MATCHINGS AND COVERINGS

UNIT STRUCTURE

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10.1 LEARNING OBJECTIVES

After going through this unit, you will be able to-

- define matching in a graph and distinguish between maximum matching and perfect matching.
- understand Berge's Theorem and Hall's Marriage problem.
- understand the concepts of vertex cover and edge cover.
- define dominating sets, minimal dominating set and domination number.
- define factorization and identify '1-factor' (perfect matching) and '2-factor' (cycle cover).

10.2 INTRODUCTION

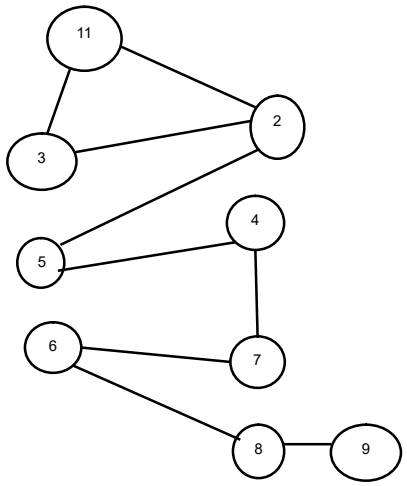
Matchings and Coverings on graphs is very important for the study of graphs. These topics has numerous applications. Matching refers to a set of edges where no two edges share a common vertex, it is a method of pairing up the vertices. Covering involves finding a set of vertices or edge that "covers" all other edges or vertices in the graph respectively. This chapter explores these concepts together with some other concepts like dominating sets, dominating numbers, critical vertices and edges and factorization. We shall also come across terms like maximum matching, perfect matching and we will also discus the famous Hall's Marriage Problem which is related to matching.

10.3 MATCHINGS

Definition: A matching of graph G is a subgraph of G such that every edge shares no vertex with any other edge i.e., each vertex matching M has degree 1.

Size of a matching:

The size of a matching is the number of edges in that matching.



Let us denote the edge that connects vertex i and j by (i, j).

e.g., (5, 4) is a matching of size one as there is one element.

{(5,4), (6,7)} is a matching of size two.

{(3,2), (5,4), (6, 7)} is a matching of size three.

The question is can we get a matching of size greater than three.

The answer is, yes.

{(1, 3), (2, 5), (4, 7), (6, 8)} is a matching of size greater than three.

But we cannot get a matching of size greater than four. We call this matching the maximum matching.

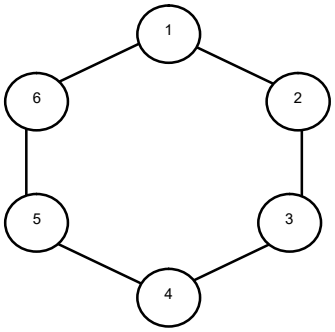
Maximum matching and Matching number:

A matching is maximum when it has the largest possible size. There may be several maximum matchings.

The matching number of a graph is the size of a maximum matching of that graph.

Complete/perfect matching:

A matching of a graph G is complete/perfect if it contains all the vertices of graph G.

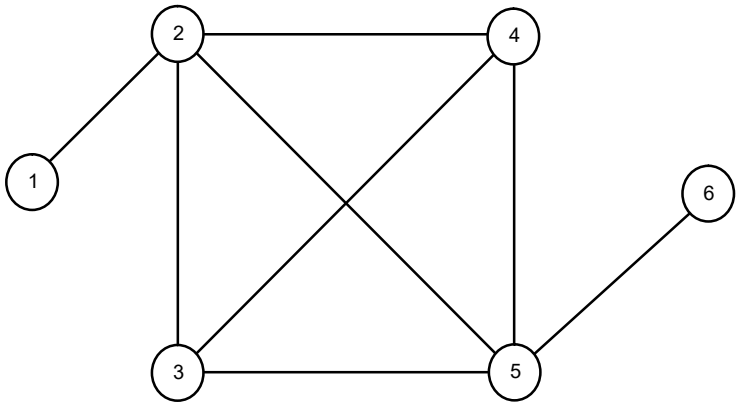


M= {(1, 2), (3, 4), (5, 6)}

Saturated and unsaturated vertex:

A vertex is saturated if it is incident to M.
A vertex is unsaturated if it is not incident to M.

Example:



Let,

M= {(1, 2), (4, 5)}

Then,

1,2,4,5 are saturated vertices.

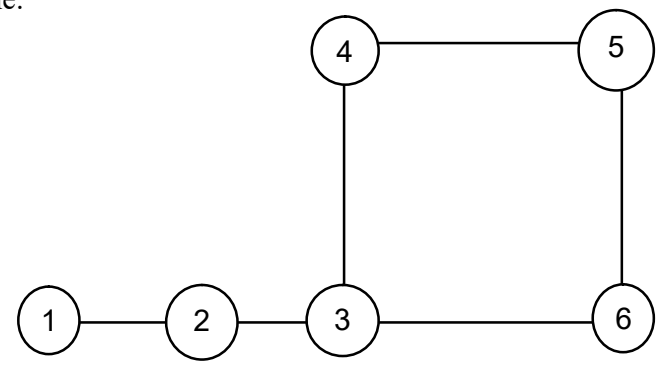
3,6 are unsaturated vertices.

Note: In a perfect matching all the vertices are saturated.

Alternating and paths:

A M-alternating path that alternates between edges in M and not in M, all the vertices except the start and end must be saturated.

Example:



Let,

M= {(2,3), (5, 6)} be a matching in G.

M-alternating paths for this graph are:

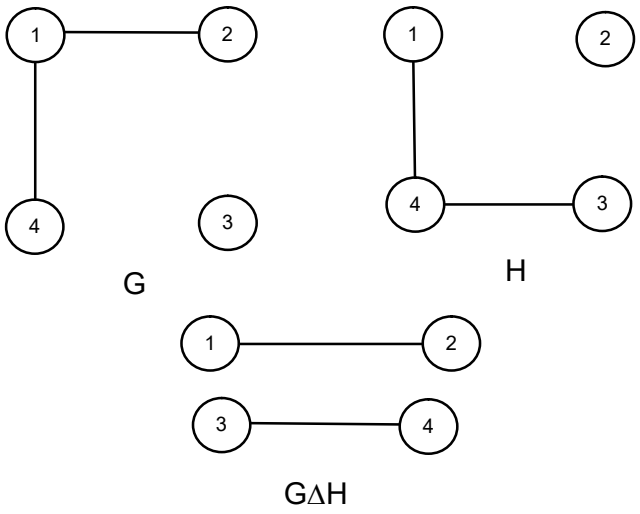
a) 12, 23, 34, 45, 56
b) 23, 36, 65
c) 45, 56, 63rgumenting path:

A M-augmenting path is an M-alternating path that starts and ends with an unsaturated vertex.

In previous example, a) and c) are M-augmenting paths for matching M of graph G.

Symmetric difference:

Symmetric difference between two graphs G and H or $G\Delta H$ is the subgraph $G \cup H$ edges are the edges that appear in only one of G and H.



Berge's theorem:

A matching M is a maximum if and only if G has no M-augmenting path.

Lemma 1:

Consider two matchings M and M'. Let, G' be $M\Delta M'$. All connected components of G' must be isolated vertices, even cycles alternating between M and M', or paths alternating between M and M'.

Proof:

Each vertex in G' can have at maximum degree 2.

$\therefore$ G' can only be made of paths and cycles.

All cycles in G' must alternate between M and M'

So they must be all even cycles.

Berge's theorem:

A matching M is a maximum if and only if G has no M-augmenting path.

Proof:

Suppose, M is maximal.

If a M-augmenting path existed there would be a larger matching, which we can form by replacing the edges in M in the path with the "complementary edges in the path".

Combining the edges in G/M gives us an M-augmenting path.

And, if we take the edges in G/M we have a larger matching, which is a contradiction.

Hence, no M-augmenting path can exist.

Conversely,

Suppose no M-augmenting paths exist.

Suppose, N is a large matching.

Consider, the edge-induced symmetric difference subgraph MΔN, which will consist of all edges in M that are not in N, and all edges in N that are not in M.

Since, this subgraph is limited to edges in M, N.

Thus, every vertex will have a degree at most 2.

Consequently, every connected component is a path or cycle.

∵ the edges from N and edges from M are disconnected, then any path or cycle in MΔN must alternate between edges in N and edges in M

∵|N|>|M|, therefore at least one of the connected components will have one or more from M.

But this will be an M-augmenting path, which contradicts our assumption that there is no M-augmenting path.

Berge's theorem provides a way to create a maximum matching as follows:

- Start with any matching.
- Attempt to augment it.
- Later rinse, repeat.

Hall's marriage problem

Hall's Marriage Theorem, named after and proven by English mathematician Philip Hall in 1935, is a fundamental result in graph theory that provides a condition for determining whether a bipartite graph has a perfect matching—that is, a way to pair elements from one set to elements in another set such that each element is paired exactly once.

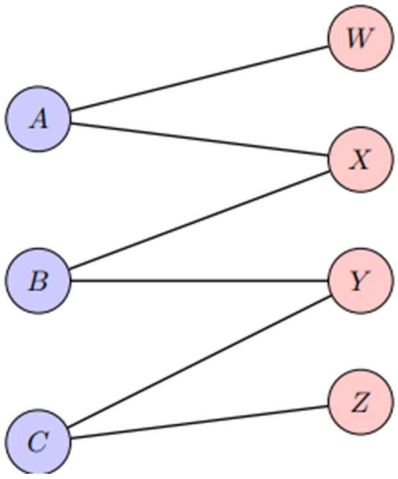
Hall's Marriage Theorem applies to a bipartite graph $G = (X \cup Y, E)$, where X and Y are two disjoint sets of vertices, and edges exist only between vertices in X and vertices in Y. The theorem gives a condition for the existence of a *matching* that covers all vertices in X, meaning each vertex in X is paired with exactly one vertex in Y.

Hall's Marriage Theorem, states that For a bipartite graph $G = (X \cup Y, E)$, there exists a matching that covers all vertices in X (i.e., matches every vertex in $X$to a unique vertex in Y) if and only if for every subset $S \subseteq X$, the number of neighbours of S in Y, denoted $|N(S)|$, is at least as large as $|S|$. Mathematically:

$\forall S \subseteq X, |N(S)| \geq |S|$,

where $N(S) = \{y \in Y \mid \exists x \in S \; such \; that \; (x, y) \in E\}$

Bipartite Graph for Hall's Marriage Theorem



The graph illustrates a scenario where Hall's condition is satisfied, as verified above. In the "marriage problem" context:

- $X = \{A, B, C\}$ could represent three people seeking partners.
- $Y = \{W, X, Y, Z\}$ could represent four potential partners.
- Edges represent acceptable matches (e.g., A is willing to match with W or X).
- The theorem guarantees that each person in X can be paired with a unique partner in Y, as demonstrated by the matching $A \to W, B \to X, C \to Y$.

If we were to modify the graph to violate Hall's condition (e.g., reduce Y to $\{W, X\}$ with edges $(A, W), (A, X), (B, X), (C, X)$, we'd find a subset like $S = \{A, B, C\}$ with $N(S) = \{W, X\}$, so $| N(S) | = 2 < 3 = | S |$, and no matching could cover all of X, illustrating the necessity of the condition.

This graph is a concrete example of Hall's Marriage Theorem in action. It shows how the theorem's condition ensures a matching exists, which is useful in real-world problems like assigning tasks, scheduling, or pairing resources.

10.4 COVERINGS

In graph theory, a covering graph is a subgraph which contains either all the vertices or all the edges corresponding to some other graph.

A subgraph which contains all the vertices is called an edge covering. A subgraph which contains all the edges is called vertex covering.

A vertex cover of an undirected graph G= (V, E) is a subset $V' \subseteq V$ such that if (u, v) is an edge of G, then either $u \in V'$ or $v \in V'$ (or both)

The size of vertex cover is the number of vertices in it.

The set of all the vertices in the graph is trivially a vertex cover for that graph. The problem is to find the minimum size of a vertex cover in a graph G.

Independent set

Independent set is a subset of vertices in a graph where no two vertices in the subset are connected by an edge.

The number of vertices in the independent set is the size of the independent set.

A subset with just a single vertex of the given graph is trivially an Independent set of the graph. The problem is to find an independent set with maximum size.

Theorem

In a graph G= (V, E), if S is an independent set, then V\S is a vertex cover and vice versa.

Proof:

Let,

S be an independent set.

$\Rightarrow$ No edge e= (u, v) in G, such that $u \in S \; and \; v \in S$

$\Rightarrow$ one of the u or v must lie in V\S

$\Rightarrow$ V\S is a vertex cover.

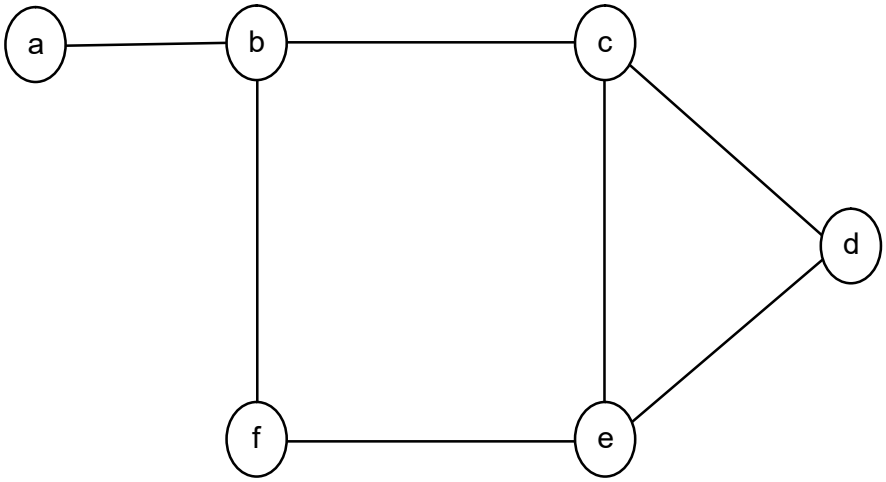
Conversely,

V\S is a vertex cover.

$\Rightarrow \forall$ (u, v) edge in G, at least one of the u or v lies in V\S.

$\Rightarrow$ No edge e= (u, v) in G, such that $u \in S \; and \; v \in S$

$\Rightarrow$ S is an independent set.



Vertex cover = {b, e, c}

Independent set = {a, f, d}

The number of vertices in a maximum independent set of G is called independence number (or the stability number) of G and is denoted by $\alpha(G)$. The number of vertices in a minimum covering is called the covering number denoted by $\beta(G)$.

Corollary:

For any graph $\alpha+\beta = \eta$

Edge covering

A non-empty subset S of E is called an edge covering or an edge covering of G if every non- isolated vertex in G is incident with at least one edge in S.

Trivial edge cover

E is itself an edge cover of G.

Minimal edge cover

An edge covering S of G is called a minimal edge cover if no proper subset of S is an edge covering of G.

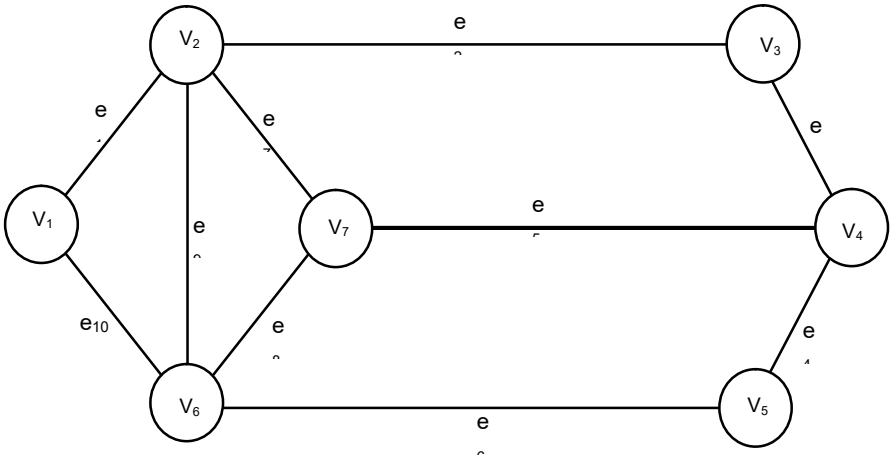


Fig. 7

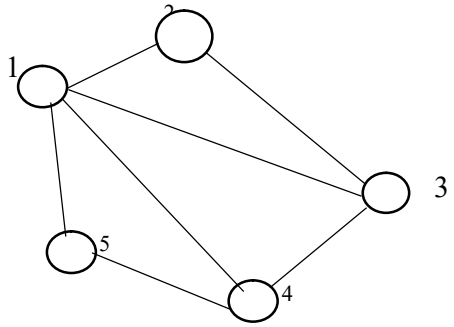
W= $\{v_2, v_4, v_6\}$ is a vertex cover

S= $\{e_2, e_4, e_6, e_8\}$ is an edge cover.

10.5 DOMINATING SETS

A Dominating set in a graph G is a set of vertices that dominates every vertex v in the graph G in the following sense: Either v is included in the dominating set or is adjacent to one or more vertices included in the dominating set. A graph can have many dominating set.

Example:



The dominating set in this graph can be: {1,5}, {4,2,3}, {1,4,5} and many others.

A dominating set need not be independent. For example, the set of all vertices of a given graph is a dominating set (trivially), but it is not an independent set unless all the vertices are isolated.

Minimal Dominating set:

Minimal Dominating set is a dominating set from which no vertex can be removed without destroying its dominance property.

For example: {1,5}, {4,3} are minimal dominating sets for the graph

But {1,4,5} is not a minimal dominating set, as the vertex '5' can be removed from this dominating set without violating the dominance property.

Some observations:

1. Any one vertex in a complete graph constitutes a minimal dominating set.
2. Every dominating set contains atleast one minimal dominating set.
3. A minimal dominating set may or may not be independent set.

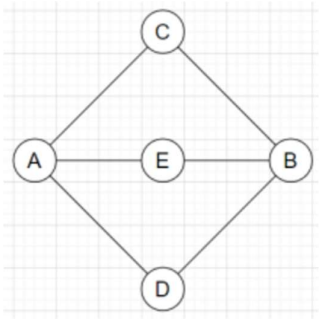
Domination Number:

A graph can have many minimal dominating sets of different sizes. The number of vertices in the smallest minimal dominating set of a graph G is called the domination number and is denoted by α(G).

In the above graph, the domination number is 2, i.e, α(G) = 2. Therefore, the size of the minimal dominating set can be 2.

W CHECK YOUR PROGRESS:

Q 1. What is the domination number of the graph given below?



Q 2. What is the size of the minimum vertex cover of a complete bipartite graph $K_{m,n}$

10.6 FACTORIZATION

The decomposition of a graph G into edge disjoint subgraphs $G_1, G_2, G_3, \ldots G_r$ such that

$$E(G) = \bigcup_{i=1}^{r} E(G_i)$$

is called graph's decomposition, and is denoted as

$$G = G_1 \oplus G_2 \oplus \ldots \oplus G_r$$

A factorization of a graph is a spanning subgraph of G, not necessarily connected.

If G's edges can be divided into edge- disjoint factors $G_1, G_2, G_3, \ldots G_r$, this collection $\{G_1, G_2, G_3, \ldots G_r\}$ is termed as factorization of G.

A k-factor is a spanning subgraph H of G which every vertex F has degree k.

A k-factorization of a graph G is a set of k-factors $G_1, G_2, G_3, \ldots . G_l$ such that

$G = G_1 \cup G_2 \cup \ldots . \cup G_l$ and $E(G_i) \cap E(G_j) = \phi$

1-factorization

When all the vertices in the disjoint factors of G have degree one.

A 1-factor is almost the same as perfect matching.

When G has 1-factor, say G_1 ,

→ |v| is even

→ the edges of G_1 are edge disjoint.

K_{2n+1} (complete graph with 2n+1 vertices) cannot have 1-factor,

∵ number of vertices is odd.

Theorem: The complete graph k_{2n} is 1-factorable.

Proof: We need to show that a partition of the set E into 2n-1factors.

Consider,

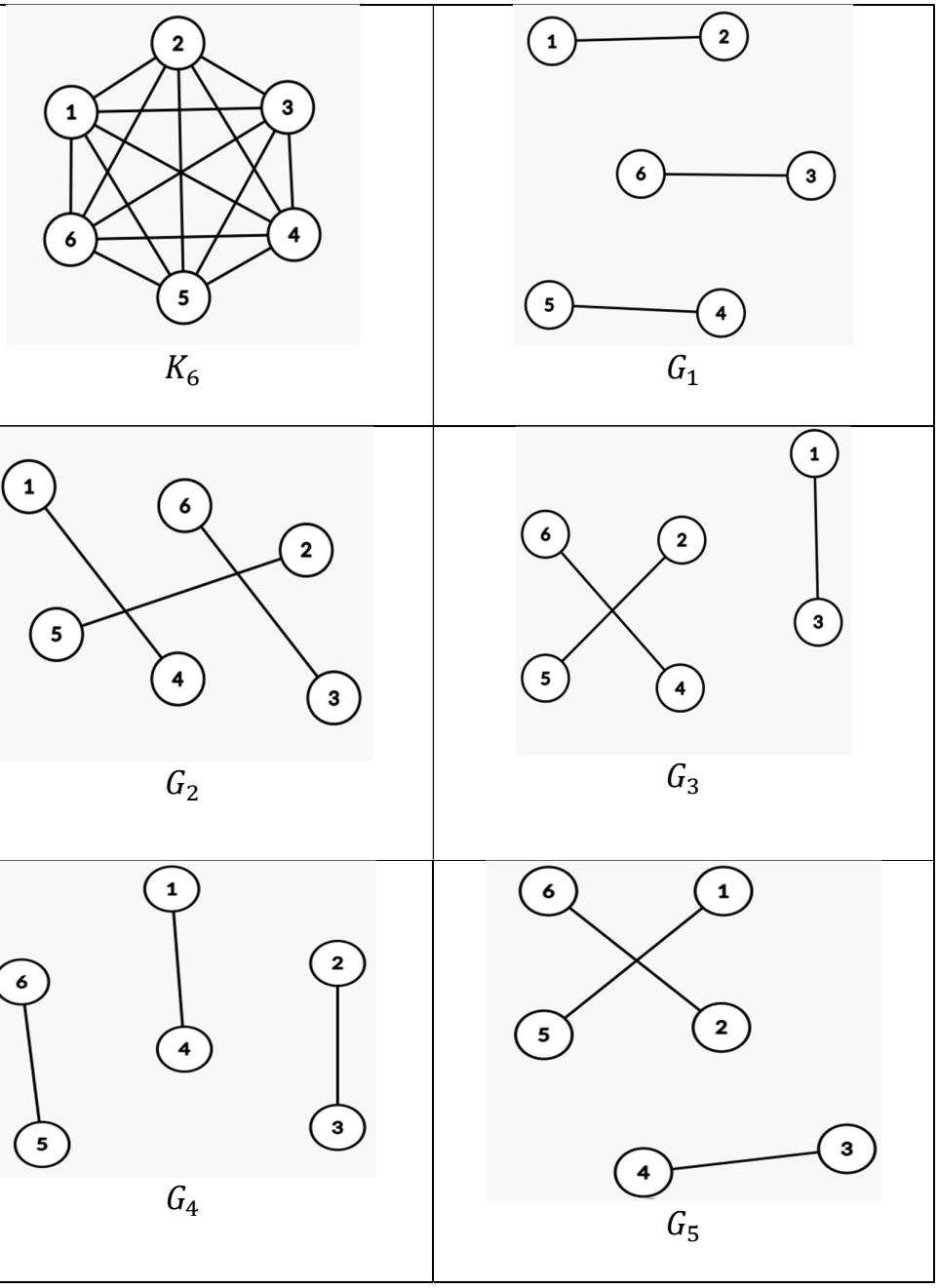
V= $\{v_1, v_2, \ldots, v_{2n}\}$be the vertices of k_{2n}

Define i=1, 2, 3, …,(2n-1)

The sets $E_i = \{v_i v_{2n}\} \cup \{v_{i-j} v_{i+j} \mid j = 1, 2, \ldots, (n-1)\}$ i+j and i-j are modulo (2n-1) operations.

The collection $\{E_i\}$ is easily seen to give an appropriate partition of E, and the union of the subgraphs G_i induced by the E_i is a 1-factorization of k_{2n}

Example:



K_6 G_1 G_2 G_3 G_4 G_5

2-factorization

If a graph is 2-factorable, then each factor must be union of disjoint cycles. If a 2-factor is connected, it is a spanning cycle.

- k_{2n} is not 2-factorable $\forall$ n
- k_{2n+1} is 2-factorable

Theorem: The graph k_{2n+1} is the sum of n-spanning cycles.

Proof: Let us label the vertices $v_1, v_2, \ldots, v_{2n+1}$

Then we construct n paths P_i on the points $v_1, v_2, \ldots, v_{2n}$ as follows:

$$P_i = v_i v_{i+1} v_i v_{i-2} \ldots v_{i+n-1} v_{i-n}$$

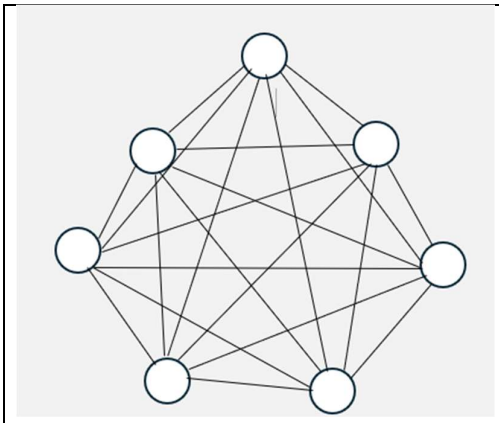
Thus,

j^{th} point of P_i is v_k, where

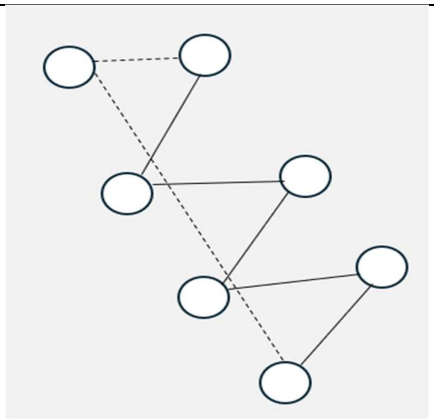
$$k = i + (-1)^{j+1} \left[\frac{j}{2}\right]$$

and all subscripts are taken as the integers 1, 2, …, 2n(mod 2n).

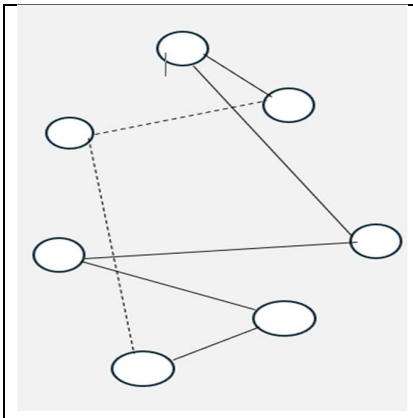
The spanning cycle z_i is then constructed by joining v_{2n+1} to the end point of P_i.



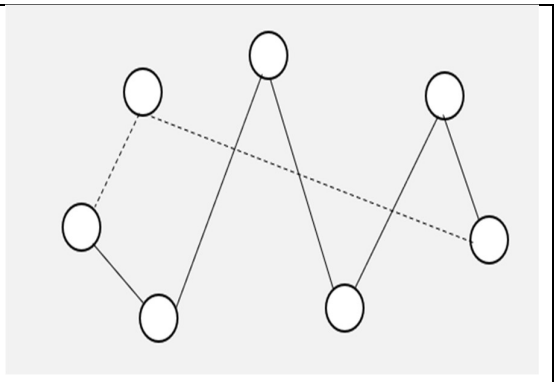
K_7



z_1



z_2

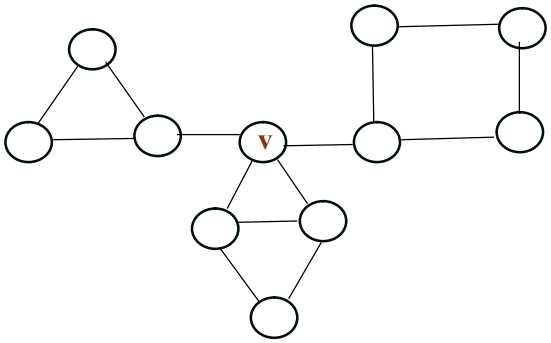


z_3

10.7 CRITICAL VERTICES AND EDGES

A critical vertex in a connected graph is a vertex whose removal (removal of that vertex and edges connected to it) increases the number of components. If v is a critical vertex of a connected graph, then G-{v} is disconnected. Removal of a critical vertex also lowers the chromatic number of the graph by one.

Example:



In this example if we remove the vertex 'v' from the graph and all the edges connected to the vertex 'v', we get three components.

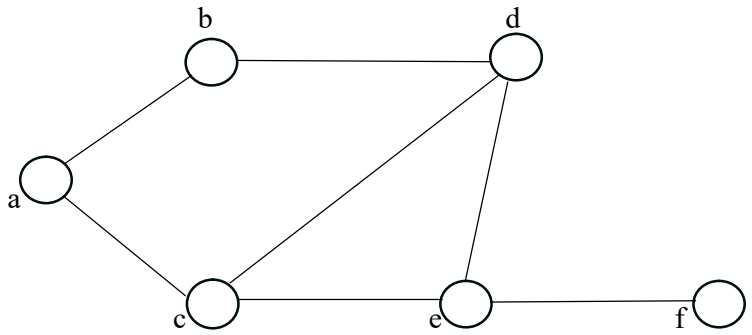
A critical edge is an edge whose removal produces a graph with more than one component then the original graph. It is also known as the bridge edge.

The set of all minimum number of edges of G whose removal disconnects a graph G is called a cut set of the graph G. The cut set S of a graph must satisfy the following:

- S is a subset of the edge set E of the graph G

- Removal of edges from a connected graph disconnects the graph
- No proper subset of G satisfy the condition

Example:



In the above graph, each of the sets {{b,d}, {c,d}, {c,e}} and {{e,f}} is a cut set. The edge {e,f} is the only critical edge or bridge edge.

210.8 LET US SUM UP:

- This unit covers some fundamental graph theory concepts like matchings, coverings, dominating sets, factorizations etc. Matching in a graph is a set of edges without any common vertex. Maximum matching is the largest possible matching in a graph while perfect matching is a matching in a graph that covers all vertices (every vertex gets saturated). Key Theorems involving matching are: Berge's Theorem which states that a matching is maximum iff no augmenting path exists. Hall's Marriage Theorem states that a bipartite graph has a perfect matching iff every subset of one partition has enough neighbors in the other.
- Vertex cover is a subset of vertices of that covers all edges while edge cover is s subset of edges of the graph that covers all the vertices of the graph. Dominating set is a subset of vertices of the graph such that every vertex of the graph is in the subset or adjacent to a vertex of the subset. The size of the smallest dominating set is the domination number.
- A k-factorization refers to decomposing a graph into edge-disjoint k-regular spanning subgraphs. 1-factorization is equivalent to perfect matching whereas 2-factorization is the union of disjoint cycles.
- Critical vertex is a vertex whose removal increases the number of components in the graph while critical edge (Bridge edge) is an edge in the graph whose removal increase the number of components in the graph.

K 10.9 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q1: The domination number of the graph is given by the smallest size of a minimal dominating set. Following the greedy approach for the given graph, vertices A and B have the maximum degree and cover all the adjacent vertices of the graph. The minimal dominating set of the graph will be {A, B}. Hence, the domination number of the graph is 2.

Ans to Q2: In a complete bipartite graph $K_{m, n}$, all the vertices of one side are connected to all the vertices of another side. Hence, the minimum number of edges can be covered by selecting the minimum number of vertices present on any of the sides. Thus the size of the minimum vertex cover is min{m,n}.

O 10.10 FURTHER READINGS:

1. F. Harary, Graph Theory, Narosa Publishing House, New Delhi, 1988.

2. R. Balakrishnan and K. Rengananthan, a textbook of Graph Theory, Springer, 2000.

3. G. Chartrand, L. Lesniak, Graphs & digraphs (Fourth edition), Chapman & hall/CRC, 2005.

4. Narasingh Deo, Graph Theory with applications to engineering and computer science, 2011.

5. R. J. Wilson, Introduction to Graph Theory (5^{th} Edition), Prentice Hall, 2010.

M 10.11 MODEL QUESTIONS

1. Compute the number of perfect matchings in a complete graph on 2n vertices.
2. Prove that if a tree has a perfect matching, then that matching is unique.
3. Prove that every pendant edge of a graph must belong to every edge cokvering of the graph.
4. Prove that every regular biograph is 1-factorable.
5. Prove Every bridgeless cubic graph is the sum of 1-factor and 2-factor.

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