UNIT 12: PLANAR GRAPHS

UNIT STRUCTURE

12.1 Learning Objective

12.2 Introduction to Planar graph

12.3 Kuratowski's Theorem

12.4 Outerplanar Graph

12.5 The five Colour problem

12.6 Dual graphs

12.7 Let us sum up

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12.1 LEARNING OBJECTIVE

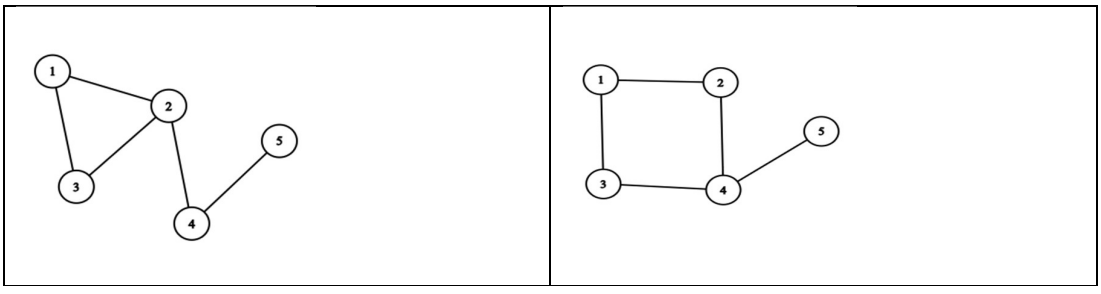
After completing this unit, students should be able to:

- understand and identify planar graphs.
- recognize Kuratowski's graphs.
- understand the five colour Theorem
- identify Outerplanar graphs
- understand the concept of dual graphs.

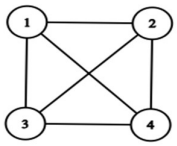
12.2 INTRODUCTION TO PLANAR GRAPH

A planar graph is a graph that can be drawn in the plane without any edge crossing, i.e, no two of its edges intersect. A graph that cannot be drawn on a plane without a crossover between its edges is called non-planar graph.

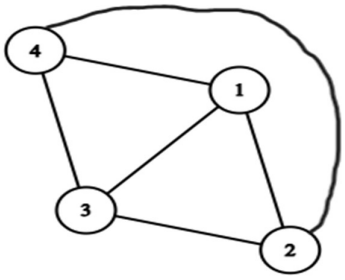
Examples:



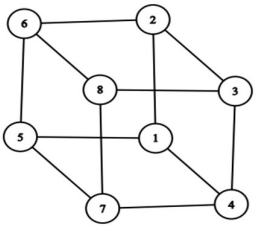
The graph:

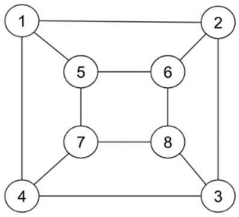


seems non-planar, but if we draw the graph in the given way:

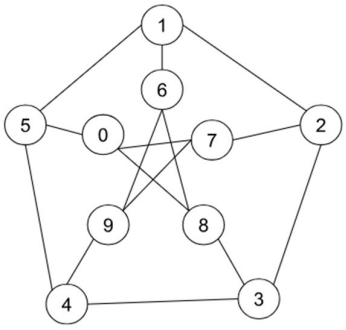


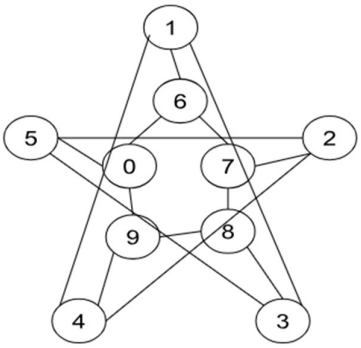
Then it is a planar graph, as no edge cross each other.





The above two graphs are different representations of the same graph. The first one is a planar representation but the second one is not. However, the graph is planar as there exist a planar representation.



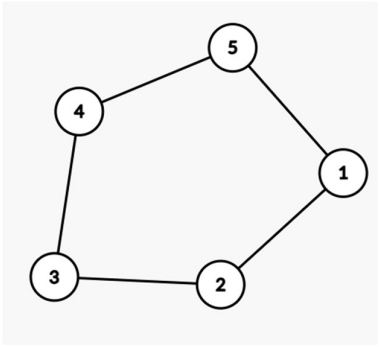


Any of the above representation cannot be drawn on a plane without edge cross-over. Hence, the given graph is non-planar.

12.3 KURATOWSKI'S TWO GRAPHS

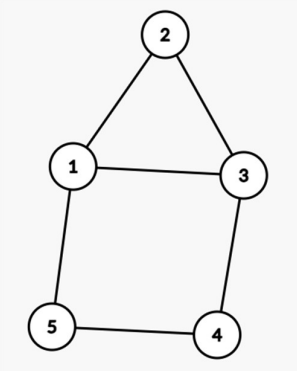
Theorem 12.1 : The complete graph of five vertices is nonplanar

Proof: Let the five vertices in the complete graph be named 1,2,3,4,5. A complete graph is a simple graph in which every vertex is joined to every other vertex by means of an edge. This being the case, we must have a circuit going from 1 to 2, 2 to 3, 3 to 4, 4 to 5, 5 to 1, i.e, a pentagon



This pentagon divides the plane into two regions, one inside and one outside.

Let us now connect the vertex 1 4 with 3 by an edge, this edge can be drawn inside or outside the pentagon. Let, us draw the edge inside the pentagon.



Now, we draw an edge from 2 to 4 and one from 2 to 5. Since, neither of the edge can be drawn inside the pentagon without crossing, we draw these edges outside the pentagon. Now, we draw an edge from 3 to 5.

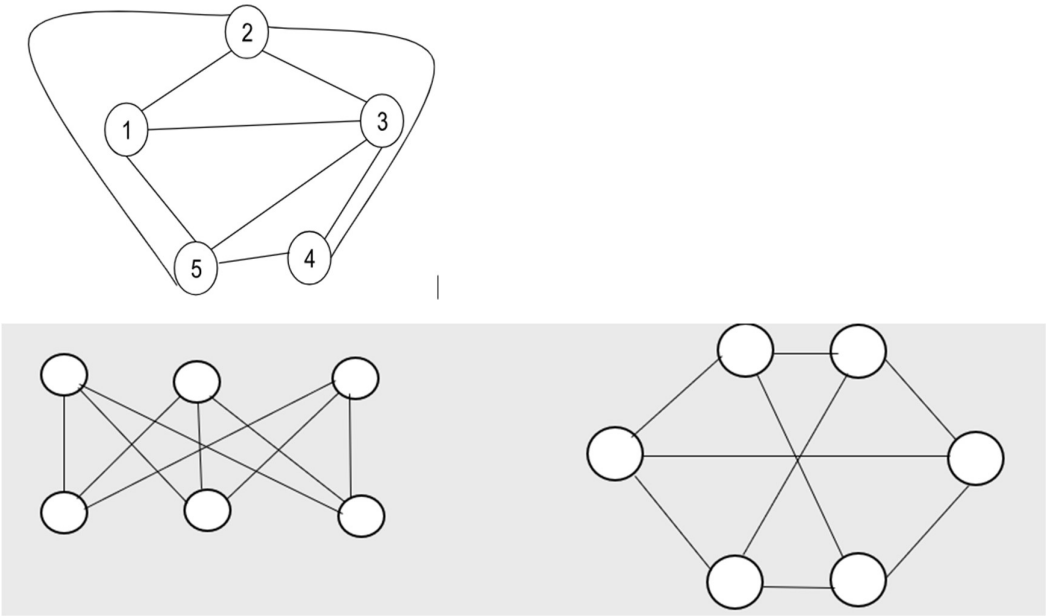


Fig: Kuratowski's second graph

Theorem 12.2: Kuratowski's second graph is also nonplanar.

(left as an exercise)

Common properties to the two graphs of Kuratowski's:

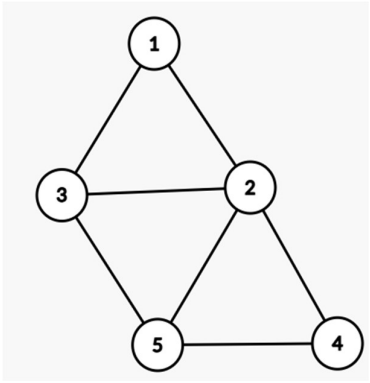
1) Both are regular graphs.
2) Both are nonplanar.
3) Removal of one edge or a vertex makes each a planar graph.

4) Kuratowski's first graph is the nonplanar graph with the smallest number of vertices, and Kuratowski;s second graph is the nonplanar graph with the smallest number of edges. Thus both are the simplest nonplanar graphs.

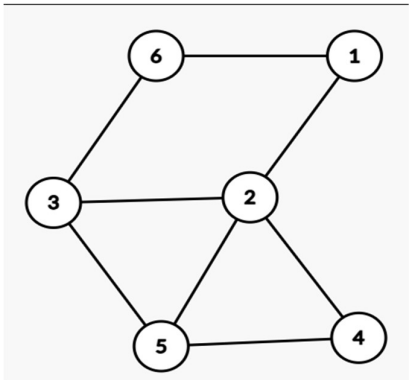
Homeomorphic graphs:

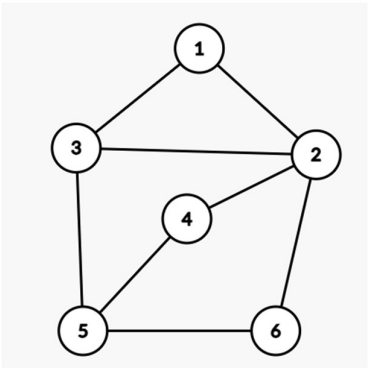
Two graphs are said to be Homeomorphic iff each can be obtained from the same graph by adding vertices (necessarily of degree 2) to edges.

The graph G_1 and G_2 are Homeomorphic graphs, since both are obtainable from the graph G in that Figure by adding a vertex to one of its edges.



Homeomorphic graphs to these graphs are:





Representation of a graph:

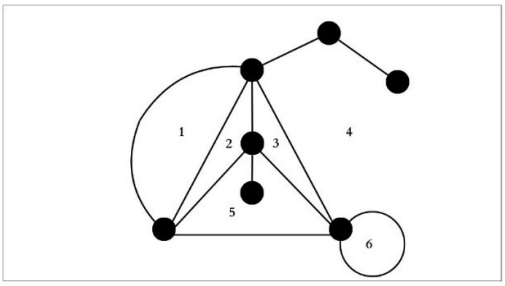
Region or Faces:

It refers to the areas enclosed by the edges and vertices when the graph has a planar representation. A region is characterised by the sdet of edges (or vertices) forming its boundary.

Infinite Region:

The portion of the plane lying outside a graph is called infinite, unbounded, outer or exterior region.

Graph:

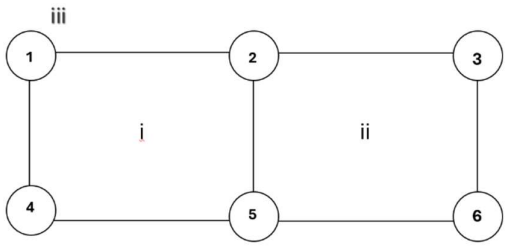


In the above graphical representation, Region 1,2,5,6,3 are finite regions while region 4 is infinite region.

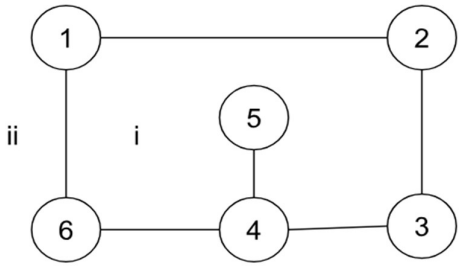
Degree of faces:

The boundary of a face is the subgraph containing all edges adjacent to that face and a boundary walk is a closed walk containing all of these edges. The degree of the face is the minimum length of a boundary walk.

Example:



The boundary of region I has edges 12, 25, 54, 41 and a boundary walk is 12, 25, 54, 41. Therefore, the degree of region i is 4. Similarly degree of region ii is 4. Degree of region iii is 6 as the boundary walk consist of edges 12, 23, 36, 65, 54, 41.

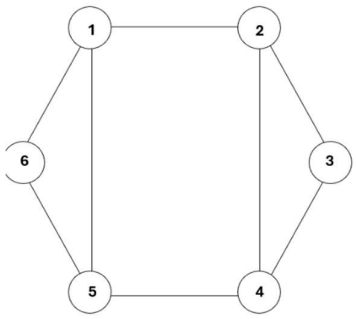


The boundary of the face 1 has edge 12, 23, 34, 45, 64, 61and a boundary walk is 12, 23, 34, 45, 54, 46, 61 so the degree of face 1 is 7.

12.4 OUTER PLANAR GRAPH

A planar graph G is said to be Outer Planar graph if all the vertices lie on the outer face.

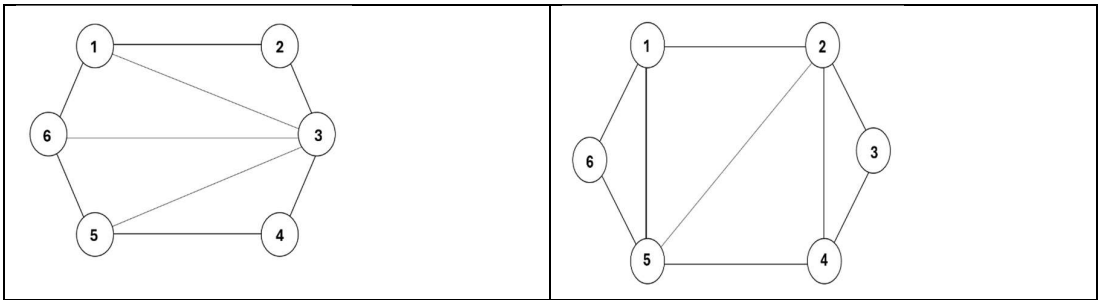
Example:



Maximal Outer Planar Graph

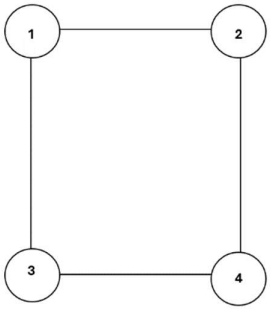
An Outer Planar graph G is called maximal Outer planar graph if no edge can be added without losing outer planarity.

Example:



The above graphs are Maximal outer planar.

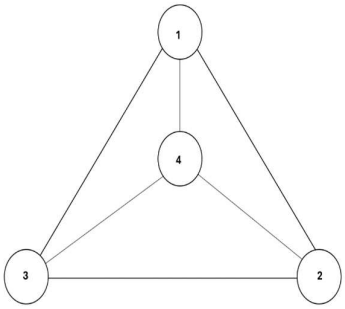
The graph:



is not maximal outer planar as if we add an edge (14) or (23) the graph still remains outer planar.

Minimally non outer planar: A graph is said to be minimally non outer planarif one one vertex lies in the interior region.

Example:



Theorem 12.3: Euler's Theorem:

If G is a connected plane graph with n-vertices, m edges and r-regions, then

$$n - m + r = 2 \qquad (1)$$

Proof: We will prove this theorem by Induction on m.

If m = 0: Then the graph is a connected graph with no edges, i.e, the graph can have only one vertex, so n = 1, m = 0, and r = 1.

Therefore, n - m + r = 1 - 0 + 1 = 2

Let, us suppose (1) holds when the number of edges in the graph is less than m (2)

To show that: (1) holds when the number of edges is exactly m.

Two cases may arise:

Case 1: When the graph is a tree.

⇒ There are no cycles. So, if the number of edges is m - 1, the number of vertices is m and the number of region is one.

Thus, m - (m - 1) + 1 = m - m + 1 + 1 = 2

Case 2: When the graph is not a tree.

⇒ There are cycles

Let, C be a cycle in the graph. If we remove an edge from C, the graph will still be connected but the number of edges will decrease by one and the number of regions will also decrease by one.So, if there were m edges and r regions, now the number of edges will be m - 1, and the number of regions will be r - 1.

By induction hypothesis,

n - (m - 1) + (r - 1) = 2

⇒ n - m + r = 2

Thus, the theorem is true when the graph has n vertices and m-edges.

Corollaries:

Corollary 1: Suppose G is a connected planar graph with n vertices, m edges and r faces where $n \geq 3$. Then $m \leq 3n - 6$.

Proof: The sum of the degrees of the faces is equal to twice the number of edges. But, each face must have degree greater than or equal to three.

So, we have $3r \leq 2m$.

According to Euler's formula: n - m + r = 2

→ r = m - n + 2

→ 3r = 3m - 3n +6

So, $3r \leq 2m$

→$3m - 3n + 6 \leq 2m$

→$m \leq 3n - 6$

W CHECK YOUR PROGRESS

1. A connected plane graph has 10 vertices each of degree 3. In how many regions does a representation of this planar graph split the plane?
2. Show that the complete graph with five vertices K_5 is not coplanar.

12.5 FIVE COLOR THEOREM

Theorem 12.4 : Every planar graph can be 5-colured.

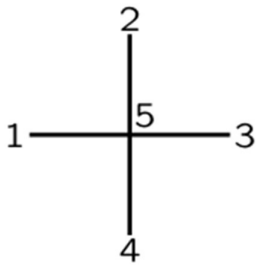
Proof: We will use induction on the number of vertices to prove the theorem

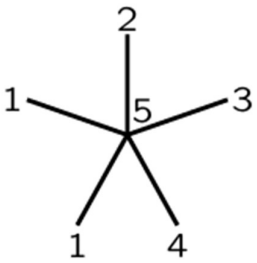
Base Case: If the graph contains only one vertex then pick a colour for that vertex and we are done.

Inductive step: Assume k>=1, and assume that every planar graph with k or lesser number of vertices can be 5-coloured.

Now, consider a graph with k+1 vertices. We know that the the graph has a vertex of degree 5 or fewer. Remove that vertex and all the edges connected to that vertex. Now the graph contains k vertices and by induction hypothesis, we can 5-colour the remaining graph. Let us now put the vertex and edges back. Thus, now we have a graph with all the vertices coloured except one.

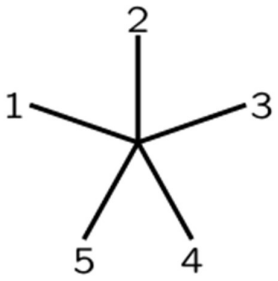
If that vertex(uncoloured) has degree less than 5 or has degree 5 and only four or fewer colours are used to colour the rest of the vertices then colour that vertex with the fifth available colour which is unused.

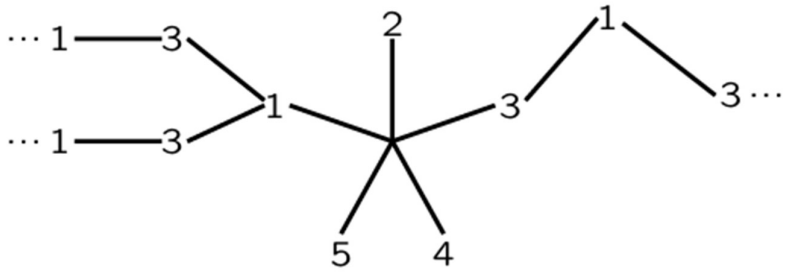




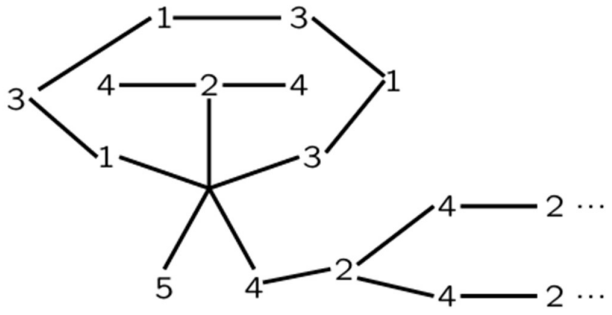
If that vertex has degree 5 and all the five colours have been used: Let us represent the colours with number 1, 2, 3 , 4, 5 for the five vertices connected to the uncoloured vertex.

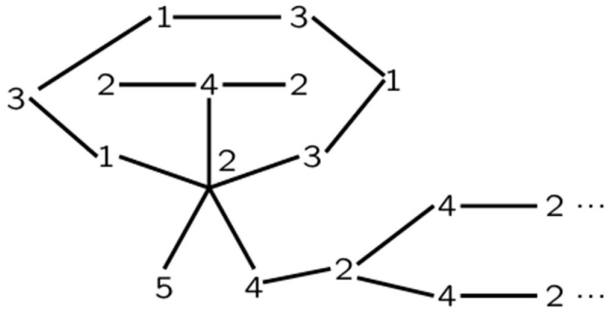
Now make a subgraph of the main graph of all vertices coloured 1 or 3 which are connected to vertices coloured 1 and 3 which are adjacent to the uncoloured vertex.





If the adjacent vertex(adjacent to uncoloured vertex) coloured 1 and the adjacent vertex(adjacent to uncoloured vertex) coloured 3 are not connected by a simple path in this subgraph, simply exchange the colours 1 and 3 throughout the subgraph connected to vertex coloured 1(adjacent to uncoloured vertex).





Thus, colour 1 can be used for the uncoloured vertex.

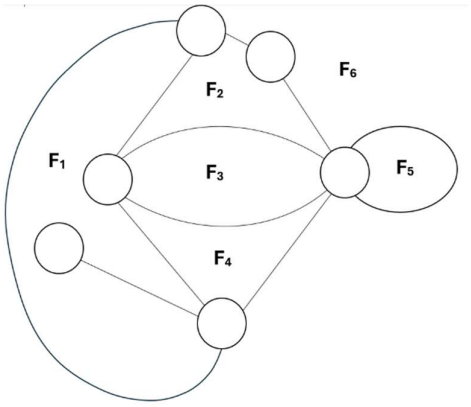
But if vertex 1 and 3 are connected via a simple path in the subgraph, we do the same subgraph process with the vertices coloured 2 and 4 adjacent the ‘special’ vertex. Note

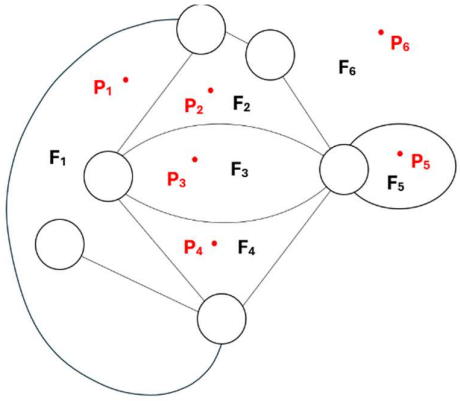
that this will be a disconnected pair of subgraphs, separated by a path connecting the vertices coloured 1 and 3. Now, we can exchange the coloured 2 and 4 in the subgraph connected to the adjacent vertex labelled 2. This will leave colour 2 for the 'uncoloured' vertex.

12.6 DUAL OF PLANAR GRAPHS

Consider the plane representation of a graph G (fig) with faces F1, F2, F3, …., F6. Let us place points p1, p2, p3, ….p6 in each of the six regions. Let us join the points pi's according to the following procedure:

Example:





Let, us join the points p1, p2, …, p6 in the following procedure:

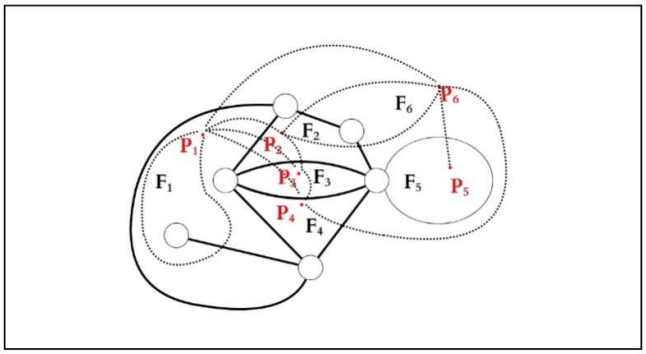
i) If two regions Fi and Fj are adjacent (i.e, have a common edge) draw a line between pi and pj that intersects the common edge between Fi and Fj.

ii) If more than one edge is common between face Fi and Fj, then draw one line between pi and pj for each of the common edge.

iii) For an edge lying completely in one region, say Fk draw a self-loop at point pk intersecting e exactly once.

By this procedure we obtain a graph G* in broken lines. Such a graph is dual.

Graph:



Observations:

i) An edge forming a self-loop in G yields a pendant edge in G*.

(Pendant edge: An edge incident to a pendant vertex. Pendant vertex: A pendant vertex in a graph is a vertex that is connected to only one edge, i.e, the vertex has degree one)

ii) A pendent edge in G yields a self-loop in G*.

iii) Edges that are in series in G, produces parallel edges in G*.

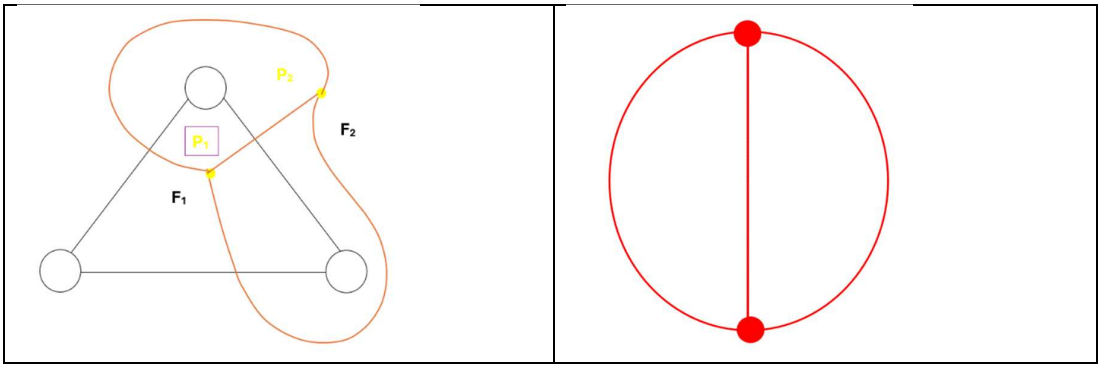
iv) Parallel edges in G produces series edge in G*.

v) The number of edges constituting teh boundary of a region Fi in G is equal to the degree of the corresponding vertex pi in G* and vice-versa.

vi) G* can also be embedded in the plane and therefore its planar.

vii) The number of vertices, edges are same in the graph and its dual.

Example:



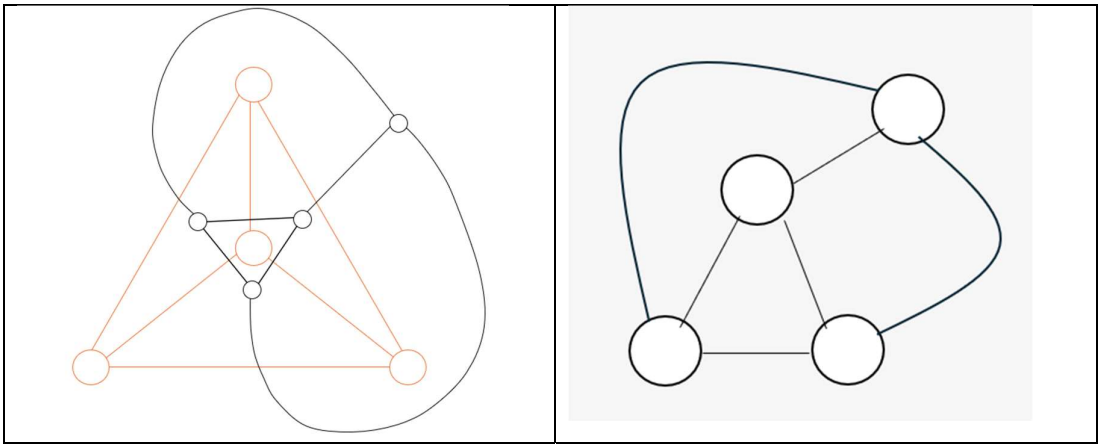
G G*

Uniqueness of Dual graphs: Given a planar graph G, we can construct more than one geometric dual of G. all duals of a planar graph G are 2-isomorphic and every 2-isomorphic to a geometric dual of G is also a geometric dual of G.

Double Dual: Given a planar graph G, suppose we construct a Geometric dual G* and the geometric dual G** of G*. Then, G** is called the double dual of G. If G is planar, then G** and G are 2-isomorphic.

Self-dual graphs: A planar graph G is said to be self-dual if G is isomorphic to its geometric dual G*, i.e, if G =' G*

Example:



K_4 K_4^*

Both K_4 and K_4^*have four vertices and six edges. Every vertex in K4* is joined by an edge same as in K_4. ThusK_4 and $K_4^*$are isomorphic. Hence, Both K_4 is a self-dual graph.

Dual of a subgraph:

Let, G be a planar graph and G* be the geometric dual. Let, e be the edge in G and e* be its dual edge in G*. Consider the subgraph G-e got by deleting e from G. Then the geometric dual of G-e can be constructed as explained in the two possible cases:

i) If e is common to the boundary of two regions:

Then removal of e from G will merge these two regions into one, amd the corresponding two vertices get merged into one, and the edge e* gets deleted.

ii) Suppose e is not on the boundary of two regions in G.

This implies e is a pendant edge in G, so e* is a self loop in G*

This implies dual of G-e is same as dual of G*-e*.

The dual of G-e can be constructed for any choice of e.

Since, Every subgraph H of a graph is of the form G-'s' where s is the set of edges.

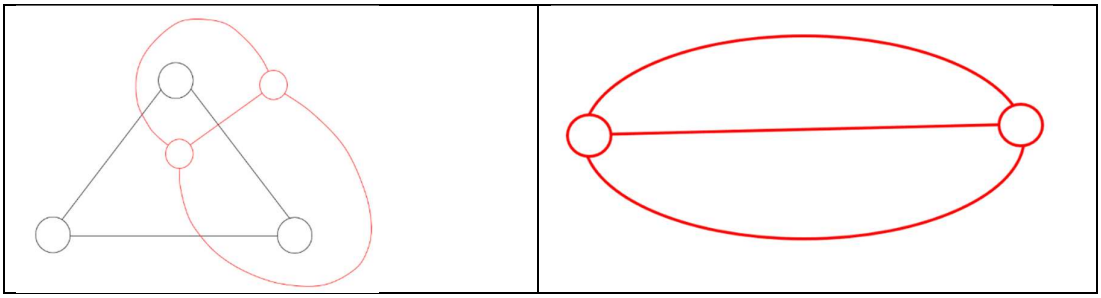
Abstract duals

A graph G* is said to be an abstract dual of graph G if there is one to one correspondence between the edges of G and those of edges of G* form a cut set.

Duality Theorem

If G* is an abstract dual of G, then G is an abstract dual of G*

Example:



G G*

If G is a planar graph then any geometric dual serves as an abstract dual.

212.7 LET US SUM UP

- A planar graph refers to a graph which has a planar representation i.e, it can be drawn on a plane without any edge crossing whereas a non-planar graph (eg., Kuratowski's K_5 and $K_{3,3}$) cannot. Some major theorems related to it are:

- Euler's Formula: For a planar graph G, n-m+r=2, where n is the number of vertices, m is the number of edges and r is the number of regions in G.
- Five-Colour Problem: Every planar graph can be coloured with atmost five colours.
- An Outerplanar graph is a graph that has all its vertices on the outer face. A dual graph G* is formed from the main graph by representing each face/region as a vertex and edge crossing shared boundaries. We have also discussed double dual which is the dual of the dual (G**), self dual graph (when G is isomorphic to G*), Abstract dual.

M 12.8 ANSWERS TO CHECK YOUR PROGRESS

An to Q1: Here, n = 10 and degree of each vertex is 3.

So, the sum of degree of all the vertices = 3*10 = 30

But, the sum of degree of all vertices = 2*(number of edges)

the sum of degree of all vertices = 2m

Therefore, 30 = 2m

Hence, m =15. The total number of edges is 15.

By Euler's Formula, we have n-m+r = 2

So, 10 -15 + r = 2

r = 7

Ans to Q2:

K_5 has five vertices and 10 edges. Therefore, n = 5, m = 10
So, 3n -6 = 3*5 – 6 = 9 < e
Thus, the graph violates the inequality $m \leq 3n - 6$ and hence it is not coplanar

O 12.9 FURTHER READINGS

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2. R. Balakrishnan and K. Rengananthan, a textbook of Graph Theory, Springer, 2000.

3. G. Chartrand, L. Lesniak, Graphs & digraphs (Fourth edition), Chapman & hall/CRC, 2005.

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M 12.10 MODEL QUESTIONS

Q1. Show that the graph K_5 and $K_{3,3}$ is non coplanar.

Q2. Prove that every subgraph of a planar graph is planar.

Q3. Prove that a planar graph G is isomorphic to G^{**} if and only if G is connected.

Q4. A simple planar graph to which no edge can be added without destroying its planarity is called maximal planar graph. Prove that every region in a maximal planar graph is a triangle.

Q5. For which values of r, s is the complete bipartite graph $K_{r,s}$ non-planar?

Q6. A connected graph has nine vertices having degrees 2, 2, 2, 3, 3, 3, 4, 4, 5. How many edges are there? How many faces are there?

Q7. Show that the condition $m \leq 3n - 6$ is not a sufficient condition for a connected simple graph with n vertices and m edges to be planar.

Q8. What is the minimum number of vertices necessary for a simple connected graphs with 11 edges to be planar.

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