UNIT 14: DIGRAPHS

UNIT STRUCTURE

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14.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- understand the concept of a digraph, including its structure and basic terminology like vertices, edges, arcs, in-degree, and out-degree.
- identify and differentiate types of digraphs, such as simple, asymmetric, symmetric, isomorphic, balanced, regular, complete, and connected digraphs.
- apply the concept of directional duality and understand acyclic digraphs and their properties.
- understand matrices associated with digraphs, including incidence, circuit, and adjacency matrices, and how these represent and analyze digraphs.

14.2 INTRODUCTION

A digraph, or directed graph, is a mathematical structure consisting of a finite set of vertices and a collection of ordered pairs of distinct vertices called arcs or directed edges. Each arc represents a connection from one vertex to another, indicating directionality. This concept is fundamental in graph theory and has applications in various fields such as computer science, engineering, and social networks. Digraphs are used to model relationships where direction matters, such as in web links, flow of information, or one-way streets. The study of digraphs involves analyzing properties like in-degree, out-degree, types of digraphs, connectivity, and associated matrices, which together provide deep insights into their structural behavior.

14.3 DIAGRAPH

A digraph D consists of a finite set V of points and a collection of ordered pairs of distinct points. Any such pair (u, v) is referred to as a directed line or an arc, and is typically represented by the notation uv. The arc uv is incident with u and v and extends from u to v. When v is adjacent from u, we say that u is next to v.

In other words, A collection of vertices $V = \{v_1, v_2, \ldots\}$, a set of edges $E = \{e_1, e_2, \ldots\}$, and a mapping ψ that maps each edge onto an ordered pair of vertices (v_i, v_j) make up a directed graph, also known as a digraph G. For Example, a digraph with five vertices and ten edges is depicted in the fig. 14.1.

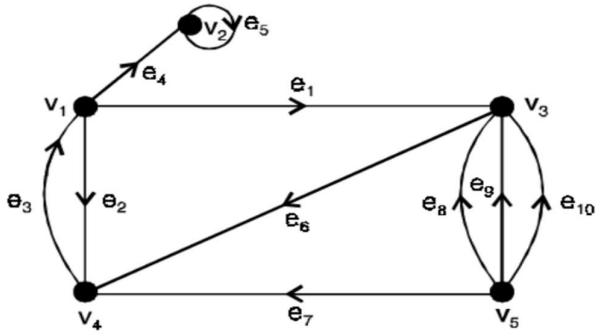


Fig. 14.1

In-Degree and Out-Degree

The out-degree of a vertex (v) in a digraph D is the number of edges that incident out of it, and the in-degree of a vertex (v) is the number of edges that incident into it. The symbols $d^+(v)$ and $d^-(v)$ represent the out-degree and in-degree of v, respectively.

14.3.1 Types of Diagraphs

Simple Digraphs: A digraphs that has no self-loop or parallel edges is called a simple digraph. Although the digraph in the fig. 14.2(a) is simple, the underlying graph in the fig. 14.2(b) is not simple.

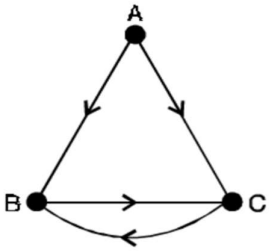


Fig 14.2(a)

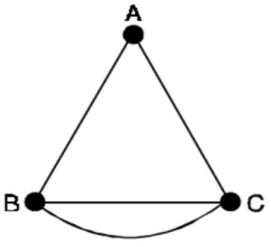


Fig 14.2(b)

Asymmetric Digraphs: Asymmetric or antisymmetric digraphs are those that allow for self-loops but only have one directed edge connecting a pair of vertices. For example, the digraph in the fig. 14.3(a), is asymmetric. The digraph in the fig. 14.3(b) is neither symmetric nor asymmetric. The digraph in the fig. 14.3(b) is simple and asymmetric. The digraph in the fig. 14.3(b) is simple but not asymmetric.

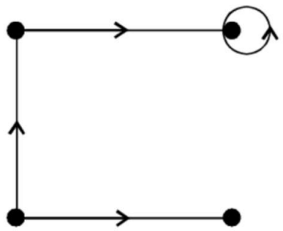


Fig 14.3(a)

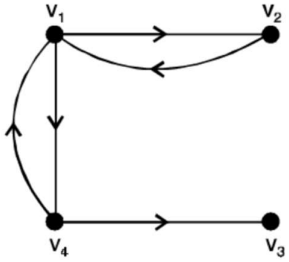


Fig 14.3(b)

Symmetric Digraph: Digraphs in which for every edge (a, b) (i.e., from vertex a to b) there is also an edge (b, a). For example, the digraph in the fig. 14.4 is a symmetric digraph. The digraph in the fig. 14.3(b) is not symmetric. This digraph has (v_4, v_3) as an edge but does not have (v_3, v_4) as an edge. The digraph in the fig. 14.4 is simple also. Such a digraph is called a symmetric simple digraph. The digraph in the fig. 14.3(b) is simple and non-symmetric.

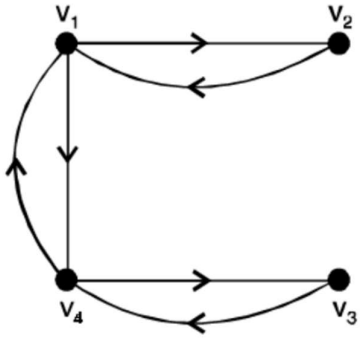


Fig. 14.4

Isomorphic Digraphs: In terms of graph properties, isomorphic graphs are defined as having the same behavior. In other words, two isomorphic graphs are identical if their labels are removed. Two digraphs must be isomorphic if the orientations of their respective edges agree in addition to the corresponding undirected graphs being isomorphic. For example, fig. 14.5, shows two digraphs

that are not isomorphic, although they are orientations of the same undirected graph.

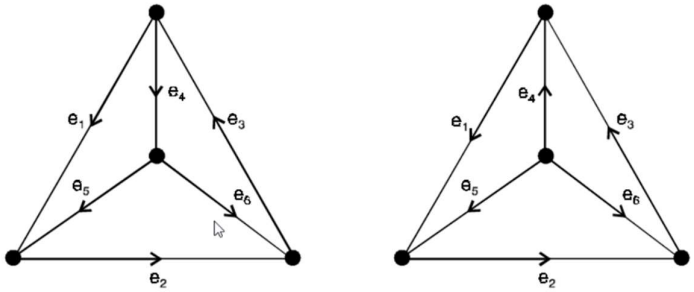


Fig. 14.5

Balanced Digraphs: A digraph D is said to be a balanced digraph or an isograph if $d^+(v) = d^-(v)$ for every vertex v of D.

Regular Digraph: A balanced digraph is said to be regular if every vertex has the same in-degree and out-degree as every other vertex.

Complete Digraphs: A complete undirected graph was defined as a simple graph in which every vertex is joined to every other vertex exactly by one edge.
Complete Symmetric Digraph: A complete symmetric digraph is a simple digraph in which there is exactly one edge directed from every vertex to every other vertex (see fig. 14.6).

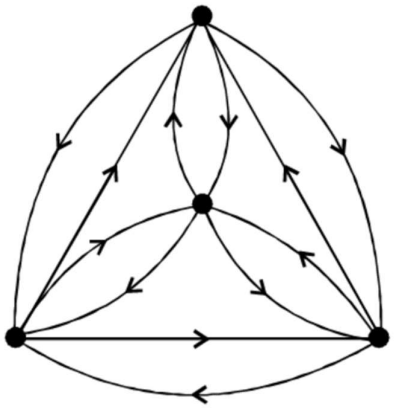


Fig. 14.6

Complete Asymmetric Digraph: A complete asymmetric digraph is an asymmetric digraph in which there is exactly one edge between every pair of vertices (see fig. 14.5).

14.3.2 Connected Digraphs

Strongly Connected: A digraph G is said to be strongly connected if there is at least one directed path from every vertex to every other vertex.

Weakly Connected: A digraph G is said to be weakly connected if its corresponding undirected graph is connected but G is not strongly connected. Fig. 14.5, one of the digraphs is strongly connected, and the other one is weakly connected.

Component and Fragments: Each maximal connected (weakly or strongly) subgraph of a digraph G is called a component of G. But within each component of G the maximal strongly connected subgraphs are called the fragments (or strongly connected fragments) of G. For example, the digraph in the fig. 14.7, consists of two components. The component g_1 contains three fragments $\{e_1, e_2\}, \{e_5, e_6, e_7, e_8\}$ and $\{e_{10}\}$. We observe that e_3, e_4 and e_9 do not appear in any fragment of g_1.

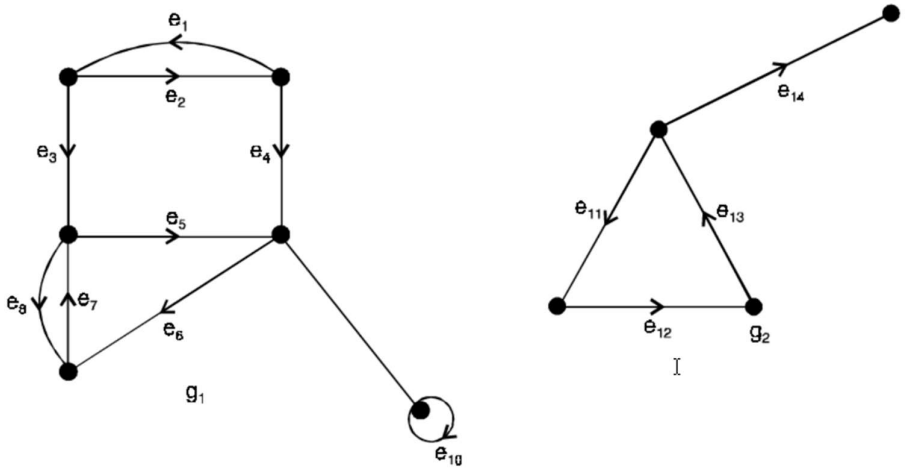


Fig. 14.7

Condensation

The condensation Gc of a digraph G is a digraph where all directed edges connecting strongly connected components are replaced by a single directed edge, and each strongly connected fragment is replaced by a vertex. For example, the condensation of the digraph G in Fig. 14.7 is shown in Fig. 14.8.

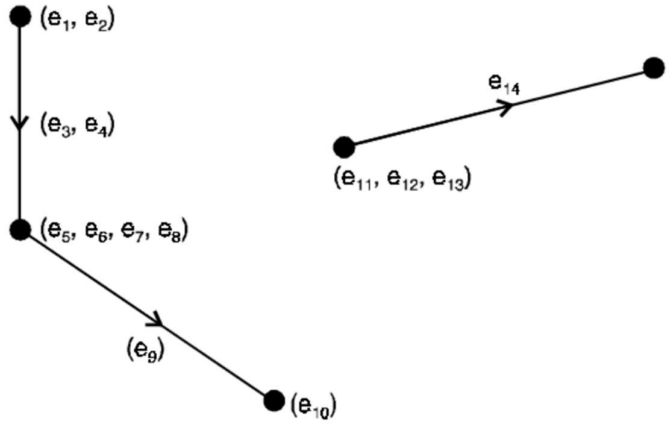


Fig. 14.8

Observations:

(i) The condensation of a strongly connected digraph is simply a vertex.

(ii) The condensation of a digraph has no directed circuit.

Reachability

A digraph D has two vertices, u and v. If there is at least one directed path in D connecting u and v, then v is reachable (or accessible) from u.

Orientable Graph

A graph G is said to be orientable if there exists a strongly connected digraph D for which G is the underlying graph. For example, the graph is fig. 14.9(a) is orientable, a strongly directed digraph for which this graph is the underlying graph is illustrated in fig. 14.9(b).

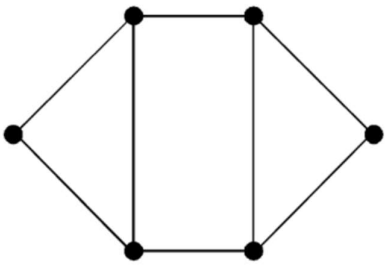


Fig 14.9(a)

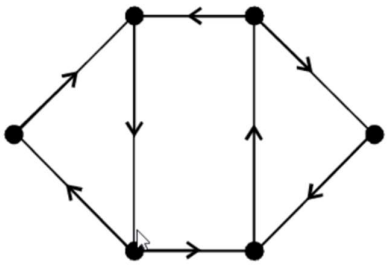


Fig 14.9(b)

Euler Digraphs

In a digraph G a closed directed walk (i.e., a directed walk that starts and ends at the same vertex) which transverses every edge of G exactly once is called a directed Euler line. A graph containing a directed Euler line is termed an Euler digraph. For example, the graph in the fig. 14.10, is an Euler digraph, in which the walk $a\ b\ c\ d\ e\ f$ is an Euler line.

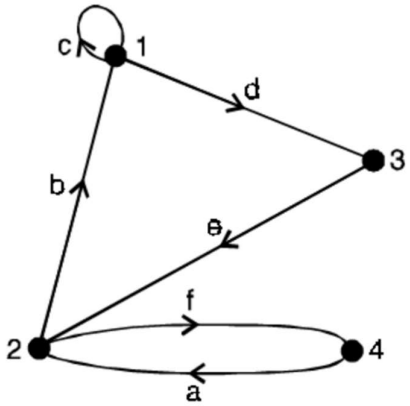


Fig. 14.10

Hand Shaking Dilemma

In a digraph D, the sum of the out-degree of all vertices is equal to the sum of the in-degrees of all vertices, each sum being equal to the number of edges in D.

Directed walk

A directed walk or a directed trail in D is a finite sequence whose terms are alternately vertices and edges in D such that each edge is incident out of the vertex preceding it in the sequence and incident into the vertex following it. In a directed walk, a vertex can occur more than once, but not an edge.

Directed path

A directed path is an open directed walk that does not repeat any vertex.

Directed circuit

A directed circuit, also known as a directed cycle, is a closed directed walk in which no vertices, except the initial and final vertices are repeated.

Length

The number of edges present in a directed walk, directed path, directed circuit is called its length.

Semi-Walk

A walk on the underlying graph of a digraph D is called a semi-walk; however, it is not a directed walk in D. A walk in D can mean either a directed walk or a semi-walk in D.

Semi-path

A path in the underlying graph of a digraph D that is not a directed path in D is called a semi-path. A path in D can be interpreted as either a semi-path or a directed path.

Semi-circuit

A circuit in the underlying graph of a digraph D is called a semi-circuit, although it is not a directed circuit in D itself. A circuit in D may refer to a semi-circuit or a directed circuit.

Tournaments

An orientated complete graph is a tournament. Fig. 14.11 displays all tournaments with two, three, and four points. It is referred to as a transitive triple for the first three points and a cycle triple for the second.

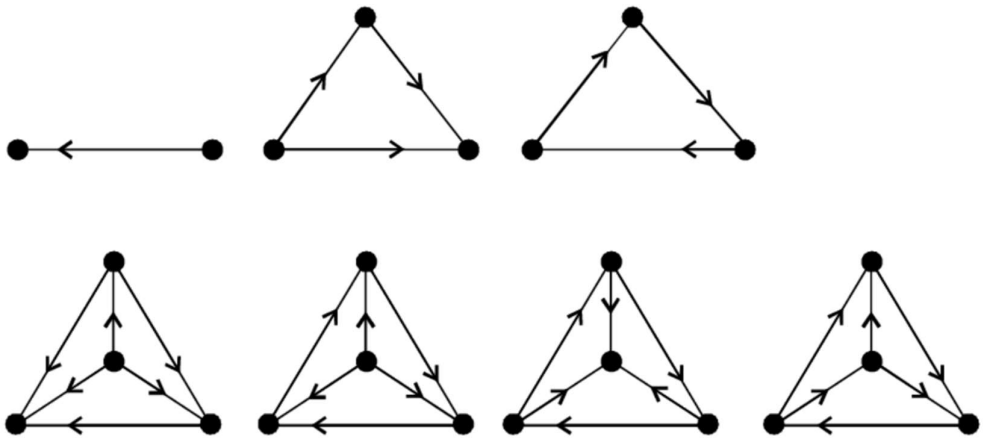


Fig. 14.11

Theorem 1: A digraph is strong if and only if it has a spanning closed walk, it is unilateral if and only if it has a spanning walk, and it is weak if and only if it has a spanning semi-walk.

Example 1: Let D be a digraph with an odd number of vertices prove that if each vertex of D has an odd out-degree then D has an odd number of vertices with odd in-degree.

Solution:

Let $v_1, v_2, \ldots v_n$ be the n vertices of D, where n is odd. Also let m be the number of edges in D.

They by handshaking dilemma

$$d^+(v_1) + d^+(v_2) + \ldots + d^+(v_n) = m \quad \ldots \quad (1)$$

$$d^-(v_1) + d^-(v_2) + \ldots + d^-(v_n) = m \quad \ldots \quad (2)$$

If each vertex v_i has odd out-degree, then the left-hand side of (1) is a sum of n odd numbers. Since n is odd, this sum must also be odd. Thus m is odd.

Let k be the number of vertices with odd in-degree. Then $n - k$ number of vertices have even in-degree. Without loss of generality, let us take $v_1, v_2, \ldots v_k$ to be the vertices with odd in-degree and $v_{k+1}, v_{k+2}, \ldots v_n$ to be the vertices with even in-degree.

Then, (2) may be rewritten as

$$\sum_{i=1}^{k} d^-(v_i) + \sum_{i=k+1}^{n} d^-(v_i) = m \quad \ldots (3)$$

Now the second sum on the left-hand side of this expression is even. Also, m is odd. Therefore, the first sum must be odd. That is,

$$d^-(v_1) + d^-(v_2) + \ldots + d^-(v_k) = \text{odd} \quad \ldots \quad (4)$$

But, each of $d^-(v_1), d^-(v_2), \ldots, d^-(v_k)$ is odd.

Therefore, the number of terms in the left-hand side of (4) must be odd, that is; k is odd.

Example 2: Prove that a complete symmetric digraph of n vertices contain $n(n-1)$ edges and a complete asymmetric digraph of n vertices contains $\frac{n(n-1)}{2}$ edges.

Solution:

In a complete asymmetric digraph, there is exactly one edge between every pair of

vertices. Therefore, the number of edges in such a digraph is precisely equal to the number of pairs of vertices. The number of pairs of vertices that can be chosen from n vertices is ${}^{n}C_2 = \frac{1}{2}n(n-1)$.

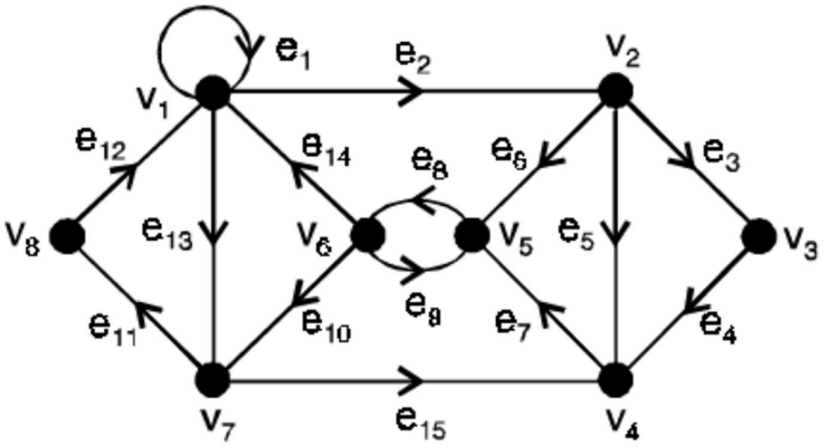
Thus, a complete asymmetric digraph with n vertices has exactly $\frac{1}{2}n(n-1)$ edges.

In a complete symmetric digraph, there exist two edges with opposite directions between every pair of vertices.

Therefore, the number of edges in such a digraph with n vertices is

$2 \times \frac{1}{2}n(n-1) = n(n-1)$

Example 3: Find the sequence of vertices and edges of the longest walk in the digraph shown in the below fig.:



Solution:

We check that in the given digraph, for each vertex, the in-degree is equal to the out

degree. Therefore, the digraph is an Euler digraph. The longest walk in the digraph is a directed Euler line, a directed walk which includes all the edges of the digraph.

The digraph reveals that the directed Euler line is shown below:

$V_1\ e_1\ V_1\ e_{13}\ V_7\ e_{11}\ V_8\ e_{12} V_1\ e_2\ V_2\ e_3\ V_3\ e_4\ V_4\ e_5\ V_2\ e_6\ V_5\ e_8\ V_6\ e_{10}\ V_7\ e_{15}\ V_4\ e_7\ V_5 e_9\ V_6\ e_{14}\ V_1$

This is the required sequence.

Example 4: Prove that a connected digraph D that does not contain a closed directed walk must have a source and a sink.

Solution:

Consider a directed walk q in D, which contains a maximum number of vertices.

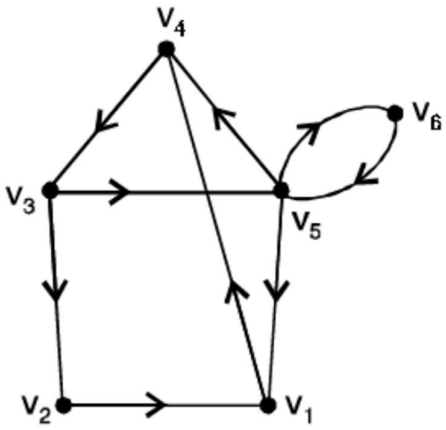
Let u be the initial vertex of q and v be the terminal vertex. Suppose v is not a sink. Then there must be an edge that begins at v. Since D has no closed walk, this edge cannot end at u.

Hence it must end at some vertex v'.

Consequently, there is created a directed walk q' that contains all vertices of q and v'.

This contradicts the maximality of q. Hence v has to be a sink. Similarly, u has to be a source. Thus, D contains at least one sink and at least one source.

Example 5: Show that the digraph given below is strongly connected.



Solution:

A digraph D is strongly connected if every vertex of D is reachable from every other

vertex of D.

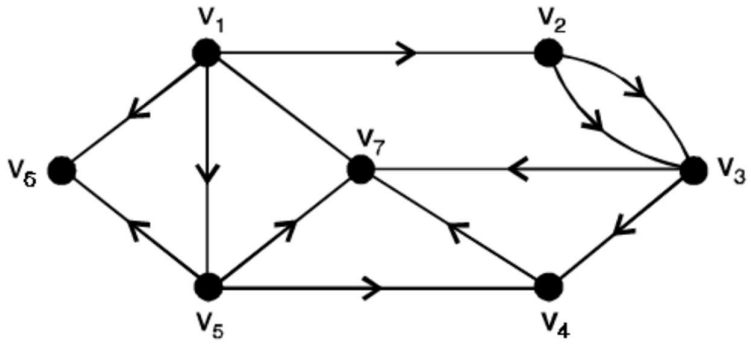
Let us verify whether this condition holds for the given digraph.

The following table indicates the directed paths from each of the vertices V_1, V_2, V_3, V_4, V_5, V_6 to every other vertex. The existence of these directed paths proves that the given digraph is strongly connected.

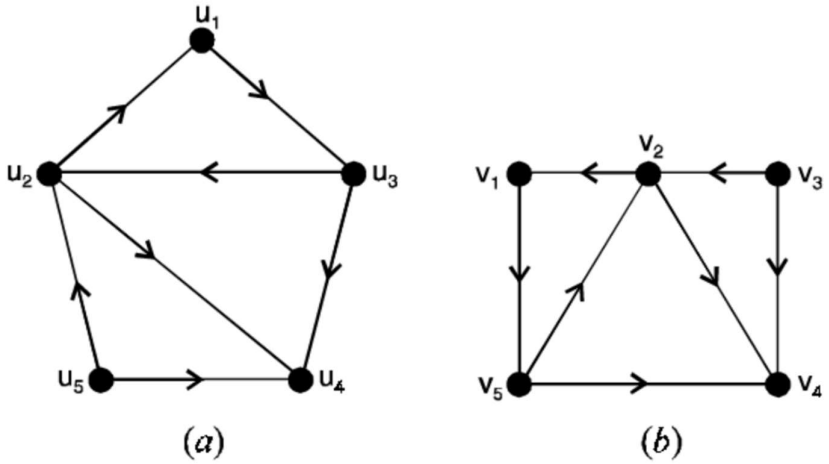
	V_1	V_2	V_3	V_4	V_5	V_6
V_1	–	$V_1V_4V_3V_2$	$V_1V_4V_3$	V_1V_4	$V_1V_4V_3V_5$	$V_1V_4V_3V_5V_6$
V_2	V_2V_1	–	$V_2V_1V_4V_3$	$V_2V_1V_4$	$V_2V_1V_4V_3V_5$	$V_2V_1V_4V_3V_5V_6$
V_3	$V_3V_2V_1$	V_3V_2	–	$V_3V_5V_4$	V_3V_5	$V_3V_5V_6$
V_4	$V_4V_3V_2V_1$	$V_4V_3V_2$	V_4V_3	–	$V_4V_3V_5$	$V_4V_3V_5V_6$
V_5	V_5V_1	$V_5V_4V_3V_2$	$V_5V_4V_3$	V_5V_4	–	V_5V_6
V_6	$V_6V_5V_1$	$V_6V_5V_4V_3V_2$	$V_6V_5V_4V_3$	$V_6V_5V_4$	V_6V_5	–

W CHECK YOUR PROGRESS 1

Q1. Find the in-degrees and out-degrees of the vertices of the digraphs shown in the fig. below. Also, verify the handshaking dilemma.



Q2. Verify that the following two digraphs are isomorphic



Q3. Let D be a connected simple digraph with n vertices and m edges. Prove that

$$n - 1 \leq m \leq n(n - 1)$$

14.4 DIRECTIONAL DUALITY AND ACYCLIC DIGRAPH

The converse digraph D' of D has the same points as D and the arc uv is in D' if and only if the arc uv is in D. Thus, the converse of D is obtained by reversing the direction of every arc of D. We have already seen other converse ideas, such as indegree and outdegree, and these concepts associated with direction are linked by a fairly powerful principle. In the theory of binary relations, this is a traditional outcome.

Principle of Directional Duality

Every theorem concerning digraphs has a corresponding theorem that can be found by substituting its converse for each concept. We now demonstrate how this idea leads to novel outcomes. There are no directed cycles in an acyclic digraph.

Theorem 2: An acyclic digraph has at least one point of outdegree zero.

Proof:

Consider the last point of any maximal path in the digraph. This point can have no points adjacent from it since otherwise there would be a cycle or the path would not be maximal.

The dual theorem follows immediately by applying the principle of Directional Duality. In keeping with the use of D' to denote the converse of digraph D, we shall use primes to denote dual results.

Theorem 3: An acyclic digraph D has at least one point of indegree zero.

Theorem 4: The following properties of a digraph D are equivalent

(i) D is acyclic.

(ii) $D*$ is isomorphic to D.

(iii) Every walk of D is a path.

(iv) It is possible to order the points of D so that the adjacency matrix $A(D)$ is upper triangular.

Theorem 5: A weak digraph is an out-tree if and only if exactly one point has indegree 0 and all others have indegree 1.

Theorem 6: A weak digraph is an in-tree if and only if exactly one point has outdegree 0 and all others have outdegree 1.

Theorem 7: The following are equivalent for a weak digraph D

(i) D is functional.

(ii) D has exactly one cycle, the removal of whose arcs results in a digraph in which each weak component is an in-tree.

(iii) D has exactly one cycle Z, and the removal of any arcs of Z results in an in-tree.

Theorem 8: Every acyclic digraph has a unique point basis consisting of all points of indegree 0.

Corollary: Every point basis of a digraph D consists of exactly one point from each of those strong components in D which from the point basis of D^*.

Theorem 9: Every digraph with no odd cycles has a 1-basis.

Corollary: Every acyclic digraph has a 1-basis.

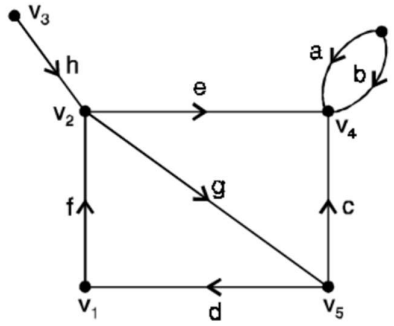
14.5 MATRICES ASSOCIATED WITH DIGRAPHS

14.5.1 Incidence Matrix of a Digraph

The incidence matrix $A = \{a_{ij}\}$ of digraph D with n vertices and e edges and no self-loops in an $n \times n$ matrix whose rows correspond to vertices and columns correspond to edges such that

$$a_{ij} = \begin{cases} 1, & \text{if } j^{th} \text{ edge is incident out of } i^{th} \text{ vertex} \\ -1, & \text{if } j^{th} \text{ edge is incident into } i^{th} \text{ vertex} \\ 0, & \text{if } j^{th} \text{ edge is not incident on } i^{th} \text{ vertex} \end{cases}$$

For example, A digraph and its incidence matrix are shown in the fig. 14.12



$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array}\begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Fig. 14.12

14.5.2 Circuit Matrix of a Digraph

Let G be a digraph with e edges and q circuits. An arbitrary orientation is assigned to each of the q circuits. Then a circuit matrix $B = \{b_{ij}\}$ of the digraph G is a q by e matrix defined as

$$b_{ij} = \begin{cases} 1, & \text{if } i^{th} \text{circuit includes } j^{th} \text{ edge, and the orientations of the edge and circuit coincide} \\ -1, & \text{if } i^{th} \text{circuit includes } j^{th} \text{ edge, but the orientations of the two are opposite} \\ 0, & \text{if } i^{th} \text{ circuit does not include the } j^{th} \text{ edge} \end{cases}$$

For example, a circuit matrix of the digraph in fig. 14.12 is

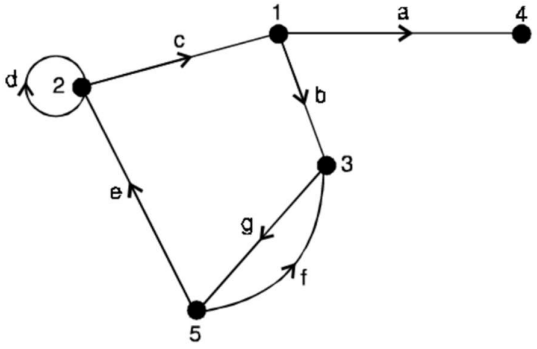
$$\begin{array}{c} \begin{array}{ccccccccc} a & b & c & & d & e & & f & g & h \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & & 1 & 0 & & 1 & 1 & 0 \\ 0 & 0 & 1 & & 0 & -1 & & 0 & 1 & 0 \\ 0 & 0 & 1 & & -1 & -1 & & -1 & 0 & 0 \\ -1 & 1 & 0 & & 0 & 0 & & 0 & 0 & 0 \end{bmatrix} \end{array}$$

14.5.3 Adjacency Matrix of a Digraph

Let G be a digraph with n vertices and without parallel edges has an adjacency matrix $X = \{x_{ij}\}$, which is a $n \times n$ matrix with elements provided by

$$x_{ij} = \begin{cases} 1, & \text{if there is an edge directed from } i^{th} \text{ vertex to } j^{th} \text{ vertex} \\ 0, & \text{otherwise} \end{cases}$$

For example, a digraph and its adjacency matrix are shown in the below fig.



$$X = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Observations:

(i) X is a symmetric matrix if and only if G is a symmetric digraph.

(ii) Every non-zero element on the main diagonal represents a self-loop at the corresponding vertex.

(iii) Parallel edges in X cannot be displayed. The adjacency matrix is therefore only defined for digraphs with non-parallel edges.

(iv) The out-degree of the corresponding vertex is equal to the sum of the rows, and the in-degree of the corresponding vertex is equal to the sum of the columns. The number of edges in G is equal to the number of non-zero entries in X.

(v) If the adjacency matrix of a digraph G is X, then the adjacency matrix of a digraph G^R that results from reversing the direction of each edge in G is the transposed matrix X^T.

(vi) There is a unique digraph G of n vertices such that Q is the adjacency matrix of G for any square $(0,1)$-matrix Q of order n.

Theorem 10: Let G be a connected graph. Then G is orientable if and only if each edge of G is contained in at least one cycle.

Proof:

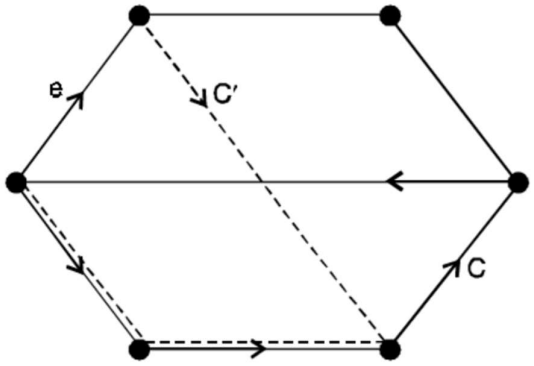
The necessity of the condition is clear. We have to prove sufficiency.

We select any cycle C and apply cyclic directionality to its edges.

The proof is finished if every edge of G is contained in C. If not, we select any edge e that is next to an edge of C but is not in C.

According to our hypothesis, e is a part of a cycle C', the edges of which we can direct cyclically, with the exception of those edges that have already been directed, or those of C' that are also in C.

The situation is shown in the fig. below, where dashed lines indicate the edges of C', making it easy to see that the resulting digraph is strongly connected.



In this manner, we continue until all edges are directed, directing at least one new edge at a time. The result follows because the digraph remains strongly connected at every level.

Theorem 11: A connected digraph is Eulerian if and only if for each vertex of D out deg$(v) =$ in deg(v).

Theorem 12: Let D be a strongly connected digraph with n vertices. If out deg $(v) \geq \frac{n}{2}$ and in deg $(v) \geq \frac{n}{2}$ for each vertex v, then D is Hamiltonian.

Theorem 13: If $A(G)$ is the incidence matrix of a connected digraph of n vertices, the rank of $A(G) = n - 1$.

Theorem 14: The $(i, j)^{th}$ entry in X^r equals the number of different directed edge sequences of r edges from the i^{th} vertex to the j^{th}.

Proof. (By induction)

The theorem is trivially true for $r = 1$.

As the inductive hypothesis, assume that the theorem holds for X^{r-1}. The $(i, j)^{th}$ entry in

$$X^r (= X^{r-1}.X) = \sum_{k=1}^{n} [(i,\ k)^{th} \text{ entry in } X^{r-1}].x_{kj}$$

$$= \sum_{k=1}^{n} (\text{number of all directed edge sequences of length } r - 1 \text{ from vertex } i \text{ to } k).\, x_{kj} \qquad (1)$$

according to the induction hypothesis. In (1), $x_{kj} = 1$ or 0 depending on whether or not there is a directed edge from k to j. Thus, a term in the sum (1) is non zero if and only if there is a directed edge sequence of length r from i to j, whose last edge is from k to j.

If the term is non zero, its value equals the number of such edge sequences from i to j via k. This holds for every $k, 1 \leq k \leq n$. Therefore (1) is equal to the number of all possible directed edge sequence from i to j.

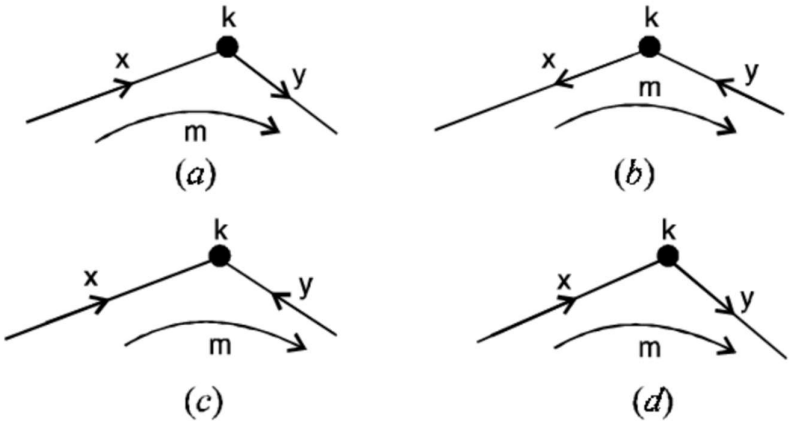
Theorem 15: Let B and A be respectively, the circuit matrix and incidence matrix of a self-loop-free digraph such that the columns in B and A are arranged using the same order of edges. Then

$$A.B^T = B.A^T = 0.$$

Where superscript T denotes the transposed Matrix.

Proof: Consider the m^{th} row in B and the kth row in A. If the circuit m does not include any edge incident on vertex k, the product of the two rows is clearly zero. If, on the other hand, vertex k is in circuit m, there are exactly two edges (say x and y) incident on k that are also in circuit m.

This situation can occur in only four different ways, as shown in the fig. below.



The possible entries in row k of A and row m of B in column positions x and y are tabulated for each of these four cases.

Case	Row k		Row m		Dot product
	column x	column y	column x	column y	Row k. Row m
(i)	-1	1	1	1	0
(ii)	1	-1	-1	-1	0
(iii)	-1	-1	1	-1	0
(iv)	1	1	-1	1	0

In each case, the dot product is zero. Therefore, the theorem.

Theorem 16: The i,j entry $a_{ij}^{(n)}$ of A^n is the number of walks of length n from v_i to v_j.

Corollary: The entries of the reachability and distance matrices can be obtained from the powers of A as follows:

(i) for all i, $r_{ii} = 1$ and $d_{ii} = 0$

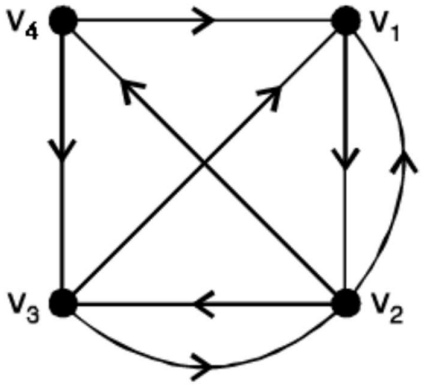
(ii) $r_{ij} = 1$ if and only if for some n, $a_{ij}^{(n)} > 0$

(iii) $d(v_i, v_j)$ is the least n(if any) such that $a_{ij}^{(n)} > 0$, and is ∞ otherwise.

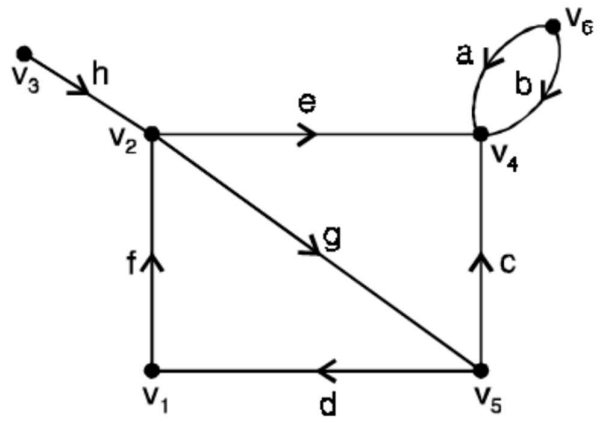
Corollary: Let v_i be a point of a digraph D. The strong component of D containing v_i is determined by the entries of 1 in the *i*th row (or column) of the matrix $R \times R^T$

W CHECK YOUR PROGRESS 2

Q1. Find the adjacency matrix for the digraph shown in the fig. below



Q2. For the digraph in the fig. below, find the incidence matrix:



214.6 LET US SUM UP

- **Digraph Definition:** A digraph is composed of a set of vertices and directed edges (arcs) that connect ordered pairs of vertices.
- **Degrees of Vertices:**
  - *Out-degree:* Number of edges leaving a vertex.
  - *In-degree:* Number of edges entering a vertex.
  - **Types of Digraphs:**
  - *Simple Digraph:* No self-loops or parallel edges.
  - *Asymmetric Digraph:* One-way edges only.
  - *Symmetric Digraph:* If there is an edge from A to B, then there is also one from B to A.
  - *Isomorphic Digraphs:* Structurally identical despite different labeling.
  - *Balanced and Regular Digraphs:* Equal in-degree and out-degree.
  - *Complete Digraphs:* Every vertex is connected to every other vertex.
  - *Tournaments:* Oriented complete graphs.
- **Connectivity:**
- *Strongly Connected:* Path exists from every vertex to every other vertex.
- *Weakly Connected:* The underlying undirected graph is connected, but not all paths are directed.
- *Components and Fragments:* Subsections based on connectivity.
- **Special Concepts:**

- *Condensation:* Collapsing strongly connected components into single vertices.
- *Reachability:* One vertex can be reached from another via a directed path.
- *Euler Digraph:* A digraph with a closed walk that includes every edge exactly once.
- *Handshaking Dilemma:* Sum of in-degrees equals sum of out-degrees, both equal to number of edges.

- **Walks and Paths:**
- *Directed Walk:* Sequence of edges with directional continuity.
- *Directed Path:* Directed walk with no repeated vertices.
- *Directed Circuit:* Closed directed walk with distinct vertices except the start/end.
- *Semi-walks, Semi-paths, Semi-circuits:* In the underlying undirected graph.
- **Directional Duality:**
  - Every theorem has a dual counterpart by reversing arc directions.
  - Acyclic Digraphs have no cycles and possess specific properties like points with in- or out-degree zero.
- **Matrices:**
  - *Incidence Matrix:* Represents vertex-edge incidence with direction.
  - *Circuit Matrix:* Represents circuits using assigned orientations.
  - *Adjacency Matrix:* Represents presence and direction of edges between vertices.

M 14.7 ANSWERS TO CHECK YOUR PROGRESS

Check your progress 1

Ans to Q 1. The given digraph has 7 vertices and 12 edges. The out-degree of a vertex is got by counting the number of edges that go out of the vertex and the in-degree of a vertex is got by counting the number of edges that end at the vertex. Thus, we obtain the following data.

Vertex	Out-degree	In-degree
V_1	4	0
V_2	2	1

V_3	2	2
V_4	1	2
V_5	3	1
V_6	0	2
V_7	0	4

This table gives the out-degrees and in-degrees of all vertices. We note that v_1 is a source and v_6 and v_7 are sinks.

Also, check that

sum of out-degrees $=$ sum of in-degrees $= 12 =$ No. of edges.

Ans to Q 2. Let us consider the following one-to-one correspondence between the directed edges in the two digraphs:

$$(u_2, u_1) \leftrightarrow (v_2, v_1), \quad (u_2, u_4) \leftrightarrow (v_2, v_4), \quad (u_3, u_4) \leftrightarrow (v_5, v_4),$$

$$(u_1, u_3) \leftrightarrow (v_1, v_5), \quad (u_5, u_2) \leftrightarrow (v_3, v_2), \quad (u_3, u_2) \leftrightarrow (v_5, v_2),$$

$$(u_5, u_4) \leftrightarrow (v_3, v_4)$$

These yield the following one-to-one correspondence between the vertices in the two digraphs:

$$u_1 \leftrightarrow v_1, \quad u_2 \leftrightarrow v_2, \quad u_3 \leftrightarrow v_5, \quad u_4 \leftrightarrow v_4, \quad u_5 \leftrightarrow v_3$$

The above-mentioned one-to-one correspondences between the vertices and the directed edges establish the isomorphism between the given digraphs.

Ans to Q 3.

Since D is connected, its underlying graph G is connected. Therefore, $m \geq n - 1$.

In a simple digraph, there exists at most two edges in opposite directions between every pair of vertices.

Therefore, the number of edges in such a digraph cannot exceed $2 \times {}^nC_2 = n(n-1)$.

i.e., $m \leq n(n-1)$. Thus $n - 1 \leq m \leq n(n-1)$.

Check your progress 2

Ans to Q 1. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

Ans to Q 2. $\begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

M 14.8 FURTHER READING

1. C. Vasudev, Graph Theory with Applications, New Age International (P) Limited, Publishers, 2006.
2. F. Harary, Graph theory, Narosa Publishing House, New Delhi, 1988.
3. R. Balakrishnan and K. Renganathan, A textbook of Graph theory, Springer, 2000.
4. R. J. Wilson, Introduction to Graph Theory (5th Edition), Prentice Hall, 2010

M 14.9 MODEL QUESTIONS

Q1. Show that an acyclic digraph has at least one point of outdegree zero.

Q2. Consider the digraph D shown in the fig. 14.13. In this digraph find

(i) a directed walk of length 8. Is this walk a directed path?

(ii) a directed trial of length 10

(iii) a directed trial of longest length

(iv) a directed path of longest length.

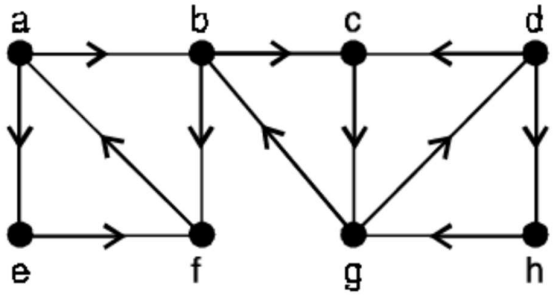


Fig 14.13

Q3. Show that the digraph shown in the fig 14.14., is an Euler digraph. Indicate a directed Euler line in it.

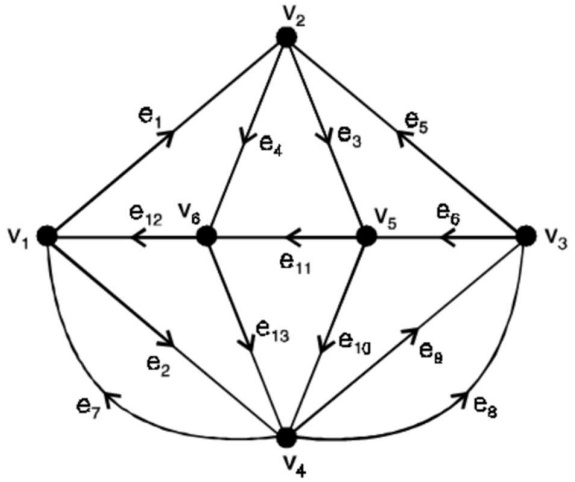


Fig 14.14

Q4. Prove that the connected graph G is orientable if and only if each edge of G is contained in at least one cycle.

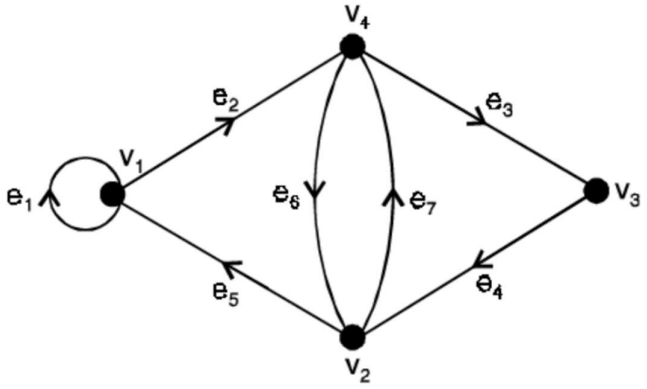
5. Show that the $(i, j)^{th}$ entry in X^r equals the number of different directed edge sequences of r edges from the i^{th} vertex to the j^{th}.

Q6. Let B and A be respectively, the circuit matrix and incidence matrix of a self-loop-free digraph such that the columns in B and A are arranged using the same order of edges. Then prove that

$$A.B^T = B.A^T = 0.$$

Where superscript T denotes the transposed Matrix

Q7. Show that the following is an Euler digraph. Find a directed Euler line in it.



*** *** ***