UNIT 13: MATRICES: ASSOCIATED WITH GRAPHS

UNIT STRUCTURE

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13.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- learn about matrices associated with graphs
- learn the properties of these matrices
- explore the applications of matrices to solve problems in graph theory

13.2 INTRODUCTION

The study of graph theory involves the analysis of relationships between objects, represented as vertices connected by edges. Matrices provide a powerful tool for representing and analyzing graphs. This unit explores various matrices associated with graphs, including the adjacency matrix, incidence matrix, cycle matrix, path matrix, and cut-set matrix. Each of these matrices captures specific aspects of the graph's structure, enabling us to understand and solve problems related to connectivity, paths, cycles, and other graph properties.

13.3 ADJACENCY MATRIX

13.3.1 Representation of undirected graph

A graph G with n vertices and without parallel edges has an adjacency matrix $A = \{a_{ij}\}$, which is a $n \times n$ matrix with elements provided by

$$a_{ij} = \begin{cases} 1, & \text{if there is an edge between } i\text{th and } j\text{th vertices} \\ 0, & \text{if there is no edge between them} \end{cases}$$

The adjacency matrix for a particular graph is determined by the ordering of the vertices. For a graph of n vertices, there can be up to $n!$ different adjacency matrices due to the different ordering of vertices. Adjacency matrices can be easily obtained from one another by swapping rows and columns. The following are some observations regarding a graph G's adjacency matrix A:

(i) A is symmetric i.e., $a_{ij} = a_{ji}$ for all i and j

(ii) If the graph has no self-loops, then all of the entries along its principal diagonal are zeros. A self-loop at the vertex corresponding to $a_{ij} = 1$.

(iii) The degree of a vertex in a simple graph (one without parallel edges or a self-loop) is equal to the number of 1's in the corresponding row or column of A.

(iv) The number of pathways of length (number of edge occurrences) m from vertex v_i to vertex v_j is the (i, j) entry of A^m.

(v) Let G be a graph with n vertices $(v_1, v_2, \ldots, v_n)$, and let A be G's adjacency matrix with regard to this vertex listing. Let B be the matrix.

$$B = A + A^2 + A^3 + \ldots + A^{n-1}$$

If there are no zero entries in the main diagonal of B, then G is a connected graph. This result can also be used to determine a graph's connectedness using its adjacency matrix. Additionally, undirected graphs with loops and many edges can be represented using adjacency. A loop at vertex v_1 is indicated by a 1 at the (i, j)th position of the adjacency matrix. The adjacency matrix is no longer a zero-one matrix when there are numerous edges since the number of edges associated with $\{v_i - v_j\}$ is equal to the (i, j)th element. All undirected graphs, including multigraphs and pseudographs, have symmetric adjacency matrices.

13.3.2 Representation of directed graph

The matrix $A = \{a_{ij}\}_{n \times n}$ is the adjacency matrix of a diagonal D with n vertices, where

$$a_{ij} = \begin{cases} 1, & \text{if arc } \{v_i - v_j\} \text{ is in D} \\ 0, & \text{otherwise} \end{cases}$$

There are several observations that may be drawn about the adjacency matrix of a diagonal are as follows

(i) A may not be symmetric if there are no edges from v_i to v_j, but there may be an edge from v_j to v_i.

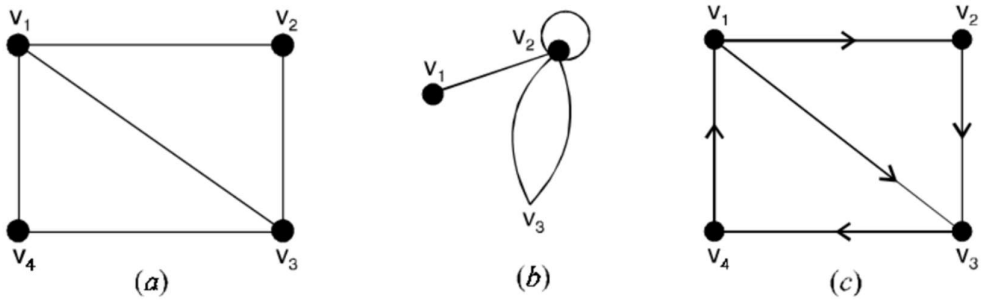
(ii) The number of arcs pointing towards v_j is equal to the sum of any column of j of A.

(iii) The number of arcs pointing away from vertex v_i (out degree of vertex v_i) equals the total of the entries in row i.

(iv). The number of paths of length m from vertex v_i to vertex v_j elements of A^T equals the (i, j) entry of A^m. A displays the vertices' degree.

Directed multigraphs can alternatively be represented by the adjacency matrices. Once more, when two vertices have several edges linking them in the same direction, the resulting matrices are not zero-one matrices. A directed multigraph's adjacency matrix has a_{ij} equal to the number of edges connected to (v_i, v_j).

Example 1: Use adjacency matrix to represent the graphs shown in Figure below



Solution:

(a) We arrange the vertices as follows: v_1, v_2, v_3 and v_4. Given that the graph has four vertices, its adjacency matrix will be a square matrix of order four. The required adjacency matrix A is

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

(b) We arrange the vertices as follows: v_1, v_2 and v_3. The adjacency matrix for the graph is as follows:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 0 \end{bmatrix}$$

(c) We arrange the vertices as follows: v_1, v_2, v_3 and v_4. The adjacency matrix for the graph is as follows:

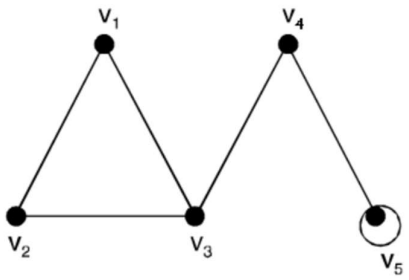
$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Example 2: Draw the undirected graphs corresponding to adjacency matrix

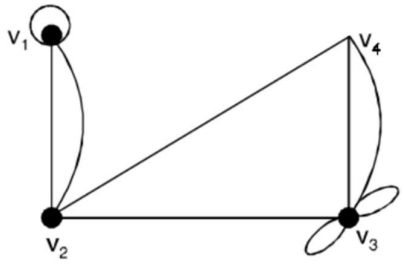
$$(a)\ A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \qquad (b)\ B = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

Solution:

(a) Since the given matrix A is a square of order 5, the graph G has five vertices v_1, v_2, v_3, v_4 and v_5. Draw an edge from v_i to v_j where $a_{ij} = 1$. The required graph is presented in Figure below.



(b) Since the adjacency matrix B is a square of order 4, the graph G has five vertices v_1, v_2, v_3 and v_4. Draw n edges from v_i to v_j where $a_{ij} = n$ and draw n loop at v_i where $a_{ii} = n$. The required graph is presented in Figure below.

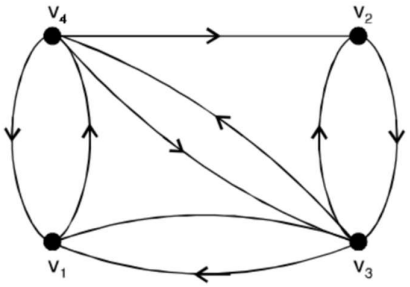


Example 3: Draw the directed graph G corresponding to adjacency matrix

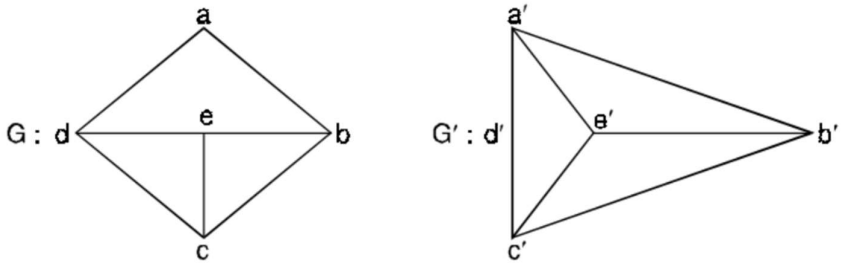
$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Solution:

Since the adjacency matrix A is a square of order 4, the graph G has five vertices v_1, v_2, v_3 and v_4. Draw an edge from v_i to v_j where $a_{ij} = 1$. The required graph is presented in figure below.



Example 4: Show that the graphs G and G' are isomorphic



Solution:

Consider the map $f: G \to G'$ define as $f(a) = d', f(b) = a', f(c) = b', f(d) = c'$ and $f(e) = e'$. The adjacency matrix of G for the ordering a, b, c, d and e is

$$A(G) = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

The adjacency matrix of G' for the ordering d', a', b', c' and e' is

$$\begin{array}{ccccc} d' & a' & b' & c' & e' \end{array}$$

$$A(G') = \begin{matrix} d' \\ a' \\ b' \\ c' \\ e' \end{matrix} \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

We get, $A(G) = A(G')$

$\therefore G$ and G' are isomorphic.

13.4 THE INCIDENCE MATRIX

13.4.1 Representation of undirected graph

Let $G = (V, E)$ be an undirected graph with n labelled vertices and m labelled edges. The incidence matrix $B = \{b_{ij}\}$ is a $n \times m$ matrix, where

$$b_{ij} = \begin{cases} 1, & \text{when edge } e_j \text{ is incident with } v_i \\ 0, & \text{otherwise} \end{cases}$$

The following are some observations we can make regarding the incidence matrix B of G:

(i) Each column in B contains exactly two-unit entries.

(ii) An isolated vertex is represented by a row that contains all 0 elements.

(iii) A pendent vertex is represented by a row with a single unit entry.

(iv) The degree of the corresponding vertex v_i is equal to the number of unit entries in row i of B.

(v) The labelling of the vertices (edges) of G is equivalent to the permutation of any two rows (any two columns) of B.

(vi) Two graphs are isomorphic if and only if there is only a permutation of rows and columns between their respective incidence matrices.

(vii) The rank of B is $n - 1$ if G is connected to n vertices.

Multiple edges and loops can also be represented by incidence matrices. The incidence matrix uses columns with similar entries to indicate many edges. since the same pair of vertices is incident with these edges. Loops are represented by a column with a single entry equal to 1, which corresponds to the vertex incident with the loop.

13.4.2 Representation of directed graph

The incidence matrix $B = \{b_{ij}\}$ of digraph D with n vertices and m edges is the $n \times m$ matrix in which

$$b_{ij} = \begin{cases} 1, & \text{if arc } j \text{ is directed away from a vertex } v_i \\ -1, & \text{if arc } j \text{ is directed towards vertex } v_i \\ 0, & \text{otherwise} \end{cases}$$

Theorem 1: Two graphs G_1 and G_2 are isomorphic if and only if their incidence matrices $A(G_1)$ and $A(G_2)$ differ only by permutation of rows and columns.

Proof:

Let the graphs G_1 and G_2 be isomorphic. Then there is a one-one correspondence between the vertices and edges in G_1 and G_2 such that the incidence relation is preserved. Thus $A(G_1)$ and $A(G_2)$ are either same or differ only by permutation of rows and columns.

The converse follows, since permutation of any two rows or columns in an incidence matrix simply corresponds to relabeling the vertices and edges of the same graph.

Rank of the incidence matrix

Let G be a graph and let $A(G)$ be its incidence matrix. Now each row in $A(G)$ is a vector over $GF(2)$ in the vector space of graph G. Let the row vectors be denoted by $A_1, A_2, \ldots, A_n$. Then,

$$A(G) = \begin{bmatrix} A_1 \\ A_2 \\ . \\ . \\ . \\ A_n \end{bmatrix}$$

Since there are exactly two ones in every column of A, the sum of all these vectors is 0 (this being a modulo 2 sum of the corresponding entries).

Thus, the vectors $A_1, A_2, \ldots, A_n$ are linearly dependent. Therefore, rank $A < n$.

Hence, rank $A \leq n - 1$.

Theorem 2: If $A(G)$ is an incidence matrix of a connected graph G with n vertices, then rank of $A(G)$ is $n - 1$.

Proof:

Let G be a connected graph with n vertices and let the number of edges in G be m. Let $A(G)$ be the incidence matrix and let $A_1, A_2, \ldots, A_n$ be the row vector of $A(G)$. Then,

$$A(G) = \begin{bmatrix} A_1 \\ A_2 \\ . \\ . \\ . \\ A_n \end{bmatrix}$$

Clearly, rank $A(G) \leq n-1$ … (1)

Consider the sum of any m of these row vectors, $m \leq n-1$. Since G is connected, $A(G)$ cannot be partitioned in the form

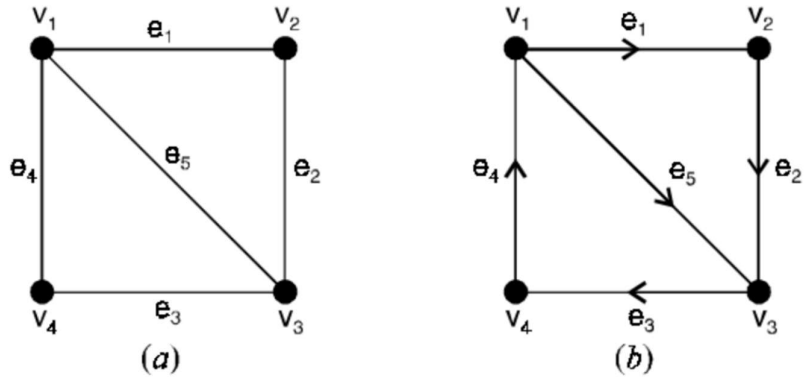
$$A(G) = \begin{bmatrix} A(G_1) & 0 \\ 0 & A(G_2) \end{bmatrix}$$

such that $A(G_1)$ has m rows and $A(G_2)$ has $n-m$ rows. Thus, there exists no $m \times m$ submatrix of $A(G)$ for $m \leq n-1$, such that the modulo 2 sum of these m rows is equal to zero. As there are only two elements 0 and 1 in this field, the additions of all vectors taken m at a time for $m = 1,2,..,n-1$ gives all possible linear combinations of $n-1$ row vectors. Thus, no linear combinations of m row vectors of A, for $m \leq n-1$, is zero.

Therefore, rank $A(G) \geq n-1$ … (2)

Combining (1) and (2), it follows that rank $A(G) = n-1$

Example 5: Find the incidence matrix to represent the graph shown in Figure below:



Solution:

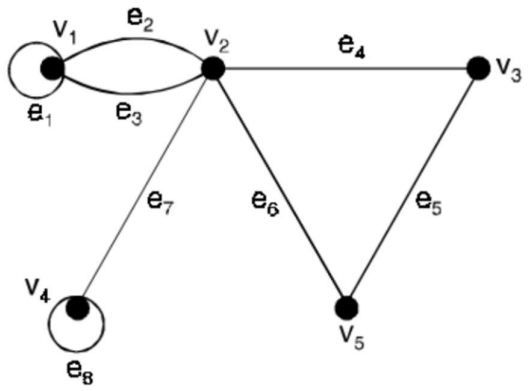
(a) The incidence matrix is obtained by entering for row v and column e is 1 if e is incident on v and 0 otherwise. The incidence matrix is

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

(b) The incidence matrix is

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix}$$

Example 6: Represent the Pseudograph shown in Figure below, using an incidence matrix

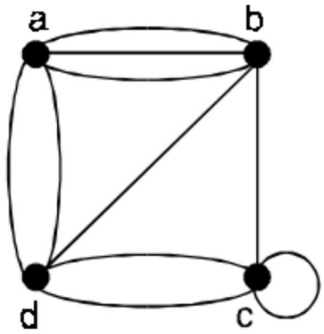


Solution: The incidence matrix is

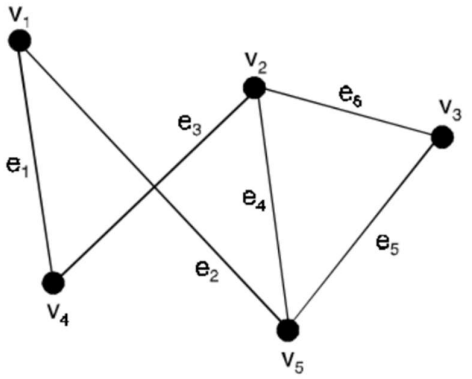
$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

W CHECK YOUR PROGRESS 1

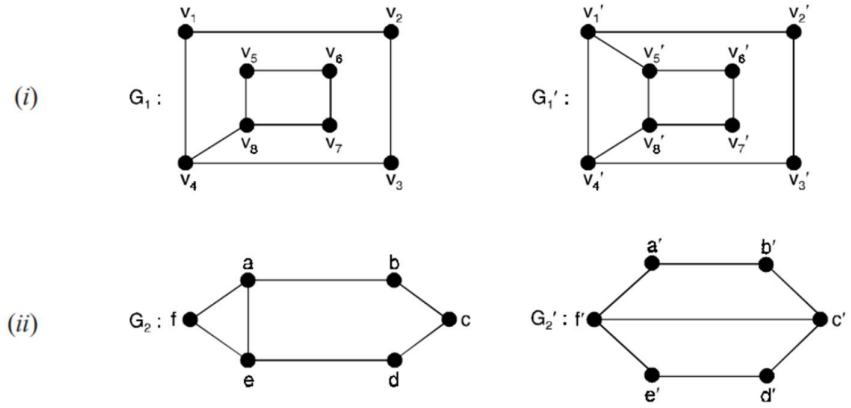
Q1. Use an adjacency matrix to represent the pseudograph shown in Figure below:



Q2. Represent the graph shown in Figure below, with an incidence matrix



Q3. Determine whether the following graphs are isomorphic



13.5 CYCLE MATRIX

Let the graph G have m edges and let q be the number of different cycles in G. The cycle matrix $B = \{b_{ij}\}_{q \times m}$ of G is a (0,1) matrix of order $q \times m$, where

$$b_{ij} = \begin{cases} 1, & \text{if the } i\text{th cycle includes } j\text{th edge} \\ 0, & \text{otherwise} \end{cases}$$

The cycle matrix B of a graph G is denoted by $B(G)$.

The circuit matrix observations include the following:

(i) A column of all zeros corresponds to a non-cycle edge, i.e. an edge which does not belong to any cycle.

(ii) Each row of $B(G)$ is a cycle vector.

(iii) A cycle matrix has the property of representing a self-loop and the corresponding row has a single one.

(iv) The number of ones in a row is equal to the number of edges in the corresponding cycle.

(v) If the graph G is separable (or-disconnected) and consists of two blocks (or components) H_1 and H_2, then the cycle matrix $B(G)$ can be written in a block-diagonal form as

$$B(G) = \begin{bmatrix} B(H_1) & 0 \\ 0 & B(H_2) \end{bmatrix}$$

Where $B(H_1)$ and $B(H_2)$ are the cycle matrices of H_1 and H_2. This follows from the fact that cycles in H_1 have no edges belonging to H_2 and vice versa.

(vi) Permutation of any two rows or columns in a cycle matrix corresponds to relabeling the cycles and the edges.

(vii) We know two graphs G_1 and G_2 are 2-isomorphic if and only if they have cycle correspondence. Thus, two graphs G_1 and G_2 have the same cycle matrix if and only if G_1 and G_2 are 2-isomorphic. This implies that the cycle matrix does not specify a graph completely, but only specifies the graph within 2-isomorphism.

Theorem 3: If G is a graph without self-loops, with incidence matrix A, and cycle matrix B whose columns are arranged using the same order of edges, then every row of B is orthogonal to every row of A, i.e.

$$AB^T = BA^T \equiv 0 \ (mod\ 2)$$

Where A^T and B^T are the transposes of A and B respectively.

Proof:

Consider a vertex v and a circuit Γ in the graph G. Either v is in Γ or it is not. If v is not in Γ, there is no edge in the circuit Γ i.e. incident on v. On the other hand, if v is in Γ,the number of those edges in the circuit Γ that are incident on v is exactly two.

With this remark in mind, consider the ith row in A and the jth row in B. Since the edges are arranged in the same order, the nonzero entries in the

corresponding positions occur only of the particular edge is incident on the ith vertex and is also in the jth circuit.

If the ith vertex is not the jth circuit, there is no such nonzero entry, and the dot product of the two rows is zero. If the ith vertex is in the jth circuit, there will be exactly two 1's in the sum of the products of individual entries. Since $1 + 1 = 0(mod\ 2)$, the dot product of the two arbitrary rows one from A and the other from B is zero. Hence, the theorem.

Theorem 4: Sylvester's Theorem- If A and B are matrices of order $k \times n$ and $n \times p$ respectively, then $nullity(AB) \leq nullity(A) + nullity(B)$

Theorem 5: If B is a cycle matrix of a connected graph G, with n vertices and m edges, then rank $B = m - n + 1$

Proof: Let A be the incidence matrix of the connected graph G.

Then $AB^T \equiv 0\ (mod\ 2)$

Using Sylvester's theorem, we have rank A+ rank $B^T \leq m$ so that

rank A+ rank $B \leq m$

Therefore, rank $B \leq m$ –rank A

As rank $A = n - 1$, we get rank $B \leq m - (n - 1) = m - n + 1$

But, rank $B \geq m - n + 1$

Combining, we get rank $B = m - n + 1$

Theorem 6: If B is a cycle matrix of a disconnected graph G, with n vertices, m edges and k components, then rank $B = m - n + k$

Proof:

Let B be the cycle matrix of the disconnected graph G with n vertices, m edges and k components. Let the k components $G_1, G_2, G_3, \ldots, G_k$ have the number of vertices $n_1, n_2, \ldots, n_k$ vertices respectively and the number of edges $m_1, m_2, \ldots, m_k$ respectively.

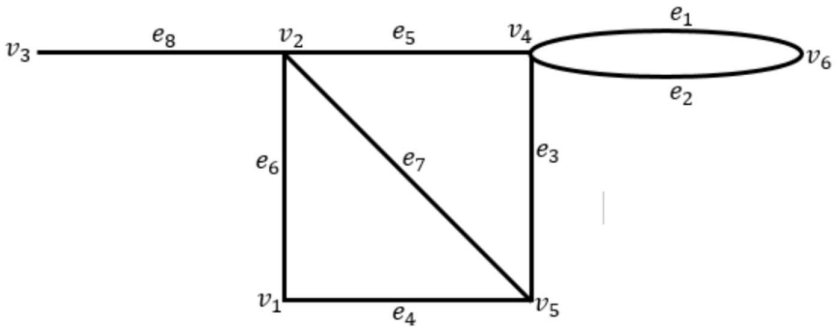
Then, we have

$$\sum_{i=1}^{k} n_i = n \text{ and } \sum_{i=1}^{k} m_i = m$$

Since each component G_i is connected, from the above theorem, we have $\text{rank}\big(B(G_i)\big) = m_i - n_i + 1$. Therefore,

$$rank\big(B(G)\big) = \sum_{i=1}^{k} rank\big(B(G_i)\big) = \sum_{i=1}^{k} m_i - n_i + 1 = m - n + k$$

Example 7: Find the cycle matrix of the given Graph G



Solution:

The Graph G has 4 different cycles

$z_1 = \{e_1, e_2\}$, $z_2 = \{e_3, e_5, e_7\}$, $z_3 = \{e_4, e_6, e_7\}$, $z_4 = \{e_3, e_4, e_6, e_5\}$

The cycle matrix is

$$B(G) = \begin{array}{c} \\ z_1 \\ z_2 \\ z_3 \\ z_4 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

13.6 PATH MATRIX

Let G be a graph with m edges, and u and v be any two vertices in G. The path matrix for vertices u and v denoted by $P(u,v) = \big[p_{ij}\big]_{q \times m}$ where q is the number of different paths between u and v, is defined as

$$p_{ij} = \begin{cases} 1, & \text{if } j\text{th edge lies in the } i\text{th path} \\ 0, & \text{otherwise} \end{cases}$$

Clearly, a path matrix is defined for a particular pair of vertices, the rows in $P(u,v)$ correspond to different paths between u and v, and the columns correspond to different edges in G.

The path matrix observations are given in the following:

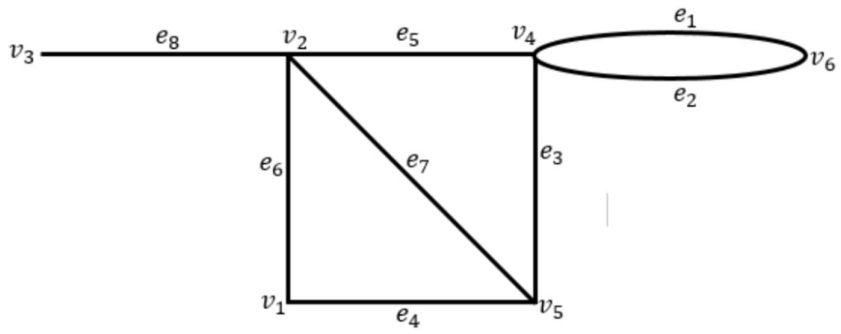
(i) A column of all zeros corresponds to an edge that does not lie in any path between u and v.

(ii) A column of all ones corresponds to an edge that lies in every path between u and v.

(iii) There is no row with all zeros.

(iv) The ring sum of any two rows in $P(u,v)$ corresponds to a cycle or an edge-disjoint union of cycles.

Example 8: Find the path matrix for vertices v_3 and v_4.



Solution:

The different path between the vertices v_3 and v_4

$$p_1 = \{e_8, e_5\}, \qquad p_2 = \{e_8, e_7, e_3\}, \qquad p_3 = \{e_8, e_6, e_4, e_3\}$$

The path matrix for v_3, v_4 is given by

$$P(v_3, v_4) = \begin{matrix} \\ \\ \\ \end{matrix} \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{matrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

13.7 CUT-SET MATRIX

Let the graph G have m edges and q cut sets. The cut set matrix $C = \{c_{ij}\}_{q \times m}$ of G is a (0,1) matrix of order $q \times m$, with

$$c_{ij} = \begin{cases} 1, & \text{if } i\text{th cut set contains } j\text{th edge} \\ 0, & \text{otherwise} \end{cases}$$

The circuit matrix observations include the following:

(i) The permutation of rows or columns in a cut set matrix corresponds simply to renaming of the cut sets and edges respectively.

(ii) Each row of $C(G)$ is a cut set vector.

(iii) A column with all zeros corresponds to an edge forming a self-loop.

(iv) Parallel edges from identical columns in the cut set matrix.

(v) In a non-separable graph, since every set of edges incident on a vertex is a cut-set, therefore every row of incidence matrix $A(G)$ is included as a row in the cut-set matrix $C(G)$. i.e. for a non-separable graph $G, C(G)$ contains $A(G)$. For a separable graph, the incidence matrix of each block is contained in the cut-set matrix.

(vi) It follows from observation (v), that rank $C(G) \geq$ rank $A(G)$. Therefore, for a connected graph with n vertices, rank $C(G) \geq n - 1$.

Theorem 7: If G is a connected graph, then the rank of a cut-set matrix $C(G)$ is equal to the rank of incidence matrix $A(G)$, which equals the rank of graph G.

Proof:

Let $A(G)$, $B(G)$ and $C(G)$ be the incidence, cycle and cut-set matrix of the connected graph G. Then we have

rank $C(G) \geq n - 1$ … (1)

Since the number of edges common to a cut-set and a cycle is always even, every row in C is orthogonal to every row in B, provided the edges in both B and C are arranged in the same order.

Thus, $BC^T = CB^T \equiv 0 (mod\ 2)$ … (2)

Now, applying Sylvester's theorem to equation (2), we have

rank B+ rank $C \leq m$

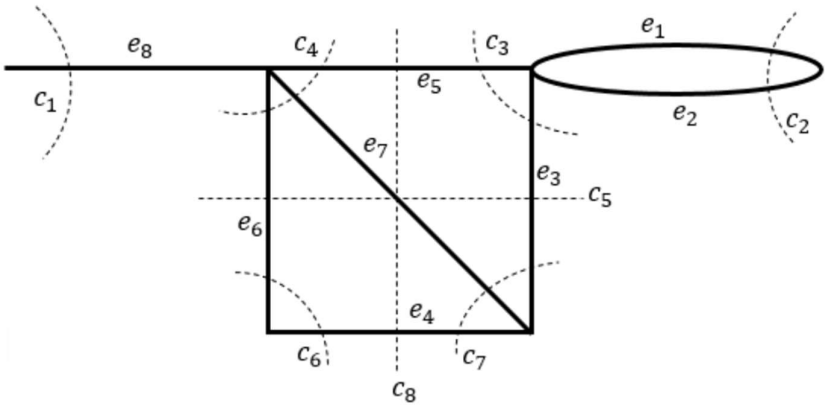
For a connected graph, we have rank $B = m - n + 1$

Therefore, rank $C \leq m$ –rank $B = m - (m - n + 1) = n - 1$

So, rank $C \leq n - 1$ … (3)

It follows from (1) and (3) that rank $C = n - 1$

Example 9: Find the cut-set matrix of the given graph G



Solution: In the graph G, $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$

The cut set are

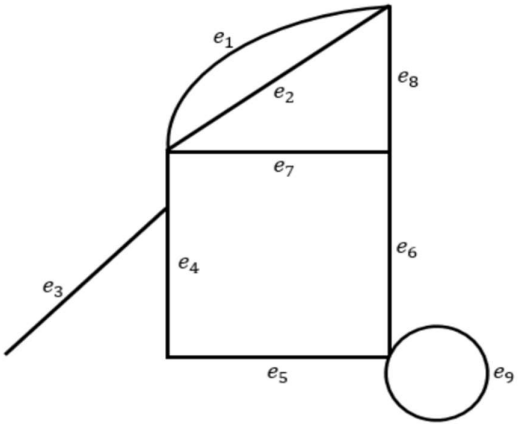
$$c_1 = \{e_8\}, \quad c_2 = \{e_1, e_2\}, \quad c_3 = \{e_3, e_5\}, \quad c_4 = \{e_5, e_6, e_7\},$$
$$c_5 = \{e_3, e_6, e_7\},$$
$$c_6 = \{e_4, e_6\}, \quad c_7 = \{e_3, e_4, e_7\}, \quad c_8 = \{e_4, e_5, e_7\}$$

The cut set matrix is given by

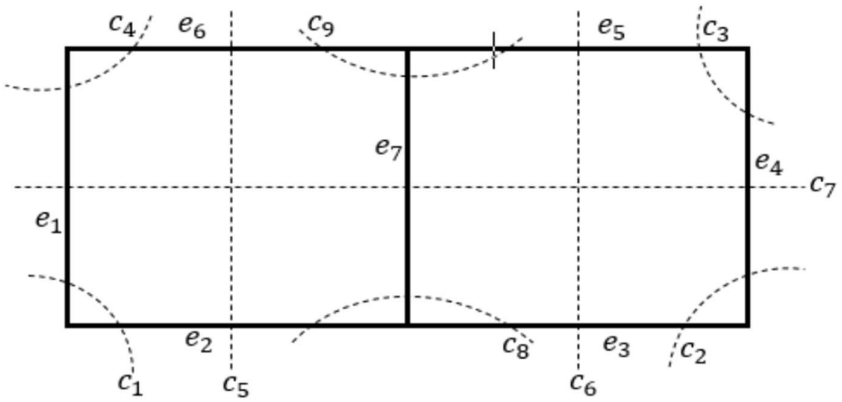
$$C(G) = \begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

W CHECK YOUR PROGRESS 2

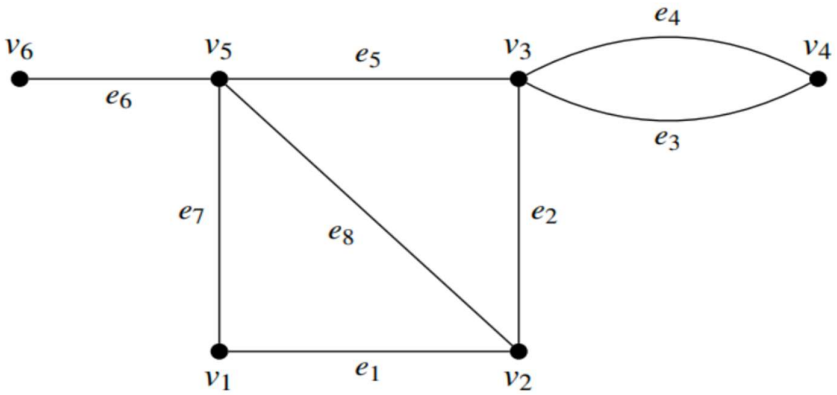
1. Find the Cycle matrix of the given graph G



2. Find the cut set matrix of the given graph G



3. Find the path matrix of the given graph G for vertices v_1 and v_3



13.8 LET US SUM UP

- The adjacency matrix represents the connections between vertices in a graph. A graph G with n vertices and without parallel edges has an adjacency matrix $A = \{a_{ij}\}$, which is a $n \times n$ matrix with

$$a_{ij} = \begin{cases} 1, & \text{if there is an edge between } i\text{th and } j\text{th vertices} \\ 0, & \text{if there is no edge between them} \end{cases}$$

- The incidence matrix represents the relationship between vertices and edges in a graph. The incidence matrix $B = \{b_{ij}\}$ is a $n \times m$ matrix, where

$$b_{ij} = \begin{cases} 1, & \text{when edge } e_j \text{ is incident with } v_i \\ 0, & \text{otherwise} \end{cases}$$

- The cycle matrix represents the cycles in a graph. A cycle is a closed path in a graph, and the cycle matrix indicates which edges are part of which cycles. The cycle matrix $B = \{b_{ij}\}_{q \times m}$ of G is a (0,1) matrix of order $q \times m$, where

$$b_{ij} = \begin{cases} 1, & \text{if the } i\text{th cycle includes } j\text{th edge} \\ 0, & \text{otherwise} \end{cases}$$

- The path matrix represents the paths between two vertices in a graph. A path is a sequence of vertices and edges that connects two vertices. The path matrix for vertices u and v denoted by $P(u, v) = [p_{ij}]_{q \times m}$ where q is the number of different paths between u and v, is defined as

$$p_{ij} = \begin{cases} 1, & \text{if } j\text{th edge lies in the } i\text{th path} \\ 0, & \text{otherwise} \end{cases}$$

- The cut-set matrix represents the cut-sets in a graph. A cut-set is a set of edges whose removal disconnects the graph. The cut set matrix $C = \{c_{ij}\}_{q \times m}$ of G is a (0,1) matrix of order $q \times m$, with

$$c_{ij} = \begin{cases} 1, & \text{if } i\text{th cut set contains } j\text{th edge} \\ 0, & \text{otherwise} \end{cases}$$

K 13.9 ANSWERS TO CHECK YOUR PROGRESS

Check your progress 1

Ans to Q 1. $\begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix}$

Ans to Q 2. $\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$

Ans to Q 3. (i) Not isomorphic (ii) Not isomorphic

Check your progress 2

Ans to Q 1.

$$B(G) = \begin{array}{c} \\ z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \end{array} \begin{array}{c} \begin{array}{ccccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Ans to Q 2.

$$C(G) = \begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \\ c_9 \end{array} \begin{array}{c} \begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

Ans to Q 3.

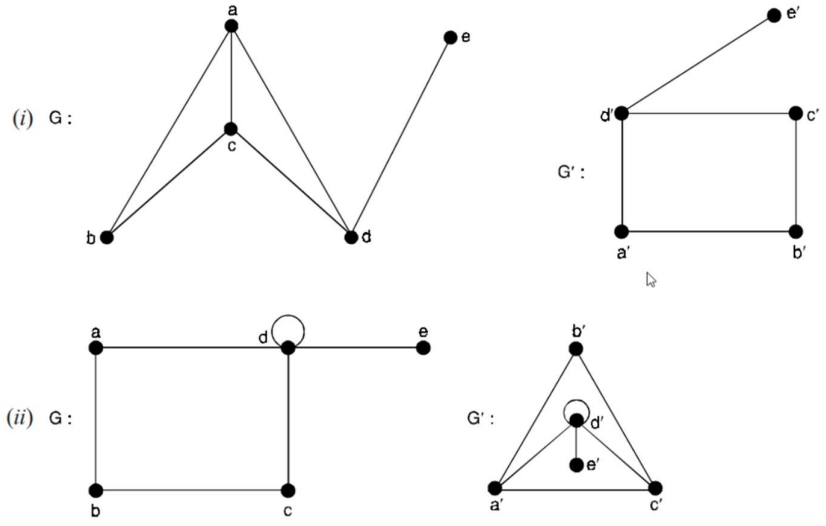
$$P(v_1, v_3) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

O 13.10 FURTHER READING

1. C. Vasudev, Graph Theory with Applications, New Age International (P) Limited, Publishers, 2006.
2. Sudev Naduvath, Lecture Notes on Graph Theory, Centre for Studies in Discrete Mathematics, 2017.
3. R. Balakrishnan and K. Renganathan, A textbook of Graph theory, Springer, 2000.
4. R. J. Wilson, Introduction to Graph Theory (5th Edition), Prentice Hall, 2010

M 13.11 MODEL QUESTIONS

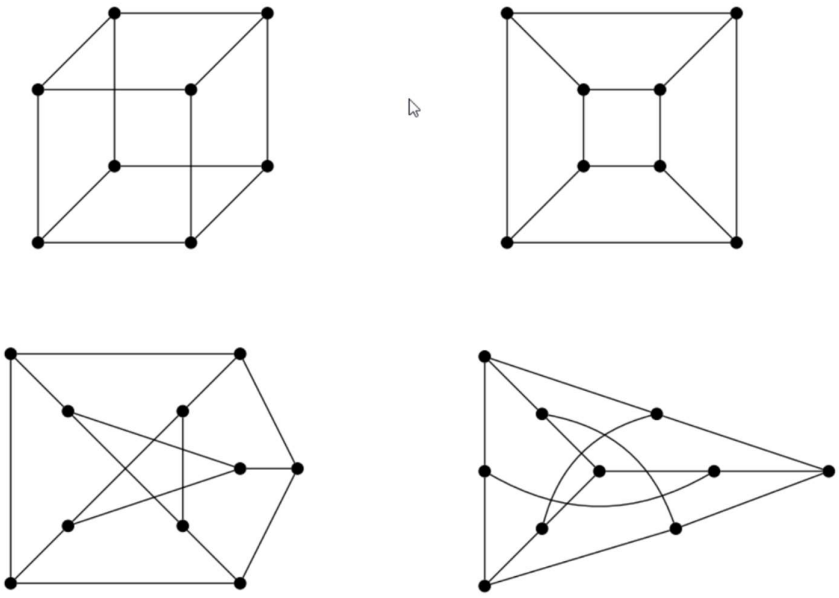
Q1. Show that the following graphs are isomorphic



Q2. Prove that two graphs are isomorphic if and only if their incidence matrices differ only by permutation of rows and columns.

Q3. Show that, if $A(G)$ is an incidence matrix of a connected graph G with n vertices, then rank of $A(G)$ is $n-1$.

Q4. Find the incidence matrix, cycle matrix, cut-set matrix and adjacency matrix for the following graphs.



Q5. Show that the rank of the cycle matrix B a connected graph G with n vertices, ε edges and k components is $\varepsilon - n + k$.

Q6. Prove that the rank of a cut-set matrix $C(G)$ of a connected graph G is equal to the rank of incidence matrix $A(G)$, which equals the rank of graph G.

*** *** ***