UNIT 11 : COLOURABILITY

UNIT STRUCTURE

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11.1 LEARNING OBJECTIVES:

After going through this unit, you will be able to-

- to understand the techniques for assigning colours to certain elements like vertices and edges of a graph with some constraints and to apply them in real life world.
- to understand the concept of Chromatic polynomial and chromatic number.
- to understand the applications of colouring problems in various fields like network design, scheduling problems etc.

11.2 INTRODUCTION:

Graph colouring problem involves assigning colours to certain elements of a graph subject to certain restrictions and constraints such as: no two adjacent elements have the same colour.

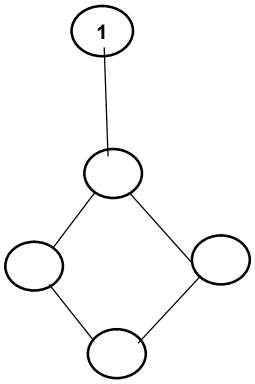
Vertex colouring is colouring vertices of a graph such that no two adjacent vertex have the same colour and edge colouring is colouring edges of a graph such that no two adjacent edge have the same colour.

11.3 CHROMATIC NUMBER

The Chromatic number of a graph is the minimum number of colours needed to colour the vertices of the graph so that no two adjacent vertices (vertices connected by an edge) have the same colour. A graph that requires κ different colours for proper colouring and cannot be properly coloured using less than κ colours is called κ −chromatic graph and the number κ is called chromatic number.

In colouring graphs there is no point in discussing about disconnected graphs as colouring one component has no effect in the other component. So, we generally consider only connected graphs.

Example:

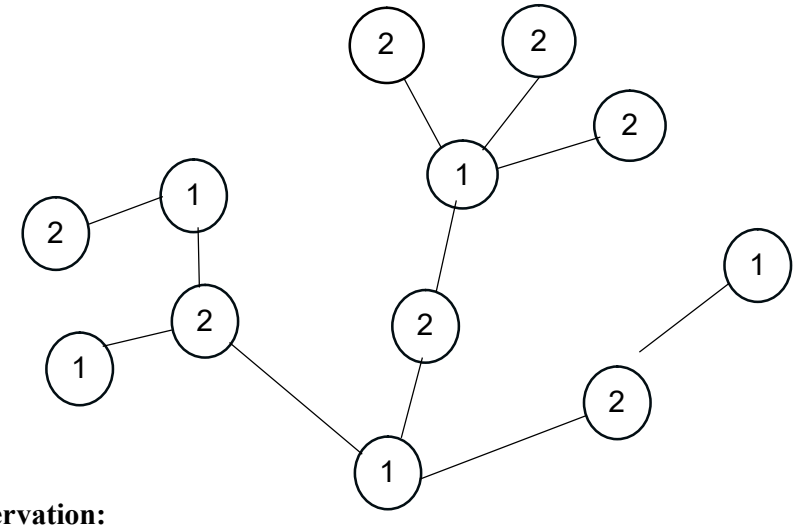


Observations:

i) A graph consisting of only isolated vertices is 1-chromatic
ii) A graph with one or more edges is at least 2-chromatic
iii) A complete graph K_n is n-chromatic.

Theorem 11.1: Every tree with two or more vertices is 2-chromatic

Proof: Select any vertex v in the given tree T. Consider T as the root and paint it with colour 1. Paint all vertices adjacent to vertex v with colour 2. Then paint all vertices adjacent to those vertices with colour 1. Repeat the procedure until the graph is properly coloured.



Observation:

A tree is 2-chromatic, not every 2-chromatic graph is a tree.

Theorem 11.2: A graph with atleast one edge is 2-chromatic if and only if it has no circuit of odd length.

Proof: Let G be a connected graph with circuits of only even lengths. Consider a spanning tree T in G. Using the colouring procedure and the result in Theorem 11.1, let us properly colour T with two colours. Now add the chords to T one by one. Since G had no circuits of odd length, the end vertices of every chord being replaced are differently coloured in T. Thus, G is coloured with two colours, with no adjacent vertices having the same colour. That is G is 2-chromatic.

Conversely, if G has a circuit of odd length, we would need at least three colours just for that circuit. Hence, Proved.

11.4 CHROMATIC PARTITIONING

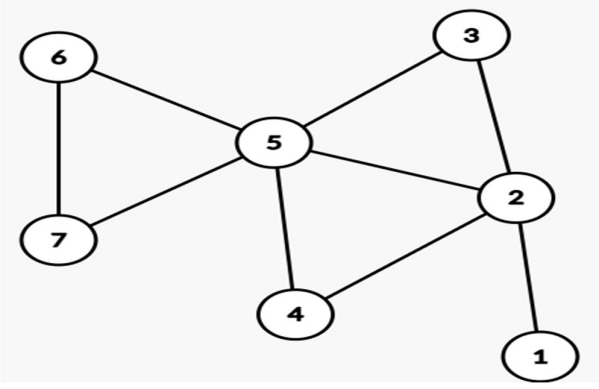
Given a simple, connected graph G, partition all vertices of G into smallest possible number of disjoint, independent sets. This problem is known as chromatic partitioning.

Observations:

1) No two vertices in any of these subsets are adjacent. Such a subset of vertices is called independent set.

2) The number of vertices in the largest independent set of a graph is called independence number and is denoted by β(G).

3) Consider a κ- chromatic graph G of n-vertices properly coloured with κ- different colours.
Since, the largest number of vertices in G with the same colour cannot exceed the independence number β(G). Therefore, $\beta(G) \geq \frac{n}{\kappa}$



The following four are some are chromatic partitioning of the graph:
{(a,c,d,f), (b,g), (e)}
{(a,c,d,g), (b,f), (e)}

{(c,d,f), (b,g), (a,e)}
{(c,d,g), (b,f), (a,e)}

11.5 CHROMATIC POLYNOMIAL

The chromatic polynomial $P_n(\lambda)$ of a graph with 'n' number of vertices gives the number of ways of properly colouring the graph using λ or fewer number of colours.

Let, c_i be the different ways of properly colouring the graph G using exactly i different colours.

Since i colours can be chosen out of λ colours in $\binom{\lambda}{i}$ ways.

There are $c_i\binom{\lambda}{i}$ different ways of properly colouring the graph G using exactly i colours out of λ colours.

Since $i \in [1, n]$ (as it is not possible to use more than n colours on n vertices).

The Chromatic polynomial is a sum of these terms.

$$P_n(\lambda) = \sum_{i=1}^{n} c_i\binom{\lambda}{i}$$
$$= c_1\frac{\lambda}{1!} + c_2\frac{\lambda(\lambda-1)}{2!} + \cdots . + c_n\frac{\lambda(\lambda-1)\ldots(\lambda-n+1)}{n!}$$

For a given graph, c_i can be evaluated.

For example, for any graph with atleast one edges requires atleast two colours for proper colouring and therefore, $c_1 = 0$

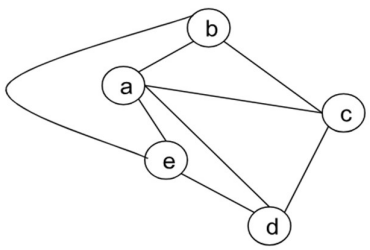
A graph with n vertices and using n different colours can be properly coloured in $n!$ Ways, i.e,

$$c_n = n!$$

Theorem 11.3 : A graph of n vertices is a complete graph if and only if its chromatic polynomial is $P_n(\lambda) = \lambda(\lambda-1)(\lambda-2)\ldots(\lambda-n+1)$.

Proof: With λ colours, there are λ different ways of colouring any selected vertex of a graph. A second vertex can be coloured properly in exactly $\lambda - 1$ ways, the third in $\lambda - 2$ ways, and so on, and the nth in $\lambda - n + 1$ ways if and only if every vertex is adjacent to every other. That is, iff the graph is complete.

Example:



Solution:

$$P_5(\lambda) = C_1\lambda + C_2\frac{\lambda(\lambda-1)}{2!} + \cdots + C_5\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}{5!}$$

Since the graph in the question has a triangle it will require atleast three different colours for proper colouring.

Therefore, $C_1 = 0, C_2 = 0$ and $C_5 = 5!$

To evaluate C_3, suppose we have three different colours x, y, z. These three colours can be properly assigned to vertices a, b, c in 3! = 6 ways.

After this, we have no colours left, vertex e must have the same colour as vertex c and vertex d must have the same colour as vertex b.

Therefore, $C_3 = 6$

Similarly, with four colours, vertex a, b, c can be properly coloured in 4*6 ways, i.e, 24 ways.

The fourth colour can be assigned to vertex d or e in 2 different ways.

Therefore $C_4 = 24 * 2 = 48$

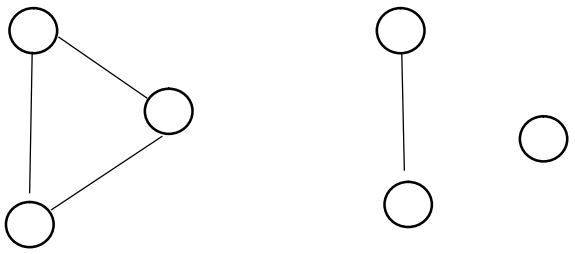
Substituting these values in $P_5(\lambda)$, we get

$$P_5(\lambda) = \lambda(\lambda-1)(\lambda-2) + 2\lambda(\lambda-1)(\lambda-2)(\lambda-3) + \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)$$

$$= \lambda(\lambda-1)(\lambda-2)(\lambda^2-5\lambda+7)$$

W CHECK YOUR PROGRESS:

Q1: Find the chromatic polynomial for the graph shown below:



Q2: What is the chromatic number of the complete bipartite graph $K_{m,n}$ where m and n are positive integers.

11.6 EDGE COLOURING

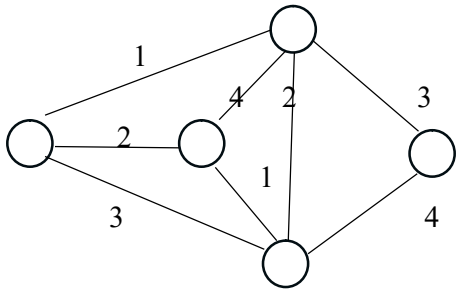
An edge colouring of a graph G is a function that assigns colour to each edge of the graph. If E(G) is the set of edges of the graph G, C is the set of distinct colours, then $f: E(G) \to C$. For a positive integer k, a k-edge coluring assigns k-different colours to the edges of the graph.

A proper edge colouring of a graph is an edge colouring such that no two adjacent edges are assigned the same colour. Thus, a proper edge colouring f of G is a function $f: E(G) \to C$ such that $f(e) \neq f(e')$ whenever e and e' are adjacent.

11.7 CHROMATIC INDEX

The chromatic index of a graph G, denoted by $X'(G)$ is the minimum number of different colours required for a proper edge colouring of G. G is a k-edge chromatic if $X'(G) = k$

Example: The graph given below has chromatic index equal to 4



Now we will state a Theorem without its proof.

If Δ is the largest vertex degree of G, then $X'(G) \geq \Delta$.

The following Theorem is known as Vizing's Theorem, it gives very sharp bounds for the chromatic index of a simple graph.

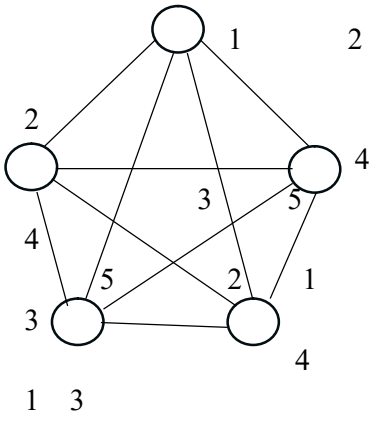
Theorem 11.4: If G is a simple graph with largest vertex degree Δ, then $\Delta \leq X'(G) \leq \Delta + 1$

Graphs that can be coloured with Δ colours are called class one graphs. Graphs that required $\Delta + 1$ colours are called class two graphs.

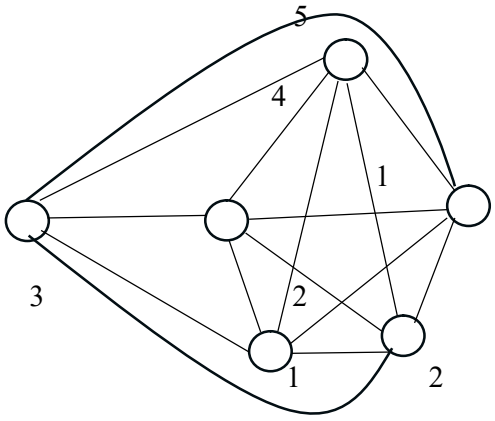
Theorem 11.5: $\boldsymbol{X'(K_n) = n}$ if n is odd $(n \neq 1)$, and $X'(K_n) = n - 1$ if n is even.

Proof: The result is trivial if the number of vertices is less than or equal to 2. Let us assume that the number of vertices is greater than or equal to 3.

If n is odd, then we can n-colour the edges of K_n by placing the vertices of K_n in the form of a regular n-gon, colouring the edges around the boundary with a different colour for each edge, and then colouring each remaining edge with the colour used for the boundary edge parallel to it (Fig. a) The fact that K_n is not (n-1) colourable follows by observing that the largest possible number of edges of the same colour is (n-1)/2 and so K_n has atmost (n-1)/2 * $X'(K_n)$ edges.



(Fig. a)



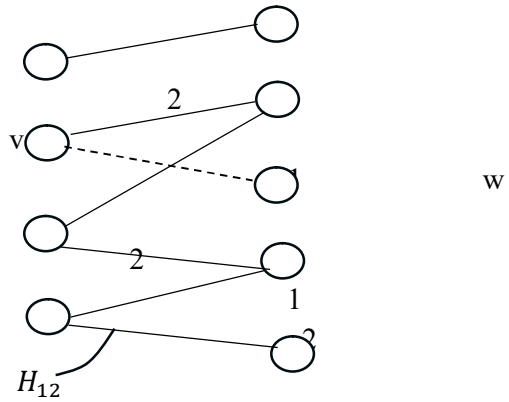
(Fig. b)

If n is even, we first obtain K_n by joining the complete graph K_{n-1} to a single vertex. If we now colour the edges of K_{n-1} as above, then there is one colour missing at each vertex, and these missing colours are different. We complete the edge colouring of K_n by colouring the remaining edges with these missing colours (fig. b)

Theorem 11.6: (Konig 1916). If G is a bipartite graph with largest vertex-degree Δ, then $X'(G) = \Delta$.

Proof: We use induction on the number of edges of G, and prove that if all but one of the edges have been coloured with at most Δ colours, then there is a Δ −colouring of the edges of G.

So, suppose that each edge of G has been coloured, except for the edge vw. Then there is at least one colour missing at the vertex v, and at least one missing at the vertex w. If some colour is missing from both v and w, then we colour the edge vw with this colour. If this is not the case, then let 1 be the colour missing at v, and 2 be a colour missing at w, and let H_{12}
be the connected subgraph of g consisting of the vertex v and those edges and vertices of G that can be reached from v by a path consisting entirely of edges coloured 1 or 2.



Since G is a bipartite graph, the subgraph H_{12} cannot contain the vertex w, and so we can interchange the colours 1 and 2 in this subgraph without affecting w or the rest of the colourings. The edge vw can now be coloured 2, thereby completing the colouring of the edges of G.

Corollary: $X'(K_{r,s})$ = max(r,s)

211.8 LET US SUM UP:

- In this unit we have discussed about vertex colouring and edge colouring. We have also defined the term chromatic number which is the minimum number of colours required to colour the vertices of a graph such that no two adjacent vertices have the same colour. We have also seen the chromatic polynomial of various graphs like complete graph and bipartite graph.
- As in the case of vertex colouring, in edge colouring the minimum number of colours required to colour all the edges of graph such that no two adjacent edge has the same colour is known as the chromatic index of the graph and is denoted by $X'(G)$. We have found the chromatic index of complete graph and bipartite graphs.

K 11.9 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q No.1 Solution: Since G is made up of three components where the first one is a complete graph with three vertices, the second one is a linear graph and the third one is an isolated vertex.

The complete graph (first component) can be coloured in $\lambda(\lambda-1)(\lambda-2)$ ways, the second component can be coloured in $\lambda(\lambda-1)$ ways and the thired component can be coloured in λ ways.

Therefore, by the rule of product, the chromatic polynomial of graph is

$$\lambda(\lambda-1)(\lambda-2)\lambda(\lambda-1)\lambda = \lambda^3(\lambda-1)^2(\lambda-3)$$

Ans to Q No.2 Solution: The number of colours needed may seem to depend on m and n.

However, we can colour the graph using only two colours. Colour the set of m vertices with one colour and the set of n vertices with the second colour.

Since, edges connect only a vertex from the set of m vertices and a vertex from the set of n vertices, no two adjacent vertices have the same colour.

O 12.10 FURTHER READINGS

1. F. Harary, Graph Theory, Narosa Publishing House, New Delhi, 1988.

2. R. Balakrishnan and K. Rengananthan, a textbook of Graph Theory, Springer, 2000.

3. G. Chartrand, L. Lesniak, Graphs & digraphs (Fourth edition), Chapman & hall/CRC, 2005.

4. Narasingh Deo, Graph Theory with applications to engineering and computer science, 2011.

5. R. J. Wilson, Introduction to Graph Theory (5th Edition), Prentice Hall, 2010.

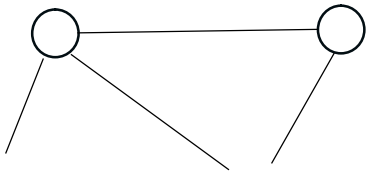
M 12.11 MODEL QUESTIONS

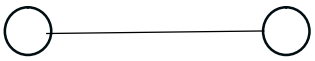
Q1. Prove that: An n-vertex graph is a tree if and only if its chromatic polynomial is:

$$P_n(\lambda) = \lambda(\lambda-1)^{n-1}$$

Q.2. In how many ways a tree on five vertices can be properly coloured with atmost four different colours.

Q3. rite down the chromatic polynomial of the graph:

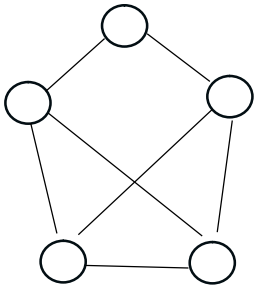


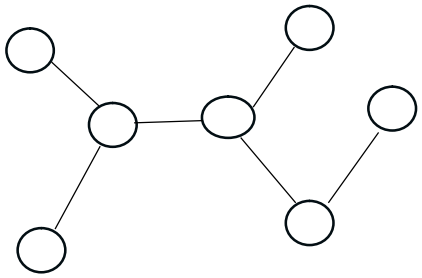


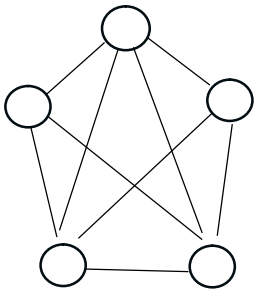
Q 3. What is the Chromatic number of the graph C_n?

Q.4,Find the chromatic polynomial and chromatic number for the graph $K_{3,3}$

Q5. Find the chromatic index of each graph:







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