous functions is also F-continuous in fuzzy topological spaces.

Q.3. Define the product fuzzy topological space of two fuzzy topological spaces. How it is the product topology generated in this context?

Q.4. Given two fuzzy topological spaces (X, T_X) and (Y, T_Y), construct the product fuzzy

topological space $(X \times Y, T)$ and describe its basic open sets.

Q.5. Prove that the projecting mappings from a product fuzzy topological space onto its component space are F-continuous.

Q.6. Define an induced fuzzy topological space. Given a crisp topological space (X, T) and a fuzzy set $\mu: X \to [0,1]$, describe the method to induce a fuzzy topology on X from μ. Illustrate your answer with an example.

Q.7. Discuss the properties of an induced fuzzy topology. Under what conditions does an induced fuzzy topological space satisfy the axioms of a fuzzy topological space.

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